Study of Multidimensional Systems of Conservation Laws: Problems, Difficulties and Progress

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Abstract
In the study of multidimensional systems of conservation laws people confront more difficulties than that for one-dimensional systems. The difficulties include characteristic boundary, free boundary associated with unknown nonlinear waves, various nonlinear wave structure, mixed type equations, strong singularities, etc. Most of them come from the complexity of characteristics. We will give a survey on the progress obtained in the study of this topic with the applications in various physical problems, and will also emphasize some crucial points for the further development of this theory in future.

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1. Introduction
The system of conservation laws originates from the study of compressible fluid dynamics, shallow water waves, elastodynamics, magnetohydrodynamics etc. The extensive study of it started from the middle of the last century. The problems in its first studies are one-space-dimensional. Since the real physical problems occur in the three-dimensional space, then in the study of one-space-dimensional problems it is automatically assumed that all quantities under consideration is uniform with respect to two space-variables among three space-variables in regular physical space. However, many physical problems do not
have such symmetry, so that they are generally multidimensional. Therefore, the study of multidimensional conservation laws is inevitable and is indisputably important.

In the study of multidimensional systems of conservation laws people often confronts more difficulties than that for one-dimensional systems. Most of difficulties come from the complicated structure of their characteristics. As a matter of fact, the characteristics are the path of the propagation of perturbations in the motion of media. In one-space-dimensional case the characteristics of the system under consideration are characteristic curves in time-space plane, but in multidimensional case the characteristic varieties are surfaces or more general manifolds, so that the location of characteristics or their intersections will be rather complicated. This situation is also reflected in the descriptions of various nonlinear waves associated with characteristics, including shocks, rarefaction waves and contact discontinuities. As for the methodology, the integration along characteristic curves is a very efficient method in treating various one-space-dimensional problems. Based on it many techniques are developed. However, due to the complexity of characteristics one needs to develop totally different methods to deal with corresponding problems in multidimensional case. Next we will recall the progress obtained in the study of multidimensional conservation laws with showing the main difficulties, which have been overcome or to be overcome.

In this paper the main prototype of the system under consideration is the Euler system for inviscid compressible flow with the form

\begin{equation}
\begin{aligned}
\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) &= 0, \\
\frac{\partial (\rho \vec{v})}{\partial t} + \text{div}(\rho \vec{v} \times \vec{v}) + \nabla p &= 0, \\
\frac{\partial E}{\partial t} + \text{div}(\vec{v}(E + p)) &= 0,
\end{aligned}
\end{equation}

where \( \rho, \vec{v}, p, E \) represent the density, the velocity, the pressure and the total energy respectively. Meanwhile, when the flow is isentropic and irrotational, the Euler system can be reduced to a second order equation – potential flow equation (see [34])

\begin{equation}
(\rho(\nabla \phi))_t + \sum_{j=1}^{3} (\phi_{x_j} \rho(\nabla \phi))_{x_j} = 0,
\end{equation}

where \( \phi \) is the potential of velocity, satisfying \( \vec{v} = \nabla \phi \).

2. Characteristic Boundary Value Problems

The first task in the study of the multidimensional conservation laws is to establish the local theory of classical solutions. The main tool in this stage is
the energy estimates. In most cases the existence and the stability of solutions to partial differential equations rely on various energy estimates. The main estimates for multidimensional system of conservation laws are measured by Sobolev norms. The usual Sobolev space is good in treating Cauchy problems and boundary value problems when the boundary is not characteristic. However, when the boundary is characteristic, the derivatives of the system on the normal direction at the boundary are not “complete”. Hence it is inevitable to meet the smoothness loss at the boundary when a linearization or an iterative process is designed to solve a boundary value problem for nonlinear equations with characteristic boundary. On the other hand, the characteristic boundary appears very often in the systems of conservation laws. For instance, if one consider an inviscid flow in a domain with a rigid wall as its boundary (or a part of its boundary), then the impermeable condition let the boundary be characteristic for the Euler system describing the inviscid flow.

