PROCEEDINGS

OF THE

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OF

MATHEMATICIANS

1954

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ONE HOUR LECTURES - SHORT LECTURES

This book, listed as Volume I, is the last to appear of the three volumes of the Proceedings. It contains organizational items such as the lists of officers, delegates, and members; the scientific program; the secretary's report (including official speeches); the address by the chairman of the Fields Medal Committee 1954; the one hour lectures except those held in joint sessions with the symposia (see Volume III); and those abstracts of short lectures which could not be printed before the Congress in Volume II. At the end is added a list giving information about publication in full of the subjects dealt with in the short lectures (Volume I and II).

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SCIENTIFIC PROGRAM

THURSDAY, 2nd SEPTEMBER

10.30 Concertgebouw, van Baerlestraat

OPENING SESSION *)

1. A few words of welcome by the President of the „Wiskundig Genootschap" and of the Organizing Committee, Prof. Dr J. A. Schouten.
2. Opening of the Congress by the Burgomaster of Amsterdam.
3. Election of the President of the Congress.
4. Presidential Address.
5. Musical interlude: — Piano Solo by Fania Chapiro

   F. Chopin:
   Impromptu in a flat major op. 29
   Nocturne in c minor op. 48
   Scherzo no. 2 in b flat minor op. 31

6. Address by the President of the Fields' Medal Committee 1954, Prof. Dr H. Weyl. Presentation of medals.
7. Musical interlude: — Piano Solo by Fania Chapiro

   C. Debussy:
   Suite pour le Piano
   (prélude – sarabande – toccata)

8. Communications by the Executive Committee of the International Mathematical Union.
9. Final address by the representative of the Government.

3.00–4.00 Concertgebouw, van Baerlestraat

In the chair: J. A. SCHOUTEN

J. VON NEUMANN, On unsolved problems in mathematics.
(Address by invitation of the Organizing Committee).

*) See also j. 138 ff.

FRIDAY, 3rd SEPTEMBER, MORNING

9.00–10.00 room 10

In the chair: O. PERRON

HARISH–CHANDRA, Representations of semisimple Lie groups.
(Address by invitation of the Organizing Committee).

9.00–10.00 room 2

In the chair: R. COURANT

E. C. TITCHMARSH, Eigenfunction problems arising from differential equations.
(Address by invitation of the Organizing Committee).

10.20–11.20 room 10

In the chair: O. PERRON

R. BRAUER, On the structure of groups of finite order.
(Address by invitation of the Organizing Committee).

10.20–11.20 room 2

In the chair: R. COURANT

E. STIEFEL, Recent developments in relaxation techniques.
(Address by invitation of the Organizing Committee).

Section I room 10

In the chair: C. VISSER

11.40–12.10 A. NÉRON, Propriétés arithmétiques de certaines familles de courbes algébriques.
(Address by invitation of the Organizing Committee)

12.20–12.35 S. IYANAGA, Sur la loi générale d'associativité.

12.40–12.55 H. GRELL, Erhaltungssätze der Idealtheorie und ihre Folgerungen.

Section II room 2

In the chair: J. RIDDER

11.40–12.10 L. CESARI, Retraction, Homotopy, Integral.
(Address by invitation of the Organizing Committee).

12.20–12.50 H. A. L. BEHNKE, Funktionentheorie auf komplexen Mannigfaltigkeiten.
(Address by invitation of the Organizing Committee).

Section IIIa room 16

In the chair: G. DE RHAM

11.40–11.55 G. HIRSCH, Sur des invariants attachés aux sections dans les espaces fibrés.

12.00–12.15 TH. J. WILLMORE, Parallel distributions.

12.20–12.35 A. DOLD, Über fasernweise Homotopieäquivalenz von Faserräumen.

12.40–12.55 C. EHRESMANN, Sur les structures infinitésimales régulières.

Section IIIb room 21

In the chair: H. BUSEMANN

11.40–11.55 C. EHRESMANN, Sur les pseudogroupes de transformations de Lie.

12.00–12.15 W. BARTHEL, Zur Flächentheorie in Finslerschen Räumen.

12.20–12.35 Y. C. WONG, Subflat affinely connected spaces.

12.40–12.55 P. LIBERMANN, Sur les sous-pseudogroupes et prolongements d'un pseudogroupe de Lie.

Section IIIc room 15

In the chair: J. A. TODD

11.40–11.55 H. GERMER, Einige kubische und quadratische Cremona-Transformationen des projektiven R_n in sich.

12.00–12.15 L. GAUTHIER, Sur certaines transformations Crémonniennes associées aux congruences de droites d'ordre un.

12.20–12.35 L. GODEAUX, Faisceaux de surfaces algébriques irrégulières.

12.40–12.55 W. MAIER, Simplex-Inhalte im elliptischen R_3.

Section IV room 24

In the chair: D. VAN DANTZIG

11.40–11.55 E. LUKACS, On certain periodic characteristic functions.

12.00–12.15 D. R. COX, A note on the formal use of complex probabilities in the theory of stochastic processes.

12.20–12.35 H. BERGSTRÖM, On limiting distributions for normed sums.

12.40–12.55 R. D. LORD, Applications of Hankel transforms in the theory of probability.

Section Va room 8

In the chair: J. W. MACCOLL

11.40–11.55 V. BAUMANN, Eine nichtlineare Integrodifferentialgleichung der Thermodynamik.

12.00–12.15 P. TH. LANDSBERG, A partial quantum statistical proof of the third law of thermodynamics.

12.20–12.35 F. L. ALT, Recent results on the equation of burning.

12.40–12.55 C. L. PERRY, Temperature distributions for fluids moving in hot cylindrical pipes.

Section Vb room 7

In the chair: R. GRAN-OLSSON

11.40–11.55 M. REINER, Second order effects in infinitesimal elasticity.

12.00–12.15 P. G. BORDONI, On the invariants of finite strains.

12.40–1.10 A. WEINSTEIN, Axially elliptic and hyperbolic problems. *(Address by invitation of the Organizing Committee).*

Section VII room 11

In the chair: L. N. H. BUNT

11.40–12.10 K. PIENE, School mathematics for Universities and for life. *(Address by invitation of the Organizing Committee).*

12.20–12.35 H. ATHEN, Die Vektorrechnung im deutschen Schulunterricht.

12.40–12.55 K. G. WOLFF, Das projektive Denken im mathematischen Unterricht.

FRIDAY, 3rd SEPTEMBER, AFTERNOON

Section I room 13

In the chair: H. RADEMACHER

2.40–3.10 H. MAASZ, Differentialgleichungen und Modulfunktionen. *(Address by invitation of the Organizing Committee).*

3.20–3.35 M. KNESER, Anwendung eines Satzes von Mann auf die Geometrie der Zahlen.

3.40–3.55 L. J. MORDELL, Integer solutions of the equation $ax^3 + ay^3 + bz^3 = bc^3$.

4.00–4.15 TH. SCHNEIDER, Arithmetische Bedingungen für algebraische Funktionen.

4.20–4.35 K. F. ROTH, On irregularities of distribution.

4.40–4.55 E. WITT, Verlagerung von Gruppen und Hauptidealsatz.

5.00–5.15 P. SAMUEL, Remarques sur le lemme de Hensel.

5.20–5.35 M. EICHLER, Modulfunktionen und Riemannsche Vermutung für die Kongruenzzetafunktion.

Section IIa room 15

In the chair: A. C. ZAANEN

2.40–2.55 W. W. ROGOSINSKI, Functionals on subspaces of L^α.

3.00–3.15 J. HORVATH, Hilbert transforms of distributions in R^n.

3.20–3.35 W. H. J. FUCHS, On the magnitude of Fourier transforms.

3.40–3.55 CH. FOX, Chain transforms.

4.00–4.15 E. HEWITT, Fourier transforms of the class L_p.

4.20–4.35 D. WATERMAN, On an integral of Marcinkiewicz.

4.40–4.55 H. J. ETTLINGER, The Holmgren-Riesz integral transform.

Section IIb room 18

In the chair: H. MILLOUX

2.40–2.55 O. LEHTO, A method in the value distribution theory of meromorphic functions.

3.00–3.15 N. TERZIOGLU, Über das Phragmén-Lindelöfsche Prinzip.

3.20–3.35 P. SEIBERT, Eine Verschärfung des Satzes von Denjoy-Carleman-Ahlfors für eine Klasse von ganzen Funktionen.

3.40–3.55 E. F. COLLINGWOOD, On the radial cluster sets of analytic functions.

4.00–4.15 H. GRUNSKY, Flächendifferenzenrechnung in der Funktionentheorie.

4.20–4.35 J. TEGHEM, Quelques propriétés relatives aux séries entières et aux fonctions analytiques correspondantes.

4.40–4.55 R. SAN-JUAN, Classes sémianalytiques.

5.00–5.15 V. G. AVAKUMOVIĆ, Ein Lückensatz für Dirichletsche Reihen.

Section IIc room 16

In the chair: M. BRELOT

2.40–2.55 W. STOLL, Die Erzeugung von Modifikationen komplexer Mannigfaltigkeiten durch σ-Prozesse.

3.00–3.15 G. DE RHAM, La notion de valeur à la frontière pour un courant.

3.20–3.35 L. J. MYRBERG, Über die Integration der Poissonschen Gleichung auf Riemannschen Flächen.

3.40–3.55 H. W. J. GRAUERT, Charakterisierung der Holomorph-konvexität durch Kählersche Metrik.

4.00–4.15 G. L. PAPY, Sur la notion de différentielle extérieure.

4.20–4.35 K. W. ENDL, Zum Typenproblem Riemannscher Flächen.

4.40–4.55 J. LELONG-FERRAND, Utilisation de métriques non euclidiennes dans l'étude des transformations conformes.

5.00–5.15 A. PFLUGER, Über die Riemann'sche Periodenrelation für offene Flächen.

5.20–5.35 Z. NEHARI, An integral equation associated with a function-theoretic extremal problem.

Section IId room 21

In the chair: R. G. COOKE

2.40–2.55 P. CH. ROSENBLOOM, Partial differential equations of parabolic type.

3.00–3.15 G. HELLWIG, Das Anfangswertproblem bei partiellen Differentialgleichungen von gemischtem Typus.

3.20–3.35 S. AGMON, The fundamental solution and Tricomi's problem for a class of equations of mixed type.

3.40–3.55 J. NITSCHE, Über die linearen Randwertprobleme elliptischer Differentialgleichungssysteme.

4.00–4.15 L. BERS and L. NIRENBERG (speaker), Boundary value problems for non-linear elliptic equations in two independent variables.

4.20–4.35 E. NETANYAHU (to be read by S. BERGMAN), On the singularities of solutions of linear partial differential equations of the elliptic type.

4.40–4.55 C. PUCCI, Il problema di Cauchy per le equazioni a derivate parziali.

5.00–5.15 N. SALTYKOW, Domaine d'existence des intégrales d'équations aux derivées partielles d'ordre supérieur au premier.

Section IIe room 20

In the chair: E. J. MACSHANE

2.40–2.55 S. CINQUINI, Sopra una forma più ampia del problema di Cauchy per i sistemi di equazioni a derivate parziali del primo ordine.

3.00–3.15 W. VELTE, Zur Variationsrechnung mehrfacher Integrale.

3.20–3.35 F. D. FAULKNER, Pfaffian equations and the problem of Bolza.

3.40–3.55 R. W. E. HEINZ, Über die Existenz einer Fläche konstanter mittlerer Krümmung bei vorgegebener Berandung.

4.00–4.15 S. FAEDO, Conditions nécessaires pour le minimum dans le problème de la digue à gravité de moindre volume.

4.20–4.35 G. STAMPACCHIA, Problèmes de Neumann relatifs aux équations du calcul des variations.

4.40–4.55 M. CINQUINI-CIBRARIO, Una estensione nello studio dei sistemi di equazioni a derivate parziali.

5.00–5.15 A. FINZI, On the generating of a transformation on a closed curve by an infinitesimal transformation.

Section III room 2

In the chair: H. FREUDENTHAL

2.40–3.10 H. S. M. COXETER, Regular honeycombs in hyperbolic space.
(Address by invitation of the Organizing Committee).

3.20–3.50 D. MONTGOMERY, Topological transformation groups.
(Address by invitation of the Organizing Committee).

4.00–4.30 B. ECKMANN, Zur Homologietheorie von Räumen und Gruppen.
(Address by invitation of the Organizing Committee).

4.40–5.10 J. P. SERRE, Cohomologie et géométrie algébrique.
(Address by invitation of the Organizing Committee).

Section IV room 7

In the chair: A. C. AITKEN

2.40–2.55 J. NEYMAN, Probabilistic theory of clustering of galaxies with particular reference to the hypothesis of an expanding universe.

3.00–3.15 E. L. SCOTT, Distribution of certain characteristics of clusters of galaxies, with particular reference to the hypothesis of an expanding universe.

3.20–3.35 F. N. DAVID and N. L. JOHNSON (speaker), Tests for skewness and kurtosis with ordered variates.

3.40–3.55 H. KNESER, Wertfunktion und Versicherung.

4.00–4.15 G. ELFVING, A unified approach to the allocation problem in sampling theory.

4.40–4.55 A. RENYI, On a new axiomatic foundation of the theory of probability.

Section V room 10

In the chair: S. C. VAN VEEN

2.40–3.10 J. KAMPÉ DE FÉRIET, Problèmes mathématiques posés par la mécanique statistique de la turbulence.
(Address by invitation of the Organizing Committee).

3.20–3.50 J. J. STOKER, Recent progress in the theory of surface waves.
(Address by invitation of the Organizing Committee).

4.00–4.30 M. R. HESTENES, Hilbert space methods in calculus of variations and numerical analysis.
(Address by invitation of the Organizing Committee).

4.40–5.10 F. RELLICH, Halbbeschränkte Differentialoperatoren höherer Ordnung.
(Address by invitation of the Organizing Committee).

Section VI room 8

In the chair: A. HEYTING

2.40–2.55 R. L. GOODSTEIN, A free variable function theory.

3.00–3.15 TH. A. SKOLEM, A critical remark on foundational research.

3.20–3.35 G. KREISEL, Bases for systems of analysis.

4.00–4.15 A. ROSE, A Gödel theorem for an infinitevalued erweiterter Aussagenkalkül.

4.20–4.35 A. SCHMIDT, Ein rein aussagenlogischer Zugang zu den Modalitäten der strikten Logik.

4.40–4.55 H. KARL, Zur Weiterentwicklung der Mengenlehre.

5.00–5.15 S. KURODA, On the intuitionistic and formalistic theory of real numbers.

Section VII room 11

In the chair: W. O. STORER

2.40–2.55 P. SENGENHORST, Arbeitsunterrichtliche Methoden in der Mathematik der höheren Schulen.

3.00–3.15 M. BRIDGER, The mathematical laboratory in the grammar school and the technical college.

3.20–3.35 K. WIGAND, Intuitive Methoden im mathematischen Unterricht.

3.40–3.55 FR. DRENCKHAHN, Strukturstufen der Schulmathematik in Anpassung an alterstypische Auffassungsweisen.

4.00–4.15 A. BAUR, Anschauligkeit und Strenge im mathematischen Unterricht der deutschen Oberschule.

4.40–4.55 H. CRAMER, Mathematik an Gymnasien (Höheren Schulen).

5.00–5.15 L. N. H. BUNT, Didactical research in the field of mathematics at the Institute of Education of the University of Utrecht.

(At 5.20 the President of the International Committee on Mathematical Instruction, Prof. A. Châtelet, inaugurated the „Exposition de Pédagogie Mathématique").

SATURDAY, 4th SEPTEMBER, MORNING

9.00–10.00 room 10

In the chair: K. KURATOWSKI

J. A. DIEUDONNÉ, Le calcul différentiel dans les corps de caractéristique $p > 0$. *(Address by invitation of the Organizing Committee).*

9.00–10.00 room 2

In the chair: K. KNOPP

B. JESSEN, Some aspects of the theory of almost periodic functions. *(Address by invitation of the Organizing Committee).*

Section Ia room 16

In the chair: A. A. ALBERT

10.30–10.45 R. CROISOT, Sur la classification des demi-groupes.

10.50–11.05 H. NEUMANN, Near-rings connected with free groups.

11.10–11.25 H. A. THURSTON, Reduction of finitary operations to binary and singulary operations.

11.30–11.45 J. RIGUET, Applications de la théorie des relations binaires à l'algèbre et à la théorie des machines.

11.50–12.05 W. PEREMANS, Some remarks on the notion of a free algebraic system.

12.10–12.25 A. L. FOSTER, On a unique subdirect factorization in universal algebras and their characterization by their identities.

12.30–12.45 L. LESIEUR, Sur un problème d'immersion.

12.50–1.05 D. A. KAPPOS, Einbettung eines beliebigen Verbandes in einem *o*-topologischen Verband.

Section Ib room 21

In the chair: T. SCHNEIDER

10.30–10.45 M. KRAITCHIK, Sur les cuboides rationnels.

10.50–11.05 W. KNÖDEL, Carmichael-Zahlen und neuere holländische Arbeiten.

11.10–11.25 A. JAEGER, Lineare Differentialgleichungen in algebraischen Funktionenkörpern mehrerer Unbestimmter bei Primzahlcharakteristik.

11.30–11.45 C. ARF, Über die Galoissche Gruppe der algebraisch abgeschlossenen Hülle eines Potenzreihenkörpers über $GF(p)$.

11.50–12.05 E. LAMPRECHT, Allgemeine Gausssche Summen in endlichen Ringen.

12.10–12.25 J. T. TATE, The cohomology groups of algebraic number fields.

12.30–12.45 M. KOECHER, Zur Operatorentheorie der Modulformen n-ten Grades.

12.50–1.05 H. HALBERSTAM, On the distribution of strongly additive number theoretic functions.

Section II room 2

In the chair: G. PÓLYA

10.30–11.00 W. K. HAYMAN, The coefficients of schlicht and allied functions. *(Address by invitation of the Organizing Committee).*

11.10–11.40 A. ZYGMUND, On the Hilbert transform in E^n. *(Address by invitation of the Organizing Committee).*

11.50–12.20 P. J. MYRBERG, Über automorphe Funktionen. *(Address by invitation of the Organizing Committee).*

12.30–1.00 K. KODAIRA, Some results in the transcendental theory of algebraic varieties. *(Address by invitation of the Organizing Committee).*

Section IIIa room 13

In the chair: H. CARTAN

10.30–10.45 M. KERVAIRE, Generalization of a theorem of G. de Rham and expression of Hopf invariant as an integral.

10.50–11.05 P. E. DOLBEAULT, Sur la cohomologie des variétés analytiques complexes.

11.10–11.25 H. GUGGENHEIMER, Une suite exacte dans la théorie des variétés analytiques complexes.

11.30–11.45 K. YANO, Some remarks on almost complex manifolds.

11.50–12.05 E. M. PATTERSON, Kähler spaces which are Riemann extensions.

12.10–12.25 E. CALABI, The space of Kähler metrics.

12.30–12.45 A. H. WALLACE, Homology on algebraic varieties.

12.50–1.05 M. BERGER, Groupes d'holonomie homogène des variétés riemanniennes.

Section IIIb room 15

In the chair: H. KNESER

10.30–10.45 H. HERRMANN, Morphologie der Figuren und der Konfigurationen.

10.50–11.05 J. SEIDEL, An approach to n-dimensional euclidean and non-euclidean geometry.

11.10–11.25 G. PICKERT, Sechseckgewebe und potenz-assoziative Loops.

11.30–11.45 W. KLINGENBERG, Projektive und affine Ebenen mit Nachbarelementen.

11.50–12.05 F. BACHMANN, Begründung der Geometrie aus dem Spiegelungsbegriff.

12.10–12.25 P. BERGAU, Zur Geometrie einer Klasse von Gruppen.

12.30–12.45 FR. BENNHOLD, Zur synthetischen Begrundung der projektiven Geometrie der Ebene mit Hilfe des Archimedischen Postulates.

Section Va room 7

In the chair: J. M. BURGERS

10.30–10.45 J. L. SYNGE, Maxwellian fields in vacuo without singularities and with finite total energy.

10.50–11.05 D. P. RIABOUCHINSKY, On the correlation between the fundamental equations of the hydrodynamical and electromagnetic fields.

11.10–11.25 D. GRAFFI, Su un problema di induzione magnetica.

11.30–11.45 M. L. DE SOCIO, Propagazione di un'onda elettromagnetica in una guida con dielettrico eterogeneo.

11.50–12.05 R. NARDINI, Comportamento asintotico della soluzione di un problema al contorno della magnetoidrodinamica.

12.10–12.25 E. P. MILES (speaker) and E. WILLIAMS, A basic set of homogeneous harmonic polynomials in three variables.

12.30–12.45 O. BJÖRGUM, On the analytic representation of Beltrami vector fields $\triangle \times \boldsymbol{v} = \Omega \boldsymbol{v}$.

12.50–1.05 E. HÖLDER, Über die Differentialgleichungen der Supraleitung.

Section Vb room 24

In the chair: A. WALTHER

10.30–10.45 A. S. HOUSEHOLDER, Generation of error in computations with continued fractions.

10.50–11.05 W. E. MILNE, The error of the trapezoidal formula.

11.10–11.25 S. J. TUPPER, Some new results in numerical analysis.

11.30–11.45 J. KUNTZMANN, Représentation approchée de dérivées.

11.50–12.05 J. TODD, The condition of matrices.

12.10–12.25 E. G. KOGBETLIANTZ, Diagonalization of general complex matrices as a new method for solution of linear equations.

12.50–1.05 A. GONZALES DEL VALLE, La dinamica isostatica de las redes electricas y sus aplicaciones al dimensionado automatico de estructuras mecanicas.

Section Vc room 8

In the chair: L. ROSENHEAD

10.30–10.45 D. GILBARG, Comparison methods in fluid dynamics.

10.50–11.05 V. G. SZEBEHELY, Hydrodynamic application of some new integral transformations.

11.10–11.25 A. Ghaffari, On some fundamental solutions of axially symmetric flows.

11.30–11.45 S. D. Nigam, Motion of a body of revolution in rotating fluid. (read by B. R. Seth).

11.50–12.05 A. R. Mitchell, Rotational flow past cylinders.

12.10–12.25 K. Maruhn, Eine hydrodynamische Existenzbetrachtung.

12.30–12.45 T. P. Angelitch, Über die Bewegung starrer Körper mit nicht holonomen Bindungen in einer inkompressiblen Flüssigkeit.

12.50–1.05 G. Bandyopadhyay, Plane flow of compressible fluid in a nonrigid tube adapting itself instantaneously to pressure (read by B. R. Seth).

Section VI room 10

In the chair: A. Schmidt

10.30–11.00 A. Mostowski, Development and applications of the „projective" classification of sets of integers.
(Address by invitation of the Organizing Committee).

11.10–11.25 K. Schröter, Zur Theorie des bestimmten Artikels.

11.30–11.45 L. Henkin, Γ-Completeness.

11.50–12.05 R. Fraissé, Obtention en théorie des relations de certaines classes d'origine logique.

12.10–12.25 R. O. Gandy, On the possibility of proving the consistency of the simple theory of types.

12.30–12.45 J. R. Büchi (speaker) and J. B. Wright, Abstraction versus generalization.

12.50– 1.05 M. C. Badillo Barallat, Esquemas representativos de sistemas regidos por una lógica polivalente.

Section VII room 11

In the chair: G. Ascoli

Session dedicated to inquiring of the International Committee on Mathematical Instruction on: „Mathematical Instruction for students between 16 and 21 years of age."

10.30–10.35 H. A. L. Behnke, Opening speech.

10.35–10.50 A. Châtelet, Education in France.

10.55–11.10 H. A. L. Behnke, Der mathematische Unterricht der 16–21-jährigen Jugend in der Bundesrepublik Deutschland.

11.15–11.30 S. Mac Lane, Intermediate Mathematical Instruction in the United States.

11.35–11.50 M. VILLA, L'insegnamento della matematica in Italia per i giovani dai 16 ai 21 anni.

11.55–12.10 Miss M. L. CARTWRIGHT, Report of the British Sub-Commission on Mathematical Instruction for students between 16 and 21 years of age.

12.15–12.30 L. N. H. BUNT, Mathematical Instruction for Students between 16 and 21 years of age in the Netherlands.

12.35–12.50 O. FROSTMANN, Summary of a Report on the Mathematical Instruction in Sweden for Students between 16 and 21 years of age.

MONDAY, 6th SEPTEMBER, MORNING

9.00–10.00 room 10

In the chair: H. HOPF

K. BORSUK, Sur l'élimination des phénomènes paradoxaux en topologie générale.

(Address by invitation of the Organizing Committee).

9.00–10.00 room 2

In the chair: W. V. D. HODGE

J. NEYMAN, Current problems of mathematical statistics.

(Address by invitation of the Organizing Committee).

10.20–11.20 room 2

In the chair: W. V. D. HODGE

D. VAN DANTZIG, Mathematical problems raised by the flood disaster 1953.

(Address by invitation of the Organizing Committee).

10.20–11.20 room 13

In the chair: A. CHURCH

A. TARSKI, Mathematics and metamathematics.

(Address by invitation of the Organizing Committee).

Section I room 10

In the chair: M. DEURING

11.40–12.10 N. JACOBSON, Some aspects of the theory of representations of Jordan algebras.
(Address by invitation of the Organizing Committee).

12.20–12.50 H. DAVENPORT, Simultaneous diophantine approximation.
(Address by invitation of the Organizing Committee).

Section II room 2

In the chair: J. KARAMATA

11.40–12.10 K. CHANDRASEKHARAN, Some problems in Fourier analysis.
(Address by invitation of the Organizing Committee).

12.20–12.50 M. L. CARTWRIGHT, Some aspects of the theory of non-linear differential equations.
(Address by invitation of the Organizing Committee).

Section IIIa room 15

In the chair: H. WHITNEY

11.40–11.55 P. J. HILTON, On the homotopy groups of the union of spheres.

12.00–12.15 E. E. MOISE, The invariance of the knot-types.

12.20–12.35 D. PUPPE, Zur Homotopie von Abbildungen eines Polyeders in eine Sphäre.

12.40–12.55 E. BURGER, Bemerkungen zur Homotopietheorie.

Section IIIb room 16

In the chair: L. GODEAUX

11.40–11.55 C. LONGO, On the classification of linear complexes of planes.

12.00–12.15 E. MARCHIONNA, Un théorème d'unicité birationelle pour les fonctions algébriques de plusieurs variables.

12.20–12.35 R. CALAPSO, Le reti 0 di Guichard e il teorema di permatbilità di Bianchi.

12.40–12.55 F. BERNSTEIN, The mathematics of the human bloodgroups and the algebraic line-congruences.

Section IV room 24

In the chair: F. J. WEYL

11.40–11.55 B. DE FINETTI, Une façon d'introduire la notion de „mesure", particulièrement convenable pour la théorie des probabilités.

12.00–12.15 W. L. SMITH (read by D. V. LINDLEY), Regenerative stochastic processes.

12.20–12.35 C. L. SCHEFFER, Wald's fundamental identity for general stochastic processes.

12.40–12.55 A. J. L. BLANC-LAPIERRE, Application de la notion de fonction caractéristique à l'étude de quelques problèmes de mécanique statistique. (read by J. Kampé de Fériet).

Section Va room 7

In the chair: D. RIABOUCHINSKY

11.40–11.55 F. GALLISOT, Les formes extérieures en mécanique.

12.00–12.15 C. DE LOSADA Y PUGA, Sur l'accélération séculaire de la lune.

12.20–12.35 C. AGOSTINELLI, Ricerche sulle soluzioni periodiche del problema ristretto dei tre corpi.

12.40–12.55 G. GRIOLI, On the precessions of a rigid heavy body fixed in a point.

Section Vb room 8

In the chair: M. REINER

11.40–11.55 R. GRAN-OLSSON, Some remarks on a paper by C. Carathéodory. (Z. angew. Math. Mech. 13 (1933), p. 71–76).

12.40–12.55 G. LAMPARIELLO, Allgemeine Betrachtungen über Fortpflanzung der elektromagnetischen Wellen in den bewegten Körpern.

Section VI room 13

In the chair: A. A. FRAENKEL

11.40–12.10 P. LORENZEN, Die Fiktion der Überabzählbarkeit. *(Address by invitation of the Organizing Committee).*

12.20–12.35 P. BERNAYS, Über den Zusammenhang des Herbrand'schen Satzes mit den neueren Ergebnissen von Schütte und Stenius.

12.40–12.55 K. SCHRÖTER, Grundzüge einer systematischen Theorie formalisierter mathematischen Disziplinen.

Section VII room 11

In the chair: E. W. BETH

11.40–11.55 R. TATON, Quelques remarques relatives à l'influence des techniques sur l'évolution de la géometrie.

12.00–12.15 Ž. MARKOVIČ, La théorie de Platon sur l'Un et la Dyade indéfinie et ses traces dans la mathématique grecque.

12.20–12.35 H. H. HANSEN, Geometrie und Wirklichkeit.

12.40–12.55 L. GUGGENBUHL, Henri Brocard and the geometry of triangle.

MONDAY, 6th SEPTEMBER, AFTERNOON

Section Ia room 13

In the chair: H. K. T. BRANDT

2.40–2.55 H. ORSINGER, Resultantensysteme aus Koeffizienten algebraischer Relationen.

3.00–3.15 K. E. AUBERT, Some applications of r-ideals to valuation theory.

3.20–3.35 Š. SCHWARZ, Characters of commutative semigroups.

3.40–3.55 F. KASCH, Grundlagen einer Theorie der Frobeniuserweiterungen.

4.00–4.15 P. M. COHN, Homomorphic images of special Jordan-algebras.

4.20–4.35 A. W. GOLDIE, Decompositions of semi-simple rings.

5.00–5.15 GOVERDAN LAL BAKHSHI, Theory of remainders in the algebra of Grassmann.

5.20–5.35 D. TAMARI, A refined classification of semigroups leading to generalized polynomial rings with a generalized degree concept.

Section Ib room 15

In the chair: M. KRAITCHIK

2.40–2.55 A. GLODEN, Questions d'analyse diophantienne multigrade.

3.00–3.15 M. GUT, Relativquadratische Zahlkörper, deren Klassenzahl durch eine vorgegebene ungerade Primzahl teilbar ist.

3.20–3.35 A. FRÖHLICH, Non abelian laws of prime decomposition.

3.40–3.55 I. A. BARNETT, Fermat's last theorem in binary integral matrices.

4.00–4.15 H. J. A. DUPARC, Periodicity properties of some recurring sets of integers.

4.20–4.35 C. PISOT, Sur un ensemble fermé d'entiers algébriques.

4.40–4.55 C. ORLOFF, Mathematisches Spektrum der Wurzeln einer algebraischen Gleichung.

5.00–5.15 H. F. SANDHAM, The perimeter of an ellipse.

5.20–5.35 H. ROHRBACH (speaker) and B. VOLKMANN, Verallgemeinerte asymptotische Dichten.

Section II room 16

In the chair: A. ROSENTHAL

2.40–3.10 CHR. PAUC, Contributions à la théorie de la différentiation de fonctions d'ensemble.
(Address by invitation of the Organizing Committee).

Section IIa room 16

In the chair: O. HAUPT

3.20–3.35 M. G. ARSOVE, The Looman-Menchoff theorem and some subharmonic function analogues.

3.40–3.55 W. J. TRJITZINSKY, Non summable generalized Laplacians.

4.20–4.35 A. ROSENTHAL, On the continuity of functions of several variables.

4.40–4.55 B. VAN ROOTSELAAR, Intuitionistic theory of integration.

5.20–5.35 G. ALEXITS, Sur la caractérisation des fonctions dérivables par leur séries de Fourier.

Section IIb room 21

In the chair: L. Fantappiè

3.20–3.35 F. WOLF, Spectral decomposition of operators with a linear spectrum.

3.40–3.55 H. REITER, L^1-spaces on locally compact groups.

4.00–4.15 H. O. CORDES, Die Spektralzerlegung von hypermaximalen Operatoren Hilbertscher Räume, die durch Separation zerfallen.

4.20–4.35 R. G. COOKE, Reciprocals of infinite matrices and inverses of linear operators.

5.00–5.15 P. VERMES, Infinite matrices associated with basic sets of polynomials.

Section IIc room 18

In the chair: C. S. MEIJER

3.20–3.35 R. CONTI and G. SANSONE (speaker), On an equation of T. Uno and R. Yokomi.

3.40–3.55 F. W. J. OLVER, The asymptotic expansion of Legendre functions of large order.

4.00–4.15 F. G. TRICOMI, Asymptotische Eigenschaften der konfluenten hypergeometrischen Funktionen.

4.20–4.35 R. E. LANGER, On the asymptotic solutions of ordinary linear differential equations of the third order in a region containing a turning point.

4.40–4.55 Z. JANKOVIĆ, On solutions of the generalized Laguerre differential equation.

5.00–5.15 T. POPOVICIU, Les polynomes de S. N. Bernstein et le problème de l'interpolation.

5.20–5.35 R. WEYRICH, Analytische Theorie der Differentialgleichungen auf der Grundlage von Entwicklungen nach Besselschen Funktionen.

Section IId room 20

In the chair: J. LERAY

3.20–3.35 C. MIRANDA, Sull' integrazione delle forme differenziali.

3.40–3.55 A. GHIZZETTI, Une méthode générale pour obtenir des formules de quadrature.

4.00–4.15 H. J. BREMERMANN, Subharmonic functions in several complex variables.

4.20–4.35 M. FEKETE, Transfinite diameter and Fourier series.

4.40–4.55 A. GHAFFARI, Etude globale d'une équation différentielle non linéaire.

5.00–5.15 G. R. MORRIS, A differential equation for forced undamped non-linear oscillations.

Section IIIa room 7

In the chair: R. L. WILDER

3.00–3.15 K. KURATOWSKI, Sur les fonctions rationnalles homotopes à des homéomorphies.

3.20–3.35 H. SCHUBERT, Über Brückendarstellungen von Knoten.

3.40–3.55 A. STÖHR, Zerlegung von Flächen vom Geschlecht eins in ähnliche Rechtecke.

4.00–4.15 T. DEKKER and J. DE GROOT (speaker), Decomposition of a sphere.

4.20–4.35 E. HEMMINGSEN, Plane continua and homeomorphisms thereof with equicontinuous, non periodic iterates.

4.40–4.55 O. HANNER, Retraction of metric and non-metric spaces.

5.00–5.15 E. A. MICHAEL, Selection theorems for continuous functions.

5.20–5.35 J. L. TITS, Espaces homogènes et groupes de Lie exceptionnels.

Section IIIb room 8

In the chair: K. STRUBECKER

3.20–3.35 G. EWALD, A new foundation of geometry of circles.

3.40–3.55 E. BLANC, Une axiomatique élémentaire de géométrie euclidienne.

4.00–4.15 R. M. F. SAUER, Projektiv-geometrische Sätze über lineare partielle Differentialgleichungen 2. Ordnung.

4.20–4.35 K. LEICHTWEISS, Existenz und Eindeutigkeit in der mehrdimensionalen Differentialgeometrie.

5.00–5.15 M. VILLA, Problemi integrali sulle trasformazioni puntuali.

Section IV room 24

In the chair: S. S. WILKS

2.40–2.55 Z. W. BIRNBAUM, On the power of a distribution-free test of fit.

3.00–3.15 S. H. ABDEL-ATY, Ordered variables in binomial and Poisson samples.

3.20–3.35 A. BENARD and PH. VAN ELTEREN (speaker), A generalization of the method of m rankings.

3.40–3.55 P. DE MUNTER, Tests non-paramétriques pour la comparaison de deux ou plusieurs échantillons.

4.00–4.15 H. R. VAN DER VAART, On a basic distribution-free multi-decision solution of a certain k-sample problem.

4.20–4.35 D. J. STOKER, An upper bound for the deviation from normality of Wilcoxon's test statistic for the two-sample problem in the general case.

4.40–4.55 J. CHASTENET DE GÉRY (speaker) and J. M. SOURIAU, Étude de la correlation de deux espaces vectoriels aléatoires.

5.00–5.15 A. RENYI, On the theory of order statistics.

Section VII room 11

In the chair: O. A. FROSTMAN

2.40–3.10 C. T. DALTRY, Self-education by children in mathematics using Gestalt methods (i.e. learning-through-insight).
(Address by invitation of the Organizing Committee).

3.20–3.35 J. L. P. CHEVRIER, Les notions de structure en mathématiques élémentaires.

3.40–3.55 H. RÜPING, Philosophische Vertiefung des mathematischen Unterrichts.

4.00–4.15 R. CRESPO PEREIRA, The teaching of mathematics.

4.20–4.35 H. F. FEHR, Administration of mathematical education in the United States of America.

4.40–4.55 E. PALAZZO, Pedagogia scientifica inquadrata in una teoria matematica.

5.00–5.15 R. DOLINSKY, Determinanten im Unterricht allgemeinbildender Schulen.

5.20–5.35 V. AMATO, Sur insegnamento matematico nelle scuole secondarie e sui testi scolastici.

TUESDAY, 7th SEPTEMBER, MORNING

9.00–10.00 room 10

In the chair: E. BOMPIANI

A. LICHNEROVICZ, Les groupes d'holonomie et leurs applications.
(Address by invitation of the Organizing Committee).

9.00–10.00 room 2

In the chair: M. Plancherel

S. Goldstein, On some methods of approximation in Fluid Mechanics. *(Address by invitation of the Organizing Committee).*

9.00–10.00 room 13

In the chair: N. E. Nørlund

S. M. Nikolskii, Nekotorue woprosu priblichenia funkzii polynomami. (Einige Fragen der Approximation durch Polynome).
(Address by invitation of the Organizing Committee).

Section I room 13

In the chair: V. Brun

10.25–10.55 E. Hlawka, Das inhomogene Problem in der Geometrie der Zahlen.
(Address by invitation of the Organizing Committee).

11.05–11.35 P. Erdös, Additive number theoretic functions and applications of probability to number theory.
(Address by invitation of the Organizing Committee).

Section IIa room 15

In the chair: A. J. Ward

10.25–10.40 W. Nef, Über eine Ausdehnung von A. Tarski's algebraischer Inhaltstheorie.

10.45–11.00 H. Bauer, Topologische Kennzeichnung des total-additiven und rein-endlich-additiven Teils einer additiven Mengenfunktion.

11.05–11.20 K. Krickeberg, Characterization of integrals as set functions.

Section IIb room 16

In the chair: G. M. Köthe

10.25–10.40 R. E. Fullerton, Geometric properties of a basis in a Banach space.

11.05–11.20 A. F. Ruston, Fredholm formulae and the Riesz theory.

11.25–11.40 H. F. Bohnenblust (speaker) and S. Karlin, Geometrical notions in Banach Algebras.

Section IIc room 21

In the chair: H. Grunsky

10.25–10.40 M. Brelot, Sur l'allure des fonctions harmoniques à la frontière.

10.45–11.00 C. ULUÇAY, Bloch functions and the definition of a new constant.

11.05–11.20 M. OHTSUKA, Gross's star theorems and their applications.

11.25–11.40 F. W. PERKINS, Properties of polygonal means of functions.

Section IIIa room 10

In the chair: CH. EHRESMANN

10.25–10.40 D. GALE, Irreducible convex sets.

11.05–11.20 I. RATIB, Sur le problème des quatre couleurs.

11.25–11.40 W. HAKEN, Über Flächen in 3-dimensionalen Mannigfaltigkeiten. Lösung des Isotopieproblems für den Kreisknoten.

Section IIIb room 11

In the chair: W. SÜSS

10.25–10.40 K. L. STELLMACHER, Eine klasse von Huygens-scher Differentialgleichungen.

10.45–11.00 H. KUNLE, Zur projektiven Kinematik einparametriger Quadrikscharen.

11.05–11.20 S. BILINSKI, Eine Verallgemeinerung der Formeln von Frenet und eine Isomorphie gewisser Teile der Differentialgeometrie der Raumkurven.

11.25–11.40 M. BARNER, Kinematik in der projektiven Differentialgeometrie.

Section IV room 7

In the chair: B. DE FINETTI

10.25–10.55 J. L. DOOB, Interrelations between Brownian Motion and Potential Theory.
(Address by invitation of the Organizing Committee).

Section V room 2

In the chair: W. E. MILNE

10.25–10.55 L. COLLATZ, Fehlermasz-Prinzipien in der praktischen Analysis.
(Address by invitation of the Organizing Committee).

11.05–11.35 G. FICHERA, Methods of functional linear analysis in mathematical physics.
(Address by invitation of the Organizing Committee).

Section VI room 8

In the chair: P. C. ROSENBLOOM

10.25–10.55 J. BARKLEY ROSSER, The relative strength of Zermelo's set theory and Quine's new foundations.
(Address by invitation of the Organizing Committee).

11.05–11.20 H. B. CURRY, Generalizations of the deduction theorem.
11.25–11.40 M. H. LÖB, Solution of a problem by Leon Henkin.

WEDNESDAY, 8th SEPTEMBER, MORNING

9.00–10.00 room 10

In the chair: L. J. MORDELL

B. SEGRE, Geometry upon an algebraic variety.
(Address by invitation of the Organizing Committee).

9.00–10.00 room 2

In the chair: M. GUT

I. M. GELFAND (read by Harish Chandra), Some aspects of functional analysis and algebra.
(Address by invitation of the Organizing Committee).

9.00–10.00 room 13

In the chair: A. DENJOY

K. YOSIDA, Semi-group theory and the integration problem of diffusion equations.
(Address by invitation of the Organizing Committee).

Section I room 7

In the chair: M. KRASNER

10.25–10.55 D. G. NORTHCOTT, Specialization methods in algebraic geometry.
(Address by invitation of the Organizing Committee).

Section Ia room 7

In the chair: W. KRULL

11.05–11.20 I. M. H. ETHERINGTON, Entropic functions of non-associative algebras.
11.25–11.40 R. M. COHN, Specializations over difference fields.
11.45–12.00 E. A. BEHRENS, Nichtassoziative Ringe.
12.05–12.20 S. K. ZAREMBA, Spacing problems in abelian groups.
12.25–12.40 P. E. KUSTAANHEIMO, An axiomatic definition of the tensor calculus.
12.45–1.00 I. FLEISCHER, On the extension theory for modules.

Section Ib room 8

In the chair: V. JARNIK

11.05–11.20 K. MAHLER, A problem in diophantine approximations.
11.25–11.40 C. A. ROGERS, The Minkowski-Hlawka theorem.
11.45–12.00 K. B. GUNDLACH, Über eine Abschätzung der Fourier-koeffizienten ganzer Spitzenformen zur Hilbertschen Modulgruppe.
12.05–12.20 M. CUGIANI, On the „chains" of consecutive prime numbers.
12.25–12.40 G. J. RIEGER, Neuere Ergebnisse beim Waringschen Problem.
12.45–1.00 H. E. RICHERT, Anwendungen von Mittelwertsätzen Dirichletscher Reihen in der Zahlentheorie.

Section II room 2

In the chair: A. OSTROWSKI

10.25–10.55 A. ERDÉLYI, Asymptotic solutions of differential equations with transition points.
(Address by invitation of the Organizing Committee).
11.05–11.35 F. BUREAU, Les solutions élémentaires et le problème de Cauchy.
(Address by invitation of the Organizing Committee).
11.45–12.15 E. HILLE, Some aspects of Cauchy's problem.
(Address by invitation of the Organizing Committee).
12.25–12.55 T. WAŻEWSKI, Sur une méthode topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles.
(Address by invitation of the Organizing Committee).

Section IIIa room 15

In the chair: F. W. LEVI

10.25–10.40 G. M. KÖTHE, Lineare Räume mit linearer Topologie.
10.45–11.00 G. NÖBELING, Über die Erweiterungen topologischer Räume.
11.25–11.40 L. M. BLUMENTHAL, Boolean geometry II.
11.45–12.00 C. BERGE, Théorie des jeux et structures topologiques.
12.05–12.20 H. SCHIRMER, Mindestzahlen von Koinzidenzpunkten.
12.25–12.40 H. B. GRIFFITHS, Products of fundamental groups.
12.45–1.00 W. H. COCKCROFT, On two dimensional aspherical complexes.

Section IIIb room 18

In the chair: J. HAANTJES

10.25–10.40 L. A. SANTALÓ, On the kinematic formula in spaces of constant curvature.

10.45–11.00 H. HADWIGER, Zur kinematischen Hauptformel der Integralgeometrie.

11.05–11.20 K. STRUBECKER, Minimalflächen des isotropen Raumes.

11.25–11.40 K. P. GROTEMEYER, Kongruenzsätze für isometrische, offene vollständige Flächen positiver Krümmung.

11.45–12.00 G. FERNÁNDEZ, On developable surfaces in the four dimensional space of constant curvature.

12.05–12.20 D. BLANUŠA, Einige Resultate über die Einbettung zweidimensionaler Raumformen konstanter Krümmung in höhere Räume konstanter Krümmung.

12.45– 1.00 V. DALLA VOLTA, On Siegel-Hua spaces; plane elements of zero sectional curvature.

Section IIIc room 10

In the chair: J. C. H. GERRETSEN

10.25–10.40 P. VINCENSINI, Sur une représentation de l'espace réglé dans E_4.

10.45–11.00 A. MARCHAUD, Points singuliers des surfaces du troisième ordre de la géométrie finie.

11.05–11.20 F. GHERARDELLI, Birational covariants of linear systems of curves on algebraic surfaces.

11.25–11.40 E. LLUIS R., Sur certaines projections des variétés algébriques.

11.45–12.00 C. FUSA, Osservazioni intorno alle classificazioni delle superficie ellittiche.

12.05–12.20 B. A. ROSINA, Sur les quadriques généralisées.

Section IV room 24

In the chair: F. N. DAVID

10.25–10.40 F. N. FRENKIEL (speaker) and J. KAMPÉ DE FÉRIET, Correlation for truncated samples of a random function.

10.45–11.00 S. RUSHTON, Sequential procedures in the analysis of variance.

11.05–11.20 C. VAN EEDEN, Sequential test with three possible decisions for the comparison of two unknown probabilities.

11.25–11.40 V. BOHUN-CHUDYNIV, On a general method for constructing completely orthogonal, $2^k \times 2^k$ squares $(k \geqq 1)$ by using closed orthogonal systems of K-nions.

11.45–12.00 G. KLERK-GROBBEN, Confidence limits for the ratio of two means.

12.05–12.20 F. AZORIN, On the sum of independent k-dimensional rectangular variates.

Section Va room 20

In the chair: A. González Dominguez

10.25–10.40 F. Roger, Méthode matricielle de couplage des constituants en spectroscopie moléculaire.

10.45–11.00 A. Pignedoli, About some researches in diffusion problems of mathematical physics.

11.05–11.20 N. G. van Kampen, The analytic behavior of the scattering matrix.

11.25–11.40 G. Bodiou, Sur la correspondance entre bivecteurs et spineurs.

11.45–12.00 R. Faure, Transformation conforme en mécanique ondulatoire généralisation de la notion de valeur propre.

12.25–12.40 R. G. Campbell, A generalization of Fejér's formula in a series of orthogonal polynomials.

12.45–1.00 A. Gonzales Dominguez, On some distributions of quantum electrodynamics.

Section Vb room 16

In the chair: T. M. Vogel

10.25–10.40 W. Quade, Numerische Integration gewöhnlicher Differentialgleichungen durch Interpolation nach Hermite.

10.45–11.00 W. A. Mersman, Numerical calculation of certain inverse Laplace transforms.

11.05–11.20 J. W. Green, The solution of parabolic partial differential equations by difference methods I.

11.25–11.40 S. Kirkby (speaker) and T. R. F. Nonweiler, The numerical solution of certain differential equations occurring in Crocco's theory of the laminar boundary layer.

11.45–12.00 J. V. Garwick, The optimal approximation of functions by polynomials.

12.05–12.20 R. A. Fairthorne, Generating functions of number languages.

12.25–12.40 H. Wallman, An electronic computing machine for the solution of differential and integral equations.

12.45–1.00 C. Ross, Use and abuse of modern computers.

Section Vc room 21

In the chair: R. Timman

10.25–10.40 I. Proudman and W. H. Reid (speaker), On the production of vorticity in a normally distributed and isotropic turbulent velocity field.

10.45–11.00 J. Bass, Sur les solutions aléatoires de certaines équations aux dérivées partielles.

11.05–11.20 Tchen, Chan-Mou, Hydrodynamical stability of interfacial oscillations of two superposed streams.

11.45–12.00 W. Eckhaus and A. I. van de Vooren (speaker), Aerodynamic forces on oscillating swept wings of large aspect ratio in incompressible flow.

12.05–12.20 A. P. Burger (speaker) and R. Timman, Asymptotic solution at high frequencies of the boundary problem of diffraction by a strip.

12.25–12.40 C. K. Thornhill, The diffraction of a shock of moderate strength around a right-angled corner.

12.45–1.00 D. C. Pack, Oscillations of an axially symmetrical supersonic jet of gas embedded in a supersonic stream.

Section VII room 11

In the chair: S. MacLane

Session dedicated to inquiring of the International Committee on Mathematical Instruction on: „The Part of Mathematics and the Mathematician in Contemporary Life".

10.25–10.55 G. Kurepa, Le rôle des mathématiques et du mathématicien à l'époque actuelle.
(*Address by invitation of the International Committee on Mathematical Instruction*).

11.05–11.20 G. Ascoli, Le rôle de la mathématique et du mathématicien dans la vie contemporaine.

11.25–11.40 G. Darmois, Le rôle du mathematicien dans la vie contemporaine.

11.45–12.00 D. v. Dantzig, The function of mathematics in modern society and its consequences for the teaching of mathematics.

Continuation of reports on: „Mathematical Instruction for students between 16 and 21 years of age."

12.25–12.40 F. Hohenberg, Der mathematische Unterricht in Österreich.

12.45–1.00 S. Bundgaard, Report on Mathematics Instruction in Denmark.

WEDNESDAY, 8th SEPTEMBER, AFTERNOON

Section Ia room 7

In the chair: W. Ledermann

2.40–2.55 G. de Beauregard Robinson, The modular representation theory of the symmetric group.

3.00–3.15 F. WEVER, Darstellung von Gruppen als Faktorgruppen von invariant zugeordneten Gruppen.

3.20–3.35 S. PICCARD, Quelques invariants des groupes d'ordre fini.

3.40–3.55 B. H. NEUMANN, Groups covered by permutable subsets.

4.00–4.15 F. LOONSTRA, The groupextension of the group of the integers by that same group.

4.20–4.35 M. LAZARD, Sur une méthode de démonstration de certaines identités dans les groupes.

4.40–4.55 P. JAFFARD, Sur certains groupes réticulés.

5.00–5.15 B. HUPPERT, Überauflösbare Gruppen.

5.40–5.55 G. PICK, Sur une nouvelle généralisation de la notion de nilpotence d'un groupe.

Section Ib room 8

In the chair: F. K. SCHMIDT

2.40–2.55 W. H. H. PETERSSON, Das asymptotische Verhalten von kombinierten Partitionenfunktionen.

3.00–3.15 A. AIGNER, Die kubische Fermat-gleichung in quadratischen Körpern.

3.20–3.35 V. JARNÍK, Approximations diophantiennes linéaires et homogènes.

3.40–3.55 G. RICCI, Sur la différence entre nombres premiers consécutifs.

4.00–4.15 B. VOLKMANN, On the fractional dimension of certain sets in number theory.

4.20–4.35 K. PRACHAR, On a result of Walfisz.

4.40–4.55 P. KUHN, Über die Primteiler eines Polynoms.

5.00–5.15 S. SARANTOPOULOS, Sur le premier cas du (dernier) théorème de Fermat.

Section IIa room 16

In the chair: J. C. BURKILL

3.00–3.15 E. R. LORCH, The concept of volume for convex bodies in Hilbert space.

3.20–3.35 E. THOMA, Über vollständige Erweiterungen linearer, stetiger Abbildungen.

4.00–4.15 M. DOLCHER, Exceptions to n-covering for continuous mappings of a plane region.

4.20–4.35 B. FUGLEDE, Closed extensions of partial differential operators.

4.40–4.55 J. SEBASTIÃO E SILVA, Sur certains espaces vectoriels localement convexes.

5.00–5.15 E. de Giorgi, Una nuova definizione di varietà k-dimensionale orientata, e di misura k-dimensionale di un insieme di uno spazio r-dimensionale.

Section IIb room 18

In the chair: M. M. Schiffer

2.40–2.55 F. W. Schäfke, Beiträge zur Theorie der speziellen Funktionen der mathematischen Physik.

3.00–3.15 H. Wittich, Funktionentheoretische Eigenschaften der Lösungen gewöhnlicher Differentialgleichungen.

3.20–3.35 N. A. Bowen, The relation between a lemma of J. M. Whittaker and convergence theorems of Vitali, Blaschke and Montel type.

3.40–3.55 F. M. Reza, Some geometrical properties of the „Root Locus" curve.

4.00–4.15 G. E. Forsythe (speaker) and E. G. Straus, On best conditioned matrices.

4.20–4.35 W. B. Jurkat, Vorzeichenverteilungen in Matrizen.

4.40–4.55 G. Alexits, La convergence presque partout des développements de fonctions continues en séries de polynomes orthogonaux.

5.00–5.15 S. Schottlaender, Über eine analytische Methode zur Untersuchung automatisch gesteuerter Bewegungen.

Section IIc room 21

In the chair: A. Pfluger

2.40–2.55 R. Osserman, On a conjecture in the problem of type for simply-connected Riemann surfaces.

3.00–3.15 M. Maschler, Properties of minimal domains, (read by S. Bergmann).

3.20–3.35 G. af Hällström, Ein eindeutiger Ordnungsbegriff bei Funktionen mit nullberandetem Existenzgebiet.

3.40–3.55 F. Huckemann, Über den Einfluss von Randstellen Riemannscher Flächen auf die Wertverteilung.

4.00–4.15 L. Cattabriga, Bemerkungen über das verallgemeinerte Dirichletproblem.

4.20–4.35 H. P. Künzi, Neue Beiträge zur Wertverteilungslehre.

4.40–4.55 E. Martinelli, Sur les intersections des variétés analytiques complexes.

5.00–5.15 M. Radojčić, Sur les suites de fonctions algébriques et l'existence des fonctions analytiques ayant un domaine d'existence quelconque.

Section IId room 22

In the chair: W. J. TRJITZINSKY

2.40–2.55 H. PAILLOUX, Calcul symbolique et équations aux dérivées partielles.

3.00–3.15 F. BERTOLINI, Sul problema di Cauchy per la equazione di Laplace in più variabili indipendenti.

3.20–3.35 E. MAGENES, Sur les problèmes aux limites mixtes relatifs aux équations linéaires aux dérivées partielles du second ordre.

3.40–3.55 G. E. HEILBRONN, Invariant relatif à la caractéristique implicite des équations $s = f(x, y, z, p, q, r)$.

4.00–4.15 N. SALTIKOV, Recherches des intégrales de S. Lie d'équations aux dérivées partielles du premier ordre.

4.20–4.35 M. VOLPATO, Sopra un problema di valori al contorno per l'equazione differenziale

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}, \lambda).$$

4.40–4.55 L. AMERIO. Varietà analitiche chiuse trasformate in sè dai sistemi differenziali periodici.

5.00–5.15 M. SCE, Sulla validità di formule integrali nelle algebre.

5.20–5.35 A. M. KOLMOGOROV, Über Eigenfunktionen partieller Differentialgleichungen auf abgeschlossenen Mannigfaltigkeiten.

Section IIIa room 13

In the chair: L. A. SANTALÓ

2.40–2.55 F. SEMIN, Sur une propriété caractéristique des surfaces à courbure moyenne constante.

3.20–3.35 G. SABAN, On a class of cylindrical congruences.

3.40–3.55 C. PAPAÏOANNOU, Sur la théorie des complexes de courbes.

4.00–4.15 A. DOU, Le rang des 4-réseaux de courbes dans le plan.

4.20–4.35 M. DECUYPER, Sur des surfaces particulières.

Section IIIb room 24

In the chair: I. JOHANSSON

2.40–2.55 M. VACCARO, Sulla caratteristica dei complessi simpliciali n-dimensionali χ-omogenei.

3.00–3.15 S. WYLIE, Abstract cell complexes.

3.20–3.35 I. FÁRY, Sur la catégorie des classes de cohomologie d'un espace.

3.40–3.55 A. HELLER, Homological resolutions of complexes with operators.

4.00–4.15 R. L. WILDER, A type of connectivity.

4.20–4.35 E. CH. ZEEMAN, Dihomology.

4.40–4.55 M. KATĚTOV, K nekotorym voprosam teorii razmernosti. (Probleme der Dimensionstheorie).

5.00–5.15 S. LEMOINE, Sur les variétés localement déformables d'un espace complet.

5.20–5.35 P. S. ALEXANDROV, O gomeomorfisme tocesnyh mnozestv. (Über die Homeomorphie von Punktmengen).

Section IIIc room 15

In the chair: F. CHÂTELET

2.40–2.55 M. BENEDICTY, Neutral fields on algebraic curves.

3.00–3.15 F. E. P. HIRZEBRUCH, Der Satz von Riemann-Roch und das Toddsche arithmetische Geschlecht für algebraische Mannigfaltigkeiten.

3.40–3.55 I. BARSOTTI, Structure of group-varieties.

4.00–4.15 L. ROTH, Pseudo-abelian varieties.

4.20–4.35 W. L. EDGE, The projective orthogonal and linear fractional representations of the simple group of order 360.

5.20–5.35 P. DEDECKER, La suite spectrale d'un système différentiel extérieur.

Section IV room 2

In the chair: E. LUKACS

2.40–3.10 D. BLACKWELL, Controlled Random Walks.
(Address by invitation of the Organizing Committee).

3.20–3.50 R. FORTET, Lois des grands nombres pour des éléments aléatoires généraux.
(Address by invitation of the Organizing Committee).

4.00–4.15 A. H. COPELAND SR., A process related to certain visual phenomena.

4.20–4.35 P. K. BOSE, Normalization of frequency function.

4.40–4.55 R. FORTET, Convergence de la répartition empirique vers la répartition théorique, pour les éléments aléatoires généraux.

5.00–5.15

Section Va room 14

In the chair: J. L. SYNGE

2.40–2.55 Rev. J. R. MCCONNELL, The negative proton problem.

3.20–3.35 B. M. SEN, A new classical theory of light and matter.

3.40–3.55 W. B. BONNOR, Stability of the expanding universe.

4.00–4.15 V. KNICHAL, Cartesische Darstellungen des Minkowskischen Raumes.

4.20–4.35 B. Finzi, Deduzione variazionale delle equazioni del campo relativistico unitario.

4.40–4.55 P. Udeschini, Successiva linearizzazione delle equazioni del campo unitario einsteiniano.

5.00–5.15 M. Pastori, Sul tensore elettromagnetico della teoria unitaria di Einstein.

5.20–5.35 E. B. Schieldrop, A classical principle with a „relativistic" path element containing least action as a specialisation.

5.40–5.55 E. Storchi, Sul principio dell'azione potenziale.

Section Vb room 20

In the chair: E. T. Copson

2.40–2.55 W. Magnus, On the exponential solution of differential equations for a linear operator.

3.00–3.15 S. D. Daymond, The evaluation of certain eigenvalues of the equation $\nabla^2\Phi + \nu\Phi = 0$, where the domain is an ellipse and when (*i*) $\Phi = 0$, (ii) $\delta\Phi/\delta n = 0$ on the boundary.

3.20–3.35 P. Brousse, Sur une classe d'équations elliptiques présentant une ligne singulière.

4.00–4.15 C. J. Tranter, Dual integral equations.

4.20–4.35 H. W. Wielandt, Error bounds for eigenvalues of hermitian integral equations and infinite matrices.

4.40–4.55 R. Guy, Sur une équation intégrale opératorielle dans un espace abstrait de Hilbert.

5.00–5.15 J. Peltier, Analyse spectrale de certaines familles de matrices.

5.20–5.35 E. Isaacson, On the numerical solution of mixed initial boundary value problems for hyperbolic equations.

5.40–5.55 H. J. Greenberg, Variational theorems and the eigenvalue problem for plane strain in perfect plasticity.

Section Vc room 11

In the chair: R. Sauer

2.40–2.55 A. D. Wasel, A method for determination of subsonic flow patterns (read by H. J. Bremermann).

3.00–3.15 B. R. Seth, Synthetic method for compressible flow.

3.20–3.35 W. H. McCrea, Compressible flow in a gravitational field.

3.40–3.55 J. B. Diaz and G. S. S. Ludford (speaker), An approximation for the transonic flow of a gas.

4.00–4.15 O. H. B. Behrbohm, Zur Theorie der homogenen linearisierten Überschallströmungen.

4.20–4.35 A. H. TAUB, Singularities on shocks.

4.40–4.55 C. W. JONES, One-dimensional non-homentropic gas flow.

5.00–5.15 R. BERKER, Sur les équations du mouvement d'un gaz.

5.20–5.35 R. DE POSSEL (speaker) and J. VALENSI, Sur le sillage d'un obstacle perméable.

5.40–5.55 H. E. SALZER, The use of Poisson's formula in pattern synthesis.

THURSDAY, 9th SEPTEMBER, MORNING

Section Ia room 3

In the chair: R. BAER

9.00–9.15 K. W. GRUENBERG, Residual properties of groups.

9.20–9.35 W. GASCHÜTZ, Modulare Darstellungen endlicher Gruppen, die durch freie Gruppen induziert werden.

9.40–9.55 F. L. BAUER, Zur Darstellungstheorie der Spingruppe.

10.20–10.35 A. M. MACBEATH, The class-number of the symmetric semigroup.

10.40–10.55 G. B. PRESTON, Inverse semigroups.

Section Ib room 11

In the chair: H. A. HEILBRONN

9.20–9.35 O. TAUSSKY-TODD, Normal matrices in some problems in algebraic number theory.

9.40–9.55 L. S. GODDARD, An extension of a matrix theorem of A. Brauer.

10.00–10.15 V. BOHUN-CHUDYNIV, On methods of constructing orthogonal square matrices of every order composed of differing integers (also complex numbers, quaternions, octonions, sedecimions, etc.)

10.20–10.35 B. W. JONES, On balanced incomplete block designs.

10.40–10.55 R. RADO, A partition calculus.

Section IIa room 18

In the chair: G. NÖBELING

9.00–9.15 J. C. BURKILL, An integral for distributions.

9.20–9.35 H. KÖNIG, Multiplikation von Distributionen.

9.40–9.55 A. C. OFFORD, Some applications of the theory of probability in analysis.

10.20–10.35 G. KUREPA, Some induction principles.

10.40–10.55 K. WAGNER, Über ein abgeschwächtes Auswahlpostulat.

Section IIb room 20

In the chair: W. W. ROGOSINSKI

9.00–9.15 L. S. BOSANQUET, On convergence and summability factors in a sequence.

9.20–9.35 H. DELANGE, Sur un procédé de sommation des séries divergentes.

9.40–9.55 G. A. GARREAU, Some types of infinite matrices.

10.00–10.15 W. B. JURKAT, Gliedweise Integration und Einzigkeitssätze bei trigonometrischen Reihen.

10.20–10.35 J. KARAMATA, Remarque relative à la sommation des séries de Fourier par les procédés de Nörlund.

10.40–10.55 A. PEYERIMHOFF, Lokalisationssätze für absolute Cesàrosummierbarkeit von trigonometrischen Reihen.

Section IIc room 16

In the chair: F. SMITHIES

9.00–9.15 L. FANTAPPIÈ, Calcolo degli autovalori e autofunzioni degli operatori „Fisici" su un gruppo topologico compatto. (Le calcul des valeurs propres des opérateurs physiques).

9.20–9.35 F. F. BONSALL, Endomorphisms of partially ordered vector spaces.

9.40–9.55 I. HALPERIN, On the non-reflexive L-function spaces.

10.00–10.15 A. SARD, Linear functionals on $K_{p,q}$, $B_{p,q}$.

10.20–10.35 J. SCHMIDT, The existence of orthogonal bases in abstract spaces.

10.40–10.55 K. ZELLER, *FK*-Räume.

Section IId room 21

In the chair: S. MANDELBROJT

9.00–9.15 P. LELONG, Mesures de Radon associées à une fonction plurisousharmonique. Application au calcul des fonctions entières de n variables complexes ayant des zéros donnés.

9.20–9.35 R. BADER, Fonctions à singularités polaires sur les surfaces de Riemann ouvertes.

9.40–9.55 J. HEINHOLD, Zur konformen Abbildung einfach zusammenhängender schlichter Gebiete.

10.00–10.15 S. M. NIKOLSKII, Graničnye svoĭstva funktsiĭ mnogih peremennyh i primenenie ih k variatsionnym zadačam (Boundary properties of functions of several variables with application to variational problems).

10.20–10.35 G. B. RIZZA, On Dirichlet's problem for components of analytic functions of several complex variables.

Section IIe room 15

In the chair: J. FAVARD

9.00–9.15 W. MAAK, Fastperiodische Funktionen und Ergodensatz.

9.20–9.35 N. STULOFF, Total monotone fastperiodische Funktionen.

9.40–9.55 A. P. GUINAND, Concordance and the Riemann Zeta-function.

10.00–10.15 C. OBI, Uniformly almost periodic solutions of non-linear differential equations of the second order.

10.20–10.35 O. EMERSLEBEN, Über Funktionalgleichungen zwischen Epsteinschen Zetafunktionen gleichen Arguments und verschiedener Parameter.

10.40–10.55 L. TCHAKALOFF, Über eine Art Faktorenfolgen in der Theorie der algebraischen Gleichungen.

Section III room 2

In the chair: R. MOUFANG

9.40–10.10 K. YANO, On pseudo-Hermitian and pseudo-Kaehlerian manifolds. *(Address by invitation of the Organizing Committee).*

10.20–10.50 H. FREUDENTHAL, La topologie dans les fondements de la géometrie.
(Address by invitation of the Organizing Committee).

Section IV room 13

In the chair: D. G. KENDALL

9.20–9.35 CH. BLANC, Evaluations stochastiques d'erreurs.

9.40–9.55 D. E. BARTON, Neyman's ψ_k^2 test of goodness of fit when the null hypothesis is composite.

10.00–10.15 C. MISHRA, On alternative methods of representing univariate distributions by mathematical curves.

10.20–10.35 M. ZIAUD-DIN, On development of symmetric functions and symmetric functional statistics.

10.40–10.55 G. POMPILJ, Lo schema di Coolidge generalizzato.

Section Va room 7

In the chair: L. AMERIO

9.00–9.15 L. G. CHAMBERS, The geometrical interpretation of Rayleigh's principle and Schwinger's variational principle.

9.20–9.35 M. J. BECKMANN, On a variational problem in the mathematical theory of production.

9.40–9.55 R. Bellman and J. M. Danskin (speaker), On differential-difference equations and control problems.

10.20–10.35 L. Caprioli, Sul comportamento energetico di alcuni sistemi non-lineari autoefficienti.

Section Vb room 8

In the chair: O. H. B. Behrbohm

9.00–9.15 M. Holt, Linear perturbations of hyperbolic problems in three independent variables.

9.20–9.35 R. Gibrat, Etudes mathématiques nécessitées par l'utilisation de l'énergie des marées.

9.40–9.55 J.–L. Destouches, Sur un problème mathématique posé par la physique théorique.

10.00–10.15 R. F. Dressler, Entropy changes in the equations for rarefaction waves.

10.20–10.35 F. Stallmann, Über konforme Abbildung von Kreisbogen-polygonen.

10.40–10.55 H. K. Dettmar, Bemerkungen über eine Klasse symmetrisierbarer Eigenwertaufgaben.

11.25–12.25 room 13

In the chair: S. Iyanaga

A. Weil, Abstract vs. classical algebraic geometry.
(Address by invitation of the Organizing Committee).

11.15–12.15 room 2

In the chair: O. Veblen

P. S. Alexandrov, Isbrannije woprosy koretiko mnogeotwennij topologii sa poslednie 20 let (Aus der mengentheoretischen Topologie der letzten zwanzig Jahren).
(Address by invitation of the Organizing Committee).

THURSDAY, 9th SEPTEMBER, AFTERNOON

2.30–3.30 Concertgebouw, van Baerlestraat

In the chair: J. A. Schouten

A. N. Kolmogorov, Obščaja teorija dinamičeskih sistem i klassičeskaja mehanika (Théorie générale des systèmes dynamiques et mécanique classique).
(Address by invitation of the Organizing Committee).

3.45 Concertgebouw, van Baerlestraat

CLOSING SESSION*)

*) See also p. 151 f.f.

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS AMSTERDAM 1954

(Secretary's Report)

I. The preparations

At the final plenary session of the International Congress of Mathematicians 1950, held in Cambridge, Mass., the Congress accepted the invitation of the delegation from the Netherlands to hold the next Congress in Amsterdam.

Immediately after its return, the Netherlands delegation reported this decision to the Netherlands Mathematical Society (Het Wiskundig Genootschap). It was obvious that the International Congress of Mathematicians 1954 should be held under the auspices of the ,,Wiskundig Genootschap", the society which unites all Netherlands mathematicians and whose 175th anniversary was to be celebrated in 1953.

In October 1950 the Board of the "Wiskundig Genootschap" appointed a small committee with the task to draw up a report on structure and regulations of the Congress. This Committee consisted of the professors:

H. D. Kloosterman (chairman)
J. Haantjes (secretary)
O. Bottema
J. F. Koksma
J. Popken;

Professor D. van Dantzig, at that time President of the Wiskundig Genootschap, took part in the discussions.

The Committee submitted its report at the beginning of 1951. Its main points may be summarized as follows:

a. A description of the aims of the Congress:

The most important aim of the Congress is to stimulate scientific research in the various branches of mathematics and to further good understanding and more intensive cooperation among the mathematicians of all countries. In order to attain this aim, mathematicians of all nations should be invited to meet for a short period for exchange of ideas, which should be realized by organizing lectures and by creating an informal contact between the participants in the Congress.

b. An advice on the lectures:

1. About 20 mathematicians should be invited by the Organizing Committee for such *lectures* which may be of interest to large groups of participants, e.g. lectures which furnish a survey of the recent developments of mathematics in various not too limited domains of mathematical science. Each of these lectures should last about *one hour*.

2. About 40 experts in the various main branches of mathematics should be invited to give *half-hour lectures*.

3. *Short lectures* to be given by members of the Congress expressing a wish to do so: If possible, the time to be allotted to a short lecture should not be less than a quarter of an hour, the final decision depending on the number of applications. The Program Committee should be requested to group the short lectures in such a way that lectures on the same or mutually related problems should be held in the same sessions where related half-hour lectures were to be held.

After the one-hour lectures there should be no discussion; after the half-hour lectures and the short lectures there might be a discussion.

c. The recreational program should be in accordance with the aims of the Congress as expressed above, a special ladies program being desirable.

d. The Congress should last about a week, e.g. from Thursday to Thursday. On Sunday the Congress office should be closed and no scientific sessions should take place.

e. It being undesirable that the scientific program should be known to members only at the beginning of the Congress, the program should be sent to them some time before the Congress.

It was thought desirable that the abstracts of the short lectures should be offered to the members in the form of preprints. If possible, the volume of the preprints should be identical with one of the volumes of the Proceedings.

f. The Board of the "Wiskundig Genootschap" was requested to establish as soon as possible:

A. an Organizing Committee and
B. a Financial Committee

and later, on proposal of the Organizing Committee, the necessary subcommittees.

The chairmen of the sub-committees should belong to the Organizing Committee; on the other hand the members of the Organizing Committee should be spread at least over all Netherlands Universities with a mathematical department.

g. The "Wiskundig Genootschap" might appoint a president designate of the Congress on the proposal of the Organizing Committee and might also establish a committee of honour, etc.

h. all working committees mentioned, should be responsible to the Board of the "Wiskundig Genootschap". The "Wiskundig Genootschap" should fix definite dates for the Congress on a proposal of the Organizing Committee. The Congress should take place in Amsterdam.

