ODEs
Exam 2 **(50 points)**

Instructions: Show all of your work. *No credit will be awarded for an answer without the necessary work.*

1. (**9 points**) Solve the initial value problem $t^2x'' + tx' - x = 0$, $x(0) = 1$, $x'(0) = 0$.
2. (**9 points**) Solve the initial value problem $x'' + 3x' + 2x = 4t$, $x(0) = -2$, $x'(0) = 1$.
3. (**9 points**) Find the general solution of $u' = Au + f(t)$, where
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; f(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
4. (**9 points**) Find a formula for the solution of the IVP $x'' + x = f(t)$, $x(0) = x'(0) = 0$ and identify the Green's function.
5. (**9 points**) Apply the power series method to solve the ODE $x'' + 3tx' + 3x = 0$. Use a power series centered at $t = 0$. (Find **one** solution for full credit; if you find **two** linearly independent solutions, you will receive extra credit.)
6. (**5 points**) Here is a theorem stated in class: *Suppose the power series*
$$f(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n$$
has a positive radius of convergence. Then f is differentiable on the interior of its interval of convergence and
$$f'(t) = \sum_{n=1}^{\infty} na_n(t - t_0)^{n-1}.$$
Moreover, the power series for f' has the same radius of convergence as does the power series for f. You may assume this theorem to be true.
   (a) Suppose f is defined by $f(x) = \sum_{n=0}^{\infty} a_n(t - t_0)^n$ and that the series has a positive radius of convergence. Prove that
   $$a_n = \frac{f^{(n)}(t_0)}{n!}, \; n = 0, 1, 2, \ldots.$$
   (b) Suppose $\sum_{n=0}^{\infty} a_n(t - t_0)^n$ is a power series with a positive radius of convergence, and suppose that $\sum_{n=0}^{\infty} a_n(t - t_0)^n = 0$ for all t in the interval of convergence. Prove that $a_n = 0$ for all $n = 0, 1, 2, \ldots$.