People found that in the proof of the regularity of the possible solutions (or approximate solutions) to the characteristic boundary value problems for multidimensional systems of conservation laws, it is necessary to introduce a new weighted Sobolev space to deal with these problems (e.g. see [9]). This space needs to have such a property: “one order gain of differentiation in normal direction to the boundary should be compensated by two orders loss of differentiation in tangential directions”. More precisely, let \( D^p_q u \) denote the derivative of the function \( u \), where \( p \) is the order in tangential directions, \( q \) is the order in the normal direction, let \( H^{p,t}_s \) denote the Sobolev space of \( u \), satisfying \( D^p_q u \in L^2([0,h] \times \Omega) \) (\( p \leq s, q \leq t \)). Define

\[
B_p = \bigcap_{d \leq \frac{p}{2}} H^{p-2d}_d([0,h] \times \Omega),
\]

which is the Sobolev space describing the above-mentioned property (the space is also denoted by \( H^p_t \) in [38],[41] etc.). Such a space has been employed to prove the existence and uniqueness of the local existence of solutions to characteristic boundary value problems [9],[42]. The Sobolev spaces with such a property is also applied to other physical problems involving characteristic boundaries (for instance, [2],[10]).

On characteristic boundary value problems of symmetric hyperbolic systems we would like also mention the contributions given by R.Agemi[1], D.G.Ebin [25] and P.Secchi [41].

3. 1-D Like Problems with M-D Perturbation – Fan-shaped Wave Structure

Like the nonlinear systems of conservation laws in one space-dimension a general solution of a nonlinear multidimensional system of conservation laws may
develop singularities, no matter how smooth the initial data are (e.g. see [43]). Therefore, in the next stage people must study the theory of weak solutions. The main nonlinear waves for multidimensional systems of conservation laws are shock wave, simple wave and contact discontinuity, like in the one-space-dimensional case. However, the structure of nonlinear waves is much more complicated. It should be noticed that the front of nonlinear waves often plays the role of the boundary of a domain where the solution is smooth. Generally, the location of the wave front should be determined together with the solution. Hence to find the weak solution involving various nonlinear waves often require to solve a free boundary value problem.

Due to the complexity of characteristic varieties of multidimensional systems the nonlinear wave structure for these systems is plentiful. The simplest case is the 1-d like structure with m-d perturbations. In 1-d case, besides a single wave propagates along a curve, one often meets such a wave structure: several waves represented by a set of curves issue from a point. Similarly, a common wave structure in multidimensional case is that several nonlinear waves issue from a smooth curve. Such a wave structure is called fan-shaped wave structure. For instance, consider a Cauchy problem of multidimensional system of conservation laws with initial data, which is discontinuous along a smooth curve (for simplicity we only consider two-space-dimensional case here). Then there will possibly be shock waves, simple waves and contact discontinuities issuing from the curve bearing the discontinuity of the initial data. Especially, under some restrictions on the initial data the solution will only contain one wave among these three kinds of nonlinear waves.

In 1983 A.Majda started the study of weak solutions to multidimensional system of conservation laws with fan-shaped wave structure. He first proved the existence of the solution to the initial value problem of a nonlinear multidimensional system involving a shock front, which issues from a curve carrying the discontinuity of the initial data ([33]). In his work the Cauchy problem with discontinuous data is treated as a free boundary value problem, and the shock front is the free boundary, which is to be determined together with the solution. In the free boundary value problem the Rankine-Hugoniot conditions give a differential relation of the function describing the unknown shock front. An important fact is that the relation is an elliptic differential system for the unknown function on the free boundary. Meanwhile, as a boundary value problem for the original system of conservation laws the uniform Kreiss-Lopatinski condition on the boundary is fulfilled. Such a property let suitable estimates, which dominates the variation of the solution with a shock front, can be established. Then the estimates directly lead to the stability of the solution to the linearized problem and the existence of the local solution of the nonlinear problem near the curve describing the discontinuity of the initial data.