The report of the Structure Committee was approved of by the "Wiskundig Genootschap". In fact most of its proposals have been realized although in the light of later experience on some points the Organizing Committee made some variations. Thus, the Structure Committee had raised the question whether there should be official languages in the Congress or not and had suggested a number of languages to be chosen as official ones. In view of the fact, however, that in any case the organizers would be unable to handle all languages which might be appointed as official ones, it was decided later that all languages should be admitted. On the other hand, for technical reasons the secretariat in its foreign correspondence had to restrict itself mainly to English, French and German.

On its own advice, the Structure Committee was discharged of its task and a small committee consisting of

H. D. Kloosterman
J. F. Koksma
J. P. van Rooijen

was appointed to take the first practical steps and to prepare the institution of further committees by the Board of the Mathematical Society.

In the course of 1951 several steps were prepared and several contacts were laid.

In view of the fact that a large attendance might be expected, there was some doubt whether the Congress could be held in Amsterdam in view of the hotel accommodation.

The first task, therefore, was a thorough investigation of the possibilities in Amsterdam and in the Hague (Scheveningen). Also the choice of the dates of the Congress was connected with this problem, as both cities are overcrowded in the summer season which lasts till September 1st at least.

A report on these problems was drawn up by a group of mathematicians for discussion by the Organizing Committee.

In view of regulation and limitation of financial and juridical responsibilities, in October 1952 a foundation: "Stichting Internationaal Mathematisch Congres 1954" was established. In the same year several committees started their work, viz.:

The Central Organizing Committee
The Financial Committee

The Technical Committee
The Program Committee
The Budget Committee
The Proceedings Committee
The Section Committees
The Entertainment Committee

the constitution of which is to be found on pages 10–12 of this Volume, and the work of which was coordinated by the Executive Committee of the Central Organizing Committee, consisting of

J. A. Schouten, President
H. D. Kloosterman, Vice-President
J. F. Koksma, Secretary
F. Loonstra, 2nd Secretary
J. Haantjes, Treasurer.

In the discussions of this Committee representatives of the various subcommittees also took part; thus Prof. Dr N. G. de Bruijn as secretary of the Program Committee, Prof. Dr A. Heyting as President of the Technical Committee, Dr J. Seidel as President of the Entertainment Committee regularly attended the about 30 official meetings of the Executive Committee.

In order to draw the attention of authorities and commercial and industrial circles to the Congress, the Board of the Wiskundig Genootschap and the Executive Committee of the Organizing Committee distributed a booklet, in which the history, the aims and the importance of the International Congress of Mathematicians were exposed.

His Royal Highness, Prince Bernhard of the Netherlands, consented to give his Patronage to the Congress.

Moreover, a Committee of Honour including several high authorities was established (see pages 7–8 of this Volume). The Financial Committee was extended with several representatives from industrial and commercial circles. Owing to the work of this Committee considerable subsidies from commercial and industrial bodies were granted to the Congress (see the list of donors, page 9 of this Volume).

As to the Congress funds, they consisted of a) the membershipfees (D.Gld. 50,— for regular members and D.Gld. 20,— for associate members) [1]), b) a

[1]) A. A special arrangement was made in case the wife of a regular member being a mathematician herself, wished to have the rights of a regular member as well as the rights of an associate member, however, without the right of receiving a copy of the Proceedings. In such a case her fee was fixed on D.Gld. 30,—.

B. For those not wishing to attend the banquet, all fees mentioned here were diminished by D.Gld. 10,—.

subsidy by the Netherlands Government, c) a subsidy by the Municipality of Amsterdam, d) several grants mainly by commercial and industrial bodies as mentioned above, e) a grant by UNESCO through the intermediary of IMU, exclusively intended as a subvention to the cost of editing the Congress Proceedings.

At the end of 1952 it was definitely decided that the Congress would take place in Amsterdam from September 2nd till September 9th, so that the first communication could be prepared. It appeared at the beginning of 1953. The second communication was sent out at the end of 1953. Both communications were sent to the Executive Committees of all mathematical societies known to the Organizing Committee; in addition they were sent to the editors of mathematical journals, to the mathematical departments of Universities, to the Senates of Universities and to the Academies of Science. Moreover, the Academies of Science, the Universities, the Mathematical Societies and some other Institutions were invited to appoint delegates to the Congress.

In the meantime, the Ministry of Foreign Affairs sent out invitations to foreign Governments to send representatives to the Congress. The authorities of this Ministry and of other Ministries rendered all possible help to the Organizers of the Congress with respect to difficulties concerning visas of the Congressists, etc.

The Program Committee, after several sessions, had drawn up a list of mathematicians who might be invited to give the one-hour lectures (about 20) and the half-hour lectures (about 40). The following mathematicians accepted the invitation to give a one-hour lecture:

P. S. Alexandrov
K. Borsuk
R. Brauer
D. van Dantzig
J. A. Dieudonné
I. M. Gelfand
S. Goldstein
Harish Chandra
B. Jessen
A. N. Kolmogorov
A. Lichnerowicz
J. von Neumann
J. Neyman
S. M. Nikolskii
B. Segre

E. Stiefel
A. Tarski
E. C. Titchmarsh
A. Weil
K. Yosida

ctions

It was decided that the Congress should have seven sections, viz.:

I. Algebra and Theory of Numbers
II. Analysis
III. Geometry and Topology
IV. Probability and Statistics
V. Mathematical Physics and Applied Mathematics
VI. Logic and Foundations
VII. Philosophy, History and Education.

alf-hour ctures

The following mathematicians accepted the invitation to give a half-hour lecture:

Section I

H. Davenport
P. Erdös
E. Hlawka
N. Jacobson
H. Maasz
A. Néron
D. G. Northcott

Section II

H. A. L. Behnke
F. Bureau
M. L. Cartwright
L. Cesari
K. Chandrasekharan
A. Erdélyi
W. K. Hayman
E. Hille
K. Kodaira
P. J. Myrberg
Chr. Pauc
T. Ważewski
A. Zygmund

Section III

H. S. M. Coxeter
B. Eckmann
H. Freudenthal
D. Montgomery
J. P. Serre
K. Yano

Section IV

D. Blackwell
J. L. Doob
R. Fortet

Section V

L. Collatz
G. Fichera
M. R. Hestenes
J. Kampé de Fériet
F. Rellich
J. J. Stoker
A. Weinstein

Section VI	*Section VII*
P. Lorenzen	C. T. Daltry
A. Mostowski	K. Piene
J. Barkley Rosser	D. Kurepa

(Professor Kurepa was invited on behalf of the Executive Committee of ICMI).

In drawing up the program, the one-hour lectures were placed at the beginning of the day. In view of their number it generally was necessary to have two of them at the same time, but on topics chosen from different fields. The half-hour lectures took place after the one-hour lectures in the rooms where the short lectures, belonging to related fields, were to be held during the remainder of the morning or during the afternoon.

More than 550 members applied to the Congress Committee for giving a short lecture. Photocopies of the abstracts of those lectures were made in the Mathematical Centre and sent to the secretaries of the section committees concerned, who forwarded them to the referees. Most of them could be accepted without further correspondence with the authors. 443 abstracts could be admitted in time and were printed in Volume II of the Proceedings which appeared before the Congress and was handed to the members. Several manuscripts came too late to be printed before the Congress.

In Volume I were published 108 further abstracts, so in all there were accepted 551 manuscripts. The number of short lectures actually given was 496 as at the last moment 55 authors were prevented from attending the Congress. In addition to the above-mentioned lectures, organized by the International Congress of Mathematicians, in Section VII two series of lectures viz. on "Mathematical Instruction for students between 16 and 21 years of age" and "The Part of Mathematics and the Mathematician in Contemporary Life" respectively, should be mentioned. These lectures were organized by the International Committee on Mathematical Instruction (I.C.M.I.) in cooperation with the Organizing Committee of the Congress.

During 1953 contracts were made with several hotels in view of possible difficulties with the housing problem, with the Royal Tropical Institute and other bodies with respect to conference rooms, with travelling agencies, etc. in connection with entertainment and excursions.

During 1953 the staff of the secretariat which was housed in the rooms of the Mathematical Centre was quite small, consisting of a professional typist who, if necessary, was assisted by her colleagues of the Mathematical Centre.

In the second half of 1953 two ladies were engaged, Miss L. J. Noordstar as assistant-secretary, and Miss A. C. de Bruijn van Melis- en Mariekerke to deal exclusively with the hotel problem. During 1954 this staff was gradually increased by the appointment of some typists, in addition several students got half-time jobs in the Congress office. When necessary, the staff of the Mathematical Centre put itself at the disposal of the Congress; in particular the IBM-set of the Centre was used for registrational purposes.

Among the technical items may be mentioned the Congress emblem which was designed by Mr D. Kortenoeven, assistant of the Mathematical Centre at Amsterdam. It showed the step-gable of a so-called Amsterdam "Grachtenhuis": the main type of a merchant's house, dating from the 17th century, as one still finds along the canals of Amsterdam. The same design was used for the badge which was carried out in Dutch crockery by the ceramist Mr Marcel B. Keezer at Leiden.

Further, in most lecture-rooms there were traffic-lights operated by the chairman in order to keep the speakers to the time allotted. Yellow light meant: you can speak for another two minutes, red light meant: stop.

Finally may be mentioned the arrangements with the Amsterdam Municipal Tramways, which furnished a passe-partout to the Congressists, allowing unlimited free use of the local trams, busses and ferries during the Congress. Similar arrangements were made with some musea and the Zoological garden "Artis".

II. Some facts and figures

The Congress was attended by 2120 persons, viz. 1553 regular members (1436 men and 117 women) and 567 associate members (47 men and 520 women) [1].

The following list may illustrate the geographical distribution of the 1553 regular members:

Column A gives the total number of members having the nationality of the country concerned;

Column B gives the number of regular members living in the country concerned in 1954;

Column C gives the number among those listed in B having moreover the nationality of the country concerned.

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[1]) Originally 1605 mathematicians applied for regular membership, but 52 cancelled their application; 642 persons applied for associate membership, but 75 cancelled their application.

The (17) wives of mathematicians for whom the special arrangement mentioned in the footnote on page 130 of this report, was made, are listed here under the regular members and not under the associate members.

	A	B	C
Algeria	—	3	—
Argentina	10	10	10
Australia	3	1	—
Austria	19	15	13
Belgium	27	29	26
Brazil	3	3	3
Bulgaria	1	1	1
Canada	17	18	11
China	4	1	1
Columbia	—	1	—
Cuba	1	—	—
Czecho-Slovakia	4	4	4
Denmark	14	18	14
Egypt	3	2	2
Finland	9	9	9
France	138	158	128
Germany	207	212	197
Gold-Coast	—	2	—
Great-Britain	261	261	237
Greece	11	8	8
Hungary	3	2	2
Iceland	1	1	1
India	18	12	11
Indonesia	2	3	2
Iran	3	2	2
Ireland	11	16	11
Israel	10	10	8
Italy	89	91	87
Japan	10	3	3
Latvia	1	—	—
Luxemburg	2	2	2
Mexico	3	3	2
The Netherlands	212	223	210
Nigeria	2	1	1
Norway	21	20	20
Pakistan	2	3	2
Peru	2	2	2
Philippines	2	2	2
Poland	7	6	6

	A	B	C
Portugal	5	4	4
Roumania	5	4	4
Russia	5	5	5
Siam.	1	—	—
Soudan	—	1	—
South-Africa	10	5	4
Spain	8	8	8
Sweden	39	41	38
Switzerland.	41	52	38
Tunisia	—	3	—
Turkey.	12	12	10
United States of America	228	242	198
Uruguay	1	2	1
Venezuela	1	1	1
Vietnam	1	—	—
West-Africa	—	2	—
Yugoslavia	15	13	13
Stateless	12		
No statement.	36		
	1553	1553	1362

For organizational purposes all participants were requested to state whether they were able to understand English, French or German.

The following list gives the numbers of those regular members speaking or understanding the languages mentioned in column 1. The last column refers to associate members only.

Language(s)	number of men	number of women	total	ass. members
French	64	2	66	67
German	34	3	37	25
English	189	16	205	133
French, German	36	2	38	5
German, English	148	9	157	67
French, English	193	29	222	84
French, German, English	339	32	371	71
No statement	433	24	457	115

The following list may illustrate the distribution of the regular members over the main professions:

Profession	Number of men	number of women	total
Univ. Teacher . .	772	53	825
Assistant	71	8	79
Student	78	14	92
Researchman. . .	56	8	64
Teacher	125	11	136

The following list concerns the ages of the 1553 regular members:

Age	number of men	number of women
10—19	—	1
20—29	317	31
30—39	406	32
40—49	347	17
50—59	194	17
60—69	94	5
70—79	26	—
no statement	52	14

All regular members were requested beforehand to state their preference for the different sections. The result may be listed as follows:

Sections	1	2	3	4	5	6	7	no statement
1st choice	374	373	177	104	148	42	22	196
	27	24	15	15	11	2	5	18
	401	397	192	119	159	44	27	214
2nd choice	49	236	197	94	170	73	59	558
	7	11	21	8	9	8	5	48
	56	247	218	102	179	81	64	606
3rd choice	11	10	78	37	83	67	75	1075
	1	6	0	1	2	2	11	94
	12	16	78	38	85	69	86	1169
4th choice	3	2	1	10	19	28	30	1343
	1	0	1	0	1	0	0	114
	4	2	2	10	20	28	30	1457
5th choice	1	0	0	0	4	2	13	1416
	0	0	0	0	0	1	0	116
	1	0	0	0	4	3	13	1532

6th choice	0	0	1	0	0	2	0	1433
	0	0	0	0	0	0	1	116
	0	0	1	0	0	2	1	1549
7th choice	0	0	0	0	0	0	2	1434
	0	0	0	0	0	0	0	117
	0	0	0	0	0	0	2	1551
Totally	438	621	454	245	424	214	201	
	36	41	37	24	23	13	22	
	474	662	491	269	447	227	223	

Each column of three numbers after each choice indicates the preference of respectively: men, women and the total of both.

The Headquarters of the Congress were located in the Royal Tropical Institute. The Conference rooms were situated in the neighbourhood of the Headquarters, whereas opening- and closing sessions took place in the "Concertgebouw" of Amsterdam.

In the Hall of the Institute several Congress services were concentrated: a reception office, where the port-folios (containing Congress documents) and forthcoming communications for members were issued at sight of membership-card, a General Information desk, a Hotel Information desk, a branch-office of the Amsterdam Bank, a post-office with telephone- and telegraph-accommodation for members, and an Excursion desk.

The ladies of the Netherlands members recognisable by a special badge kindly had consented to the Organizing Committee to serve as guides for their foreign colleagues.

III. A bird's eye view of the Congress

On Wednesday, September 1st, a considerable number of members collected their portfolios containing the Congress documents, on presentation of the membership card which was sent to them beforehand.

In the evening an informal meeting was held in the Restaurant rooms of the Amsterdam Zoological Garden "Natura Artis Magistra" (Artis); the Dutch calculating prodigy W. Klein, alias Pascal, gave some performances.

At 10.30 A.M., Thursday, September 2nd, the opening session of the Congress was held in the Amsterdam Concerthall (Concertgebouw). For this occasion the podium was decorated with flowers and with the flags of all countries which were represented in the Congress by their mathematicians.

H. R. H. Prince Bernhard of the Netherlands, who was unable to attend the session, sent a message which was reproduced in the Congress Guide and which also is to be found on page 141 of this Volume.

The President of the "Wiskundig Genootschap" and of the Organizing Committee, Prof. Dr. Ir. J. A. Schouten, spoke the following words of welcome.

OPENING SPEECH BY PROF. J. A. SCHOUTEN

"Mijnheer de Vertegenwoordiger van Zijne Excellentie de Minister van Onderwijs, Kunsten en Wetenschappen, Mijnheer de Vertegenwoordiger van de Commissaris der Koningin in de Provincie Noord-Holland, Mijnheer de Burgemeester, Dames en Heren,

Als Voorzitter van het Nederlands Wiskundig Genootschap en van het Organisatie Comité van dit congres, zij het mij vergund allereerst in de landstaal alle congresbezoekers, die deze taal verstaan, hartelijk welkom te heten en de hoop uit te spreken, dat zij hier een vruchtbare en aangename tijd zullen doorbrengen. Het is niet doenlijk hier alle officiële instanties, instellingen en particulieren op te noemen, die hetzij financieel, hetzij op andere wijze aan de tot stand koming van dit congres hebben meegewerkt en ik moge derhalve er mede volstaan tot hen allen een algemeen woord van hartelijke dank te richten.

As President of the Mathematical Society of the Netherlands and of the Organizing Committee of this Congress, my first duty is to welcome all participants of this Congress and to express the sincere hope that their stay in Amsterdam will be a most successful and most pleasant one. It would be impossible to mention here the names of all official bodies, all institutions and all private persons, who by financial help or in other ways have made the organization of this Congress possible and I can only express here our warmest thanks to all.

En ma qualité de Président de la Société Mathématique Néerlandaise et du Comité Organisateur du présent Congrès je commence par souhaiter la bienvenue à tous les membres du Congrès.

J'espère que vous aurez en cette ville un séjour agréable qui puisse stimuler vos travaux scientifiques. Il n'est pas possible d'énumérer ici les autorités, les institutions et les particuliers qui par leur appui financier ou par un concours d'autre nature ont rendu possible l'organisation du présent Congrès. Qu'il me suffise donc d'exprimer mes vifs remerciements à eux tous.

Als Vorsitzender des Niederländischen Mathematischen Vereins und des Organisationskomitees dieses Kongresses heisse ich alle Teilnehmer herzlichst willkommen. Ich spreche die Hoffnung aus, dass Sie hier eine Woche verleben mögen, die gleichzeitig angenehm und wissenschaftlich anregend ist. Es ist mir nicht möglich hier alle offiziellen Instanzen, alle Institutionen und alle Privatpersonen zu nennen, die uns bei der Organisation sei es finanziell, sei es auf anderem Wege unterstützt haben, und muss ich mich darauf beschränken allen von dieser Stelle aus herzlichst Dank zu sagen.

Come presidente della Società Matematica Olandese e del Comitato d'Organizzazione di questo congresso do il benvenuto a tutti i congressisti. Spero che essi avranno qui un soggiorno piacevole e otterranno qui uno stimulo scientifico. Non è possibile enumerare in questo luogo tutte le istanze uffiziali, tutte le istituzioni e persone, per il cui appoggio, finanziario o personale, questo congresso si è potuto realizzare. Posso quindi soltanto esprimere la mia profonda gratitudine a tutti.

Som ordförande för Matematiska Sällskapet i Nederländerna och för Kongressens organiserande kommitté vill jag först och främst hälsa alla kongressdeltagarna hjärtligt välkomna. Jag hoppas, att vistelsen i Amsterdam skall bli bade fruktbärande och angenäm. Det skulle vara omöjligt att här nämna namnen av alla offentliga och enskilda institutioner och av alla privatpersoner, som genom sin finansiella hjälp eller pa annat sätt möjliggjort organisationen av denna kongress och jag kan endast framföra vart hjärtligaste tack till alla.

Председателю Голландского Математического Общества и организационного комитета настоящего конгресса надлежит первым долгом приветствовать всех участников конгресса и выразить надежду, что их пребывание в Амстердаме будет особенно плодотворным и приятным. Немыслимо упомянуть здесь все официальные инстанции, все учреждения и всех частных лиц, которые посредством финансовой помощи или каким-нибудь иным образом сделали возможным созыв этого конгресса, и я могу только выразить всем им нашу самую сердечную благодарность.

Of course I can not go on in this way because I have already gone beyond the limit of those languages that I can really speak to the limit of the languages that I can only pronounce a little. So you will excuse me if I use from now on the English language.

But first I have to use once more my own language to address Mr d'Ailly Burgomaster of the city of Amsterdam, who will do us the honour to open this Congress."

Professor Schouten then invited Mr Arn. J. d'Ailly to open the Congress.

After the Congress was opened Professor Oswald Veblen asked for permission to speak and addressed the Congress in the following speech:

SPEECH BY PROF. O. VEBLEN

"The series of International Congresses of which this is to be one are very loosely held together. They are not congresses of mathematics, that highly organized body of knowledge, but of mathematicians, those rather chaotic

Paleis Soestdijk
29 Juli 1954

Niet in de gelegenheid zijnde persoonlijk aanwezig te zijn bij de opening van het Internationaal Mathematisch Congres 1954, roep ik langs deze weg een hartelijk welkom toe aan de vele honderden mathematici uit alle delen der wereld, die thans in Nederland zijn samengekomen.

Het doel van dit Congres is gezamenlijk een vak te beoefenen en te bevorderen, dat als weinig andere de eenheid van het menselijk geslacht demonstreert en dat -mede in het licht van recente toepassingen- nog steeds aan belangrijkheid wint.

Ik wens alle deelnemers goede en vruchtbare dagen toe.

Prins der Nederlanden

individuals who create and conserve it. At the end of each congress they somehow agree on the country where the next one is to be held and then leave it to their colleagues in this country to work out a program.

To symbolize the tenuous continuity which is thus achieved, the President of the old congress emerges for a moment from the obscurity in which he belongs, to propose the name of the person selected by the hosts of the new congress to preside over it.

In this case, it is a pleasure for me to utter the name of an old friend and colleague who has distinguished himself by long and successful devotion to one of the most important branches of our sciences. He has also displayed what is not too common a trait, a willingness to do more than his share of the drudgery which is necessary to our common effort. I present the name of Professor J. A. Schouten for President of the Congres of Amsterdam. Will you manifest your approval?"

The Congress applauded Professor Veblen's proposal and Professor Schouten, accepting his election, delivered the following presidential address:

PRESIDENTIAL ADDRESS BY PROF. J. A. SCHOUTEN

"My first task as President is a sad one. I have to ask your attention for the memory of two of our congress members: Professor Fabio Conforto from the University of Rome, who died on the 24th of February and Professor Rodolphe Henri Joseph Germay from the University of Liège, who died on the 16th of May. Their work will live in our minds and our thoughts are with their families.

Then I have to inform you that our Patron, His Royal Highness Prince Bernhard of the Netherlands, is abroad and cannot attend this opening-session. His Royal Highness gave us a message for the Congress, the text of which, with some translations, you will find in the program. I propose to send His Royal Highness the following telegram:

"Aan Z.K.H. Prins Bernhard der Nederlanden.

Het Internationaal Mathematisch Congres 1954 te Amsterdam, bijeen in zijn openingszitting, spreekt zijn welgemeende dank uit voor de hartelijke woorden van welkom en aanmoediging, vervat in de boodschap die Uwe Koninklijke Hoogheid als Beschermheer tot het Congres heeft willen richten. J. A. Schouten Voorzitter".

"To His Royal Highness Prince Bernhard of the Netherlands.

The International Congress of Mathematicians 1954 at Amsterdam, at the opening session, expresses its warmest thanks for the kind words of welcome and encouragement contained in the message of Your Royal Highness as Patron of this Congress".

Finally I wish to draw your attention to a fact which was perhaps not so clear four years ago, but which is absolutely clear now: *the place of mathematics in the world has changed entirely after the second war.* Before, mathematics had an honourable place among the sciences because of its central position, its history and its traditions, but there were in those times not many mathematicians and most people had only some bad memories from their school years and the comforting idea that in real life they would meet mathematics never more. Even some older engineers propagated the idea that the mathematical training of technical students was only a kind of quasi-scientific ornament that could be dropped before long for the greater part because technical methods themselves had by now developed into real sciences!

It was not really necessary to discuss such ideas because real life gave the answer to questions of this kind in a very short time and with the utmost clearness.

During and after the war it became obvious to every one that nearly all branches of modern society in war and in peace need a lot of mathematics of all kinds, from the simplest school arithmetics up to the highest developed theoretical parts. In fact, there is nowadays no big factory without its computing machines and no investigation involving series of experiments or observations is possible without an elaborate application of modern statistics. But computing machines do not work without a staff of very good mathematicians for the programming and modern statistics need also mathematicians of a very high standard. Also the so-called "applied mathematics" came to new life and asked for more men well trained in mathematics and physics, because modern computing machines had made it possible to make use of solutions that formerly only had theoretical value on account of the impossibility of doing the computing work in a reasonable time. It is very remarkable that in connection with all these activities a development of many parts of pure mathematics was necessary, thus making true the word of Felix Klein that all which is mathematically pure will find sooner or later some practical application.

Thus mathematicians of all kinds are needed in numbers our ancestors could not have dreamt of and universities all over the world are constantly busy producing more of them. Even technical universities, instead of dropping a great deal of their mathematics, are now training mathematical engineers, who have to fill the gap between mathematics and practical engineering sciences.

Now this is all very satisfying and we could be content that our science got so prominent a place in the structure of modern society. But some difficulties arise. Sixty one years ago the first mathematical congress at Chicago was attended by 25 mathematicians. In 1936 we had the congress in Oslo with 500 and after the war Cambridge (Mass.) with 2316 and this congress at Amsterdam with 1550 attendants, notwithstanding the fact that in Cambridge there were 1410 Americans and among us only 240.

On the one hand we may be happy with this progress, but on the other hand it is wise not to shut the eyes for the fact already pointed out by Professor Veblen in his opening address at Cambridge, that there is a limit to congresses of this kind. This limit will perhaps be reached very soon if the number of mathematicians goes on increasing as rapidly as it does now and if in the future, as I fervently hope, big countries with a great number of good mathematicians will break with the system of sending a very small delegation, the extent of which is in no way proportional to the mathematical importance of the country involved. This sytem of sending a small delegation only is entirely wrong, the chief aims of a mathematical congress being, as Professor Störmer pointed out in his presidential address at Oslo, to enable the direct exchange of ideas from man to man and to give a great number of younger people the opportunity to get the personal contacts they need for orientation and stimulation. The average age of participants at our congress is 40½ and that is too old.

As Professor Veblen put it, mathematics is so "terribly individual" that a man practically can only speak for himself and this means that instead of a small delegation we need an adequate number of scientists and among them many younger men, to get all the personal contacts we are so urgently looking for.

But if the number of participants increases the question arises: shall we have in future one big congress or instead several smaller meetings on definite topics. In the years after the war we have already had several very small meetings called colloquia and as far as I can see they were a great success. But what I mean here is a splitting up of the big congress into a small number of parts to be held separately but with one central organization. Personally I think that the mutual induction of the several branches of mathematics is so very important that we should try as long as possible to save the idea of one big congress. But, if from purely technical considerations such a congress would become impossible, the splitting up should be done very carefully and with an open eye for the structure of the science of mathematics as a whole.

It is good to remember here a word of Poincaré's, who stated explicitly that in mathematics there are two kinds of mental acting, one above all occupied with logical deduction and the other guided by a more intuitive faculty for arranging or rearranging known facts in a new way satisfying some principle of aesthetics or of unification. Poincaré laid particular stress on the point that the choice of the method is by no means fixed by the matter treated and that it has nothing to do with the difference between analysis and geometry. There are famous analysts using largely the more intuitive faculty and famous geometers working as a rule with deductional methods.

Exaggerating one aspect of mathematics and neglecting the other part invariably leads in the end to undesirable results. It is my personal opinion that the lack of interest in mathematics among young people is for the greater part due to the fact that the more intuitive aspect of mathematics is sometimes neglected, for instance where geometry is reduced to a system of axioms and deductions only, thus overstressing just that aspect of geometry which is most uninteresting for young people of an age between 12 and 18. I am glad that in section VII of this congress this point will be discussed.

In 1905 "L'Enseignement Mathématique" started an inquiry into the methods of working of mathematicians. The results of this inquiry augmented and developed later by several authors, for instance Carmichael and Hadamard, can be expressed shortly as follows. The faculty of deduction belongs more to the conscious mind, the subconscious being in general only able to perform very simple and trivial deductions. On the contrary the faculty of rearranging is typical of the work of the subconscious and is described by Carmichael as consisting of an extremely rapid passing over of innumerable useless combinations till a vital one or some vital ones rise to consciousness, to bring, after a severe control of the conscious mind, new truth to light.

It is remarkable that our modern computing machines can imitate some of the lower parts of both faculties of our mind. In fact, there are machines, effecting a few simple logical deductions, and other machines, especially constructed for the investigation of big molecules, which are able to pass in a short time over say a million possible combinations of phases in order to single out some twenty five most suitable ones for a more detailed examination.

The development of any part of mathematics always involved the action of both faculties of the mind in the same or in different investigations. This we should take into serious consideration if we wish to organize congresses in the future, be it a big one or several smaller ones. It is certainly very important to create the possibility of mutual induction of several sections during a congress, but it is far more important that both faculties of the mind come into their right in every section. So a section for analysis should try to stimulate the influence of the more intuitive faculty and a section for geometry should make sure that sufficient place is reserved for deductive investigation. In order to come to a practical result I should like to ask all of you to give your attention to this point during this congress and especially to observe how the two faculties of mind are really working in every section and to give your opinion as to the sufficiency of their interaction. In this way we might get a scientific inquiry, the results of which could be very valuable for an efficient organization of future congresses."

After the presidential address a musical interlude took place, viz. a piano solo by Mrs Fania Chapiro:

F. Chopin
Impromptu in a flat major op. 29
Nocturne in c minor op. 48
Scherzo no. 2 in b flat minor op. 31

Then Professor Herman Weyl, President of the Fields Medal Committee 1954,[1]) addressed the Congress on behalf of the Fields Medal Committee, expounding the grounds on which the Fields Medals 1954 were awarded to Mr K. Kodaira and Mr J. P. Serre.

After his address, which is printed in full in this Volume on pages 161–174, he presented the Medals to the winners.

A second musical interlude by Mrs Fania Chapiro then followed:

C. Debussy
Suite pour le Piano
(prélude — sarabande — toccata)

Professor Bompiani, Secretary of the Executive Committee of I.M.U. then addressed the Congress with the following speech on behalf of I.M.U., which had held its General Assembly at the Hague on August 31 and September 1.

SPEECH BY PROF. E. BOMPIANI

"The International Mathematical Union has the honour to announce the election of officers and other holders of office for the period January 1, 1955—December 31, 1958, as follows:

President of the International Mathematical Union:
H. Hopf (Switzerland)
1st Vice-President: A. Denjoy (France)
2nd Vice-President: W. V. D. Hodge (U.K.)
Secretary: E. Bompiani (Italy)
Elected Members: K. Chandrasekharan (India), J. F. Koksma (The Netherlands), S. MacLane (U.S.A.)

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[1]) The Fields Medal Committee 1954 consisted of: Prof. H. Weyl (President), Prof. E. Bompiani, Prof. F. Bureau, Prof. H. Cartan, Prof. A. Ostrowski, Prof. Å. Pleijel, Prof. G. Szegö, Prof. E. C. Titchmarsh.

Chairmen of Committees and Commissions:
International Committee on Mathematical Instruction:
H. Behnke (Germany)
Committee on a World Directory:
M. H. Stone (U.S.A.)
Commission on Scientific Publication:
E. Hille (U.S.A.)
Commission on Exchange of Mathematicians:
H. Davenport (U.K.)

The Organizing Committee of this Congress and the Union have chosen a joint committee to recommend the time and place of the next International Congress of Mathematicians — its recommendation will be announced at the closing session of this Congress, on Thursday, September 9th, 1954."

The final address in the opening session was delivered by the representative of the Dutch Government, Mr H. R. Woltjer, Head of the Department of Advanced Education and Sciences of the Ministry of Education, Arts and Sciences:

ADDRESS BY Mr. H. R. WOLTJER

"Mr. President, Mr. Burgomaster of the City of Amsterdam, Ladies and Gentlemen,

The Minister of Education, Arts and Sciences, His Excellency Mr. Cals deeply regrets that official duties prevent his being personally among you this morning. His Excellency has asked me to state that the Netherlands Government feels greatly honoured about this highly important Congress being held in the Netherlands, and to convey to you his best wishes for its success.

The Government may express her gratitude to the Board of the Wiskundig Genootschap and the Organizing Committee who gave all their energy and time for the organization.

Besides, I want to offer my heartfelt congratulations to the young winners of the Fields Medals on obtaining this high distinction granted for outstanding scientific merits. I could imagine that a real scientist attaches but little importance to congratulations offered by a Government's official, not so much so, because activity of the public authorities with science so often implies a threat to its autonomy — the attention and respect of politics, business and finance is directed often wholly to its results — as because science finds its reward in itself. In the spirit of science resides its chief value. This can be asserted without abating anything of the claim for the value of its results. Knowledge for the sake of knowledge, as the history of science proves, is an aim with an irresistible fascination for mankind, which needs no defence. The mere fact that science does, to a great extent, gratify our intellectual curiosity, is a sufficient reason for its existence. Science gives therefore a great satisfaction to those who apply themselves to it.

Keppler, telling about a discovery he had made, says the intense pleasure he had received from his discovery never could be told in words. He regretted no more the time wasted; he tired of no labour; he shunned no toil of reckoning days and night spent in calculating until he could see whether his hypothesis would agree with the orbits of Copernicus or whether his joy was to vanish into air. Every scientist, exerting himself to advance science, will experience something like that. It is the aesthetic value of science which gives this inner satisfaction in all scientific work. If scientific knowledge consisted of a mere inventory of facts, it might still be interesting and even useful, but it would not be one of the major activities of the mind. It would not be pursued with passion. For science to have inspired such ardour and devotion in men it is obvious that it must meet one of the deepest needs of human nature. This need manifests itself as the desire for beauty. It is in its aesthetic aspect that the chief charm of science resides. This is true for scientific men themselves. To the majority of laymen, science is valuable chiefly for its practical application. But to all the greatest men of science practical applications have emerged incidentally, as a sort of by-product. This is, perhaps, most obviously true of the men who created the mathematical sciences. In the work of mathematicians, in particular, the aesthetic motive is very apparent. Many mathematicians have written about their work in a sort of prose poetry, and the satisfactions they get from it seem indistinguishable from those of an artist. The language of aesthetics is never far to seek in the writing of mathematicians. In their frequent references to the "elegance", "beauty", and so on of mathematical theorems they evidently imagine themselves to be appealing to sensibilities that all mathematicians share. Nearly all mathematicians show themselves uneasy in presence of a proof which is inelegant, however convincing, and, sooner or later, endeavour to replace it by one which approaches closer to their aesthetic ideal. Some of them have gone so far as to remark that the actual solution of problems interests them much less than the beauty of the methods by which they found that solution. If mathematics is to be ranked as a science, then it is, of all the sciences, the one most akin to the arts.

I hope that in the strain of these thoughts, derived from Sullivan, the congratulations with the Fields Medals will be understood and accepted.

This involves also some of the reasons why the Dutch Government has such a vivid interest in this International Congress of Mathematics and highly appreciates that so many mathematicians from all over the world assemble here. In this circle I may deem myself discharged of going into the value and the significance of mathematics in science and society. It is fascinating to see how history is showing again and again the mutual influence which always existed between mathematics and the other sciences and how mathematics have changed the world outlook. Nothing is more impressive than the fact that as mathematics withdrew increasingly into the upper regions of ever greater extremes of abstract thought, it returned back to earth with a corresponding growth of importance for the analysis of concrete fact. The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact. For this very reason it makes mathematics a living science. It was Whitehead who warned for dead knowledge. He says, What is wanted is "activity in the presence of knowledge". In all abstract thinking there must be a final connection with the living reality. This activity in the presence of knowledge, is an activity of the mind, in which the human factor is more important than an outsider would expect. Poincaré has told us that even in pure mathematics, where reason, one would think, is most pure and undefiled, a proof which is quite satisfactory to one mathematician is often not at all satisfactory to another. Indeed, Poincaré was led to divide mathematicians into psychological types, and to point out that a kind of reasoning which would convince one type would never convince another. In view of these facts it is obviously misleading to present science as differing fundamentally from the arts by its "impersonal" character. Therefore I can't see this Congress as an efficient organization of exchange of scientific information only, but also of human understanding. Seen in this way the social and artistic part of the program of this Congress has also a more profound sense. Also scientifically human contacts can still strenghthen considerably the success of the Congress. Lucien Price in his recent book "Dialogues of Whitehead" reported that Whitehead once said: "By myself I am only one more professor, but with my wife, I am first-rate".

May human understanding contribute also to the success of this Congress and to the strenghthening of the bonds between the countries you have come from and the Netherlands."

After the closing of the session a buffet-lunch for authorities, invited guests, Congress speakers and officials took place. In the afternoon, Professor John von Neumann delivered the first one-hour lecture, on the invitation of the Congress Committee, for the plenary session speaking on "Unsolved problems in mathematics". A photograph of the Congressists was taken in front of the Concertgebouw after Professor Von Neumann's lecture.

In the evening of the opening-day, the Netherlands Government represented by His Excellency Mr. J. J. Cals, Minister of Education, and the Amsterdam Municipality, represented by the Burgomaster, Mr Arn. J. d'Ailly, held an official reception in the rooms of the Rijksmuseum.

During Friday, September 3rd, scientific sessions took place; in the meantime there were several excursions for associate members. In the evening a concert by the Concertgebouw orchestra took place in the Concertgebouw for the members of the Congress. The program of this concert, which was conducted by Eduard van Beinum, consisted of the following works:

1. Second Suite in b minor, Johann Sebastian Bach
   (Ouverture, Rondeau, Sarabande, Bourrée I, Bourrée II, Polonaise, Menuet, Badinerie)
   Solo flute: Hubert Barwahser
2. Concerto in f major, KV 459, for piano and orchestra, Wolfgang Amadeus Mozart
   (Allegro, Allegretto, Allegro assai)
   Piano Soloist: Hans Henkemans
3. Symphonische etude, Hendrik Andriessen
   (Quasi adagio, allegro, con spirito, adagio, allegro vivace)
4. La Mer, Claude Debussy
   Trois esquisses symphoniques
   (De l'aube à midi sur la mer; Jeux de vagues; Dialogue du vent et de la mer).

On Saturday, September 4th, scientific sessions took place in the morning. During the afternoon a boattrip through the canals and harbours of Amsterdam was offered to the members of the Congress by the Municipality of Amsterdam. The evening was free. For those who wanted to spend their evening in the company of their fellow-members, a number of cafés and restaurants were suggested.

On Sunday, September 5th, the Headquarters of the Congress were closed and no sessions took place. A day-trip through the Dutch waterlandscape and visit to the bird park "Avifauna", partly by boat and partly by bus, was offered to the Congressists.

Monday, September 6th was filled with scientific sessions. Morning excursions had been arranged for associate members. During the evening all members could make their choice from several entertainments, viz.:

1. Concert of Chambermusic by the "Hollands Strijkkwartet".
2. Chansons evening by Georgette Hagedoorn.

3. Show of historical costumes by Cruys Voorbergh.
4. Simultaneous chess game by Dr Max Euwe and Mr S. Flohr.
5. Show of Dutch cultural films.

On Tuesday-morning, September 7th, scientific sessions took place. During the afternoon all members could make their choice from several excursions, viz.

1. Royal blast furnaces and steelworks at Velsen and the North Sea Locks at IJmuiden.
2. Cruquius museum and the Teyler Foundation at Haarlem.
3. Marken and Volendam.
4. The Hague and Scheveningen.
5. The dike, damming up the "Zuiderzee" and the polder of the Wieringermeer.
6. Delft.
7. Enkhuizen and the "Zuiderzee" museum.

During the evening a party was given in the Complex "Bellevue". The program contained several diversions, e.g. the calculating prodigy Wim Klein (alias Pascal) and the illusionist Driebeek, and in some rooms opportunity was given for quiet conversation.

On Wednesday, September 8th, scientific sessions took place in the morning and afternoon. Morning excursions had been organized for associate members.

In the afternoon of this day, at 3.30 P.M. a score of members of various nationalities, under the guidance of Prof. Dr J. A. Schouten, were received by Her Majesty, Queen Juliana of the Netherlands at the Royal Palace in Soestdijk.

In the evening the official Congress Banquet took place with 1500 guests in the Wintergardens of the Grand Hotel "Krasnapolsky". The Banquet was presided by Prof. Dr O. Bottema. At the end of the dinner several speakers addressed the participants.

On Thursday, September 9th, scientific sessions took place, and the closing session of the Congress was held in the afternoon of that day. The session was opened by an address of the last one-hour speaker, Prof. Dr A. N. Kolmogoroff, who spoke on "General theories of dynamical systems in classical mechanics".

After Professor Kolmogoroff's lecture, Professor Schouten addressed the Congress and read first the following telegram, with a translation in English, from the Royal Palace in Soestdijk:

"In opdracht van Z.K.H. de Prins der Nederlanden dank ik U zeer voor Uw telegram en breng U de beste wensen van Z.K.H. voor het welslagen van het Congres over."

"At the request of His Royal Highness, Prince of the Netherlands, I thank you very much for your telegram and offer you the best wishes from His Royal Highness for the success of the Congress."

Then Professor Schouten spoke as follows:

"As Professor Bompiani told in the opening session there is a joint committee consisting of the President and Secretary of the I.M.U., the President and Secretary of the Organizing Committee of the Congress and Prof. Iyanaga as a fifth member for the preparation of the discussion on the place of the next Congress. This joint committee has received one letter only from Professor W. V. D. Hodge, authorized by the Royal Society, the Lord Provost and Town Council of the City of Edinburgh, and the Principal of the University of Edinburgh, inviting the International Congress of Mathematicians to meet in Edinburgh in 1958. There was also a letter from the Department of Mathematics of the Hebrew University in Jerusalem concerning the Congress 1962. But as this matter belongs to the competence of the Congress 1958 it will be forwarded to the Organizing Committee of this latter Congress. The joint committee recommended that Professor Hodge from Cambridge be invited to speak."

Professor Hodge spoke as follows:

SPEECH BY PROF. W. V. D. HODGE

"I am speaking as the representative of the delegates from Great Britain and Northern Ireland. I have the honour to convey an invitation from the mathematicians of Great Britain and Northern Ireland, sponsored by the City of Edinburgh, the University of Edinburgh, the Royal Society of London and the Royal Society of Edinburgh, to the mathematicians of the world to hold their next Congress in Edinburgh in 1958. If this invitation is accepted, I can assure mathematicians of every country of a warm welcome, and we in Britain will do our best to organise a Congress worthy of the high standard set by our present hosts.

In issuing this invitation, I want to mention two particular points. As is well known, a Festival of Music and Drama is held each year in Edinburgh, over a period which includes the first week of September, which seems to have become the accepted time for our congresses, and this will necessitate some departure from recent practice. While we cannot be expected to fix the dates of the Congress precisely at this moment, it is our intention to try to arrange to hold it during the earlier part of August. From enquiries I have made, I have formed the impression that this change of date will be welcomed by many mathematicians, and we hope that it will be possible to time the Congress so that those mathematicians who are also interested in the Musical Festival will be able with as little inconvenience as possible to attend both gatherings.

The second point which I have to make refers to something which our President, Professor Schouten, said in his address, and which was also mentioned by Professor Veblen. The steadily increasing size of our Congresses has caused some people to wonder whether they are not in fact becoming too big, and in danger of getting out of control. It would be most improper for me to enter into any argument here, and I shall only express my personal conviction that the purpose of international congresses and of specialised colloquia are quite different, and that there is a real danger that if the complexities and cost of organisation continue to increase it will become more and more difficult to find countries able and willing to undertake the burden of arranging a congress, and eventually there might only be one or two of the few remaining rich countries able to do so. There can be no doubt that this would be very bad for mathematics.

Therefore, if the invitation to meet in Edinburgh is accepted, we propose to give serious thought to the question whether we can in some way give a lead towards achieving some simplifaction in congress arrangements, without sacrificing anything essential in the scientific value of the Congress, and at the same time providing, in some measure, for the social side of our activities. But I can at the same time promise that we shall make every endeavour to ensure that the Edinburgh Congress will be both successful and enjoyable.

Ladies and gentlemen, I have the honour to invite you to meet in Edinburgh in 1958."

The invitation of Professor Hodge was received with great applause. The Chairman stated that the Congress had unanimously decided that the Congress 1958 will take place in Edinburgh.

After this Professor Hopf, Zürich, addressed the Congress as follows:

SPEECH BY PROF. H. HOPF

"Herr Präsident, meine Damen und Herren!

Bereits sechs unter uns ausländischen Mathematikern haben gestern abend beim Bankett unseren Dank an die Niederlande — ihre Königin, ihr Volk, ihre Behörden — und besonders an unsere niederländischen Kollegen ausgesprochen. Trotzdem verlangt es wohl die gegenwärtige Stunde, daß noch ein siebenter zu demselben Thema das Wort nimmt. Denn erst in diesem Augenblick, also am wirklichen Schluß des Kongresses, dürfen wir, wenn wir mathematisch korrekt sind, einiges als bewiesen anerkennen, was gestern noch Vermutung

— allerdings bereits sehr plausible Vermutung — war. Der zentrale hierhergehörige Satz lautet: *"Der Internationale Mathematikerkongress 1954 in Amsterdam ist in jeder Hinsicht aufs allerbeste gelungen."* Und wir fügen sogleich den historischen Kommentar hinzu: *"Diesen Satz und seinen Beweis verdankt man den holländischen Mathematikern und den Behörden und Personen, die ihnen bei der Organisation des Kongresses geholfen haben."*

Daneben gibt es auch Sätze, die prinzipiell allgemeiner sind als der soeben formulierte, da sie nicht nur von einem einzigen Ereignis, sondern von einer ganzen Klasse von Ereignissen handeln; besonders wichtig ist der folgende: *"Die Abhaltung grosser und allgemeiner internationaler Mathematikerkongresse ist auch für die Zukunft wünschenswert."* Daß wir auch diesen Satz heute als bewiesen ansehen dürfen — wieder dank unseren holländischen Kollegen — scheint mir klar genug zu sein: denn wer von uns würde nicht wünschen, daß sich Erlebnisse, wie wir sie in diesen Tagen hier gehabt haben, wiederholen mögen? Aber trivial ist dieser Satz durchaus nicht; es wird in letzter Zeit, mit guten Gründen pro und contra, darüber diskutiert, ob man nicht die großen allgemeinen Kongresse mit ihren riesigen Teilnehmerzahlen und ihren fast ebenso großen Zahlen von Vorträgen ganz ersetzen solle durch häufigere, in kleinen Kreisen stattfindende Symposien über Spezialgebiete. Der außerordentliche Wert dieser Symposien für die Förderung der behandelten Gebiete steht außer Frage; andererseits sollen und können die allgemeinen Kongresse gerade ein Gegengewicht gegen die notgedrungene Spezialisierung bilden und der Einheit der Mathematik dienen: der Sinn und die Bedeutung der auf ihnen gehaltenen "allgemeinen Vorträge" liegt doch wohl darin, daß diese Vorträge über die Fortschritte in einzelnen Gebieten berichten, sich aber in erster Linie gerade an diejenigen Mathematiker wenden, welche nicht Spezialisten in diesen Gebieten sind; und ein großer Kongress gibt dem einzelnen Mathematiker Gelegenheit zur persönlichen Aussprache auch mit solchen Kollegen, mit denen er nicht ohnehin in Verbindung steht. Ich glaube, daß man eine Koordinierung der großen allgemeinen Kongresse und der kleinen speziellen Symposien anstreben muß, mit dem Ziel, die großen Kongresse von einem Teil der Sektionsvorträge zu entlasten. — Aber es ist hier nicht Zeit und Ort für eine solche Diskussion; was hier festgestellt werden soll, ist dieses: der Amsterdamer Kongress stellt eine außerordentlich gute Approximation des *idealen* allgemeinen Kongresses dar. Die Anziehungskraft der Niederlande und das allgemeine Vertrauen in die niederländischen Kollegen und Behörden haben sich als stark genug erwiesen, um viele der besten Mathematiker aus allen Teilen der Welt hier zusammenkommen zu lassen — auch aus Teilen, die den meisten von uns heute nicht zugänglich sind — und die Organisatoren des Kongresses haben diese glücklichen Umstände klug benutzt, um ein Vortragsprogramm von ungewöhnlicher

Reichhaltigkeit und Vielseitigkeit und von höchstem Niveau aufzustellen und durchzuführen. Es dürfte wohl keinen Teilnehmer geben, der in dieser Woche nicht mit Genuß Neues gelernt und neue Anregungen empfangen hätte.

Aber es kommt zu dem rein Wissenschaftlichen noch etwas anderes hinzu. Viele Laien glauben ja, die Mathematiker seien nicht nur unverständlich, sondern in gewissem Sinne auch unmenschlich: unsere Tätigkeit sei so unpersönlich und spiele in einer so überaus kühlen Atmosphäre, daß menschliche Regungen und Beziehungen auf unsere Arbeit überhaupt keinen Einfluß hätten. Wir Mathematiker wissen, daß diese Ansicht ganz falsch ist, und wir freuen uns, wenn das, was wir wissen, durch Erfahrungen bestätigt wird. Der Amsterdamer Kongress war eine solche Bestätigung, wie wir sie uns besser und schöner gar nicht wünschen können. Der Zustrom von Kollegen aus aller Welt brachte alte Freunde, die durch die Ungunst der Zeit lange getrennt gewesen waren, wieder zusammen; mathematische Gespräche legten den Grund zu neuen Freundschaften; die kühle wissenschaftliche Atmosphäre verschmolz, ohne an nachhaltiger Wirkung einzubüßen, mit einer warmen und geselligen. Mit der Erinnerung an die mathematischen Ereignisse dieses Kongresses wird, neben manchem anderen, auch die Erinnerung an den eindrucksvollen festlichen Abend im Rijksmuseum und an erholsame heitere Mittage unter der Devise "Natura Artis Magistra" bestehen bleiben. Aber es war so, wie es immer ist: die Atmosphäre, in der sich die Stimmung der Gäste entwickelt, wird geschaffen und bestimmt durch die Persönlichkeiten der Gastgeber.

Für Vieles und Vielartiges also haben wir unseren Gastgebern zu danken, und es wäre nicht schwierig, dieses Viele im einzelnen noch genauer aufzuzählen; aber das ist gewiß nicht nötig. In größere Verlegenheit bringt mich die Frage, *wem* wir zu danken haben. Es sind dies in erster Linie natürlich die holländischen Mathematiker; ihnen können wir nachher beim Abschied noch einmal dankend die Hand drücken. Aber ihnen wiederum haben bei ihrer enormen Arbeit andere mit Rat und Tat beigestanden, denen wir auch zu danken wünschen, die wir aber nicht oder nur zum Teil kennen, sodaß ich den Versuch, sie hier zu nennen, nicht wage, um nicht peinliche Irrtümer und Unterlassungen zu begehen. Ich halte es daher für das Beste, jetzt am Schlusse des Kongresses noch einmal die Hilfe unseres Präsidenten, Herrn *Schouten*, in Anspruch zu nehmen: ich bitte ihn, unseren aufrichtigen Dank, den ich ihm hiermit gleichsam als ein Paket übergebe, zu verwalten und an alle diejenigen Personen, Behörden und Körperschaften zu verteilen, die an der Organisation des Kongresses mitgewirkt und die dazu beigetragen haben, daß dieser Kongress in einem so hohen Maße und in jeder Hinsicht wohlgelungen ist.

Meine Damen und Herren, ich habe mich hier meiner eigenen Sprache bedient, also deutsch gesprochen. Dem internationalen Charakter dieser Versammlung wäre es angemessen, wenn ich dem Beispiel, das Herr *Schouten* in der Eröffnungssitzung gegeben hat, folgen und das, was ich eben gesagt habe, in mehreren anderen Sprachen wiederholen würde. Aber das verbieten meine mangelhaften Sprachkenntnisse und auch die vorgerückte Zeit. Immerhin kann und will ich wenigstens eine Zusammenfassung, ein Résumé des Gesagten noch in einigen Sprachen vorbringen. Dieses Résumé darf kurz sein; denn schließlich läßt sich ja das, worauf es uns ankommt, in einem einzigen Worte ausdrücken. Dieses Wort, das man in allen Sprachen leicht und gern lernt, heißt: Danke — merci — grazie — thank you — spasibo — tak — *van harte bedankt*!"

The last words of the Congress, spoken by the Chairman, were the following:

CLOSING SPEECH BY PROF. J. A. SCHOUTEN

"First speaking as President of the Organizing Committee, I wish to distribute the general thanks given by Prof. Hopf. There are a great number of persons and organisations that have to be thanked for their kind interest and help. I remind you first of the interest that was taken in the Congress by Her Majesty the Queen who was kind enough to receive a delegation of the Congress and of the honour His Royal Highness Prince Bernhard of the Netherlands did us by consenting to be our Patron. The Ministry of Education helped us not only financially but in every way possible. The City of Amsterdam gave us its financial help and the never failing assistance of all local authorities during the whole period of preparation. Without this assistance a Congress like this with so many participants and with all its excursions and entertainments could not have been organized.

A great number of industries in the Netherlands helped financially. From the great number of other organizations that have to be thanked for their kind cooperation only allow me to mention the I.M.U., the C.I.E.M., I.C.M.I., or I.M.U.K., the Committee for the Fields Medals and the Organizing Committees of the three symposia that took place in connection with the Congress.

The Organizing Committee formed sub-committees and sub-sub-committees, till at last nearly every mathematician in the Netherlands belonged in some way to the big committee resulting in this way. To all these cooperators thanks must be given. Last but not least all mathematicians who have come from abroad to Amsterdam must be thanked, because they made this Congress. The Organizing Committee only provided the opportunity of doing it. I express the hope that they have all found here something interesting and something that gives them much stimulus for their future work.

In the opening session I drew attention to the fact that mathematics nowadays has a much more important place in the world than before. Now something about the consequences. It is an experimental fact that mutual understanding is very difficult as long as problems can not be formulated in an exact way. For instance it is much easier to avoid misunderstanding in the field of physics than in the field of economics or political sciences. Now mathematics is in one of its aspects the science of exact formulation and that means that better understanding may arise where problems can be formulated mathematically.

In our days several branches of mathematics as for instance mathematical statistics, can be applied not only to physical problems but also to problems from many other fields, for instance, economics. Of course, problems in real life can never be reduced to calculation alone, but exact formulation and some calculations, where they are possible, can do much good. So we may hope that mathematics in its modern development will become more and more useful for the mutual understanding of mankind as a whole. With these words full of hope for the future I close the Congress 1954.

IV. Demonstrations and Exhibitions

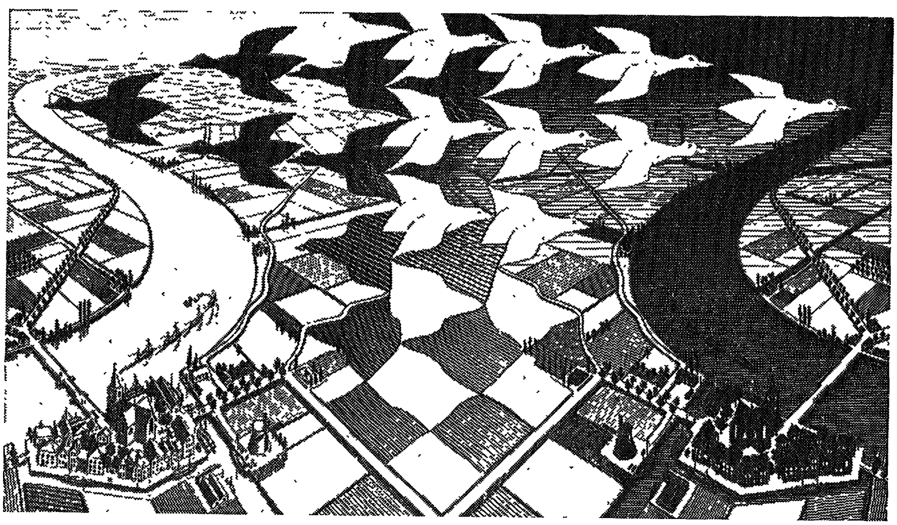
In the scope of Section V there were demonstrations of some electronic devices. These demonstrations were made possible by the kindness of the directors of the I.B.M. (International Business Machines Corp.), of the Mathematical Centre, and of the Royal Shell Laboratories (B.P.M.). The following machines were exhibited:

1. the IBM electronic calculators 604 and 626.
2. the electronic computer ARRA built in the Mathematical Centre at Amsterdam.
3. the electronic computer "Miracle" built by Ferranti and belonging to the Royal Shell Research Laboratories (B.P.M.) at Amsterdam.

During the Congress an exhibition of mathematical books was held in the Royal Tropical Institute. This exhibition was organized, on behalf of the Congress Committee, by Messrs Swets & Zeitlinger, booksellers at Amsterdam with the aid of the Mathematical Centre.

A complete catalogue of the exhibition was handed to the members together with the other Congress documents. For the organization of the exhibition, the Organizing Committee is especially indebted to Mr J. Sibeijn of Swets & Zeitlinger and Mr J. Verhoeff of the Mathematical Centre.

In connection with I.C.M.I. lectures in Section VII mentioned above, under the auspices of I.C.M.I. an interesting exhibition of didactical and pedagogical works in the field of mathematics was held in the building where the sessions of Section VII took place. The exhibition was prepared by Mr A. Cardot from Paris in cooperation with Professor E. W. Beth, President of the Netherlands sub-committee of the I.C.M.I.. The exhibition was opened by Professor A. Châtelet, President of the I.C.M.I. on Friday-afternoon, September 3rd.



In the Royal Tropical Institute the opportunity was given by the Organizing Committee to Messrs E. J. F. van Dissel & Zn., Eindhoven, Linen-factories, to display and sell linen table-cloths, showing the Gaussian primes in the complex plane. These cloths were woven from a design by Professor Balth. van der Pol.

On the occasion of the Congress, an exhibition was organized in the Municipal Museum of Amsterdam, showing the graphical work of Mr M. C. Escher, which shows many mathematical tendencies and is connected in a remarkable way with the mathematical way of thought. [1]). The exhibition was opened by Professor N. G. de Bruijn.

For their readiness to organize this exposition, the Organizing Committee is indebted to Mr Escher as well as to the Director of the Municipal Museum, Jkhr W. J. H. B. Sandberg.

[1]) See the above reproduction („Day and Night").

V. Symposia

In connection with the Congress although organized independently by the "Wiskundig Genootschap" three symposia took place with the moral and financial aid of UNESCO, ICSU and IMU, viz.

A. a symposium on Stochastic Processes
under the care of Professor D. van Dantzig
Honorary President Professor Maurice Fréchet.

B. a symposium on Algebraic Geometry
under the care of Professor H. D. Kloosterman
Honorary President Professor Francesco Severi.

C. a symposium on Mathematical Interpretation of Formal Systems
under the care of Professor A. Heyting
Honorary President Professor Emile Borel.

Symposium A was held on Wednesday, September 1st, Saturday, September 4th and Wednesday, September 8th.

Speakers in this symposium were: J. L. Doob, P. Lévy, D. V. Lindley, R. Fortet, D. G. Kendall, E. Hille, A. J. L. Blanc-Lapierre and D. Blackwell. Some of the lectures were held in joint sessions of Congress and symposium.

All lectures held in symposium A will be found in Volume III of these Proceedings.

Symposium B was held on Monday, September 6th, Tuesday, September 7th, Friday, September 10th, whereas joint sessions of Congress and symposium were held on Friday, September 3rd, Saturday, September 4th and Wednesday, September 8th.

Speakers in this symposium were: J. G. Semple, M. Rosenlicht, F. Severi, F. Hirzebruch, B. L. van der Waerden, W. Gröbner, A. Néron, J. P. Serre, K. Kodaira, B. Segre, D. G. Northcott.

All lectures held in symposium B will be found in Volume III of these Proceedings.

Symposium C was held on Thursday, September 9th and Friday, September 10th.

Speakers in this symposium were: Th. Skolem, G. Kreisel, A. Robinson, Hao Wang, L. Henkin and J. Løs. There were no joint sessions of this symposium and the Congress.

The lectures held in symposium C are edited in a separate book under the title *Mathematical Interpretation of formal Systems, Amsterdam 1955*.

VI. Centenary H. Poincaré

After the Congress a session took place in the Hague, under the presidency of Professor Gaston Julia, which session was dedicated to the commemoration of Henri Poincaré, who was born 100 years ago. Although that manifestation was independent of the Congress, the Congress Committee had rendered her help with great pleasure. In collaboration with the Executive Committee of the Organizing Committee of the Congress, a special program committee consisting of Prof. G. Julia, Prof. H. Freudenthal and Prof. C. Visser had drawn up the program. Prof. F. Loonstra had been found willing to charge himself with the technical preparations. After the session the participants were received by the Ambassador of France in the Netherlands.

J. F. Koksma, *Secretary,*
International Congress of Mathematicians 1954.

IN MEMORIAM

Nine months after having adopted the above Report, the Organizing Committee was grieved by the death of Prof. Johannes Haantjes, who as its treasurer had taken an essential part in the organization of the International Congress of Mathematicians 1954.
He died on the 8th of February 1956, at the age of 46.

On the 9th of December 1955 Prof. Hermann Weyl, President of the Fields Medal Committee 1954, died at the age of 70.

On the 8th of February 1957 Prof. John von Neumann, invited speaker to the Congress, died at the age of 53.

On the 25th of September 1955 Prof. Franz Rellich, invited speaker to the Congress, died at the age of 49.

ADDRESS OF THE PRESIDENT OF THE FIELDS MEDAL COMMITTEE 1954

Professor HERMANN WEYL

That at each International Mathematical Congress two gold medals be presented to two young mathematicians who have won distinction in recent years by outstanding work in our science: this was the intention of the late Professor J. C. Fields, the donor of the trust fund for these medals, and such the resolution adopted at the International Congresses in Toronto 1924 and Zurich 1932. In his Toronto memorandum Professor Fields expressed the wish that scientific merit be the only guide for the award of the medals, and he expressed the hope that the prizes would be considered by the recipients not only as an acknowledgment of past, but also as an encouragement for future, work.

Owing to the turbulent conditions of the world, award of the Fields Medals has up to now taken place but twice, at the Oslo Congress in 1936 and at Harvard, in 1950. As Chairman of a Committee consisting of the following members: E. Bompiani, F. Bureau, H. Cartan, A. Ostrowski, Å. Pleijel, G. Szegö, E. C. Titchmarsh and the speaker, I have the great honor and pleasure to perform this ceremony here-now for the third time. Our Committee owes its origin to the action of Prof. J. G. van der Corput as nominee of Het Wiskundig Genootschap for the presidency of this Congress, Prof. J. A. Schouten as Chairman, Prof. H. D. Kloosterman as Vice-Chairman and Prof. J. F. Koksma as Secretary of its Organizing Committee. Not for the composition, but for the judgment of our Committee the responsibility is ours, and we assume it gladly and with a good conscience indeed. After much deliberation during which many names were discussed and which made us fully aware of the arbitrariness involved in the selection of just two from the array of mathematicians of outstanding merit, we finally agreed, and this decision was reached as unanimously as anyone could reasonably expect, to present the two gold medals, together with an honorarium of $ 1,500 each, to:

Professor *Kunihiko Kodaira,* and

Dr. *Jean-Pierre Serre.*

I hope the Congress as a whole will approve our choice. In justification of it let me say this: by study and information we became convinced that Serre and Kodaira had not only made highly original and important contributions to mathematics in recent years, but that these hold out great promises for future fruitful non-analytic (will say: non-foreseeable) continuation. Carrying out the Committee's resolution, I now call upon Professor Kodaira and Dr. Serre to receive from my hands the prizes awarded them — awarded them, to repeat the donor's words, as recognition of past, and encouragement for future, research work. In the name of the Committee I extend to you, Professor Kodaira, and to you, Dr. Serre, my heartiest congratulations.

Two precedents have established the custom to combine with the award of the Fields Medals a brief survey of the recipients' main mathematical achievements, in particular of those which most attracted the Committee's attention. In Oslo it was not its Chairman, Elie Cartan, but another of the Committee's members, Carathéodory, who discharged this duty. In 1950, at Harvard, Harald Bohr was Chairman of the Fields Medals Committee, and he did both: handed over the medals to Atle Selberg and Laurent Schwartz and delivered the *laudatio*. I am going to follow his example, though with considerable hesitation; for I realize how difficult it is for a man of my age to keep abreast of the rapid development in methods, problems and results which the young generation forces upon our old science; and without the help of friends inside and outside the Committee I could not have shouldered this burden at all. It rests more heavily on my than on my predecessors' shoulders; for while they reported on things within the circle of classical analysis, where every mathematician is at home, I must speak on achievements that have a less familiar conceptual basis. A report like this cannot help reflecting personal impressions. Hence I now speak for myself and no longer as Chairman of the Committee. Thus freed from irksome bonds, I start by confessing that I am deeply satisfied by the Committee's choice — though I hope they will admit that it was reached without undue pressure from my side.

In view of the difficulties mentioned and in view of certain common foundations of Serre's and Kodaira's work, I find it convenient to explain as briefly as I can a number of universal concepts before entering upon some of our laureates' individual achievements. Be prepared then to have to listen now to a short lecture on cohomology, linear differential forms, faisceaux or sheaves Kähler manifolds and complex line bundles [1]).

An n-dimensional manifold M may be covered by a finite or denumerable sequence of neighborhoods U^ϱ. Their non-empty intersections are denoted by $U^{\varrho\sigma} = U^\varrho \cap U^\sigma$, $U^{\varrho\sigma\tau} = U^\varrho \cap U^\sigma \cap U^\tau, \ldots$. *Co-chains* of dimension $0, 1, 2, \ldots$ are defined by associating with each $U^\varrho, U^{\varrho\sigma}, U^{\varrho\sigma\tau}, \ldots$ numbers $c^\varrho, c^{\varrho\sigma}, c^{\varrho\sigma\tau}, \ldots$ depending skew-symmetrically on the indices. Take a 1-dimensional cochain c with the coefficients $c^{\varrho\sigma}$ as example. Its 2-dimensional *co-boundary* ∂c is defined by associating with each $U^{\varrho\sigma\tau}$ the number

$$(\partial c)^{\varrho\sigma\tau} = c^{\varrho\sigma} - c^{\varrho\tau} + c^{\sigma\tau}.$$

A cochain c is *co-closed* and called a *co-cycle* if its co-boundary is zero, $\partial c = 0$.

[1]) Only an abbreviated version of this part was read in the oral address.

The co-boundary $\partial c'$ of a $(q-1)$-dimensional cochain c' is a q-dimensional cocycle, of which we say that it is *co-homolog* zero (~ 0) or a *bounding* cocycle. The linear manifold of the cocycles of dimension q modulo the bounding cocycles form the additive abelian cohomology group $\mathfrak{C}_q$. We write $\mathfrak{C}_q(R)$ or $\mathfrak{C}_q(I)$ according as to whether we admit to the "sheaf" of numbers c all real numbers or only the integers. (Unfortunately history forbids us to drop the prefix *co*- in all these terms.) The groups $\mathfrak{C}_q$ apparently depend on a definite coverage of the manifold by overlapping neighborhoods U^ϱ. But either by normalizing it in a suitable way, or by passing to ever more refined coverings one makes himself independent thereof.

The n-dimensional manifold M_n may be locally referred to n realvalued coordinates $x_1, x_2, \ldots, x_n$. With respect to them we can define covariant skew-symmetric tensors (or rather: tensor fields) of rank $q = 0, 1, 2, \ldots$. For instance, such a field f of rank 2 has skew-symmetric numerical components f_{ik} which are functions of the coordinates (while the indices i, k, of course, run from 1 to n). It is convenient to write f as the "linear differential form"

$$f = \frac{1}{2!}\sum_{i,k} f_{ik}(dx_i \wedge dx_k)$$

with a skew-symmetric product $dx_i \wedge dx_k$ of the differentials, because this suggests the transformation under transition to other coordinates. We therefore speak simply of *forms* of rank $q = 0, 1, 2, \ldots$. The invariant *derivative* df, called *rotation* in tensor analysis, of a form f of rank 2, is defined by

$$(df)_{ikl} = \frac{\partial f_{ik}}{\partial x_l} - \frac{\partial f_{il}}{\partial x_k} + \frac{\partial f_{kl}}{\partial x_i}$$

and is of the next higher rank 3. Similarly for all ranks. f is said to be *closed* if $df = 0$. The derivative $d\varphi$ of a form φ of rank $q-1$ is a closed form of rank q, which we call a *derived* form (or ~ 0). The closed forms of rank q modulo the derived forms constitute a linear manifold, the additive abelian group Γ_q of rank q. The parallelism to the theory of cochains is obvious. In his thesis de Rham replaced this parallelism by a true group-isomorphism. Today this connection is best established in terms of the new-fangled notion of faisceau first introduced by Leray. Princeton has decreed that 'sheaf' should be the English equivalent of the French 'faisceau'. Sheaves play an important part in Kodaira's as well as Serre's investigations.

The fact that a form f of rank q is closed may also be expressed by saying that the equation $d\varphi = f$ has a *local* solution φ everywhere; i.e. in each neighborhood U^ϱ there is a $(q-1)$-form φ^ϱ such that $f = d\varphi^\varrho$. Such a form φ^ϱ defined only in a neighborhood U^ϱ I call, slightly modifying Chern's terminology, germ of a form, or simply *germ*. The closed germs of rank q belong, so I will say, to the sheaf G_q. The local φ^ϱ is not uniquely determined by f, but one may add to it any closed germ ω^ϱ of rank $q-1$, $\omega^\varrho \epsilon\, G_{q-1}$. Under what conditions will f be homolog zero? Provided this arbitrariness may be used in such a way as to make $\varphi^\varrho = \varphi^\sigma$ in each $U^{\varrho\sigma}$. One therefore forms $\Phi^{\varrho\sigma} = \varphi^\varrho - \varphi^\sigma$ in $U^{\varrho\sigma}$; being closed, this germ belongs to G_{q-1}. But it is determined only up to an additive term $(\partial\omega)^{\varrho\sigma} = \omega^\varrho - \omega^\sigma$. The several $\Phi^{\varrho\sigma}$ and $(\partial\omega)^{\varrho\sigma}$ make up a 1-dimensional cocycle Φ and bounding cocycle $\partial\omega$ respectively, not in the sheaf of numbers however, but in G_{q-1}. Thus f determines uniquely an element of the 1-dimensional cohomology group $\mathfrak{C}_1(G_{q-1})$ in the sheaf of germs G_{q-1}, in such a way that the corresponding element vanishes if and only if f is homolog zero. Following the argument in the opposite direction, one realizes that the resulting homomorphism $\Gamma_q \to \mathfrak{C}_1(G_{q-1})$ is in fact an isomorphism: $\Gamma_q \simeq \mathfrak{C}_1(G_{q-1})$. The isomorphic mapping of the one group onto the other has been constructed explicitly.

A similar argument leads to the sequence of isomorphisms

$$\mathfrak{C}_1(G_{q-1}) \simeq \mathfrak{C}_2(G_{q-2}) \simeq \ldots \simeq \mathfrak{C}_q(G_0),$$

and since G_0 is identical with the domain R of real numbers, we thus arrive at de Rham's result

$$\Gamma_q \simeq \mathfrak{C}_q(R). \tag{1}$$

The usefulness of faisceaux for algebraic geometry was recognized early in 1953 by Dr. Serre and D. C. Spencer.

I need a little more of tensor analysis in the form in which Einstein used it for his general relativity theory. On our n-dimensional manifold M we may also consider skew-symmetric contravariant tensor densities or *"co-forms"* of rank 0, 1, 2, ... I denote their components by German letters and upper indices: $\mathfrak{g}, \mathfrak{g}^i, \mathfrak{g}^{ik}, \ldots$. For them the invariant operator ∂ of derivation is the one called *divergence* in tensor analysis, leading for instance from a tensor density $\mathfrak{g}$ of rank 2 to a vector density $\partial\mathfrak{g}$ according to the equation

$$(\partial\mathfrak{g})^i = \sum_k \frac{\partial\mathfrak{g}^{ik}}{\partial x_k}.$$

If M is compact, a scalar density $\mathfrak{j}$ has an invariant integral $\int\mathfrak{j}$ over M.

