Non Uniformly Hyperbolic Dynamics: Hénon Maps and Related Dynamical Systems

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Abstract

In the 1960s and 1970s a large part of the theory of dynamical systems concerned the case of uniformly hyperbolic or Axiom A dynamical system and abstract ergodic theory of smooth dynamical systems. However since around 1980 an emphasize has been on concrete examples of one-dimensional dynamical systems with abundance of chaotic behavior (Collet & Eckmann and Jakobson). New proofs of Jakobson’s one-dimensional results were given by Benedicks and Carleson [5] and were considerably extended to apply to the case of Hénon maps by the same authors [6]. Since then there has been a considerable development of these techniques and the methods have been extended to the ergodic theory and also to other dynamical systems (work by Viana, Young, Benedicks and many others). In the cases when it applies one can now say that this theory is now almost as complete as the Axiom A theory.

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1. Introduction

Dynamical systems as a discipline was born in Henri Poincaré’s famous treatise of the three body problem. In retrospect arguably one can view as his most remarkable discovery of the homoclinic phenomenon. Stable and unstable manifolds of a fixed point or periodic point may intersect at a homoclinic point thereby producing a very complicated dynamic behavior—what we now often call chaotic.

In my opinion in the development of theory of dynamical systems—like in the development of all good mathematics—one can clearly see two stages. The first

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stage is the understanding of concrete examples and the second stage is generalization. A large part of the second encompasses the introduction of the right concepts which makes the arguments in the concrete examples into a theory.

One such development starts with the two famous papers by M. L. Cartwright and J. Littlewood “On non-linear differential equations of the second order I and II”, which was among the first to treat nonlinear differential equations in depth. Littlewood was astonished by the difficulties that arose in studying these model problems and called the second of these papers “the monster”.

S. Smale gives in “Finding a horseshoe on the beaches of Rio” an entertaining account of how he was lead to his now ubiquitous horseshoe model for chaotic dynamics. In fact in his first paper in dynamical systems he made the conjecture that “chaotic dynamics does not exist” but received a letter from N. Levinson with a paper clarifying the previous work by Cartwright and Littlewood and which effectively contained a counterexample to Smale’s conjecture. Levinson’s paper contained extensive calculations which Smale found difficult follow and this lead him to construct a model with minimal complexity but still with the main features of Levinson’s ODE model.

Starting from the horseshoe model, Smale and his group at Berkeley started to develop the theory of uniformly hyperbolic or Axiom A dynamical systems in the seventies. One central concept in this theory is that of an axiom A attractor for a diffeomorphism $f$ of a manifold $M$.

Let $A$ be an invariant set for a diffeomorphism of a manifold $M$. $A$ is said to be a hyperbolic set for $f$ if there is a continuous splitting of the tangent bundle of $M$ restricted to $A$, $TM|_A$, which is invariant under the derivative map $Df$: $TM|_A = E^s \oplus E^u$; $Df(E^s) = E^s$; $Df(E^u) = E^u$; and for which there are constants $C > 0$ and $c > 0$, such that $||Df^n|| \leq Ce^{-cn}$ and $||Df^n|| \leq Ce^{-cn}$, for all $n \geq 0$ and there is a uniform lower bound for the angle between stable and unstable manifolds: $\angle(E^u(x), E^s(x)) > C$, $\forall x \in A$.

$A$ is called an attractor (in the sense of Conley) if there is a neighborhood $U \supset A$ such that $f(U) \subset A$ and $A = \cap_{n=1}^{\infty} f^n(U)$. This attractor is topologically transitive if there is $x \in A$ with dense orbit in $A$. If moreover $f|_A$ is uniformly hyperbolic $A$ is an Axiom A attractor.

The ergodic theory of Axiom A attractors was developed by Sinai, Ruelle and Bowen in the 1970. In particular for an Axiom A attractor they constructed so called SRB-measures: measures with absolutely continuous conditional measures on unstable manifolds. This measures $\mu$ are also physical measures in the sense of Ruelle since for $z_0$ in a set of initial points of positive Riemannian measure the Birkhoff sums $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^jz_0} \to \mu$ as $n \to \infty$.