Under some other restrictions on the initial data one can find a solution to the Cauchy problem containing a centered rarefaction wave. In this case the
rarefaction wave is formed by a family of characteristic surfaces in two-space-dimensional case. All these characteristic surfaces issue from a curve carrying the discontinuity of the initial data. The rarefaction wave likes a fan with a front surface and a back surface. The solution is differentiable inside the rarefaction wave region, and it is only continuous on the front surface and the back surface of the fan. Moreover, the solution is discontinuous at the edge of the fan. The front surface and the back surface are unknown and has to be determined with the solution together, hence they can also be regarded as free boundaries. Different from the shock front solution case the front and the back of the rarefaction wave are characteristics. Hence in the corresponding iterative process to establish the existence of the solution to the nonlinear problems one will also confront the “derivative loss” difficulty, which happens even for fixed boundary value problems with characteristic boundary, as indicated in the previous section. An delicate treatment is given in [2], where the Nash-Moser iterative scheme is applied to overcome this difficulty. Meanwhile, the weighted Sobolev space \( B_p \) defined in the previous section is also employed once more. In [2] the local existence of the solution with a rarefaction wave are proved.

The most difficult case in solving Cauchy problem of multi-dimensional system of conservation laws with discontinuous initial data is the case of contact discontinuity, which is a surface formed by stream lines, and is also called compressible vortex sheets. This surface is obviously characteristic. However, different from the rarefaction wave case the solution on this characteristic surface is also discontinuous. On the other hand, different from the shock front case the uniform Kreiss-Lopatinskii condition on the boundary is not satisfied. In other words, in the study of the compressible vortex sheets one will confront the difficulties, which appear in either the shock front case or the rarefaction wave case. J-F.Coulombel and P.Secchi studied the vortex sheets case for multidimensional system of conservation laws in [23]. They noticed that the normal component of the unknown vector on the boundary satisfies the weak Lopatinskii conditions. When the non-degenerate part of the Lopatinskii condition is extracted, the microlocal symmetrizer can then be constructed. In the meantime, the Nash-Moser iterative scheme is also applied to avoid the derivatives loss. These techniques help them to derive the energy estimates, which ensure the stability of solutions to linearized problems and the existence of solutions to nonlinear problems.

When the initial data do not satisfy the restrictions given in the above three special cases the solution to the multidimensional systems under consideration may contain more nonlinear waves, like two shocks (see [35],[37]), one shock and one rarefaction wave (see [29]) etc. General data with discontinuity on a smooth curve may develop all three kinds of nonlinear waves (shock, rarefaction wave and contact discontinuity). Although the main difficulties in the three individual cases have been overcome, the result in most general cases has not been established so far. Obviously, such a result is significant and is anticipated.
People may also find fan-shaped configurations of nonlinear waves in many other problems in fluid dynamics. For instance, an important problem in gas dynamics is the study of supersonic flow past a wedge. For a steady supersonic flow past a three-dimensional wedge, when the attack angle and the vertex angle of the wedge are well controlled, an attached shock front at the edge of the wedge will be formed. Such a physical problem can also be reduced to a boundary value problem in a domain between the attached shock front and the surface of the wedge. The shock front is the free boundary for this boundary value problem. The local existence of the solution with the attached shock front was proved in [11]. Other results on physical problems with fan-shaped wave configurations can be found in [10] for shock reflection by a smooth surface, in [36] for propagation of sound waves etc.

4. Essentially M-D Problems – Flower-shaped Wave Structure

It should be emphasized that in the multidimensional space many complicated wave structure are not 1-d like ones with m-d perturbations. Such structures are essentially multidimensional. Based on the progress in the study of various multidimensional problems with fan-shaped wave structure the researchers gradually concentrate their concerns on more complicated cases.

A good example of the essentially multidimensional problem is the problem on supersonic flow past a pointed body. Like the problem on supersonic flow past a wedge, when the vertex angle of the pointed body is less than a critical value, the shock front is attached at the tip of the body, forming a bigger conical surface. Obviously, such a shock front is not a perturbation of a plane shock, and the state between the shock and the surface of the body is not a perturbation of a constant state either. Here the shock front, as well as the surface of the body, issues from a single point – the tip of the conical body. Hence such a wave structure is called flower-shaped wave structure. We notice that the domain is formed by two conical surfaces with strong singularity at the tip, which will cause new difficulties.

In [12] the author gives a proof of the existence of shock front solution near the tip. The problem is first approximated by the straight version of the original problem, i.e. the pointed body is replaced by a conical body with straight generating lines, and the coming flow is assumed to be constant. For the problem of straight version one can make analysis in self-similar coordinates, by which the problem can be reduced to a free boundary value problem of an elliptic equation.