Assume now that we are dealing with a Riemannian manifold M_n carrying a metric by dint of an invariant quadratic differential form

$$ds^2 = \sum_{i,k} \gamma_{ik}dx_i dx_k \quad (\gamma_{ik} = \gamma_{ki}) \tag{*}$$

with symmetric coefficients γ_{ik} depending on the variables x. Then one has the algebraic operation $g \leftrightarrows \mathfrak{g}$, which changes a form g into a co-form $\mathfrak{g}$ of the same rank, and vice versa, so that we can identify the forms with the co-forms. A relation like $\partial \mathfrak{f} = \mathfrak{s}$ for co-forms $\mathfrak{f}$, $\mathfrak{s}$ may therefore be written as $\partial f = s$ for the corresponding forms f, s. The derivative d *increases*, the co-derivative ∂ *decreases* the rank by 1. Incidentally, in this symbolism Maxwell's equations for the electromagnetic field f of rank 2 in the empty four-dimensional world may be simply written as $df = 0$, $\partial f = s$, where s is the electric current. With a form f and a co-form $\mathfrak{g}$ of rank 2 one can construct the scalar density

$$f \cdot \mathfrak{g} = \frac{1}{2!} \sum f_{ik} \mathfrak{g}^{ik}$$

and thus form the integral

$$\int f \cdot \mathfrak{g} = (f, g).$$

In the functional "space" of infinitely many dimensions, the elements or "vectors" of which are the forms f of rank p, we may introduce this number (f, g) as the scalar product of the two "vectors" f, g. It depends symmetrically on f and g, and if the metric ground form (*) is positive-definite (not indefinite as in the physical four-dimensional world), then (f, f) is also positive-definite, i.e. $(f, f) > 0$ except for $f = 0$. The fact that the integral over M of the divergence of a vector density is zero, entails the equation

$$(2) \qquad (d\varphi, g) + (\varphi, \partial g) = 0$$

for any $(p-1)$-form φ and p-form g.

A form h satisfying both conditions $dh = 0$ and $\partial h = 0$ is called *harmonic*. The central existence theorem in Hodge's book on the "Theory and application of harmonic integrals" published in 1941 asserts that every closed form f, $df = 0$, is homolog to a uniquely determined harmonic form h, $f - h \sim 0$. Consequently de Rham's result (1) leads to a definite *isomorphic mapping between the cohomology group* $\mathfrak{C}_p(R)$ *and the additive abelian group of harmonic forms of rank* p. A proof of Hodge's theorem is ready at hand if one operates in the space Σ of all closed forms f of rank p. The closed forms of rank p which are homolog zero, i.e. derivatives $d\varphi$ of forms φ of rank $p - 1$, form a linear subspace Σ_0 of Σ. As in a vector space of finite dimension, we split any vector f in Σ into one, f_0, lying in Σ_0, $f_0 \sim 0$, and one, h, perpendicular to Σ_0, i.e. satisfying the condition $(d\varphi, h) = 0$ for every form φ of rank $p - 1$. According to (2) the latter equation amounts to $(\varphi, \partial h) = 0$ for every φ, hence to $\partial h = 0$. Thus as a vector in Σ, the form h satisfies the condition $dh = 0$, as one perpendicular to Σ_0 the condition $\partial h = 0$; it is therefore harmonic, and we have

$$f = f_0 + h, \qquad f_0 \sim 0, \qquad \text{or } f \sim h.$$

Of course, this method of orthogonal projection needs justification in a functional space of infinite dimension, a justification that is essentially identical with that of Dirichlet's minimal principle. In passing you may notice that I have avoided speaking about cycles and integrals of closed forms over cycles (periods); instead I talked of co-cycles only.

Hodge applied the theory of harmonic forms to algebraic varieties. These are special *complex-analytic varieties*. Such a variety V_n of n complex and $2n$ real dimensions can be locally referred to n complex-valued coordinates $z_r = x_r + iy_r$; and two sets of such coordinates, z and z', in two overlapping neighborhoods U and U', are related to each other in $U \cap U'$ by a regular-analytic transformation of non-vanishing Jacobian. A *metric* can be introduced into the complex variety by an invariant positive-definite Hermitean form

$$\alpha \doteq \sum_{r,s} \alpha_{rs} dz_r d\bar{z}_s \qquad (\bar{\alpha}_{rs} = \alpha_{sr}). \tag{3}$$

The coefficients α_{rs} are functions of the real variables x_r and y_r, or of z_r and $\bar{z}_r$. The conditions of symmetry $\bar{\alpha}_{rs} = \alpha_{sr}$ guarantee that the form is *real*. Such a metric is clearly a special Riemannian metric. The Hermitean symmetry of the coefficients α_{rs} evidently implies skew symmetry for the quantities $\omega_{rs} = i \cdot \alpha_{rs}$ arising through multiplication by $i = \sqrt{-1}$. Hence with the metric ground form α there is associated a real linear differential form of rank 2,

$$\omega = \sum_{r,s} \omega_{rs}(dz_r \wedge d\bar{z}_s) \qquad (\bar{\omega}_{rs} = -\omega_{sr})$$

If ω satisfies the condition $d\omega = 0$, we speak of a *Kähler metric*. In introducing this notion, Kähler observed that any manifold imbedded into a Kähler manifold by substituting for $z_1, \ldots, z_n$ analytic functions of new variables is again a Kähler manifold. By the way, the equation $d\omega = 0$ implies $\partial\omega = 0$; ω is therefore harmonic. Since the complex projective space is obviously Kählerian, so is any algebraic variety imbedded in this space in a non-singular fashion. From the transcendental standpoint it is easier to deal with Kähler varieties in general than with the special algebraic ones, and experience has shown that many notions and results of algebraic geometry apply to them. In the complex domain we must distinguish not only ranks, but also *types* among the linear differential forms. For instance, the form

$$\sum \varphi_{rst}(dz_r \wedge dz_s \wedge d\bar{z}_t)$$

is of type (2, 1) and rank $2 + 1 = 3$, because of 2 factors dz and 1 factor $d\bar{z}$ in its general expression. Of special importance are the harmonic forms of type $(p, 0)$, $p = 0, 1, \ldots, n$. If V_n is compact, there exists only a finite number g_p of linearly independent such forms. (By the way, for a form f of type $(p, 0)$ on a Kähler variety the equation $df = 0$ implies $\partial f = 0$.)

As a last preparation I speak of fiber spaces F with the basis V_n, the fiber of which is the "line" of all complex numbers ζ with the understanding that the automorphic mappings of the individual fiber consist in the multiplications of ζ by an arbitrary complex constant $\neq 0$. I call these special fiber spaces *complex line bundles*. Thus if U, U' are two overlapping neighborhoods and the points of the part of the fiber space with basis U are represented by $z_1, \ldots, z_n; \zeta$, those with basis U' by $z'_1, \ldots, z'_n; \zeta'$, then in $U \cap U'$ the coordinates $z'_1, \ldots, z'_n$ are related to $z_1, \ldots, z_n$ by an invertible holomorphic transformation, and ζ and ζ' by a relation $\zeta' = f \cdot \zeta$ where f is a nowhere-vanishing holomorphic function of the z (or the z') in $U \cap U'$. If V_n is covered by neighborhoods U^ϱ we have such transition factors $f = f_{\varrho\sigma}$ for each $U^{\varrho\sigma}$ and it is

$$(4) \qquad f_{\varrho\sigma} f_{\sigma\tau} f_{\tau\varrho} = 1 \qquad \text{in } U^{\varrho\sigma\tau}.$$

These $f_{\varrho\sigma}$ determine the fiber bundle. But because of the relativity of the fiber coordinate ζ the factor system $f_{\varrho\sigma}$ may be replaced without changing the bundle by any equivalent factor system $f^*_{\varrho\sigma} = g_\varrho f_{\varrho\sigma} g_\sigma^{-1}$ where g_ϱ is any nowhere-vanishing holomorphic function in U^ϱ. (Equation (4) is a condition of multiplicative closure. Hence we may speak of the line bundles as elements of the multiplicative 1-dimensional cohomology group in the sheaf of non-vanishing holomorphic germs.) A *divisor* D on a complex variety V_n is locally, in a neighborhood U^ϱ, defined by a meromorphic function Φ_ϱ in U^ϱ, but in such manner that replacement of Φ_ϱ by $g_\varrho \cdot \Phi_\varrho$ does not change the divisor. By $\Phi_\sigma = f_{\varrho\sigma} \cdot \Phi_\varrho$ we obtain transition functions $f_{\varrho\sigma}$ in $U^{\varrho\sigma}$ of the nature described before. Thus a divisor determines a line bundle, and two divisors the same line bundle if and only if they are equivalent. It is therefore convenient, with Spencer and Serre, to replace the notion of a class of equivalent divisors by that of a complex line bundle F over V_n. Kodaira introduced forms φ with coefficients in F. By this he means that form-germs ζ, ζ' defined in two overlapping neighborhoods $U = U^\varrho$, $U' = U^\sigma$ are not related by the equation $\zeta' = \zeta$, but by the equation $\zeta' = f_{\varrho\sigma} \cdot \zeta$ which describes the fiber coincidence in F.

I am sorry for this long introduction. But it will enable me to speak a little more intelligently, or intelligibly, on the most important of Kodaira's and Serre's achievements than I otherwise could have done.

Our Committee did not adopt a strict definition of what is meant by a 'young-mathematician'. But if the limit is put at forty, then Kodaira is young — while Dr. Serre, born in 1926, is young by any definition. Kunihiko Kodaira is a native of Japan and grew up in Tokyo. He studied mathematics and later also theoretical physics at Tokyo University and became in due time lecturer, assistant professor and finally full professor at that university, a position still held by him today. In 1949 he came to the Institute for Advanced Study in Princeton and has remained in America since then, at the Institute, at Johns

Hopkins and at Princeton University, to which he now belongs as a Research Associate. In a short biography written at my request he mentioned as the two main influences in his mathematical development: the teaching of Prof. S. Iyanaga and my book on Riemann surfaces. Of that I am very proud.

Kodaira's early papers deal with a number of questions in algebra, topology, group theory, almost periodic functions. They show good craftsmanship and originality, testifying also to the width of his mathematical interests; but I shall put here all the emphasis on his later work. There is first his fine treatment of the eigenfunction expansions for ordinary differential equations of order 2 and more generally of any even order, with singular ends. This work he began in Japan while still cut off from Western mathematics. After the distinction between the limit point and the limit circle case for a singular end, established as early as 1909, the main problem was how to determine explicitly the non-negative weight differential as function of the spectral parameter. This was accomplished independently by Titchmarsh and Kodaira; however I find Kodaira's approach more direct. The generalization to arbitrary even order, especially of the basic distinction between limit point and limit circle, is far from trivial.

But Kodaira's outstanding achievement lies in the theory of harmonic integrals and the numerous profound applications he made of it to Kählerian and more specially to algebraic varieties.

Riemann had based the theory of algebraic functions of one variable and their integrals on topology and construction of potentials on his Riemann surfaces by means of the transcendental Dirichlet principle. Later on, algebraic methods, Dedekind-Weber, Hensel-Landsberg, came to the fore. Weierstrass' function-theoretic treatment and the geometric approach by Clebsch and Max Noether occupy an intermediary position, but their constructions are also fundamentally algebraic. These methods carried the day when in the splendid development of the Italian school one passed from algebraic curves to algebraic varieties of higher dimensions. Algebra naturally will ask how much of the results originally stated for the coefficient field of all complex numbers survive if this field is replaced by other fields, in particular by fields of prime characteristic. Riemann's transcendental method, however, was not entirely forgotten, it suffices to mention the names of E. Picard and S. Lefschetz. But only with Hodge's book on Harmonic Integrals, in my opinion one of the great landmarks in the history of our science in the present century, its vast possibilities became evident; and Kodaira, beside Hodge himself, was the first to exploit them.

The central existence theorem in Hodge's book states that there is a uniquely determined harmonic form f of rank p with pre-assigned periods, or as I prefer to say, corresponding to a given element of the cohomology group

$\mathfrak{C}_p(R)$. Hodge's proof by means of the parametrix method, which E. E. Levi and D. Hilbert had applied to elliptic partial differential equations of the second order, contained a gap which at the time looked pretty serious. The gap was filled in 1942/43 by two people: Kodaira and the speaker; of course, independently of each other: for, as you may remember, our two nations, Japan and the USA, were at war in these years. While I stuck to the parametrix method, which had been suggested to Hodge by H. Kneser — to this proof de Rham soon gave the most elegant form —, Kodaira returned to the method of orthogonal projection, tried before by Hodge and essentially equivalent to the Dirichlet principle. It has the great advantage of being also applicable to non-compact Riemannian manifolds. In an impressive paper "Harmonic fields in Riemannian manifolds (generalized potential theory)" published in the Annals of Mathematics 1949 immediately after his arrival in the United States, Kodaira proves the existence of harmonic forms with prescribed singularities (and periods). A certain fundamental lemma needed for the justification of the principle of orthogonal projection is based here on the existence in the small of a fundamental solution; de Rham simplified this part in constructing that solution by the parametrix method. In the same paper Kodaira also gave the analog of Riemann-Roch's theorem for harmonic forms on a compact Riemannian manifold.

However, the relationship between harmonic and analytic functions is not as simple for several variables as it is for one. When turning from real Riemannian to complex Kählerian manifolds, Kodaira attacked the problem from its most difficult end, that of establishing the true Riemann-Roch theorem on algebraic varieties. From de Rham he had learnt that generalization of forms in the direction of Laurent Schwartz's *distributions* which de Rham called *currents*, and making use of a formalism developed by B. Eckmann and H. Guggenheimer, he derived a general existence theorem concerning complex currents. Then, in that wise limitation which, to use a word of Goethe's, shows the master, he first turned to the lowest case of complex dimension $n = 2$. The Riemann-Roch theorem which he obtained for a curve on the surface without multiple irreducible components, coincided with one derived by O. Goldman in algebraic fashion. It shortly afterwards enabled Kodaira and W. L. Chow to prove that a compact Kähler surface possessing two algebraically independent meromorphic functions is an algebraic surface without singularities. In '52 Kodaira passed on to the next higher case $n = 3$, — hard pioneering work that had to be done before one could hope for mastering algebraic varieties V_n of arbitrary dimension n.

The arithmetic genus γ of a compact Kähler variety V_n may be defined by means of the numbers g_p of the linearly independent harmonic forms of type

$(p, 0)$ (differentials of the first kind) as the alternating sum

$$\gamma(V_n) = g_0 - g_1 + g_2 - \ldots \pm g_n.$$

F. Severi had, by a so-called postulation formula, introduced two genera p_a and P_a for *algebraic* varieties V_n. (I tread here on hot soil, where I do not feel at home at all.) Let V_n be imbedded without singularities into an ambient complex projective space S. The hyperplanes in S define by their intersection with the variety V_n in S a class of equivalent divisors E. The genus $p_a(V_n)$ is obtained from studying the number of linearly independent meromorphic functions on V_n which are multiples of the power E^s of a divisor E of this class in its dependence on the variable positive integral exponent s. In a similar fashion $P_a(V_n)$ is obtained by means of the linear manifold of all multiples of E^s among the meromorphic n-fold differentials on V_n. (What is such a differential? A quantity represented in terms of local complex coordinates z on V_n by a meromorphic function Φ of z in such a way that, under the influence of an analytic coordinate transformation, Φ gets multiplied by the Jacobian.) — It is possible to generalize the genera γ, p_a, P_a from the underlying algebraic variety V_n to any divisor D on V_n.

By clever combination of algebraic computations with existential theorems ultimately derived from his theory of harmonic forms, Kodaira succeeded in proving one of Severi's conjectures, namely the identity of γ and P_a. I omit another famous conjecture settled by him in the same paper, concerning the so-called continuous systems on algebraic varieties. The identity of γ and p_a, also surmised by Severi, was proved not long afterwards by Kodaira and D. C. Spencer, after they had first investigated the structure of complex line bundles F over a Kähler variety V_n in several joint papers and Spencer and Serre had introduced the instrument of sheaves into this field of research. Replacing the divisor D by such a bundle F, one considers $\mathfrak{C}_q(\Omega_p(F))$, i.e. the q-dimensional cohomology group ($\mathfrak{C}_q$) in the sheaf of germs of holomorphic p-forms (Ω_p) with coefficients in F. In generalization of results due to P. Dolbeault, Kodaira showed that this group is isomorphic to the linear manifold $\Gamma_{p,q}(F)$ consisting of all harmonic forms of type (p, q) with coefficients in F and therefore is of finite dimension $d_{p,q}(F)$. The proof of the equation $\gamma = p_a$ operates with the Euler characteristic $\chi_p(F)$ defined as the alternating sum over q of these dimensions $d_{p,q}(F)$.

Finally I come to one of the most exciting results of Kodaira's investigations, published in the last number of the Ann. of Math. Hodge had remarked that algebraic varieties are not only Kähler varieties but that, in the general isomorphism between harmonic forms and cocycles modulo bounding cocycles, the harmonic ω of rank 2 connected with the Kähler metric corresponds to a

2-dimensional cocycle with *integral* coefficients. He called such Kähler varieties *of restricted type* and proved several properties for them that were known to hold for algebraic varieties. But now Kodaira has succeeded in showing that any Hodge variety, i.e. Kähler variety of restricted type, is *algebraic*, thus arriving at a very satisfactory intrinsic characterization of algebraic varieties. One can approach the same question also in opposite direction. You remember that a complex line bundle over a complex-analytic variety V_n was characterized by non-vanishing holomorphic transition factors $f_{\varrho\sigma}$ defined in $U^{\varrho\sigma}$. The automorphisms of a fiber consisted of the multiplications of the fiber variable ζ by an arbitrary non-vanishing complex constant. Multiplication can be changed into addition by passing to the logarithm. But the logarithm is not single-valued, and this has the consequence that the sum

$$\log f_{\varrho\sigma} + \log f_{\sigma\tau} + \log f_{\tau\varrho}$$

in $U^{\varrho\sigma\tau}$ is not necessarily zero but equals $2\pi i$ times an integer $n^{\varrho\sigma\tau}$; these integers are not uniquely determined, but just to that extent in which a 2-dimensional cocycle with integral coefficients modulo the bounding cocycles is determined; in other words, the bundle determines a "characteristic" element of the cohomology group $\mathfrak{C}_2(I)$. To it there corresponds a harmonic form ω of rank 2. It turns out that it is of type (1, 1) and real. But of course, the corresponding Hermitean form α with the coefficients $\alpha_{rs} = \omega_{rs}/i$ need not be positive-definite. However *if it is*, then we have exactly the situation considered by Hodge. Under these circumstances (I simplify his results a bit) Kodaira shows by an ingenious argument that all the cohomology groups $\mathfrak{C}_q(\Omega_p(F))$ reduce to zero for $q \geqq 1$. Making use of this fact for $p = 0$, he then succeeds in constructing a sufficient number of meromorphic functions on the variety, by means of which he effects its non-singular embedment in a projective space. His theorem is a profound generalization of the well-known fact that every compact Riemann surface belongs to an algebraic function field.

Kodaira has travelled a long and arduous road since he first developed the general theory of harmonic forms on a Riemannian manifold. But only if someone has the courage of attacking the primary concrete problems in all their complexity, will the general concepts gradually emerge which resolve the difficulties and ease the further progress; now the promised land seems to lie before him and his collaborators like an open plain. The last phase is characterized by a remarkable confluence of ideas to which, next to Kodaira, D. C. Spencer, F. Hirzebruch, R. Thom and also, last but not least, Serre contributed.

With this salto mortale I turn to the other of our two laureates of today, Dr. Jean-Pierre Serre. He is an ancien élève de l'Ecole Normale Supérieure, Docteur ès Sciences and at present Chargé de cours at Nancy University. In spring '52 he made his pilgrimage to Princeton. Considering the impressive line of first rank mathematicians France has in recent times given to our science, one need not look far for the formative influences that determined the beginnings of Dr. Serre's career. If one name is to be mentioned, I think, it would be that of Henri Cartan.

I have done Dr. Serre a real injustice, by passing over in silence in my report on Kodaira several results of Serre's which bore on the last phase of Kodaira's and Spencer's work. Let me try to make up for it! During 51/52 Serre in collaboration with H. Cartan succeeded in giving a brilliant reformulation and extension of some of the main results of complex variable theory in terms of cohomology in a complex-analytic sheaf. One of Serre's theorems in this line carries over de Rham's fundamental proposition referred to at the beginning of my address from real differential forms to complex holomorphic forms on a complex variety — provided this is of the type investigated by K. Stein. This shows at once that the Stein varieties have quite special topological properties, e.g. their cohomology groups $\mathfrak{C}_p(R)$ are zero for $n < p \leqq 2n$. Serre moreover proved a duality theorem concerning the cohomologies $\mathfrak{C}_q(\Omega_p(F))$ mentioned before — even in the more general case where the line of one complex variable ζ is replaced by a multi-dimensional vector space whose automorphisms are the non-singular linear transformations with complex coefficients. Serre also proved a theorem of finiteness of Kodaira's type, in one respect more limited, but in another respect much wider than his. By means of it Serre formulated a generalization of the Riemann-Roch theorem for algebraic varieties of arbitrary dimension. There is, beside the genera γ, p_a, P_a, a fourth, the Todd genus $T(D)$. Since '52, Hirzebruch had made an intensive investigation of the Todd genus after reducing its definition to a much more satisfactory form than originally given for it. By making use of Kodaira's results, including his most recent that every Hodge variety is algebraic, and Thom's theory of 'cobordisme', Hirzebruch was able, by ingenious calculations, to prove Serre's conjecture and to establish the identity of the Todd genus T with the arithmetic genus γ.

Hearing this, you may get the impression that our Committee did wrong in awarding the Fields Medals to two men whose research runs on such closely neighboring lines. This contact, however, has been established only during the last year and may well be a transient phenomenon. Serre's work before, which above all fascinated our Committee by the wealth of its surprising numerical results, is concerned with quite a different problem, the *homotopy theory of spheres*. Two mappings of the n-dimensional sphere S_n into a simply connected space M are *homotop* or equivalent to each other, and belong to the same *class*, if one can be continuously deformed into the other. In 1935 W. Hurewicz had given topology a new impetus by showing how homotopy classes may be composed; the resulting group of homotopy classes, the homotopy group, is abelian (except in the lowest case $n = 1$). Let us assume M also to be a sphere S_{n+k}, and denote then the homotopy group by $\pi(n + k, n)$. Since the case $k = 0$ has been settled long ago, we assume $k > 0$. At the Harvard Congress in 1950 Heinz Hopf chose this problem to illustrate the type of questions with which at the time topologists were mainly concerned. Said he: "Die gegenwärtige Situation in diesem Gebiet ist sehr unübersichtlich, und es ist verständlich und erfreulich, dass sich viele Geometer bemühen, hier Klarheit zu schaffen und ein Gesetz zu erkennen." Well, the man who has been most successful in carrying out this program during the intervening years is Serre, partly in collaboration with H. Cartan. Hopf had obtained quite early the result that $\pi(2n - 1, n)$ is infinite for even n. Let Z_r denote the cyclic group of order r. H. Freudenthal found that the structure of the group $\pi(n + k, n)$ depends but on the difference k of the two dimensions, I denote it therefore by $\pi'(k)$, as soon as $n \geqq k + 2$. Some isolated results had been obtained by various disparate methods, as e.g $\pi'(1) \simeq \pi'(2) \simeq Z_2$, Hurewicz' isomorphism $\pi(m, 2) \simeq \pi(m, 3)$ for $m \geqq 3$ and such strange ones as Eckmann's $\pi(m, 4) \simeq \pi(m - 1, 3) + \pi(m, 7)$; $\pi(m, 8) \simeq \pi(m - 1, 7) + \pi(m, 15)$. Serre's Comptes Rendus note of January '51, soon followed by his thesis, approached the problem by a new universal method which at once bore manifold fruit. The mystery began to be pierced. One now knows that $\pi(2n - 1, n)$ for even n are the *only* infinite homotopy groups, and that each of them is the direct sum of Z_∞ and a finite group. The groups $\pi'(4)$, $\pi'(5)$, $\pi'(12)$ are zero, and one knows explicitly all $\pi(n + k, n)$ for $k \leqq 8$ and arbitrary n. Some strange laws in which the prime numbers p play a role have emerged; e.g. the p-primary component of the abelian group $\pi(n + k, n)$ is zero for all $k < 4p - 6$, except $k = 2p - 3$ where it is Z_p. Serre has not remained the only one working in this field, but his intervention has been decisive, both with respect to methods as well as results.

The crux of his method is the reduction of the *homotopy* group of a space M to the *homology* group of suitably constructed auxiliary spaces. Since for the investigation of homology groups strong algebraic methods have recently been developed, the homotopy problem thus comes within the range of algebraic processes, and this is probably the reason for the greater success of Serre's method in comparison to those previously applied.

Let M be a connected space. Over it as a base we construct a sort of fiber space by associating with each point a of M, as the fiber corresponding to a, the totality of all continuous curves starting from a and returning to a. Unfortunately these fibers are not homeomorphic to each other as in the classical fiber spaces. Nevertheless Eckmann's fundamental construction for fiber spaces which the French call *relèvement des homotopies* carries over to them, and this is the essential point. (For an unbounded, unramified covering manifold $\overline{M}$ over M where the fiber consists of isolated points this fundamental construction is simply the unique determination of a line $\bar{\gamma}$ on $\overline{M}$ of which initial point $\bar{a}$ and trace γ are given.) It remained to study the relationship between the homologies of this generalized fiber space, its base and its fiber. Here Serre makes use of Leray's notion of "suite spectrale". The technical difficulties in carrying out this program are considerable, but Serre overcomes them all. Thus he proved for instance that for a simply connected space M of which all homology groups are finite or have a finite set of generators, the same statements hold for their homotopy groups. He shows that one can treat their p-primary components independently of each other for each prime p; etc.

Serre applied his results to the theory of Marston Morse concerning geodesics on Riemannian manifolds; joining hands with Armand Borel and another time with G. P. Hochschild, he won interesting results about Lie groups and algebras, thus e.g. settling a problem posed by Hopf to the effect that no sphere with the exception of S_2 and perhaps S_6 can be endowed with a complex-analytic structure. There is no time to discuss these things, and I prefer to let all the light of my lamp fall on Serre's investigations in the homotopy theory and the theory of sheaves. They are rich enough to show the enormous fecundity of his mathematical mind.

Here ends my report. If I omitted essential parts or misrepresented others, I ask for your pardon, Dr. Serre and Dr. Kodaira; it is not easy for an older man to follow your striding paces. Dear Kodaira: Your work has more than one connection with what I tried to do in my younger years; but you reached heights of which I never dreamt. Since you came to Princeton in 1949 it has been one of the greatest joys of my life to watch your mathematical development. I have no such close personal relation to you, Dr. Serre, and your research; but let me say this that never before have I witnessed such a brilliant ascension of a star in the mathematical sky as yours. The mathematical community is proud of the work you both have done. It shows that the old gnarled tree of mathematics is still full of sap and life. Carry on as you began!

ONE-HOUR LECTURES

AUS DER MENGENTHEORETISCHEN TOPOLOGIE DER LETZTEN ZWANZIG JAHREN.

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Dieser Vortrag soll kein Bericht werden: Nichts liegt mir ferner, als der Anspruch auf eine vollständige Darstellung — sei es auch nur der wichtigsten Ereignisse in der Entwickelung der mengentheoretischen Topologie in diesen letzten zwei Jahrzehnten. Meine Aufgabe ist ganz bescheiden, und auch ziemlich subjektiv: ich will bloss an wenigen, mir besonders nahe stehenden, aber auch mir wichtig erscheinenden Beispielen zu zeigen versuchen, dass die mengentheoretische Topologie kein bereits erschöpftes Gebiet mathematischer Forschung darstellt, dass es vielmehr unter den neueren Errungenschaften auf diesem Gebiet solche gibt, die auch ein allgemeinmathematisches Interesse beanspruchen dürfen.

I

Von den Ergebnissen, die sozusagen der „reinen" mengentheoretischen Topologie angehören, d.h. nicht nur ihrem Inhalt, sondern auch ihren Beweisen nach in den Bereich der Mengenlehre fallen, will ich in diesem Vortrage zweierlei berühren: erstens, die 1951 von Jury Smirnov [1]) erbrachte Lösung des allgemeinen Metrisationsproblems für beliebige topologische Räume [1] [2]), zweitens, das Problem der sogenannten „gleichmässigen Topologie", auf dessen Lösung einerseits die „structures uniformes" von André Weil, andererseits aber die viel weniger bekannten „Nachbarschaftsräume" von Ephrämowich auf zwei verschiedene Weisen hinauszielen. Die Theorie der Nachbarschaftsräume hängt auf eine bemerkenswerte Art mit der Theorie der bikompakten Erweiterungen topologischer Räume zusammen, die ihrerseits eins der wichtigsten Kapitel der allgemeinen Topologie bildet.

1. Das Metrisationsproblem entstand gleichzeitig mit dem Begriff des topologischen Raumes selbst: sobald man im Besitze dieses Begriffes ist, kann man nicht umhin, nach den Bedingungen zu fragen, die für die Möglichkeit, in

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[1]) Und — wie ich nachträglich erfahre — unabhängig von ihm auch vom japanischen Mathematiker Nagata.

[2]) Ziffern in eckigen Klammern bedeuten Hinweise auf das sich am Ende des Artikels befindende Literaturverzeichnis.

den gegebenen topologischen Raum eine Metrik einzuführen, notwendig und hinreichend sind.

Es handelt sich mit anderen Worten um die folgende Frage:

Wann ist ein topologischer Raum metrisierbar, das heisst, einem metrischen Raume homöomorph?

In wichtigen Spezialfällen wurde diese Frage bereits in den Jahren 1922–1923 von Urysohn (für kompakte Räume und Räume mit abzählbarer Basis) und (für im kleinen kompakte Räume von mir) beantwortet. Eine allgemeine und befriedigende Lösung des Metrisationsproblems blieb aber aus; nach mehreren Bemühungen ist man, wie mir scheint, sogar zu der Meinung gekommen, eine solche sei überhaupt ziemlich aussichtslos. Umso überraschender wirkt die überaus einfache Lösung, die diesem berühmten Problem Jury Smirnov, und Nagata 1951 gegeben haben.

Der klassische Urysohnsche Metrisationssatz lautet bekanntlich: Notwendig und hinreichend dafür, dass ein topologischer Raum mit abzählbarer Basis metrisierbar sei, ist, dass dieser Raum normal sei. Tychonoff hat den Urysohnschen Satz dahin etwas verschärft, dass er gezeigt hat, dass für die Metrisierbarkeit eines Raumes mit abzählbarer Basis bereits die Regularitätsforderung ausreicht.

Um den Satz von Smirnov zu formulieren brauchen wir nur einen Begriff: denjenigen der lokalen Endlichkeit eines Mengensystem: ein System von Punktmengen eines topologischen Raumes heisst lokal endlich, wenn jeder Raumpunkt eine Umgebung besitzt, die nur mit endlich vielen Punktmengen des gegebenen Mengensystems gemeinsame Punkte hat.

Sodann gilt folgender

Allgemeiner Metrisationssatz. *Die notwendige und hinreichende Bedingung für die Metrisierbarkeit eines topologischen Raumes R lautet:*

Der Raum R soll regulär sein und eine Basis besitzen, welche die Vereinigung von höchstens abzählbarvielen lokal endlichen Mengensystemen ist.

Offenbar enthält der Satz von Nagata – Smirnov den Urysohnschen Metrisationssatz als Spezialfall.

2. Bekanntlich kann man den Begriff des topologischen Raumes unmittelbar auf den Begriff des Berührungspunktes einer beliebigen Punktmenge des Raumes gründen (der Punkt a ist Berührungspunkt der Menge $\mathscr{A}$, wenn er zu der abgeschlossenen Hülle von $\mathscr{A}$ gehört); insbesondere ist eine Abbildung eines topologischen Raumes X in einen topologischen Raum Y stetig, wenn sie die Berührungspunkte sämtlicher Punktmengen erhält; die Topologisierung einer Menge (d.h. die Verwandlung der Menge in einen topologischen Raum) besteht somit darin, dass zwischen gewissen Teilmengen der gegebenen Menge und gewissen Elementen der Menge das Vorhandensein einer „Berührung"

oder einer „Nachbarschaft" erklärt wird. Dieser Sachverhalt legt die Frage nahe, ob es Möglichkeiten gibt, unter Geltung gewisser natürlicher Voraussetzungen nicht nur Element und Teilmenge, sondern auch zwei Teilmengen einer gegebenen Menge als benachbart zu erklären und dann als gleichmässig stetige diejenigen Abbildungen besonders auszuzeichnen, bei denen diese Nachbarschaftsbeziehung erhalten bleibt. Man kommt auf diese Weise zu *Nachbarschaftsräumen* und „Nachbarschaftstopologie". Das Vorbild ist durch die metrischen Räume gegeben: zwei Punktmengen eines metrische Raumes dürften „benachbart" heissen, wenn die Entfernung zwischen diesen Punktmengen gleich Null ist. Man überzeugt sich leicht davon, dass die Abbildungen eines metrischen Raumes X in einen metrischen Raum Y, die je zwei in diesem Sinne benachbarte Punktmengen des Raumes X in zwei benachbarte Punktmengen des Raumes Y überführen, nichts anderes als die (im gewöhlichen Sinne) gleichmässig stetigen Abbildungen des einen Raumes in den anderen sind. Auch in jeder topologischen Gruppe lässt sich ein natürlicher Nachbarschaftsbegriff einführen.

Der Begriff des Nachbarschaftsraumes rührt von *Ephrämowich* her und bildet offenbar einen anderen Ansatz zur Lösung derselben Aufgabe, die zu lösen auch die „structures uniformes" von André Weil berufen sind. Der Nachbarschaftsbegriff von Ephrämowich wird unter folgenden Bedingungen eingeführt, die als Axiome der Nachbarschaftsräume anzusprechen sind:

Axiom I (Symmetrieaxiom). Falls A mit B benachbart ist, so auch B mit A.

Axiom II (Summenaxiom). Die Vereinigungsmenge $A \cup B$ zweier Mengen A und B ist dann und nur dann mit einer dritten Menge C benachbart, wenn mindestens eine unter den Mengen A und B mit C benachbart ist.

Axiom III (Identitätsaxiom). Zwei einpunktige Mengen a und b sind dann und nur dann benachbart, falls sie identisch sind.

Axiom IV (von der leeren Menge). Die leere Punktmenge ist mit keiner Punktmenge benachbart.

Um das letzte Axiom — das Trennungsaxiom — zu formulieren, bezeichnen wir als „Nachbarschaft" einer Punktmenge $\mathscr{A}$ im Nachbarschaftraume R jede Menge $O\mathscr{A}$ von der Eigenschaft, dat $\mathscr{A}$ mit dem Komplement $R - O\mathscr{A}$ zu der Menge $O\mathscr{A}$ nicht benachbart ist. Somit lautet

Axiom V. („Trennungsaxiom"). Je zwei „voneinander ferne" (d.h. miteinander nicht benachbarte) Mengen besitzen zueinader fremde Nachbarschaften.

Nun ist jeder Nachbarschaftsraum R auf eine und nur eine Weise als topologischer Raum aufzufassen: die Topologie im Raume R wird dadurch definiert, dass man als abgeschlossene Hülle einer beliebigen Menge $\mathscr{A}$ die Menge derjenigen Punkte erklärt, die mit der Menge $\mathscr{A}$ benachbart sind. Der auf diese

Weise gewonnene topologische Raum ist ein nicht nur regulärer, sondern sogar vollständig regulärer Raum (ein Hausdorffscher Raum R heisst vollständig regulär, wenn man zu jedem seiner Punkte a und jeder diesen Punkt nicht enthaltenden abgescholossenen Menge F eine im ganzen R definierte und dortselbst stetige Funktion $f(x)$ finden kann, die im Punkte a etwa den Wert 0, in sämtlichen Punkten von F den Wert 1 annimmt und überall der Ungleihung $0 \leqq f(x) \leqq 1$ genügt). Umgekehrt, lässt sich jeder vollständig reguläre Raum R — und zwar im allgemeinen auf unendlichviele Weisen — als ein auf dem topologischen Raum R „gedeihender'' Nachbarschaftsraum auffassen.

Bis dahin läuft die Theorie der Nachbarschaftsräume mit der der Weilschen „Gleichmässigkeitsräumen'' (structures uniformes) parallel. Allerdings ist die Zerspaltung in Gleichmässigkeitsräume feiner: jeder Gleichmässigkeitsraum lässt sich auf einer einzigen Weise als Nachbarschaftsraum auffassen, während jeder Nachbarschaftsraum auf mindestens eine, im allgemeinen aber auf unendlich viele verschiedene Weisen als Weilscher Gleichmässigkeitsraum aufgefasst werden kann.

Jetzt kommen wir zu dem vielleicht wichtigsten unter den ziemlich zahlreichen Sätzen, die die Theorie der Nachbarschaftsräume bilden.

Bekanntlich sind (nach einem alten Satz von Tychonoff) die vollständig regulären Räume mit den Teilmengen der bikompakten Hausdorffschen Räumen identisch. Und zwar besitzt jeder vollständig reguläre Raum eine bikompakte Erweiterung (d.h. er ist überalldichte Teilmenge von mindestens einem Bikompaktum).

Es lässt sich in die Menge der bikompakten Erweiterungen eines gegebenen vollständig regulären Raumes R eine Ordnungsrelation einführen und zwar sagt man, dass die bikompakte Erweiterung b_2R des Raumes R stärker ist, als die bikompakte Erweiterung b_1R desselben Raumes falls man b_2R stetig unter Festhaltung sämtlicher Punkte von R auf b_1R abbilden kann. Ein berühmter Satz von Eduard Čech (1936) besagt, dass es in der so gewonnenen teilweise geordneten Menge der bikompakten Erweiterungen des gegebenen vollständig regulären Raumes R ein einziges maximales Element d.h. eine bikompakte Erweiterung βR gibt, die auf jede bikompakte Erweiterung bR desselben Raumes stetig unter Festhaltung aller Punkte von R abgebildet werden kann.

Ed. Čech hat seinen Raum βR als einen gewissen Raum stetiger Funktionen definiert. Ich habe [2] 1939 eine andere Konstruktion desselben Raumes gegeben, die sich bei weiteren Untersuchungen (von Fomin, Katetov, Smirnov u.a.) als besonders bequem und fruchtbar erwiesen hat, die aber scheinbar wenig bekannt geblieben ist. Deswegen werde ich sie hier kurz auseinandersetzen.

Ich sage, die offene Mange Γ' (des topologischen Raumes R) sei in der offenen Menge Γ (desselben Raumes) vollständig regulär enthalten, falls es eine im ganzen Raume R stetige Funktion $f(x)$, $0 \leq f(x) \leq 1$ gibt, die auf $\overline{\Gamma'}$ gleich 0 und in $R - \Gamma$ gleich 1 ist.

Ein Mengensystem heisst bekanntlich zentriert, falls je endlichviele Mengen dieses Systems einen nichtleeren Durchschnitt haben. Ein zentriertes System von offenen Mengen eines vollständig regulären Raumes R heisse vollständig regulär, wenn jede Menge dieses Systems mindestens eine Menge desselben Systems vollständig regulär enthält. Schliesslich heisst ein System offener Mengen des vollständig regulären Raumes R ein Ende dieses Raumes, falls es zentriert, und vollständig regulär ist und in keinem von ihm verschiedenen zentrierten vollständig regulären System von in R offenen Mengen enthalten ist. Als Beispiel eines Endes kann die Gesamtheit aller Umgebungen eines Punktes (des volständig regulären Raumes R) dienen. Ich betrachte jetzt die Menge αR aller Enden $\xi = \{\Gamma\}$ des Raumes R und erkläre für irgendein $\xi \in \alpha R$ als Umgebung $U_\Gamma \xi$ des Ende ξ die Menge aller Enden, die das festgewählte Element Γ des Endes ξ enthalten. Auf diese Weise wird αR zu einem Umgebungsraum, und zwar zu einem Bikompaktum. Da man einen Punkt von R mit dem aus sämtlichen Umgebungen dieses Punktes bestehenden Ende identifizieren kann, lässt sich R als eine — und zwar überalldichte — Teilmenge des Bikompaktums αR auffassen, so das αR als bikompakte Erweiterung von R anzusprechen ist; nun zeigt man, dass die bikompakte Erweiterung αR maximal und infolgendessen mit der Čechschen Erweiterung βR identisch ist.

Der Endenbegriff lässt sich dahin modifizieren, dass man mit seiner Hilfe nicht nur die maximale Erweiterung, sondern sämtliche bikompakten Erweiterungen des Raumes R erhält. Und diese Tatsache bildet die Grundlage des folgenden Smirnovschen Hauptsatzes über Nachbarschaftsräume:

Satz von Smirnov [3]. *Ist bR eine bikompakte Erweiterung des vollständig regulären Raumes R, so erhält man einen auf R definierten Nachbarschaftsraum R_b sobald man zwei Teilmengen von R dann und nur dann als benachbart erklärt, wenn ihre abgeschlossenen Hüllen in bR gemeinsame Punkte haben. Dabei kann jeder Nachbarschaftsraum, der sich auf dem topologischen Raume R definieren lässt, auf diese Weise erzeugt werden, und zwar entsteht eine eineindeutige Beziehung zwischen den bikompakten Erweiterungen des vollständig-regulären Raumes R und allen „auf diesem Raume gedeihenden" Nachbarschaftsräumen. Die gleichmässigstetigen Abbildungen eines Nachbarschaftsraumes R_b auf einen Nachbarschaftsraum R'_b sind identisch mit denjenigen stetigen Abbildungen des topologischen Raumes R auf den topologischen Raum R', die sich zu stetigen Abbildungen von bR auf bR' erweitern lassen.*

Mit anderen Worten: Es besteht eine vollständige Äquivalenz zwischen der Theorie der Nachbarschaftsräume und der Theorie der bikompakten Erweiterungen vollständig regulärer topologischer Räume.

II

3. Die Schöpfung der Dimensionstheorie bildet einen Wendepunkt in der Entwicklung der mengentheoretischen Topologie. Diese besondere Stellung der Dimensionstheorie lässt sich leicht verstehen. Erstens beginnt wohl jede geometrische Betrachtung mit der Frage nach der Dimension der zu behandelnden geometrischen Gebilde; zweitens ist historisch gerade die Dimensionstheorie zum Treffpunkt verschiedener — mengentheoretischer und kombinatorisch-algebraischer Methoden geworden; denn einerseits wurde im Laufe der Entwicklung der Dimensionstheorie die mengentheoretische Topologie als solche durch neue Fragestellungen und Methoden bereichert, während andererseits dimensionstheoretische Probleme auf die Entwicklung auch derjenigen Teile der Topologie, die mit dem Dimensionsbegriff selbst scheinbar nichts zu tun hatten, eine überaus anregende Wirkung ausgeübt haben; man denke etwa daran, dass eine dimensionstheoretische Frage den ersten Anlass zur Untersuchung der wesentlichen Abbildungen von Polyedern auf gleichdimensionale Sphären gegeben hat — einer Untersuchung, die zu dem berühmten Hopfschen Klassifikationssatz und mit ihm zur Eröffnung einer neuen Periode in der Geschichte der Topologie führte — der Periode, die man als Homotopie-Periode bezeichnen dürfte.

4. In der Entwicklung der Dimensionstheorie lassen sich deutlich zwei Perioden unterscheiden: in der ersten werden allgemeine dimensionstheoretische Satze ausschliesslich mit punktmengentheoretischen Methoden bewiesen, während sich die zweite Periode, beginnend am Anfang der dreissiger Jahren mit dem Begriff der wesentlichen Abbildung, sich durch systemathische Benutzung kombinatorisch-algebraischer Methoden auszeichnet. Die Fragen, die in dieser zweiten Periode behandelt werden, haben einen ausgesprochen geometrischen Inhalt.

Die erste Periode überliess der Zukunft unter manchen ungelösten Aufgaben die folgenden drei, die nach meiner Meinung ein besonderes Interesse beanspruchen dürfen:

1. Ist die Dimension des topologischen Produktes zweier Kompakta gleich der Summe der Dimensionen dieser Kompakta? Mit anderen Worten, ist die sogenannte „dimensionstheoretische Produkthypothese" richtig? (Menger).

2. Gibt es zu jeder $(n-1)$ dimensionalen Menge $\mathscr{A}$ im n-dimensionalen Raum R^n eine Kugel U^n, die durch diese Menge zerschnitten wird — in dem Sinne, dass es in $U^n - \mathscr{A}$ ein Punktepaar gibt, welches durch kein in $U^n - \mathscr{A}$ liegendes Kontinuum verbunden werden kann? (Urysohn).

3. Ist in jedem unendlichdimensionalen Kompaktum ein Teilkompaktum von beliebiger endlicher Dimension enthalten? (Tumarkin).

Die dritte dieser Fragen bleibt bis heute offen; das Einzige, was hier bekannt ist, ist der Satz des holländischen Mathematikers van Heemert [18] welcher behauptet, dass jedes unendlichdimensionale Kompaktum ein zweidimensionales Teilkompaktum enthält. Die vollständige Lösung des Tumarkinschen Problems dürfte von solchen Teilen der Topologie wie z.B. die Theorie der Sphärenabbildungen abhängen.

Die beiden ersteren Fragen sind vollständig gelöst, und ihre Lösung erfolgte im Rahmen der allgemeinen kombinatorischen Topologie der Punktmengen, zu der insbesondere die homologische oder „geometrische" Dimensionstheorie als ein wesentliches Kapitel gehört.

Das wesentliche in der letztgenannten Theorie (vgl. [4]), besteht im Folgenden. Zyklen und Homologieen im Sinne der kombinatorischen Topologie lassen sich nicht nur auf Polyedern, sondern — mittels eines gewissen Grenzüberganges — auch auf beliebigen Kompakten definieren [3]). Unter den verschiedenen abgeschlossenen Teilmengen eines Kompaktums, auf denen ein gegebener Zyklus liegt, gibt es eine kleinste, und diese heisst der Träger des Zyklus. Ein Zyklus heisst wesentlich, wenn er auf seinem Träger nicht berandet (nicht homolog Null ist). Nun ist auf einem n-dimensionalen Kompaktum φ jeder berandende Zyklus von einer Dimension $\geqq n$ unwesentlich (er „klappt" — in einem naheliegenden Sinne — „zusammen"), während es immer auf φ berandende wesentliche $(n-1)$—dimensionale Zyklen gibt. Beispiel: Es sei φ eine abgeschlossene Kreisscheibe mit dem Rande φ^*. Die Kreisperipherie φ^* ist Träger eines eindimensionalen auf φ^* nichtberandenden, infolgedessen wesentlichen Zyklus, welcher aber auf der Kreisscheibe φ berandet. Einen wesentlichen zweidimensionalen Zyklus gibt es auf φ nicht. Also ist die Kreisscheibe zweidimensional.

Das Hauptresultat der vorangehenden Betrachtung besteht darin, dass sich die Dimension durch Begriffe der kombinatorischen Topologie — Zyklus und Berandung — ausdrücken lässt. Diese Begriffe sind aber algebraischer Art — ihnen liegt eine Abelsche Gruppe, die Koeffizientengruppe, oder der Koeffizientenbereich zugrunde. Folglich hängt auch der kombinatorische (oder homologische) Dimensionsbegriff von der Wahl des zugrundegelegten Koeffizientenbereiches ab. Und tatsächlich gibt es unendlichviele homologische Dimensionen; die gewöhnliche, Brouwer-Urysohn-Mengersche Dimension ist die grösste unter allen homologischen Dimensionen; man erhält sie, wenn man als Koeffizientgruppe die additive Gruppe der modulo 1 reduzierten rationalen Zahlen nimmt [4]). Die Existenz, neben der gewöhnlichen Brouwer-Urysohnschen

[3]) Man vergl. hierüber den § 6 dieses Vortrages.

[4]) Geht man zu der dualen „∇-Theorie" über (∇-Zyklen — auch cozyklen genannt, ∇-Ränder), so tritt als Koeffizientenbereich die Gruppe der ganzen Zahlen auf.

Dimension noch unendlich vieler — wenigstens vom algebraischen Standpunkte — gleichberechtigter Dimensionsfunktionen, die alle eine Reihe von Eigenschaften der gewöhnlichen Dimension gemein haben, erlaubt eine viel feinere Untersuchung der dimensionalen Struktur eines Kompaktums. Und als eins der ersten Resultate dieser feineren Untersuchung erhielt man die Widerlegung (durch Pontrjagin) der Produkthypothese. Falls namlich ein im gewöhnlichen Sinne n-dimensionales [5]) Kompaktum nach irgendeinem Koeffizientenbereich eine Dimension $< n$ hat, (sie kann ja nach dem eben gesagten nicht $> n$ sein) so ist man berechtigt, dieses Kompaktum als im gewissen Sinne „schwach-dimensional" zu betrachten und unter diesen schwach-dimensionalen Kompakten diejenigen zu suchen, die die negative Antwort auf die Produkthypothese liefern. Dieser Ansatz hat sich als erfolgreich erwiesen: es ist Pontrjagin gelungen, für jede Primzahl p eine zweidimensionale abgeschlossene Menge Π_p im R^4 zu konstruiren, deren Dimension nach dieser Zahl p (d.h. nach der zyklischen Gruppe der Ordnung p als Koeffizientengruppe) gleich 2, nach allen anderen Primzahlen aber gleich 1 ist. Die gewöhnliche Dimension aller dieser Π_p ist 2, während die Dimension des topologischen Produktes $\Pi_p \times \Pi_{p'}$ (für $p \neq p'$) gleich 3 ausfällt. Zum Beispiel, ist schon $\dim(\Pi_2 \times \Pi_3) = 3$ und die Mengersche Produktvermutung ist hiermit widerlegt!

Dieses Resultat bedeutet u.a., dass die Dimensionen zweier endlichdimensionaler Kompakte die Dimension ihres Produktes noch nicht bestimmen. Es erhebt sich somit die Frage nach denjenigen topologischen Invarianten, deren Kenntnis für die beiden gegebenen Kompakta die Dimension des Produktes dieser Kompakta eindeutig bestimmt.

Es liegt nahe, zu vermuten, dass die Gesamtheit der Dimensionen nach allen möglichen Koeffizientengruppen die Dimension des Produktes bestimmt. Diese Vermutung wurde vor einigen Jahren von Bockstein bewiesen [6]. Bockstein beweist mehr: er zeigt, dass wenn man für zwei Kompakta φ, φ' die Dimensionen nach sämtlichen Koeffizientengruppen kennt, so kennt man sie alle auch für das Produkt $\varphi \times \varphi'$. Des weiteren zeigt Bockstein auch, dass die Dimensionen nach allen möglichen Koeffizientengruppen bereits bestimmt sind, wenn man bloss die Dimensionen nach gewissen abzälbar-vielen Koeffizientengruppen kennt. Es sind dies (in der Sprache der V-Theorie) die folgenden Gruppen:

1. Die additive Gruppe der rationalen Zahlen $\mathfrak{R}$,

2. Die Untergruppen $\mathfrak{R}_p \subset \mathfrak{R}$ derjenigen rationalen Zahlen, die (als unkürzbare Brüche dargestellt) im Nenner eine gegebene Primzahl p nicht enthalten;

[5]) „Dimension" schlechthin bedeutet immer die gewöhnliche Dimension; sonst wird die Koeffizientengruppe erwähnt, nach der die Dimension zu verstehen ist.

3. Die zyklischen Gruppen von Primzahlordnung;

4. Die sogenannten Gruppen Q^p; unter Q^p versteht man die additive Gruppe der modulo 1 reduzierten Rationalzahlen von der Form $\frac{k}{2^p}$, wobei p eine feste Primzahl ist und k die Gesamtheit aller ganzen Zahlen durchläuft.

Bockstein gibt explizite Formeln an, die es für ein gegebenes beliebiges Kompaktum φ erlauben, nach Angabe einer Abelschen Gruppe $\mathfrak{A}$ und der Dimensionen von φ nach den soeben angeführten Koeffizientenbereichen die Dimension von φ nach dem Koeffizientenbereich $\mathfrak{A}$ zu berechnen.

Von mehreren weiteren interessanten Sätzen aus demselben Ideenkreis seien hier die folgenden erwähnt:

es gibt unendlich viele Koeffizientenbereiche, für die die Produkthypothese gilt; und zwar gilt sie insbesondere für alle Koeffizienten*körper* (Pontrjagin);

es lässt sich ein zweidimensionales Kompaktum Π konstruieren, dessen topologisches Quadrat $\Pi \times \Pi$ dreidimensional ist [7] (Boltjanski);

Nennen wir jetzt ein Kompaktum φ dimensionell-vollwertig falls es der folgenden Bedingung genügt: bei jeder Wahl des Kompaktums φ' ist

$$\dim (\varphi \times \varphi') = \dim \varphi + \dim \varphi'.$$

Dann lautet eine notwendige und hinreichende Bedingung für die dimensionelle Vollwertigkeit eines Kompaktums φ folgendermassen: für jede Primzahl p ist $\dim \varphi$ gleich der Dimension von φ nach der Koeffizientengruppe Q^p (Boltjanski). Schliesslich sei noch Folgendes erwähnt: es gibt eine abzählbare Folge von zweidimensionalen Kompakten

$$\Pi_1, \Pi_2, \dots, \Pi_k, \dots$$

von der Eigenschaft, dass jedes Kompaktum φ welches für alle Π_k der Bedingung

$$\dim (\varphi \times \Pi_k) = \dim \varphi + \dim \Pi_k = \dim \varphi + 2$$

genügt, dimensionell vollwertig ist (d.h. der Bedingung $\dim (\varphi \times \varphi') = \dim \varphi + \dim \varphi'$ für jedes Kompaktum φ' genügt).

5. Die Urysohnsche Vermutung, dass eine $(n-1)$-dimensionale Menge im R^n den Raum im Kleinen zerschneidet hat sich für abgeschlossene Mengen bestätigt: sie wurde von mir [4] 1931 im Rahmen des folgenden allgemeineren Verschlingungssatzes bewiesen:

Dimensioneller Hindernissatz. *Es sei φ ein r-dimensionales Kompaktum im R^n. Dann gibt es im R^n eine n-dimensionale Vollkugel U^n von der Eigenschaft, dass in $U^n - \varphi$ (Polyeder)-zyklen von der Dimension $n - r - 1$ und von beliebig kleinem Durchmesser liegen, die in $U^n - \varphi$ nicht beranden. Für $s < n - r - 1$ ist dagegen bei beliebiger Wahl der Vollkugel U^n jeder in $U^n - \varphi$ gelegene s-dimensionale Zyklus in $U^n - \varphi$ homolog Null.*

Mit etwas anderen Worten: Die zu einem r-dimensionalen Kompaktum φ komplementäre offene Menge $\Gamma = R^n - \varphi$ ist für $s < n - r - 1$ gleichmässig lokal zusammenhängend; für $s = n - r - 1$ ist sie es bestimmt nicht.

Nun bedeutet der gleichmässige lokale Zusammenhang in einer bestimmten Dimension, dass „kleine" Zyklen „kleine" Ketten beranden, wobei „klein" einen kleinen Durchmesser bedeutet.

Das Attribut „klein", auf Zyklen angewendet, kann aber auch andere Bedeutungen haben, wie die folgenden Figuren und die folgenden Definitionen lehren:

Fig. 1

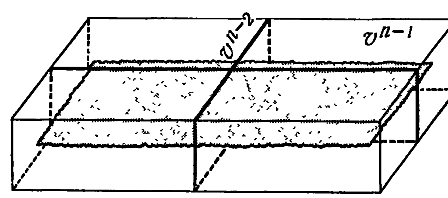

Fig. 2

Definition I. Es sei x eine r-dimensionale Kette in R^n. Wir bezeichnen mit τx die untere Grenze derjenigen $\varepsilon > 0$, für die es eine ε-Verschiebung der Menge der Eckpunkte der Kette x gibt, welche diese Kette in eine identisch verschwindende Kette überführt. Die Zahl τx heisse etwa die Wesentlichkeitsschranke der Kette x. (Im Folgenden werden nur Wesentlichkeitsschranken von Zyklen betrachtet).

Definition II. Es sei wieder x eine r-dimensionale Kette im R^n und $p \leqq r$ eine nicht negative ganze Zahl. Wir bezeichnen mit $\alpha^p x$ die untere Grenze derjenigen ε, für die es es eine ε-Verschiebung der Menge der Eckpunkte der Kette x, gibt, welche sämtliche Simplexe dieser Kette in Simplexe von einer Dimension $\leqq p$ überfuhrt. Die Zahl $\alpha^p x$ heisse etwa der p-dimensionale (oder p-te) Halbmesser der Kette x. Für $p < r$ ist offenbar $\tau x \leqq \alpha^p x$ und die obigen Figuren zeigen, dass tatsächlich $\tau x \leqq \alpha^p x$ auch für einen Zyklus x sein kann.

Schliesslich ist $\alpha^0 x$ gleich der Hälfte des Durchmessers von x.

Unter Benutzung dieser Begriffe lässt sich der dimensionelle Hindernissatz folgendermassen aussprechen:

Es sei $\varphi \subset R^n$ ein r-dimensionales Kompaktum. Dann existiert ein festes $\gamma > 0$ von der Eigenschaft, dass bei beliebigem $\varepsilon > 0$ es $(n - r - 1)$-dimensionale Zyklen v^{n-r-1} mit $\alpha^n v^{n-r-1} < \varepsilon$ in $\Gamma = R^n - \varphi$ gibt, die aber so beschaffen sind, dass für jede von einem solchen Zyklus berandete Kette x die Ungleichung $\alpha^0 x > \gamma$ gilt. Andererseits berandet jeder s-dimensionale Zyklus, $s < n - r - 1$, in Γ, dessen Halbmesser $\alpha^0 z^1 < \varepsilon$ ist, eine Kette x mit $\alpha^0 x < \varepsilon$.

Somit spiegelt sich die Tatsache, dass dim $\varphi = r$ ist, in gewissen metrischen Eigenschaften der in Γ gelegenen Zyklen einer bestimmten Dimensionszahl — nähmlich, der Dimensionszahl $n - r - 1$ wieder. Sie spiegelt sich aber auch wieder in den metrischen Eigenschaften der $(n - 1)$-dimensionalen Zyklen von Γ. Nennen wir einen in Γ gelegenen $(n - 1)$-dimensionalen Zyklus v^{n-1} einen „Sack" um φ, falls v^{n-1} mit jedem Punkt von φ verschlungen ist. Es ist zunächst klar, dass für ein r-dimensionales Kompaktum $\varphi \subset R^n, r < n$, es Säcke v^{n-1} mit beliebig kleinem τv^{n-1} und sogar beliebig kleinem $\alpha^r v^{n-1}$ gibt. Andererseits hat aber Sitnikov [8] folgendes tiefliegendes Theorem bewiesen:

Zu jedem r-dimensionalen $\Phi \subset R^n$ gibt es eine feste positive Zahl γ von der Eigenschaft, dass für jeden Sack v^{n-1} um φ die Ungleichung

$$\alpha^{r-1}\, v^{n-1} > \gamma$$

gilt.

Dieser Satz liefert offenbar — neben dem Hinernissatze — eine neue Charakterisierung der Dimension eines Kompaktums $\varphi \subset R^n$ mittels Eigenschaften des Komplementes $\Gamma = R^n - \varphi$; sein anschaulicher Inhalt ist aus den beigefügten Figuren klar. Diese Figuren machen es aber auch plausibel, dass auf dem Sack v^{n-1} (auf Figur 2 ist $n = 3$, $r = 2$) ein $(n - 2)$-dimensionaler „Gürtel" v^{n-2} liegt, dessen $\alpha^{r-1} v^{n-2}$ beliebig klein vorausgesetzt werden kann, der aber in Γ nur Ketten x mit einem hinreichend grossen $\alpha^{r-1}\, x$ berandet, usw, bis man schliesslich zu den Zyklen v^{n-r-1} des Hindernissatzes gelangt.

Man erhält somit folgenden allgemeinen Satz [8]:

Sitnikovscher Sack-und Gürtelsatz. *Es sei φ ein r-dimensionales Kompaktum im R^n, $r \leqq n - 1$ Es gibt ein festes $\gamma > 0$ mit der folgenden Eigenschaft: zu jedem $k = 1, 2, \ldots, r + 1$ und jedem $\varepsilon > 0$ existiert in $\Gamma = R^n - \varphi$ ein $(n - k)$-dimensionaler „Gürtel" um φ, d.h. ein $(n - k)$-dimensionaler (bei $k > 1$ in Γ berandender) Zyklus v^{n-k} mit*

$$\alpha^{r-k+1}\, v^{n-k} < \varepsilon \;\; ; \;\; \tau\, v^{n-k} < \varepsilon \text{ [6])}$$

welcher die folgenden beiden Eigenschaften besitzt:

1⁰ *Für jede durch v^{n-k} in Γ berandete Kette x^{n-k+1} gilt*

$$\alpha^{r-k+1}\, x^{n-k+1} > \gamma$$

2⁰ *Für jeden dem Zyklus v^{n-k} in dessen γ-Umgebung (in bezug auf Γ) homologen Zyklus w^{n-k} gilt $\alpha^{r-k} w > \gamma$.*

Es sei andererseits $s > r$ und $k = 1, 2, \ldots, s + 1$; dann ist bei jedem $\varepsilon > 0$ jeder $(n - k)$-dimensionale Zyklus z^{n-k} in Γ mit $\tau z^{n-k} < \varepsilon$ in seiner ε-Umgebung (in bezug auf Γ) *einem Zyklus z' mit beliebig kleinem $\alpha^{s-k}\, z'$ homolog*

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[6]) Bei $r < n - 1$ folgt die zweite dieser Ungleichungen aus der ersten, bei $r = n - 1$ ist die erste trivial.

und für $s > r$ *und* $k = 2, \ldots, s+1$ *berandet jeder* $(n-k)$*-dimensionale* (in Γ berandende) *Zyklus* z^{n-k} *mit* $\alpha^{s-k+1} z^{n-k} < \varepsilon$ (und $\tau z < \varepsilon$ bei $s = n-1$) *eine in* Γ *liegende Kette* x^{n-k+1}, *für die ebenfalls* $\alpha^{s-k+1} x^{n-k+1} < \varepsilon$ *ist.*

Bei $k = 1$ sind in diesem Theorem die Sitnikovschen Säcke, bei $k = r+1$ die alten dimensionellen Hindernisse enthalten.

Dieser Satz führt zu neuartigen geometrischen Fragen, von denen ich hier nur die folgende erwähne:

Wir bezeichnen als den *p-dimensionalen Halbmesser* $\alpha^p\varphi$ eines Kompaktums φ die untere Grenze derjenigen $\varepsilon > 0$, für die es eine ε-Überführung des Kompaktums φ in ein Polyeder von einer Dimension $\leqq p$ gibt.

Sodann sei φ ein n-dimensionales Polyeder im R^n, welches homogendimensional ist in dem Sinne, dass es die abgeschlossene Hülle seines offenen Kernes bildet — m.a.W. φ ist Summe von endlichvielen n-dimensionalen abgeschlossenen Simplexen des R^n. Es sei φ^* der Rand von φ. Es handelt sich um den Beweis oder um die Widerlegung der folgenden Vermutung: Für $p < n-1$ ist

$$\alpha^p \varphi = \alpha^p \varphi^*.$$

Die Frage bleibt offen sogar für $n = 3$, $p = 1$ und auch fur den Spezialfall, dass φ ein dem Volltorus homöomorphes Polyeder ist.

An diesem Problem, welches sich aus dem Sitnikovschen Satz und seinem Beweise fast zwangsmässig herausschälen lässt, sieht man, wie sehr der um diesen Satz gravitierende Gedankenkreis unserern einfachsten geometrischen Vorstellungen nahekommt. Das Wesen des Sitnikovschen Satzes ist eben Geometrie viel mehr als eigentliche Mengenlehre; setzt man in der Aussage des Gürtel-Theorems von vorneherein voraus, das Kompaktum sei ein Polyeder, so entstehen fast gar keine Vereinfachungen in der Beweisführung, so dass der Satz auch im Gebiete der Polyedergeometrie als neu, und sicher interessant erscheint.

III

6. Sehr grosse Fortschritte zeigt in diesen letzten Jahren die Topologie der nicht abgeschlossenen Punktmengen der Euklidischen Räume. Kurz gesagt, handelt es sich um nichts Geringers als um das Zustandekommen eines neuen Gebietes topologischer Forschung — der kombinatorischen Topologie beliebiger Punktmengen Euklidischer Räume — ein Resultat, welches vor zwanzig Jahren ziemlich aussichtslos erschien. (Vgl. etwa meinen Vortrag an der Moskauer Internationalen Topologischen Konferenz 1935, abgedruckt in [9], insbesondere § 1, Nr. 6 und § 4, Nr. 1) [7]).

Es gibt im Wesentlichen zwei Methoden, Homologie-Invarianten nicht abgeschlossener Mengen aufzustellen und zu untersuchen. Die eine Methode besteht in der Betrachtung des teilweise geordneten („gerichteten") Systems der in der gegebenen Menge $\mathscr{A}$ liegenden Kompakta; die Homologie-bzw. Kohomologiegruppen dieser Kompakte bilden dann ein direktes bzw. inverses Spektrum, dessen Limesgruppe die betreffende Gruppe der Menge $\mathscr{A}$ ist. Wir formulieren nochmals die Definition für die Homologie-oder Δ-Gruppen; ein Δ-Zyklus der Menge $\mathscr{A}$ ist ein Δ-Zyklus irgendeines Kompaktums $\varphi \subseteqq \mathscr{A}$, dieser Zyklus berandet (oder ist homolog Null in $\mathscr{A}$), falls er in einem Kompaktum φ', $\varphi \subseteqq \varphi' \subseteqq \mathscr{A}$, berandet; die Faktorgruppe der Gruppe aller p-dimensionalen Δ-Zyklen von $\mathscr{A}$ nach der Untergruppe der berandenden Zyklen ist die p-dimensionale Δ-Gruppe von $\mathscr{A}$. Man erhält dabei zwei verschiedene Gruppe: die Gruppe $\Delta^p_0 \mathscr{A}$ (die „*Vietorissche*" Gruppe) und die *Sitnikovsche* Gruppe $\Delta^p \mathscr{A}$ je nachdem man Zyklen und Berandungen in Kompakten im „schwachen" (d.h. gewöhnlichen Vietorisschen) oder im neueren von Sitnikov [10] herrührenden „starken" Sinne versteht [8]).

Die zweite Methode Homologiegruppen einer beliebigen Menge $\mathscr{A}$ zu definieren erläutern wir zuerst für den Fall der ∇-Gruppen (der „Kohomologiegruppen"). Man betrachte zu diesem Zweck das gerichtete System sämtlicher offenen, sternendlichen Überdeckungen α, β ... von $\mathscr{A}$ und bezeichne den Nerv jeder Überdeckung mit demselben Buchstaben wie die Überdeckung selbst. Falls β auf α folgt (in Zeichen: $\beta > \alpha$) dh. jedes Element der Überdeckung β in mindestens einem Element der Überdeckung α enthalten ist, sind

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[7]) Es dürfte bei dieser Gelegenheit erwähnt werden, dass es K. Kuratowski war, der als erster meinen „Überführungssatz" von Kompaktheitsvoraussetzungen befreit hat.

[8]) Ein schwacher oder Vietorisscher Zyklus des Kompaktums φ ist eine Folge

$$z^p = (z^p_1, z^p_2, \ldots, z^p_k, \ldots) \tag{1}$$

wobei z^p_k ein ε_k-Zyklus in φ ist und es ε_k-Ketten x^{p+1}_k in φ mit $\Delta x^{p+1}_k = z^p_{k+1} - z^p_k$ gibt, $\varepsilon_k \to 0$. Der Zyklus (1) berandet in φ („schwache" Berandung), falls dortselbst ε'_k-Ketten y^{p+1}_k existieren, $\varepsilon'_k \to 0$, welche durch z^p_k berandet sind: $\Delta y^{p+1}_k = z^p_k$.

Ein *starker* oder *Sitnikov*scher Zyklus ist eine Folge

$$z^p = (z^p_k, x^{p+1}_k) \tag{2}$$

von ε_k-Zyklen z^p_k und ε_k-Ketten x^{p+1}_k in φ, $\varepsilon^k \to 0$, wobei $\Delta x^{p+1}_k = z^p_{k+1} - z^p_k$ ist (d.h. es wird eine bestimmte Wahl der Ketten x^{p+1}_k, deren Existenz bereits in der Vietorisschen Definition behauptet wird, auch wirklich getroffen).

Der Sitnikovsche Zyklus (2) berandet im Kompaktum („starke" Berandung), falls es ε'_k-Ketten, $\varepsilon'_k \to 0$, y^{p+1}_k und x^{p+2}_k derart gibt, dass $\Delta y^{p+1}_k = z^p_k$ und

$$\Delta x^{p+2}_k = y^{p+1}_{k+1} - y^{p+1}_k - x^{p+1}_k \text{ gilt.}$$

Der Sitnikovsche Zyklus (2) berandet in φ schwach, wenn bloss die Existenz der von den durch z^p_k berandeten ε'_k-Ketten y^{p+1}_k feststeht.

„Projektionen" $\tilde{\omega}^{\beta}_{\alpha}$ von β in α definiert: das sind simpliziale Abbildungen des Nerven β in den Nerv α, die man erhält, wenn man jedem Element der Überdeckung β ein, dieses Element enthaltendes Element der Überdeckung α entsprechen lässt. Diese Projektionen definieren bekanntlich einen und denselben Homomorphismus π^{α}_{β} (ebenfalls „Projektion" genannt) der Gruppe $\nabla^p\alpha$ in die Gruppe $\nabla^p\beta$ (bei der Definition dieser Gruppen werden durchweg unendliche Ketten — insbesondere unendliche ∇-Zyklen benutzt). Es entsteht ein direktes Gruppenspektrum

$$\{\nabla^p\alpha,\ \pi^{\alpha}_{\beta}\},$$

dessen Limes natürlicherweise als die Gruppe $\nabla^p A$ definiert wird (Koeffizientenbereich — beliebig).

Ohne explizite Bezugnahme auf Limesgruppen von Gruppenspektra lässt sich die Definition der Gruppe $\nabla^p A$ wie folgt formulieren. Ein ∇-Zyklus der Menge $\mathscr{A}$ ist definitionsgemäss ein ∇-Zyklus des Nerven irgendeiner Überdeckung α. Zwei ∇-Zyklen z_α und z_β (die beziehungsweise auf den Nerven α und β liegen) heissen untereinander homolog in $\mathscr{A}$, wenn es eine Überdeckung γ von $\mathscr{A}$ gibt, die sowohl auf α als auch auf β folgt, und für die die Homologie

$$\pi^{\alpha}_{\gamma} z_\alpha \sim \pi^{\beta}_{\gamma} z_\beta \text{ in } \gamma$$

gilt. Somit wird die Gesamtheit der p-dimensionalen ∇-Zyklen der Menge $\mathscr{A}$ in Klassen untereinander homologer ∇-Zyklen — wir sagen kurz: in ∇-Klassen eingeteilt, und diese ∇-Klassen sind Elemente der Gruppe $\nabla^p A$. Die Addition zweier ∇-Klassen $\mathfrak{z}^p$ und $\mathfrak{z}'^p$ wird folgendermassen definiert: man wähle in diesen Klassen je einen ∇-Zyklus z^p_α und z'^p_α unter der Bedingung, dass beide Zyklen auf einem und demselben Nerven α liegen (man überzeugt sich leicht, dass eine solche Wahl stets möglich ist); sodann definiert man $\mathfrak{z}^p + \mathfrak{z}'^p$ als diejenige ∇-Klasse, die den Zyklus $z^p_\alpha + z'^p_\alpha$ enthält. Der Beweis der Unabhängigkeit der so gewonnenen ∇-Klasse $\mathfrak{z}^p + \mathfrak{z}'^p$ von der speziellen Wahl der Zyklen z^p_α und z'^p_α ist leicht zu führen.