In fact for a topologically transitive Axiom A attractor $\mu$ is unique and the Birkhoff sums converges for a.e. $z_0 \in B$, where $B$ is the basin of attraction $B = \cup_{j \geq 0} f^{-j}(U)$. Although the Axiom A theory is quite satisfactory and complete it is not applicable to very many concrete dynamical systems. There is a general theory of ergodic theory of smooth dynamical systems due to among others Pesin, Katok, Ledrappier and others but concrete examples were lacking. Around 1980 the theory of chaotic one-dimensional maps was started. M. Misiurewicz studied multimodal
maps $f$ of the interval, whose critical set or set of turning points, $C$, has the property that for all $z_0 \in C$ and all $j \geq 1$, $\text{dist}(f^jz_0, C) \geq \delta > 0$ and proved existence of absolutely continuous invariant measures. Then Collet and Eckmann [12] proved abundance, i.e. positive Lebesgue measure, of aperiodic behavior for a family of unimodal maps of the interval and, Jakobson in [20] proved abundance of existence of absolutely continuous invariant measures for the quadratic family. A new proof of Jakobson’s theorem was then given by Lennart Carleson and myself in [5] and the methods from this paper were later used by us in [6] to prove aperiodic, chaotic behavior for a class of Hénon maps, which are small perturbations of quadratic maps. The methods of [6] have turned out to be useful for several other dynamical systems and the corresponding ergodic theory has been developed. There have been other accounts of this development, in particular by my collaborators Lai- Sang Young (see e.g. [35]) and Marcelo Viana (see e.g. the proceedings of ICM98 [30]).

2. Hénon maps

In 1978, M. Hénon proposed as a model for non-linear two-dimensional dynamical systems the map

$$(x, y) \mapsto (1 + y - ax^2, bx) \quad 0 < a < 2, \ b > 0.$$ 

He chose the parameters $a = 1.4$ and $b = 0.3$ and proved that $f = f_{a,b}$ has an attractor in the sense of Conley.

He also verified numerically that this Hénon map has sensitive dependence on initial conditions and produced his well-known computer pictures of the attractor. Hénon proposed that this dynamical system should have a “strange attractor” and that it should be more eligible to analysis than the ubiquitous Lorenz system.

In principle most initial points could be attracted to a long periodic cycle. In view of the famous result of S. Newhouse, [24], periodic attractors are topologically generic, so it was not at all a priori clear that the attractor seen by Hénon was not a long stable periodic orbit.

However Lennart Carleson and I, [6], managed to prove that what Hénon conjectured was true—not for the parameters $(a, b) = (1.4, 0.3)$ that Hénon studied—but for small $b > 0$. In fact we managed to prove the following result:

**Theorem 1.** There is a constant $b_0 > 0$ such that for all $b$, $0 < b < b_0$ there is a set $A_b$ of parameters $a$, such that its one-dimensional Lebesgue measure $|A_b| > 0$ and such that for all $a \in A_b$, $f = f_{a,b}$, has the following properties

1. There is an open set $U = U_{a,b}$ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n=0}^{\infty} f^n(U) = W^u(\hat{z})$, where $W^u(\hat{z})$ is the unstable manifold of the fixed point $\hat{z}$ of $f$ in the first quadrant.

2. There is a point $z_0 = z_0(a,b)$ such that $\{f^j(z_0)\}_{j=0}^{\infty}$ is dense on $\Lambda$, and there is $c > 0$ such that $|Df^j(z_0)(0,1)| \geq c^j$, $j = 1, 2, \ldots$.

Hence $\Lambda$ is a topological transitive attractor with sensitive dependence on initial conditions. An immediate consequence of Fubinis theorem is that the “good
parameter set" $A = \bigcup_{b>0} A_b \times \{b\}$ is (a Cantor set) of positive two-dimensional Lebesgue measure.

The first part of the theorem is easy to prove and the result is true for an open set of parameters, i.e. for a small rectangle close to $a = 2$ and $b = 0$ contained in $\{(a,b) : 0 < a < 2, b > 0\}$. The system is dissipative since $|\det Df_{a,b}| = |b| < 1$. In this case applying an argument of Palis and Takens, [28], it follows that a region that is enclosed by pieces of stable and unstable manifolds of the same fixed point $\hat{z}$ is attracted to the unstable manifold $W^u(\hat{z})$. With some additional arguments one can see that also a neighborhood of the closure of the unstable manifold is attracted. However the attractor could a priori be a proper subset of $W^u(\hat{z})$.