The main result obtained in [12] is

**Theorem 4.1.** Assume that a pointed body is given by \( r = b(z, \theta) \), where \((r, \theta, z)\) is the relative cylindrical coordinates, \( r = R/z \) is the ratio of the regular
cylindrical coordinates $R$ and $z$, $b(z, \theta)$ is a small perturbation of a constant $b_0$ in the sense of [12]. Assume that a supersonic flow parallel to the $z$-axis comes from infinity with speed $q = q_\infty$ satisfying $q_\infty > a_\infty \left( = \left( \frac{\gamma p_\infty}{\rho_\infty} \right)^\frac{1}{2} \right)$, where $p_\infty, \rho_\infty$ are the pressure and the density at infinity respectively. Besides, $b_0$ is less than a critical value determined by $q_\infty, p_\infty, \rho_\infty$ introduced in [24]. Then the problem of the supersonic flow past the pointed body admits a local weak entropy solution with a pointed shock front attached at the origin.

The result on the existence of the solution with its shock front structure near the tip of the body enable us to study global existence and the asymptotic behavior of the flow behind the shock waves [22],[31],[47].

Another well-known problem involving flower-shaped wave structure is “shock reflection by a wedge”. This is a problem on unsteady flow in two-dimensional space.

When a plane shock front hits a wedge, then a reflected shock will move outward from the edge of the wedge, while the incident shock moves forward in time. By symmetry the problem amounts to consider a shock front hitting a ramp (hence the problem is also called “shock reflection by a ramp”). Let the instant, when the shock front touches the edge of the wedge, be the time $t = 0$, the problem is invariant under the dilation of the time coordinate and the space coordinates. Hence one can look for the self-similar solution of the problem. The corresponding flow in the self-similar coordinates $\xi = x/t, \eta = y/t$ is called pseudo-steady flow, because all parameters of the flow depend only on the coordinates $(\xi, \eta)$, and does not depend on the time $t$ explicitly.

Since all possible waves in this problem are invariant under the dilation, they can be viewed as a “flower” generated from the origin. Hence in the $(t, x, y)$ physical space we obtain a flower-shaped wave structure. Depending on the vertex angle of the wedge (or the angle between the ramp and the horizon) the shock reflection may have various patterns. Among them the simplest case is the regular reflection, for which only a smooth curved reflected shock moves outward like an expanding bubble. Since in the region behind the reflected shock both relatively supersonic flow and relatively subsonic flow will occur, to prove the existence of the solution to the regular shock reflection problem needs to solve a nonlinear mixed type equation, or at least a nonlinear degenerate elliptic equation. B.L.Keyfitz and her collaborators [5],[6] first use UTSD (unsteady transonic small disturbance) equation as the model to establish a result on the existence of solution to the shock reflection problem. Later, G.Q. Chen and M.Feldman [8] use the potential flow equation as the model to discuss regular reflection. Since the coefficients of the potential flow equation depend on the derivatives of the unknown function (the gradient of the flow potential), the proof for the latter case is more difficult. Indeed, the authors of [8] indicated the following conclusion:
For a given supersonic incoming flow, one can find a suitable angle $\theta_c$ and a number $\alpha$, such that, if the angle of the inclination $\theta_w$ of the ramp is in $\left(\theta_c, \frac{\pi}{2}\right)$, then there is a global self-similar solution of the potential flow equation, satisfying the assigned boundary conditions. The solution is globally in $C^{1,\alpha}$, and is $C^\infty$ outside of the shock front and the sonic line. Moreover, the solution is stable with respect to the change of the angle and converges to the normal reflection as $\theta_w \to \pi/2$.

For some combinations of parameters of the upstream flow and the angle of the ramp the regular reflection is impossible. The corresponding wave patterns in these cases are called irregular shock reflection. Among them the most important case is the Mach reflection, which is composed of three shock fronts (incident, reflected and Mach stem) and a contact discontinuity. Near the intersection of these waves the above wave configuration is called Mach configuration. It is interesting that in the self-similar coordinate plane we again confront such a wave structure, in which several nonlinear waves issue from a point.