Man definiert mit Hilfe von Überdeckungen auch Δ-Gruppen einer Punktmenge $\mathscr{A}$. Zu diesem Zweck betrachte man auf jedem Nerven α die mit Hilfe von nur endlichen Ketten definierte gewöhnliche Δ-Gruppe; wir bezeichnen sie mit $\delta^p\alpha$. Die Projektionen $\tilde{\omega}^{\beta}_{\alpha}$ ergeben Homomorphismen $\tilde{\omega}^{\beta}_{\alpha}$ von $\delta^p\beta$ in $\delta^p\alpha$ und somit ein inverses Spectrum

$$\{\delta^p\alpha,\ \tilde{\omega}^{\beta}_{\alpha}\}$$

dessen Limesgruppe mit $\delta^p A$ bezeichnet wird. Diese Definition kann auch so formuliert werden. Man wähle auf jedem α je einen (endlichen) Δ-Zyklus z^p_α unter der Bedingung, dass für $\beta > \alpha$ stets

$$\tilde{\omega}^{\beta}_{\alpha} z^p_\beta \sim z^p_\alpha \text{ in } \alpha$$

sei (Homologie im Sinne der endlichen Ketten). Ein dieser Bedingung genügendes Zyklensystem

$$z^p = \{z^p_\alpha\}$$

heisse ein Projektionszyklus der Menge $\mathscr{A}$. Auf Grund der Regel

$$\{z^p_\alpha\} + \{z'^p_\alpha\} = \{z^p_\alpha + z'^p_\alpha\}$$

bilden die Projektionszyklen eine Gruppe Z^pA, in der als Untergruppe die Gruppe der berandenden Projektionszyklen enthalten ist; dabei berandet der Projektionszyklus $z^p = \{z^p_\alpha\}$ in $\mathscr{A}$, falls z^p_α in α berandet, und dies für alle Überdeckungen α. Die Gruppe δ^pA ist die Faktorgruppe der Gruppe Z^pA nach der Untergruppe der berandenden Projektionszyklen.

Mann kann schliesslich zur Gruppe δ^pA auch noch auf die folgende Weise gelangen. In jeder Umgebung λ der Menge $\mathscr{A}$ wähle man einen (gewöhnlichen endlichen Polyeder-)Zyklus z^p_λ unter der Bedingung, dass falls die Umgebung λ' in λ enthalten ist, stets die Homologie $z^p_{\lambda'} \sim z^p_\lambda$ in λ besteht. Ein dieser Bedingung genügendes Zyklensystem $\{z^p_\lambda\}$ heisse etwa ein *schwebender* Zyklus der Menge $\mathscr{A}$. Ein solcher Zyklus berandet in $\mathscr{A}$ falls jeder Polyederzyklus z^p_λ in der betreffenden offenen Menge λ (polyedral) berandet. Addition ist wieder durch die Formel

$$\{z^p_\lambda\} + \{z'^p_\lambda\} = \{z^p_\lambda + z'^p_\lambda\}$$

definiert. Die Faktorgruppe der Gruppe aller schwebenden Zyklen der Dimension p nach der Untergruppe der berandenden schwebenden Zyklen erweist sich als der Gruppe δ^pA isomorph (dieser Isomorphiesatz wurde gleichzeitig von S. Kaplan und mir im Jahre **1947** bewiesen [11], [12]).

7. Ein wesentlicher Teil der kombinatorischen Topologie der nicht abgeschlossenen Punktmengen besteht in einer Reihe von Dualitätssätzen, die die Dualisierbarkeit der verschiedenen Homologiegruppen, d.h. die Eigenschaft der betreffenden Gruppen einer Menge A sich durch topologische Invarianten der zu A komplementären Menge ausdrücken zu lassen, beweisen. Der erste allgemeine Dualitätssatz dieser Art wurde von mir im Jahre 1947 in [11] bewiesen [9]). Ich benutzte dabei wesentlich die von G. Chogoshvili herrührenden algebraischen Hilfssätze über Gruppenspektra. Noch umfassendere Dualitätssätze wurden nachher von Sitnikov [10] gefunden, dem man überhaupt die grössten Fortschritte auf diesem Gebiet verdankt.

[9]) In seiner Arbeit [12] beweist S. Kaplan interessante Sätze aus der Homologietheorie nicht abgeschlossener Mengen (darunter den Satz von der Gleichheit entsprechender Bettischen Zahlen zweier zueinander komplementären Mengen im S^n), aber keinen Dualitätssatz im modernen Sinne des Wortes, d.h. keinen Satz, welcher die Homologiegruppen einer Menge $A \subset S^n$ durch Homologiegruppen der Komplementärmenge $B = S^n - A$ bestimmen liesse.

Der von mir bewiesene Dualitätssatz bezieht sich auf die Gruppen $\delta^p A$. Bevor wir ihn und weitere Sätze von Sitnikov formulieren, machen wir ein für allemal folgende Bemerkungen. Alle Punktmengen, die wir betrachten, sollen im n-dimensionalen sphärischen Raum S^n (und nicht wie früher im R^n) liegen. Das Komplement $S^n - A$ zu einer gegebenen (beliebigen) Punktmenge A soll stets mit B bezeichnet werden; p und q seien immer zwei ganze nichtnegative Zahlen mit der Summe $p + q = n - 1$. Schliesslich werden mit $\mathfrak{A}$, $\mathfrak{B}$ zwei beliebige im Sinne von Pontrjagin zueinander duale Abelsche Gruppen bezeichnet, von denen $\mathfrak{A}$ immer diskret, und $\mathfrak{B}$ bikompakt sein soll: Mit K wird die stetige zyklische Gruppe, d.h. die additive Gruppe der modulo 1 reduzierten reellen Zahlen bezeichnet.

Es sei nun z^q_B irgendein Vietorisscher Zyklus in B nach dem Koeffizientenbereich $\mathfrak{B}$; dann ist klar, dass für jeden schwebenden und wie leich ersichtlich auch für jeden Projektionszyklus z^p_A der Menge A (nach dem Koeffizientenbereich $\mathfrak{A}$) die Verschlingungszahl $w(z^p_A, z^q_B) \in K$ definiert ist. Ist $z^q_B \sim 0$ in B, so ist $w(z^p_A, z^q_B) = 0$ für jeden Zyklus z^p_A. Aber die Umkehrung braucht nicht richtig zu sein, und darin liegt ein an Folgerungen reicher Unterschied zwischen der Topologie der Kompakta und der der nicht abgeschlossenen Mengen: im Allgemeinen, gibt es Vietorissche Zyklen z^q_B in B, die in B nicht homolog Null sind und die trotzem die Verschlingungszahl Null mit jedem Projektionszyklus (oder, was auf das Gleiche hinauskommt, mit jedem schwebenden Zyklus) der Menge $\mathscr{A}$ haben. Es lässt sich beweisen, dass die Gruppe dieser „unverschlingbaren" Zyklen der Menge B (Koeffizientenbereich — bikompakte Gruppe $\mathfrak{B}$) eine topologische Invariante der Menge B ist. Folglich ist sowohl die Untergruppe $N^q_\Delta B \subseteqq \Delta^q_C B$ (der Gruppe $\Delta^q_C B$) deren Elemente Klassen von unverschlingbaren Zyklen sind, als auch die Faktorgruppe $\Delta'^q B = \Delta^q_C B - N^q_\Delta B$ eine topologische Invariante von B. Nun zeigt sich, dass man in die Gruppe $\Delta'^q_C B$ eine natürliche Topologie einführen kann, und in dieser Topologie ist die Gruppe $\Delta'^q_C B$ überalldichte Untergruppe einer eindeutig bestimmten Gruppe, die mit $\overline{\Delta}^q B$ bezeichnet wird und eine mit B invariant verbundene Homologiegruppe vom Δ-Typus darstellt.

Der von mir [11] bewiesene Dualitätssatz lautet:

Die Gruppe $\overline{\Delta}^q B$ (nach dem Koeffizientenbereich $\mathfrak{B}$) *und die Gruppe* $\delta^p A$ (nach dem Koeffizientenbereich $\mathfrak{A}$) sind ***zuneiander dual*** (dh. jede dieser Gruppen ist die Charakterengruppe der anderen).

Wie gesagt, ist die Gruppe $\overline{\Delta}^q B$ die bikompakte Hülle der (auf eine gewisse natürliche Weise topologisierten) Faktor-Gruppe der Gruppe $\Delta^q_C B$ nach der "Unverschlingbarkeitsgruppe" $N^q_\Delta B$. Letztere Gruppe ist ebenfalls dualisierbar. Bevor wir versuchen, diese Tatsache plausibel zu machen, bemerken wir, dass es in $\Delta^q_C B$ noch eine „erweiterte Unverschlingbarkeitsgruppe" $N^q_{\Delta C} B \subseteqq N^q_\Delta B$ gibt welche aus denjenigen Homologieklassen $\zeta^q \varepsilon \Delta^q_C B$ besteht, die eine verschwindende Verschlingungszahl mit jedem Vietorisschen Zyklus der Menge A haben. Die Homologieklassen, die die Elemente von $N^q_{\Delta C} B$ sind, bestehen aus solchen Zyklen der Menge$B$, die in jeder Umgebung dieser Menge beranden. Die Gruppe $N^q_{\Delta C} B$ bildet ebenfalls eine topologische Invariante der Menge B und diese Invariante ist dualisierbar (*Koeffizientenbereich — beliebig*).

8. Was sind die ∇-Analoga der Gruppen $N^q_\Delta B$ und $N^q_{\Delta C} B$, definiert für die Menge A? Um diese Frage zu beantworten definieren wir zwei Gruppen, $N^p_\nabla A$ und $N^p_{\nabla C} A$.

Die Gruppe $N^p_{\nabla C} A$ wird in gleicher Weise für die beiden Arten von Koeffizientenbereichen definiert (für diskrete Koeffizientenbereiche $\mathfrak{A}$ und für bikompakte $\mathfrak{B}$): in beiden Fällen besteht $N^p_{\nabla C} A$ aus denjenigen ∇-Klassen, deren Skalarprodukt mit jedem kompakten Projektionszyklus (= mit jedem Vietorisschen) Zyklus z^p_A der Menge A den Wert Null hat (dabei muss als Koeffizientenbereich die Gruppe genommen werden, die zu dem für $\nabla^p A \supseteqq N^p_{\nabla C} A$ genommenen Koeffizientenbereich dual ist).

Die Definition der Gruppe $N^p_\nabla A$ hängt von dem jeweiligen Koeffizientenbereich ab: $N^p_\nabla (A, \mathfrak{A})$ wird definiert als diejenige Untergruppe von $\nabla^p(A, \mathfrak{A})$ deren Elemente ein verschwindendes Skalarprodukt mit allen Elementen von $\Delta^p(A, \mathfrak{A})$ (d.h. mit jedem Sitnikovschen Δ-Zyklus) haben.

Die Gruppe $N^p_\nabla(A, \mathfrak{B})$wird dagegen als die Untergruppe von $\nabla^p(A, \mathfrak{B})$ definiert, deren Elemente ein verschwindendes Skalarprodukt mit allen Elementen von $\delta^p(A, \mathfrak{A})$ (d.h. mit jedem Projektionszyklus) haben.

Schliesslich zeichnen wir noch in der Sitnikovschen Gruppe $\Delta^p B$ die Unterguppe $H^q B$ derjenigen Homologieklassen aus, deren Elemente schwach berandete Sitnikovsche Zyklen sind. Man sieht leicht ein, dass die Vietorissche Gruppe $\Delta^q_C B$ der Faktorgruppe der Sitnikovschen Gruppe $\Delta^q B$ nach der Untergruppe $H^q B$ isomorph ist (Koeffizientenbereich beliebig).

Jetzt können wir den universellen Sitnikovschen Dualitätssatz in voller Schärfe und Allgemeinheit formulieren:

Dualitätstheorem von Sitnikov [10], [13]. *Die Gruppen* $\nabla^p A$ *und* $\Delta^p B$ (Koeffizientenbereich — beliebig) *sind untereinander isomorph*; *und zwar gibt es zwischen diesen Gruppen einen solchen Isomorphismus, welcher — im Falle des diskreten Koeffizientenbereiches* $\mathfrak{A}$ *die Gruppen*

$$\nabla^p A \supseteqq N^p_{\nabla C} A \supseteqq N^p_\nabla A$$

respektive auf die Gruppen

$$\Delta^q B \supseteqq N^q_{\nabla C} B \supseteqq H^q B$$

und im Falle des bikompakten Koeffizientenbereiches $\mathfrak{B}$ *die Gruppen*

$$\nabla^p A \supseteqq N^p_{\nabla C} A \supseteqq N^p_\nabla A$$

der Reihe nach auf die Gruppen

$$\Delta^q B \supseteqq N^q_{\Delta C} B \supseteqq N^q_{\Delta} B$$

abbildet.

Hieraus folgt, dass sämtliche hier genannten Gruppen, sowie ihre Faktorgruppen dualisierbar sind [9a]).

Insbesondere ist die Vietorissche Gruppe $\Delta^p_C A$ dualisierbar: sie ist (da der Faktorgruppe $\Delta^p A - H^p A$ isomorph) zu der Gruppe $\Delta^q B - N^q_{\nabla} B$ isomorph. Man kann kurz zusammenfassen: *alle auf die oder andere natürliche Weise definierten Homologiegruppen nichtabgeschlossener Mengen sind dualisierbar* [10]).

Mein Bericht über die Sitnikovsche Dualitätstheorie ist lange nicht vollständig; aber auch das Gesagte genügt wohl, um zur Überzeugung zu gelangen, dass diese Theorie einen ganz gewaltigen Fortschritt der allgemeinen kombinatorischen Topologie darstellt. Man sieht auch, dass die Homologie-Verhältnisse im Bereiche der nicht abgeschlossenen Mengen wesentlich anders sind als im Bereiche der Kompakta. Auf die verschiedenen spezielleren Sätze dieser Theorie [11]), auf Beispiele, die sie illustrieren, auf zahlreiche sich darbietende Probleme [12]) kann ich hier nicht weiter eingehen; es möge derentwegen auf die ausführlichen Sitnikovschen Publikationen [10] im Matematičeski Sbornik unter dem Gesamttitel „Kombinatorische Topologie der nicht abgeschlossenen Mengen" verwiesen werden.

[9a]) Im allgemeinen Sitnikovschen Dualitätssatz ist auch der von mir bewiesene Dualitätssatz enthalten.

[10]) Dagegen sind die mit Hilfe endlicher Überdeckungen definierten Homologiegruppen nicht dualisierbar.

[11]) Ich möchte hier nur noch folgenden von mir kürzlich bewiesenen Satz [14] erwähnen. Jeder Vietorissche Zyklus kann (offenbar) als ein schwebender, folglich, auch als ein Projektionszyklus aufgefasst werden. Deshalb liegt ein natürlicher Homomorphismus L der Gruppe $\Delta^p_C A$ in die Gruppe $\delta^p A$ vor (ist A ein Kompaktum, so ist dieser Homomorphismus die Identität zwischen den beiden Gruppen). *Alle Eigenschaften dieses Homomorphismus* (insbesondere seine Kern-und Bildgruppe) *sind dualisierbar.*

[12]) Von diesen seien die folgenden erwähnt.

1. Der von mir bewiesene Dualitätssatz ist eine „Verschlingungsdualität"

$$\delta A \,|\, \bar{\Delta}^q B;$$

es fragt sich, ob der Sitnikovsche Dualitätssatz

$$\nabla^p A = \Delta^q B$$

auch in die Form einer Verschlingungsdualität gebracht werden kann. Verwandt damit ist die Frage, ob die Charakterengruppe der Gruppe $\nabla^p A$ einen selbständigen geometrischen Sinn (als topologische Invariante der Menge A) hat.

9. Es ist von vornherein klar, dass die soeben geschilderten wesentlichen Fortschritte in der kombinatorischen Topologie der Punktmengen nicht ohne Einfluss auf die Dimensionstheorie bleiben konnten [13]. Und tatsächtlich hat das von mir bereits vor fast 20 Jahren formulierte [9] Problem der Charakterisierung der Dimension einer beliebigen Punktmenge im R^n durch lokale Homologie-Eigenschaften des Komlementärraumes seine vollständige Lösung durch Sitnikov [10], [15] gefunden: *es gilt im vollem Masse der dimensionelle Hindernissatz auch für beliebige* (nicht nur fur die abgeschlossenen) *Punktmengen des* R^n. Nur muss man jetzt unter „Zyklus" und „Berandung" Sitnikovsche Berandungen verstehen. Mit anderen Worten gilt der folgende Satz:

Algemeiner Hindernissatz. *Es sei* $\mathscr{A}$ *eine beliebige r-dimensionale Punktmenge im* R^n. *Dann existiert eine Vollkugel* U^n *des Raumes* R^n *von der Eigenschaft, dass es in* $U^n - \mathscr{A}$ *ein* $(n - r - 1)$*-dimensionaler Sitnikovscher Zyklus liegt, welcher in* $U^n - \mathscr{A}$ *nicht berandet. Wenn aber* $s < n - r - 1$ *und* U^n *eine beliebige Vollkugel des* R^n *ist, so berandet in* $U^n - \mathscr{A}$ *jeder dortselbst liegende Sitnikovsche Zyklus. Dabei ist „Berandung" immer im „starken" (d.h. Sitnikovschen) Sinne zu verstehen.*

Dass dabei die Berandungen im Sinne Sitnikovs zu verstehen sind — ist wesentlich: sonst gilt der Satz nicht. Insbesondere konstruiert Sitnikov [16] im R^3 eine zweidimensionale Menge $\mathscr{A}$, die kein Gebiet dieses Raumes zerschneidet (obwohl man, dem Hindernissatz gemäss, eine solche Vollkugel $U^3 \subset R^3$ finden kann, dass es in $U^3 - \mathscr{A}$ einen nulldimensionalen Zyklus (sogar einen solchen, der aus einem einzigen Punktepaar besteht) finden kann, welcher in $U^3 - \mathscr{A}$ nicht (im Sinne Sitnikovs) berandet.

Somit stimmt die Urysohnsche Vermutung für nicht abgeschlossene Punktmengen (sogar des dreidimensionalen Raumes) nicht mehr! Das heisst, diese Vermutung stimmt nicht, wenn man sie im buchstäblichen, naiv-mengentheoretischen Sinne versteht. Sie stimmt aber wohl — und das lehrt der allge-

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2. Gibt man dem Pontrjaginschen Dualitätssatz die Form einer (von Alexander und Kolmogoroff herrührenden) Isomorphie

$$\Delta^p A \approx \Delta^{p+1} B$$

(A — abgeschlossen, B — offen), so bleibt sie bekanntlich richtig, wenn man als umfassenden Raum anstatt der Sphäre S^n ein beliebiges in den Dimensionzahlen p und $p + 1$ azyklisches Polyeder (sogar einen beliebigen im kleinen bikompakten normalen Raum) nimmt.

Gelten analoge Verallgemeinerungen für den von mir, bzw. von Sitnikov bewiesenen Dualitätssatz, oder ist in diesen Sätzen die Voraussetzung, der umfassende Raum sei eine Mannigfaltigkeit, wesentlich?

[13]) Es seien in diesem Zusammenhange eine wichtige Arbeit von G. H. Dowker [20] sowie meine Arbeit [4b] und die Arbeit von Hemmingsen [21] erwähnt.

meine Hindernissatz — wenn man den Begriff „Raumzerschneidung" in sachgemässem Sinne algebraisiert, d.h. ihn als Auftreten eines nicht berandenden nulldimensionalen Zyklus auffasst. Und die Berandung muss dabei eben im Sinne Sitnikovs verstanden werden.

Der Beweis des Sitnikovschen Hindernissatzes erfordert sehr feine Hilfsmittel der modernen Homologietheorie. Ein wesentlicher Bestandteil des Beweises besteht im folgenden Satze, der wegen seines anschaulichen Inhaltes ein Interesse für sich beanspruchen darf, der aber trotz dieser Anschaulichkeit recht schwer zu beweisen ist:

Deformationssatz. Es sei $\mathscr{A}$ *eine beliebige Punktmenge des* R^n *und* z *ein auf einem Kompaktum* $\varphi \subset R^n - A$ *liegender Sitnikovscher Zyklus. Es seien* g_θ *und* h_θ, $0 \leqq \theta \leqq 1$, *stetige Deformationen der Mengen* $\mathscr{A}$ *bzw.* φ *im* R^n, *die der einzigen Bedingung genügen, dass bei beliebigen* θ, $0 \leqq \theta \leqq 1$ *die Mengen* $g_\theta \mathscr{A}$ *und* $h_\theta \varphi$ *keine gemeinsamen Punkte haben. Falls dann* $h_1 z \sim 0$ *in* $R^n - g_1 \mathscr{A}$ *ist, so ist auch* $z \sim 0$ *in* $R^n - \mathscr{A}$ *(Homologie im Sitnikovschen Sinne).* Der Satz bleibt richtig wenn man den Raum R^n durch eine beliebige Mannigfaltigkeit M^n ersetzt.

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SUR L'ÉLIMINATION DE PHÉNOMÈNES PARADOXAUX EN TOPOLOGIE GÉNÉRALE

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1. Les notions de la géométrie élémentaire sont intimement liées aux propriétés des corps matériels. Aussi les théorèmes géométriques peuvent-ils être interprétés et vérifiés sur des modèles convenablement construits. C'est notamment le cas des chapitres de la géométrie élémentaire qui concernent les polyèdres. En vue des considérations qui vont suivre, j'entends ici par *polyèdres* les images homéomorphes de ceux de la géométrie élémentaire, donc aussi les polyèdres dits *curvilignes*.

Tout polyèdre se laisse *trianguler*, c'est-à-dire décomposer en un système fini de simplexes de manière que la partie commune de deux quelconques soit le simplexe tendu sur leurs sommets communs. L'effet en est que la structure topologique de tout polyèdre se trouve parfaitement déterminée par la table finie des incidences entre les simplexes de sa triangulation. C'est à ce caractère finitiste de leur structure que les polyèdres doivent bien la simplicité de leurs propriétés topologiques. Aussi, malgré la richesse considérable de formes, nous ne rencontrons parmi les propriétés intrinsèques de polyèdres aucune propriété paradoxale ou — comme on dit parfois — aucun phénomène pathologique, aucune monstruosité.

Mais l'espace euclidien et les polyèdres n'épuisent pas l'objet des recherches géométriques. Déjà en mathématique classique l'on voit apparaître, à coté de l'espace de géométrie élémentaire, des espaces beaucoup plus généraux, dus aux idées de Grassman, Lobatchevsky, Riemann et autres. L'attention des géomètres se concentrait au début sur les figures qui, tout en étant situées dans ces espaces, ne s'éloignent en général pas de l'intuition, grâce à la simplicité de leur structure. La situation a changé lorsqu'on a été amené, par suite du développement de la théorie des ensembles, à considérer des ensembles tout à fait arbitraires situés dans les espaces classiques et — plus encore — lorsqu'on a introduit la notion d'*espaces abstraits*.

Le processus de généralisation de la notion d'espace joue un rôle fondamental dans la mathématique contemporaine et il est probablement loin d'être achevé. Les espaces fort généraux, introduits axiomatiquement, s'écartent bien sensiblement de ceux de la mathématique classique. On a réussi cependant d'établir des théorèmes importants qui déterminent à peu près la place que les espaces de la mathématique classique, définis d'habitude arithmétiquement, occupent dans la vaste famille des espaces abstraits, définis axiomatiquement. Tel est, en premier lieu, le théorème capital d'Urysohn [1] qui caractérise axiomatiquement les espaces homéomorphes aux ensembles situés dans l'espace de Hilbert. Tel est aussi le théorème de Menger et Nöbeling [2] (sur le plongement d'ensembles de dimension n, par homéomorphie, dans le cube de dimension $2n + 1$) qui caractérise les images homéomorphes d'ensembles situés dans les espaces euclidiens. On a réussi d'autre part, surtout grâce aux travaux de Brouwer, Alexandroff, Vietoris, Čech, Lefschetz et autres d'étendre aux certains espaces abstraits très généraux une partie notable de notions qui se sont formées dans le domaine de la topologie des polyèdres. On a fait aussi ressortir l'élément géométrique contenu dans la notion d'espace abstrait. Enfin, les relations entre les notions topologiques générales et les figures géométriques classiques ont trouvé leur expression dans le peu de théorèmes connus qui caractérisent topologiquement l'*arc simple* [3], la *courbe simple fermée* [4], le *disque fermé*, la *sphère* de dimension 2 [6] et, plus généralement, les *variétés* de même dimension [7]. C'est à ceux théorèmes qu'appartient la caractérisation topologique des polyèdres de dimension 2, trouvée récemment à Varsovie par Kosiński [8].

Les généralisations topologiques des espaces de géométrie ayant introduit des propriétés paradoxales, il est naturel de se poser la question si l'on ne peut quand même définir à l'aide des notions de topologie générale une classe d'espace qui, tout en restant à l'abri des paradoxes, serait assez générale pour renfermer tous les polyèdres. Ce problème ne passionne peut-être pas, en ce moment, la plupart de topologues. Il ne se trouve même pas sur la route magistrale de leurs recherches actuelles. Néanmoins, il me semble mériter l'attention et pouvoir susciter l'intéret de tout mathématicien. C'est pourquoi je l'ai choisi pour sujet de ma conférence.

2. Le besoin d'avoir une notion extrêmement générale d'espace se manifeste dans divers domaines de mathématique. Il est clair cependant qu'en faisant croitre la généralité de cette notion, sa teneur géométrique décroit. Elle semble déjà fort pauvre lorsqu'on s'avance au delà des espaces métrisables. C'est pourquoi il est plausible de postuler ici la

I *Métrisabilité.*

Ce postulat, associé à celui de séparabilité, reduit (grâce au théorème précité d'Urysohn) les espaces abstraits généraux aux ensembles situés dans celui de Hilbert, donc dans un espace défini arithmétiquement.

En y ajoutant le postulat de la

II *Dimension finie.*

et en appliquant le théorème précité de Menger et Nöbeling, on parvient aux ensembles situés dans les espaces euclidiens en s'approchant ainsi du domaine classique de la géométrie.

Pourtant, ces ensembles étant arbitraires, il y a parmi eux qui ne ressemblent guère, par ses propriétés, aux figures classiques, voire aux polyèdres. Pour s'approcher d'elles — ou, si l'on veut, des notions aux propriétés possiblement voisines de celles d'objets matériels, il parait donc opportun d'ajouter à I et II le postulat de

III *Compacité.*

On sait que la compacité d'un espace métrique en assure la séparabilité, de sorte que les postulats I—III, pris ensemble, caractérisent topologiquement la classe de tous les ensembles compacts situés dans les espaces euclidiens. La topologie de ces ensembles s'est particulièrement développée et ses résultats témoignent de leur richesse en matière géométrique.

Mais même parmi les epaces assujettis aux postulats I—III, il y a notoirement qui sont porteurs de singularités impossibles dans le monde de polyèdres. Le *phénomène d'indécomposabilité* par exemple, decouvert par Brouwer [9] et même celui infiniment plus accentué, dit *indécomposabilité totale* ou *héréditaire* (découvert par Knaster [10]), est — comme l'a montré Mazurkiewicz [11] — la règle dans cette classe d'espaces: ceux qui en sont exempts y constituent en un certain sens des exceptions bien rares. Une autre singularité qui se présente dans cette classe d'espaces consiste dans la divergence, découverte par Pontrjagin [12], entre la notion géométrique usuelle et les notions qui lui correspondent dans la théorie de l'homologie.

La nature trop générale d'espaces métriques compacts a suggéré aux mathématiciens de soumettre les espaces dont ils s'occupaient à des conditions supplémentaires locales. Telle est par exemple la *connexité locale*, de même que ses analogues en dimensions supérieures à 1. Parmi ces classes moins générales d'espaces compacts, celle des espaces dits *rétractes absolus de voisinage* mérite une attention particulière. Ils ont été introduits d'abord pour les besoins de la théorie des prolongements de transformations continues. Il leur revient actuellement un role essentiel dans plusieurs domaines de topologie, en particulier dans la théorie de l'*homotopie* et dans celle des *espaces fibrés*. La notion de ces espaces — appelés en transcription anglaise „*absolute neighbourhood retracts*" et qui seront désignés ici par l'abréviation ANR — empruntée à Lefschetz [13] — a été plus récemment généralisée d'une manière essentielle grâce aux travaux de Kuratowski, Hu, Saito, Hanner et autres [14]. Pour nos buts cependant, il est plutôt utile de rétrécir cette notion, à savoir en la soumettant au postulat II (dimension finie), qui n'entre pas dans la définition primitive de ANR. Alors, les espaces ANR peuvent être définis comme assujettis aux postulats I—III et à celui de

IV *Contractilité locale.*

L'espace X est dit *localement contractile* lorsque, quel que soit le point x de X tout voisinage U de x (c'est-à-dire ensemble dont x est un point intérieur) contient un voisinage V de x qui se laisse réduire à un point par une déformation continue affectuée dans U.

Il est bien connu que les propriétés des espaces qui sont des ANR ressemblent beaucoup à celles des polyèdres [15]. En particulier, les groupes d'homologie de ces espaces sont isomorphes à ceux de certains polyèdres et plusieurs théorèmes établis d'abord pour les polyèdres sont entièrement valables pour les ANR. Telle est toute la théorie homologique des points invariants par exemple. Les ANR sont d'ailleurs, dans certain sens, le domaine naturel de la théorie de l'homotopie.

Mais, contrairement aux polyèdres, dont les définitions sont *constructives* (par les moyens de la géométrie élémentaire) les ANR sont définis *axiomatiquement* (par ceux de la topologie générale). C'est pourquoi les ANR sont souvent plus commodes dans les considérations topologiques que les polyèdres. La démonstration qu'un espace est ANR se réduit à vérifier qu'il satisfait aux postulats I—IV, tandis que le problème de montrer topologiquement qu'il est un polyèdre comporte en général des difficultés insurmontables à l'heure actuelle, car nous ignorons les propriétés topologiques qui caractérisent les polyèdres. Il est manifeste par exemple que le produit cartésien de deux espaces n'est un ANR que s'ils le sont tous les deux; autrement dit, un diviseur cartésien d'un ANR est toujours un ANR. Par contre, la question si un diviseur cartésien d'un polyèdre est toujours un polyèdre parait extrêmement difficile. Elle n'est résolue jusqu'à présent que pour les dimensions 1 et 2 par Kosiński [16] (à savoir par l'affirmative). Notre ignorance des propriétés topologiques caractérisant les polyèdres est l'une des raisons que le problème classique de triangulation concernant les variétés comporte toujours de si considérables difficultés. Il est possible que les résultats récents de Bing et de Moise [17] sur la triangulation des variétés à 3 dimensions contribueront à caractériser topologiquement tous les polyèdres à 3 dimensions.

3. Au point de vue du sujet proposé, il importe surtout d'étudier la question si les propriétés topologiques des ANR sont déjà suffisamment proches de celles des polyèdres; en d'autres termes, s'il existe même parmi les ANR des espaces

ayant des propriétés paradoxales. Oui, il en existe et on le sait depuis des dizaines d'années. Je vais m'arrêter un peu sur trois de ces propriétés, car elles me semblent les plus importantes [18].

Tout polyèdre est un ANR, il est donc localement contractile. Or, il est aisé de montrer que tout polyèdre X satisfait à la condition suivante, qui est plus restrictive que la contractilité locale [19]:

V *Quel que soit le point $x \in X$, tout voisinage U de x contient un tel voisinage V de x que tout sous-ensemble compact B de V se laisse réduire à un point par une déformation continue effectuée dans un sous-ensemble A de U assujetti à la condition $\dim A \leqq \dim B + 1$.*

Voici l'exemple [20] d'un espace X qui est ANR, mais ne satisfait pas à V: considérons une fonction continue f qui transforme un segment rectiligne I situé à l'intérieur du carré Q_2 en un cube Q_3 et identifions les points de I transformés par f en un même point de Q_3; l'espace X est ce que devient Q_2 par cette identification. Vu la liaison étroite de cet exemple avec l'existence de transformations continues f qui élèvent la dimension, découvertes en 1890 par Peano [21], j'appelle *singularité de Peano* l'absence de la propriété V chez un ANR.

La deuxième singularité est de nature homologique. Désignons par $p_n(X)$ le nombre n-dimensionnel de Betti de l'espace X. Il est facile de démontrer que tous les polyèdres satisfont à la condition suivante:

VI *Si $p_n(X) > 0$, il existe un sous-ensemble compact F de X pour lequel $p_n(X) = p_n(F) + 1$.*

Il n'en est pas ainsi pour tous les ensembles compacts. En 1910 Brouwer [22] a construit un continu plan X qui était la frontière commune à 3 régions. On a alors $p_1(X) = 2$ et en même temps $p_1(F) = 0$ pour tout sous-ensemble compact F de X qui est distinct de X. La condition VI est donc en défaut pour ce continu, ce que j'appelle *singularité de Brouwer*. Sur le plan, elle n'est possible que dans les ensembles de structure topologique très compliquée [23], mais il en est autrement déjà dans l'espace euclidien de dimension 3. Lubański [24] y a construit récemment un ANR constituant la frontière commune à 3 régions et par conséquent jouissant de la singularité de Brouwer.

La troisième singularité est liée à la notion de *rétracte absolu*. On appelle ainsi — et on désigne par AR — tout ANR contractile (à un point) dans lui-même. En particulier, tout polyèdre contractile dans lui-même est un AR. Plus encore: X étant un polyèdre, il existe manifestement, pour chacun de ses points, un voisinage aussi petit que l'on veut et qui est contractile dans lui-même. Tout polyèdre X satisfait par conséquent à la condition:

VII *Tout point x de X a un voisinage arbitrairement petit qui est un AR.*

Il existe cependant, déjà dans l'espace euclidien de dimension 3, des ensembles X qui sont des ANR ne satisfaisant pas à la condition VII. J'appelle *singularité de Mazurkiewicz* le phénomène en question, car c'est à lui que remonte la première idée d'un exemple de ce genre [25]. Un example plus singulier a été construit par moi en 1948 [26], à savoir un AR de dimension 2, situé dans l'espace euclidien de dimension 3 et dont tout vrai sous-ensemble de dimension 2 a le premier nombre de Betti infini; en conséquence, il n'est pas un ANR. Cet AR est donc en quelque sorte une surface sans trou, mais dont toute surface partielle en a une infinité.

4. Evidemment, pour se débarasser des singularités de Peano, Brouwer et Mazurkiewicz, il suffit tout simplement d'ajouter aux postulats I—IV, qui définissent les ANR de dimension finie, les conditions V—VII comme de nouveaux postulats. Ce mode de procéder serait justifié si chacune des conditions V—VII enrichissait essentiellement la teneur géométrique de l'espace. On ignore cependant, du moins à l'heure actuelle, s'il en est exactement ainsi. On sait, il est vrai, que ces trois conditions sont indépendantes l'une des autres. On sait aussi que la condition V a des conséquences intéressantes, dont surtout la suivante mérite d'être rappelé ici, à savoir que pour les espaces ANR assujettis à la condition V les dimensions homologiques coïncident avec la dimension usuelle. Mais on ne sait point si cette coincidence n'est générale pour les ANR, c'est-à-dire si elle ne résulte des postulats I—IV seuls. On sait moins encore sur les conséquences des conditions VI et VII.

On peut montrer que chacune des conditions V—VII est un invariant de la multiplication cartésienne. Mais on n'a pas réussi jusqu'à présent de résoudre le problème si ces conditions se transmettent de l'espace à ces diviseurs cartésiens. Aussi, aucune classe intéressante de transformations (sauf celle des homéomorphies) n'est connue qui laissent les conditions V—VII invariantes.

Toutes ces circonstances me font douter que la voie directe pour dégager la classe cherchée d'espaces conduise par les conditions V—VII. Je suis plutôt enclin de croire qu'elles doivent être remplaceés par d'autres conditions supprimant avec autant d'efficacité les singularités de Peano, Brouwer et Mazurkiewicz, mais convenant mieux au but proposé. Les conditions V—VII sont donc à considérer comme des sondages préliminaires destinés à faciliter l'orientation convenable de recherches ultérieures.

5. Le problème de déterminer une classe d'espaces ayant les propriétés proches de celles des polyèdres n'a pas été précisé. La notion de „proximité des

propriétés" est subjective et peut être conçue, par conséquent, de bien de manières diverses. On peut toutefois chercher de la préciser.

On sait par exemple que les espaces satisfaisant aux postulats I—IV coïncident avec les ensembles homéomorphes aux rétractes de polyèdres. Autrement dit, ces espaces ne diffèrent pas des polyèdres au point de vue des invariants des transformations dites *rétractions*. On peut dire aussi que la classe de ces espaces approche celle des polyèdres aux invariants de la rétraction près.

Un autre mode de préciser la notion de propriétés voisines a été indiqué par Hurewicz [27]. Il a introduit en 1936 la notion importante de type d'homotopie, qui est telle que l'identité de types d'homotopie de deux espaces entraîne l'isomorphie de leurs groupes d'homologie et de ceux d'homotopie. On est donc porté à préciser la proximité des espaces aux polyèdres par la simple condition que ces espaces aient le type d'homotopie des polyèdres. Or, malgré les remarquables et profonds résultats de J. H. C. Whitehead [28] sur les types d'homotopie des ANR, le problème si tout espace satisfaisant aux postulats I—IV a le type d'homotopie d'un polyèdre me semble d'être ouvert. Cependant, on sait que les relations entre le type d'homotopie d'un espace X et ceux de polyèdres deviennent plus étroites en soumettant cet espace à la condition suivante qui va plus loin que la condition VII:

VII* *L'espace X se laisse décomposer en une somme*

(*) $$X = X_1 + X_2 + \ldots + X_k$$

telle que tout ensemble de la forme $X_{i_1}, X_{i_2} \ldots X_{i_j}$, où $j = 1, 2, \ldots, k$ est soit vide, soit un AR.

Cette décomposition de X peut être regardée comme généralisation de la décomposition simpliciale d'un polyèdre. Tout comme la table des incidences simpliciales détermine complètement la structure topologique du polyèdre, la table analogue relative à la décomposition (*) détermine le type d'homotopie de X. Ce type se montre identique à celui du polyèdre qui est le nerf au sens de P. Alexandroff [29] de la décomposition (*). Je l'ai établi en 1948 [30] et A. Weil l'a étendu en 1951 [31] aux espaces de dimension infinie.

Notons que le problème de l'existence des décompositions (*) pour les variétés au sens classique peut être considéré comme une simplification du problème classique de la triangulation. Il me semble que même le problème ainsi simplifié reste toujours ouvert.

Il est clair que le problème de préciser convenablement la notion d'espaces s'approchant des polyèdres par leurs propriétés topologiques ne se résoud pas en postulant tout simplement que l'espace ait le type d'homotopie d'un polyèdre. Tous les exemples connus des ANR ne diffèrent pas des polyèdres par leurs types d'homotopie. Par conséquent, aucune des singularités envisagées ne peut pas être éliminée en ajoutant aux postulates I—IV celui que l'espace ait le type d'homotopie d'un polyèdre. Enfin, les ensembles de structure topologique bien différente peuvent avoir le même type d'homotopie. Tous les AR par exemple ont le type d'homotopie de l'ensemble composé d'un seul point.

6. Je vais essayer d'indiquer ici une voie différente qui me semble nous avancer vers le but. Il s'agit de faire correspondre au couples d'ensembles (situés dans un espace fixe) d'une distance particulière *mesurant la dissemblance* de leurs propriétés topologiques. [32]

On retrouve l'idée de mesurer les différences entre ensembles déjà dans la notion d'espace des ensembles compacts introduite en 1914 par Hausdorff [33]. Cependant la distance de Hausdorff ne dit rien sur les différences entre les propriétés topologiques d'ensembles. Les ensembles proches au sens de Hausdorff peuvent avoir la structure topologique fort différente; en particulier, tout ensemble se laisse en ce sens approcher indéfiniment par des ensembles finis.

Pour parvenir à une définition de la distance entre ensembles qui tienne compte de leurs propriétés topologiques, il faut d'abord qu'elle laisse invariantes au passage à la limite les propriétés d'homologie [34] et d'homotopie les plus importantes. C'est pourquoi j'ajoute à la distance de Hausdorff un sommande supplémentaire mesurant, pour ainsi dire le degré de contractilité d'un espace et ayant pour effet qu'une suite d'arcs par exemple approchant la circonférence, au sens de la distance hausdorfienne ne converge pas vers cette circonférence, mais diverge et ne constitue même pas une suite de Cauchy. Il en est ainsi avec une suite de circonférence concentriques dont les rayons décroissants indéfiniment convergent vers leur centre commun.

Comme il ne s'agit dans ces considérations que d' ensembles de structure assez régulière, on peut se borner ici aux ANR non vides situés dans un espace fixe, dans le cube Q de dimension n par exemple. Soit $R(Q)$ la classe des ANR en question. Pour saisir le degré de contractilité d'un espace, j'introduis la notion d'équicontractilité locale. Je dis que les ensembles d'une suite $\{X_\nu\} \subset R(Q)$ sont *équicontractiles localement* lorsque, pour tout $\varepsilon > 0$ il existe un $\eta > 0$ tel que pour tout $\nu = 1, 2, \ldots$ tout ensemble $B \subset X_\nu$ dont le diamètre $\delta(B)$ est plus petit que η est contractile (à un point) dans un ensemble $A \subset X$ de diamètre $\delta(A) < \varepsilon$ [35]. On obtient alors comme il suit la définition de la limite conforme au but proposé:

Une suite $\{X_\nu\} \subset R(Q)$ converge vers $X_0 \in R(Q)$ lorsque elle converge vers X_0 au sens de la métrique de Hausdorff et que les X_ν sont équicontractiles localement.

L'espace topologique $R(Q)$ ainsi défini peut être alors métrisé de la façon suivante:

X étant un des ensembles appartenant à $R(Q)$, désignons pour tout

$t \geqq 0$ par $E_X(t)$ l'ensemble de nombres composé de 1 et de tous les $\tau \geqq t$ pour lesquels tout ensemble $B \subset X$ de diamètre $\delta(B) \leqq t$ est contractile (à un point) dans un ensemble $A \subset X$ de diamètre $\delta(A) \leqq \tau$ (on a donc $B \subset A \subset X$ comme dans la condition V). La fonction

$$\varphi_X(t) = \inf E_X(t)$$

satisfait donc par définition à la condition

$$0 \leqq \varphi_X(t) \leqq 1 \quad \text{pour tout } t \geqq 0$$

et l'on déduit de la contractilité locale de X que

$$\lim_{t=+0} \varphi_X(t) = 0.$$

Pour X fixe, le parcours de φ_X (comme fonction de la variable t) au voisinage de $t = 0$ est, pour ainsi dire, une expression quantitative de la contractilité locale de X. Mais $\varphi_X(t)$, tout en étant une fonction non-décroissante de t, n'en est pas, en général une fonction continue. C'est pourquoi il est commode de l'améliorer en la majorant par la plus petite fonction $\psi_X(t)$ assujettie à la condition

$$\varphi_X(t) \leqq \psi_X(t) \quad \text{pour tout } t \geqq 0$$

et à celle de concavité

$$\psi_X[\lambda \cdot t_1 + (1-\lambda)t_2] \geqq \lambda \cdot \psi_X(t_1) + (1-\lambda) \cdot \psi_X(t_2),$$

où $t_1 \geqq 0$, $t_2 \geqq 0$ et $0 \leqq \lambda \leqq 1$. L'existence d'une telle fonction ψ_X se démontre aisément.

Ceci fait, en posant

$$\varrho(X_1, X_2) = \varrho_H(X_1, X_2) + \sup_{t \geqq 0} |\psi_{X_1}(t) - \psi_{X_2}(t)| \quad \text{pour } X_1, X_2 \in R(Q),$$

où ϱ_H désigne la distance de Hausdorff, la classe $R(Q)$ d'ensembles devient un espace métrique. *Cet espace est séparable et complet.* Plus encore, on constate que la distance ϱ saisit en général au juste les différences topologiques entre les ANR. En particulier, lorsque X_0 est un ANR non vide situé dans le cube Q, c'est-à-dire que X_0 est un point de l'espace $R(Q)$, *tout ensemble $X \in R(Q)$ suffisamment proche de X_0 a le même type d'homotopie que X_0* [36]. Par conséquent, tous les ANR qui appartiennent à la même composante de l'espace $R(Q)$ ont le même type d'homotopie. Mais la métrique ne fait pas ressortir que les différences entre les types d'homotopie. Elle est sensible aux différences entre maintes autres propriétés topologiques. Il est facile de montrer par exemple qu'un *X de dimension supérieur n'est jamais la limite* (au sens de la métrique ϱ) *d'une suite des X de dimensions inférieures.* De même, *le nombre fini de points de ramification d'une courbe ne peut augmenter à la limite et celle d'une suite convergente de courbes simples fermées ne peut être qu'une courbe simple fermée*[37]. On peut aussi démontrer que *la limite d'une suite convergente de surfaces au sens classique* c'est-à-dire des variétés fermées de dimension 2, *est toujours une surface* (au même sens) [38]. Ce théorème me semble mériter l'intéret du point de vue de l'analyse et de la géométrie différentielle. On peut supposer que, plus généralement, la limite de toute suite convergente de variétés en est toujours une. C'est à titre d'exemple, l'une d'une série de problèmes ouverts.

7. Il est difficile de juger en se moment si l'espace $R(Q)$ peut contribuer d'une façon essentielle à résoudre le problème de la définition topologique d'une classe d'ensembles ayant la structure particulièrement régulière. On sait encore trop peu sur cet espace. On ignore en particulier *si les polyèdres y constituent un ensemble dense*. S'il n'en est pas ainsi, la fermeture de l'ensemble des polyèdres, donc la classe des ANR approchables (au sens de la métrique ϱ) par les polyèdres, semble mériter une attention toute speciale. Si, par contre, l'ensemble des polyèdres est dense dans l'espace $R(Q)$, il est permis plutôt de douter que la métrique ϱ soit déjà assez fine pour qu'elle permette de distinguer les ensembles ayant des propriétés paradoxales des polyèdres. Quoi qu'il en soit, il semble intéressant d'examiner en tout cas la question si les ensembles affectés des singularités de Peano, Brouwer et Mazurkiewicz forment des sous-ensembles non-denses de $R(Q)$. Enfin, cet espace étant complet, il est légitime de s'attendre qu'il trouve application à des démonstrations d'existence procédant par la méthode dite de catégorie (et dont l'origine remonte au théorème de Baire).

8. La topologie générale a contribué jusqu'à présent relativement peu à saisir par les moyens qu'elle a élaboré le domaine important des êtres géométriques de mathématique classique. En particulier, le problème de caractériser en toute généralité les polyèdres par leur propriétés topologiques reste ouvert, malgré les beaux résultats de Bing et de Moise [17] concernant les variétés de dimension 3 et ceux de Kosiński [8] apportant une caractérisation topologique des polyèdres de dimension 2.

Le problème abordé ici est, bien entendu, beaucoup plus modeste, car il n'exige pas une caractérisation complète de l'ensemble des polyèdres, mais seulement un choix convenable de l'un de ses sur-ensembles, à savoir qui soit libre de phénomènes paradoxaux, étrangers aux propriétés géométriques des objets réels. Les relations entre elles et les conceptions abstraites de la topologie générale, qui engendrent si souvent des paradoxes géométriques, méritent à mon avis d'être étudiées et élucidées. Mais alors il faut les attaquer.

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ON THE STRUCTURE OF GROUPS OF FINITE ORDER

RICHARD BRAUER

The theory of groups of finite order has been rather in a state of stagnation in recent years. This has certainly not been due to a lack of unsolved problems. As in the theory of numbers, it is easier to ask questions in the theory of groups than to answer them. If I present here some investigations on groups of finite order, it is with the hope of raising new interest in this field. The results which I am going to give will be incomplete and new unanswered questions will be added to the old ones. There is, however, hope that further progress will be possible along the lines suggested.

Let me start with the following question. If $\mathfrak{G}$ is a group of order g which is not cyclic of prime order, what can be said about the existence of proper subgroups $\mathfrak{H}$ of relatively large order h? It seems probable that the following conjecture is true. *There always exist proper subgroups $\mathfrak{H}$ of an order $h > \sqrt[3]{g}$.* For instance, if $\mathfrak{G}$ is soluble, it is easy to see that there exist proper subgroups $\mathfrak{H}$ even with $h \geqq \sqrt{g}$. There is a famous conjecture to the effect that all groups of odd order are soluble. Thus, if we accept this old conjecture, we have to consider only groups of even order. It is somewhat surprising that for groups of even order our conjecture can be proved. In other words, we have the following theorem

Theorem. If $\mathfrak{G}$ is a group of even order $g > 2$, there exists a proper subgroup $\mathfrak{H}$ of an order $h > \sqrt[3]{g}$.

The proof is quite elementary and I will sketch it, since it forms a preparation for some of the later developments. It will be convenient to work with the group algebra Γ of $\mathfrak{G}$ over the field of rational numbers. This is an associative algebra consisting of all formal expressions

$$\gamma = \sum_{\sigma} a_{\sigma}\sigma$$

where σ ranges over the elements of $\mathfrak{G}$ and where the a_{σ} are rational numbers. The operations in Γ are defined in the obvious manner.

By an *involution* of $\mathfrak{G}$, we shall mean an element J of $\mathfrak{G}$ of order 2. Let $\mathfrak{M}$ denote the set of all involutions in $\mathfrak{G}$. Since g is even, $\mathfrak{M}$ is not empty. Set

$$M = \sum_{J \in M} J$$

taken as an element of Γ. Then $M^2 \in \Gamma$ and we have a formula

$$(1) \qquad M^2 = \sum_\sigma a_\sigma \sigma.$$

It is clear that a_σ here is the number of ordered pairs (X, Y) such that

$$(2) \qquad XY = \sigma, \qquad (X, Y \in \mathfrak{M}).$$

It is also clear that a_σ will depend only on the class of conjugate elements of $\mathfrak{G}$ to which σ belongs. If then the classes of conjugate elements of $\mathfrak{G}$ are denoted by $\mathfrak{K}_0, \mathfrak{K}_1, \ldots, \mathfrak{K}_{k-1}$ and if $K_i \in \Gamma$ is the sum of the elements of $\mathfrak{K}_i$, the formula (1) can be rewritten in the form

$$(1^*) \qquad M^2 = \sum_{i=0}^{k-1} a_i K_i$$

where we set $a_i = a_\sigma$ for $\sigma \in \mathfrak{K}_i$.

If (2) holds, it follows at once that $X^{-1}\sigma X = \sigma^{-1}$. We shall say that an element σ is *real* in $\mathfrak{G}$, if σ and σ^{-1} are conjugate in $\mathfrak{G}$. Thus, if σ is non-real in $\mathfrak{G}$, (2) is impossible and $a_i = 0$ for non-real classes $\mathfrak{K}_i$. Moreover, if $\mathfrak{N}(\sigma)$ is the normalizer of σ in $\mathfrak{G}$, (i.e. the subgroup of $\mathfrak{G}$ consisting of the elements commuting with σ), and if $n(\sigma)$ is the order of $\mathfrak{N}(\sigma)$, there exist at most $n(\sigma)$ elements X with $X^{-1}\sigma X = \sigma^{-1}$. Thus, if σ_i is an element of $\mathfrak{K}_i$ we have $a_i \leqq n(\sigma_i)$. If σ_i is an involution, one sees easily that this inequality can be replaced by $a_i \leqq n(\sigma_i) - 2$. Actually, it will be clear that much more accurate statements concerning the values of a_i can be given, but this is not important for the moment.

Let us now compare the number of group elements appearing on both sides of (1*). If m denotes the total number of involutions in $\mathfrak{G}$, the left side of (1*) is the sum of m^2 elements of $\mathfrak{G}$. On the right, we have a sum of $\sum a_i g/n(\sigma_i)$ elements of $\mathfrak{G}$, since $\mathfrak{K}_i$ consists of $g/n(\sigma_i)$ elements. Hence

$$m^2 = \sum_{i=0}^{k-1} a_i g/n(\sigma_i).$$

Let $\mathfrak{K}_0$ denote the class containing the identity 1. Clearly, $a_0 = m$. Let $\mathfrak{K}_1, \ldots, \mathfrak{K}_r$ designate the classes consisting of involutions and let $\mathfrak{K}_{r+1}, \ldots, \mathfrak{K}_l$ designate the other classes consisting of real elements of $\mathfrak{G}$. If the estimates for the a_i are used, we find

$$m^2 \leqq m + \sum_{i=1}^{r} (n(\sigma_i) - 2)g/n(\sigma_i) + \sum_{j=r+1}^{l} n(\sigma_i)g/n(\sigma_i).$$

Each of the m involutions appears in one of the classes $\mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_r$, and hence

$$m = \sum_{i=1}^{r} g/n(\sigma_i).$$

Thus,

$$(3) \qquad m^2 < \sum_{i=1}^{l} n(\sigma_i)g/n(\sigma_i) = lg.$$

On the other hand, the total number of elements in $\mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_l$ is less than g, that is,

$$\sum_{i=1}^{l} g/n(\sigma_i) < g. \tag{4}$$

Let h be the maximal value of $n(\sigma_i)$ appearing in (4). Then $g/h \leqq g/n(\sigma_i)$ and (4) yields

$$lg/h < g.$$

If this is used in (3), we have $m^2 < gh$. On the other hand, $m \geqq g/n(\sigma_1) \geqq g/h$. On combining these inequalities, we find $g < h^3$. If $h = n(\sigma_j) \neq g$, then $\mathfrak{N}(\sigma_j)$ is a proper subgroup $\mathfrak{H}$ of an order $h > \sqrt[3]{g}$. It remains to discuss the case that the maximal $n(\sigma_j)$ in (4) is equal to g. In this case, $\mathfrak{K}_j$ consists of one element only. For $r + 1 \leqq i \leqq l$ the class $\mathfrak{K}_i$ contains the two distinct elements σ_i and σ_i^{-1}. Hence we must have $1 \leqq j \leqq r$, and then σ_j is an invariant involution. Now, induction can be used. If $G/\{\sigma_j\}$ has even order, the theorem holds for this quotient group and we see at once that it then holds for $\mathfrak{G}$. If $\mathfrak{G}/\{\sigma_j\}$ has odd order g', then the order g of $\mathfrak{G}$ is twice an odd number g'. It is well known [1]) that then $\mathfrak{G}$ has a subgroup or order $g/2$. Since $g/2 > \sqrt[3]{g}$ for $g \geqq 4$, the theorem holds again and this finishes the proof. Incidentally, it is possible to improve the theorem somewhat, but it does not seem to be worth while to go into this here. Instead I rather mention a related theorem which is proved by a similar method.

Theorem. Let m denote the number of involutions of the group $\mathfrak{G}$ of even order g and set $n = g/m$. There exists a subgroup $\mathfrak{H} \neq \mathfrak{G}$ of $\mathfrak{G}$ such that the index $t = (\mathfrak{G} : \mathfrak{H})$ is either 2 or t satisfies the inequality $t \leqq n(n+2)/2$.

This result has some interesting consequences. If the group $\mathfrak{G}$ has a subgroup $\mathfrak{H} \neq \mathfrak{G}$ of index t, then $\mathfrak{G}$ possesses a transitive permutation representation on t symbols. The kernel $\mathfrak{L}$ of this permutation representation is a normal subgroup of $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{L}$ is isomorphic with a transitive subgroup of the symmetric group $\mathfrak{S}_t$ on t letters. This implies that

$$1 < (\mathfrak{G} : \mathfrak{L}) \leqq t! \leqq \text{Max}\ (2, [n(n+2)/2]!).$$

In particular, if $\mathfrak{G}$ is simple, we must have $\mathfrak{L} = \{1\}$ and hence

$$g \leqq [n(n+2)/2]!.$$

If J is any involution in $\mathfrak{G}$, then $g/n(J) \leqq m = g/n$ and hence $n \leqq n(J)$, $g \leqq (n(J)(n(J)+2)/2)!$ This yields the result [2]).

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[1]) See, for instance, W. Burnside, The Theory of Groups of Finite Order, Cambridge 1911, p. 327, Theorem II.

[2]) A result of this kind with a different bound was first obtained by K. A. Fowler and the author by a somewhat different method.

Theorem. There exist only a finite number of simple groups $\mathfrak{G}$ which contain an involution J such that the normalizer $\mathfrak{N}(J)$ of J is isomorphic to a given group.

It is not known whether a similar result holds, if, instead of an involution J, we take an element of a given odd prime order p.

The last theorem suggests the following problem: Given a group $\mathfrak{N}$ containing an involution J in its center. What are the groups $\mathfrak{G}$ containing $\mathfrak{N}$ as a subgroup such that $\mathfrak{N}$ is the normalizer of J in $\mathfrak{G}$? More generally, the case can be considered that a number of groups $\mathfrak{N}_1^*, \mathfrak{N}_2^*, \ldots, \mathfrak{N}_r^*$ are given such that each $\mathfrak{N}_i$ contains an invariant involution J_i. We then can study the groups $\mathfrak{G}$ containing subgroups $\mathfrak{N}_1^*, \mathfrak{N}_2^*, \ldots, \mathfrak{N}_r^*$ such that each $\mathfrak{N}_i^*$ is isomorphic to $\mathfrak{N}_i$ and that $\mathfrak{N}_i^*$ is the normalizer of the element J_i^* corresponding to J_i. We may further require that $J_1^*, J_2^*, \ldots, J_r^*$ represent all the distinct classes of $\mathfrak{G}$ which contain involutions. Usually, it will be convenient to add further assumptions in order to characterize groups uniquely.

In this generality, the problem is far too difficult to handle. There are, however, some general methods which give at least some information and which we shall discuss next.

The first of these is closely related with what we have done above. If we use the same notation, the elements $K_1, K_2, \ldots, K_k$ form a basis of the center Λ of the group algebra Γ. We therefore have formulas

$$K_\alpha K_\beta = \sum c_{\alpha\beta\gamma} K_\gamma$$

with rational coefficients $c_{\alpha\beta\gamma}$. Actually, it is easy to see that the $c_{\alpha\beta\gamma}$ are non-negative integers. The $c_{\alpha\beta\gamma}$ can be expressed in terms of the irreducible characters of $\mathfrak{G}$ and, as a matter of fact, it is well known that a knowledge of the $c_{\alpha\beta\gamma}$ is equivalent with a knowledge of the characters.

Suppose now that g is even. The element M in (1*) is the sum of $K_1, K_2, \ldots$ K_r. Hence the a_i in (1*) are sums of certain of the $c_{\alpha\beta\gamma}$. Since we can study the a_i directly and since we can express the $c_{\alpha\beta\gamma}$ in terms of the characters, we obtain new relations for the characters.

The second idea is of a more fundamental nature. It applies to groups of even or odd order and it can be used for other questions too. Let $\chi_1, \chi_2, \ldots, \chi_k$ designate the irreducible characters of $\mathfrak{G}$. Let p be a fixed prime number. In our application, we take $p = 2$, but this is not important now. The characters χ_i are distributed into disjoint sets, called the *p-blocks* of $\mathfrak{G}$. Here, χ_α and χ_β belong to the same block, if

$$\frac{g}{n(\sigma)} \frac{\chi_\alpha(\sigma)}{Dg\chi_\alpha} \equiv \frac{g}{n(\sigma)} \frac{\chi_\beta(\sigma)}{Dg\chi_\beta} \pmod{\mathfrak{p}}$$

for all $\sigma \in \mathfrak{G}$ where $\mathfrak{p}$ is a fixed prime ideal divisor of p in the field of the g-th roots of unity. This is a technical definition of the blocks. There is a better one which belongs to the theory of arithmetic of algèbras, but it would take too long to discuss this here[3]).

Suppose then we have a block B of characters, say, consisting of the characters $\chi_1, \chi_2, \ldots, \chi_s$. We form the highest power p^d which divides one of the numbers $g/Dg\chi_i$, $1 \leqq i \leqq s$. Then d is called the *defect* of B. The first basic result is, that for given p and d and all groups of finite order, there exist only a finite number of "types" of p-blocks of defect d. For each type, the number s of characters of the block is determined. Further, for each type, one can give explicitly all relations

$$\sum c_i \chi_i(\sigma) = 0$$

which hold simultaneously for all elements σ of $\mathfrak{G}$ of order prime to p. If p^a is the highest power dividing g, one may assume that, for each type, the values of the degrees $Dg\chi_i$ (mod p^{a-d+1}) are known.

In particular, for given p and d, there must exist an upper bound for the number s of characters in the block. It is probable that $s \leqq p^d$. However, this can only be proved for $d \leqq 2$. The best known result, proved recently by W. Feit and myself, is that $s < \frac{1}{4}p^{2d}$(for $d > 1$). Actually, for $d \leqq 1$, all possible types can be enumerated. For larger d, the number of types may be uncomfortably large. It would be very important for our work if the theory of blocks could be refined.

There is another result which I have to mention. If all subgroups $\mathfrak{D}$ of order p^d of a p-Sylow group $\mathfrak{P}$ of $\mathfrak{G}$ and their normalizers $\mathfrak{N}(\mathfrak{D})$, in $\mathfrak{G}$ are known, the number of p-blocks of defect d can be determined and information about the possible types obtained. For $d = 0$, we have $\mathfrak{G} = \mathfrak{N}(\mathfrak{D})$ and nothing can be learned about the number of p-blocks of defect $d = 0$. For $d > 0$, a real reduction can be achieved.

These results can be applied, if we have information about $\mathfrak{G}$ as outlined in the program above. The prime p is to be taken as 2. The method will give us a wealth of information about the characters of $\mathfrak{G}$ and this again can be applied to study the group $\mathfrak{G}$ further.

Let me now discuss some special cases. I first consider a relatively simple case. Consider a group $\mathfrak{G}$ of an order $g = 4g'$ where g' is odd. In order to have more definite results, we shall assume that no element outside a Sylow subgroup $\mathfrak{S}$ of order 4 commutes with all the elements of $\mathfrak{S}$. Further, we shall assume that $\mathfrak{G}$ does not have a normal subgroup of index 2. The

[3]) Some of the results on blocks mentioned here have been stated without proof in the following papers: R. Brauer, On the arithmetic in a group ring, Proc. Nat. Acad. Sci. U.S.A. 30, 109—114 (1944) and On blocks of characters of groups of finite order I, II, Proc. Nat. Acad. Sci. U.S.A. 32, pp. 182—186, 215—219 (1946).

normalizer $\mathfrak{N}(J)$ of an involution J has an order $n(J)$ of the form $n(J) = 4v$ with an odd v. If our methods are applied here, the result is obtained that the order g can be expressed by two integral rational parameters f_1, f_2 and a sign $\delta = \pm 1$. We have

$$g = 32v^3 \frac{f_1 f_2 f_3}{(f_1 + 1)(f_2 + \delta)(f_3 - 1)}$$

where we set $f_3 = 1 + f_1 + \delta f_2$. We must also have $f_1 \equiv 1 \pmod 4$, $f_2 \equiv \delta \pmod 4$. Actually, $1, f_1, d_2, f_3$ are degrees of irreducible representations and all other such degrees are divisible by 4. As a strange consequence of our result we see that g will lie very close to $32v^3$. In particular, if $\mathfrak{G}$ is to be simple, then f_1, f_2, f_3 will increase with g and g is asymptotically equal to $32v^3$.

There do exist infinitely many simple groups of the type in question. They are groups $PSL(2, q)$ [4]) i.e. groups of unimodular projectivities of the projective line of a finite geometry belonging to a field with q elements. The prime power $q > 3$ has to satisfy the conditions $q \equiv 3$ or $5 \pmod 8$ in order that $\mathfrak{G}$ meets the requirements. Also, q may be taken equal to 4. In the latter case, the icosahedral group is obtained which is the only group possible for $v = 1$.

Probably, the groups mentioned here are the only simple groups of the type in question and, possibly, the only simple groups of an order $g = 4g'$ with an odd g'. However, this question has to be left open.

In any case, we come close here to a characterization of some of the simple groups $PSL(2, q)$. Actually, characterizations of all these groups can be given within the framework of our program. For instance, we have the following theorem first conjectured by M. Suzuki

Theorem. Suppose that $\mathfrak{G}$ is a group of even order g such that if two cyclic subgroups $\mathfrak{A}$ and $\mathfrak{B}$ of even order have an intersection different from $\{1\}$, then $\mathfrak{A}$ and $\mathfrak{B}$ both are subgroups of a cyclic subgroup of $\mathfrak{G}$. If $\mathfrak{G}$ does not have a normal subgroup of index 2, then $\mathfrak{G}$ is a group $PSL(2, q)$ (with $q > 2$) [5]).

Actually, under the assumptions of this theorem the characters of $\mathfrak{G}$ can be determined completely. In particular, this yields the order g of $\mathfrak{G}$ and now a characterization of the groups in question given by H. Zassenhaus [6]) can be used. There are various extensions and generalizations.

There is some hope that, for the other known simple groups, there exist similar characterizations. However, the continuation of this work becomes

[4]) The notation here is that used in B. L. van der Waerden, Gruppen von linearen Transformationen, Ergebnisse der Math., vol. 4 (1935).

[5]) This theorem was proved more or less independently by M. Suzuki and the author and by G. E. Wall.

[6]) H. Zassenhaus, Abh. Math. Sem. Univ. Hamburg, 11, 17—40 (1936).

more and more difficult and, for this reason, I have been able to complete the work only in the case of the groups $PSL(3, q)$, the group of unimodular collineations of a projective plane under special assumptions for q. The result here is as follows

Theorem. Suppose that $\mathfrak{G}$ *is a group of finite order which satisfies the following conditions*

(I) $\mathfrak{G}$ *contains an involution* J *whose normalizer* $\mathfrak{N}(J)$ *is isomorphic to* $GL(2, q)$, *(the full linear group of degree* 2 *over a Galois field with* q *elements).*

(II) *If* C *is an element* $\neq 1$ *of the center* $\mathfrak{C}$ *of* $\mathfrak{N}(J)$, *the normalizer* $\mathfrak{N}(C)$ *of* C *in* $\mathfrak{G}$ *is* $\mathfrak{N}(J)$ *(and not larger than* $\mathfrak{N}(J)$*).*

(III) $\mathfrak{G}$ *is its own commutator subgroup.*

If $q \equiv -1$ *(mod* 4*),* $q \not\equiv 1$ (mod 3), *and* $q \neq 3$ *then* $\mathfrak{G}$ *is isomorphic to* $PSL(3, q)$. *If* $q = 3$, *we haven the additional case that* $\mathfrak{G}$ *can be the simple Mathieu group of order* 7912.

The latter group appears here as a kind of distorted version of the group $PSL(3, 3)$ of order 5616.

Since the group $GL(2, q)$ can be characterized by means of the preceding work, this theorem gives a characterization of the group $PSL(3, q)$ for the values of q in question. There seems to be little doubt that similar characterizations exist for the other values of q too. Some preliminary work indicates that the hyperorthogonal groups in three dimensions can be treated in a similar fashion. As was already stated, there is some hope that similar characterizations exist for the other simple groups though it is not clear whether these results are accessible to our methods.

I should like now to discuss briefly the ideas of the proof of the last theorem. If $\mathfrak{G}$ is actually equal to the group of collineations of a projective plane π, every involution J of $\mathfrak{G}$ leaves fixed a point P of π and all points of a line l not through P. Thus, to every J there corresponds a pair (P, l) of a point and a line. Instead of building up geometry using points as the fundamental elements, we can use such pairs (P, l) as the fundamental concept. The various incidence relations can then be described in terms of the involutions J corresponding to the pair (P, l). The axioms take the form of group theoretical statements. The projective plane is thus replaced by the set $\mathfrak{M}$ of all involutions $J \in \mathfrak{G}$. The projectivities are given by the transformations

$$T_\sigma : J \to \sigma^{-1} J \sigma \qquad (J \in \mathfrak{M})$$

of $\mathfrak{M}$ onto itself where σ is a fixed element of $\mathfrak{G}$. Of course, these T_σ form the full projective group $PGL(3, q)$ which for $q \not\equiv 1$ (mod 3) coincides with the special projective group.

Suppose now that $\mathfrak{G}$ is an arbitrary group which satisfies the conditions of the theorem and consider the set $\mathfrak{M}$ of involutions. If we can show that the axioms of projective geometry hold (in terms of involutions), then $\mathfrak{M}$ becomes a projective plane. If we can show further that the projectivities are given by the transformation T_σ, it follows that $\mathfrak{G}$ is isomorphic to $PSL(3, q)$ and we are finished.

The real difficulty then is to prove certain properties of the set $\mathfrak{M}$ of involutions which are in no way evident from the given assumptions. As a matter of fact, we have to use a long detour in order to obtain them. The methods described above suffice in our present case again to determine the characters of $\mathfrak{G}$ and, in particular, the order g. Now the knowledge of the characters provides us with the necessary information about the involutions and the proof can be finished.

The problem treated here of characterizing special simple groups is of interest in connection with the important unsolved problem of determining all simple groups of finite order. In order to be able to recognize the known simple groups, we need workable characterizations of these groups. Though we are certainly very far from a solution of the general problem of the simple groups of finite order, at least some kind of plan seems to evolve according to which the problem might be attacked. One hope would be that the only non-cyclic simple groups are the alternating groups, the finite analogues of the simple Lie groups and some distorted versions of a few of the latter groups. It would be necessary to see where these groups fit into our program and to show that no other cases can arise. It is not possible to say whether this plan will be workable, since some of the most important links are missing.

As a matter of fact, it is necessary to mention in this connection again the old conjecture that all groups of odd order are soluble. Of course, this would now obtain an added significance. No progress has been made on this question since it was first formulated.

Under these circumstances, it is perhaps of interest to state an equivalent conjecture. This is based on the following remark which can be proved rather easily. If $\mathfrak{G}$ is a group, if $\mathfrak{K}_0, \mathfrak{K}_1, \ldots, \mathfrak{K}_{k-1}$ are all the classes and if we use the same notation as above, it can be shown that $\mathfrak{G}$ is equal to its commutator group $\mathfrak{G}'$, if and only if the formula holds

$$(5) \qquad g \cdot K_0 K_1 \cdots K_{k-1} = \prod_{i=0}^{k-1} g_i \cdot (K_0 + K_1 + \ldots + K_{k-1})$$

where g_i is the number of elements in the class K_i. Thus the conjecture on groups of odd order is equivalent with the fact that the equation (5) can never hold for a group of odd order [7]).

[7]) I was informed by H. Wielandt that he also found this result.

Let me finish this talk with a number of remarks which are of some interest in connection with our problems. Let $\mathfrak{G}_1$ denote the set $\mathfrak{G} - \{1\}$ obtained from $\mathfrak{G}$ by removing the identity 1. Then the distance $d(\sigma, \tau)$ of two elements of $\mathfrak{G}_1$ can be defined as follows.For $\sigma = \tau$, we set $d(\sigma, \tau) = 0$. If σ and τ commute, but if $\sigma \neq \tau$, we set $d(\sigma, \tau) = 1$. If there exists a chain of elements σ_i of $\mathfrak{G}_1$,

$$\sigma_0 = \sigma, \sigma_1, \sigma_2, \ldots, \sigma_r = \tau,$$

of length r such that any two consecutive elements σ_{i-1} and σ_i commute, let $d(\sigma, \tau)$ be the length of the shortest such chain. Finally, if no such chain exists, set $d(\sigma, \tau) = \infty$. It is clear that $\mathfrak{G}_1$ then becomes a metric space (in which infinite distances are permitted).