The second part of the theorem is only true for parameters $(a,b) \in A$. The main ingredient in the proof of the second part of the theorem is the identification of a critical set $C$ for these Hénon attractors. The set $C$ is countable and located on $W^u(\hat{z})$ but it is natural to expect that the Hausdorff dimension of $C$ is positive.

For each $z_0 \in C$ the following holds: (i) $|D^2 f(z_0)\tau(z_0)| \leq e^{-\varepsilon_0}$ $\forall j \geq 1$, where $\tau(z_0)$ is the tangent vector of the unstable manifold at $z_0$; (ii) trough each $z_0 \in C$ there is a local unstable manifold $W^u(z_0)$ tangent to $W^u(\hat{z})$.

The proof of the theorem is a huge induction in time $n$. Successively preliminary critical points or sets, precritical points, are defined on higher and higher generations of the unstable manifold. We roughly say that $z \in W^u(\hat{z})$ is of generation $G$ if $z \in f^G(\gamma) \setminus f^{G-1}(\gamma)$, where $\gamma$ is the horizontal segment of the local unstable manifold $W^u(z)$ through the fixed point.

In analogy with the methods from the one-dimensional case of [5] parameters are chosen so that inductively $d(f^j z_0, C_G) \geq e^{-\varepsilon_0} \forall j \leq n$, $\forall z_0 \in C_G$, where $C_G$ is the set of precritical points of generation $\leq G = \theta(b) \cdot n$, where $\theta(b) = C/\log(1/b)$ and $\alpha > 0$ is a suitably chosen numerical constant. Moreover a further parameter selection is made so that, informally speaking, too deep returns close to the critical set do not occur too often. The estimate of the measure of the set, deleted because of this condition, is made by a large deviation argument.

3. The ergodic theory

Existence of SRB-measures. For the set of "good parameters" $A$ of Theorem 1, Lai-Sang Young and I proved in [9], the following result

**Theorem 2.** For all $(a,b) \in A$, $f_{a,b}$ has a unique SRB measure supported on the attractor.

As a consequence it follows by general smooth ergodic theory that there is a set of initial points $E$ of positive two-dimensional Lebesgue measure, a subset of the topological basin $B$, such that for all $z_0 \in E$ the Birkhoff sums $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j z_0} \to \mu$, weak-*$ as $n \to \infty$.

Decay of correlation and the central limit theorem. For the same class of Hénon maps as in Theorem 1, L.S. Young and I, [10], managed to prove decay of correlation and a version of the central limit theorem. Our main results may be summarized in the following theorem:
Theorem 3. Suppose \( \varphi \) and \( \psi \) are Hölder observables, i.e. they are functions on the plane that belong to some Hölder class \( \alpha \), \( 0 < \alpha \leq 1 \). Then for some \( C > 0 \) and \( c > 0 \)

1. \( \left| \int \varphi(f^j(x)) \psi(x) \, d\mu - \left( \int \varphi \, d\mu \right) \left( \int \psi \, d\mu \right) \right| \leq Ce^{-cj} \quad \forall j \geq 0; \)
2. \( \mu \left( \left\{ x : \frac{1}{n} \left( \sum_{j=0}^{n-1} \varphi(f^j x) - n \int \varphi \, d\mu \right) \leq t \right\} \right) \to \Phi_{\alpha}(t) \) as \( n \to \infty \), where \( \Phi_{\alpha}(t) \) is the normal distribution function \( \mathcal{N}(0, \sigma) \).