Even for the steady compressible flow the shock reflection can also be distinguished as regular reflection (oblique shock reflection) and irregular reflection (including Mach reflection). In fact, it is von Neumann, who first found the wave structure in Mach reflection and proposed the concept of Mach configuration in 1943 based on the numerous physical experiments and mathematical analysis for the shock reflection in steady compressible flow [3], [39].

In the study of Mach configuration, if all shock fronts and the contact discontinuity are straight lines and the states in each domains separated by these nonlinear waves are constant, the configuration is called flat Mach configuration. Locally at the triple point, the Mach configuration always can be approximately viewed as flat configuration. The related problem, which people are deeply concerned with, is the stability of Mach configuration, because only stable wave configuration can actually occur in physics.

Generally, the flow behind the Mach stem is always subsonic, but the flow passing across the incident shock and then the reflected shock can be either subsonic or supersonic. Therefore, referring to the flow in the downstream part we classify the Mach configurations as E-E type and E-H type. For E-E type Mach configuration the flow in the downstream part is composed of two branches of subsonic flow separated by a streamline. For E-H type Mach configuration the flow in the downstream part is composed of a supersonic flow and a subsonic flow adjacent to each other with a streamline separating them.

In [14], [15] we proved the stability of the E-E type Mach configuration for both steady case and unsteady case. For example, the result in the steady case is:

**Theorem 4.2.** Assume that the constant states $U_0^i(0 \leq i \leq 3)$, the shock fronts $S_i(i = 1, 2, 3)$ and the contact discontinuity $D$ form a flat Mach configuration, where $U_0^i$ is the state of the coming supersonic flow, $U_1^0$ is the state behind
the incident shock $S_1$, $U_2^0$, $U_3^0$ are the states behind the Mach stem $S_2$ and the reflected shock $S_3$ respectively, $U_2^0$ and $U_3^0$ are both subsonic and are separated by a contact discontinuity $D$. Assume that $U_0^0$ and $\tilde{S}_1$ are non-flat perturbations of $U_0^0$ and $S_1$, then one can find a non-flat Mach configuration in a neighborhood of the triple point, where the shock fronts $S_2, S_3$ and the contact discontinuity $D$ are slightly perturbed. Correspondingly, the states $U_1, U_2, U_3$ in each domain separated by these nonlinear waves are the perturbation of the corresponding states for the original flat Mach configuration.

The stability of the E-H type Mach configuration is also proved in [20] under an additional assumption that the reflected shock is weak.

The above result supported the reasonableness of the Mach configuration proposed by von Neumann [39]. However, the global existence of the Mach reflection is still quite open.

V.Elling and T.P.Liu studied the problem on a moving wedge hitting a static gas in [26]. Suppose there is a uniform static gas filling up the whole space outside a given wedge. The wedge suddenly moves into the air with a constant speed in the direction of its symmetric axis, then the gas flow caused by the motion of the wedge is also self-similar. Obviously, in the time-space coordinate system the wave configuration is flower-shaped. If the speed of the wedge is supersonic, then near the head of the wedge the motion of the air is not influenced by the initial state, so that there is a shock front attached at the edge of the wedge, provided the vertex angle of the wedge is less than a critical value. Meanwhile, far from the vertex of the wedge, the motion of the air amounts to one dimensional: a plane wall moves into the domain filled with static gas along the normal direction. According to the theory of one-dimensional conservation laws there will be a plane shock, moving along the direction normal to the plane wall. On the self-similar coordinate plane the latter is called tail shock. The straight head shock attached at the edge and the straight tail shock are connected by a curved shock. In accordance, behind the shock the flow is a mixture of a relatively supersonic flow and a relatively subsonic flow. The existence of the solution for this problem was established in [26].

Another interesting physical problem involving flower-shaped wave structure is the dam-collapse problem [44]. Assume that a wedge-shaped reservoir is filled with water. At the time $t = 0$ the dam suddenly collapsed so that the water floods outside of the reservoir. The problem is to determine the flow in $t > 0$. Since the motion of the water is governed by the shallow water equation, which is quite similar to the Euler equation in gas dynamics, the dam-collapse problem is also similar to the problem of expansion of gas contained in a wedge-shaped domain into vacuum. In both cases the motion of the fluid is given by an interaction of two rarefaction waves indeed. The problem was solved in [32].