Let $\mathfrak{G}$ now be an arbitrary group of even order g. The geometric part of the proof of the characterization of $PSL(3, q)$ shows that it will be of importance in the general case to consider the set $\mathfrak{M}$ of involutions (or the set $\mathfrak{G}_1$) as a kind of geometry in which the fundamental group is given by the transformations $T_\sigma : J \to \sigma^{-1} J \sigma$ where σ is a fixed element of $\mathfrak{G}$ while J ranges over $\mathfrak{M}$ (or over $\mathfrak{G}_1$). A very first step in this direction is given by the consideration of the distances. The following results can be proved without much difficulty.

Theorem. If $\sigma \neq 1$ is a real element of a group $\mathfrak{G}$ of even order and if the distance $d(\sigma, \mathfrak{M})$ of σ from the set of involutions is larger than 3, then the distance $d(\sigma, \mathfrak{M})$ is infinite. In this case, the normalizer $\mathfrak{N}(\sigma)$ of σ is an abelian group $\mathfrak{H}$ whose order is relatively prime to the index $(\mathfrak{G} : \mathfrak{H})$. The group $\mathfrak{H}$ consists only of real elements. It is the normalizer of each of its elements different from 1.

Theorem. If the group $\mathfrak{G}$ of even order contains more than one class of involutions, then any two involutions of $\mathfrak{G}$ have at most distance 3.

By combining these two statements, it is seen that if a group of even order contains more than one class of involutions, then any two real elements σ, $\tau \neq 1$ of $\mathfrak{G}$ either have infinite distance or distance $d(\sigma, \tau) \leqq 9$.

HARVARD UNIVERSITY.

MATHEMATICAL PROBLEMS RAISED BY THE FLOOD DISASTER 1953.

D. VAN DANTZIG

Mr Chairman, Ladies and Gentlemen,

Human fate and human will are not computable. Still, mathematics can soften nature's impact on human fate, and strengthen the effect of human response.

In the small hours of February 1st 1953 the South Western part of the Netherlands was stricken by a flood disaster unsurpassed in the memory of this country. Big floods, some of which causing heavy losses had occurred before, e.g. in 1916, when lack of manpower in wartime had partly been responsible for neglect of dikes. But we have at least to go back to the floods of 1825 and 1775, and perhaps even to those of 1570 and 1421 to find anything comparable with last year's one (for details see "De Ingenieur" [1953]).

According to data provided by Ir A. G. Maris over 150000 hectares of land were flooded. It caused a loss of over 1800 human lives; about 9000 buildings were demolished and 38000 damaged; there were 67 breaks of dikes, and hundreds of kilometers of dikes were heavily damaged. The total economic loss is estimated at 1,5 till 2 milliards of guilders. Also to a lesser extent parts of Great Britain and Belgium were galestricken and suffered considerable losses.

On the other hand it gave rise to perhaps the finest example of spontaneous national and international helpfulness hitherto seen in history. Help, in the form of voluntary labour and material goods, flowed in, not only from every part of our country, but from a large number of foreign countries also, and to such an enormous extent that we can not think of this proof of real unity of mankind without the deepest emotion.

In order to design measures for preventing similar disasters in future the government appointed a committee, consisting of prominent engineers with Ir A. G. Maris as Chairman. It is called the Delta-committee, because its realm is the whole delta formed by the rivers Rhine, Meuse and Scheldt. The renowned hydrogolist J. Th. Thijsse was immediately recalled from a lecturing tour in the U.S.A. The Delta-committee decided to base its proposals on a broad scientific basis, and therefore took several scientific institutions as advisers, like the Central Planning Bureau, the Royal Dutch Meteorological Institute, the Hydrological Laboratory of the Technical University at Delft, and the Mathematical Centre, and of course, several divisions of the Public Works Department itself. Between these institutions a fertile cooperation was established, in the form of working- and discussion groups, exchange of problems and results, etc.

Since then the breaks in the dikes have all been closed, already before the winter fell, the land has been reclaimed and drained, and an energetic beginning has been made to repair the other material damage. The Delta-committee has advised to close completely four of the six sea-arms in the South. Along the two other ones which lead to the harbours of Rotterdam and Antwerp, the dikes will be heightened, whereas a number of other works will be carried out.

The mathematical problems raised by the flood fall into three sets, belonging to different branches of mathematics, viz.

1. Statistical extrapolation problems concerning the distribution of the height of the sea-level;
2. Econometric decision problems concerning the height to which dikes should be heightened;
3. Problems in applied mathematics concerning the question which heightening of the sea-level is caused by a given depression moving over the North Sea.

The problems belonging to the first two groups are trivial from the mathematical point of view. Their interest for the mathematician lies solely in the logical analysis of the applicability of special mathemathical models. Those belonging to the third group, on the contrary, lead to rather difficult questions on partial differential equations, but are, as yet, still far from completely solved. Lack of time prevents me to go into the econometric problems, which I have investigated with the help of J. Kriens, and discussed extensively last month before the European Meeting of the Econometric Society at Upsala (D. van Dantzig [1954*b*]).

I shall give a short survey of the most important results hitherto obtained in the institutions mentioned before, giving somewhat more details about those, obtained in the Mathematical Centre, viz. in its Statistics Department in cooperative work under Hemelrijk and myself and in the Applied Mathematics Department under my own guidance.

The question might be asked why I venture to speak before an International Mathematical Congress about problems which partly are so elementary as to be almost trivial from a purely mathematical point of view, and, inasfar as they are mathematically interesting, have hitherto led to partial results only, most of which, moreover, are not my own.

Big floods, even far bigger and more disastrous ones, have occurred in many other countries also, even quite recently, and an event which has shaken our country to its roots may look small on a world wide scale. We, who have so recently suffered from such an event, are first to sympathize with the victims in other parts of the world. So we rather would not miss any possibly existing chance that results obtained and methods used here for solving the mathematical problems involved might be of some use in other similar cases also.

Although it might be considered as a national honorary duty of the Netherlands to further the solution of this problem, it is, as we shall see, so big that with restricted intellectual manpower available in a small country like ours during the time we are working on it (one year) only some modest results could be obtained. Because of the urgency of flood prevention in so large parts of Europe, Asia and America, this could be regarded as a typical instance of a problem where some form of international cooperation would be useful. Uniting internationally intellectual capacity and pooling mathematical knowledge would doubtless shorten considerably the time needed for solving the underlying problems.

This is not a matter of large scale computing only. In 1932 (D. van Dantzig [1932]) I have expressed the hope that there is one difference, at least, between mathematicians and horses. Horses have been abolished and replaced by cars and tractors, but mathematicians, I said, need not be afraid of being abolished in order to be replaced by computation machines. Although since then the area of mechanization of mathematics has set in, I still think this to be true, and the hydrodynamical problems needed for flood prevention offer a typical instance where the job of a mathematician does not consist of just "feeding a machine", and which, although it ultimately certainly will require large scale computing, in the present phase rather has need of what might be called "large scale mathematics".

These combined national and international aspects of the problem may, I hope, be considered as a justification of my speaking before this congress.

Statistical problems.

Formerly, engineers based the dike building on the highest flood hitherto observed. In the thirties, however, engineers became aware of the weakness of this method. In 1939 the government appointed a committee to investigate the question, which measures had to be taken in order to increase the security offered by our dike-system. Had not the german invasion and its aftermath, the reconstruction period needed after the war, prevented to carry out the measures it proposed, the 1953 flood would have caused no disaster. It struck us unprepared because it came too soon after the war.

In his by now renowned paper of 1939 P. J. Wemelsfelder [1939] determined a statistical estimate of the (cumulative) distribution of the sea-level heights during high tide at Hoek van Holland. Based on the observations during the period 1888—1937 he found that the exceedance frequencies $n(h)/n$, where $n(h)$ is the number of exceedances of the level h during n years, when plotted on logarithmic paper, very closely followed a straight line (fig. 1) and drew important conclusions from this result.

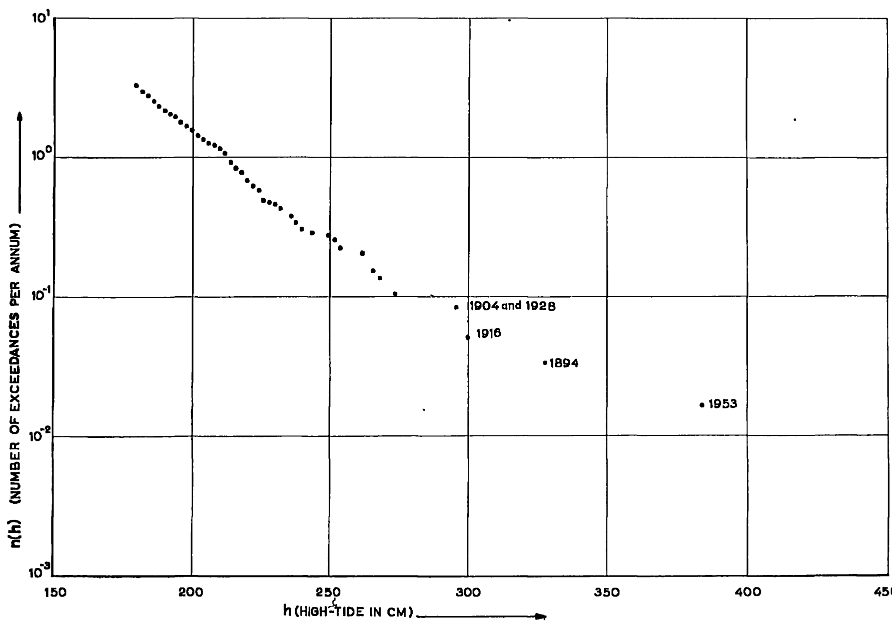

Fig. 1. Exceedance frequencies $\left(\text{plotted at } \frac{i}{60}\right)$ of high tide at Hoek van Holland in the 60 winters of 1888—1939, 1945—1954.

It states (neglecting for the moment the fact that the sea-level at succesive high tides are not independent) that the exceedance probability $1 - F(h)$, where h is the height of the sea-level at high tide above an appropriate zero level, and $F(h)$ its (cumulative) distribution function, is an exponential function $e^{-\alpha h}$. It may also be expressed as a power with any base $g : g^{-h/a_g}$, where $a_g = \alpha^{-1} \ln g$, the difference in height corresponding with probabilities in ratio $1/g$, is a convenient measure, which for $g = 2$, we shortly call the "halving height". At Hoek van Holland it is about 18 centimetres.

When one looks at the graphs one finds that the highest values of h have a tendency to deviate to the right.

One may want therefore to test whether this deviation is significant. The difficulty, caused by the dependence of the order statistics may be overcome in the following way.

Let $\underline{x}_1, \ldots, \underline{x}_n$ [1]) be n independent [2]) continuous "isomorous" variates, i.e. variates all having the same continuous distribution function $F(x)$; let $Q(x) = 1 - F(x)$ be its complement, and $\underline{x}_{(1)}, \ldots, \underline{x}_{(n)}$ the same values arranged according to increasing magnitude:

$$\underline{x}_{(1)} < \underline{x}_{(2)} < \ldots < \underline{x}_{(n)}$$

i.e. the order statistics. For abbreviation we put

$$Q_{(i)} = Q(x_{(i)}).$$

Then, putting for
$$n \geqq i > 1$$
$$H_i = \left(\frac{Q_{(i)}}{Q_{(i-1)}}\right)^{n-i+1}$$

and $H_1 = Q_{(1)}^n$, we have (cf. D. van Dantzig [1947—1950] p. 282) for the element of the simultaneous distribution

$$dF_1 \cdot dF_2 \cdot \ldots \cdot dF_n = dQ_1 \cdot dQ_2 \cdot \ldots \cdot dQ_n = n!\, dQ_{(1)} \cdot dQ_{(2)} \cdot \ldots \cdot dQ_{(n)} =$$
$$= dH_1 \cdot dH_2 \cdot \ldots \cdot dH_n$$

so that $\underline{H}_1, \underline{H}_2, \ldots, \underline{H}_n$ are independent isomorous variates, all homogeneously distributed over the range (0,1) [3]). In the case where the $\underline{x}_i$ have an exponential distribution, say for $x > 0$:

$$Q(x) = e^{-\alpha x}$$

we have for $i > 1$:

$$H_i = e^{-(n-i+1)\alpha(x_{(i)} - x_{(i-1)})} \text{ and } H_1 = e^{-n\alpha x_{(1)}}$$

so that this implies that the n variates

$$n\underline{x}_{(1)},\ (n-1)(\underline{x}_{(2)} - \underline{x}_{(1)}), \ldots,\ 2(\underline{x}_{(n-1)} - \underline{x}_{(n-2)}),\ \underline{x}_{(n)} - \underline{x}_{(n-1)}$$

are independent and all isomorous with the original $\underline{x}_i$, a special consequence which may more easily be derived directly from the properties of the exponential distribution itself.

As the difference $x_{(n)} - x_{(n-1)}$ between the highest two order statistics in our case is 57 cm, and $2(x_{(n-1)} - x_{(n-2)}) = 56$ cm, these values, with a

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[1]) The random character of a variable is denoted by underlining its symbol.

[2]) "Mutually completely independent" according to J. Neyman's terminology.

[3]) Since this conference was held, I learned from A. Rényi that an equivalent result has also been obtained independently by S. Malmquist [1950] and A. Rényi [1953].

halving height of 18 cm, have exceedance probabilities 0,11 and therefore are by no means out of the common. On the other hand the estimate of the halving height, based on the two differences of the highest values alone (which, of course, because of the small number of data used is very uncertain) would be $\frac{1}{2}(56 + 57) \ln 2 = 39$ cm, i.e. more than double the value found from the lowest values.

According to a remark due to J. Hemelrijk, one has to take account of the fact that this statistical investigation was partly begun because of the flood disaster, i.e. that the series of observations was not terminated at a random moment, but just after a record was reached. This fact might lead to a spurious significance. In order to overcome this difficulty one might either drop the 1953 observation, or otherwise add a number of fictitious observations representing e.g. another ten years. Assuming that then nothing out of the common will happen, these may be drawn at random from the distribution hitherto found. As, however, the highest value does not significantly fall out of the distribution, Hemelrijk's methodologically so important remark can be disregarded in the present case.

Anyhow, the upward tendency of the highest values suggest that the population might be a mixture of two or more different ones, and one may try to unmix these. To this purpose one can first sift out the most dangerous months, which appear to be November, December and January and give rise to a distribution significantly differing from that belonging to the other months. In the second place a selection can be made on meteorological grounds. C. J. van der Ham in the Royal Dutch Meteorological Institute has selected a set of depressions from the period 1888—1939, 1945—1954, which followed paths within a definite geographical strip. His data were analysed in the Mathematical Centre and gave rise to a distribution significantly different from the previous one.

Possibilities of other selections have been investigated in the Mathematical Centre, namely with regard to sun-spot maxima, and to "dangerous years", showing, apart from high maxima, a higher overall average. The first result was completely negative, the latter one, although some indication for its reality was found, could not be established as a significant selection. Like most other statistical results obtained in the Mathematical Centre, this was done in a cooperative study under J. Hemelrijk, to which among others H. Kesten en J. Th. Runnenburg have greatly contributed.

Finally, as A. Benard and Mrs. E. Bos-Levenbach, following a suggestion by J. Hemelrijk, pointed out, the upward tendency of the highest values, which is very often found in similar cases, is, at least partly, a spurious effect, caused by drawing the empirical distribution function, as customary, at the levels $\frac{i}{n+1}$ $\left(\text{or } \frac{i}{n}\right)$. The former choice is usually justified by saying that the distribution function $F(x_{(i)})$ of the i^{th} order statistic $\underline{x}_{(i)}$ has expectation $\frac{i}{n+1}$. As, however, the distributions of the order-statistics are very skew, the median is, for $i > \frac{1}{2}n$ larger and for $i < \frac{1}{2}n$ smaller than the expectation, so that more often than not the empirical distribution will have an S-form rather than be approximately straight. This is a well known phenomenon in most cases where probability paper of any kind is used, and may be overcome by replacing the levels $\frac{i}{n+1}$ by the *medians* instead of the expectations of the $F(x_{(i)})$. These, being the medians of beta-distributions, can easily be computed for any given i and n, and it is found that they differ very little from their asymptotic values, which are

$$\frac{i - \frac{1}{3}}{n + \frac{1}{3}}.$$

Pıactically it is only important when i or $n - i$ are very small. For $i = 1$ $\frac{\ln 2}{n + 1 - \ln 4} \approx \frac{1 - 0{,}3}{n + 0{,}4}$ is a better approximation.

This replacement of the expectations by the medians may be useful in numerous other cases where some kind of probability paper is used. In our case the deviation to the right at the high values of h does not disappear completely. By the selection mentioned the halving height is raised from 18 to about 23 cm, with a 0,01 upper confidence limit of 28,9 cm. The result is shown in figure 2, again on a logarithmic scale. It is evident that the straight line fits excellently and that the tendency to the right in the highest values is of no importance anymore, so that this must be considered the best estimate of the distribution hitherto available.

One might perhaps object that the emphasis laid here on the deviation of the highest values is somewhat heavy, but it must be remarked firstly, that the guiding principle during the investigation was not connected with these highest values, but with the meteorological background and secondly that the two selected groups (dangerous months, and dangerous types of depressions) both show clearly significant differences with the rest (although, of course, no one would want to suggest that all dangerous cases and only these have been selected). Hence the selections are justified from a purely statistical point of view, even without taking into account *here* the fact that in problems like the present one it is always desirable to remain on the safe side. This fact ought to be (and has been) taken account of in the econometric decision problem.

The question arises whether extrapolation of the distribution now is reliable. From a purely mathematical point of view it is almost trivial that this

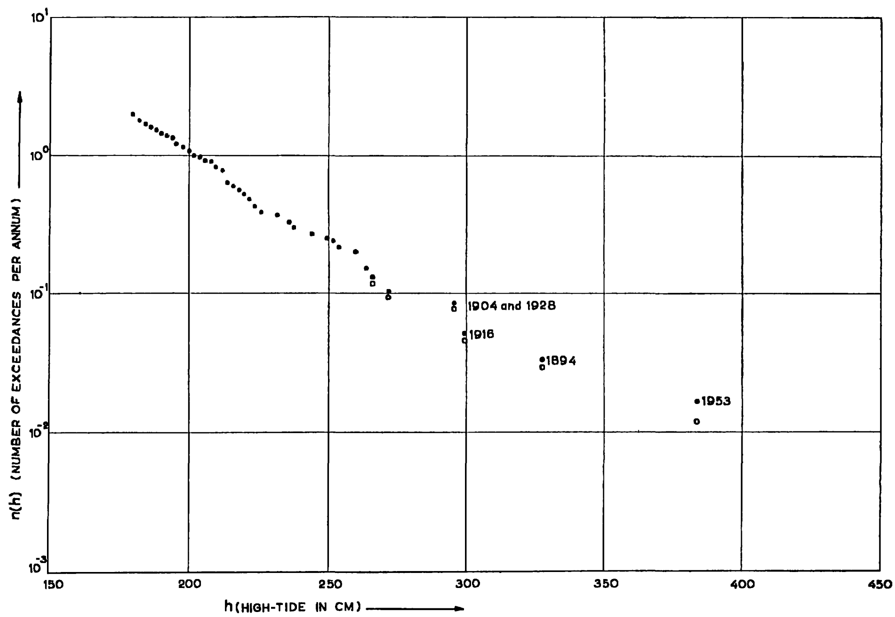

Fig. 2. Exceedance frequencies of high tide during dangerous depressions in the winters of 1888—1939, 1945—1954 at Hoek van Holland. The points are plotted like in fig. 1; the circles denote the plotting positions according to the method of Bos-Benard.

is by no means the case. Even if the most perfect agreement has been found between a finite number of observations and a mathematical distribution function, there is not the slightest guarantee that extrapolation towards the "tails" is allowed. As an example we assume that a perfectly symmetrical die has been thrown n times on a table, and that, instead of the number of eyes, the greatest height above the table is observed. We admitt a small error of measurement, which is normally distributed. If the standard deviation, expressed in the edge of the die as unit is 0,01, the probability that a height $> 1{,}05$ will be found is for all practical purposes negligible. If, however, the edges and angles of the cube are ground off, there is a positive and constant probability that a height $\approx \sqrt{2}$ or even $\approx \sqrt{3}$ will be found. If it has never occurred during the n observations that the die comes to rest on an edge, there is not the slightest statistical evidence for the "bubbles in the tail" of the probability density, which actually are present, and the probability of a height e.g. $> 1{,}4 = 1 + 40\sigma$ is far greater than the best possible estimate. We can only assert for any small positive α with a probability $\geqq 1 - \alpha$ that the probability of the never observed event will be

$$\leqq 1 - \alpha^{\frac{1}{n}} \approx \frac{1}{n} \ln \frac{1}{\alpha}.$$

This perhaps somewhat "pathological" example can be replaced by a more realistic one (cf. D. van Dantzig and J. Hemelrijk [1953]).

Assume that a parameter x on which a windfield depends has a probability density as indicated in fig. 3 below the x-axis, and that the sea level $h = \varphi(x)$

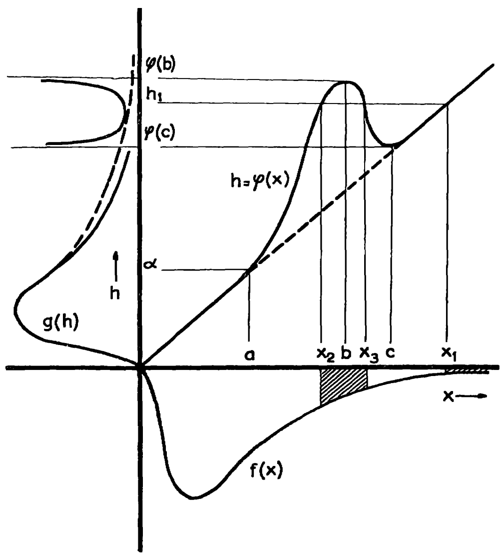

Fig. 3. Influence on a probability distribution of a non-linear transformation.

was found for small values of x ($x \leqq a$) hitherto observed to be proportional with x. If this relation would hold for *all* x, the probability distribution of h would (except for a scale factor) be the same as that of x. If, however, $h = \varphi(x)$ actually has a relative maximum at $x = b$ and a relative minimum at $x = c$, the probability density $g(h)$ of h will be different. In fact, it will become infinite at the relative maximal and minimal values of h, i.e. at $h = \varphi(b)$ and $h = \varphi(c)$. The probability that $\underline{h} > h_1$ would then be grossly underestimated by extrapolation of $h = \varphi(x)$ beyond the region of available observations: E.g. $P\{\underline{h} > h_1\}$ would be estimated as $\int_{x_1}^{\infty} f(x)dx$ whereas actually it is $\int_{x_2}^{x_3} f(x)dx$ greater.

Of course in reality the windfield will depend on more than one parameter so that the probability density $g(h)$ of $\underline{h}$ will not really become infinite but only suffer from "tail-bubbling".

A similar form of $\varphi(x)$ will be found in the case where resonance phenomena occur. For his reason it is of the greatest importance to understand the physical causes of the high values of h. In particular we want to know, which types of motion of a depression over the North Sea will be in resonance with the proper oscillations of the basin.

The probability that the height h will not be exceeded during the n independent tides occurring in a year is $(1 - Q(h))^h \approx e^{-nQ(h)} \approx 1 - nQ(h)$ as not only $Q(h)$, but also $nQ(h)$ is small for the larger values of h. By the restriction to independent tides n is not equal to 706, the number of high tides per year, but it is stochastic. We may replace n by its expected value. If only the wintermonths or dangerous depressions are taken, n must accordingly be reduced.

In our case of an exponential distribution the second expression becomes $\exp(-ne^{-\alpha h})$, which has the well-known form of the distribution of extreme values, first derived by R. A. Fisher and L. H. C. Tippett [1928] and extensively studied, generalized and applied to many problems by E. J. Gumbel [1933], [1954].

Fig. 4 shows the distribution of the selected data along the Gumbel-line, which evidently fits well. Determination of the constants by Gumbel's methods, however, gives less accurate results than the direct one. Moreover we see that the last form of the approximation $1 - nQ(h)$ becomes in our case $1 - ne^{-\alpha h}$, i.e. is itself an exponential distribution with the same parameter α (but a different origin), so that we can use this one throughout without going further into the theory of extreme values.

Other distributions than the exponential one, for the high tides have been tried. So J. J. Dronkers [1953] in the Public Works Department has tried a.o. the logarithmically normal distribution, also often used in similar cases. Also the distribution of the difference of the sea level that really occurred and the predicted sea level was investigated by the Public Works Department (cf. F. Volker [1951]). Moreover, the effect of the local wind force has been investigated in the K.N.M.I. by P. J. Rijkoort [1954].

Resuming our statistical results we may say:

The best approximation hitherto obtained to the distributions of high tides and of storm levels is the exponential one, as already found by Wemelsfelder. Although it has no theoretical foundation, and no guarantee at all exists that it will remain valid after considerable extrapolation, there are on the other hand no significant deviations, if the selection of dangerous months and of Van der Ham depressions is used. In particular the 1953 flood does not fall significantly out of the distibution. For these reasons the best thing to do is to use it as long as no significantly deviating results have been found, meanwhile continuing research in this direction, bearing in mind that we have reached here the limits of applicability of statistical methods. For, the relatively

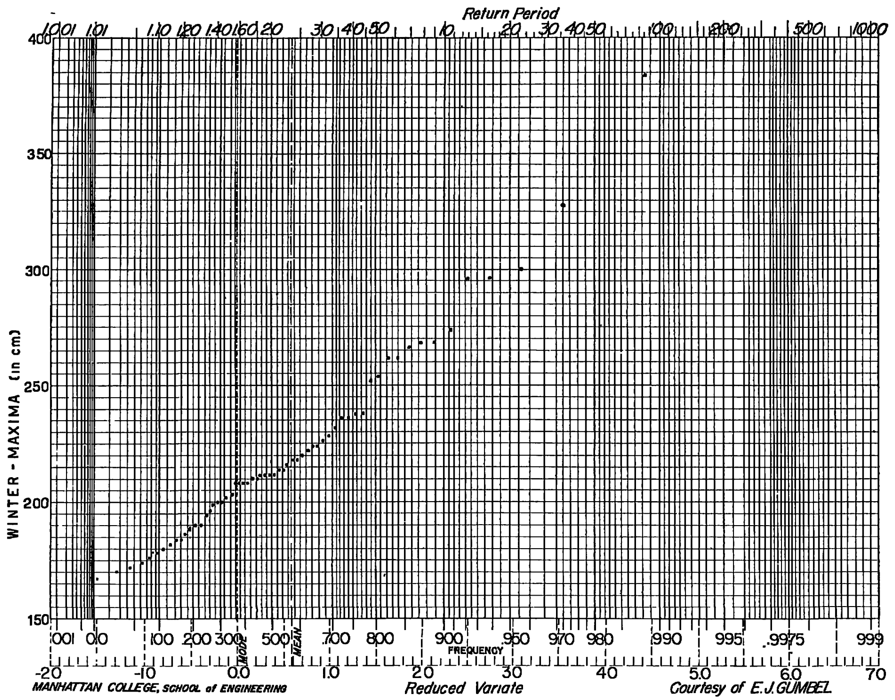

Fig. 4. Winter maxima of high-tide at Hoek van Holland during the period 1888—1939, 1946—1954, plotted according to the Bos-Benard method.

small statistical material which is available makes it impossible to exclude the possibility of slow secular changes in the probability distributions, which nevertheless, if present, might play havoc with all statistical predictions. Apart from known — though insufficiently known — secular phenomena, like the sea level rising and the land level sinking, these can also not be excluded because of the large rôle instabilities play in atmospheric conditions. Thereby e.g. small changes of average temperature in the polar regions might lead to relatively large changes of the probability distribution of the courses and depths of depressions.

It will therefore be necessary to take the physical background into account, in particular to study more closely possible resonance and interference phenomena, which are known to exist. Whereas statistical methods suffer from the drawback that they can be applied to *past experience* only, physical methods allow us to find out, which phenomena could possibly occur in future under any imaginable realistic circumstances, and under which circumstances these might become dangerous.

So this is the point where statistics must pass the torch to hydrodynamics.

The hydrodynamic problem.

The hydrodynamic problem is: to determine the elevation of the sea-level if a depression of given form and intensity moves over the North Sea, and, in particular, to find those motions of depressions which give rise to especially high elevations, i.e. to obtain a better insight in the physical background of storm surges on the North Sea. This problem had already been studied before the 1953 flood by W. F. Schalkwijk [1947] in the Meteorological Institute. His paper has become fundamental to all later research. He considered more in particular the stationary solutions of the linearized equations for a wind-field, constant in space and time.

The problem can be attacked only by means of considerable simplifications. In the first place the non-linear hydrodynamical equations are replaced by their linear approximation. Although some non-linear effects are known, e.g. the fact that the influence of wind is greater at low tide than at high tide, they are also known to be rather small. Because of this linearization we may consider the difference only between the actual elevation and the one due to the astronomical tide, i.e. we may leave the astronomical tide out of consideration. In the second place the vertical component of the velocity of the water is neglected, and the horizontal components are replaced by their average values over a vertical column reaching from the bottom to the level of the sea. Thirdly internal friction and friction caused by the coasts are neglected, and only friction due to the bottom is taken account of. Moreover the irregular form of the North Sea is replaced by a rectangular basin, attached to an ocean represented by a half plane. This approximation is quite reasonable, if the "leak" at the Dover Straits is neglected.

The non-linear equations governing the problem were recently discussed anew in the Public Works Department by J. C. Schönfeld [1954]. We shall here, however, consider the twodimensional linearized equations only.

Then, be ζ the elevation of the sea level above a given zero level, x and y the rectangular coordinates on the surface of the sea, u and v the average components of the velocity, h the depth of the sea, gX and gY the components of the external force per unit of mass, consisting of the gradient of the atmospheric pressure, and the shearing force exerted by the windfield. The latter is *not* proportional to the wind velocity, but approximately to its square. We shall, however, not go into this relationship, but assume X and Y to be *given* functions of x, y, t. The equations of continuity and of motion have the form

$$\begin{aligned} &\frac{\partial(hu)}{\partial x}+\frac{\partial(hv)}{\partial y}+\frac{\partial\zeta}{\partial t}=0\\ (1)\qquad &\frac{\partial u}{\partial t}+\lambda u-\Omega v+g\frac{\partial\zeta}{\partial x}=gX\\ &\frac{\partial v}{\partial t}+\lambda v+\Omega u+g\frac{\partial\zeta}{\partial y}=gY. \end{aligned}$$

Here g is the acceleration of gravity, λ a friction coefficient, roughly proportional with h^{-1}, and Ω the coefficient of Coriolis, viz. $\Omega = 2\omega \sin \varphi$ where ω is the angular velocity of the rotation of the earth and φ the latitude. Although Ω and h, hence also λ depend on x and y, we shall consider the case only where they are constant (small basin of constant depth). Because of the linearity of the equations the tidal motion may be omitted.

The North Sea is considered (fig. 5) as the rectangle

$$R: \quad |x| \leqq a, \quad 0 < y < b$$

where $x = -a$, $y = 0$, $x = +a$ roughly represent the English, the Dutch and the Scandinavian coasts respectively, whereas $y = b$ is the open end of the sea at the Atlantic Ocean. Roughly $2a \approx 450$ km, $b \approx 4a \approx 900$ km. The "leak" at the Dover Straits is at present left out of consideration.

The boundary conditions state that the normal component of the velocity vanishes along the coasts, and that ζ is continuous along the ocean frontier. If the ocean is considered as infinitely deep, we may assume $\zeta = 0$ there.

Hence

$$(2)\qquad \Gamma \begin{cases} \Gamma_1 : \ |x| < a,\ y = b & \zeta = 0 \\ \Gamma_2 \begin{cases} x = \pm a,\ 0 < y < b & u = 0 \\ |x| < a,\ y = 0 & v = 0. \end{cases} \end{cases}$$

We shall not introduce dimensionless variables, as sometimes one type of normalization, e.g. $\Omega = gh = 1$, sometimes another one, e.g. $a = \frac{1}{2}\pi$, is the more useful one.

We consider the stationary case first. The equations reduce to

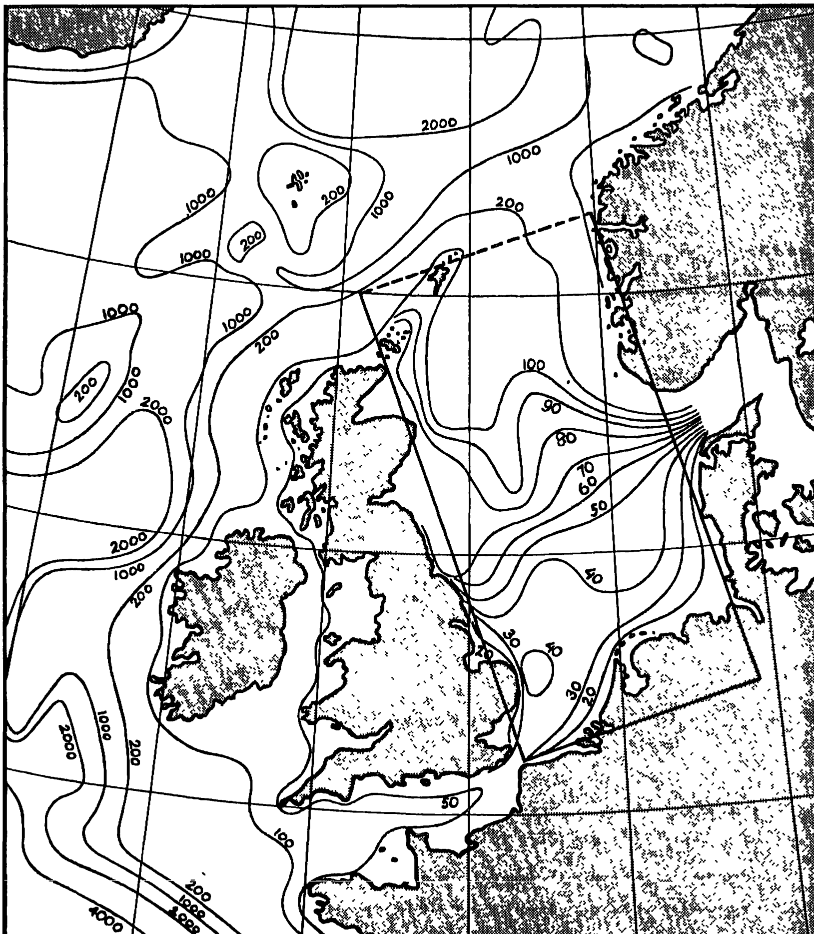

Fig. 5. The North Sea with approximating rectangle and depth in metres.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\lambda u - \Omega v + g\,\frac{\partial \zeta}{\partial x} = gX \tag{3}$$

$$\lambda v + \Omega u + g\,\frac{\partial \zeta}{\partial y} = gY.$$

Whereas Schalkwijk had already studied the case of constant X and Y, M. P. H. Weenink [1954], working in the Meteorological Institute with P. Groen, recently studied the case where $X = 0$ throughout, $Y = \text{const} = 2W$ on the Western half of the North Sea, and $Y = 0$ on the Eastern half, supposing the sea 1^0 to be closed, or 2^0 to border an ocean, represented by a half plane of a depth equal to that of the sea. In both cases he found ζ in the origin (which is the point most important for our country) to have the *same* value as in the case of a constant windfield, having the same overall average value (hence $Y = W$ everywhere on the North Sea). The conformal mapping of a half plane with adjoining rectangle needed in this case had already been studied in the Mathematical Centre by G. W. Veltkamp [1953a].

The general theory is being developed in the Applied Mathematics Department of the Mathematical Centre under my direction. Firstly the case of stationary windfields on a sea of constant or sectionally constant depth was studied by G. W. Veltkamp [1953b, 1954]. As a two dimensional vectorfield (X, Y) can always be described by means of two potentials U and V, determining its divergence-free and curl-free parts respectively:

$$X = -\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}, \; Y = \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y},$$

the solution can be obtained in the case of a closed sea from a complex analytic function L of $z = x + iy$ as

$$\zeta = \operatorname{Re} \frac{\Omega}{\lambda} + i\,(L + U + iV), \tag{4}$$

where L must satisfy the boundary condition

$$U + \operatorname{Re} L = \text{const.} \tag{5}$$

along each closed part of the boundary. The computations could be carried out for some special examples of circular windfiels above an ocean, represented by a halfplane. As an example fig. 6 shows the lines of constant ζ for the case $V = 0, U = -\frac{1}{2}b^3\,(|z - ia|^2 + b^2)^{-1}$, representing a divergence-free windfield with a non-singular centre. In the case of a sea with sectionally constant depth, Veltkamp found that L must be replaced by a sectionally holomorphic function, and that the Plemelj formulae induce a singular integral equation along the line of discontinuity of the depth. If a shallow bay (the North Sea) borders an ocean (the Atlantic) of finite but large depth, an iteration procedure using the ratio of the two depths (both assumed to be constant) can be applied.

If in the case of a rectangular bay as mentioned before (where one side is about twice as long as the other, which implies that the Jacobian elliptic function which maps the rectangle on a half-plane nearly degenerates) the potentials U and V are chosen in such a way that $U = 0$ on Γ (which is always possible) then the function $L(z)$ proves to be nearly a constant in the vicinity of "the South Coast", viz.

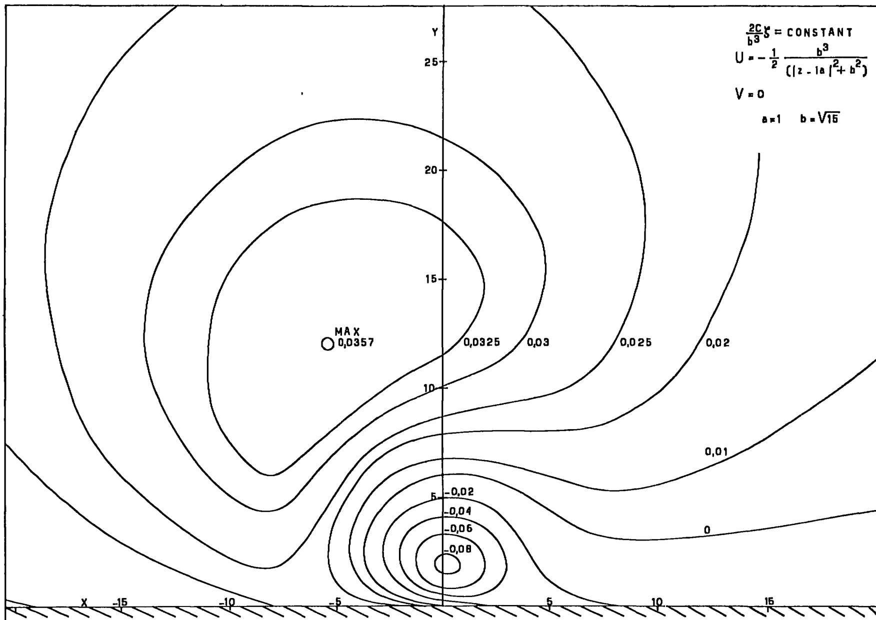

Fig. 6. A stationary windfield on a half plane. Lines of constant ζ.

$$L \sim -\tfrac{1}{2} i \cos \pi\gamma \int_0^a \left[\left(\operatorname{tg} \frac{\pi t}{4a} \right)^{-2\gamma} V(t-a, b) - \left(\operatorname{tg} \frac{\pi t}{4a} \right)^{2\gamma} V(a-t, b) \right] dt$$

with $$\gamma = \frac{1}{\pi} \operatorname{arc\,tg} \frac{\Omega}{\lambda}, \quad 0 < \gamma < \tfrac{1}{2}.$$

Accordingly ζ can be computed in the point $x = y = 0$. For a few types of windfields the numerical work was carried out in the Computation Department of the Mathematical Centre.

For the case of a linear windfield

$$X = \tfrac{1}{2} W \left(1 - \frac{x}{2a} - \frac{2y - b}{b} \right)$$

$$Y = -W \left(1 - \frac{5}{4} \frac{x}{a} \right),$$

with $b = 4a$, sketched in fig. 7, the result was

$$\zeta = 1{,}20 \frac{bW}{h},$$

whereas the homogeneous windfield

$$X = \tfrac{1}{2} W \quad Y = -W$$

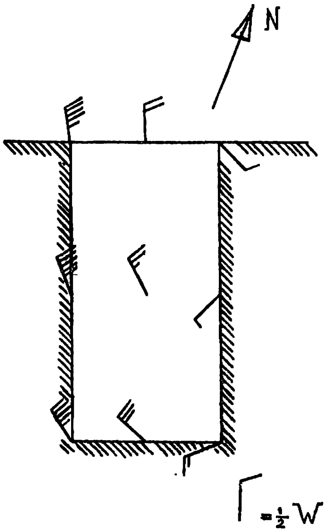

Fig. 7. Linear windfield on rectangular sea.

would have given $\zeta = 1{,}105\,\dfrac{bW}{h}$,

and the field

$$X = 0 \quad Y = -\,W$$

led to $\zeta = 1 \cdot \dfrac{bW}{h}$. Hence we see that the cross component $x = \frac{1}{2}W$ adds a $10\frac{1}{2}\%$ to the elevation and the non-homogeneity another $9\frac{1}{2}\%$.

I now come to the non-stationary case which was attacked a few months ago in the Mathematical Centre.

By means of a Laplace transformation

$$\bar{\zeta}(x, y, p) = \int_0^\infty e^{-pt}\zeta(x, y, t)dt$$

(6)

$$\zeta(x, y, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\bar{\zeta}(x, y, p)dp,$$

and similarly for u and v, we obtain the equations

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{p}{h}\,\bar{\zeta} = 0$$

(7)

$$(p + \lambda)\bar{u} - \Omega\bar{v} + g\,\frac{\partial\bar{\zeta}}{\partial x} = g\bar{X}$$

$$(p + \lambda)\bar{v} + \Omega\bar{u} + g\,\frac{\partial\bar{\zeta}}{\partial y} = g\bar{Y}.$$

In the first place it is of importance to determine the proper oscillations of the basin by putting $X = Y = 0$, $p = -\lambda + i\omega$. Normalizing the dimensions by putting

$$gh = 1, \quad a = \tfrac{1}{2}\pi$$

we find that $\bar{u}$, $\bar{v}$ and $\bar{\zeta}$ all satisfy the differential equation

$$\Delta\bar{u} - k^2\bar{u} = 0 \tag{8}$$

with

$$k^2 = \frac{1}{gh}\left(1 + \frac{i\lambda}{\omega}\right)(\Omega^2 - \omega^2) \quad (\mathrm{Re}\, k > 0). \tag{9}$$

Without friction and without Coriolis force we would simply have $k = \dfrac{i\omega}{\sqrt{gh}}$. Coriolis changes the relation into $k^2 = \dfrac{\Omega^2 - \omega^2}{gh}$, and the friction is the cause of the occurrance of the denominator ω. In other terms: because of the friction the differential equation in x, y, t is no longer of the second order, but of the third order with respect to time. If the variation of depth had also been taken into account first order derivatives with respect to x and y would have occurred too, and the equations would have been of fifth order with respect to time.

The proper oscillations can be found by generalizing slightly a method introduced by G. I. Taylor [1922] in 1922 already for an infinitely long channel, closed at one end, without friction ($\lambda = 0$).

Putting formally

$$\bar{u} = \sum_{-\infty}^{\infty}{}_n \; i^n e^{inx}\left\{u_n^+ e^{y\sqrt{k^2+n^2}} + u_n^- e^{-y\sqrt{k^2+n^2}}\right\} \tag{10}$$

($\mathrm{Re}\sqrt{k^2 + n^2} > 0$), the differential equation $(\Delta - k^2)\bar{u} = 0$ is satisfied, as well as the boundary conditions $\bar{u} = 0$ if $x = \pm\frac{1}{2}\pi$, provided u_n^+ and u_n^- are odd functions of n. The analogous quantities $v_n^\pm$ can easily be expressed in the $u_n^\pm$ by means of factors

$$\frac{n\Omega + \omega\sqrt{k^2 + n^2}}{n\omega + \Omega\sqrt{k^2 + n^2}},$$

and in the two "Kelvin-waves", i.e. the solutions of (7) (for $X = Y = 0$), together with $\bar{u} = 0$.

By means of the identities for $|z| < \frac{1}{2}\pi$

$$e^{(2m-1)iz} = \frac{1}{\pi} \sum_{-\infty}^{\infty}{}_n \frac{(-1)^{m-n-1}}{m-n-\frac{1}{2}} e^{2inz},$$

the two other boundary conditions can be introduced, the $u_n^{\pm}$ are found to vanish for all even n, and the other values have to be found from an infinite sequence of equations. The characteristic values of ω, i.e. the poles of the Laplace transform have to be computed as the roots of an infinite determinant (D. van Dantzig [1954a]). The computation has not yet been carried out.

We now return to the non-homogeneous equations (7). By elimination of $\bar{u}$ and $\bar{v}$ we obtain the differential equation

$$\Delta\bar{\zeta} - k^2\bar{\zeta} = \bar{F} \tag{11}$$

$$\begin{gathered} k^2 = \frac{p}{gh}\left(p + \lambda + \frac{\Omega^2}{p+\lambda}\right) \\ F = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} - \frac{\Omega}{p+\lambda}\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right), \end{gathered} \tag{12}$$

whereas the boundary conditions become: $\bar{\zeta} = 0$ along the ocean Γ_1 and

$$\frac{\partial\bar{\zeta}}{\partial\nu} + \frac{\Omega}{p+\lambda}\frac{\partial\bar{\zeta}}{\partial s} = \bar{f} \tag{13}$$

with

$$f = \begin{cases} X + \dfrac{\Omega}{p+\lambda} Y \text{ for } x = \pm a,\ 0 < y < b \\[2ex] \dfrac{\Omega}{p+\lambda} X - Y \text{ for } |x| < a,\ y = 0. \end{cases} \tag{14}$$

The main difficulties are caused by the presence of the force of Coriolis, and consist of the occurrence of the skew boundary conditions (13) instead of the usual normal ones. If the force of Coriolis were negligible we should have $\Omega = 0$, and the problems were relatively simple, but it is not, and sometimes it is even the greatest term determining the motion.

Application of Green's theorem with an as yet undetermined function $G = G(x, y, \xi, \eta)$ having a logarithmic singularity at $x = \xi$, $y = \eta$, use of the boundary conditions (13) and partial integration gives

$$\begin{aligned} \bar{\zeta} = \bar{\varphi} - \iint_R \bar{\zeta}(\Delta - k^2)G\, d\xi d\eta\, + \\ + \int_{\Gamma_2} \bar{\zeta}\left(\frac{\partial G}{\partial\nu} - \frac{\Omega}{p+\lambda}\frac{\partial G}{\partial s}\right) ds - \int_{\Gamma_1} G\frac{\partial\bar{\zeta}}{\partial\nu}\, ds, \end{aligned} \tag{15}$$

$$\varphi = \iint_R G\bar{F}\, d\xi d\eta - \int_{\Gamma_2} G\bar{f} ds.$$

If we could find a Green's function G, satisfying

$$\Delta G - k^2 G = 0 \text{ in } R$$

$$(16) \qquad G = 0 \text{ on } \Gamma_1$$

$$\frac{\partial G}{\partial \nu} - \frac{\Omega}{p + \lambda} \frac{\partial G}{\partial s} = 0 \text{ on } \Gamma_2,$$

(15) would yield the solution explicitly. We did, however, not succeed in obtaining this G explicitly. So we have to be content with less.

The following methods offer themselves.

1. Taking

$$G = \frac{1}{2\pi} \{K_0(kr) - K_0(kr')\},$$

where K_0 is Bessel's function of the third kind, zero order and imaginary argument, whereas

$$r^2 = (x - \xi)^2 + (y - \eta)^2, \; r'^2 = (x - \xi)^2 + (y + \eta - 2b)^2,$$

the differential equation and the boundary condition along the ocean are satisfied. Thereby (15) becomes a Cauchy-singular integral equation along the coast, which may be solved by the methods Poincaré [1910] introduced for similar problems in the theory of tides. These methods have been studied extensively by the Tiflis school under I. N. Mushkhelishvili [1953]. Notably results obtained by I. N. Vekua [1939] and B. V. Khvedelidze [1943] are important for our purpose.

2. A solution of the differential equations (16), satisfying the boundary conditions along the two long sides $x = \pm a$ of the rectangle may be found; it yields a singular integral equation along the two other sides.

3. The same can be done for the two other sides of the rectangle. As Veltkamp pointed out, this is better, as the main effects in the more important cases will be a current in the North-South direction.

4. Finally we can satisfy all boundary conditions by a harmonic function G. Then (15) becomes

$$\bar{\zeta} = \bar{\varphi} + k^2 \iint \bar{\zeta}\, G\, d\xi d\eta,$$

i.e. a *non-singular* integral equation, which for small k^2 may be solved explicitly by the method of iterated kernels. The solutions G in the three last-mentioned cases have all been obtained in the Mathematical Centre by H. A. Lauwerier [1954]. In the last case we have

$$G = \mathrm{Re}\,\frac{1}{4\pi}\int_{M(z)}^{\infty}\left\{\frac{A}{s-M(\zeta)}-\frac{A^*}{s-M(\zeta)^*}\right\}s^{-\frac{\omega}{\pi}}\,ds$$

where $z = x + iy$, $\zeta = \xi + i\eta$ and $M(z)$ a double-periodic function, the periods of which are twice the sides of the rectangle. The asterisk denotes the complex conjugate. For the North Sea, where roughly $b \approx 4a$ the elliptic function $M(z)$ very nearly degenerates into trigonometric functions.

The solution for small k^2, i.e. small p, which we obtain in this way may be transformed back into an asymptotic expansion of ζ, valid for large t. This, however, is of relatively little importance, as we are not particularly interested in the height the sea-level reaches after the storm is over. In combination, however, with the determination of the proper oscillations the path of the inverse Laplace transformation can be drawn over one or more of the corresponding points $p = -\lambda + i\omega$ and thereby will lead to solutions in which the main effect is incorporated.

Lauwerier also clarified the main cause of the difficulty of the problem, due to the skew boundary condition. Taking first the case of a half plane $x > 0$ with a skew boundary condition in a constant (real) direction, Green's function has not only a logarithmic singularity in the image $(-\xi, \eta)$ or (ξ, η), but it makes also a finite jump along a half line ending in this point, which can be considered as being covered by dipoles.

If we pass to a quarter plane we get apart from the logarithmic singularities obtained by reflection also an algebraic singularity at the corner.

This, Mr Chairman, shows how far we had come just before the Congress began.

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LE CALCUL DIFFÉRENTIEL DANS LES CORPS DE CARACTÉRISTIQUE $p > 0$.

JEAN DIEUDONNÉ

La théorie dont j'ai l'honneur de parler ici est le développement d'idées que je n'ai réussi à formuler de façon précise que depuis assez peu de temps; aussi serait-il plus juste de parler d'un embryon de théorie, et je m'excuse d'avance du caractère très incomplet des résultats obtenus jusqu'ici. Ce ne sont pas par contre les problèmes qui manquent, et qui, comme j'espère vous en convaincre, se présentent de la façon la plus naturelle; et s'il est vrai que les problèmes sont la nourriture même du mathématicien, ce sera là ma justification.

1. *Dérivations et semi-dérivations* [1]). On sait que, depuis sa fondation, le caractère formel du Calcul différentiel n'a cessé de frapper les mathématiciens, jusqu'à en constituer à leurs yeux, à certaines époques, l'élément essentiel. Mais ce n'est guère que depuis une cinquantaine d'années qu'il a été possible de donner à ces idées une forme rigoureuse, grâce à l'introduction des notions de polynôme abstrait et de série formelle. Bornons-nous au cas le plus simple; dans l'anneau $K[[X]]$ des séries formelles à une indéterminée sur un corps commutatif quelconque K, la „dérivation par rapport à X" est l'opération qui à toute série formelle $f(X) = \sum_{k=0}^{\infty} a_k X^k$, fait correspondre la série formelle

$$(1) \qquad Df = \frac{df}{dX} = \sum_{k=0}^{\infty} k a_k X^{k-1},$$

et on constate immédiatement que cette opération jouit des trois propriétés essentielles de la dérivée classique, savoir

$$(2) \quad D(f+g) = Df + Dg, \quad D(\lambda f) = \lambda Df \text{ pour } \lambda \in K, \quad D(fg) = fDg + gDf.$$

Je ne m'étendrai pas ici sur l'extension de cette notion de „dérivation" aux algèbres quelconques, ni sur les diverses généralisations qui en ont été introduites, et dont l'importance, dans les travaux modernes d'Algèbre, de Géométrie algébrique et de Topologie algébrique, n'a plus à être soulignée aujourd'hui. Signalons seulement la généralisation évidente de la définition (1) aux séries

[1]) Les numéros entre crochets renvoient à la bibliographie placée à la fin de ce travail.

formelles à un nombre quelconque d'indéterminées X_i $(1 \leqq i \leqq n)$, avec la propriété immédiate de permutabilité des „dérivées partielles" $\partial/\partial X_i$ $(1 \leqq i \leqq n)$.

Dans les propriétés fondamentales (2), la caractéristique du corps K n'intervient pas. Mais elle fait son apparition dès que l'on s'occupe des „dérivées d'ordre supérieur" et de ce qu'on peut considérer comme la formule fondamentale du Calcul différentiel classique, la formule de Taylor. Si K est de caractéristique 0, cette dernière s'établit pour les séries formelles comme dans le cas classique, et on a donc

$$f(X+Y) = \sum_{k=0}^{\infty} (k!)^{-1} D^k f(X) Y^k \tag{3}$$

(Y étant une seconde indéterminée). Mais si K est de caractéristique $p > 0$, on a $k! = 0$ pour $k \geqq p$, et d'autre part il résulte immédiatement de (1) que l'on a $D^p = 0$ (et par suite $D^k = 0$ pour $k \geqq p$); autrement dit, les termes de (3) n'ont plus de sens après celui de degré $p-1$.

Mais il est clair que l'on peut toujours „développer" $f(X+Y)$ suivant les monômes Y^k, quelle que soit la caractéristique de K, et par suite écrire

$$f(X+Y) = \sum_{k=0}^{\infty} \Delta_k f(X) Y^k \tag{4}$$

ou les Δ_k sont des opérateurs linéaires dans l'anneau $K[[X]]$; d'où l'idée naturelle d'étudier ces opérateurs, qui, dans le cas où la caractéristique est $\neq 0$, jouent en quelque sorte le rôle des „dérivées d'ordre supérieur". Cette idée a sans doute été introduite pour la première fois par H. Hasse, F. K. Schmidt et O. Teichmüller ([6], [7], [9]), qui, ainsi que leurs élèves, ont commencé l'étude des opérateurs Δ_k et ont cherché à s'en servir dans diverses questions d'algèbre. On constate aussitôt, par multiplication des formules (4) pour f et g, que la propriété fondamentale des dérivations, la troisième relation (2), est remplacée par une „formule de Leibniz"

$$\Delta_m(fg) = \sum_{k=0}^{m} \Delta_k f \Delta_{m-k} g; \tag{5}$$

d'autre part, en exprimant $f(X+Y+Z)$ de deux manières, on a les relations

$$\Delta_m \Delta_n = \binom{m+n}{n} \Delta_{m+n}. \tag{6}$$

Il semble donc (et c'est le point de vue des auteurs précités) que l'on ne puisse étudier les opérateurs Δ_k isolément, mais uniquement le système qu'ils forment, avec les relations (5) et (6) qui les relient; et on conçoit sans peine que le maniement d'un tel système présente des difficultés considérables. Mais il est possible d'envisager ces opérateurs d'un autre point de vue: de même qu'en caractéristique 0, les Δ_k ne sont autre que les monômes $D^k/k!$, on peut chercher un „système de générateurs" des opérateurs Δ_k en caractéristique p. On constate aisément que, si on pose $D_h = \Delta_{p^h}$ $(h = 0, 1, \ldots,)$, les D_h sont de tels générateurs; et de façon précise, si $m = \sum_{h=0}^{\infty} \alpha_h p^h$ avec $0 \leqq \alpha_h \leqq p - 1$ et $\alpha_h = 0$ à partir d'un certain rang (développement p-adique de m), on a [3]

$$\Delta_m = \prod_{h=0}^{\infty} (\alpha_h!)^{-1} D_h^{\alpha_h}. \tag{7}$$

On est donc ramené à l'étude des D_h, et on vérifie aussitôt les propriétés suivantes de ces opérateurs: 1) restreint à l'anneau $K[[X^{p^h}]]$ des séries formelles en X^{p^h}, D_h est une *dérivation*; 2) plus généralement, on a

$$D_h(fg) = fD_h g + gD_h f \tag{8}$$

pourvu que l'une au moins des séries formelles f, g appartienne à $K[[X^{p^h}]]$. L'intérêt de ces propriétés est que, contrairement à (5) et (6), elles font intervenir l'opérateur D_h *individuellement*, et sont par suite beaucoup plus maniables; mais elles ne déterminent pas D_h et on est donc conduit à étudier tous les opérateurs ayant ces propriétés.

Plaçons-nous tout de suite dans l'algèbre $\mathfrak{o} = K[[x_1, \ldots, x_n]]$ des séries formelles à n indéterminées x_i $(1 \leqq i \leqq n)$ sur un corps K de caractéristique $p > 0$; on désignera par $\mathfrak{o}_r$ la sous-algèbre des séries formelles en $x_1^{p^r}, \ldots, x_n^{p^r}$. Nous dirons qu'un opérateur Δ dans $\mathfrak{o}$ est une *semi-dérivation de hauteur r* si $\Delta(\mathfrak{o}_r) \subset \mathfrak{o}_r$ et si l'on a

$$\Delta(fg) = f\Delta g + g\Delta f \tag{9}$$

pourvu que l'une au moins des deux séries formelles f, g appartienne à $\mathfrak{o}_r$. Il est immédiat que $\Delta(1) = 0$ et que la restriction de Δ à $\mathfrak{o}_r$ est une *dérivation*; la formule de Leibniz montre alors que Δ s'annule dans $\mathfrak{o}_{r+1}$. Nous dirons que Δ est une semi-dérivation *spéciale* de hauteur r si elle s'annule dans $\mathfrak{o}_r$; toute semi-dérivation de hauteur r est donc une semi-dérivation spéciale de hauteur $r + 1$. On vérifie aisément les propriétés suivantes:

3) *Si Δ est une semi-dérivation de hauteur r, il en est de même de Δ^p et de $f\Delta$, où $f \in \mathfrak{o}_r$.*

4) *Si Δ_1, Δ_2 sont deux semi-dérivations de hauteur r, il en est de même de $[\Delta_1, \Delta_2] = \Delta_1\Delta_2 - \Delta_2\Delta_1$; en outre, $[\Delta_1, \Delta_2]$ est spéciale si Δ_1 ou Δ_2 l'est.*

5) *Si Δ_1, Δ_2 sont deux semi-dérivations spéciales de hauteur r, il en est de même de $\Delta_1\Delta_2$ et de $g\Delta_1$ pour toute série formelle g.*

On peut encore dire que les semi-dérivations de hauteur r forment une *algèbre de Lie* $\mathfrak{D}_r$ sur l'anneau $\mathfrak{o}_r$, dans laquelle les semi-dérivations spéciales forment un *idéal* $\mathfrak{S}_r$; en outre, $\mathfrak{S}_r$ est une *algèbre associative* sur l'anneau $\mathfrak{o}$.

Désignons en outre par $D_{hi}f$ le coefficient de $y_i^{p^h}$ dans le „développement de Taylor" de $f(x_1 + y_1, \ldots, x_n + y_n)$, où les y_i sont n indéterminées; D_{hi} est une semi-dérivation de hauteur h, telle que $D_{hi}(x_j^{p^h}) = \delta_{ij}$ (indice de Kronecker), $D_{hi}(x_1^{\alpha_1} \ldots x_n^{\alpha_n}) = 0$ si tous les α_i sont $< p^h$; on a $D_{hi}D_{kj} = D_{kj}D_{hi}$ quels que soient les indices, et $D_{hi}^p = 0$. On montre alors que *le $\mathfrak{o}$-module $\mathfrak{S}_r$ a une base formée des produits* $\prod_{h=0}^{r-1} \prod_{i=1}^{n} D_{hi}^{\lambda_{hi}}$, *où* $0 \leqq \lambda_{hi} < p$ *et les* λ_{hi} *ne sont pas tous nuls*; et d'autre part, *le $\mathfrak{o}_r$-module $\mathfrak{D}_r$ est somme directe de $\mathfrak{S}_r$ et du $\mathfrak{o}_r$-module ayant pour base les* D_{ri} $(1 \leqq i \leqq n)$. Il est commode ici d'introduire des notations abrégées; pour tout système $\alpha = (\alpha_1, \ldots, \alpha_n)$ de n entiers $\geqq 0$, on pose $x_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$; si $\alpha_i = \sum_{h=0}^{\infty} \lambda_{hi} p^h$ est le développement p-adique de α_i $(1 \leqq i \leqq n)$, on pose $D_\alpha = \prod_{h=0}^{\infty} \prod_{i=1}^{n} D_{hi}^{\lambda_{hi}}$. On dit que α est *de hauteur* r si $r + 1$ est le plus petit entier tel que les λ_{hi} soient tous nuls pour $h \geqq r + 1$, et on pose $r = h(\alpha)$. On écrit aussi $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ („degré total" de α, ou de x_α), et $\alpha! = \prod_{h=0}^{\infty} \prod_{i=1}^{n} \lambda_{hi}!$, si bien que la „formule de Taylor" prend la forme

$$f(x_1 + y_1, \ldots, x_n + y_n) = \sum_\alpha \frac{1}{\alpha!} y_\alpha D_\alpha f. \tag{10}$$

2. *Groupes de Lie formels et hyperalgèbres de Lie.* Il paraît vraisemblable que les notions introduites ci-dessus puissent avoir de nombreux usages en algèbre; j'ai indiqué comment on est amené assez naturellement à les introduire dans la théorie des extensions radicielles d'exposant quelconque [2]. Je vais me borner ici à montrer comment elles servent à bâtir une théorie de Lie pour les „groupes de Lie formels" sur un corps K de caractéristique $p > 0$ (voir [4] et [5]).

Un *groupe de Lie formel de dimension* n (qu'il vaudrait peut-être mieux appeler une „loi de groupe"[2]) sur un corps commutatif K consiste en la donnée de n séries formelles à coefficients dans K, sans terme constant, par rapport à $2n$ indéterminées, $\varphi_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$, satisfaisant aux conditions

$$\varphi(\varphi(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \varphi(\mathbf{x}, \varphi(\mathbf{y}, \mathbf{z})) \tag{11}$$

$$\varphi(\mathbf{x}, \mathbf{e}) = \varphi(\mathbf{e}, \mathbf{x}) = \mathbf{x} \tag{12}$$

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[2]) Dans cette conception, il ne subsiste en effet plus rien de la notion „ensembliste" de groupe; on ne peut même plus parler d'un „élément" du groupe, et il n'est pas question, en général, de „substituer" des valeurs quelconques aux indéterminées x_i.

où on a posé $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{z} = (z_1, \ldots, z_n)$ (z_i nouvelles indéterminées), $\varphi = (\varphi_1, \ldots, \varphi_n)$ et $\mathbf{e} = (0, \ldots, 0)$. On tire aussitôt de là que les termes du premier degré de φ_i sont $x_i + y_i$, et qu'il existe n séries formelles $\theta_i(\mathbf{x})$ telles que $\varphi(\mathbf{x}, \theta(\mathbf{x})) = \varphi(\theta(\mathbf{x}), \mathbf{x}) = \mathbf{x}$ (autrement dit, l',,associativité'' et l'existence d'un ,,élément neutre'' entraînent l'existence d'un ,,inverse''). La loi de composition d'un groupe de Lie au sens classique, considérée dans un voisinage de l'élément neutre, donne un exemple de groupe de Lie formel (sur le corps des nombres réels ou des nombres complexes); de même, les ,,groupes de Lie algébriques'' au sens de Weil-Chevalley ([10], [1]), sur un corps quelconque K, donnent des groupes de Lie formels lorsqu'on les considère "localement", c'est-à-dire que l'on identifie l'anneau local de la variété du groupe en l'élément neutre avec un sous-anneau d'un anneau de séries formelles.

A partir d'un groupe de Lie formel G, on peut en déduire d'autres par le procédé du ,,changement de variables'': étant données n séries formelles u_i $(1 \leqq i \leqq n)$ sans terme constant, dont le jacobien a un terme constant $\neq 0$, on considère le système des n séries formelles $\overline{\varphi}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) = \mathbf{u}\,(\varphi(\mathbf{u}^{-1}(\overline{\mathbf{x}}), \mathbf{u}^{-1}(\overline{\mathbf{y}})))$ par rapport à $2n$ indéterminées $\overline{x}_1, \ldots, \overline{x}_n, \overline{y}_1, \ldots, \overline{y}_n$; il est immédiat que cela définit un nouveau groupe de Lie formel $\overline{G}$, dont la structure n'est pas essentiellement distincte de celle de G; on dira que les n séries formelles u_i constituent un *isomorphisme* $\mathbf{u}$ de G sur $\overline{G}$, et les résultats intéressants de la théorie seront évidemment ceux qui restent invariants par tout isomorphisme.

Si on veut étudier les groupes de Lie formels sur le modèle de la théorie classique de Lie, on est conduit à introduire les *opérateurs différentiels invariants* par le groupe. Par ,,opérateur différentiel'' au sens strict, on entend une combinaison linéaire finie $\sum_\alpha c_\alpha D_\alpha$, où les c_α appartiennent à l'anneau $\mathfrak{o}$; plus généralement, si A est un $\mathfrak{o}$-module (par exemple un anneau de séries formelles par rapport aux x_i et à certaines autres indéterminées), un ,,opérateur différentiel'' au sens large, à valeurs dans A, sera une somme $\sum_\alpha c_\alpha D_\alpha$, où les coefficients c_α appartiennent à A et sont tels que la somme $\sum_\alpha c_\alpha D_\alpha f$ ait un sens pour toute série formelle $f \in \mathfrak{o}$; on introduira en outre ici l'opérateur différentiel impropre $D_0 = I$ (identité). Des exemples importants de tels opérateurs différentiels au sens large sont donnés par les *opérateurs de translation à gauche et à droite* dans G

$$L_{\mathbf{y}} f = f(\varphi(\mathbf{y}, \mathbf{x})), \qquad R_{\mathbf{y}} f = f(\varphi(\mathbf{x}, \mathbf{y})) \tag{13}$$

A étant ici l'anneau des séries formelles à $2n$ indéterminées x_i, y_i, comme il résulte aisément de la formule de Taylor et de (12). Cela étant, un opérateur différentiel $D = \sum_\alpha c_\alpha D_\alpha$ est dit *invariant à gauche* par G (ou simplement *in-*

variant) s'il satisfait à la relation

(14) $$L_{\mathbf{y}} D = D L_{\mathbf{y}}.$$

Un tel opérateur est entièrement déterminé par sa „valeur à l'origine'' $D(\mathbf{e})$, où $D(\mathbf{e})f$ est la série formelle Df où on substitue $\mathbf{e}$ à $\mathbf{x}$, c'est-à-dire $\sum_\alpha c_\alpha(\mathbf{e}) D_\alpha(\mathbf{e}) f$; il résulte en effet de (14) que l'on a

$$(Df)(\mathbf{y}) = D(\mathbf{e})(L_{\mathbf{y}} f).$$

En outre, les $a_\alpha = c_\alpha(\mathbf{e})$ peuvent être pris arbitrairement dans le module A, pourvu que la somme $\sum_\alpha a_\alpha D_\alpha(\mathbf{e})$ ait un sens. Parmi les opérateurs invariants obtenus ainsi, nous considérerons en particulier ceux dont la „valeur à l'origine'' est celle d'une semi-dérivation D_{hi}, opérateurs que nous désignerons par X_{hi} $(1 \leqq i \leqq n, 0 \leqq h < +\infty)$; ces opérateurs sont des *semi-dérivations de hauteur* h, qui en général ne commutent plus entre eux; pour tout indice $\alpha = (\alpha_1, \ldots, \alpha_n)$ de hauteur r, on posera

$$X_\alpha = X_{01}^{\lambda_{01}} X_{02}^{\lambda_{02}} \ldots X_{0n}^{\lambda_{0n}} X_{11}^{\lambda_{11}} \ldots X_{1n}^{\lambda_{1n}} \ldots X_{r1}^{\lambda_{r1}} \ldots X_{rn}^{\lambda_{rn}}$$

(les notations étant celles du no 2); on notera qu'en général, la „valeur à l'origine'' $X_\alpha(\mathbf{e})$ est distincte de $D_\alpha(\mathbf{e})$ si X_α n'est pas l'un des X_{hi}. Cela étant, on montre que *le* $\mathfrak{o}$-*module* $\mathfrak{S}_r$ *a une base formée des* X_α *où* $h(\alpha) < r$, et que *le* $\mathfrak{o}_r$-*module* $\mathfrak{D}_r$ *est somme directe de* $\mathfrak{S}_r$ *et du* $\mathfrak{o}_r$-*module ayant pour base les* X_{ri} $(1 \leqq i \leqq n)$. En outre pour qu'un opérateur différentiel (au sens large) $\sum_\alpha c_\alpha X_\alpha$, où les c_α sont des séries formelles par rapport aux x_i (et éventuellement à d'autres indéterminées) soit *invariant*, il faut et il suffit que les c_α *ne contiennent pas les* x_i. En particulier, considérons l'ensemble $\mathfrak{G}$ des opérateurs différentiels (au sens strict) invariants $\sum_\alpha c_\alpha D_\alpha$, où les c_α sont dans $\mathfrak{o}$; cet ensemble est évidemment une *algèbre associative sur* K, que nous appellerons l'*hyperalgèbre de Lie* de G; elle admet pour *base sur* K l'ensemble des X_α. De façon plus précise, $\mathfrak{g}_r = \mathfrak{D}_r \cap \mathfrak{g}$ est une *p-algèbre de Lie*[3]) (pour l'opération $[X, Y] = XY\text{-}YX$) formée des *semi-dérivations invariantes de hauteur* r, dans laquelle $\mathfrak{s}_r = \mathfrak{S}_r \cap \mathfrak{G}$ est un *idéal*, qui est aussi une *sous-algèbre associative* de $\mathfrak{G}$, formée des *semi-dérivations spéciales invariantes* (*de hauteur* r), et qu'on peut identifier à l'*algèbre enveloppante* de l'algèbre de Lie $\mathfrak{g}_{r-1}$; les X_α de hauteur $h(\alpha) < r$ forment une *base de* $\mathfrak{s}_r$ *sur* K, et $\mathfrak{g}_r$ est somme directe de $\mathfrak{s}_r$ et du sous-espace vectoriel ayant pour base (sur K) les X_{ri} $(1 \leqq i \leqq n)$; $\mathfrak{g}_0$ n'est autre que l'*algèbre de Lie* (au sens usuel) du groupe G, formée des *dérivations invariantes*.

La condition d'associativité (11) peut s'écrire $L_{\mathbf{y}} R_{\mathbf{z}} = R_{\mathbf{z}} L_{\mathbf{y}}$, et signifie donc que $R_{\mathbf{z}}$ est un opérateur différentiel *invariant à gauche*. On en conclut que, pour

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[3]) *„Restricted Lie algebra of characteristic p''* dans la terminologie de Jacobson [8].

toute série formelle $f \in \mathfrak{o}$, on a la „formule de Taylor" dans le groupe G

$$f(\varphi(\mathbf{x}, \mathbf{y})) = \sum_{\alpha} P_{\alpha}(\mathbf{y}) X_{\alpha} f \tag{15}$$

où les P_{α} sont des séries formelles par rapport aux y_i seuls, dont le terme de plus bas degré total est de degré $|\alpha|$ ($P_0 = 1$). On démontre que toute relation $\sum_{\alpha} c_{\alpha} P_{\alpha}(\mathbf{x}) = 0$, où les c_{α} ne contiennent pas les x_i, entraîne $c_{\alpha} = 0$ pour tout α („indépendance linéaire" au sens fort des P_{α}).