The methods used to prove this theorem involved the definition of a return set \( X = X_u \cup X_s \), where \( X_u \) is a set of approximately horizontal long unstable manifolds \( \gamma_u \) and \( X_s \) is a set of approximately vertical stable manifolds \( \gamma_s \), indexed, say, by an arclength coordinate of its intersection with one of the unstable manifolds of \( X_u \). A dynamical tower construction was made and a Markov extension (Markov partition on the tower) was constructed. One of the key estimates concerns the distribution of the return time \( R(x) = R_i \) for \( x \in X \), which is defined as the time the image \( X \) returns to the base of the tower. \( R_i \) may be defined as the first time such that \( f^{R_i}(X) \subset (\gamma_u \cap X) \) for all \( \gamma_u \) such that \( f^{R_i}(X) \cap \gamma_u \neq \emptyset \) and \( R(x) \in X \). Our estimate is that there are constants \( C \) and \( c > 0 \) such that for each \( \gamma_u \), \( \left| \{ x \in X \cap \gamma_u : R(x) > t \} \right| \leq C^{-ct} \). L.S. Young was then able to give a generalization of this setting of dynamical towers, which applies to other dynamical systems. In particular she managed to prove exponential decay of correlation for dissipative billiards ([34]).

The metric basin problem. A natural question that arises in connection with Theorem 2 is for which set of initial points \( z_0 \) the Birkhoff sums \( n^{-1} \sum_{j=0}^{n-1} f^j z_0 \) converge weak-\( * \) to the SRB-measure \( \mu \). As previously mentioned from the smooth ergodic theory it only follows that this is true for a set of initial points of positive two-dimensional Lebesgue measure.

In [8], Marcelo Viana and I were able to complete the picture: in fact almost all points of the topological basin are generic for the SRB-measure and the basin is foliated by stable manifolds.

Theorem 4. Let us consider the set of Hénon maps \( f_{a,b} \), where \( (a,b) \in A \) (the set of good parameters of Theorem 1). Then the following holds

1. Through Lebesgue a.e. \( z_0 \in B \) there is an infinitely long stable manifold \( W^s(z_0) \) that hits the attractor.
2. For a.e. initial point \( z_0 \in B \), \( \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j z_0} \to \mu \) weak-\( * \) as \( n \to \infty \), where \( \mu \) is the SRB-measure of Theorem 2.

This work was in fact carried out in the more general setting of the Hénon-like maps of Mora & Viana (see Section 4. below). We were also able to characterize the topological basin in this setting: its boundary is the stable manifold of the fixed point in the third quadrant (this was proved independently by Y. Cao [11]).

4. Other dynamical systems

Hénon-like maps. After a rescaling of the second coordinate, the Hénon maps may be written as \( (x, y) \mapsto (1 - \alpha x^2, 0) + \sqrt{b}(y, x) \). More generally Mora & Viana,
con-sided maps of the form $f_\alpha(x, y) = (1 - \alpha x^2, 0) + \psi(x, y)$, where $c_1 b \leq |\det(Df(x, y))| \leq c_2 b$, $||D(\log |\det Df|)||_{C^1} \leq C$ for some $c_1, c_2, C$ and $||\psi||_{C^3} = O(b^{1/2})$. They managed to carry out the same program as in Theorem 1 for these class of maps, which they called Hénon-like maps, and managed to prove prevalence of strange attractors, i.e. that there is a set of positive measure of parameters $\alpha$ so that $f_\alpha$ exhibits a strange attractor.

Mora & Viana used a one-parameter family of maps $g_\mu$, $-\varepsilon < \mu < \varepsilon$, such that $g_0$ has a homoclinic tangency and proved that there is a positive measure set of parameters $\mu$ and a neighborhood $U_\mu$ such that $g_\mu|_{U_\mu}$ has a Hénon-like strange attractor. This is done following Palis & Takens [27], by proving that there is a linear change of variables and in the parameters $\Phi_N$ so that $\Phi_N^{-1} \circ g_\mu \circ \Phi_N(\xi, \eta) = (1 - a\xi^2, 0) + \psi(\xi, \eta)$, where $\psi$ satisfies the appropriate estimates of the Hénon-like maps. Note that the consequence of this is not the existence of a global attractor as in Theorem 1 but the existence of a local attractor close to the homoclinic tangency.

Saddle node bifurcations. Another case where methods based on those in [6] turned out to be useful is the case of saddle node bifurcations treated by Diaz, Rocha and Viana in [16]. In that paper they show that when unfolding a one-parameter family with a critical saddle node cycle Hénon like strange attractors appear with positive density at the bifurcation value. Moreover they prove that in an open class of such families the strange attractors are of global type.