More complicated wave structures with flower-shaped wave configuration appear in the study of multidimensional Riemann problems. Consider
two-dimensional Riemann problem, which is a Cauchy problem of the two-dimensional system of conservation laws with piecewise constant initial data, which takes different constants in different angular domains. In [49] the various cases for 2-d Riemann problems are mentioned and classified, where the initial data take different constants in four quadrants. It is well known that the Riemann problems in one-space-dimensional case has been well studied. Particularly, the results on 1-d Riemann problem play the fundamental role in the theory of conservation laws (see [27]). However, the study on Riemann problems in multidimensional case is only at its beginning.

The Riemann problem is invariant under the dilation of the coordinates. Its solution also has flower-shaped wave configuration. By using self-similar coordinates all waves becomes fixed in the new coordinate plane. Hence a $d+1$ dimensional unsteady problem (1 time-dimension plus $d$ space-dimensions) becomes a $d$-dimensional problem in self-similar coordinate system. The latter is usually called pseudo-steady problem. Therefore, to determine a flower-shaped wave structure for $d+1$ dimensional unsteady problem is then reduced to look for a global solution on the self-similar coordinate system. Far away from the origin the influence of the origin vanishes, so that the $d+1$ dimensional unsteady problem is one space-dimensional, which can be solved by using the theory of one-dimensional system of conservation laws. In accordance, for a given multidimensional Riemann problem one may have many nonlinear waves (formed by straight characteristics) coming from infinity in different directions, and these nonlinear waves will interact when they meet together. The plentiful phenomena of interaction of these waves lead to the great complexity of the nonlinear wave structure either in the self-similar coordinates or in the original physical coordinates.

The Riemann problem is a special initial value problem for the hyperbolic system of conservation laws. Like the setting of the Riemann problems we can also consider some initial boundary value problems invariant under dilation of time coordinate and space coordinates. Many physical problems, including the above-mentioned “shock reflection by a ramp” and “dam-collapse”, can be derived in such a way, that the initial data take different constants in different sectors, while some sectors are solid, where no flow could go into. Such problems are called initial boundary value problems of Riemann type. For instance, we can take initial data as follows. The whole plane is separated by the rays $\theta = \theta_0$ ($0 < \theta_0 < \pi/2$), $\theta = \pi/2$ and $\theta = -\pi$ to three sectors. The sector $-\pi < \theta < \theta_0$ are solid and no gas can go into. Meanwhile, the gas is assumed to take different constant states in $\theta_0 < \theta < \pi/2$ and $\pi/2 < \theta < \pi$. Moreover, the flow parameters in both sides of $\theta = \pi/2$ can determine a single plane shock moving forward to the ramp $\theta = \theta_0$. Then, the initial boundary value problem with such data amounts to the physical problem “shock reflection by a ramp”.

Like the above setting one can also discuss other initial boundary value problems of Riemann type. For instance, in the above example, if the flow
parameters on the both side of $\theta = \pi/2$ can determine a single rarefaction wave moving forward to the ramp, then we obtain a problem “reflection of rarefaction wave by a ramp”. Similarly, if $\theta_0 < 0$ we can obtain the problem “shock diffraction by a convex angle”. In the latter case there may not be any reflected shock, but there is a sonic wave propagating from the origin to infinity.

The dam-collapse problem can also be considered as such an initial boundary value problem of Riemann type. The problem corresponds to the case with initial data: the gas takes non-vacuum constant state in a sector $-\theta_0 < \theta < \theta_0$, while the domain $\theta_0 < \theta < 2\pi - \theta_0$ at the initial time is vacuum. Similarly, one can also consider the case corresponding to the initial data: the gas takes non-vacuum state in the domain $\theta_0 < \theta < \pi/2$, while the domain $\pi/2 < \theta < \pi$ is vacuum and the domain $\pi < \theta < 2\pi + \theta_0$ is solid. The problem will give a wave pattern of the reflection of a “full” rarefaction wave (expanding up to vacuum) by a ramp.

5. Global Theory and Mixed Type Equations

The previous section shows that in the study of compressible flow many problems involve both supersonic flow and subsonic flow (or relatively supersonic flow and relatively subsonic flow in pseudo-steady case), then the system describing the flow has complex characteristics in subsonic region, while its all characteristics in supersonic region are real. Therefore, in order to study the flow globally, we have to consider transonic flows and mixed type equations.