Pour tout couple d'indices α, β, on peut écrire

$$X_{\alpha} X_{\beta} = \sum_{\gamma} c_{\alpha\beta\gamma} X_{\gamma} \tag{16}$$

où les *constantes de structure* $c_{\alpha\beta\gamma} \in K$ définissent complètement la structure de l'algèbre associative $\mathfrak{G}$.

Il est clair que ces constantes ne sont pas indépendantes, et sont entièrement déterminées lorsqu'on connaît l'expression: 1⁰ des crochets $[X_{hi}, X_{kj}]$ ($k \leqq h$) comme combinaisons linéaires des X_{α} appartenant à $\mathfrak{g}_h$ si $h = k$, à $\mathfrak{s}_h$ si $k < h$; 2⁰ des puissances X_{hi}^p comme combinaisons linéaires des X_{α} appartenant à $\mathfrak{g}_h$.

Ces constantes sont liées à la loi de groupe par les deux formules fondamentales

$$P_{\gamma}(\varphi(\mathbf{x}, \mathbf{y})) = \sum_{\alpha,\beta} c_{\alpha\beta\gamma} P_{\alpha}(\mathbf{x}) P_{\beta}(\mathbf{y}) \tag{17}$$

$$X_{\beta} P_{\alpha} = \sum_{\gamma} c_{\gamma\beta\alpha} P_{\gamma}. \tag{18}$$

On notera aussi que l'on a

$$X_{\alpha}(\mathbf{e}) P_{\beta} = \delta_{\alpha\beta} \quad \text{(indice de Kronecker)} \tag{19}$$

et en particulier que, dans la série formelle $P_{\alpha}(\mathbf{x})$, il n'y a pas de terme en $x_j^{p^h}$, sauf si $\alpha = p^h \varepsilon_j$, en désignant par ε_j l'indice $(0, \ldots, 0, 1, 0, \ldots 0)$ où le 1 est à la j-ème place.

On montre que *pour que G soit abélien il faut et il suffit que l'algèbre* $\mathfrak{G}$ *soit commutative*. La structure de $\mathfrak{G}$ est alors entièrement déterminée par l'expression des X_{hi}^p. Par exemple pour le groupe additif ($n = 1$) de loi

$$(x, y) \to x + y$$

on a $X_h = D_h$, $X_h^p = 0$ pour tout h (on supprime ici l'indice $i = 1$); pour le groupe multiplicatif ($n = 1$), de loi

$$(x, y) \to x + y + xy$$

on a $X_h = (1 + x^{p^h}) D_h$, et $X_h^p = X_h$ pour tout h. Pour le „second groupe additif de Witt" de dimension $n = 2$, défini par

$$\begin{cases} \varphi_1(\mathbf{x}, \mathbf{y}) = x_1 + y_1 \\ \varphi_2(\mathbf{x}, \mathbf{y}) = x_2 + y_2 - \sum_{k=1}^{p-1} (-1)^k x_1^k y_1^{p-k}/k \end{cases}$$

on a les relations

$$X_{h1}^p = X_{h2}, \qquad X_{h2}^p = 0.$$

On notera que l'*algèbre de Lie* $\mathfrak{g}_0$ peut par contre être abélienne sans que le groupe G soit abélien [4]).

Il n'est peut-être pas sans intérêt de remarquer que la théorie des groupes de Lie formels sur un corps de caractéristique 0 peut être présentée de façon exactement analogue, mais que dans ce cas l'algèbre $\mathfrak{G}$ est *engendrée* par l'algèbre de Lie $\mathfrak{g}_0$, d'où le caractère beaucoup plus simple de la théorie.

3. *Homomorphismes et sous-groupes*. Soient G et $\overline{G}$ deux groupes de Lie formels, de dimensions n et m respectivement, et soient $\varphi(\mathbf{x}, \mathbf{y})$, $\psi(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ leurs lois de composition respectives. Par définition, un „*homomorphisme*" de G dans $\overline{G}$ est un système $\mathbf{u} = (u_1, \ldots, u_m)$ de m séries formelles en $\mathbf{x} = (x_1, \ldots, x_n)$ telles que l'on ait les m relations $\psi(\mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{y})) = \mathbf{u}(\varphi(\mathbf{x}, \mathbf{y}))$. A partir d'un tel homomorphisme, on définit un *homomorphisme* $\mathbf{u}'$ *de l'hyperalgèbre de G dans celle de* $\overline{G}$, dit *dérivé* de $\mathbf{u}$, de la façon suivante: à toute semi-dérivation invariante $X \in \mathfrak{G}$, on fait correspondre la semi-dérivation invariante $\overline{X} = \mathbf{u}'(X)$ dont la „valeur à l'origine" est, pour une série formelle $\bar{f}$ par rapport au système $\overline{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_m)$ d'indéterminées, le terme constant de la série formelle $X(\bar{f}_o \mathbf{u})$. On démontre que l'on a

$$(20) \qquad \bar{f}(\mathbf{u}(\mathbf{x})) = \sum_\alpha P_\alpha(\mathbf{x}) \,.\, (\mathbf{u}'(X_\alpha)\,(\bar{\mathbf{e}}))\bar{f}$$

et par suite que la donnée de $\mathbf{u}'$ *détermine complètement* l'homomorphisme $\mathbf{u}$; en outre, si

$$(21) \qquad \mathbf{u}'(X_\alpha) = \sum_\lambda a_{\alpha\lambda} \overline{X}_\lambda$$

on a

$$(22) \qquad \overline{P}_\lambda(\mathbf{u}(\mathbf{x})) = \sum_\alpha a_{\alpha\lambda} P_\alpha(\mathbf{x}).$$

Il est immédiat, par ailleurs, que $\mathbf{u}'$ transforme une semi-dérivation invariante

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[4]) Par suite d'une erreur de transcription, l'exemple de groupe pour lequel se présente ce phénomène, dû à Chevalley, a été remplacé dans [4, no 11] par un groupe dont l'algèbre de Lie n'est pas abélienne; ce passage doit être rectifié comme suit. La loi de groupe est

$$(x_1, x_2)\ (y_1, y_2) = (x_1 + y_1 + x_1 y_1,\ x_2 + y_2 + x_1^p y_2)$$

ce qui donne

$$X_{01} = (1 + x_1) D_{01}, \quad X_{02} = (1 + x_1^p) D_{02}$$
$$X_{11} = (1 + x_1^p) D_{11}, \quad X_{12} = (1 + x_1^{p^2}) D_{12}$$

s'où $[X_{01}, X_{02}] = 0$, mais $[X_{11}, X_{02}] = X_{02}$.

de hauteur r (resp. une semi-dérivation spéciale invariante de hauteur r) en une semi-dérivation de même nature.

Toutefois, on se heurte ici au premier des problèmes non résolus, et qui n'ont pas d'analogue dans la théorie classique; toutes les conditions nécessaires précédantes, même jointes à d'autres conditions qui seront indiquées plus loin, *ne sont pas suffisantes* pour qu'un homomorphisme de $\mathfrak{G}$ dans $\overline{\mathfrak{G}}$ soit de la forme $\mathbf{u}'$. On ne connaît même pas de telles conditions lorsqu'on suppose que l'application considérée est un *isomorphisme*; ce qui, en particulier, interdit pour le moment de caractériser les „changements de variables" par leur effet sur l'hyperalgèbre du groupe.

Les formules précédentes permettent cependant de définir, par analogie avec le cas classique, la notion de *„loi de composition canonique"*. On dit que la loi φ est *canonique* pour le groupe G, si les „premiers termes" P_{ε_i} dans les développements (15) sont tels que

$$P_{\varepsilon_i}(\mathbf{x}) = x_i \quad (1 \leqq i \leqq n). \tag{23}$$

En général, une loi φ n'est pas canonique; le problème des *„variables canoniques"* consiste à trouver un changement de variables $\bar{\mathbf{x}} = \mathbf{u}(\mathbf{x})$ du type $\bar{x}_i = x_i + \ldots$ $(1 \leqq i \leqq n)$ les termes non écrits étant de degré > 1, tel que la loi de groupe pour ces nouvelles variables soit canonique. Dans le cas classique, le problème est immédiatement résolu, en prenant $\bar{x}_i = P_{\varepsilon_i}(\mathbf{x})$; j'ai d'abord cru par erreur que ce résultat était encore valable en caractéristique $p > 0$ [4, n⁰ 16]; en fait, il existe bien un changement de variables qui donne une loi canonique, mais on ne peut l'obtenir qu'en *itérant* indéfiniment le changement de variables précédent [5]. Même pour les groupes les plus simples, la détermination explicite des variables canoniques est un problème qui paraît ardu. Pour le groupe additif $(n = 1)$ et le second groupe additif de Witt, la loi est canonique; il en est de même pour le groupe multiplicatif $(n = 1)$ lorsque $p = 2$, mais cela n'est plus exact dès que $p > 2$; pour $p = 3$, le changement de variables canoniques est $\bar{x} = (x - x^2)/(1 + x)$, et j'ignore l'expression de ce changement de variables pour $p > 3$.

On notera ici que, par passage aux „variables canoniques", les constantes de structure de l'*algèbre de Lie* $\mathfrak{g}_0$ restent inchangées, mais *non* les $c_{\alpha\beta\gamma}$ dont les indices sont de hauteur $\geqq 1$.

L'intérêt de la notion de loi canonique est qu'elle permet de formuler des théorèmes qui „remontent" de l'hyperalgèbre au groupe. En premier lieu, on a la „moitié" du troisième théorème de Lie, celle qui affirme l'*unicité* du groupe; de façon précise, *il ne peut exister qu'une seule loi canonique correspondant à une hyperalgèbre de Lie donnée* [5]. Bien entendu, cela laisse intact le problème fondamental de la théorie, l'*existence* d'une telle loi; on sait, par des exemples, que les $c_{\alpha\beta\gamma}$ doivent vérifier certaines conditions pour que le problème admette une solution, mais il semble difficile de trouver le système complet de ces conditions. On ne sait même pas attaquer le problème dans les cas les plus simples: par exemple, on ignore si un groupe de dimension 1 est nécessairement abélien, et on ne sait pas davantage énumérer tous les types de groupes abéliens de dimension 1. [4bis])

La notion de „sous-groupe" qui paraît la plus naturelle dans la théorie (et qui correspond à la notion classique dans les groupes de Lie réels ou complexes, ainsi que dans les groupes de Lie algébriques sur un corps quelconque) est celle d'une „variété" de dimension $m < n$ ayant un point simple à l'origine et stable pour la multiplication. Cela revient à la définition suivante: *après au besoin un changement de variables*, le sous-groupe H de G doit s'obtenir en *annulant* les $n - m$ dernières coordonnées x_i de $\mathbf{x}$; le fait que H est stable signifie que l'on doit avoir identiquement

$$(24) \qquad \varphi_i(x_1, \ldots, x_m, 0, \ldots, 0, y_1, \ldots y_m, 0, \ldots, 0) = 0, \qquad m + 1 \leqq i \leqq n.$$

On en conclut aisément que les X_α tels que $\alpha = (\alpha_1, \ldots, \alpha_m, 0, \ldots, 0)$ forment une *sous-algèbre* de $\mathfrak{G}$. En outre, on peut montrer que les formules correspondant à (24) sont encore vraies lorsqu'on passe en variables canoniques pour la loi de G. Mais ici, on a une *réciproque*: si les X_α tels que $\alpha = (\alpha_1, \ldots, \alpha_m, 0, \ldots, 0)$ forment une sous-algèbre de G et si la loi de G est *canonique*, alors *les relations* (24) *sont vérifiées*; en d'autres termes, à la sous-algèbre de $\mathfrak{G}$ considérée correspond un sous-groupe [5]. Malheureusement, l'intérêt de ce résultat est fortement limité par le fait qu'on ignore quels sont les „changements de base" permis dans $\mathfrak{G}$ (i.e., provenant d'un changement de variables dans G).

Lorsqu'on se tourne vers les relations entre homomorphismes et sous-groupes, on rencontre des phénomènes tout à fait nouveaux. Par exemple, soit $G^{(1)}$ le groupe dont la loi est obtenue en élevant à la puissance p-ème tous les coefficients des φ_i; il est immédiat que si on pose $p_i(\mathbf{x}) = x_i^p$, $\mathbf{p} = (p_1, \ldots, p_n)$ est un homomorphisme „canonique" de G *sur* $G^{(1)}$ (qu'il convient d'appeler „l'homomorphisme de Frobenius"), qui *n'est pas un isomorphisme*, mais dont le „noyau" est *réduit à* **e**. L'homomorphisme „dérivé" $\mathbf{p}'$ est tel que

$$(25) \qquad \begin{cases} \mathbf{p}'(X_\alpha) = 0 \text{ si } \alpha \text{ n'est pas de la forme } p\beta = (p\beta_1, \ldots, p\beta_n), \\ \mathbf{p}'(X_{p\alpha}) = X_\alpha. \end{cases}$$

[4bis]) (*Note ajoutée pendant la correction des épreuves*). Depuis l'èpoque où a été prononcée cette conférence, je suis parvenu à déterminer la structure des groupes abéliens de dimension quelconque, sur un corps quelconque de caractéristique p (voir deux articles dans *American Journal of Mathematics* et un untre dans *Math. Zeitschrift*). D'autre part, M. Lazard a prouvé que tout groupe formel de dimension 1 est abélien (*C. R. Acad. Sci- Paris*, 239 (1954), p. 942).

De l'existence d'un tel homomorphisme, on déduit que l'on a

$$P_{p\alpha}(\mathbf{x}) = \left(P_\alpha(\mathbf{x})\right)^p; \tag{26}$$

en outre, l'algèbre de Lie $\mathfrak{g}_r/\mathfrak{z}_r$ est *isomorphe* à l'algèbre de Lie $(\mathfrak{g}_0)^{(r)}$, obtenue en élévant à la puissance p^r les constantes de structure de $\mathfrak{g}_0$. De même, les constantes $a_{\alpha\lambda}$ de la formule (21) sont telles que

$$\begin{cases} a_{\alpha,p\lambda} = 0 \text{ si } \alpha \text{ n'est pas de la forme } p\beta \\ a_{p\alpha,p\lambda} = a^p_{\alpha\lambda}. \end{cases} \tag{27}$$

Supposons maintenant le corps K *parfait*. On peut alors montrer [4] que, si $\mathbf{u}$ est un homomorphisme de G dans $\overline{G}$, on peut faire des changements de variables dans ces deux groupes, de sorte que $\mathbf{u}$ soit donné par les relations

$$\begin{cases} u_i(\mathbf{x}) = x_i^{p^k} & \text{pour } r_0 + \ldots + r_{k-1} + 1 \leqq i \leqq r_0 + \ldots + r_k \quad (0 \leqq k \leqq t) \\ u_i(\mathbf{x}) = 0 & \text{pour } i > \varrho = r_0 + r_1 + \ldots + r_t. \end{cases} \tag{28}$$

Les nombres r_k (qui peuvent être nuls) sont indépendants de tout changement de variables; en annulant les ϱ premières coordonnées dans G, on a un sous-groupe, le *noyau* H de $\mathbf{u}$, et en annulant les $m - \varrho$ dernières coordonnées dans $\overline{G}$ (m dimension de $\overline{G}$) un sous-groupe L, l'*image* de $\mathbf{u}$. Les X_{0i} tels que $i > r_0 + \ldots + r_k$ engendrent un *p-idéal* $\mathfrak{a}_k$ de l'algèbre de Lie $\mathfrak{g}_0$ de G; l'idéal $\mathfrak{a}_0$ est le *noyau* de la restriction de $\mathbf{u}'$ à $\mathfrak{g}_0$, mais ce n'est ici que le premier élément d'une suite d'idéaux

$$\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \ldots \subset \mathfrak{a}_t$$

dont le *dernier* seulement est l'*algèbre de Lie* de H. De même, dans l'algèbre de Lie $\overline{\mathfrak{g}}_0$ de $\overline{G}$, les $\overline{X}_{0i}$ tels que $i \leqq r_0 + \ldots + r_k$ engendrent une *sous-algèbre* $\mathfrak{b}_k$, et on a ainsi une suite croissante de sous-algèbres

$$\mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \ldots \subset \mathfrak{b}_t$$

dont chacune est un *p-idéal* de la suivante; $\mathfrak{b}_t$ est l'algèbre de Lie de L, $\mathfrak{b}_0$ est isomorphe à $\mathfrak{g}_0/\mathfrak{a}_0$ et $\mathfrak{b}_k/\mathfrak{b}_{k-1}$ à l'algèbre de Lie $(\mathfrak{a}_{k-1}/\mathfrak{a}_k)^{(k)}$.

Une conséquence facile de ces résultats est que si l'algèbre de Lie $\mathfrak{g}_0$ est *simple*, le groupe G est *simple* au sens suivant: il n'existe pas d'homomorphisme

[5]) Pour les groupes de Lie algébriques, cette notion de „simplicité" est plus stricte que la notion „ naturelle" dans cette théorie, c'est-à-dire l' inexistence d'homomorphismes *rationnels* non triviaux; on pourrait en effet concevoir qu'il existe des homomorphismes au sens „local" considéré ici, mais que ces homomorphismes ne proviennent pas d'homomorphismes rationnels. Il serait donc concevable que des groupes algébriques „simples" ne soient pas simples au sens de la définition donnée ici; cette éventualité me paraît toutefois peu probable, même lorsque l'algèbre de Lie du groupe algébrique considéré n'est pas simple, comme dans l'exemple qui suit. Il y a encore là un problème à élucider, qui est peut-être d'ailleurs moins inabordable que les autres.

de G autre que des isomorphismes, composés avec des homomorphismes de Frobenius itérés [5]). La réciproque est inexacte; on sait par exemple [8] que pour $p = 2$, l'algèbre de Lie du groupe unimodulaire $SL_2(K)$ est *résoluble*, mais on peut montrer que ce groupe est cependant *simple* au sens précédent, en considérant la „seconde" algèbre de Lie $\mathfrak{g}_1$ de ce groupe.

4. *Conclusion.* Les résultats que je viens d'esquisser me semblent permettre de conclure que s'il doit jamais exister une „théorie de Lie" pour les groupes formels en caractéristique $p > 0$, elle ne peut guère être fondée d'autre façon. Mais pratiquement tout reste à faire pour qu'on puisse vraiment parler d'une telle théorie, et peut-être un jour l'appliquer à d'autres branches des mathématiques. Je voudrais en terminant indiquer un mode d'attaque possible qui „fractionne" en quelque sorte les difficultés considérables que semblent présenter les problèmes dont j'ai parlé. Il s'agit d'une idée qui est substantiellement due à M. Lazard: désignons, dans l'anneau $\mathfrak{o}$, par $\mathfrak{u}_r$ l'idéal engendré par les puissances $x_j^{p^r}$ des indéterminées, et au lieu des relations d'associativité (11), considérons ces relations *modulo* $\mathfrak{u}_r$ (on peut évidemment supposer alors que les φ_i sont des polynômes de degré $< p^r$ par rapport à chaque indéterminée); on peut dire, avec M. Lazard, qu'on définit ainsi un *bourgeon de groupe formel de hauteur* $r - 1$. On vérifie alors aisément qu'on peut développer *toute* la théorie précédente, en calculant constamment „modulo $\mathfrak{u}_r$" sur les séries formelles, et en ne considérant que les semi-dérivations *de hauteur* $< r$. En particulier, on voit exactement à quoi correspondent les *p-algèbres de Lie*: ce ne sont pas aux groupes de Lie formels, mais bien aux *bourgeons de hauteur* 0. On peut espérer arriver sans trop de peine à donner une théorie satisfaisante de ces bourgeons, puis essayer de „monter" dans l'échelle des bourgeons successifs, peut-être en introduisant ici des idées analogues aux „obstructions" de la Topologie algébrique? Pour le moment, il est bien certain qu'il nous manque avant tout, dans ce nouveau „calcul différentiel", la technique du maniement des formules, qu'une familiarité de 3 siècles nous a permis d'acquérir en calcul différentiel classique; c'est sans doute seulement lorsque nous en aurons acquis l'habitude que nous pourrons, suivant l'expression de Galois, „sauter à pieds joints par dessus les calculs" et parvenir peut-être ainsi aux résultats décisifs.

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SOME ASPECTS OF FUNCTIONAL ANALYSIS AND ALGEBRA

I. GELFAND.

1. Introduction
2. Representations of Groups
3. Generalized Functions
4. Generalized Functions and Representations of Groups
5. Differential Operators
6. Analysis in Functional Spaces

1. *Introduction.*

Functional analysis, which has become an independent branch of mathematics at the beginning of this century, occupies one of the central positions in contemporary mathematics. This is explained on the one hand by the fact that functional analysis made use of the main classical methods of analysis and algebra, and on the other hand by the rôle which functional analysis plays in contemporary physical science, especially in quantum physics.

The study of mathematical problems connected with quantum mechanics was a turning point in the development of functional anlysis itself, and at the present time to a great extent it determines the main paths for its development.

It can be said without exaggeration that contemporary functional analysis represents a new and serious step in the development of mathematics.

In the last few years a number of new branches have arisen in functional analysis. Although relatively recently (about 20 to 30 years ago) functional analysis was thought of mainly as the theory of linear normed spaces, at the present time that theory, which is important and, roughly speaking, is completed, cannot even be considered as one of the basic branches of functional analysis.

In general, functional analysis is still far from being completed, but the basic tendencies in its development are considerably clearer now than they were some 15 to 20 years ago.

It is of course impossible to discuss in this paper all the basic questions of functional analysis. Therefore, we will limit ourselves to the consideration of a few selected problems. Although at a first glance the problems considered differ in character, common to all of them is, for instance, the fact that the development of each of these branches is closely connected with and is stimulated by the development of quantum mechanics and the quantum field theory. Perhaps not all the problems which will be discussed lie on the main path of development of functional analysis. However we hope that they will at least aid in determing that path.

2. *Representations of groups.*

One of the main branches of functional analysis in which the cooperation of analytic and algebraic methods typical for that field of mathematics, is clearly seen, is the theory of representations of groups.

The theory of representations developed as a branch of algebra in the study of representations of finite groups. The connection between the theory of representations and analysis was discovered after the study of representations of compact topological groups had begun. From that time on the theory of representations, both in the character of the problems with which it was concerned and in its methods, developed essentially as a branch of functional analysis.

It is sufficient to point out, for instance, the connection of the theory of representations with the almost periodic functions, the spherical functions, or the generalized spherical functions, which arise in the representations of the group of rotations of threedimensional space [1], [2].

The development of the theory of representations of groups was stimulated first by the quantum theory and later by the quantum field theory. In particular, from these theories it became clear that the theory of representations is one of the basic mathematical methods for the study of symmetries (invariancies) occurring in physics.

The analytic character of that theory is, naturally, most clearly seen in the representations of Lie groups. We shall limit ourselves to the best known case of semi-simple Lie groups [5], [7], which includes such important groups as, for example, the group of unimodular matrices, the orthogonal group, the Lorentz group, etc.

An important example of representation is the so-called regular representation. This representation acts in the space of all square integrable functions on a group and consists in making correspond to each element h the linear transformation T_h given by the formula: $T_h\varphi(g) = \varphi(gh)$. Decomposing this representation into irreducible ones, in the case of a compact group one obtains all the irreducible representations of the group under consideration. In the case of semi-simple Lie groups the decomposition of the regular representation gives rise to an important class of irreducible representations, the so-called basic series of irreducible representations. However (and in this respect locally compact groups differ essentially from compact ones), not all the irreducible representations of the semi-simple Lie groups are contained in this class.

The reason for the appearance of the so-called supplementary series may be illustrated by a simple example. Let us consider two similar groups: the group of the rotations of a sphere, i.e. the group of the motions of a surface of constant positive curvature and the group of the motions of a surface of constant negative curvature (Lobachevsky planes).

All irreducible representations of rotation groups are given by spherical functions and can be obtained as follows. Let us consider a Laplace operator on the surface of a sphere (the second differential Beltrami parameter). The eigenfunctions of this operator corresponding to a given eigenvalue (i.e. spherical functions) form the space in which the required irreducible unitary representations act. Let us now consider the Laplace operator in the space of the square integrable functions on a Lobachevsky plane. Its spectrum consists of all numbers from $-\infty$ to 0, and spectral expansion in this case also gives irreducible representations, more explicitely the main series of representations of the group of the motions of the Lobachevsky plane. All the eigenfunctions decrease in this case as $e^{-\sqrt{k}r}$, where $-k$ is the curvature of the Lobachevsky plane. However, if we should only require that the eigenfunctions considered be bounded, then more eigenfunctions might arise, corresponding to the supplementary part of the spectrum — the segment from 0 to k. Such a difference between the case of square integrability and the case of boundedness is connected with the fact that the circumference of a circle on a Lobachevsky plane increases with the radius r as $e^{-\sqrt{k}r}$, and, therefore, the class of square integrable functions is much narrower than the class of bounded functions. These supplementary eigenfunctions do not occur in the expansion of square integrable functions, and give rise to a supplementary series of irreducible representations. More exactly, eigenfunctions depending only on the radius are positive definite functions, and with their aid supplementary series of representations may be realized. This example shows that for locally compact homogeneous spaces, in contrast to compact ones, the collection of those invariant elementary positive definite functions, which are required for the Plancherel expansion theorem (the expansion of square integrable functions into an integral analogous to the Fourier integral) is essentially different from the collection of all invariant elementary positive definite functions.

Apparently, supplementary series of representations appear in those cases where there is such a neighbourhood U of the unity in the group, that the measure of U^n increases as a geometric progression.

Before exposing the results related to the representations of the basic series it is desirable to make a few remarks on the so-called dimensions of functional spaces. As is well known, spaces of square integrable functions of any number of variables are isomorphic. However, as far as we know, in concrete questions of analysis it never happens that the space of „all'' the functions of a certain number of variables is effectively transformed one-to-one into the space of „all'' the functions of another number of variables. It may be said without exaggeration that for analysis the isomorphism of all Hilbert spaces has no more importance than, let us say, for algebraic geometry the fact that a curve and a surface are sets of equal power.

We shall attempt to determine in a given space of functions the number of variables occurring in the functions of the space where the representations of the semi-simple Lie groups are defined. It is possible here to make a comparison with fundamental Burnside theorems on finite groups, according to which the number of nonequivalent representations of a finite group G is equal to the number of classes of conjugated elements in a given group and the sum of the squares of the dimensions of all (nonequivalent) representations is equal to the order of the group. The proof of these theorems is based on the decomposition of regular representations into irreducible ones. Similar facts hold for representations of the Lie groups if the dimensions of functional spaces are conceived in the corresponding way. We shall illustrate this taking as an example the group of complex unimodular matrices with determinant equal to 1.

An element of this group is determined by the values of $n^2 - 1$ complex parameters. A class of conjugated elements is in general determined by the values of $n - 1$ parameters (it consists of all the matrices with a given set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1 \lambda_2 \ldots \lambda_n = 1$). The analogue of the first Burnside theorem learns that the representations of the groups mentioned above are determined by $n - 1$ complex (i.e. $2n - 2$ real) parameters. And the representations of the basic series are, in fact, determined by the values of $2n-2$ parameters ($n - 1$ integral ones and $n - 1$ real ones).

Let us determine now in a functional space the number of variables occurring in the functions of the space on which the irreducible representations are defined. The regular representation acts in the space of the functions on the group, i.e. in the space of functions of $n^2 - 1$ complex parameters; it is decomposed into irreducible representations. Let l represent the dimension of the functional space on which an irreducible representation of a given group acts. The analogue of Burnside's second theorem in this case will be expressed by the relationship $n^2 - 1 = n - 1 + 2l$, from which it follows that $l = \frac{n(n-1)}{2}$, i.e. that each of the irreducible representations is realized in a space of functions depending on $\frac{n(n-1)}{2}$ parameters. This is what really occurs. It is highly improbable that there exists a realization of these representations in a space of „all'' the functions depending on a different number of variables, which in some way is natural. It would be interesting to develop a theory in which these considerations would become exact.

Let us now return to the description of the irreducible representations of semi-simple Lie groups. We shall limit ourselves to the case of the group of complex unimodular matrices, as the picture in that case is typical for the general case. The representation of the group of second order matrices operates in the space of functions of a complex variable z subject to fractional linear transformations $\frac{\alpha z+\beta}{\gamma z+\delta}$ given by the second order matrices $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$. In constructing the representations of the group of matrices of the nth order it is important, in the first place, to define correctly the conception of fractional linear transformation for this case. The conventional definition of fractional linear transformations is unsatisfactory, since it would lead to representations in the space of functions of $n-1$ variables and not in the space of functions of $\frac{n(n-1)}{2}$ variables, as should occur in accordance with what has been said above. The generalization of the fractional linear transformations necessary in this case is as follows: let us take a point in the n-dimensional projective space, a straight line passing through this point, a plane passing through this straight line, etc. We shall call such a combination a *generalized linear element*. It is not difficult to check that in the $(n-1)$-dimensional projective space the generalized linear element is determined by the values of $\frac{n(n-1)}{2}$ parameters. The representations of the group under consideration are obtained as follows: a measure is introduced in some way in the space of the linear elements, and in the space of the functions of "z" which are square integrable with this measure the operator T_g corresponding to the element "g" is determined by means of the formula: $T_g f(z) = f(zg)\,\alpha(zg)$, where "$zg$" is the image of the linear element "z" in the projective transformation with the matrix "g", and $\alpha(zg)$ is a fixed function, specific for each representation. The function $\alpha(zg)$ is determined from the requirement that $Tg_1\, Tg_2$ should be equal to $Tg_1 g_2$ and the requirement that the representation should be unitary. The basic series differ from the supplementary one in the manner of choosing the scalar product.

It should be noted that the function α is determined apart from equivalences by the character of the group of diagonal matrices, and for an arbitrary semi-simple group, by the character of a similar (Cartan) subgroup. This character itself is a generalization of Cartan's highest weight for the case of infinite-dimensional representation.

Considering the space of functions dependent on linear elements "with omissions" (for example now a point, then a plane, etc.) we obtain a degenerated series of representations. Let us assume $z_{n_1 n_2 \ldots n_k}$ to be the respective "incomplete" linear element, $n_1, n_2, \ldots, n_k$ being the dimensions of the manifolds comprising the given linear element. Each set of dimensions has as degenerated series of its own.

The spaces of linear elements "z" and $z_{n_1 n_2 \ldots n_k}$, as well as the respective representation series may be constructed in a similar way for any semi-simple Lie group. It is possible to establish which of these representations are equivalent.

Continuing to develop the procedures expounded in [5], [6] and in [70], M. A. Naimark proved that the representations described here are all representations of classical complex groups.

The problem of the equivalence of representations can be solved using the classical theory of characters which, and this was a surprise at the time of discovery, is fully applicable to the case of infinite-dimensional representations. It is interesting to note that the formulae of the theory of characters are, in this case, not at all more complicated and, in some cases, are simpler than the formulae of finite-dimensional representations. The existence of characters for complex semi-simple groups was proved by direct calculation on the basis of the apparent type of representation for these groups which was already known at that time [5]. Harish Chandra [7] and Godement [13] proved remarkable theorems about the existence of characters, for any semi-simple group, complex or not.

Interesting new questions arise in studying the representations of real semi-simple Lie groups. When considering, for example, any of the real forms of the group of complex unimodular nth order matrices, the same calculation of the number of parameters given above shows that irreducible unitary representations should be realized in a space of functions depending on $\frac{n(n-1)}{2}$ real parameters $\left(\text{i.e. in a functional space of real dimension } \frac{n(n-1)}{2}\right)$. In order to obtain actually these representations, let us again consider the space "Z" of linear elements which correspond to a complex group. The space "Z" decomposes into parts, which are transitive with respect to the real form. One of these parts is a manifold of real dimension $\frac{n(n-1)}{2}$. On this manifold a functional space is constructed, in which representations of the real group are assigned in a fashion similar to the one described above for a complex group.

For the transitive components whose real dimension is $\frac{n(n-1)}{2}+r$, where $r>0$, the representation is realized in a space of functions which are arbitrary functions of $\frac{n(n-1)}{2}-r$ real parameters and depend analytically on r complex parameters. The transitive manifold of the highest possible dimension $\left(\frac{n(n-1)}{2}\right.$ complex variables, i.e. $n(n-1)$ real variables$\Big)$ is of special interest. The corresponding irreducible representations of the real group are constructed in a space of functions which are analytic in all $\frac{n(n-1)}{2}$ parameters. A similar construction may be used for degenerated series as well. In doing so one should consider the decomposition of the respective spaces of "incomplete linear elements" into transitive components. In this manner we can describe, for example, the representations of the real unimodular group [11] previously described by Bargman [1] for $n=2$.

The analyticity with respect to some of the variables becomes manifest in the theory of representations in the following manner. Every transitive homogeneous space is a space of the classes of conjugate elements with respect to some subgroup. Functions on a homogeneous space may be considered as functions on a group G, which are constant on classes of conjugate elements with respect to some subgroup K. In passing from the group G to its real form the intersection of this real form with the subgroup K or with a conjugate of it may be partially imaginary. The requirement of constancy on the classes of conjugate elements is replaced by the requirement of analyticity, just as constancy on straight lines parallel to the straight line $x=y$ $\left(\text{i.e. the fulfilment of the equation } \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}\right)$ becomes analyticity of f $\left(\text{i.e. the fulfilment of the Cauchy-Riemann conditions } \frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}\right)$, when that line is replaced by the straight line $x=iy$. The regular way of determining for which parameters analyticity is required is to write the condition that the functions be constant on the classes of conjugate elements, in the complex case, as $X_j f=0$, where f is a function on the group, and X_j are Lie operators corresponding to infinitesimal transformations in the subgroup. By introducing the parameters in the required way we find that f is independent of some of them and satisfies the Cauchy-Riemann condition for pairs of others. Of special interest in connection with the theory of automorphic functions are the transitive manifolds of the highest dimensions already mentioned, on which the functions are analytic in all parameters. General automorphic functions should be defined as functions on these manifolds, invariant with respect to some discrete subgroup of the corresponding semi-simple group. The study of such automorphic functions would probably be of interest.

For the special case of a unimodular matrix group of the second order a somewhat different connection with the theory of automorphic functions was discovered by S. V. Fomin and the author of this report in a paper [12] on the theory of dynamic systems. Automorphic functions are obtained in this paper by decomposing into irreducible components the representation acting in the space of the functions on G, constant on the classes of conjugate elements with respect to a certain discrete subgroup. In each of the irreducible representations of the discrete series contained in this representation there is a certain automorphic form determining uniquely the given irreducible representation. The full decomposition of the representation of the group of second order matrices mentioned above into irreducible ones, as well as the analogous decomposition for the case when the representation acts in a space of cosets of the discrete subgroup of the semi-simple group G, would without doubt be of interest.

3. *Generalized functions.*

In one way or another generalized functions have been considered in mathematics and its applications for rather a long time.

A substantial contribution to the formation of the concept of generalized functions was made in the works of Hadamard [15], and later of M. Riesz [16], on finite parts of divergent integrals. Generalized functions (Dirac's δ-function, etc.) were systematically used in quantum mechanics beginning with the nineteen twenties. The general concept of generalized functions as functionals was developed by S. L. Sobolev in connection with his investigations on equations of the hyperbolic type. The treatment of generalized functions as linear functionals on some space of sufficiently smooth functions is the most convenient and natural one.

Much was done in developing the theory of generalized functions by L. Schwartz [18] who combined and systematized the material at hand and presented it from a unified standpoint. The appearance of his book fostered the penetration of these concepts into various branches of mathematics. Schwartz has also introduced, with the aid of generalized functions, the notion of the Fourier transform of functions which do not grow faster than a certain power.

It is, however, necessary to generalize the definition of the Fourier transform for a broader class of functions. This can be seen from the following example which is traditional for the Fourier method. In considering the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, as was shown by A. N. Tichonov, the class of functions, which do not increase faster than $e^{|x|^2}$, is the natural class of functions in which the existence and uniqueness of the solution of the Cauchy problem is ensured. However, the various generalizations of the definition of the Fourier integral, given by Bochner [20], Carleman [19] and Schwartz [18], allow the Cauchy problem to be solved only for the class of functions which do not increase faster than a certain power of x. A further generalization of the Fourier transform requires a further development of the idea of generalized function [60], which, incidentally, is important from other points of view as well. Starting with Schwartz's definition we shall generalize the definition of Fourier transform so that it shall fit some classes of fast growing functions [60]. It is important that in order to define the Fourier transform we must, as a rule, use analytic functions.

By the term generalized functions we shall understand functionals over some topological space S of infinitely differentiable functions which approach zero faster than any power of x. We shall call this space the basic space. The Fourier transforms of all the functions of the basic space also form a basic space $\tilde{S}$, to which we shall refer as the dual of S.

If we are given the generalized function T, i.e. the functional $T(\varphi)$ over some basic space S, then in generalizing Schwartz's definition we shall introduce the Fourier transform $\tilde{T}$ as a functional over $\tilde{S}$ determined by the equation $\tilde{T}(\tilde{\varphi}) = T(\varphi)$ $(\varphi \in S, \tilde{\varphi} \in \tilde{S})$. The space Z_p^p of all entire analytic functions $\varphi(z)$ of order of growth not higher than p and of order of decrease along the real axis not lower than p: $|\varphi(z)| \leqq C_1 e^{c_2|z|^p}$, $|\varphi(x)| < C_3 e^{-c_4|x|^p}$ may serve as an example of a basic space. It is possible to mention a number of other spaces, for example the space K of all finite infinitely differentiable functions introduced by Schwartz, and its dual space Z^1 of entire analytic functions $\varphi(z)$ of order of growth not higher than 1, decreasing along each straight line parallel to the axis of x, faster than any power of x:

$$\sup_{-\infty < x < \infty} |x^n \varphi(x + iy)| \leqq C_n e^{c_n|y|} \quad (n = 1, 2, \ldots).$$

For applying the Fourier transform method to a problem (for example to a given type of differential equations) it is important to select the basic space (and, consequently, the set of generalized functions). For instance, if we have a system of differential equations of order p,

$$\frac{\partial u_i}{\partial t} = \sum P_{ik}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) u_k,$$

where the P_{ik} represent polynomials of degree not higher than p, then by introducing the basic space of analytic functions Z_q^q $(p < q < p + \varepsilon)$ we can, with the help of the Fourier transform, prove the uniqueness of the solution of the Cauchy problem for the class of functions which do not grow faster than $e^{|z|^{p'-\varepsilon}}$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. One of the principal points in the proof of this theorem, the fact that $\tilde{Z}_p^p = Z_{p'}^{p'}$, is based on the Phragmén-Lindelöf theorem. Thus, the use of a complex variable plays an important part in this case and this is evidently connected with the very essence of the problem. And indeed from the general theorem formulated here follows the uniqueness of the solution of the Schroedinger equation $\frac{\partial u}{\partial t} = i \triangle u$ in the class of functions which do not grow faster than $e^{|z|^{2-\varepsilon}}$. While for the equation of heat transfer the proof of uniqueness in the same class of functions may be obtained in numerous ways, it is hard to conceive how one could prove this theorem for the Schroedinger equation without using the theory of functions of a complex variable.

Lack of time does not permit us to discuss other applications of generalized functions. We shall only emphasize once more the following two points:

1) In order to be able to use the method of generalized functions it is important that for each problem, or class of problems, one can construct the corresponding basic space. A "universal" basic space valid for all problems does not exist, and it is senseless to attempt to construct it.

2) For a broad class of problems in which generalized functions are used it is necessary to pass to the complex area. This implies the consideration of a basic space consisting of analytic functions (or a space consisting of the solutions of some differential equations).

Another significant aspect of the theory of generalized functions should be mentioned, viz. the calculation of divergent integrals and series. The calculation of divergent integrals is carried out by Hadamard, essentially by dropping the divergent part (or, as a physicist would put it, by the "regularization" of the given divergent integral). M. Riesz obtains the convergence of some types of integrals by applying the method of analytic continuation. It must be underlined that we are speaking here not of summing oscillating integrals, but of giving finite values to integrals which really tend to $+\infty$.

In connection with this group of questions the following two problems are of interest:

I. Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial. Consider the area in which $P > 0$. Let $\varphi(x_1, \ldots, x_n)$ be an infinitely many differentiable function equal to zero outside a certain finite area. We shall examine the functional

$$(P^\lambda \cdot \varphi) = \int\limits_{P>0} P^\lambda(x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n)dx_1 \ldots dx_n.$$

It is necessary to prove that this is a meromorphic function of λ (it would be natural to call it a ζ-function of the given polynomial), whose poles are located in points forming several arithmetical progressions, as well as to calculate the residues of this function.

II. The second problem may be stated as follows: What should one understand by a rational function (as a generalized function) and how might one find its Fourier transform? We shall consider a rational function as a functional on Z^1, i.e. we shall understand the value of this functional to be an integral along some surface parallel at infinity to the real subspace of Z^1 (different surfaces produce different functionals). The Fourier transform of such a functional is the functional on K (a space of finite functions), dual with respect to Z^1. The general case of these Fourier transforms has been studied little. For the special case where the denominator of the rational function is a polynomial of the so called hyperbolic type, they were investigated, in connection with hyperbolic differential equations, by Herglotz [22], Petrovski [69], Garding [23] and Leray [24]. In connection with elliptic equations Fourier transforms of rational functions with "elliptic" denominators were studied by Shapiro [25], Lopatinski [26] and Bochner [27].

4. *Generalized Functions and Representations of Groups.*

The development of the theory of representations of Lie groups in the last few years shows that here, too, generalized functions are a convenient and useful apparatus. The use of generalized functions has, in particular, been proved useful in the investigation of the equivalence and irreducibility of representations (Brouchat [21], Mackey [30] and Mautner [37]). Generalized functions may also be employed conveniently in assigning scalar products which fit the supplementary series (see [6], § 18).

One of the specific questions of the theory of representations, where generalized functions can be used conveniently, is the so-called Plancherel theorem which gives the expansion of the function $f(g)$ on the Lie group into the analogue of the Fourier integral [9, 10]. This is equivalent to the calculation of $f(e)$ (e is the unit of the group), if for each character $\chi(g)$ of this group $\int f(g)\chi(g)dg$ is known.

For classical Lie groups, from the values of this integral, we can easily find the integral of the function $f(g)$, taken in any class of conjugate elements in general position. Thus in the case of compact groups $f(e)$ is determined. For instance, for the group of unitary matrices the conjugate element class is the set of all matrices with given eigenvalues. When all these eigenvalues tend to 1, the respective classes of conjugate elements tend to the unit, just like concentric spheres $x^2 + y^2 + z^2 = c$ shrink into their centre when $c \to 0$.

In the case of a noncompact group, for instance a group of all nth order matrices with determinant 1, the class of conjugate elements in general position is again the set of all matrices with given unequal eigenvalues. In this case, however, when the eigenvalues approach 1, the corresponding class does not reduce to a single matrix at all, but approaches the set of all matrices with the only eigenvalue 1, just as hyperboloids $x^2 + y^2 - z^2 = c$, when $c \to 0$, do not reduce to the origin, but become a cone. The unit matrix itself constitutes a rather complicated singular point in the manifold of all matrices, which have all eigenvalues equal to 1, and it is, therefore, far from obvious how one can find $f(e)$ if one knows the integral of the function on the class of conjugate elements. This problem may be solved with the aid of generalized functions by applying the method due to M. Riesz [9, 10].

In order to explain the essential idea of the method, we shall illustrate it by a simple model problem. Let f be a finite, sufficiently smooth function. Let I_c denote the integral of the function f on the hyperboloid $x^2 + y^2 - z^2 = c$. The problem is to calculate $f(0,0,0)$, if, for each c, I_c is known. Note that for this purpose the integral $\int f(x,y,z)(x^2 + y^2 - z^2)^\lambda \, dxdydz$ can be calculated if one knows only I_c, as on each hyperboloid the second factor is constant. Now, if λ approaches $-\frac{3}{2}$, the integral will approach $f(0,0,0)$. Since on the other hand the same integral can be written in terms of I_c, we obtain an expression for $f(0,0,0)$ in terms of I_c. The general problem stated above for groups is solved in a similar way.

The following problem is closely connected with the one examined. Let $P(x,y,z, \ldots)$ be a polynomial and $f(x,y,z, \ldots)$ a finite function. It is necessary to find the values of f in singular points, the integrals of f along singular lines of the surfaces $P = C$, etc., if the integrals of f over all surfaces of constant level of the polynomial P are known.

Let us consider a few problems arising in connection with the application of the theory of generalized functions to representations.

It is well-known from the theory of representations of compact groups that finding the representations of a compact group is equivalent to finding the representations of the centre of the group ring consisting of those functions which are constant on the classes of conjugate elements, i.e. which satisfy the condition $f(g) = f(g_0 g g_0^{-1})$ for all $g, g_0 \in G$. The product (convolution) of such functions is determined by the formula: $(f_1 \cdot f_2)(g) = \int f_1(g g_1^{-1}) f_2(g_1) \, d\mu(g_1)$.

The representation of the centre of the group ring is given by the formula $f(g) \to \int f(g)\chi(g)dg$ where $\chi(g)$ is a character of the group.

As mentioned above, characters for arbitrary classical groups also exist and also determine uniquely the representation. It would be interesting, in this case, too, to construct the centre of the group ring and thus to obtain characters.

The direct determination of the centre is hindered by the fact that any function belonging to the centre is constant on the classes of conjugate elements, and, consequently, as a rule, is not integrable; therefore the direct calculation of the convolution leads to divergent integrals. A more general problem of the same type is the problem of the construction of a general theory of spherical functions. It is known that general spherical functions are connected with the group of the transformations of a manifold in the same manner as the conventional spherical functions are connected with the group of the rotations of a sphere. Instead of a group of transformations of a manifold one may speak of a group and a given stationary subgroup. Then the representations of the group prove to be connected in a natural way with the ring of functions which are constant on two-sided cosets of the group on this subgroup [3].

So far only the case has been studied, where the respective stationary group is compact [31, 13, 70]. In the general case the functions of the ring, as a rule, are not integrable and the calculation of their convolution leads to divergent integrals. It is quite probable that the application of the theory of generalized functions will allow the construction of a theory of spherical functions for noncompact Lie subgroups as well.

For compact Lie groups there are two ways of proving that the representations constructed are all the possible representations of this group. One of them consists in using the theory of characters, the other in using the Cartan theory of highest weights. We have already mentioned the possibility of applying the first of them to local compact Lie groups. It is interesting that the wider use of the second method is also connected with the theory of spherical functions on noncompact stationary subgroups. For instance, in the case when G is a group of matrices with determinant 1 the subgroup Z of triangular matrices with all diagonal elements equal to one is such a subgroup.

It seems to me that, if the theory of generalized functions is used correctly, the main problems of the theory of representations of locally compact Lie groups will be not more complicated than for compact groups.

5. *Differential Operators.*

The theory of differential equations is one of the sources of modern functional analysis. In 1910 H. Weyl published one of the first works on the spectral properties of differential operators. Several related concepts such as the eigenfunction and the eigenvalue came into use at the beginning of the last century (Fourier method). A comprehensive exposition of the main questions in differential operator theory should require at least a separate report. Therefore, we shall occupy ourselves only with a few special questions.

At the present time the main progress in differential operator theory has been achieved in the following topics grouped somewhat arbitrarily:

1) Differential operators and boundary value problems
2) Spectral theory of differential operators
3) Inverse problems.

To the most important topics of the theory of differential equations belong boundary value problems for various types of equations. The connection between these problems and spectral theory is clearly indicated in the works of Friedrichs. In the last few years new important results have been obtained by Vishik [32, 33, 34, 35]. Unfortunately, due to lack of time, we cannot discuss here these works in detail.

We shall now treat more in detail the spectral theory of differential operators. Let us consider first of all selfadjoint differential operators. The existence of a general theorem for the spectral expansion of an arbitrary selfadjoint operator in differential operator theory cannot be considered as a final solution to the problem, as it is desirable to obtain this expansion as an expansion by ordinary or generalized eigenfunctions and not as an abstract resolution of the unity E_λ.

The expansion by eigenfunctions over a finite interval, in the case of ordinary differential equations, has been known for a long time. In 1910 Weyl considered the case of an expansion over an infinite interval for a differential operator of the second order. The spectral expansion has further been extended to the case of ordinary selfadjoint differential operators of higher orders by M. G. Krein [28] and Kodaira [29], and to the case of partial elliptic differential equations by A. J. Powsner [65] who used Carleman's results for this purpose. Mautner [37] presented an interesting study filling the gap between general spectral expansions and expansions by eigenfunctions. In this study restrictions are imposed on an operator in functional space, allowing the spectral expansion to be accomplished by functions. From Mautner's results it is possible to obtain for elliptic operators the expansions by eigenfunctions (see also Garding [30]).

In the field of partial differential operators progress has been made only for elliptic operators. A solution to the problem of a spectral expansion of an arbitrary (e.g. hyperbolic) operator has not been found as yet. Generalized functions do not give any new results for elliptic operators. However, eigenfunctions should be sought for as generalized functions in the case of hyperbolic differential operators.

Let us consider the problem of a spectral expansion of a dynamic system. Let a system of differential equations be given in a certain analytic manifold:

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n),$$

where the X_i are analytic functions. In the manifold there exists an invariant integral (invariant volume).

This system may be compared with the partial differential operator

$$Z = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \ldots + X_n \frac{\partial}{\partial x_n}$$

(which can be called the Lie operator for this system), characterizing an infinitely small displacement along the trajectory. If a Hilbert space consisting of square integrable functions is considered, the operator iL will be a selfadjoint operator in this space having, in accordance with the general theory, a certain spectral expansion. The rather trivial case where the spectrum of this operator is a pure point spectrum has been investigated completely. As far as we know, the continuous spectrum in a spectral expansion has been investigated only for the so-called geodesic streams in manifolds of constant negative curvature [12]. However, even in this case, the investigation of the corresponding eigenfunctions has not yet been completed. Here it is necessary to prove the existence of a complete system of generalized eigenfunctions for any analytic dynamic system. In all known examples of transitive dynamic systems with a continuous spectrum the multiplicity of this spectrum is infinite. At the same time, the eigenfunctions (considered in the usual way and not as generalized functions) are only constant here. It is interesting to determine whether other generalized eigenfunctions (corresponding let us say to the eigenvalue $\lambda = 0$) exist in such systems.

A second problem in the field of spectral expansion of differential operators is as follows. It is well-known that the theory of eigenvalues is equivalent to the study of a pencil of quadratic forms $A + \lambda E$. The more general problem concerning the study of the pencil $A + \lambda B$, where $B(u,u)$ is a positive definite quadratic form, in the general theory of linear operators (of finite dimensions or not), is equivalent to this one, as we have only to choose a new scalar product. However, in the case of differential operators, where the theorem for the spectral expansion of the unit operator is not the final result of the whole theory but rather a leading thread, the problem for the quadratic differential form $A(u,u) + \lambda\, B(u,u)$ is a new one. An interesting example of such a study is found in the publications of Sobolev [38] and Alexandryan [39]. There a problem is considered which is essentially equivalent to the study of quadratic forms

$$\iint \frac{\partial^2 u}{\partial x^2} u dx dy + \lambda \iint \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dx dy$$

for functions equal to zero on the boundary of a region G.

An interesting and important problem in the spectral theory of differentia operators is the study of the asymptotic behaviour of the eigenfunctions and eigenvalues. Numerous results have been obtained recently in the study of partial differential equations by B. M. Levitan [41, 42]. The investigation of the asymptotic behaviour of the eigenfunctions and eigenvalues of differential operators is closely related to the so-called quasi-classical approximation in quantum mechanics.

Selfadjoint operators were considered above. Much less progress has been made in the spectral theory of non-selfadjoint operators, although it sphysical application is of great value. In this field we have the early works of Tamarkin and Hille where the case of a finite interval of a straight line is considered and an important work of Carleman where the existence of eigenfunctions for the elliptic differential operator of the second order is proved. The results obtained by M. V. Keldysh [40] in 1951 are an important contribution to this field. He proved that the system of eigenfunctions for an elliptic equation together with the so-called allied functions (the analogue of the basis with respect to which the matrix takes the Jordanform) is complete. In the works of M. V. Keldysh a wide use has been made of the theory of functions of a complex variable and also of his theorems on the general theory of operators (see also Browder [43, 44]).

These results may be considered as the beginning of a new stage in spectral theory, since here, for the first time, a theorem on the completeness of eigenfunctions has been given that goes beyond the field of selfadjoint operators.

A study by M. S. Lifshitz [45] should be mentioned where the possibility of transforming a selfadjoint and completely continuous operator into the triangular form has been proved. M. S. Lifshitz makes use of the methods of the theory of functions of a complex variable and in particular of Potapov's theorem [46], when dealing with the expansion of bounded analytic matric functions in a product, similar to the Blaschke product.

Hardly anything has been done in the spectral theory of non-selfadjoint operators in the case of an infinite region. In this case it is not even clear under what conditions for the operator a spectral expansion is possible. Let us consider the simple example of a differential equation of the first order:

$$Ly = i\frac{dy}{dx} + a(x)y,$$

where $a(x)$ represents a certain complex function. In this example we shall use the following most conservative definition of a spectrum. Let us consider the operator over a finite interval $(-N,N)$ having, for instance, periodic boundary conditions. Then a discrete spectrum $\lambda(N)$ is obtained. Now let $N \to \infty$. The limit of the set $\lambda(N)$ for $N \to \infty$ will be called the spectrum of the operator L along the whole axis. The eigenvalues of the operator L, with periodic boundary conditions, in the interval $(-N, N)$ are equal to

$$\lambda_n^N = \frac{1}{2N}\int_{-N}^{N} a(s)ds + \pi\frac{n}{N}.$$

We can see, for instance, that, if the mean value of the imaginary part of the function $a(x)$ in the interval $(-N, N)$ approaches ∞ as $N \to \infty$, then the whole spectrum goes to infinity as $N \to \infty$. Consequently, in this case (e.g. for the operator $i\frac{dy}{dx} + ix^2y$) there is no spectrum in the finite part of the plane and the very problem of the spectral expansion for such an operator has no meaning.

It is hardly possible to have a spectral expansion over an infinite interval, for example in a Hilbert space for the operator $i\frac{d^2y}{dx^2} + ixy$, which is the "Fourier transform" of the above mentioned operator. As far as we know, the only spectral expansion of non-selfadjoint differential operators in an infinite region is the expansion of the differential operators of the second order shown in the work of Naimark [47].

Finally we shall say something about inverse problems. So-called inverse problems (see also [48, 49]) have become of great importance in connection with certain problems of mathematical physics and especially of quantum mechanics. Let us examine for example Schroedinger's equation

$$-\psi''(r) + V(r)\psi = k^2\psi$$

with the conditions that $\psi(0) = 0$, $\psi'(0) = 1$. Assuming that the potential $V(r) = 0$ for $r \geqq a$ we have, if r is sufficiently large, $\psi(r) = \varrho(k)\sin(kr + \eta(k))$ where $\eta(k)$ is the phase shift. The inverse problem in this case is to find the potential from $\eta(k)$ or $\varrho(k)$. Where there is no discrete spectrum, the potential is determined uniquely by the phase shift, as was proved by Levinson [50]. At about the same time, similar solutions to inverse problems in geophysics were obtained by A. N. Tichonov [51].

The uniqueness of the solution to the inverse problem for an arbitrary

$V(r)$ has been proved by Marchenko [52], who considered, instead of the phase, the so-called spectral function $\varrho(\lambda)$, which in the preceding particular case coincides with the multiplier $\varrho(k)$.

Further topics in the field of inverse problems for equations of the second order are the following questions:

a. What can serve as spectral function and in particular what point set can serve as a spectrum for a differential operator;

b. In which way can the potential be reproduced effectively by $\varrho(k)$?

All these problems were solved by M. G. Krein [53, 54, 55], B. M. Levitan and the author [56]. In these studies the authors derived necessary and sufficient conditions for $\varrho(k)$ being the spectral function of a differential operator. The determination of the coefficients of the equation by means of $\varrho(k)$ was reduced to the solution of a certain Fredholm integral equation [56]. These results make it also possible to penetrate into the nature of the spectrum of differential operators. It follows that:

1) For any closed set in a finite interval there exists a differential operator the spectrum of which in this interval coincides with the given set.

2) Any sequence fulfilling certain asymptotic conditions can be the sequence of eigenvalues for a Sturm-Liouville operator in a finite interval (with corresponding boundary conditions).

The above mentioned integral equation for inverse problems was applied to problems on dispersion theory in quantum mechanics in the works of Jost and Kohn [57]. It was used by Corinalesi [58] in the case of a relativistic particle.

Using the same integral equation M. G. Krein derived important formulas in order to determine $V(r)$ by means of $\varrho(\lambda)$ for a large class of functions $\varrho(\lambda)$, and in particular for all rational functions.

So far we have considered the one-dimensional case of an inverse problem. Little has been done for the case of several independent variables. It would be most natural to consider in this case a problem similar to that of determining $V(r)$ by means of the phase.

Let us consider the equation:

$$-\triangle u + V(x,y,z)u + \varkappa^2 u = 0.$$

Assume that $V=0$ outside a certain finite region and that only equations without any discrete spectrum are to be considered. Let us examine the solution of this equation and also a normal derivative of the solution on a sufficiently large closed surface. The function u and its normal derivative can be divided into two components, u_1 and u_2 say, representing convergent and divergent waves. We may consider these waves (both outgoing and ingoing) without taking into account the derivatives of u_1 and u_2, because the values of the normal derivatives are determined by the values of the function itself. Let $S(k)$ be the operator which to each outgoing wave associates the corresponding ingoing wave. The problem is to determine whether the potential $V(r)$ can be defined by means of the operator $S(k)$. Instead of the operator $S(k)$ the operator $R(\lambda)$ introduced by Wigner can be considered. It establishes a relationship between the function on the surface and its normal derivative [67, 68].

In quantum mechanics this problem can be treated in terms of dispersion theory in the same way as in the one-dimensional case. We mention also an interpretation of this problem in the field of optics. A light wave coming from infinity is dispersed in the points of inhomogeneity defined by a certain function V. We observe a dispersed wave. Is it possible to reproduce the function V using the data, obtained by varying in any manner the oncoming waves and observing the corresponding dispersed waves? If it is possible, how can we do it?

A possible variant of the inverse problem is as follows. Let us examine the external boundary condition problem for the equation

$$-\triangle u + V(x_1, x_2, \dots, x_n)u = \lambda u,$$

with the boundary condition $\frac{\partial u}{\partial n} = 0$. Let us consider the resolution of the unity $E(\lambda)$ corresponding to this operator. $E(\lambda)$ is an integral operator with a kernel $\varrho(P,Q,\lambda)$. If we consider this kernel for points P and Q lying on the boundary of the region, we get the function $\varrho(S_1,S_2,\lambda)$, which is an analogue of the function $\varrho(\lambda)$ in the one-dimensional problem. Now one has to find the potential $V(r)$ knowing $\varrho(S_1,S_2,\lambda)$. Some results for this second problem were obtained by Berezanski [59].

6. *Analysis in Functional Spaces.*

Although the branches of functional analysis mentioned above are comparatively new and are continuing to develop rapidly, they nevertheless have already taken a definite form and acquired, so to say, a "personal" character. This is not true for the group of questions covered by the last section of this report. We shall deal here with problems and procedures which are just beginning to appear; it is, however, possible that in the future they will occupy a central place in functional analysis as a whole. There are a number of physical problems in which, apart from difficulties of physical character, other difficulties arise, due to the absence of a sufficiently general, adequate mathematical apparatus. Some questions of quantum electrodynamics, the theory of turbulence, etc. are of this type. Lately such a mechanism has begun to take shape. It might be called analysis in functional spaces. We shall illustrate the questions and methods arising here on a problem of quantum electrodynamics. We shall not use any data of quantum electrodynamics, but shall only consider the simplest model equation. On the basis of ideas which are to be found in the well-known paper by V.A. Fok [17], Schwinger [62] developed quantum electrodynamics with the aid of functionals in a space (referred to as the space of external sources). In doing so the equations of quantum electrodynamics become a comparatively complicated system of integral equations and lead to relationships between functionals and their variational derivatives. This system of equations could be simplified much and reduced to a linear differential equation in the functional space, or to a system of such equations [66]. We shall demonstrate this procedure by using a very simple equation, the so-called Thiring quantum equation

$$(-\square + \varkappa^2)\psi = \lambda\psi^3 + I,$$

where

$$\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial t^2}$$

and $\psi(x_1,x_2,x_3,x_4)$, as usual in quantum theory, designate certain operators. Following the method of Schwinger we add certain scalar functions $I(x)$ (called external sources) to this equation and reduce it to a system of ordinary differential equations (not containing operators) in the following way. Let us assume that at the moment $t = -\infty$ the external sources $I(x)$ are absent and let e_0 designate the state of vacuum without external sources. If then external sources are introduced, the state e_0 will alter in some way; and at $t = +\infty$ will change into some state e_0^1. Let us assume that Z is equal to the scalar product of the wave functions e_0 and e_0^1 ($|Z|^2$ indicates the probability that the vacuum has remained a vacuum). Z is a linear functional of the external sources I and satisfies the following differential equation:

$$\lambda \frac{\delta^2 Z}{\delta I(x)^2} + (\square - \varkappa^2) \frac{\delta Z}{\delta I(x)} = I(x)Z. \tag{1}$$

Here $\dfrac{\delta Z}{\delta I(x)}$ is the so-called variational-functional derivative, i.e. the limit of the ratio of the increase of Z and $\int \delta I(x)dx$, where the variation $\delta I(x)$ is concentrated in an infinitesimal neighbourhood of the fixed point x_0.

The equation (1) has a similar form as the Airy equation, which is obtained from it, if Z is considered, not as a functional, but as a function and the variational-functional derivatives are replaced by usual derivatives.

It seems natural to call the solutions of the functional equation (1) Airy functionals. They can be obtained in the following way.

Let us consider in the four-dimensional space of variables x_1,x_2,x_3,x_4 a cube of side L, divide it into cells of side τ and assume that Z depends only on the values of I at the joints of these cells. The equation (1) reduces to a system of partial differential equations

$$\frac{1}{\omega^2}\frac{\partial^2 Z}{\partial I_i^2}+\frac{1}{\omega}\sum_k R_{ik}\frac{\partial Z}{\partial I_k}=I_i Z,$$

where ω is the volume of a cell and R^{ik} is the matrix obtained by replacing the operator $\square - \varkappa^2$ by the corresponding difference operator.

We shall solve this equation with the aid of a Laplace transform, i.e. we shall search a solution of the form

$$Z(I_1,I_2,\ldots,I_n)=\int \xi(s_1,\ldots,s_n)e^{i\Sigma I_k s_k}\,ds_1\ldots ds_n, \tag{2}$$

where the integration is extended over the surface obtained as the direct product of n contours C_k chosen in the planes of the corresponding complex variables s_k. Thus we obtain the following expression for $\xi(s_1, s_2, \ldots, s_n)$:

$$\xi(s^1,\ldots,s_n)=e^{i\Sigma\frac{s_k^3}{3}+\Sigma R_{ik}s_i s_k}.$$

In order that the integral (2) be different from zero it is necessary that each of the contours C_k extend into infinity. At the same time we must ensure the convergence of the integral (2). This problem can be solved for each of the contours C_k along which one integrates. It is easy to verify that for the convergence of the integral (2) it is necessary and sufficient that each of the contours C_k extend into infinity within one of three angles, I, II and III say. In order to obtain a nonzero integral, one must take the contour C_k in the union of the angles I and II, I and III, or II and III. In the latter case the value of the integral is equal to the sum of its values in the two former cases, so there will be only two independent integrals. In integrating along all the n contours we obtain in this way 2^n independent solutions. Linear combination of the solutions furnishes the general solution of the equation (1). However, if we should now pass to the limit by letting the dimensions of the cells tend to zero ($\tau \to 0$), i.e. if we go over from indices $1,2,\ldots,n$ to a continuous variable x, then in the limit we shall obtain only two solutions (if in different planes we take the contours in different pairs of angles, then there will be no solution in the limiting case).

If the functional Z has been found, the calculation of various quantum-mechanical effects reduces to the determination of some functional derivatives of Z.

This treatment of quantum-mechanical problems is closely connected with the theory constructed by Feynman [63] on the basis of other considerations. Similar methods are also used by Edwards and Peierls [64].

For a rigorous justification of the method of solution given above it is necessary to be able to answer a number of other questions. For instance, one must justify the transition to the limit for $n \to \infty$. It would be still better to obtain the solution directly by integrating in the functional space. It is desirable to find the asymptotic behaviour of the Airy functionals (in a similar way as was done for the Airy functions). All these specific mathematical questions are of unquestionable interest not only for quantum electrodynamics, where they arise, but for many other fields as well. Questions which are very close to those just discussed, arise, for example, in the theory of turbulence in the interpretation given in a recent work by E. Hopf [61]. Let $u_\alpha(x)$ be the velocity field of a certain liquid. Let us introduce the functions $y^\alpha(x)$ and assume that $Z = \langle e^{\int y^\alpha u_\alpha dx} \rangle$ (here the simbol $\langle\ \rangle$ indicates taking the mean). Zrepresents a functional of $y(x)$. On the basis of the equations of hydrodynamics E. Hopf has given a linear equation for Z in functional derivatives. The similarity with quantum electrodynamics found here is of interest. For instance, the correlation between velocities in different points is calculated, as quantum-mechanical effects, by means of functional derivatives of Z. Here questions of the kind of integration in functional spaces may also play an essential rôle.

It should be noted that some of the concepts of analysis in functional spaces do exist already rather a long time (f.i. N. Wiener's concept of measure in a functional space). However, in the near future, they may occupy a considerably more significant place than they have so far.

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ON SOME METHODS OF APPROXIMATION IN FLUID MECHANICS

S. GOLDSTEIN

1. *Introduction.*

Certain recent work leads to the belief that it is time to consider again the Navier-Stokes equations for the steady flow of an incompressible viscous fluid, with a hope that a deeper insight into the nature of the solutions may be achieved. The investigations are concerned with an understanding of the nature, and extensions, of approximations at small and at large Reynolds numbers. I shall, however, not attempt a review of the work at small Reynolds numbers; at the present time the asymptotic approach at large Reynolds numbers seems more important and interesting, and moreover I have nothing in any way new to offer about small Reynolds numbers. First, then, some recent developments of steady, laminar, boundary-layer theory will be considered. Most modern research in fluid dynamics is concerned with turbulence or the high-speed flow of gases; after the success of Prandtl's boundary-layer theory interest died down, except for special problems and (in particular) three-dimensional effects. (See Ref. 1). I shall be concerned with a different aspect, considering boundary-layer theory as a first step towards obtaining asymptotic expansions for large Reynolds numbers. The aim at this stage is simply to show what is involved in constructing such asymptotic solutions.

Approximations at large Reynolds numbers will provide flows that are unstable, and no guidance is to be expected from experiment. However, the necessity for considering such approximations remains, not only for the sake of a deeper mathematical-physical understanding (which is also needed for considering boundary layers in high-speed gas flow), but also to begin the study of asymptotic expansions that will apply at moderate, or even fairly small, Reynolds numbers.

The simplest example is still that of two-dimensional flow past a semi-infinite flat plate parallel to the stream, and it is with this example that we shall be largely concerned. The important point, however, is that the plate is semi-infinite—there is no wake. It appears that when a wake is present the limit of the steady flow as the Reynolds number tends to infinity is not known with sufficient certainty. (Even for a finite flat plate parallel to the stream it is known only approximately).

The mathematical theory involved is that of singular perturbations: when viscosity is neglected, the governing equations are non-linear; the perturbation is linear, and contains higher derivatives with a small coefficient. Fluid mechanics is full of such problems. The Navier-Stokes equations provide one of the hardest examples, for although the non-linear part, by itself, is usually integrated at once to provide the simple linear Laplace equation when viscosity is neglected, the non-linearity makes itself strongly felt as soon as the viscous perturbation is considered. Other examples in fluid mechanics may arise from problems of heat transfer in moving fluids, from entropy variations (in the initial stages) behind a shock wave of varying strength, and from the theory of long waves on shallow water.

When the viscosity tends to zero, one of the most interesting phenomena is the appearance of singular surfaces such as vortex sheets and shock waves; we want to know more details for large finite Reynolds numbers of the flows inside the thin layers which become such singular surfaces for infinite Reynolds numbers. In particular, shock-wave phenomena in one-dimensional flows are described by nonlinear wave equations perturbed by linear higher-order terms. I shall not, however, discuss the shock-wave equations directly, but shall here, for mathematical illustration, go over to results that can be proved, by using as examples (and generalizing) two equations that have been solved explicitly (one of which has some importance in chemical engineering), and which show the nature of the phenomena quite clearly.

2. *The Navier-Stokes Equations. The Boundary-Layer Equation, and Solution for Flow along a Flat Plate.*

Non-dimensional velocities and coordinates will be used in writing the Navier-Stokes equations for two-dimensional steady motion. If U is a standard velocity, ν the kinematic viscosity of the fluid, and l a typical length, the Reynolds number is defined by

$$R = Ul/\nu. \tag{1}$$

We shall be concerned with flow past a semi-infinite flat plate parallel to the stream, lying on $y = 0$, $x \geqq 0$, where U is the undisturbed velocity of the stream; there is then no obvious length in the problem. We may simply select one arbitrarily, and keep it constant as $R \to \infty$; $R \to \infty$ means $\nu \to 0$.

Then with a usual notation the Navier-Stokes equations are

$$u = \partial\psi/\partial y, \quad v = -\,\partial\psi/\partial x, \tag{2}$$

$$\frac{1}{R}\nabla^4\psi + \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0. \tag{3}$$

For the unperturbed equation, $\nabla^2\psi$ is a function of ψ, and we have the classical theorem that with no vorticity in the undisturbed flow at infinity upstream, $\nabla^2\psi = 0$ in any region occupied by streamlines coming from infinity upstream. (This is practically correct for streamlines which have not entered a region where the perturbation terms are important).

According to boundary-layer theory, (cf. Ref. 2), we set

$$y_1 = R^{\frac{1}{2}}y, \quad \chi = R^{\frac{1}{2}}\psi, \tag{4}$$

in (3), and assume that the derivatives that occur are now all $O(1)$ as $R \to \infty$, so the boundary-layer equation is simply obtaining by putting $1/R = 0$ in the resulting equation, which is

$$\left.\begin{aligned} \frac{\partial}{\partial y_1}\left\{\frac{\partial^3\chi}{\partial y_1^3} + \frac{\partial\chi}{\partial x}\frac{\partial^2\chi}{\partial y_1^2} - \frac{\partial\chi}{\partial y_1}\frac{\partial^2\chi}{\partial x\partial y_1}\right\} + \frac{1}{R}\left\{2\,\frac{\partial^4\chi}{\partial x^2\partial y_1^2} + \frac{\partial\chi}{\partial x}\frac{\partial^3\chi}{\partial x^2\partial y_1}\right. \\ \left. - \frac{\partial\chi}{\partial y_1}\frac{\partial^3\chi}{\partial x^3}\right\} + \frac{1}{R^2}\frac{\partial^4\chi}{\partial x^4} = 0. \end{aligned}\right\} \tag{5}$$

As $y_1 \to \infty$, $\partial^3\chi/\partial y_1^3$ and $\partial^2\chi/\partial y_1^2 \to 0$, and the boundary layer equation is

$$\left.\begin{aligned} \frac{\partial\chi^3}{\partial y_1^3} + \frac{\partial\chi}{\partial x}\frac{\partial^2\chi}{\partial y_1^2} - \frac{\partial\chi}{\partial y_1}\frac{\partial^2\chi}{\partial x\partial y_1} = P(x) = -\lim_{y_1\to\infty}\frac{\partial\chi}{\partial y_1}\frac{\partial^2\chi}{\partial x\partial y_1} \\ = -\lim u\frac{\partial u}{\partial x}. \end{aligned}\right\} \tag{6}$$

The function $P(x)$, arising from the integration with respect to y_1, is simply the pressure gradient in the main stream outside the boundary layer.