This work was continued by M.J. Costa [14], [15], who proved that global strange attractors also appear when destroying a hyperbolic set (horseshoe) by collapsing it with a periodic attractor.

Viana’s dynamical systems with multiple expanding directions. In [32], M. Viana studied the dynamical system $f : T \times I \rightarrow T \times I$ given by the the following skew product $(\theta, x) \mapsto (m\theta (mod 1), a_0(\theta) - x^2)$, where $a(\theta) = a_0 + a \sin \pi(\theta - \frac{1}{2})$.

If $m$ is a sufficiently large integer ($\geq 16$ is enough), $a_0$ is such that $f_{a_0}(x) = a_0 - x^2$ is a Misiurewicz map, i.e. $|f^j(0)| \geq \delta$, $\forall j \geq 1$, and $\alpha > 0$ is sufficiently small, he managed to prove that for a.e. $(\theta, x)$ and some constants $C$ and $c > 0$, $|\partial_x f^j(\theta, x)| \geq C e^{c j}$, $\forall j \geq 1$. In the second part of this paper Viana also considers skew products of Hénon maps driven by circle endomorphisms.

An important difference with the situation in this paper compared to earlier work in the area is that the exponential approach rate condition of an orbit relative to the critical set is no longer satisfied. Instead this is replaced by a statistical property: very deep and very frequent returns to the critical region is very unlikely. The argument is based on an extension of the large deviation argument from [6].

SRB measures for the Viana maps were constructed by J.F. Alves in [1]. An important concept introduced by Alves was the notion of hyperbolic times, which are a generalization of the escape situations that were considered in [6] and they are also similar to the base in the tower construction in [34], [10].

Infinite-modal maps and flows. Motivated by the study of unfolding of saddle-focus connections for flows in three dimensions Pacifico, Rovella and Viana, [25], studied parameterized families of one-dimensional maps with infinitely many critical
points. They prove that for a positive Lebesgue measure set of parameter values the map is transitive and almost all orbits have positive Lyapunov exponent. There has been a considerable amount of work on flows, by Viana, Luzatto, Pumariño, Rodríguez and others, where techniques from [6] have played an important role. For a survey of these and related results I refer to [31].

In a different direction is the recent proof by W. Tucker, [29], of the existence of chaotic behaviour for the Lorenz attractor.

The attractors of Wang and Young. In a recent paper, D. Wang and L.S. Young, [33], carry out a theory of attractors which generalize the Hénon maps in a different direction. They consider maps of a two-dimensional manifold of the form \( f(x, y, a, b) = (F(x, y, a, b), 0) + b(u(x, y, a, b), v(x, y, a, b)) \). This class clearly differs from the Hénon-like maps of Mora & Viana. In particular the theory applies to perturbations of certain one-dimensional multimodal maps, with a transversality condition in the parameter dependence. The techniques are analogous to these in [6] but more information on the geometric structure, in particular of the critical set, is achieved. Most previous results are obtained in this setting but also new results, e.g. on some similar dynamical systems and on topological entropy.

5. Random perturbations and stochastic stability

A natural question is how the statistical properties of a dynamical system with chaotic behavior behaves when it is perturbed randomly by some small noise at each iterate. Here we will mainly consider independent additive noise and assume that the underlying ambient space \( \mathcal{M} \) is either a subset of Euclidean space or a torus but cases of more general manifolds and more general perturbations can also be considered (for this see several papers and books by Y. Kiefer).

Let \( f : \mathcal{M} \to \mathcal{M} \) and suppose that \( \xi_n, n \geq 0 \), are independent identically distributed random variables with an absolutely continuous probability density supported in a small ball \( B(0, \varepsilon) \) around the origin and consider the Markov chain \( \{X_n\}_{n=0}^{\infty} \) defined by \( X_{n+1} = f(X_n) + \xi_n \). Then there is a stationary transition probability \( p_n(E|x) = p(X_{n+1} \in E|X_n = x) \) and also at least some stationary measure \( \nu_\varepsilon \) satisfying the fixed point equation \( \nu_\varepsilon(E) = \int p_\varepsilon(E|x) \, d\nu_\varepsilon(x) \).