The study of mixed type equations was initiated by F. Tricomi in 1923. The Tricomi equation $yu_{xx} + u_{yy} = 0$ was named by his successors for his contributions to this area [45]. Later, the mixed type equations $u_{xx} + yu_{yy} = 0$ and $u_{xx} + \text{sgn} y u_{yy} = 0$, called Keldysh equation and Lavrentiev-Bitsadze equation respectively, are also proposed and studied by many authors. Both the Tricomi equation and the Keldysh equation are degenerate on the line where the type of the equation is changed. The difference is that for the Tricomi equation the characteristics in hyperbolic region is perpendicular to the degenerate line, while for the Keldysh equation the characteristics in hyperbolic region is tangential to the degenerate line. On the other hand, the Lavrentiev-Bitsadze equation has discontinuous coefficients with discontinuity on the line, where the equation changes its type. These three equations are prototypes of more complicated mixed type equations arisen in various physical problems, particularly arisen in fluid dynamics. Therefore, the study on them are important for the development of the theory of fluid dynamics. However, the change of type often causes great difficulties in the corresponding study.

Next we introduce some problems in gas dynamics related to the mixed type equations, most of them are still open. Obviously, the solution of them will promote a series of new progress in both mathematics and physics. We believe that the progress on the study of mixed type equations will bring us a breakthrough in the study of multidimensional system of conservation laws.
The first example is the E-H type Mach configuration. As mentioned in the previous section, in the discussion of Mach reflection the flow behind the reflected shock could be supersonic, which is adjacent to a subsonic flow with a slip line separating them. Such a wave structure is called E-H type Mach configuration. To determine such a (non-flat) wave configuration near the triple point one has to solve a free boundary value problem of nonlinear mixed type equation. On the curve, where the equation changes its type, the solution is continuous, while its derivatives should satisfy some consistency conditions. Since the coefficients of the equation is discontinuous due to the discontinuity of the flow parameters on the contact discontinuity, the equation belongs to Lavrentiev-Bitsadze's mixed type equation. On the other hand, the data on boundary conditions are assigned on the whole boundary of the elliptic region and on a part of the boundary of the hyperbolic region. Such a setting of boundary conditions is similar to that for Tricomi problem [45]. In [20] the local existence and stability for E-H Mach configuration is proved under the assumption that the reflected shock is weak.

Many hyperbolic problems in physical time-space variables may lead to problems for mixed type equations in self-similar coordinates. Indeed, in the self-similar coordinates plane the equation (2) can be written as

\[(c^2 - (\phi_\xi - \xi)^2)\phi_{\xi\xi} - 2(\phi_\xi - \xi)(\phi_\eta - \eta)\phi_{\xi\eta} + (c^2 - (\phi_\eta - \eta)^2)\phi_{\eta\eta} = 0, \quad (4)\]

where \(\phi\) is the potential of the velocity of the flow, \((\phi_\xi, \phi_\eta) = (u, v)\) is the velocity of the flow. The discriminant of the equation is \(c^2(c^2-(u-\xi)^2-(v-\eta)^2)\), so that the equation is elliptic for \((\xi, \eta)\) satisfying \(c^2 < (u-\xi)^2 + (v-\eta)^2\), and is hyperbolic for \((\xi, \eta)\) satisfying \(c^2 > (u-\xi)^2 + (v-\eta)^2\). Generally, the equation is elliptic as \((\xi, \eta) = (u, v)\), while it is hyperbolic as \((\xi, \eta) \to \infty\). Therefore, the equation is of mixed type generally.

The change of type causes much difficulties in various problems related to mixed type equations. While in some exceptional cases we can consider the solution in the hyperbolic region or in the elliptic region separately. One exceptional case is that the flow in the hyperbolic part is constant, and can be determined by only solving some algebraic equations. Then the problem for the mixed type equation will be reduced to a boundary value problem for a degenerate elliptic equation. The problem on the regular shock reflection or the interaction of two shock fronts belongs to such cases. By taking Chaplygin gas as model to describe the compressible flow, D.Serre studied the interaction of multidimensional shocks. The problem is finally reduced to a Dirichlet boundary value problem for a degenerate elliptic equation of second order, and the unique existence of its global solution is proved (see [40]). Meanwhile, the problems studied in [8], [26] are also reduced to a boundary value problem for degenerate elliptic equations of second order. The dam-collapse problem is another exceptional case. In this case the elliptic domain in the problem is reduced to a single point, so that the mixed type equation is reduced to a degenerate hyperbolic equation. However, in general case the interaction of
the solution in the hyperbolic region and the solution in the elliptic region is inevitable.