For flow past the semi-infinite flat plate, with the equations in cartesian coordinates, as here, the boundary conditions are

$$\chi = 0,\ \partial\chi/\partial y_1 = 0 \text{ on the plate } (y_1 = 0,\ x > 0),\ \partial\chi/\partial y_1 \to 1 \quad \text{as } y_1 \to \infty \text{ or as } x \to 0 \tag{7}$$

and the known solution is

$$\chi = x^{\frac{1}{2}}f(\eta_0), \text{ where } \eta_0 = y_1/(2x^{\frac{1}{2}}), \tag{8}$$

$$f''' + ff'' = 0,\ f(0) = f'(0) = 0,\ f' \to 2 \text{ as } \eta_0 \to \infty \tag{9}$$

$$f = \frac{\alpha\eta_0^2}{2!} - \frac{\alpha^2\eta_0^5}{5!} + \ldots \qquad (\alpha = 1.328), \tag{10}$$

$$\text{and as } \eta_0 \to \infty,\ f \backsim 2\eta_0 - \beta \qquad (\beta = 1.72) \tag{11}$$

and the error terms in (11) are exponentially small.

$$\left.\begin{aligned} \text{Also } u &= \tfrac{1}{2} f'(\eta_0),\ v = \frac{1}{2R^{\frac{1}{2}}x^{\frac{1}{2}}}\{\eta_0 f'(\eta_0) - f(\eta_0)\} \to V \\ &= \frac{\beta}{2R^{\frac{1}{2}}x^{\frac{1}{2}}} \text{ as } \eta_0 \to \infty \end{aligned}\right\} \tag{12}$$

If τ is the skin-friction,

$$\tau_1 = \frac{4\,R^{\frac{1}{2}}\tau}{\varrho U^2} = \frac{f''(0)}{x^{\frac{1}{2}}} = \frac{\alpha}{x^{\frac{1}{2}}} \tag{13}$$

and the (non-dimensional) displacement thickness is given by

$$\delta_1 = \int_0^\infty (1-u)dy = R^{-\frac{1}{2}} \lim_{y_1\to\infty} (y-\chi) = \frac{\beta x^{\frac{1}{2}}}{R^{\frac{1}{2}}} \tag{14}$$

so $V = d\delta_1/dx$, as it must from the conservation of mass.

The singularity in τ_1 at the leading edge, $x = 0$, is integrable. But v has a singularity all along the line $x = 0$. To order $R^{-\frac{1}{2}}$, $V \neq 0$, and the boundary-layer flow does not join smoothly on to the potential flow, for which the stream-function is $\Psi = y$ (or to any potential flow); although terms of order R^{-1} only have been neglected in (5), there is an error of order $R^{-\frac{1}{2}}$. The streamlines outside the boundary-layer are deflected through a distance δ_1, and to obtain a result ccrrect to order $R^{-\frac{1}{2}}$ this effect must be taken into account in the external potential flow, so as to take into account the velocity V. According to usual boundary-layer theory, the boundary condition on the potential flow is applied at the plate, so the stream-function of the potential flow is taken to be $\Psi_0 + R^{-\frac{1}{2}}\Psi_1$, where $\Psi_0 = y$ in this case, and $R^{-\frac{1}{2}}\Psi_1$ must be equal to $-\delta_1 = -R^{-\frac{1}{2}}\beta x^{\frac{1}{2}}$ at $y = 0$, $x > 0$. Then we should calculate $P(x)$ in (6) for this new potential flow, and correct χ by taking $\chi = \chi_0 + R^{-\frac{1}{2}}\chi_1$, where χ_0 is given by (8).

The potential problem is immediately solved by the use of parabolic coordinates, ξ, η, for which

$$x + iy = (\xi + i\eta)^2. \tag{15}$$

We take $\eta \geqq 0$; $\eta = 0$ on the plate. $\xi = 0$ is the negative real axis, $\xi > 0$ on the upper half plane, and $\xi < 0$ on the lower half plane. The solution for Ψ_1 is $\Psi_1 = -\beta\xi$, so for the potential flow, to order $R^{-\frac{1}{2}}$,

$$\Psi = y - \frac{\beta\xi}{R^{\frac{1}{2}}} = y - \frac{\beta}{R^{\frac{1}{2}}}\left(\frac{\tau + x}{2}\right)^{\frac{1}{2}}, \tag{16}$$

(in the upper half plane) where $\tau = (x^2 + y^2)^{\frac{1}{2}}$. [The solution has a singularity at the leading edge.]

From (16) we find that on the plate $P(x) = 0$ to order $R^{-\frac{1}{2}}$, (Ref. 3, p. 88), and so $\chi_1 = 0$; away from $x = 0$ the boundary-layer solution is correct in this case to order $R^{-\frac{1}{2}}$; the potential flow only needed correction.

Away from a neighbourhood of $x = 0$, $y = 0$, the singularity on $x = 0$ is purely artificial; near $x = 0$, $y = 0$ the difficulty is fundamental.

Next, however, let us first consider the form taken by the boundary-layer equation and solution when we use parabolic coordinates throughout.

Before leaving this section, let us note again that as $\nu \to 0$ or $R \to \infty$, boundary layers become surfaces of discontinuity; that the approach to the limit is non-uniform; and that this is of practical as well as mathematical significance. This type of behaviour is of frequent occurrence in engineering science, not only in fluid mechanics, and usually for practical purposes it is not enough to have the limit; we need quantitative descriptions of the phenomena in the transition zones. In fact, if a particular quantity in which an engineer is interested changes rapidly from one value to another, and if the transition zone becomes narrower and narrower as some parameter is decreased, then often the results for the transition zone are simply shown in a graph on a bigger and bigger scale, which is practically exactly what we do in boundary-layer theory and related theories.

3. *The Boundary-layer Equation and Solution in Parabolic Coordinates.*

Parabolic coordinates have been used by a number of authors. (Refs. 4, 5, 6, 7. Ref. 7 refers to a paper by N. E. Kochin, which I have not seen).

The Navier-Stokes equation (3) is transformed to the coordinates (ξ, η), defined in (15); the substitutions

$$\eta_1 = R^{\frac{1}{2}}\eta, \quad \chi = R^{\frac{1}{2}}\psi, \tag{17}$$

are then made, and $1/R$ is put equal to zero in the equation corresponding with (5). The leading edge is now inside the boundary layer, and the potential flow is approached everywhere as $\eta_1 \to \infty$. The boundary conditions are satisfied if

$\chi = 0$, $\partial\chi/\partial\eta_1 = 0$ on $\eta_1 = 0$, and $(1/\xi)\ (\partial\chi/\partial\eta_1) \to 2$, $(1/\eta)\ (\partial\chi/\partial\xi) \to 2$ as $\eta_1 \to \infty$.

It is easily found that

$$\chi = \xi\, f(\eta_1) \tag{18}$$

is a solution of the resulting boundary-layer equation, where f is the same function as before (eqns. (9), (10), (11)). Moreover, on the plate $\xi = x^{\frac{1}{2}}$, and

$$\tau_1 = \frac{f''(0)}{\xi} = \frac{\alpha}{x^{\frac{1}{2}}} \tag{19}$$

as before (eqn. (13)). But as $\eta_1 \to \infty$,

$$\psi \sim y - \frac{\beta\xi}{R^{\frac{1}{2}}},\ u \sim 1 - \frac{\beta\eta}{2R^{\frac{1}{2}}(\xi^2+\eta^2)},\ v \sim \frac{\beta\xi}{2R^{\frac{1}{2}}(\xi^2+\eta^2)}, \tag{20}$$

so in parabolic coordinates the external potential flow is included in the boundary-layer solution correctly to order $R^{-\frac{1}{2}}$. Away from the neighbourhood of the leading edge, the singularity on $x = 0$ has completely disappeared. In the expressions for u and v in the potential flow, η^2 must be retained; we must be careful not to put $\eta^2 = \eta_1^2/R$, and then drop this term; η is finite and non-zero in the potential flow.

The results are a particular case of a theorem lately proved by Kaplun (Ref. 7), for any flow without a wake, on the same assumptions that we have used — the usual assumptions of boundary-layer theory. Kaplun shows (i) that there are always coordinates for which the external potential flow is included in the boundary-layer solution correctly to order $R^{-\frac{1}{2}}$; these coordinates (not necessarily or usually orthogonal) are, in our notation, of the form $F_1(\Psi_1)$, $\Psi_0\, F_2(\Psi_1)$; (ii) that if $\chi = F(x,\, y_1)$ is a boundary-layer solution in *any* coordinates $(x,\, y_1)$, and x and y_1 are expressed in terms of any other coordinates ξ and η_1, then the boundary-layer solution in the coordinates ξ and η_1 is $\chi = F[x(\xi,\, 0),\ \eta_1(\partial y_1/\partial \eta_1)_{\eta_1=0}]$; (iii) that the skinfriction τ is unaltered. In our case, $\Psi_0 = y = 2\xi\eta$, $\Psi_1 = -\beta\xi$; for $F_1(\Psi_1)$ we may take ξ, and for F_2, $1/(2\xi)$. The coordinates become ξ and η, as above. Also $x(\xi,\, 0) = \xi^2$, $\partial y_1/\partial \eta_1 = 2\xi$ and $\chi = x^{\frac{1}{2}} f[y_1/(2\, x^{\frac{1}{2}})]$ becomes $\chi = \xi f[2\xi\eta_1/(2\xi)] = \xi f(\eta_1)$, as above.

Note that near $\eta_1 = 0$, $\chi = \frac{1}{2}\,\alpha\xi\eta_1^2$, $\psi = \frac{1}{2}\,\alpha R^{-\frac{1}{2}}\xi\eta_1^2$. Thus for small η_1 (where the motion is slow) ψ is a biharmonic function and a solution of Stokes's equation for creeping flow. This term is the first term of an expansion near the leading edge obtained by Carrier and Lin (Ref. 5). We return to this point later.

The error of the solution given by (18) (away from the neighbourhood of the leading edge, which we shall discuss later) is $O(R^{-1})$. But for other boundaries there will usually be an error of order $R^{-\frac{1}{2}}$, arising from the curvature of the boundary.

4. *The Boundary-layer Solution to order* R^{-1}, *and the Potential Flow to order* $R^{-3/2}$.

To consider the next approximation to a solution for χ, we may start from the equation in parabolic coordinates analogous to (5), and seek to satisfy it to order R^{-1}. This has been done for the flow in the boundary layer away from the leading edge by Alden (Ref. 4). In our notation, he sets

$$\chi = \xi f(\eta_1) + \frac{1}{R\xi}\, f_2(\eta_1) + \dots \tag{21}$$

where f_2 satisfies the linear non-homogeneous equation

$$f_2'''' + f f_2''' + 3 f' f_2'' + f'' f_2' - f''' f_2 = 2\eta_1 f''(\eta_1 f' - f) \tag{22}$$

and
$$f_2 = \alpha C \frac{\eta_1^2}{2!} - \beta^2 \frac{\eta_1^3}{3!} + \dots \qquad = C(\eta_1 f' - f) + \Delta f_2 \tag{23}$$

α and β are the constants of equations (10) and (11). Two of the constants of integration have been determined from the boundary conditions at $\eta_1 = 0$, and one from a condition as $\eta_1 \to \infty$, as Alden determined them, but the fourth constant C has been left for the present. Δf_2 is independent of C. As $\eta_1 \to \infty$,

$$f_2(\eta_1) \sim \beta C - a + \frac{b}{2\eta_1 - \beta)} \left\{ 1 + \frac{2}{(2\eta_1 - \beta)^2} + \dots \right\} \tag{24}$$

($a = 3.34$, $b = 1.66$). Because of the factor $1/\xi$, the solution cannot be valid near the leading edge, nor in the potential flow near $\xi = 0$.

For τ_1, Alden's result is

$$\tau_1 = \frac{\alpha}{x^{\frac{1}{2}}} + \frac{\alpha C}{R x^{3/2}} + \dots, \tag{25}$$

$$\text{and as } \eta_1 \to \infty,\ \psi = R^{-\frac{1}{2}} \chi \sim y - \frac{\beta \xi}{R^{\frac{1}{2}}} + \frac{\beta C - a}{R^{3/2}} \frac{1}{\xi}. \tag{26}$$

The first two terms give the same potential flow as before, but the third term is not harmonic, and Alden takes $\beta C - a = 0$, $C = 1.99$. The skin-friction would then have a non-integrable singularity at the leading edge, but in any case the solution is not valid there. Without some such condition as $\beta C - a = 0$, we should be one boundary condition short.

As regards the improvement we shall attempt, almost everything is achieved by replacing $f_2(\eta_1)/\xi$ by $\xi f_2(\eta_1)/(\xi^2 + \eta^2)$; but the argument may be made more convincing by attempting the use of a technique due to Lighthill (Ref. 8) for finding uniformly valid approximations. The method has mostly been used for hyperbolic equations (with a singular characteristic, for example), but Lighthill (Ref. 9) has used it for an elliptic equation, and Kuo (Ref. 3) has applied it to the boundary layer equation for a flat plate in cartesian coordinates, changing the x-coordinate only, and has thereby obtained results near the leading edge and in the potential flow that agree with the results from the use of parabolic coordinates. We shall use the method starting from parabolic coordinates. The straightforward application of the method does not, in this case, give an approximate solution of the Navier-Stokes equation uniformly valid in the whole field; we return to this point later.

We introduce new coordinates given implicitly in terms of the old; if we are, to begin with, content to change the ξ-coordinate only, it is not difficult to see that to order R^{-1} we should write

$$\xi = X + \frac{1}{R} \frac{g(Y_1) X}{X^2 + Y^2},\ \eta = Y,\ \eta_1 = Y_1 = R^{\frac{1}{2}} Y. \tag{27}$$

We still want to include the potential flow, so to write $Y^2 = Y_1^2/R$ and to drop this term in the denominator would be wrong. We then write

$$\chi = \chi_0 + R^{-1}\chi_1,$$

where χ_0, χ_1 are functions of X and Y_1, and seek to satisfy to order R^{-1} the equation into which (3) or (5) is transformed, and to choose $g(Y_1)$ so as to annul χ_1 completely. We find that this can be done, and we find a solution

$$\chi = Xf(Y_1), \tag{28}$$

where f is the same function as before (eqns. (9), (10), (11)). The correction of order R^{-1} is entirely in the change of coordinates.

If, in the boundary layer away from the leading edge, we carry out a formal expansion in powers of R^{-1}, we should (except for a possible change in the constant C) obtain Alden's solution. For such a formal expansion

$$\chi = \left[\xi - \frac{1}{R}\frac{g(\eta_1)\xi}{\xi^2 + \eta_1^2/R}\right] f(\eta_1) = \xi f(\eta_1) - \frac{1}{R}\frac{f(\eta_1)g(\eta_1)}{\xi}, \tag{29}$$

so we expect that we shall have

$$f(\eta_1)\, g(\eta_1) = -f_2(\eta_1), \tag{30}$$

and this can be checked and agrees. So g need not be computed independently.

Also we find that

$$\tau_1 = \left(\frac{\alpha X}{\xi^2}\right)_{Y_1=0}, \tag{31}$$

and if we assume for the present that this expression is valid at the leading edge (we return to this point later), we see that τ_1 still has a non-integrable singularity (though of a lower order than in (25)) unless X vanishes with ξ on $Y_1 = 0$. This requires $g(0) = 0$, which in turn requires $C = 0$ in (23), so if this is correct, f_2 in (30) must be replaced by Δf_2. With $C = 0$, for the external flow we find that as $Y_1 \to \infty$

$$\psi = R^{-\frac{1}{2}}\chi \sim y - \frac{\beta\xi}{R^{\frac{1}{2}}} - \frac{a\xi}{R^{3/2}(\xi^2 + \eta^2)} \tag{32}$$

to order $R^{-3/2}$. [If $C \neq 0$, the coefficient in the last term is changed from $-a$ to $\beta C - a$.] The right-hand side of (32) is harmonic, and gives a potential flow which becomes the given uniform stream as $\eta \to \infty$. It seems reasonable that there should be a correction of order $R^{-3/2}$ in the potential flow (associated with a correction to the displacement thickness), so we accept (32). Then with $C = 0$,

$$\tau_1 = \frac{\alpha}{x^{\frac{1}{2}}} \tag{33}$$

and is unaltered. There is no correction to τ_1 of order R^{-1}. Further consideration shows that this result is connected mathematically with the fact that there are no negative powers in the asymptotic expansion of f (see eqn. (11)), the error terms in (11) being exponentially small. There are negative powers in the asymptotic expansion of f_2 (eqn. (24)) or of Δf_2, and there may be a change in τ_1 of order R^{-2}.

5. *The Neighbourhood of the Leading Edge.*

Near the leading edge of a flat plate the assumptions of boundary-layer theory are not valid. With parabolic coordinates, the ξ-coordinate should also be transformed by the substitution $\xi_1 = R^{\frac{1}{2}}\xi$; in fact with cartesian coordinates the correct way to consider the neighbourhood of the leading edge is to substitute

$$x_1 = Rx, \quad y_1 = Ry, \quad \psi_1 = R\psi. \tag{34}$$

When this is done all the terms in the Navier-Stokes equations are of the same order of magnitude.

Similar remarks are true near the front stagnation point of any semi-infinite cylinder. In considering the non-uniformity of approximations to solutions for large Reynolds numbers there are three regions to be considered. In the potential flow the coordinates are left unaltered; in the boundary-layer (in more general cases, in any similar vortex-layer of high vorticity) the non-dimensional length coordinate across the layer should be magnified $R^{\frac{1}{2}}$ times, and the coordinate along the layer left unaltered (cf. eqns. (4) and (17)); in a circle whose centre is at the stagnation point and whose radius is of order R^{-1}, the non-dimensional length coordinates should all be magnified R times, as in (34).

Now in § 4, to complete the solution, we applied the condition that τ should have an integrable singularity at the leading edge. Generally, when it becomes possible to discuss more fully the integration of the Navier-Stokes equation, it is to be expected that it will not be necessary to use such a boundary condition. Meanwhile, for the problem considered in § 4, and for the method used there, no other satisfactory boundary condition presents itself; and it is to be expected that this condition will lead to the correct answer, provided that it may be applied.

Further consideration shows that there is a small sector of the circle about the leading edge in which (18) and (28) are in a certain sense valid approximations. This sector is symmetrical about the radius lying along the plate ($y = 0$, $x > 0$). It has been mentioned that very near the leading edge, $\chi = \frac{1}{2}\alpha\xi\eta_1^2$, i.e. $\psi_1 = \frac{1}{2}\alpha\xi\eta_1^2$, and that this is the first term of an expansion near the leading edge obtained by Carrier and Lin. From a consideration of the next term in the expansion, and its general form, it appears that this single term is a good approximation to the full solution of the Navier-Stokes equation, not only sufficiently near to the leading edge (x_1 and y_1 sufficiently small), but also in a sector of the form mentioned ($x_1 > 0$, $|y_1|/x_1$ sufficiently small). This error cannot be estimated exactly, but if we are content to estimate it from the second term in the expansion, we find that whereas χ itself is $O(R^{-\frac{1}{2}}\theta^2)$ in the sector, the error in χ is $O(R^{-\frac{1}{2}}\theta^3)$, where θ is the angle of the sector, and $r_1 = (x_1^2 + y_1^2)^{\frac{1}{2}}$ has been taken as $O(1)$. The error in χ in (28) may be expressed as the sum of three parts, of which this is the first; the second is the difference of $\frac{1}{2}\alpha\xi\eta_1^2$ from the result in (18), and the third the difference between the results given in (18) and (28). The second is $O(R^{-\frac{1}{2}}\theta^5)$ and the third is $O(R^{-\frac{1}{2}}\theta^3)$. Hence, within the limits we have set ourselves — terms of order R^{-1} in χ and τ_1 — the fractional error in χ (or ψ_1) is $O(\theta)$; and (33) should give, to order R^{-1}, the correct limiting result for τ_1 as the leading edge is approached along the plate; without the condition $C = 0$, (31) would give this result more generally. This is what we require.

The difficulty near the leading edge is connected with the fact that the substitution (27) is not unique, for in addition to substituting for ξ by a formula of the type shown, we may also change the η_1-coordinate by writing

$$\eta_1 = Y_1 + \frac{1}{R}\frac{G(Y_1)Y_1}{X^2 + Y^2}. \tag{35}$$

Then (30) becomes

$$fg + \eta_1 f'G = -f_2. \tag{36}$$

The solution is no longer everywhere unique. But to order R^{-1} the solution turns out to be unique, and to be the same solution as before, everywhere outside the region consisting of the small circle of radius of order R^{-1} with its centre at the leading edge and with a small sector removed about the radius along the plate; in other words, to order R^{-1} the method provides a unique answer in the region in which it is correct to start from a boundary layer solution as an approximation at large Reynolds numbers. The formula for τ_1 is unaltered.

6. *Flows with Wakes and Separation.*

If we consider the limit as $\nu \to 0$, $R \to \infty$, of the steady viscous flow past a finite flat plate ($y = 0$, $0 < x < 1$), we obtain $u = 1$ everywhere except on $y = 0$, $x \geqq 0$. Vortex sheets are present on both sides of the plate, through which u drops from 1 to 0. But there is also a singular surface on $y = 0$, $x > 1$, along which u increases from 0 to 1 as x increases from 1 to ∞. This singular surface may be regarded as the confluence of two vortex sheets. According to a calculation of the flow in the wake on boundary-layer theory (Ref. 10, § 248), the limiting value of u when $\nu \to 0$ is given by

$$u = 1 - \frac{\alpha}{2\pi^{\frac{1}{2}}x^{\frac{1}{2}}} + \ldots \tag{37}$$

along this line for large x, and by

$$u = 1.23\ (x - 1)^{1/3} - 1.18\ (x - 1)^{4/3} + \ldots \tag{38}$$

for $x - 1$ small. (The singularity at $x = 1$ may well be an artificial result of the use of cartesian coordinates). Numerical values are available to a certain extent.

Thus even for this simple case, "classical" potential flow (even with vortex sheets at the solid surface) does not give the correct limit as $\nu \to 0$.

A similar singular surface will arise in the limiting flow past any cusped, streamlined cylinder (aerofoil) for which separation of the boundary layer does not take place.

Thus even for this simple case any attempt at a construction of an accurate asymptotic expansion for large R is difficult. Kuo (Ref. 3) has made an approximate calculation of the correction of order $R^{-\frac{1}{2}}$ due to the influence of the wake on the pressure, and on $P(x)$ in eqn. (6). He uses cartesian coordinates and simply takes δ_1 constant, V zero, for $x > 1$ on $y = 0$ in calculating Ψ_1. He compares the calculated drag with experiment, and obtains fairly good agreement down to $R = 15$. If this is not accidental, it would provide a little evidence that terms of order R^{-1} are absent.

To obtain the limit as $\nu \to 0$, $R \to \infty$ of steady, two-dimensional flows past cylinders with boundary-layer separation is, of course, even more difficult. Classical theory is incorrect, but it sometimes held that the "free-streamline" theory (Ref. 10, § 10) may give the correct limit. (In general, a "free-streamline" solution is not unique, but the correct solution must satisfy the condition that the position of boundary-layer separation (calculated on boundary-layer theory and independent of R, with the external flow according to free-streamline theory) coincides with the position of separation assumed for each free streamline, and there seems little doubt that the correct solution is that for which the free streamlines have finite curvature everywhere (Refs. 11, 12, 13); so that, for example, the solution obtained long ago by Brodetsky (Ref. 14) for the flow past a circular cylinder is the correct one to choose). But there are considerable difficulties in accepting the free-streamline result as the correct limit. (The question here is purely a theoretical one at this stage, and is not that of adjusting the theory to fit more closely to experiment, which is concerned with turbulent, not steady, motion). If we start from this limit for infinite R, and seek the correction for a large finite R (small but non-zero ν), we know that the vortex sheets which are the free streamlines will diffuse, but nevertheless (as Dr. Batchelor and Professor Lagerstrom have also pointed out) we shall not return to the undisturbed flow at infinity downstream, and this contradicts the usual theory of laminar wakes (Ref. 10, § 249); one or the other or both must be wrong.

Now if we consider a motion started from rest in a viscous fluid, it is known that $\lim_{t\to\infty} \lim_{\nu\to 0}$ and $\lim_{\nu\to 0} \lim_{t\to\infty}$ are quite different. Because of the known action of viscosity in diffusing the vorticity, although vortex sheets may occur in the former, diffused vorticity may not; but restricted regions of diffused vorticity may occur in the second. A very simple example is the flow between two parallel planes, when one is held stationary and the other started moving with uniform velocity in its own plane. The first limit gives zero velocity everywhere between the planes, and a vortex sheet at the moving plane; the second gives constant vorticity between the planes. (In a limiting two-dimensional steady flow, in any finite region, the diffused vorticity must be constant in the region, or in each of two or more parts of it, as Dr. Batchelor has proved.) It is suggested that in flow past a cavity in a solid body, for example (as for the usual static-pressure hole) although in the limiting steady flow there will be a vortex sheet across the mouth of the cavity, the fluid in the cavity may well be in motion with a vorticity which, for two-dimensional motion, is constant, and not at rest. Now what we require is the second limit; it appears likely that what free streamline theory gives is, in fact, the first limit.

Dr. Batchelor, in an unpublished paper, has gone much further. He suggests that the correct limiting steady flow in two-dimensional motion in the wake behind a cylinder with separation consists of two finite regions of constant, and opposite, vorticity, bounded by two free streamlines which come together at a cusp at a finite distance downstream, and separated by a singular surface from the cusp to the rear of the cylinder. There will also be a singular surface downstream from the cusp. In this picture, the drag of the cylinder in steady flow would $\to 0$ as $\nu \to 0$. (Batchelor considers further physical details, refinements of this general picture, and flow symmetrical about an axis). The cusp would appear only in the limit.

Certain plausible physical arguments from the development of vorticity, and of circulation in any circuit, in a fluid of small viscosity may be used to support Batchelor's theory, but it is difficult to decide the question with certainty. Experiment provides no guide. It appears also that the largest Reynolds number for which a numerical solution of the Navier-Stokes equation is available is $R = 40$ for flow past a circular cylinder (Ref. 15). Such numerical calculations show a vortex pair behind the cylinder with a closed streamline, but it is not known if the region enclosed by this streamline becomes infinite as $R \to \infty$ or not.

Any satisfactory asymptotic expansions for large R for separating flows with wakes appear, therefore, to be some way away.

7. *Vortex Motion without Boundaries.*

In this section we are concerned with motion that is three-dimensional. If an assumed, three-dimensional, initial distribution of vorticity is allowed to develop according to the equations of motion of an incompressible, viscous fluid, then the statistical effect of the change of length of the vortex lines is to produce an increase in the mean square vorticity, $\overline{\omega^2}$; the effect of viscosity is to decrease $\overline{\omega^2}$, which in general first increases to a maximum and then decreases. When $\overline{\omega^2}$ is at or near its maximum, regions of high vorticity appear in the fluid, the vorticity being small elsewhere. When this is the case, in a region of high vorticity the convection terms in the equations of motion (which give rise to the effect of the stretching of the vortex lines) must be of the same order of magnitude as the viscous terms when ν is small, so that these regions will be layers whose thickness is $O(\nu^{\frac{1}{2}})$ as $\nu \to 0$, rather like boundary layers, though now no solid boundaries are assumed to be present. But in such motions as those here considered, which are more usually studied in connection with turbulent motion, the rate of dissipation of energy must remain finite and non-zero as $\nu \to 0$; in layers like boundary layers the rate of dissipation for unit area of the layer is $O(\nu^{\frac{1}{2}})$. [In regions of rapid transition of the shock-wave type, lying across the direction of the stream, in compressible fluids, the thickness is $O(\nu^{-1})$ and the rate of dissipation per unit area is $O(1)$.] After a conversation with Professors Burgers and Timman, it appeared that the only way in which the convection terms and viscous terms could be of the same order, with a rate of dissipation of $O(1)$, was that there should be layers of intense vorticity of thickness of $O(\nu^{\frac{1}{2}})$, but of area of $O(\nu^{-\frac{1}{2}})$, so their volume in any finite volume of the fluid is $O(1)$; when $\nu \to 0$ these layers become surfaces, but the surfaces still occupy a finite, non-zero fraction of the volume of the fluid. For a triply periodic distribution of the initial velocity components — i.e. with u, v, and w each given by a single term of a triple Fourier series — Taylor and Green (Ref. 16) computed by series in the time t, and showed that $\overline{\omega^2}$ would rise to a maximum before falling. The computation could be carried out only at fairly low Reynolds numbers, and a solution in powers of the Reynolds number (Ref. 17) includes the solution in powers of t and converges slightly better; but neither gives any information at high Reynolds numbers, and all attempts at finding an asymptotic solution for large Reynolds numbers have been unsuccessful. In view of the nature of the singularity as $R \to \infty$ proposed above, this is not surprising.

Recently (Ref. 18) Proudman and Reid have shown that for an infinite field of homogeneous isotropic turbulence in an inviscid fluid, with the assumption that certain fourth-order correlations of the velocity components at three points of the fluid are related to second-order correlations in the same way as for a Gaussian probability distribution, then with the initial conditions $\overline{\omega^2} = \overline{\omega_0^2}$, $d\overline{\omega^2}/dt = 0$ at $t = 0$, $\overline{\omega^2} \backsim 6/(t - t_0)^2$ near $t = t_0$, where $t_0 \propto (\overline{\omega_0^2})^{\frac{1}{2}}$. $[(\overline{\omega_0^2})^{\frac{1}{2}}t_0 = 5.9$ approximately$]$. This result has not yet been extended to a fluid of small but finite viscosity.

8. *Viscous Gases.*

When the equations of motion for a viscous compressible fluid are considered, the situation is, naturally, one of considerably greater difficulty. I shall only mention briefly two matters closely connected with subjects referred to elsewhere in this lecture.

Boundary layers have been considered by many authors, with some considerable success for the first approximation; in particular, the boundary layer along a semi-infinite flat plate along the stream has been computed. Kuo (Ref. 19) has recently made an interesting attempt to extend his work on the interaction of the boundary layer and the external stream to calculate the flow past a plate at high Mach numbers, using a similar technique of coordinate straining in Lighthill's manner. The effects now are not small, for it is imperative to take the interaction into account; there is a shock wave of finite strength before the plate, inclined at a small inclination to the plate and curved, with rotational flow behind.

The number of known exact solutions of the full equations is very small. The simplest is the steady flow between two infinite horizontal planes in relative motion parallel to themselves (shearing motion) which has been calculated by Illingworth (Ref. 20) under quite general conditions. The difference from the incompressible case is due to the variation of the viscosity with temperature and to dissipation of energy and heat conduction. If the viscosity μ is independent of the temperature T, $u = y$ as for an incompressible fluid. The next simplest case is when both plates are at the same temperature T_1, $\mu \propto T$, and the gas is a perfect gas with constant specific heats c_p, c_p/γ, and a constant Prandil number $\sigma = \mu c_p/k$, where k is the heat conductivity. Let U_1 be the velocity of the moving plate, a_1 the velocity of sound at the temperature T_1, M_1 the Mach number U_1/a_1, and

$$b = \sigma(\gamma - 1)M_1^2. \tag{39}$$

Then $$u + \frac{1}{4} bu^2 - \frac{1}{6}bu^3 = (1 + \tfrac{1}{12}b)y \tag{40}$$

and $$T/T_1 = 1 + \tfrac{1}{2}bu - \tfrac{1}{2}bu^2. \tag{41}$$

Unless the distance h between the plates is very large, so that gh/a_1^2 is not small, the pressure P between the plates is almost constant, since if P_0 is the pressure at the lower plate

$$P/P_0 = e^{-cu}, \ c = \frac{\gamma gh}{a_1^2(1+\frac{1}{12}b)}. \tag{42}$$

Because the plates are at the same temperature, the temperature, and therefore the viscosity, are greatest in the middle; $\mu\, du/dy$ is constant, so du/dy is least in the middle. The distribution is still anti-symmetrical about $y = \frac{1}{2}$.

The unsteady problem, when the moving plate is started impulsively, is one of considerable difficulty, and has not been much studied. Without a stationary plate, when the motion is produced by a single infinite flat plate started moving impulsively in its own plane with uniform velocity in an infinite gas, the problem has been considered by a number of authors (Refs. 21 to 25). Further study is justified, perhaps by a variation of mathematical methods mentioned elsewhere in this lecture. The motion is a difficult example of mixed diffusion and wave motion; the wave from the plate must culminate in a shock wave.

9. *Singular Perturbations of the Non-Linear Wave Equation.*

I pass now to my final subject, suggested by the influence of viscosity on the formation of shock waves.

Two equations have been solved explicitly which exhibit certain typical features of shock-wave theory, and I shall discuss these rather than the approximate methods used for the actual equations (similar but harder) for the motions of gases.

As a preliminary, consider the non-linear wave equation (with x as a time-like coordinate)

$$\frac{\partial u}{\partial x} + G(u)\frac{\partial u}{\partial y} = 0. \tag{43}$$

The nature of the continuous solution is well known. Let $u_0(y) = f(y)$ be the initial distribution of u when $x = 0$. For illustrative purposes, it is sufficient here to assume that $G(u)$ and $f(y)$ are monotonic. Then if there is a continuous solution it is

$$u = f(y - G(u)x), \tag{44}$$

i.e. if $y = f_1(u)$ initially, then

$$y = f_1(u) + xG(u). \tag{45}$$

Multiple values of u for the same y will occur if $G(u)$ is increasing and $f(y)$ decreasing, or if $G(u)$ is decreasing and $f(y)$ increasing. It is assumed that the solution then becomes discontinuous. When a discontinuity occurs, as y increases through the discontinuity at a given x, u decreases discontinuously if $G(u)$ is increasing, and u increases discontinuously if $G(u)$ is decreasing. There must be a discontinuity as soon as x exceeds the least possible value of $-f_1'(u)/G'(u)$. We cannot prove that there is no discontinuity for smaller x without considering the limit of a perturbed equation, but if this is assumed the resulting course of the discontinuity can be traced. Let

$$G(u) = g'(u), \tag{46}$$

and write (43) in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad v = g(u). \tag{47}$$

The discontinuity will be propagated along some curve in the (x, y) plane, and from the first of (47) and an application of Stokes's theorem it follows that along this curve

$$[u]dy - [v]dx = 0, \tag{48}$$

where $[u]$ and $[v]$ are the discontinuities in u and v; i.e., if u jumps from u_1 to u_2 across the discontinuity

$$\frac{dy}{dx} = \frac{g(u_2) - g(u_1)}{u_2 - u_1} = \frac{\int_{u_1}^{u_2} G(u)du}{u_2 - u_1}. \tag{49}$$

Also at the discontinuity

$$y = f_1(u_1) + xG(u_1) = f_1(u_2) + xG(u_2), \tag{50}$$

and these equations suffice to determine the position and strength of the discontinuity at any time. [They can be applied to determine the position and strength of a shock wave in a gas only in so far as the variations of entropy arising from the growth of the shock wave can be neglected].

The two equations which have been solved explicitly both represent singular perturbations of an equation of the type of (43) or (47). The first is Burgers's equation (Refs. 26):

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{51}$$

with ν constant.

Given an initial distribution u_0 of u the non-linear term tends (if u_0 is a decreasing function of y) to steepen the distribution, and in the absence of the viscous term of higher order, discontinuities would result. The higher-order viscous term prevents the formation of discontinuities, causes diffusion of the momentum, and a dissipation of energy which is independent of the viscosity. According to Cole (Ref. 27) the equation was first mentioned by H. Bateman in the Monthly Weather Review in 1915, and Lagerstrom, Cole and Trilling (Ref. 21) used the equation as an approximation for a weak shock wave near the steady state with dissipation neglected; the explicit solution was given by Hopf (Ref. 28) and Cole (Ref. 27); but there is no doubt that the equation is correctly named Burgers's equation.

The explicit solution is

$$u = -2\nu F_y / F \tag{52}$$

(the subscript denotes a partial derivative), where F is a solution of the heat-conduction equation:

$$\frac{\partial F}{\partial x} = \nu \frac{\partial^2 F}{\partial y^2}. \tag{53}$$

The second set of equations is

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \frac{V}{k}\frac{\partial v}{\partial y} &= u - rv + (r-1)uv. \end{aligned}\right\} \tag{54}$$

These equations arise in a number of exchange problems when a fluid flows through the voids along a column containing matter in the solid state — for example, ion exchange between a salt or acid solution and a suitable resin, with the resin particles in a fixed column through which the liquid is flowing; the exchange is of two ions A and B (sodium and hydrogen, for example) of the same valence. If c is the concentration of ion A in the fluid, measured, say, in milliequivalents per unit volume of the fluid, q the concentration of ion A in the solid (in milliequivalents per unit volume of solid), c_0 the total concentration of ions A and B entering the column, Q the total concentration capacity of the solid phase, the equations (54) are based on the conservation equation and an assumed bilinear exchange equation

$$\frac{\partial q}{\partial t} = k\left[c(Q-q) - \frac{1}{K} q(c_0 - c)\right], \tag{55}$$

with

$$r = \frac{1}{K}, \quad u = \frac{c}{c_0}, \quad v = \frac{q}{Q}. \tag{56}$$

K and k are taken as constants. V is the total rate of volume flux of the fluid, equal to αRS, where R is the linear rate of flow of the fluid, $1 - \alpha$ the fraction of the volume of the column filled by the solid, and S the cross-sectional area of the column (all assumed constant). If X is the distance along the column from the entry, and t the time from the initial entry, then x is the total ion concentration (of both ions) on the resin in the length X from the entry, and y the total ion concentration in the fluid which has passed the cross-section X when the time t is reached, so x is proportional to X, and y to $Rt - X$.

$$[x = SQ(1-\alpha)X,\ y = \alpha\, Sc_0(Rt - X).]$$

The same equations apply to other cases with different meanings of the symbols — e.g. fixed-bed adsorption, or (with $r = 1$) heat exchange between a flowing fluid and a crushed solid. When $r = 1$ the equations are linear, and the solution is the limit as $r \to 1$ of the solution for $r \neq 1$, so the case $r = 1$ will not be further considered. (See the references in Ref. 29).

We may write (54) in the form

$$u = \psi_y, \quad v = -\psi_x \tag{57}$$

and

$$\frac{V}{k}\psi_{xy} + \psi_y + r\psi_x + (1-r)\,\psi_x\psi_y = 0; \tag{58}$$

the equation for ψ is reduced to a linear form by the substitution

$$\psi = \frac{V}{k}\frac{1}{1-r}\log\ F(x, y). \tag{59}$$

Then

$$\frac{V}{k}F_{xy} + F_y + rF_x = 0. \tag{60}$$

The boundary conditions are usually such that ψ, and therefore F, are known on the positive halves of both axes, and the solution is required for all positive x and positive y. The logarithmic substitution was used by Thomas (Ref. 30). (A slightly different substitution is a little more convenient for the usual boundary conditions, but that is irrelevant here).

If we put $\nu = 0$ in (51), or $V/k = 0$ in (54), each equation reduces to an equation of the type (47), with

$$\left.\begin{aligned} g(u) &= \tfrac{1}{2}u^2 \quad \text{in (51)} \\ &= \frac{u}{r + (1-r)u} \text{ in (54)} \end{aligned}\right\} \tag{61}$$

so each equation represents a singular perturbation of a first-order non-linear wave equation.

Also, (51) may be expressed in a form similar to (57) and (58), since it may be written

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad v=\tfrac{1}{2}u^2-\nu\frac{\partial u}{\partial y}. \tag{62}$$

Hence u and v may be expressed as in (57), where

$$\psi_x+\tfrac{1}{2}(\psi_y)^2=\nu\psi_{yy} \tag{63}$$

and the solution is

$$\psi=-2\nu\log F \tag{64}$$

where F satisfies (53). Thus both equations which have been explicitly solved (by reduction to linear equations) have been solved by the same substitution. The result may be generalized. If α is a constant,

$$\alpha\{L\psi_{xx}+M\psi_{xy}+N\psi_{yy}\}+P\psi_x+Q\psi_y+R+L\psi_x^2+M\psi_x\psi_y+N\psi_y^2=0 \tag{65}$$

and

$$\psi=\alpha\log F, \tag{66}$$

then F satisfies the linear equation

$$\alpha\{LF_{xx}+MF_{xy}+NF_{yy}\}+PF_x+QF_y+\frac{R}{\alpha}F=0. \tag{67}$$

If F satisfies (67), then

$$u=\alpha F_y/F, \quad v=-\alpha F_x/F \tag{68}$$

is a solution of

$$\left.\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.\\ Lv^2-Muv+Nu^2-Pv+Qu+R=\alpha\left\{L\frac{\partial v}{\partial x}-M\frac{\partial u}{\partial x}-N\frac{\partial u}{\partial y}\right\}.\end{aligned}\right\} \tag{69}$$

Both the two equations above, the one parabolic and the other hyperbolic, are examples of this substitution. In every case in which the equations have been solved explicitly, and the limits found as $\alpha\to 0$, the limit is the solution of the unperturbed equation for u, found as previously described. Near a discontinuity the limit is non-uniform, and does not give sufficient information, particularly for (54), which is of some practical importance in a branch of chemical engineering. Since u is given by a quotient, the process of finding the required formulae for small α is far from trivial; asymptotic values must be carefully dealt with. An example, of some practical importance, from the solution of (54) may be cited. With $r > 1$, so that $G(u) = g'(u)$ is an increasing function, let u_0 be zero for $y < 0$ and for $y > Y$, and $u_0 = 1$ for $0 < y < Y$. The discontinuities are then introduced in the initial values. With the term in V/k omitted (on the "equilibrium" theory), the discontinuity at $y = Y$ is, to begin with, propagated unaltered, and for positive x is at $y = Y + x$. The discontinuity at $y = 0$ becomes diffuse and takes the form

$$u = \frac{r - (rx/y)^{\frac{1}{2}}}{r - 1} \text{ for } x/r \leqq y \leqq rx. \tag{70}$$

But when $rx = Y + x$, the head of the diffuse trailing boundary catches up the discontinuity and, for larger values of x, it eats into it, so that the strength of the discontinuity diminishes. By the methods explained (for $V/k = 0$) it may be shown that for $x \geqq Y/(r - 1)$, $u = 0$ for $y \leqq x/r$ and for $y > y^*$, and is given by the same formula as in (70) for $x/r \leqq y \leqq y^*$, where

$$(ry^*)^{\frac{1}{2}} = x^{\frac{1}{2}} + [(r - 1)Y]^{\frac{1}{2}}. \tag{71}$$

The largest value of u is

$$\frac{r}{r - 1 + [(r - 1)x/Y]^{\frac{1}{2}}}. \tag{72}$$

All this comes out as the limit of the exact solution. For the next approximation near y^*, u is given by the expression in (70) with an additional term in the denominator which, for fixed y and y^*, is exponentially large for $y > y^*$ and exponentially small for $y < y^*$ as $V/k \to 0$, but which is $O(1)$ when $y - y^* = O(V/k)$.

There are, of course, other ways of interpreting and ways of generalizing the result in (65), (66), and (67). We may increase the number of independent variables. We may, if we make a correct choice of the coefficients, consider ψ as a potential instead of a stream function. We may consider non-singular perturbation problems with α large, or general problems with α neither large nor small. There is some interest in other parabolic and hyperbolic equations we can discuss in this way, but the second-order non-linear elliptic equations do not seem to be of any general interest. An example is

$$\nabla^2\psi + \frac{1}{\alpha}\,[(\text{grad } \psi)^2 + c] = 0, \tag{73}$$

which corresponds with

$$\psi = \alpha \log F, \quad (\nabla^2 + c/\alpha^2)\, F = 0. \tag{74}$$

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REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

HARISH-CHANDRA

Let G be a Lie group and $\mathfrak{H}$ a Banach space. A representation π of G on $\mathfrak{H}$ is a mapping which assigns to every element x in G a bounded linear operator $\pi(x)$ on $\mathfrak{H}$ such that the following two conditions are fulfilled: (1) $\pi(xy) = \pi(x)\pi(y)$ $(x, y \in G), \pi(1) = I$ and (2) the mapping $(x, \psi) \to \pi(x)\psi$ of $G \times \mathfrak{H}$ into $\mathfrak{H}$ is continuous. (Here 1 is the unit element of G and I is the unit operator.) In particular if $\mathfrak{H}$ is a Hilbert space and $\pi(x)$ is a unitary operator for every $x \in G$, we say that π is a unitary representation. In the study of finite-dimensional representations the success of the infinitesimal method is well known. Our object is to make this method applicable also to the infinite-dimensional case.

First we discuss a few preliminary notions. Let f be a mapping of some neighborhood of the origin on the real line into the Banach space $\mathfrak{H}$. We say f is analytic at zero if it is possible to write it in the form

$$f(t) = \sum_{n \geqq 0} \psi_n t^n$$

where $\psi_n \in \mathfrak{H}$ and the equality means that the series on the right converges in $\mathfrak{H}$ to $f(t)$ for all values of t sufficiently near zero. The generalisation to the case of several real variables is obvious and so it is clear what is meant by an analytic mapping of a (real) analytic manifold M into $\mathfrak{H}$. Now let ψ be a fixed element in $\mathfrak{H}$ and consider the mapping $x \to \pi(x)$ of G into $\mathfrak{H}$. We shall say that ψ is well-behaved (under π) if this mapping is analytic. Let W be the space of all well-behaved elements in $\mathfrak{H}$. Then if $\mathfrak{g}_0$ is the Lie algebra of G one can prove that the limit

$$\lim_{t \to 0} \frac{1}{t} \{\pi(\exp tX)\psi - \psi\}$$

exists for every $X \in \mathfrak{g}_0$ and $\psi \in W$ and again lies in W. If we denote it by $\pi_W(X)\psi$ we get a linear mapping $\pi_W(X)$ of W into itself. It is easy to verify that

$$\pi_W([X, Y]) = \pi_W(X)\pi_W(Y) - \pi_W(Y)\pi_W(X) \qquad (X, Y \in \mathfrak{g}_0)$$

and therefore this defines a representation π_W of $\mathfrak{g}_0$ on W. Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$. We extend π_W to a representation of $\mathfrak{g}$ by linearity. The importance of the space W derives from the following result:

Let U be a subspace of W which is invariant under $\pi_W(\mathfrak{g})$. Then its closure $\overline{U}$ is invariant under $\pi(G)$.

The next step is to prove that W is dense in $\mathfrak{H}$. This can be done for a fairly large class of Lie groups. However we are primarily interested in the semisimple case. So let us suppose from now on that G is connected and semisimple. Let Z denote the center of G. We shall say that π is a permissible representation if it maps Z into scalar multiples of the unit operator. Then one can show that if π is permissible W is dense in $\mathfrak{H}$.

Let $X_1, \ldots, X_n$ be a base for $\mathfrak{g}$ over the field C of complex numbers and let $[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$, $1 \leqq i, j \leqq n$ $(c_{ij}^k \in C)$. Consider the associative algebra $\mathfrak{B}$ generated by $(1, X_1, \ldots, X_n)$ which is free except for the relations $X_i X_j - X_j X_i = \sum_{k=1}^{n} c_{ij}^k X_k$. Then $\mathfrak{B} \supset \mathfrak{g}$ and it is independent of the particular base $(X_1, \ldots, X_n)$ used in its construction. $\mathfrak{B}$ is called the universal enveloping algebra of $\mathfrak{g}$. It is obvious that every representation of $\mathfrak{g}$ can be extended uniquely to a representation of $\mathfrak{B}$ and so there is a natural $1-1$ correspondence between representations of $\mathfrak{B}$ and $\mathfrak{g}$. We shall denote two such corresponding representations usually by the same letter.

Now π_W, being a representation of $\mathfrak{g}$, may also be regarded as a representation of $\mathfrak{B}$ on W. Let $\mathfrak{Z}$ be the center of $\mathfrak{B}$. We shall say that the representation π of G is quasi-simple if $\pi(x)$ and $\pi_W(z)$ are both scalar multiples of the unit operator for $x \in Z$ and $z \in \mathfrak{Z}$. Let χ be the homomorphism of $\mathfrak{Z}$ into C such that $\pi_W(z) = \chi(z)\pi_W(1)$ $(z \in \mathfrak{Z})$. We call χ the infinitesimal character of π.

Let K be the complete inverse image in G of a maximal compact subgroup of the adjoint group of G. Then K is connected and it contains Z. Let Ω denote the set of all equivalence-classes of finite-dimensional irreducible representations of K. We denote by $\mathfrak{H}_{\mathfrak{D}}$ $(\mathfrak{D} \in \Omega)$ the subspace consisting of all those elements $\psi \in \mathfrak{H}$ which transform [1] under $\pi(K)$ according to the class $\mathfrak{D}$. Let $W_{\mathfrak{D}} = \mathfrak{H}_{\mathfrak{D}} \cap W$. One can prove that if π is permissible the space [2] $\mathfrak{H}_0 = \sum_{\mathfrak{D} \in \Omega} W_{\mathfrak{D}}$ is dense in $\mathfrak{H}$. Choose an element $\psi \neq 0$ in $\mathfrak{H}_0$ and let U be the smallest closed subspace of $\mathfrak{H}$ which is invariant under $\pi(G)$ and which contains ψ. Then if π is quasi-simple $\dim (U \cap \mathfrak{H}_{\mathfrak{D}}) < \infty$ for every $\mathfrak{D} \in \Omega$. In

[1]) This means that the linear space spanned by $\pi(u)\psi$ $(u \in K)$ is of finite dimension and the representation of K defined on it is a direct sum of representations of class $\mathfrak{D}$.

[2]) $\sum_{\mathfrak{D} \in \Omega} W_{\mathfrak{D}}$ consists of all finite linear combinations of elements in $\bigcup_{\mathfrak{D} \in \Omega} W_{\mathfrak{D}}$.

particular if π is an irreducible representation $U = \mathfrak{H}$ and therefore $\dim \mathfrak{H}_{\mathfrak{D}}$ is finite.

Now suppose $\mathfrak{H}$ is a Hilbert space and π is a quasi-simple unitary representation. Then one can deduce, from the above result, the existence of a closed subspace $U \neq \{0\}$ which is invariant and irreducible under $\pi(G)$. This fact has the following significance in relation to the theory of factors of Murray and von Neumann [6]. A unitary representation π is called a factor representation if the weakly closed algebra generated by $\pi(G)$ is a factor. It is not difficult to prove that every factor representation is quasi-simple [8] and so the above remarks are applicable to it. Hence if π is a factor representation $\mathfrak{H}$ contains a minimal closed invariant subspace and therefore from a well-known result of Murray and von Neumann [6] this factor must be of type I. This proves that only factors of type I can arise from unitary representations of a semisimple group.

Now we return to the case where $\mathfrak{H}$ is a Banach space and assume that π is quasi-simple and irreducible. Then we have seen above that $\dim \mathfrak{H}_{\mathfrak{D}} < \infty$ $(\mathfrak{D} \in \Omega)$. Actually one can prove that

$$\dim \mathfrak{H}_{\mathfrak{D}} \leqq N\big(d(\mathfrak{D})\big)^2 \quad (\mathfrak{D} \in \Omega)$$

where $d(\mathfrak{D})$ is the degree of any representation in class $\mathfrak{D}$ and N is an integer independent of $\mathfrak{D}$. This estimate can be improved under suitable assumptions on π or G. For example if we assume that G has a faithful finite-dimensional representation, the factor N can be dropped (see [5(a)] Theorem 4 and [4]). The above inequality has some interesting consequences. Suppose $\mathfrak{H}$ is a Hilbert space and $f(x)$ is a complex-valued square-integrable function on G which vanishes outside a compact set. Then it follows that the operator $\int_G f(x)\pi(x)dx$ (dx is the element of Haar measure on G) is of the Hilbert-Schmidt class. This fact had been observed by Gelfand and Naimark [3] in their study of representations of the complex classical groups.

One can also define a character of π as follows. We say that a bounded operator A on $\mathfrak{H}$ has a trace (or is of the trace class) if for every complete orthonormal set ψ_i $(i = 1, 2, \ldots)$ in $\mathfrak{H}$ the series $\sum_{i \geqq 1} (\psi_i, A\psi_i)$ is convergent and its sum is independent of the particular choice of this set. This sum is then called the trace of A and we shall denote it by $\operatorname{sp} A$. Let $C_c^\infty(G)$ denote the class of all complex-valued functions on G which are everywhere indefinitely differentiable and which vanish outside a compact set. Then if π is a quasi-simple irreducible representation of G on $\mathfrak{H}$ we consider the operator $\int f(x)\,\pi(x)dx$ $(f \in C_c^\infty(G))$. One proves that it has a trace which we denote by $T_\pi(f)$. The mapping $T_\pi : f \to T_\pi(f)$ is clearly a linear function on $C_c^\infty(G)$. It follows from the inequality $\dim \mathfrak{H}_{\mathfrak{D}} \leqq N(d(\mathfrak{D}))^2$ that it is actually a distribution in the sense of L. Schwartz [9]. We call this distribution the character of π. Equivalent representations have the same character. Conversely two irreducible unitary representations [1]) are equivalent if their characters are equal. The existence of these characters had been noticed by Gelfand and Naimark [3] for certain irreducible unitary representations of complex classical groups.

We now come to the Plancherel formula. Let $\mathfrak{E}$ be the set of all equivalence classes of irreducible unitary representations of G and let $C_c(G)$ denote the set of all complex-valued continuous functions on G which vanish outside a compact set. Choose any class $\omega \in \mathfrak{E}$ and a representation π in ω. Then we have seen that the operator $\int f(x)\pi(x)dx$ is of the Hilbert-Schmidt class. It is clear that its Hilbert-Schmidt norm depends only on the class ω of π. We denote the square of this norm by $N_\omega(f)$. Then in analogy with the Plancherel formula for locally compact abelian groups or the Peter—Weyl theorem for compact groups, we would like to obtain a positive measure $d\omega$ on $\mathfrak{E}$ such that

$$\int_G |f(x)|^2 dx = \int_{\mathfrak{E}} N_\omega(f) d\omega$$

for all $f \in C_c(G)$. In view of the fact that any factor arising from a unitary representation of G must always be of type I, it follows from the reduction theory of von Neumann [7] that such a measure $d\omega$ exists and is unique. However the problem of computing it and relating it to the structure of G still remains. In case G is a *complex* semisimple group it is possible to determine this measure explicitly. The main reason why we have to assume that G be complex, is that in that case all Cartan subgroups of G are conjugate. Since this, in general, is not true for real groups the corresponding problem for them is much more complicated. There each conjugacy class of Cartan subgroups seems to make a separate contribution to the Plancherel formula, so that the total measure $d\omega$ is the sum of the measures contributed by these various conjugacy classes. For example the group of 2×2 real unimodular matrices has two such classes and corresponding to these we get the continuous and the discrete series of representations in the Plancherel formula (see [1] and [5(b)]). Recently Gelfand and Graev [2] have made a beautiful application of the method of analytic continuation of Riesz for solving certain hyperbolic differential equations (see [9 p. 50]), towards the determination

[1]) Every irreducible unitary representation is automatically quasi-simple [8].

of $d\omega$. By this powerful method they are able to handle the case of $n \times n$ real unimodular group in detail and their results confirm the above picture. As a partial explanation of this apparently intimate connection between the conjugacy classes of Cartan subgroups and the various 'series' of representations which appear in the Plancherel formula we shall sketch here a general method of associating, in a natural way, certain irreducible unitary representations of G to each such conjugacy class. Let A be a given Cartan subgroup of G. An easy argument shows that the really important case is when $A \subset K$ and G is simple. (The general case can be reduced to this without much trouble.) Moreover in order to avoid inessential complication, let us assume that G has a faithful finite-dimensional representation. Then we can regard G as the real analytic subgroup corresponding to $\mathfrak{g}_0$, of a complex analytic group G_c with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}_0$ be the Lie algebra of A and $\mathfrak{h}$ its complexification in $\mathfrak{g}$. Then it is possible to choose a nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{h} + \mathfrak{n}$. Let A_c and N_c be the complex analytic subgroups of G_c corresponding to $\mathfrak{h}$ and $\mathfrak{n}$ respectively. Then $G_c^0 = GA_cN_c$ is an open subset of G_c and therefore it may be regarded as a complex manifold. Let ξ be a complex-analytic character of A_c. Consider the space $\mathfrak{H}_\xi$ of all holomorphic functions φ on G_c^0 such that $\varphi(xan) = \varphi(x)\,\xi(a)$ $(x \in G_c^0,\ a \in A_c,\ n \in N_c)$ and

$$||\,\varphi\,||^2 = \int_G |\,\varphi(x)\,|^2 dx < \infty.$$

It is not difficult to show that $\mathfrak{H}_\xi$ is complete under the norm $||\,\varphi\,||$ and so it is a Hilbert space. Now if $\mathfrak{H}_\xi \neq \{0\}$ we can define a unitary representation π_ξ of G on it as follows:

$$(\pi_\xi(x)\varphi)(y) = \varphi(x^{-1}y) \quad (x \in G,\ y \in G_c^0).$$

It can be proved that this representation, which is obviously unitary, is irreducible and square-integrable i.e.

$$\int_G |\,(\varphi, \pi_\xi\,(x)\varphi)\,|^2\, dx < \infty$$

for every $\varphi \in \mathfrak{H}_\xi$. (Here we have used the usual notation for the scalar product in $\mathfrak{H}_\xi$.) One can prove that $\mathfrak{H}_\xi = \{0\}$ unless the first Betti number of G is 1 or what is equivalent, unless the center of K is non-discrete. (We assume naturally that G is not compact). On the other hand if this condition is fulfilled $\mathfrak{H}_\xi \neq \{0\}$ provided ξ is suitably chosen.

As we have already mentioned above, the general case can be reduced to the one considered above and so corresponding to each Cartan subgroup A we get a series of unitary irreducible representations of G.

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SOME ASPECTS OF THE THEORY OF ALMOST PERIODIC FUNCTIONS

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1. In this address, I shall endeavour to give an account of some of the work that has been done in Copenhagen, in recent years, on almost periodic functions.

Let me first recall the main points of the theory of almost periodic functions of a real variable as developed by Bohr [1]) in 1924—25. We are concerned with complex functions $F(t) = U(t) + iV(t)$ of an unrestricted real variable t. In order to be almost periodic, the function must be continuous, and for every $\varepsilon > 0$ it must have translation numbers τ, i.e., numbers for which $| F(t + \tau) - F(t) | \leqq \varepsilon$ for all t, and not too few — more precisely, there must be a length $l(\varepsilon)$ such that every interval of this length on the real axis contains one of these numbers τ.

Every almost periodic function is bounded. The sum or product of two almost periodic functions and the limit of a uniformly convergent sequence of almost periodic functions are again almost periodic. Every almost periodic function possesses a mean value, $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T F(t)dt$. For a given almost periodic function, the mean value $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T F(t)e^{-i\lambda t}dt$, where λ is real, differs from 0 for at most an enumerable set of values of λ. Let these values be denoted by λ_n and put $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T F(t)e^{-i\lambda_n t}dt = a_n$. The trigonometric series

$$F(t) \sim \sum a_n e^{i\lambda_n t}$$

is called the Fourier series of $F(t)$ and the numbers λ_n the Fourier exponents of $F(t)$. For a continuous periodic function with period 2π, these exponents λ_n are integers and the series is the usual Fourier series. The main result of the theory is the approximation theorem according to which the class of almost periodic functions is identical with the class of those functions which can be approximated uniformly by trigonometric polynomials

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1) Acta Math. 45 (1924), 29—127, 46 (1925), 101—214; *Coll. Math. Works* II, C3, C7.

$$P(t) = c_1 e^{i\lambda_1 t} + \ldots + c_N e^{i\lambda_N t}$$

with arbitrary real exponents λ_n and arbitrary complex coefficients c_n. In other words, a function $F(t)$ is almost periodic if and only if to each ε there corresponds a trigonometric polynomial $P(t)$ such that $| F(t) - P(t) | \leqq \varepsilon$ for all t. As shown by Bochner[2]), such approximating trigonometric polynomials can be obtained from the Fourier series by a suitable summation method.

The crucial point of Bohr's definition of almost periodicity is the existence of the length $l(\varepsilon)$. We express the existence of this length by saying that, for each ε, the translation numbers form a relatively dense set on the real axis. It was therefore quite surprising when it turned out, as a consequence of a very beautiful direct proof of the approximation theorem due to Bogolyubov[3]), that the relative density condition can be replaced by a weaker condition. It suffices, indeed, to assume that for each ε there is a sequence of translation numbers τ_n such that $| \tau_n - \tau_m | \geqq \alpha > 0$ for all n and m, and $\tau_n = O(n)$. On closer examination, it turned out that actually the difference is not at all deep, since, as shown by Følner[4]), it can be proved easily that the existence of translation numbers satisfying Bogolyubov's condition implies the existence of a relatively dense set of translation numbers. The last paper of Bohr on almost periodic functions[5]) deals with the problem of whether or not Bogolyubov's condition can be replaced by a still weaker one. He proved that this is not the case — more precisely, that if, instead of $\tau_n = O(n)$, we assume $\tau_n = O(\psi(n))$, where $\psi(n)$ is any function which goes to infinity more rapidly than n, then we obtain a class of functions which is properly larger than the class of almost periodic functions.

Let us define the uniform norm of a function $F(t)$ by

$$|| F ||_U = \sup_{-\infty < t < \infty} | F(t) |.$$

It is finite for the bounded functions. By taking $|| F - G ||_U$ as distance, we obtain a metric space. The content of the approximation theorem is then that the class of almost periodic functions is the closure, with respect to this metric, of the class of trigonometric polynomials; in symbols,

$$\{\text{a.p.}\} = \text{Cl}_U \{\text{trig. pol.}\}.$$

2. Shortly after the appearance of the theory of almost periodic functions, a number of generalizations were introduced and discussed. We shall not go

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[2]) Math. Ann. 96 (1927), 119—147.

[3]) Ann. Chaire Phys. Math. Kiev 4 (1939), 195—205.

[4]) Mat. Tidsskr. B 1944, 24—27.

[5]) J. Anal. Math. 1 (1951), 11—27; *Coll. Math. Works* III, C 54.

into the definitions of these generalizations here [6]) but merely indicate how the various classes of generalized almost periodic functions are characterized through approximation by trigonometric polynomials. Instead of continuous functions, let us consider functions which are measurable in the Lebesgue sense. The first generalization was made by Stepanov. By the Stepanov norm of order $p \geqq 1$ of a measurable function $F(t)$, we mean

$$\| F \|_{S^p} = [\sup_{-\infty < t < \infty} \int_t^{t+1} |F(u)|^p du]^{1/p}.$$

The class of Stepanov almost periodic functions of order p is then the closure with respect to the distance $\| F - G \|_{S^p}$, of the class of trigonometric polynomials; in symbols,

$$\{S^p \text{a.p.}\} = \text{Cl}_{S^p} \{\text{trig. pol.}\}.$$

This generalization was also made by Wiener. Another generalization, which leads to a much wider class of functions, was given by Besicovitch who considered the norm

$$\| F \|_{B^p} = [\limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |F(t)|^p dt]^{1/p}.$$

The class of Besicovitch almost periodic functions of order p is then the closure, with respect to the distance $\| F - G \|_{B^p}$, of the class of trigonometric polynomials; in symbols

$$\{B^p \text{ a.p.}\} = \text{Cl}_{B^p} \{\text{trig. pol.}\}.$$

There is also an intermediate generalization introduced by Weyl based on a norm $\| F \|_{W^p}$. In all these cases, we have Fourier series $F(t) \sim \sum a_n e^{i\lambda_n t}$ just as in the original case.

A number of important questions concerning these generalizations were settled by Ursell [7]). Among other things, he proved that, whereas the Stepanov and Besicovitch classes are complete metric spaces, the Weyl classes are not. A very thorough examination of the relations between the different classes was made by Bohr and Følner [8]) in 1944 and continued by Følner [9]). It would lead us too far afield to go into the very detailed description they obtained.

In the case $p = 2$, we have the Parseval equality

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[6]) See, for example, the comprehensive treatment by Besicovitch and Bohr, Acta Math. 57 (1931), 203—292; Bohr, *Coll. Math. Works* II, C 27; or Besicovitch, *Almost periodic functions*, Cambridge, 1932.

[7]) Proc. London Math. Soc. (2) 32 (1931), 402—440.

[8]) Acta Math. 76 (1944), 31—155; Bohr, *Coll. Math. Works* III, C 47.

[9]) Thesis, Copenhagen, 1944.

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |F(t)|^2 dt = \sum |a_n|^2$$

in all three cases. Besicovitch's generalization is interesting because, for this generalization, the analogue of the Riesz-Fischer theorem is also valid, that is to say, any trigonometric series $\sum a_n e^{i\lambda_n t}$ with $\sum |a_n|^2$ finite is the Fourier series of a B^2 almost periodic function. This is not the case for Stepanov or Weyl functions.

One can, however, look at the question of generalization of the theorems on Fourier series of periodic functions in another way, and I would like to comment on the generalization of the Riesz-Fischer theorem by way of example. In many respects, the Stepanov generalization is the most satisfactory generalization, and one would therefore like to have analogues of the Riesz-Fischer theorem for Stepanov functions. Now, the Lebesgue class L^2 of periodic functions with period 2π can also be characterized as the class of those S^2 almost periodic functions for which the Fourier exponents λ_n are integers. Thus the Riesz-Fischer theorem holds for these functions. One can now ask: For which sequences of exponents λ_n does the Riesz-Fischer theorem hold for S^2 almost periodic functions having these exponents. This question has a very simple answer. As pointed out by Tornehave [10]), it follows from an old result of Wiener and a method of proof due to Stepanov that a sequence of exponents has the property in question if and only if there exists a constant k such that no interval of length 1 contains more than k exponents. Presumably, further interesting results could be obtained along this line.

3. The generalization of the theory of almost periodic functions which is of greatest interest for the original theory is, of course, von Neumann's generalization of the theory to functions on groups [11]). It would fall entirely outside the scope of this lecture to go into this theory which recently has received a beautiful exposition in Maak's book [12]). I would merely like to mention that the theory of almost periodic functions on groups may sometimes be put to use in the ordinary theory in cases where one would expect a direct analytical approach to be simpler. In this connection, we may mention a very simple treatment of Besicovitch almost periodic functions recently given by Doss [13]), and also the theory of linear functionals on spaces of generalized almost periodic functions. This

[10]) Math. Scand. 2 (1954), 237—242. The result is a corollary of Wiener's result and of its counterpart proved by Tornehave in this paper.

[11]) Trans. Amer. Math. Soc. 36 (1934), 445—492.

[12]) *Fastperiodische Funktionen*, Berlin/Göttingen/Heidelberg, 1950.

[13]) Bull. Sci. Math. (2) 77 (1953), 186—194.

latter theory was first considered by Doss [14]), and, recently, Følner [15]) has obtained further results generalizing the classical result of F. Riesz on the L^p spaces.

4. The remainder of this lecture will be devoted to the subject of mean motions of almost periodic functions and related topics. For a moment, let us forget about almost periodic functions and go back to a classical investigation by Lagrange [16]) from 1782 on the perturbations of the large planets. To indicate the elliptic orbit of a planet, we use certain parameters called the elements of the orbit. Owing to the influence of the other planets, the elliptic orbit of a planet changes slowly in the course of time. These parameters are therefore functions of the time t. Among these parameters is the longitude of the perihelion, i.e., a certain angular variable indicating the direction to the perihelion.

As shown by Lagrange, this angle is, in the first approximation, determined as the argument of a certain trigonometric polynomial

$$F(t) = a_1 e^{i\lambda_1 t} + \dots + a_N e^{i\lambda_N t}$$

with complex coefficients a_n and real exponents λ_n. Thus this polynomial is a sum of vectors each having a constant length and turning with a constant angular velocity. The number of terms N equals the number of planets. Thus the study of the variation of the longitude of the perihelion leads to a study of the variation of the argument $\arg F(t)$ of a trigonometric polynomial.

These polynomials were calculated by Lagrange for the different planets, and it turned out that, in most cases, the polynomial contains a preponderant term, i.e., a term whose absolute value exceeds the sum of the absolute values of the remaining terms. Suppose, for example, that the first term is preponderant, that is, $|a_1| > |a_2| + \dots + |a_N|$. Then $F(t)$ does not assume arbitrarily small values, and it is easy to see that the argument of $F(t)$ will differ by less than $\frac{1}{2}\pi$ from the argument of the first term. Consequently we have, for a continuous branch of the argument, the formula

$$\arg F(t) = \lambda_1 t + O(1).$$

Thus the argument is, in this case, the sum of a secular term $\lambda_1 t$ and a bounded remainder.

5. Lagrange formulated the problem of investigating the variation of the argument in the case when there is no preponderant term. In this case, it may happen that $F(t)$ comes arbitrarily near to 0 or even assumes the value 0. The

[14]) Amer. J. Math. 72 (1950), 81—92.

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[16]) Nouv. Mém. Acad. Berlin 1781—82, Oeuvres 5, 123—344.

line joining 0 and $F(t)$ will however vary continuously, even if $F(t)$ becomes 0, provided, that at 0, we replace the line by the tangent (which exists since the curve is analytic). It will however change its positive direction when passing through a zero of odd order. In order to be able to speak of a continuous branch of the argument, we must therefore consider the argument mod π and not as usual mod 2π. In the case of a planet, this means that instead of the perihelion itself we must consider the line of apsides.

6. After attempts by various astronomers, the first non-trivial case, $N = 3$, was completely treated by Bohl[17]) in 1909. He showed that, in this case, we always have

$$\arg F(t) = ct + o(t);$$

but the constant c is generally not. as before, one of the exponents, and the remainder is generally not bounded. This result is equivalent to the existence of the limit

$$c = \lim_{T \to \infty} \frac{\arg F(T) - \arg F(0)}{T}.$$

The constant c is called the *mean motion* of $F(t)$.

Bohl's proof depends on diophantine approximations. It is in this connection that the important notion of equidistribution mod 1 occurs for the first time. Bohl's investigation has since been extended to very general cases, notably by Weyl[18]) by means of his general theorem on equidistribution mod 1.

7. After the creation of the theory of almost periodic functions, it was natural to extend Lagrange's problem to this wider class of functions. It was Wintner who first called attention to the connection between almost periodic functions and astronomical problems. It was conjectured by Wintner and proved by Bohr[19]) in 1930 that if $F(t) \sim \sum a_n e^{i\lambda_n t}$ is an almost periodic function which does not come arbitrarily near to 0, i.e., $|F(t)| \geqq k > 0$, then

$$\arg F(t) = ct + O(1),$$

even if there is no preponderant term in the Fourier series. The mean motion c need not be one of the Fourier exponents.

8. Before continuing the discussion of Lagrange's problem, I would like to digress slightly by mentioning certain results which are connected with Bohr's theorem. If the almost periodic function $F(t)$, or, as we may also say, the almost

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[17]) J. Reine Angew. Math. 135 (1909), 189—283.