The obvious questions are here whether \( \nu_\varepsilon \) is unique and in that case if \( \nu_\varepsilon \) tends to an invariant measure of the unperturbed system when \( \varepsilon \to 0 \). This is the problem of Stochastic Stability. For the case of Axiom A attractors such results were proved by Y. Kiefer and L.S. Young.

Now such results have also been obtained for the non-uniformly hyperbolic dynamical systems described above. In the case of the quadratic interval maps of [5], L.S. Young and I proved in [9], under suitable assumptions on the density of the perturbations, that \( \nu_\varepsilon \) is unique and \( \nu_\varepsilon \to \mu \) weak*-as \( \varepsilon \to 0 \), where \( d\mu = \varphi \, dx \) is the absolutely continuous invariant measure. V. Baladi and M. Viana, [2], managed to prove this to prove that the density of \( \nu_\varepsilon \), \( \psi_\varepsilon \), converges to \( \varphi \) in \( L^1 \)-norm.

M. Viana and I have recently proved results on weak*-stochastic stability for the Hénon maps of [6] and the Hénon-like maps of [23] (see [7]). For a recent paper on decay of correlation for random skew products of quadratic maps see
6. Open problems and concluding remarks

The most important problem in this general area is the problem of positive Lyapunov exponent for the Standard Map, i.e. the map of the two-dimensional torus defined by \((x, y) \mapsto (2x - y + K \sin 2\pi x, x) \pmod{2}\).

The general belief is that at least for some parameters \(K\) there is at least one ergodic component of positive Lebesgue measure with positive Lyapunov exponent. Nothing is however rigorously known in spite of intensive work by many people. One of the most interesting results by Duarte [17] goes in the opposite direction: for a residual set of parameters \(K\) the closure of the elliptic points can have Hausdorff dimension arbitrarily close to 2.

One important difference between the Standard Map and the Hénon maps for the good parameters is that “the critical set” in the Hénon case is rather small. It has a hierarchical structure and conjecturally the Hausdorff dimension of its closure should be \(O(1/(\log(1/b)))\). The critical set for the standard map (if possible to define) should have Hausdorff-dimension \(\geq 1\).

A. Baraviera proved in his recent thesis [4], positive Lyapunov exponent for the Standard Map with parameters driven by an expanding circle endomorphism.

On natural class of problem is to consider are skew products of Viana’s type, where the parameters are driven by more general maps. One possible choice is to let the driving map be a non-uniformly hyperbolic quadratic map, either a Misiurewicz map or more generally the class of quadratic maps of [5], [6]. A more difficult project would be to study the case when driving map is a circle rotation.

The general picture for dissipative Hénon maps. It is a natural question to consider what happens for other parameters than the ones considered in [6]. One possible scenario is outlined in the following questions, which are much related to J. Palis conjectures, [26], in the \(C^r\)-generic setting.

**Question 1.** Are there for Lebesgue almost every parameter \((a, b)\) in \(\{(a, b) \in \mathbb{R}^2 : 0 < a < 2, b > 0\}\) at most finitely many coexisting strange attractors and stable periodic orbits?

If this is true the Newhouse phenomenon of infinitely coexisting sinks and Colli’s situation [13] of infinitely many coexisting Hénon-like strange attractors would only appear for a Lebesgue zero set of parameters \((a, b)\)?

**Question 2.** For the parameter values for which only finitely many Hénon-like attractors or sinks coexists: do the respective basins cover Lebesgue almost all points of the phase space?

**Question 3.** Is the set of parameters for which the Hénon map \(f_{a,b}\) is hyperbolic dense in the parameter space?

This result if true would correspond to the real Fatou conjecture proved fairly recently by Graczyk & Swiatek, [19], and Lyubich [21]. A positive answer to Question 1 would correspond to Lyubich’s result that almost all points in the quadratic family is either regular or stochastic, [22].
As can be seen from the above Dynamical Systems is a beautiful mixture of Topology and Analysis. To the hard analysis of Cartwright, Littlewood and Levinson, Smale was able to provide a topological counterpart. Recently again more analytical methods have played an important role. What will be next?

References