Next we give two interesting problems related to mixed type equations in steady compressible flow. The first one is the flow in de Laval nozzle. A de Laval nozzle contains a converging part near the entrance and a diverging part near the exit. The nozzle has a throat in the middle, where the cross sectional area takes minimum. It is known that a compressible subsonic flow will speed up as the nozzle becomes narrow and it will slow down as the nozzle becomes wide. In contrary, a compressible supersonic flow will slow down in a convergent part of a nozzle and will speed up in a divergent part of a nozzle. Therefore, the subsonic flow at the entrance of the de Laval nozzle will speed up as the nozzle is getting narrow. As Courant-Friedrichs conjectured in [24], for a suitable incoming subsonic flow and an assigned pressure at the exit the whole flow pattern could be as follows: the flow reaches sonic speed near the throat, then after passing over the throat the flow becomes supersonic and is accelerating further. Afterwards, if the pressure or other flow parameters at the exit are well controlled the supersonic flow may passes across a transonic shock front and becomes a subsonic flow again, which finally reaches the assigned pressure at the exit. Then the problem is how to determine the flow in the whole nozzle, as well as the location of the sonic line and the possible transonic shock, provided one only knows the incoming flow at the entrance and the assigned condition at the exit.

Many works based on multidimensional PDE analysis for such a problem are proceeded (see [7], [16], [17], [30], [46]). A recent result is that for a two-dimensional expanding nozzle if the supersonic flow at the entrance is given and the pressure at the exit is suitably controlled, then the transonic shock front and the subsonic flow between the shock front and the exit can be uniquely determined [17]. The result coincide with the Courant-Friedrichs’ conjecture [24] in the divergent part of the de Laval nozzle. However, how does a subsonic flow continuously transforms to a supersonic flow under the influence of the shape of the nozzle is still open. Obviously, the complete result on the existence and stability of the flow in de Laval nozzle depends on the study of mixed type equations. We notice that at least some characteristics of the equation in the hyperbolic region is tangential to the sonic line, then the nonlinear mixed type equation describing the compressible flow in the de Laval nozzle may have more similarity to the Keldysh equation.

The second example is the supersonic flow past a blunt body. It is also a long standing problem in gas dynamics (see [24], [18]). When a uniform supersonic flow attacks a fixed blunt body, there will appear a detached shock front ahead of the body. Near the head of the body the shock is almost normal, so that the flow behind the shock must be subsonic. On the other hand, if the blunt body is finite, then the flow passes around the blunt body will finally merge together. Therefore, far away from the head of the body the angle between the shock front and the vorticity of the flow will gradually become small, so
that eventually the flow behind the shock front will be supersonic. The above analysis indicates that the steady flow in the whole region between the shock front and the surface of the body must be transonic. The problem to determine the flow and the location of the shock front is also a free boundary value problem for a nonlinear mixed type equation. In the meantime, the global existence of solution must be considered. So far the analytical study on this problem is still formidable and complete open.

In the end of this paper we would like especially emphasize the importance of the study of mixed type equations. The theory of mixed type equations is much less mature than the theory of elliptic equations and hyperbolic equations. The new theory and technique to deal with various boundary value problems of mixed type equations (particularly, nonlinear mixed type equations) are crucial to the development of multidimensional conservation laws. They will also bring a breakthrough to the whole theory of partial differential equations.

We certainly have not mentioned all difficulties in the study of multidimensional systems of conservation laws. Particularly, the influence of vorticity has not been discussed. Evidently, it is a troublesome factor in the study of various physical problems. Besides, the study on viscous multidimensional conservation laws has also not been discussed. People probably have to combine the study of viscous conservation laws and Navier-Stokes equations to get better understanding on the role and influence of viscosity and vorticity.

References