[18]) Enseignem. Math. 16 (1914), 455—467, Math. Ann. 77 (1916), 313—352, Amer. J. Math. 60 (1938), 889—896, 61 (1939), 143—148.

periodic movement $F(t)$ in the plane, does not come near to two points a and b, I could prove [20]) that the mean motions c_a and c_b of $F(t) - a$ and $F(t) - b$ have a rational ratio. This result is contained in a general theorem of Fenchel and myself [21]) to the effect that an almost periodic movement $F(t)$ in a closed domain D bounded by a finite number of circles is homotopic in D to a periodic movement $G(t)$. This means that there exists a family of almost periodic movements $H_\theta(t)$ in D depending uniformly continuously on the parameter θ for $0 \leqq \theta \leqq 1$ and such that $H_0(t) = F(t)$ and $H_1(t) = G(t)$. This follows from a similar result on almost periodic movements on closed surfaces with negative Euler characteristic. A movement $f(t)$ on such a surface is called almost periodic if, for each $\varepsilon > 0$, it has a relatively dense set of translation numbers τ, i.e., numbers for which dist $[f(t+\tau), f(t)] \leqq \varepsilon$ for all t. The theorem is to the effect that such a movement is homotopic, in the sense described before, to a periodic movement on the surface. The proof depends on the simple fact that a commutative subgroup of the fundamental group of the surface is cyclic. The theorem is not true for movements on a torus.

These investigations have been continued, in various ways, by Tornehave. Recently Tornehave [22]) has discussed almost periodic movements in arbitrary metric spaces. He proves, among other things, that the theorem just mentioned also fails for movements on a sphere, a case which Fenchel and I had not decided.

9. After this digression, let us return to the discussion of Lagrange's problem. If the almost periodic function $F(t)$ comes arbitrarily near to 0, the variation of its argument may be very complicated, and if the function has zeros, it may even be impossible to choose the argument as a continuous function of t. In this case, the study of the variation of the argument seems to be of interest only when the function $F(t)$ is obtained by considering an *analytic* almost periodic function on a vertical line. Let me briefly recall the main points of the theory of analytic almost periodic functions as developed by Bohr [23]) in 1926.

We are concerned, in this theory, with functions $f(s) = f(\sigma + it)$ of a complex variable $s = \sigma + it$ in a vertical strip $\alpha < \sigma < \beta$. In order to be almost periodic, the function must be regular, and, for every $\varepsilon > 0$ and every reduced strip $\alpha_1 < \sigma < \beta_1$, it must have a relatively dense set of translation

[19]) Dan. Mat. Fys. Medd. 10, no. 10 (1930), 5—11, Comment. Math. Helv. 4 (1932), 51—64; *Coll. Math. Works* II, C 24, C 29.

[20]) Math. Ann. 111 (1935), 355—363.

[21]) Dan. Mat. Fys. Medd. 13, no. 6 (1935), 1—28.

[22]) Dan. Mat. Fys. Medd. 28, no. 13 (1954), 1—42.

[23]) Acta Math. 47 (1926), 237—281; *Coll. Math. Works* II, C 12.

numbers τ, i.e., real numbers for which $|f(s + i\tau) - f(s)| \leqq \varepsilon$ for all s in the reduced strip. On every vertical line of the strip, i.e., for every fixed σ, the function is an almost periodic function of t. The Fourier series of these functions are obtainable from a certain exponential series $f(s) \backsim \sum a_n e^{\lambda_n s}$ with complex coefficients a_n and real exponents λ_n by replacing s by $\sigma + it$. This series is called the Dirichlet series of $f(s)$ and the numbers λ_n the Dirichlet exponents of $f(s)$. The main result is the approximation theorem according to which a function is almost periodic in a strip if and only if it can be approximated uniformly in every reduced strip by exponential polynomials $p(s) = c_1 e^{\lambda_1 s} + \ldots + c_N e^{\lambda_N s}$ with complex coefficients c_n and real exponents λ_n, i.e., if to each ε and each reduced strip, there corresponds a polynomial $p(s)$ such that $|f(s) - p(s)| \leqq \varepsilon$ for all s in the reduced strip.

10. It is evident that for an analytic function the variation of the argument on vertical lines must be closely related to the distribution of the zeros of the function in vertical strips. This establishes a connection between Lagrange's problem and problems concerning the distribution of the values of analytic almost periodic functions. Such investigations were carried out by Bohr, partly in collaboration with other authors, especially in the case of the Riemann zeta function $\zeta(s) = \sum n^{-s}$. These investigations, which began about 1910, are closely related to the method of Bohl and Weyl. Historically they are at the origin of the theory of almost periodic functions [24].

It was natural to try to generalize these investigations to arbitrary analytic almost periodic functions $f(s)$. This problem was studied by myself [25] in 1933 and by Hartman [26]. The main results are as follows. For every σ between α and β, the mean value

$$\varphi(\sigma) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \log |f(\sigma + it)| \, dt$$

exists, and $\varphi(\sigma)$ is a continuous, convex function in the interval $\alpha < \sigma < \beta$. It has therefore a derivative $\varphi'(\sigma)$ at all points of the interval with the exception of at most an enumerable set. If $\varphi(\sigma)$ is differentiable at the point σ and if $\arg f(\sigma + it)$ denotes a continuous branch of the argument on the corresponding vertical line, then the mean motion $\lim\limits_{T\to\infty} \dfrac{\arg f(\sigma + iT) - \arg f(\sigma)}{T}$

[24]) See, for example, the comprehensive exposition by Bohr and the author, Acta Math. 50 (1930), 1—35, 54 (1932), 1—55; Bohr, *Coll. Math. Works* I, B 23, B 24.

[25]) Math. Ann. 108 (1933), 485—516. Certain results had been obtained previously by Favard, *Leçons sur les fonctions presque périodiques*, Paris, 1933, 129—140.

[26]) Trans. Amer. Math. Soc. 46 (1939), 64—81.

exists and is determined by the formula

$$\lim_{T\to\infty} \frac{\arg f(\sigma+iT) - \arg f(\sigma)}{T} = \varphi'(\sigma).$$

Moreover, if $\varphi(\sigma)$ is differentiable at the points σ_1 and σ_2, then the zeros of $f(s)$ in the strip $\sigma_1 < \sigma < \sigma_2$ have a relative frequency, i.e., if $N(T)$ denotes the number of zeros in the strip $\sigma_1 < \sigma < \sigma_2$ lying between the lines $t = 0$ and $t = T$, then the limit $\lim\limits_{T\to\infty} \frac{N(T)}{T}$ exists and its value is determined by the formula

$$\lim_{T\to\infty} \frac{N(T)}{T} = \frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{2\pi}.$$

For a periodic function, this is merely another form of the classical formula of Jensen. The function $\varphi(\sigma)$ is called the Jensen function of $f(s)$.

Regarding the proof of these results, I shall restrict myself to the following remarks. If we differentiate the expression for $\varphi(\sigma)$ formally and interchange the differentiation and the formation of the mean value, we obtain

$$\varphi'(\sigma) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{d}{d\sigma} \log |f(\sigma + it)| \, dt.$$

Hence by the Cauchy-Riemann equations

$$\varphi'(\sigma) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \arg f(\sigma + it) dt,$$

which is the formula for the mean motion. The formula for the relative frequency of zeros is obtained by expressing the number of zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$ by means of the variation of the argument along the contour and passing to the limit. The non-negativity of the frequency accounts for the convexity of $\varphi(\sigma)$. — The actual proof follows these lines but is complicated by the fact that $\varphi(\sigma)$ need not be differentiable.

11. By a combination of the preceding method and the method of Bohl and Weyl I obtained [27]) in 1938 a complete solution of Lagrange's problem. It turned out that every trigonometric polynomial possesses a mean motion. Subsequently, the Jensen function and its connection with mean motions and distribution of zeros has been the object of a detailed and systematic study by Tornehave and myself [28]) from 1945.

The principal result, so far as Lagrange's problem is concerned, is that, for

[27]) C. R. Acad. Sci. Paris 207 (1938), 1081—1084.

[28]) Acta Math. 77 (1945), 137—279.

an exponential polynomial $f(s) = a_1 e^{\lambda_1 s} + \ldots + a_N e^{\lambda_N s}$, the mean motion exists on every vertical line and is determined by

$$\lim_{T\to\infty} \frac{\arg f(\sigma - iT) - \arg f(\sigma)}{T} = \frac{\varphi'(\sigma - 0) + \varphi'(\sigma + 0)}{2}.$$

On the imaginary axis, we obtain the trigonometric polynomial

$$f(it) = F(t) = a_1 e^{i\lambda_1 t} + \ldots + a_N e^{i\lambda_N t},$$

which therefore possesses a mean motion.

Another important class of analytic almost periodic functions for which the last result is valid is the class of functions represented by a Dirichlet series of ordinary type $f(s) = \Sigma a_n n^{-s}$ in its half plane of uniform convergence.

In addition to these results dealing with special classes of almost periodic functions, the investigation of Tornehave and myself contains a detailed discussion of the variation of the argument on vertical lines in cases where the mean motion does not exist and a determination of all convex functions that can occur as the Jensen function of an analytic almost periodic function.

Further results have been obtained by Tornehave [29]) who, among other things, has studied the Jensen function for analytic functions of several variables.

12. As mentioned earlier, these investigations have their origin in the investigations of Bohr on the distribution of the values of the Riemann zeta function. It was natural to reconsider this question using the Jensen function. This was done by Miss Borchsenius and myself [30]) in 1948. The study of the distribution of the a-points of $\zeta(s)$, i.e., the zeros of $\zeta(s) - a$, by this method, leads to refinements of the results of Bohr. The results deal not only with the half plane $\sigma > 1$ where the function is almost periodic, but, as did the old results, with the part of the critical strip lying to the right of the critical line, i.e., the strip $\frac{1}{2} < \sigma \leqq 1$ where the function is almost periodic only in the Besicovitch sense. The treatment depends on the standard statistical method of characteristic functions, which previously had been applied by Wintner and myself [31]) to the study of the distribution of the values of the zeta function on vertical lines.

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[29]) Thesis, Copenhagen, 1944.

[30]) Acta Math. 80 (1948), 97—166.

[31]) Trans. Amer. Math. Soc. 38 (1935), 48—88.

ОБЩАЯ ТЕОРИЯ ДИНАМИЧЕСКИХ СИСТЕМ И КЛАССИЧЕСКАЯ МЕХАНИКА.

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Введение.

Для меня явилась неожиданностью необходимость сделать доклад на заключительном заседании конгресса в этом большом зале, который был ранее мне известен более как место исполнения великих произведений мировой музыки под управлением Менгельберга. Доклад, который я подготовил, не учитывая перспективы столь почетного его положения в программе нашего конгресса, будет посвящен довольно специальному кругу вопросов. Моей задачей будет выяснение путей применения основных концепций и результатов современной общей метрической и спектральной теории динамических систем к изучению консервативных динамических систем классической механики. Мне кажется, впрочем, что выбранная мною тема может иметь и более широкий интерес как один из примеров рождения новых неожиданных и глубоких связей между различными частями классической и современной математики.

В своем знаменитом докладе на конгрессе 1900-ого года Гильберт говорил, что единство математики, невозможность ее распадения на независимые друг от друга ветви, вытекают из самого существа нашей науки. Наиболее убедительным подтверждением правильности этой мысли является возникновение на каждом этапе развития математики новых, узловых пунктов, где при решении вполне конкретных проблем оказываются необходимыми и вступают в новое переплетение понятия и методы самых различных математических дисциплин. Для математики девятнадцатого века одним из таких узловых пунктов являлись вопросы интегрирования систем дифференциальных уравнений классической механики, где проблемы механики и теории дифференциальных уравнений органически переплетались с проблемами вариационного исчисления, многомерной дифференциальной геометрии, теории аналитических функций и теории непрерывных групп.

После работ Пуанкаре стала ясной фундаментальная роль для этого круга вопросов топологии. С другой стороны теорема о возвращении Пуанкаре-Каратеодори послужила началом "метрической" теории динамических систем в смысле изучения свойств движений, имеющих место при "почти всех" начальных положениях системы. Развившаяся отсюда "эргодическая теория" получила различные обобщения и сделалась самостоятельным центром притяжения и узлом переплетения методов и проблем различных новейших отделов математики /абстрактная теория меры, теория групп линейных операторов в гильбертовском и других бесконечномерных пространствах, теория случайных процессов и т.д./. На предшествующем международном конгрессе 1950-ого года общим вопросам эргодической теории был посвящен большой доклад Какутани [23].

Топологические методы, как известно, получили существенные применения в теории колебаний, в частности, при решении вполне конкретных проблем, возникающих при изучении систем автоматического регулирования, в электротехнике и т. д. Однако, эти реальные физические и технические применения относятся главным образом к неконсервативным системам. Дело сводится здесь обычно к разысканию отдельных ассимптотически устойчивых движений /в частности, устойчивых точек покоя и устойчивых предельных циклов/ и изучению пучков интегральных кривых, притягивающихся к этим ассимптотически устойчивым движениям.

В консервативных системах ассимптотически устойчивые движения невозможны. Поэтому, например, разыскание отдельных периодических движений при всем его математическом интересе имеет в случае консервативных систем лишь весьма ограниченный реальный физический интерес. Основное значение в случае консервативных систем имеет метрическая точка зрения, позволяющая изучать свойства основной массы движений. Современная общая эргодическая теория подготовила для этой цели набор понятий, обладающих очень большой физической убедительностью по своему замыслу. Однако, наши успехи в смысле анализа с этих современных точек зрения конкретных задач классической механики до настоящего времени более чем ограничены.

Дело идет в первую очередь о следующей проблеме. Допустим, что движение по s-мерному аналитическому многообразию V^s определяется канонической системой дифференциальных уравнений с аналитической функцией Гамильтона $H(q_1, \ldots, q_s, p_1, \ldots, p_s)$. Пусть при этом имеется k однозначных аналитических первых интегралов $I_1, I_2, \ldots, I_k$ и условия

$$I_1 = C_1, \ldots, I_k = C_k$$

выделяют из фазового пространства Ω^{2s} аналитическое многообразие M^{2s-k}. Как известно, при почти всех значениях $C_1, \ldots, C_k$ на M^{2s-k}

возникает естественным образом аналитическая инвариантная плотность, что дает возможность применить к движениям на M^{2s-k} общие принципы метрической теории динамических систем. Естественно обращаться к этим более современным средствам в случаях, когда кроме $I_1, \ldots, I_k$ независимых от них однозначных аналитических первых интегралов нет, или их нахождение представляется слишком трудным, а другие классические аналитические методы окончания интегрирования системы оказываются тоже неприменимыми. В таких случаях требуется при помощи тех или иных качественных рассмотрений решить вопрос о том, будет ли движение на M^{2s-k} транзитивным /т.е.будет ли почти все M^{2s-k} состоять из одного единственного эргодического множества/, в случае транзитивности-определить характер спектра, при отсутствии же транзитивности изучить с точностью до множества меры нуль /или хотя бы с точностью до множества малой меры/ характер разложения M^{2s-k} на эргодические множества и характер спектра на этих эргодических множествах.

Мне известны только две конкретные задачи классической механики, в которых эта программа в большей, или меньшей степени уже выполнена:

1/ Для движения по инерции по замкнутой поверхности V^2 всюду отрицательной кривизны*) Хопф в 1939 году установил, что движение на трехмерных многообразиях L_h^3, выделяемых требованием постоянства энергии $H = h$, транзитивно, а спектр непрерывен. /См.[8]/.

2/ Как будет указано далее, при движении по инерции по аналитическим поверхностям достаточно близким к трехосному эллипсоиду движение на L_h^3 не транзитивно и с точностью до множества малой меры разбивается на двумерные торы T^2, на каждом из которых движение транзитивно, а спектр дискретен. /См. конец § 2/.

Однако, мне представляется, что как раз сейчас наступило время, когда окажется возможным значительно более быстрое движение вперед.

§ 1. Аналитические динамические системы и их устойчивые свойства.

Динамические системы классической механики являются частным

*) Быть может, не бесполезно заметить, что в обычном эвклидовом пространстве можно задать замкнутую поверхность V^2 рода один и разместить по соседству с ней конечное число центров притяжения, или отталкивания, создающих на V^2 потенциал сил таким образом, что движение материальной точки по V^2 под действием введенных внешних сил будет математически эквивалентно движению по инерции в метрике, обладающей всюду отрицательной кривизной.

случаем аналитических динамических систем с интегральным инвариантом. Носителем такой динамической системы является аналитическое n-мерное многообразие Ω^n /фазовое пространство системы/. В соответствии с этим допустимые преобразования координат $x_1, \ldots, x_n$ точки $x \in \Omega^n$ будут всегда аналитическими.

Правые части дифференциальных уравнений, определяющих движение,

$$\frac{dx_\alpha}{dt} = F_\alpha(x_1, \ldots, x_n) \tag{1}$$

и инвариантная плотность, порождающая инвариантную меру

$$m(A) = \int_A M(x) dx_1 \ldots dx_n$$

будут считаться аналитическими функциями координат*).

В соответствии со сказанным во введении нас будут по преимуществу занимать канонические системы, т.е. системы с $n = 2s$, разделением координат точки $(q, p) \in \Omega^{2s}$ на две группы $q_1, q_2, \ldots, q_s$ и $p_1, \ldots, p_s$, преобразованиями прикосновения в качестве допустимых преобразований координат, уравнениями канонического вида

$$\frac{dq_\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q_\alpha} \tag{2}$$

и инвариантной плотностью

$$M(q, p) = 1.$$

Особое внимание будет уделено вопросу о том, какие свойства динамических систем являются при "произвольных" F_α и M /или "произвольной" функции $H(q, p)$ в случае канонических систем/ "типичными" и какие могут проявляться лишь в виде "исключения". Вопрос этот, однако, весьма деликатен. Подход со стороны категории соответствующих множеств в функциональных пространствах систем функций $\{F_\alpha, M\}$ /или функций H/ несмотря на известные успехи, полученные в этом направлении в общей теории абстрактных динамических систем, интересен более как средство для доказательств существования, чем как непосредственный ответ на сколь угодно стилизованные и идеализированные реальные запросы физиков или механиков. Подход со стороны меры, наоборот, представляется вполне здравым и естественным с физической точки зрения /как это подробно аргументирова-

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*) Всюду, где мы говорим просто о "мере" без дальнейшей специализации, мы имеем в виду меру m.

лось, например, Нейманом [1]/, но наталкивается на отсутствие естественной меры в функциональных пространствах.

Мы будем следовать двумя путями. Во-первых, для получения положительных результатов о том, что тот или иной тип динамических систем должет быть признан одним из существенных, не "исключительных" и не подлежащих ни с какой разумной точки зрения "пренебрежению" /подобно тому, как пренебрегают множествами меры нуль/, мы будем пользоваться понятием устойчивости в смысле сохранения данного типа поведения динамической системы при малом изменении функций F_α и M, или функции H. Любой тип поведения динамической системы, для которого существует хотя бы один пример его устойчивого осуществления, должен с этой точки зрения считаться существенным и не пренебрегаемым. В соответствии с принятым подходом со стороны аналитических функций "малость" изменения функции $f_0(x)$ будет пониматься в смысле перехода от функции $f_0(x)$ к функции

$$f(x) = f_0(x) + \theta\varphi(x, \theta)$$

при малом значении параметра θ и аналитичности функции φ по совокупности переменных $x_1, x_2, \ldots, x_n, \theta$. Такой подход к делу может подвергаться критике, но при нем можно получить некоторые интересные результаты. Там, где можно ограничиться близостью функций f_0 и f в смысле близости их производных ограниченного порядка, это будет указано.

Для получения отрицательных результатов о несущественном, исключительном характере какого-либо явления мы будем применять только один несколько кустарный прием: если в классе K функций $f(x)$ можно ввести конечное число функционалов

$$F_1(f),\ F_2(f), \ldots,\ F_r(f)$$

которые в том или ином смысле естественно считать принимающими "вообще говоря произвольные" значения

$$F_1(f) = C_1, \ldots,\ F_r(f) = C_r$$

из некоторой области r-мерного пространства точек $C = (C_1, \ldots, C_r)$, то мы будем считать любое явление, которое может иметь место только при C из множества, имеющего r-мерную лебеговскую меру нуль, исключительным и подлежащим "пренебрежению".

Я начну обзор конкретных результатов с применения этой идеи к исследованию динамических систем, фазовое пространство которых является двумерным тором.

§ 2. О динамических системах на двумерном торе и некоторых канонических системах с двумя степенями свободы.

Как и всюду далее, точки тора T^2 будем считать заданными круговыми координатами x_1, x_2 /точка x не меняется при переходе от x_α к $x_\alpha + 2\pi$/. Функции F_α, стоящие в правых частях уравнений

$$\frac{dx_1}{dt} = F_1(x_1, x_2), \frac{dx_2}{dt} = F_2(x_1, x_2),$$

и инвариантную плотность $M(x_1, x_2)$ будем в соответствии с ранее сказанным считать аналитическими и, кроме того, подчиним условиям

$$F_1^2 + F_2^2 > 0, \ M > 0 \tag{1}$$

и для простоты условию нормировки $m(\mathrm{T}^2) = 1$. Введем средние частоты обращения

$$\lambda_1 = \int_{T^2} F_1(x) dm \ , \ \lambda_2 = \int_{T^2} F_2(x) dm.$$

Небольшое усиление результатов Пуанкаре, Данжуа и Кнезера приводит в нашем случае к выводу, что аналитическим преобразованием координат уравнения движения можно привести к виду

$$\frac{dx_1}{dt} = \lambda_1 M(x_1, x_2), \frac{dx_2}{dt} = \lambda_2 M(x_1, x_2).$$

Хорошо известно, что в случае иррационального отношения

$$\gamma = \frac{\lambda_1}{\lambda_2}$$

все траектории оказываются всюду плотными, а мера m транзитивной. Кроме того, следуя Маркову [2], легко доказывается, что при иррациональности γ динамическая система строго эргодична, т.е. содержит одно единственное эргодическое множество E точки которого имеют собственной мерой меру

$$\mu_e = cm,$$

где c константа. Естественное утверждение, что движения на двумерном торе при условиях (1) "Вообще говоря" обладают всеми перечисленными сейчас свойствами, уже является применением упомянутого принципа пренебрежения случаями, когда некоторая конечная система функционалов /В данном случае λ_1 и λ_2/ принимает значения из некоторого множества меры нуль /в данном случае, из множества точек (λ_2, λ_2) с рациональным отношением γ/.

В заметке [3] мне удалось пойти несколько дальше. Именно, я доказал, что в предположении существования таких $c > 0$ и $h > 0$, что для всех целых r и s имеет место неравенство

$$|r - s\gamma| \geqq ch^s, \tag{2}$$

уравнения движения можно привести аналитическим преобразованием координат к виду

$$\frac{dx_1}{dt} = \lambda_1, \ \frac{dx_2}{dt} = \lambda_2. \tag{3}$$

Как известно из теории диофантовых приближений, условие (2) выполнено /при надлежащих c и h /для почти всех иррациональностей γ. Таким образом, за исключением случаев, когда γ "ненормально хорошо" приближается дробями r/s, аналитическая динамическая система с интегральным инвариантом на торе T^2 при условиях (1) неизбежно оказывается допускающей только почти периодические и даже более специально "условно периодические" движения с двумя независимыми частотами λ_1 и λ_2.

Известно много задач классической механики с двумя степенями свободы /$s = 2$, $n = 4$/, в которых из-за наличия двух однозначных на всем Ω^4 первых интегралов I_1 и I_2 четырехмерное многообразие Ω^4 распадается, за исключением некоторых исключительных многообразий не более чем трех измерений, на двумерные многообразия

$$L^2_{C_1C_2} = L^2(I_1 = C_1, \ I_2 = C_2).$$

Так как в точках покоя выполняются четыре уравнения

$$\frac{\partial H}{\partial q_1} = \frac{\partial H}{\partial q_2} = \frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial p_2} = 0,$$

то в случае аналитической функции H их множество на Ω^4 не более чем счетно. Поэтому, на многообразие L^2 они могут попасть лишь в виде исключения. Отсюда получается вывод, что почти все компактные многообразия L^2 являются именно торами /как ориентируемые, компактные, двумерные многообразия, допускающие векторное поле без нулевых векторов/.

Задачи классической механики рассматриваемого типа, как известно, всегда интегрируемы. Качественное исследование специальных задач такого рода /движение под действием силы тяжести по поверхности вращения, движение по инерции по поверхности трехосного эллипсоида и т.д., движение точки по плоскости под действием ньютоновского притяжения двух неподвижных центров и т.п./ и приводит к большому числу примеров распадения пространства Ω^4 в основном на торы T^2 с заполняющими их всюду плотно обмотками из траекторий условно периодических движений с двумя независимыми частотами λ_1 и λ_2. Между этими торами, вообще говоря, лежит всюду плотное множество торов, распадающихся из-за соизмеримости частот на замкнутые траектории, и, уже дискретным образом, не более чем трехмерные особые многообразия, на которых, в частности, помещаются точки покоя и т.н. "асимптотические" движения. Рассмотрение таких интегрируемых задач дает много интересных примеров довольно сложных разложений фазового пространства Ω на эргодические множества и остаток из "нерегулярных точек", которые лежат на траекториях асимптотических движений*).

В уже упомянутой моей заметке [3] указывается, что при исключительных /не удовлетворяющих условию (2) / иррациональных γ действительно имеется ряд новых возможностей, иногда достаточно неожиданных для аналитических систем /об этом будет сказано далее/. Но в упомянутых задачах классической механики эти исключительные случаи не появляются по весьма простой причине: переход к круговым координатам ξ_1, ξ_2 на торах T^2 и параметрам этих торов C_1, C_2 в этих задачах осуществляется преобразованиями прикосновения. Поэтому уравнения сохраняют канонический вид

$$\frac{d\xi_\alpha}{dt} = \frac{\partial}{\partial C_\alpha} H, \frac{dC_\alpha}{dt} = -\frac{\partial}{\partial \xi_\alpha} H$$

и, так как инвариантность торов T^2 получается только в случае

$$\frac{dC_1}{dt} = \frac{dC_2}{dt} = 0,$$

то H зависит только от C_1 и C_2, что приводит на каждом торе T^2 без всяких исключений к уравнениям (3) с постоянными λ_1 и λ_2.

Поэтому, реальное значение для классической механики приведенного мною анализа динамических систем на T^2 зависит от того, имеются ли достаточно важные примеры канонических систем с двумя степенями свободы, не интегрируемых классическими методами, в которых существенную роль играют инвариантные /по отношению к преобразованиям S^t/ двумерные торы.

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*) В связи с этим замечу, что весьма поучительный качественный анализ задачи о притяжении двумя неподвижными центрами, проведенный в известном трактате Шарлье, оказался неполным и частично неверным и дважды исправлялся [4], [5].

Чтобы убедиться в том, что такие примеры существуют, рассмотрим, примыкая к проведенному Биркхофом [6] исследованию окрестности эллиптического периодического движения, систему с круговыми координатами q_1, q_2 и импульсами p_1, p_2, для которой

$$H(q, p) = W(p).$$

Уравнения движения имеют вид

$$\frac{dq_\alpha}{dt} = \frac{\partial W}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = 0,$$

Очевидно, что торы T^2_c, выделяемые условиями

$$p_1 = c_1, \quad p_2 = c_2,$$

инвариантны и на каждом из них происходит условно периодическое движение

$$\frac{dq_\alpha}{dt} = \lambda_\alpha(c) = \frac{\partial}{\partial c_\alpha} W(c_1, c_2)$$

с двумя частотами, зависящими от C. Предположим, что якобиан частот λ_α по импульсам p_α отличен от нуля:

$$\left| \frac{\partial \lambda_\alpha}{\partial p_\beta} \right| = \left| \frac{\partial^2 W}{\partial p_\alpha \partial p_\beta} \right| \neq 0. \tag{4}$$

Оказывается, что в этом случае разбиение рассматриваемой области четырехмерного пространства Ω^4 на двумерные торы T^2 в основном устойчиво по отношению к малым изменениям H вида

$$H(q, p, \theta) = W(p) + \theta S(q, p, \theta).$$

Чтобы получить точную формулировку, рассмотрим область $G \subseteq \Omega^4$ определяемую условием $p \in B$, где B ограниченная область на плоскости точек p. В предположении аналитичности функций W и S и при условии (4) можно доказать следующее: для любого $\varepsilon > o$ существует такое $\delta > o$, что при $|\theta| < \delta$ в динамической системе

$$\frac{dq_\alpha}{dt} = \frac{\partial}{\partial p_\alpha} H(q, p, \theta), \quad \frac{dp_\alpha}{dt} = -\frac{\partial}{\partial p_\alpha} H(q, p, \theta)$$

вся область G кроме множества меры меньше ε состоит из инвариантных двумерных торов T^2, на каждом из которых в надлежащих /аналитически зависящих от (q, p) / круговых координатах ξ_1, ξ_2 движение определяется уравнениями

$$\frac{d\xi_1}{dt} = \lambda_1, \quad \frac{d\xi_2}{dt} = \lambda_2,$$

где λ_1 и λ_2 на каждом T^2 постоянны, т.е. являются условно периодическими с двумя периодами.

Метод доказательства заключается в том, что прослеживается судьба первоначальных торов T_c^2 с частотами $\lambda_\alpha(c)$, удовлетворяющими условию (2), при изменении θ и устанавливается, что каждый такой тор при достаточно малом θ не разрушается, а лишь смещается в Ω, сохраняя на себе траектории условно периодических движений с постоянными частотами λ_α.

Вероятно, многие слушатели уже догадались, что дело идет по существу о некоторой переработке широко дискутировавшейся в литературе по небесной механике идеи о возможности избежать появления ненормально "малых знаменателей" при расчете возмущенных орбит. Однако, в отличие от обычной теории возмущений я получаю точные результаты, а не вывод о сходимости рядов того или иного приближения конечного порядка /относительно θ/. Это достигается благодаря тому, что вместо расчета возмущенного движения при фиксированных начальных я сами начальные условия меняю так, чтобы при изменении θ все время попадать на движения с нормальными /в смысле условия (2) / частотами λ_α.

Сделаю еще три замечания по поводу сказанного.

1. Теорема о приводимости движений на T^2 к виду (3) может быть доказана и при условиях достаточно высокого порядка конечнократной дифференцируемости функций F_α и M /естественно, с соответствующим ослаблением заключения/. Теорема о сохранении торов в Ω^4, наоборот, повидимому, необходимо требует, если не аналитичности $W(p)$ и $S(q, p, \theta)$, то существования у этих функций бесконечного числа производных, подчиненных некоторым ограничениям на порядок их роста.

2. Предусмотренное во второй теореме исключительное множество меры $< \varepsilon$ действительно может оказаться всюду плотным и, вероятно, положительной меры при сколь угодно малых θ. Это явление аналогично "зонам неустойчивости", обнаруженным Биркхофом при исследовании окрестностей эллиптических периодических траекторий [6].

3. В качестве одного из специальных случаев, к которым применимо все выше сказанное, можно указать на движение по инерции по аналитической поверхности, близкой к трехосному эллипсоиду.

§ 3. Являются ли "вообще говоря" динамические системы на компактных многообразиях транзитивными и следует ли непрерывный спектр считать "общим" случаем, а дискретный —"исключительным"?

Гипотеза о преимущественном значении транзитивного случая и случая непрерывного спектра /перемешивания/ неоднократно высказывалась в связи с "эргодическими" гипотезами в физике. В применении к каноническим системам обе эти гипотезы естественно относить лишь к $2s$-1-мерным инвариантным многообразиям $L_h^{2s\text{-}1}$, которые выделяются требованием постоянства энергии

$$H = h,$$

и относить их только к случаю компактных $L_h^{2s\text{-}1}$, так как на некомпактных $L_h^{2s\text{-}1}$ в самых простых задачах имеются /и обычно господствуют по мере/ "уходящие" траектории, о которых речь будет идти далее /в § 4/. В случае отказа от первой гипотезы вторую естественно относить уже не ко всему многообразию Ω^n /или $L_h^{2s\text{-}1}$ в случае канонических систем/, а к тем эргодическим множествам, на которые распадается Ω^n /разрешая, конечно, пренебрегать эргодическими множествами, сумма которых имеет меру нуль/.

В применении к аналитическим каноническим системам на оба вопроса следует ответить отрицательно, так как теорема об устойчивости разбиения на торы, высказанная нами для систем с двумя степенями свободы, сохраняется и при любом числе степеней свободы. Если в $2s$-мерном тороидальном слое G фазового пространства Ω^{2s}

$$H(q, p, \theta) = W(p) + \theta S(q, p, \theta),$$

то при $\theta = 0$ этот слой очевидным образом распадается на инвариантные s-мерные торы T_p^s, на каждом из которых движение условно периодично с s периодами, причем в случае

$$\left| \frac{\partial^2 W}{\partial p_\alpha \partial p_\beta} \right| \neq 0$$

на почти всех торах T_p^s периоды независимы в смысле

$$(n, \lambda) = \sum_\alpha n_\alpha \lambda_\alpha \neq 0$$

при любых целых n_α и, поэтому, траектории обвивают тор всюду плотно, s-мерная лебеговская мера на T^s транзитивна, и весь тор представляет собою одно эргодическое множество. Теоремы 1 и 2 моей заметки [22] утверждают, что в описанной обстановке при малых θ вся эта картина изменяется только в том отношении, что некоторые торы, соответствующие системам частот, для которых выражения (n, λ) убывают с возрастанием

$$|n| = \sqrt{\sum n_\alpha^2},$$

слишком быстро, могут исчезнуть, большинство же торов T_p^s, сохраняя характер происходящих на них движений, несколько смещается в Ω^{2s}, продолжая при малых θ заполнять G с точностью до множества малой меры. Таким образом, при малых изменениях H, динамическая система остается не транзитивной, а область G с точностью до остатка малой меры остается распадающейся на эргодические множества с дискретным спектром /указанной специальной природы/.

В связи с этим интересно отметить, что некоторыми физиками /см. ,например, [7]/ высказывалась как раз гипотеза, что "общим случаем" канонической динамической системы без уходящих траекторий является как раз распадение Ω^{2s} на s-мерные торы T^s, несущие на себе условно периодические движения с s периодами. Идея эта, повидимому, основана только на преимущественном внимании к линейным системам и к ограниченному набору интегрируемых классических задач, но во всяком случае следует отметить, что методы доказательства приведенной выше теоремы существенно привязаны именно к расслоению Ω^{2s} на торы T^s и не применимы к расслоению на торы какой-либо иной размерности $r > s$ или $r < s$.

В общем виде указанная сейчас гипотеза вряд ли может быть поддерживаема, так как весьма вероятно, что при любом s имеются примеры канонических систем с s степенями свободы и устойчивыми транзитивностью и перемешиванием на многообразиях L_h^{2s-1}. Я имею в виду движение по геодезическим на компактных многообразиях V^s постоянной отрицательной кривизны, т.е. динамические системы с

$$H(q,p) = \sum_{\alpha\beta} g_{\alpha\beta}(q) p_\alpha p_\beta, \tag{1}$$

где g_α координаты на компактном многообразии V^s постоянной отрицательной кривизны, а $g_{\alpha\beta}$ компоненты метрического тензора на V^s.

Устойчивость отрицательной кривизны по отношению к малым изменениям функций $g_{\alpha\beta}(q)$ не требует пояснений. Затруднения заключаются лишь в том, что изменение функций $g_{\alpha\beta}(q)$ не является единственным возможным видом изменений функции $H(q,p)$ а транзитивность и перемешивание при $s > 2$ остаются доказанными лишь для случая постоянной кривизны, в то время как при варьировании $g_{\alpha\beta}$ кривизна перестает быть постоянной. Второе затруднение в случае $s = 2$, для которого транзитивность доказана и при переменной кривизне, отпадает. Первое же не существует, если ограничиться функциями $H(q, p)$ вида

$$H(q,p) = U(q) + \sum_{\alpha\beta} g_{\alpha\beta}(q) p_\alpha p_\beta \tag{2}$$

/которыми, собственно говоря, и занимается классическая механика/, так как переходом к новой метрике системы вида (2) сводятся к системам вида (1).

Если вспомнить то, что было ранее сказано о движении по инерции по поверхностям, близким к трехосному эллипсоиду, то мы приходим к выводу, что уже в простейших задачах классической механики приходится считаться как с устойчивыми и поэтому имеющими право на равное и основное внимание, по меньшей мере с двумя рассмотренными случаями, из которых один связан с транзитивностью на многообразиях постоянной энергии и непрерывным спектром, а другой– с отсутствием транзитивности и по преимуществу дискретным спектром.

Аналогичных результатов об устойчивости того или иного общего типа поведения неканонических динамических систем с интегральным инвариантом и компактным Ω^n мне неизвестно.

§ 4. Некоторые замечания о некомпактном случае.

Особенностью некомпактного случая является возможность существования траекторий, уходящих при $t \to \infty$, или при $t \to \infty$ из всякой компактной части Ω. Я изложу здесь некоторые общие положения эргодической теории, пригодные для любых непрерывных потоков S^t в локально-компактных пространствах Ω. Так как односторонний уход в бесконечность возможен лишь для траекторий, образующих множество меры нуль, то сразу определяют уходящую точку x требованием существования для любого компакта K такого T, что все точки S^t_x с $|t| > T$ лежат вне K. Через Ω'' обозначим множество всех уходящих точек. Для целей детального анализа конкретных классических динамических систем целесообразно строить "индивидуальную эргодическую теорию" не в чисто метрическом варианте, изложенном в книге Хопфа [9], а следуя более ранним работам Хопфа и Степанова [10], [11], а в некоторых пунктах непосредственно следуя изложению мемуара Крылова и Боголюбова [12], хотя он и имеет в виду компактный случай.

При таком изложении основным остается, как и в компактном случае, понятие регулярной точки. Так называется точка x, если для нее существует инвариантная мера μ, обладающая следующими свойствами:

1. $\mu(\Omega\text{-}\bar{I}_x) = 0$, где $\bar{I}_x$ замыкание траектории, проходящей через x.

2. $\mu(V_y) > 0$ для любой окрестности V_y точки $y \in I_x$.

3. Для любых двух отличных от нуля лишь на компактных множествах непрерывных функций $f(x)$ и $g(x)$

$$\lim \frac{\int_a^T f(S_x^t)dt}{\int_a^T g(S_x^t)dt} = \frac{\int_\Omega f d\mu}{\int_\Omega g d\mu},$$

если только

$$\int_\Omega g d\mu \neq 0.$$

4. Мера μ транзитивна.

Так как отсутствует требование нормировки, то мера μ определяется точкой лишь с точностью до постоянного множителя. Тем не менее, мы ее будем обозначать μ_x и называть "индивидуальной мерой" точки x. Из-за этого в определение эргодических множеств вводится небольшое изменение: две точки x и x' относятся к одному эргодическому множеству, если их индивидуальные меры совпадают в смысле совпадения с точностью до постоянного множителя. Таким образом, все множества Ω' регулярных точек представляется в виде суммы эргодических множеств

$$\Omega' = \Sigma\, \varepsilon.$$

Меры μ_ε, естественно, теперь тоже определяются эргодическим множеством лишь с точностью до постоянного множитедя.

Индивидуальная эргодическая теорема утверждает, что

$$\Omega = \Omega' + \Omega'' + N, \text{ где } \lambda(N) = 0$$

в любой инвариантной мере λ. Для нас существенно, впрочем, главным образом лишь то, что всегда

$$m(N) = 0.$$

Любая транзитивная инвариантная мера μ является или мерой μ_ε некоторого эргодического множества ε, или имеет вид

$$\mu(A) = r_I(A \cap I),$$

где r_I "временная" мера на уходящей траектории I. В отличие от второго тривиального случая, естественно, меры первого типа называть эргодическими, так как им соответствует множество ε_μ с

$$\mu_{\varepsilon_\mu} = \mu.$$

Те соображения, которые в случае компактного Ω можно привести в пользу мнения, что "общего вида" компактная динамическая система транзитивна, в применении к некомпактным динамическим системам приводит к гипотезе, что "вообще говоря" имеет место один из двух случаев: или система диссипативна /т.е. почти все ее точки уходящие/, или мера m эргодична /очевидно, что во втором случае уходящие точки образуют лишь множество меры нуль/.

Иногда эту гипотезу применяют и к отдельным классическим задачам в такой форме: если у данной задачи имеется некоторое число первых интегралов и нет основания ожидать открытия новых, то считают правдоподобным, что на многообразиях, определяемых указанием значений известных первых интегралов, имеет место транзитивность. В подтверждение такой практики можно привести то замечание, что по исследованиям Хедлунда и Хопфа эта альтернатива всегда имеет место для движений по геодезическим постоянной отрицательной кривизны.

Если заведомо известно, что существует множество положительной меры из уходящих точек, то в соответствии со сказанным возникает гипотеза о том, что система диссипативна. Повидимому, на такого рода соображениях основано предположение Биркхофа о диссипативном характере задачи трех тел.

Однако представляется вероятным, что указанными в 3 методами для канонических систем можно построить примеры устойчивого одновременного нахождения в Ω^{2s} диссипативной части положительной меры и положительной области G, заполненной в основном s-мерными инвариантными торами.

Замечу, что из более элементарных вопросов специалисты по качественной теории дифференциальных уравнений мало занимаются конкретными задачами об уходящих траекториях различных специальных типов. Ярким примером этого является то обстоятельство, что опровержение утверждений Шази о невозможности "обмена" и "захвата" в задаче трех тел [17], [18], было сначала достигнуто тяжелым /и без точных оценок ошибок логически неубедительным!/ путем численного интегрирования /Беккер [19], Шмидт [20]/ и лишь недавно пример "захвата" был построен Ситниковым весьма просто и почти без вычислений [21].

§ 5. Транзитивные меры, спектры и собственные функции аналитических систем.

Назовем меру μ в Ω^n аналитической, если она может быть задана в виде

$$\mu(A) = \int\limits_{V^k \cap A} f(\xi) d\xi_1 \dots d\xi_k,$$

где V^k некоторое локально замкнутое в Ω^n аналитическое многообразие какого-либо числа измерений $k \leqq n$ а функция f от координат ξ_α на V^k/зависящих аналитически от координат x_α в Ω^n/ аналитическая.

Многообразие V^k однозначно определяется мерой μ /если она не тождественный нуль/. Число k поэтому может быть названо и размерностью меры μ.

Нас будут специально интересовать транзитивные меры. В этом случае многообразие V^k должно быть инвариантным. Инвариантные многообразия одной и той же размерности не пересекаются, а разной размерности могут только целиком включаться одно в другое /меньшей размерности в большую/. Каждое инвариантное многообразие несет на себе не более одной транзитивной меры. В силу сказанного система аналитических транзитивных мер имеет сравнительно обозримую структуру.

До последнего времени в аналитических системах были известны только аналитические транзитивные меры. Лишь недавно Грабарь [13], построив аналитический аналог примера Маркова /аналитическую неприводимую, но не строго эргодическую динамическую систему/ и тем самым дал пример неаналитической транзитивной меры в аналитической системе. Возможно, однако, что сумма всех неаналитических эргодическких множеств всегда пренебрегаема в смысле основной меры m.

Эргодические множества однозначно определяются своими мерами μ_ε, которые по самому определению транзитивны.

Что касается эргодических множеств, соответствующих аналитическим транзитивным мерам /не сводящимся к мере μ_ε одной траектории/, то заметим только, что в случае аналитичности меры μ_ε эргодическое множество лежит на носителе V^r меры μ_ε, будучи на нем всюду плотным, но уже в некоторых простых классических примерах разность V^r-ε может быть тоже всюду плотной на V^r.

Спектральные свойства транзитивных мер на аналитических системах мало изучены.

Дискретные спектры пока получены только с конечным базисом независимых частот

$$\lambda_1, \lambda_2, \dots, \lambda_x,$$

причем для аналитических мер число независимых частот совпадает во всех известных случаях с размерностью.

Непрерывный спектр полностью определен лишь недавно Гельфандом и Фоминым [14], [15] для некоторых случаев движений по геодезическим на поверхностях постоянной отрицательной кривизны. В этих случаях он оказался счетно-кратным лебеговским.

Не исключена возможность, что только эти случаи /дискретный спектр с конечным числом независимых частот и счетно-кратный лебеговский/ являются возможными для аналитических транзитивных мер или что только они являются в том или ином смысле общими, типичными случаями.

Для неаналитических транзитивных мер представляется более вероятным их совершенно произвольное строение. Это было бы несомненным, если бы был установлен аналитический аналог теоремы Какутани [16] об изометрическом вложении произвольного потока в поток непрерывной динамической системы.

По поводу собственных функций остановимся только на примере аналитической динамической системы на двумерном торе T^2 с дискретным спектром и всюду разрывными собственными функциями. Правда, пример этот, связанный с ненормально хорошо приближаемым рациональными дробями r/s отношением $\gamma = \lambda_1/\lambda_2$ средних частот, по самому своему происхождению указывает скорее на то, что мы имеем дело не с типичным, а исключительным явлением.

Чтобы выяснить вопрос подробнее, рассмотрим вновь уравнения движения по двумерному тору, введя в них параметр θ, изменяющийся в каких-либо пределах $[\theta_1 ; \theta_2]$:

$$\frac{dx_\alpha}{dt} = F_\alpha(x_1, x_2, \theta).$$

Будем предполагать функции $F_\alpha(x_1, x_2, \theta)$ аналитическими. Очевидно, что аналитически зависеть от θ будет и отношение средних частот $\gamma(\theta)$. Если $\gamma(\theta)$ не постоянно, то множество R тех θ, для которых систему можно преобразовать аналитически к виду

$$\frac{d\xi_\alpha}{dt} = \lambda_\alpha,$$

будет занимать почти весь отрезок $[\theta_1, \theta_2]$. Собственные функции

$$\varphi_{mn} = e^{i(m\lambda_1 + n\lambda_2)}$$

при возвращении к первоначальным координатам x_1, x_2 будут для $\theta \in R$ аналитическими функциями от x_1 и x_2. Но, вообще говоря, даже на R они будут на этом множестве по θ всюду разрывными, причем эту разрывность нельзя будет уничтожить выкидыванием из R множества меры нуль. Эти обстоятельства значительно существеннее, чем то, что $\varphi_{mn}(x_1, x_2, \theta)$ можно определить и в некоторых точках остаточного множества $[\theta_1, \theta_2]$-R меры нуль за счет допущения их неаналитичности и разрывности по x_1 и x_2.

Возможно, что зависимость φ_{mn} (x_1, x_2, θ) от параметра θ на R относится к классу функций типа моногенных функций Бореля [24] и допускает несмотря на всюду разрывный характер исследование надлежащими аналитическими средствами.

Заключение.

Я буду считать свою цель достигнутой, если мне удалось убедить слушателей в том, что, несмотря на большие трудности и ограниченный характер уже полученных результатов, поставленная задача использования общих понятий современной эргодической теории для анализа качественного характера движения в аналитических и специально канонических динамических системах заслуживает большого внимания исследователей, способных охватить те многообразные связи, которые здесь обнаруживаются с самыми различными отделами математики. В заключение мне хочется поблагодарить организационный комитет с'езда за предоставленную мне возможность прочесть этот доклад и за любезную помощь в размножении конспекта с формулами и литературными ссылками, а всех собравшихся за внимание, оказаннов мне в этот последний день наших занятий, когда все уже и без того пересыщены огромным материалом докладов предшествующих дней.

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GROUPES D'HOLONOMIE

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1. *Introduction.*

Dans cette conférence, je me propose d'indiquer les principaux résultats obtenus récemment concernant la théorie des groupes d'holonomie des connexions infinitésimales et principalement des connexions affines et riemanniennes. La notion de groupe d'holonomie est due à Elie Cartan [5], mais l'intérêt actuel pour la géométrie différentielle globale et les connexions infinitésimales a fait surgir toute une série de travaux portant sur la théorie des groupes d'holonomie. En ce qui concerne la théorie générale des connexions, je signalerai les travaux d'Ehresmann (voir [8]) qui donne aussi une fort intéressante définition des groupes d'holonomie, ceux de Chern [7], André Weil et Henri Cartan [6]. Plus spécialement consacrés aux groupes d'holonomie sont différents travaux d'Armand Borel et de l'auteur de cette conférence, ceux d'Ambrose et Singer [1] de Nijenhuis [15], Nomizu [17, 18], Sasaki et Marcel Berger [2]. Un récent mémoire de G. de Rham [19] se rattache à ces questions, ainsi que des travaux de l'école japonaise (Iwamoto [9], Kentaro Yano [22]).

L'étude des groupes d'holonomie apparaît en liaison étroite avec des champs variés des mathématiques contemporaines, non seulement espaces fibrés, groupes de Lie, espaces homogènes, mais aussi variétés analytiques complexes ou kähleriennes. C'est ce qui fait, je crois, leur intérêt.

I. *Connexions infinitésimales sur un espace fibré principal.*

2. *Espace fibré principal.*

Soit $E(V, G, p, H)$ un espace fibré principal, où E et la base V sont des variétés différentiables (de classe C^∞), p la projection; les fibres étant homéomorphes au groupe structural de Lie G qui opère sur lui-même par translation à gauche. A chaque point $x \in V$, la structure d'espace fibré associe une collection $H_x \subset H$ d'homéomorphismes de G sur la fibre F_x telle que, si $h_x, k_x \in H_x$ alors $h_x^{-1} k_x = g \in G$. Un point z de E au dessus de x peut être représenté par

(2.1) $$z = h_x(\gamma) \qquad (\gamma \in G)$$

ou par

$$z = k_x(g\gamma).$$

Soit Θ_z l'espace vectoriel tangent en z à E, V_z le sous-espace tangent à la fibre en z; un vecteur de V_z est dit *vertical*. A tout champ de vecteurs de G invariant à gauche, c'est-à-dire à tout élément λ de l'algèbre de Lie L de G, on peut associer un champ de vecteurs verticaux v de E: pour z défini par (2.1), on posera

$$v(z) = h_x \lambda(\gamma) \tag{2.2}$$

et l'on vérifie immédiatement que $v(z)$ est bien indépendant du choix de h_x dans H_x. L'application $\lambda \to v$ définit un isomorphisme de l'algèbre de Lie L sur une algèbre de Lie de champs de vecteurs verticaux. Un tel champ est déterminé par la donnée d'un seul vecteur vertical et l'élément $\lambda \in L$ correspondant est dit *engendré* par ce vecteur.

D'autre part, comme il est bien connu, le group G opère sur E par „translation à droite". Si $g \in G$ et si z est défini par (2.1)

$$zg = h_x(\gamma g).$$

Nous désignerons par D_g aussi bien la translation à droite sur G que l'application $z \to zg$ de $E \to E$.

3. *Connexion infinitésimale.*

Une connexion infinitésimale $\mathfrak{H}$ sur l'espace fibré principal E est définie par la donnée sur E d'un champ de sous-espaces vectoriels $\mathfrak{H}_z$ de Θ_z tel que

a) $\mathfrak{H}_z$ dépend différentiablement de z

b) $\mathfrak{H}_z$ est supplémentaire de V_z

c) $\mathfrak{H}_z$ est invariant par G opérant sur E, autrement dit $\mathfrak{H}_{zg} = D_g \mathfrak{H}_z$.

Les vecteur de $\mathfrak{H}_z$ sont dits *horizontaux*. Tout vecteur τ de Θ_z est somme d'une partie verticale $V\tau$ et d'une partie horizontale $\mathfrak{H}\tau$. Dans la suite, les chemins envisagés sont toujours supposés continûment différentiables par morceaux. *Un chemin de E est horizontal* si toutes ses tangentes sont horizontales; G opérant sur E transforme tout chemin horizontal en un chemin horizontal. Au dessus de tout chemin $l(x_0, x_1)$ de V, il existe un chemin horizontal de E issu d'un point arbitraire de la fibre F_{x_0}.

A cette définition d'une connexion infinitésimale, on peut en substituer une autre. Une telle connexion donne en effet naissance à une 1-forme ω à valeurs dans l'algèbre de Lie L, c'est-à-dire, pour tout z, à une application linéaire de Θ_z dans cette algèbre: si $\tau \in \Theta_z$, $\omega(\tau)$ est l'élément de L engendré par $V\tau$. Cette forme ω jouit des propriétés suivantes:

a') ω est différentiable

b') si v est vertical, $\omega(v)$ est l'élément de L engendré par v

c') $\omega(D_g \tau) = (\text{adj.}\, g^{-1}) \omega(\tau)$.

Inversement toute 1-forme ω sur E à valeurs dans L jouissant des propriétés précédentes définit un champ $\mathfrak{H}_z$ de sous-espaces par $\omega(\tau) = 0$ pour $\tau \in \mathfrak{H}_z$ et le champ jouit des propriétés a, b, c; ω est dite *la forme de la connexion.*

4. *Développement-groupes d'holonomie.*

1) Considérons sur E un chemin arbitraire défini par $z(t)\, (0 \leqq t \leqq 1)$ et

soit $x(t)$ sa projection sur V. Proposons-nous, connaissant $z(t)$, de trouver un chemin horizontal de E au-dessus de $x(t)$. Un tel chemin peut être défini par

$$z'(t) = z(t)\bar{g}^1(t) \qquad \text{soit } z(t) = z'(t)g(t).$$

Avec des notations évidentes, on a

$$dz = dz'g + z\bar{g}^1dg$$

et d'après b' et c'

$$\omega_z(dz) = (\text{adj } \bar{g}^1)\omega_{z'}(dz') + \bar{g}^1dg. \tag{4.1}$$

Pour que $z'(t)$ soit horizontal, il faut et il suffit que $g(t)$ satisfasse à la relation

$$\bar{g}^1dg = \omega_z(dz). \tag{4.2}$$

La solution de (4.2) telle que $g(0) = g_0$ sera dite le *développement* de $z(t)$ sur G à partir de g_0.

2) Etant donné un point z de E, on appelle *groupe d'holonomie* Ψ_z en ce point l'ensemble des $g \in G$ tels que z et $z\bar{g}^1$ puissent être reliés par un chemin horizontal. On vérifie trivialement que Ψ_z est un sous-groupe de G. Si $\gamma \in G$, on a $\Psi_{z\gamma} = \gamma^{-1}\Psi_z\gamma$; d'autre part si z et z' sont deux points de E non-situés sur une même fibre, mais reliés par un chemin horizontal, on a $\Psi_z = \Psi_{z'}$. Par suite, si z et z' sont deux points arbitraires des E, il existe un élément γ de G tel que

$$\Psi_{z'} = \gamma^{-1}\Psi_z\gamma. \tag{4.3}$$

Etant donné un lacet en z de E, le développement de ce lacet sur G à partir de $g_0 = e$ aboutit à un élément g tel que z et $z\bar{g}^1$ soient reliés par un chemin horizontal, donc tel que $g \in \Psi_z$. Les éléments de Ψ_z peuvent ainsi être obtenus par développement des lacets en z de E.

On appelle *groupe d'holonomie restreint* en z et désigne par σ_z l'ensemble des $g \in G$ tels que z et $z\bar{g}^1$ puissent être reliés par un chemin horizontal dont la projection sur V est un lacet homotope à 0. D'après les propriétés de l'homotopie, σ_z est manifestement sous-groupe invariant dans Ψ_z; il satisfait aux différents points z de E à une relation analogue à (4.3). Enfin σ_z étant sous-groupe connexe par arcs du groupe de Lie G est lui-même *groupe de Lie*[1]).

Si l'_x et l_x sont deux lacets en x homotopes de V, il existe au-dessus de l'_x et l''_x des chemins horizontaux λ' et λ'' issus de z et aboutissant respectivement à $z\bar{g}^{1\prime}$ et $z\bar{g}^{1\prime\prime}$. Le chemin constitué par λ' suivi de $(\lambda'')^{-1}g''\bar{g}^{1\prime}$ est un chemin horizontal aboutissant à $zg''\bar{g}^{1\prime}$ et dont la projection est $l''^{-1}_x l'_x$; par suite $g''\bar{g}^{1\prime} \in \sigma_z$. Si Π_x désigne le groupe de Poincaré de V en $x = pz$, il existe aussi un homomorphisme

$$f : \Pi_x \text{ sur } \Psi_z/\sigma_z.$$

[1]) Voir Yamabé [21]. Ce résultat était connu de Borel et Licherowicz [3] qui le donnent dans le cas riemannien ainsi que de Chevalley. Voir Ambrose et Singer [1].

Soit Ψ_z^0 la composante de e du groupe Ψ_z (c'est encore un groupe de Lie); σ_z est sous-groupe de Ψ_z^0 et pour dim $\sigma_z <$ dim Ψ_z^0, serait maigre sur Ψ_z^0. Soit f^0 la restriction de f à l'ensemble Π_x^0 des classes d'homotopie donnant des éléments de Ψ_z^0; f^0 définit une application de Π_z^0 sur l'ensemble des classes à gauche de Ψ_z^0 modulo σ_z et cet ensemble est ainsi dénombrable; Ψ_z^0 serait donc maigre dans Ψ_z^0, alors qu'il est localement compact. Par connexité, il vient alors $\sigma_z = \Psi_z^0$.

Théorème [1]. *Le groupe d'holonomie restreint est la composante connexe de e du groupe d'holonomie* Ψ_z [2]).

5. *Sections locales.*

Considérons un recouvrement de V par des voisinages U_1, U_2, ... et au-dessus de chaque voisinage U_i, choisissons une section locale de E définie par $z_i(x)$. Pour $x \in U_i \cap U_j$, il existe un élément $g \in G$ tel que

$$z_j(x) = z_i(x)g(x).$$

Par un raisonnement identique à celui qui permet d'établir (4.1), on voit que, dans chaque U_i, la connexion définit une forme locale ω_i à valeurs dans l'algèbre L avec

$$\omega_j = (\text{adj } g^{-1})\omega_i + g^{-1}dg \quad (x \in U_i \cap U_j). \tag{5.1}$$

Inversement la connaissance, pour un recouvrement de V, d'un ensemble de formes locales ω_i de V à valeurs dans L, satisfaisant à (5.1), détermine une connexion infinitésimale sur E.

II. *Variétés à connexion affine.*

6. *Notion de connexion affine.*

1) Soit V_n une variété différentiable de dimension n, T_x l'espace vectoriel tangent en $x \in V_n$. Une base P relative à x cst un ensemble ordonné de n vecteurs $(\vec{e}_\alpha)$ linéairement indépendants de T_x. Sur l'ensemble $E(V_n)$ des bases relatives aux différents points x de V_n est définie une structure naturelle de variété différentiable. La projection p qui, à tout élément $P^x \in E(V_n)$ fait correspondre le point x, définit $E(V_n)$ comme espace fibré principal sur la base V_n, le groupe G étant le groupe linéaire $GL(n, R)$; $E(V_n)$ sera dit *l'espace fibré des bases.*

A toute base $P^x(\vec{e}_\alpha)$ en x correspond par dualité une cobase $\theta_x(\theta^\alpha)$, base de l'espace T_x^* dual de T_x telle que $\theta^\alpha(\vec{e}^\beta) = \delta_\beta^\alpha$.

Une connexion *affine* est, par définition, *une connexion infinitésimale sur l'espace* $E(V_n)$. La forme ω de la connexion est à valeurs dans l'algèbre de Lie du groupe linéaire. Au point $P \in E(V_n)$ cette forme peut-être représentée par une

[2]) Dans le cas riemannien voir [3].

matrice $n \times n$, désignée encore par ω, et dont les éléments sont des formes différentielles locales de $E(V_n)$

$$\omega = (\omega^\alpha_\beta).$$

La notion de différentiation absolue ∇ d'un tenseur relativement à la connexion s'en déduit aisément.

Si nous munissons les voisinages U, V, etc. ... d'un recouvrement, de sections locales P^α_U, P^α_V etc. ..., il existe pour $x \in U \cap V$ une matrice différentiable $A^U_V(x)$ telle que

$$P^\alpha_V = P^\alpha_U A^U_V(x)$$

et (5.1) peut être traduit ici par la relation:

$$\omega_V = \bar{A}^{1U}_{\ V}\omega_U A^U_V + \bar{A}^{1U}_{\ V} dA^U_V. \tag{6.1}$$

On voit ainsi que cette notion de connexion affine coïncide bien avec la notion usuelle.

2) L'espace vectoriel T_x tangent au point x de V_n admet une structure naturelle d'espace affine. Un repère R relatif au point x est un repère de l'espace affine T_x, c'est-à-dire l'ensemble d'un vecteur ξ de T_x définissant un point de T_x (qui sera dit l'origine du repère) et d'une base P de T_x. Sur l'ensemble $\mathscr{E}(V_n)$ des repères relatifs aux différents points x de V_n, est encore définie une structure naturelle de variété différentiable. La projection p qui, à tout élément $R^x \in \mathscr{E}(V_n)$ fait correspondre le point x, définit $\mathscr{E}(V_n)$ comme espace fibré principal sur V_n, le groupe G étant ici le groupe affine; $\mathscr{E}(V_n)$ sera dit *l'espace fibré des repères.*

A toute connexion infinitésimale sur $E(V_n)$ correspond naturellement une connexion infinitésimale sur $\mathscr{E}(V_n)$. La forme de la connexion, à valeurs dans l'algèbre de Lie du groupe affine peut être définie au point $R \in \mathscr{E}(V_n)$ par

$$(\omega^\alpha_\beta) \qquad (\omega^\alpha_0)$$

avec

$$\omega^\alpha_0 \equiv \theta^\alpha + \nabla\xi^\alpha [\theta(\tau) = \theta^\alpha(p\tau) \text{ pour } \tau \in \Theta_R]. \tag{6.2}$$

7. *Transport-groupes d'holonomie-développement.*

1) Soit $l(x_0, x_1)$ un chemin de V_n. Au-dessus de $l(x_0, x_1)$ dans l'espace fibré des bases, on peut construire un chemin horizontal reliant une base arbitraire P_0 en x_0 à une base P_1 en x_1. La transformation linéaire de l'espace vectoriel T_{x_0} sur l'espace vectoriel T_{x_1} qui amène P_0 sur P_1 ne dépend pas du choix de la base P_0 en x_0. Elle définit *le transport* (parallèle) le long de $l(x_0, x_1)$ de l'espace vectoriel T_{x_0} sur l'espace vectoriel T_{x_1}. On en déduit la notion de transport d'un tenseur de x_0 en x_1 le long de $l(x_0, x_1)$. De même une transformation linéaire de T_{x_0} peut être transportée, le long de $l(x_0, x_1)$, sur une transformation linéaire de T_{x_1}.

Selon une remarque d'Elie Cartan, nous pouvons procéder de même sur l'espace fibré des repères $\mathscr{E}(V_n)$ et définir le *transport, le long de* $l(x_0, x_1)$, *de l'espace affine* T_{x_0} *sur l'espace affine* T_{x_1}.

Cela posé, soit l_x un lacet de V_n en x. Ce lacet définit un transport de l'espace vectoriel (resp. affine) tangent en x sur lui-même, c'est-à-dire une transformation linéaire (resp. affine) de cet espace. La transformation inverse sera dite la transformation induite par le lacet l_x. A l'aide de ces transformations induites, on voit que *les groupes d'holonomie au-dessus de x relatifs à* $E(V_n)$ (*resp.* $\mathscr{E}(V_n)$) *peuvent être réalisés comme groupe de transformations linéaires* (*resp. affines*) *de l'espace vectoriel* (*resp. affine*) *tangent en x.*

Nous désignerons désormais par $\Phi_x(V_n)$ le groupe d'holonomie affine, par $\Psi_x(V_n)$ le groupe d'holonomie linéaire ainsi réalisés; $\Phi_x(V_n)$ sera appelé le *groupe d'holonomie* (g.h.) au point x de la variété à connexion affine V_n et $\Psi_x(V_n)$ son g.h. *homogène*. Si nous nous restreignons aux lacets de V_n homotopes à 0, nous obtenons les g.h. restreints $\varrho_x(V_n)$ et $\sigma_x(V_n)$ qui sont respectivement les composantes de e de $\Phi_x(V_n)$ et $\Psi_x(V_n)$. Le transport le long d'un chemin $l(x_0, x_1)$ définit un isomorphisme de $\Phi_{x_0}(V_n)$ sur $\Phi_{x_1}(V_n)$ etc. . . .

Sur le revêtement universel V'_n de V_n, l'image réciproque de la connexion affine par la projection définit une connexion affine; le g.h. correspondant $\Phi_{x'}(V'_n)$ est isomorphe à $\varrho_x(V_n)$. On pourra donc passer au revêtement universel pour étudier ϱ_x ou σ_x.

Tout champ de tenseurs sur V_n à différentielle absolue nulle définit au point x un tenseur invariant par Ψ_x, Inversement, de tout tenseur en x invariant par Ψ_x, on déduit par transport le long de chemins arbitraires un champ de tenseurs à différentielle absolue nulle sur V_n.

2) La construction de chemins horizontaux de $\mathscr{E}(V_n)$ au-dessus de $l(x_0, x_1)$ $[x(t); 0 \leqq t \leqq 1]$ peut s'effectuer par développement sur le groupe affine d'un chemin $R(t)$ de $\mathscr{E}(V_n)$, lui-même au-dessus de $l(x_0, x_1)$. Il est souvent commode de choisir les points $R(t)$ du chemin de $\mathscr{E}(V_n)$ tels que *l'origine de R coïncide avec x* ($\xi = 0$). Supposons qu'il en soit ainsi. Un repère initial étant choisi dans T_{x_0}, le groupe affine peut être réalisé au moyen des repères $[\xi, P]$ de l'espace affine T_{x_0} et, d'après (4.2), le développement revient alors à déterminer une famille de repères de T_{x_0} dépendant du paramètre t, telle que le long de $R(t)$

$$d\xi = P\theta \qquad dP = P\omega. \tag{7.1}$$

Les formules (7.1) sont souvent dites *les formules du repère mobile*. Les éléments du groupe d'holonomie $\Phi_{x_0}(V_n)$ peuvent ainsi être obtenus en développant, selon ces formules, les lacets λ_{R_0} ($x_0 = pR_0$) de $\mathscr{E}(V_n)$ satisfaisant à la condition posée.

3) Le lemme suivant est souvent utile en ce qui concerne les lacets de V_n homotopes à 0.

Supposons choisi, pour tout $x \in V_n$, un voisinage $U(x)$ tel que $x \in U(x)$. Un lacet est dit *petit*, relativement à ce choix, s'il est contenu dans $U(x)$; l_x est dit un *lasso* s'il existe $y \in V_n$ et un chemin $l(x, y)$ tel que

$$l_x = l(x, y)^{-1} l_y l(x, y)$$

où l_y est petit. Deux lacets en x sont équivalents, si l'on peut passer de l'un à l'autre en substituant à des produits $l(x, y)l(x, y)^{-1}$ le lacet nul en y et inversement. Cela posé, on peut établir que *tout lacet l_x homotope* à 0 *est équivalent à un produit fini de lassos d'origine x* (lemme de factorisation).

Deux lacets en x équivalents induisent la même transformation du groupe d'holonomie. Ainsi tout élément de $\varrho_x(V_n)$ [resp. $\sigma_x(V_n)$] est produit fini de transformations affines (resp. linéaires) induites par des lassos d'origine x.

8. *Groupe d'holonomie locale* [15].

Dans ce §, nous nous bornerons en principe aux transformations linéaires.

1) Soit U un voisinage de $V_n (x \in U)$. La connexion affine de V_n définit sur U une structure de variété à connexion affine et $\sigma_x(U)$ est sous-groupe de $\sigma_x(V_n)$. En tant que sous-groupe connexe de $\sigma_x(V_n)$, $\sigma_x(U)$ est complètement déterminé par son algèbre de Lie $d\sigma_x(U)$.

Nous sommes ainsi amenés à considérer le sous-groupe de $\sigma_x(V_n)$ défini par

$$\sigma_x^* = \bigcap_{x \in U} \sigma_x(U)$$

et qui est dit le groupe (homogène) d'*holonomie locale* en x. Il existe un voisinage $U(x)$ du point x tel que

$$\sigma_x^* = \sigma_x[U(x)]$$

et σ_x^* est ainsi groupe de Lie connexe. Si l'on définie les lacets petits sur V_n au moyen des voisinages $U(x)$ précédents, on déduit du lemme de factorisation que tout élément de $\sigma_x(V_n)$ est produit fini d'éléments déduits d'éléments des $\sigma_y^* (y \in V_n)$ par transport le long de chemins convenables.

Si $y \in U(x)$, on a immédiatement

$$\dim \sigma_y^* \leqq \dim \sigma_x^*$$

l'égalité n'ayant lieu que lorsque σ_y^* est isomorphe à σ_x^* par transport. On en déduit que *si $\dim \sigma_x^* =$ const. sur V_n, $\sigma_x^* = \sigma_x(V_n)$ pour tout x* [15].

2) Proposons-nous d'étudier l'algèbre de Lie $d\sigma_x^*$ de σ_x^*. Un élément de cette algèbre peut être identifié à un tenseur en x de type (1.1). On établit aisément que si a est un champ local de tenseurs de V_n qui, pour chaque x du domaine envisagé, définit un élément de $d\sigma_x^*$, la valeur de la différentielle absolue ∇a pour tout vecteur en x définit encore un tel élément.

Il est commode de choisir les voisinages $U(x)$ que nous venons d'introduire, adaptés à des coordonnées géodésiques (s, u) d'origine x, où s est le paramètre canonique sur la géodésique tangente à u en x. Soit P^x une base en x; dans $E(V_n)$, au-dessus de chaque arc géodésique d'origine x de $U(x)$, traçons le chemin horizontal issu de P^x. Nous définissons ainsi une *section locale* $(y \to P^y)$ de $E(V_n)$ au-dessus de $U(x)$ et les groupes d'holonomie $\sigma_{P^y}(U)$, identiques à $\sigma_{P^x}(U)$, peuvent être identifiés à σ^*_x. Le développement de tout chemin (fermé ou non) issu de P^x tracé sur cette section définit un élément de σ^*_x. Il en résulte que, pour tout P^y de cette section, $\omega_{P^y}(\tau)$ où τ est tangent à la section fournit un élément de l'algèbre de Lie du groupe d'holonomie locale en x [3]).

En combinant ces deux résultats, on voit que si Ω_x est la forme de courbure en x de la connexion affine, pour tout couple u, $\bar{u}$ de vecteurs, $\Omega_x(u, \bar{u})$ définit un élément de l'algèbre de Lie $d\sigma^*_x$. Par suite $\Omega_y[y \in U(x)]$ transportée de y en x le long de chemins arbitraires — notamment le long des géodésiques — donne des éléments de $d\sigma^*_x$.

Inversement, on peut obtenir des formules explicites pour le développement d'un chemin issu de P^x de la section locale précédente et on vérifie alors aisément que l'algèbre de Lie de σ^*_x est effectivement engendrée par les éléments déduits de $\Omega_y[y \in U(x)]$ par transport le long des géodésiques issues de x. On en déduit que *l'algèbre de Lie de* $\sigma_x(V_n)$ *est engendrée par les éléments déduits de* $\Omega_y(y \in V_n)$ *par transport le long des chemins* $l(x, y)$ *de* V_n.

Des raisonnements et résultats semblables sont valables pour l'holonomie affine qui fait intervenir en outre la torsion de la connexion [4]). Ambrose et Singer [1] ont établi l'analogue de l'énoncé précédent en ce qui concerne la courbure d'une connexion infinitésimale et l'algèbre de Lie du groupe d'holonomie restreint.

9. *Groupe d'holonomie infinitésimale.*

Les résultats précédents étaient valables pour une connexion différentiable de classe C^v. Dans le présent § il est nécessaire de supposer la connexion de classe C^∞.

1) En appliquant au tenseur de courbure $R^\alpha{}_{\beta,\lambda\mu}$ de la connexion affine un résultat de § 8.2, on voit que pour tout ensemble de vecteurs u, $\bar{u}$, $u_1, \ldots, u_k, \ldots$ les tenseurs en x:

$$(9.1) \qquad R^\alpha{}_{\beta,\lambda\mu} u^\lambda \bar{u}^\mu, \ldots, u_k^{\nu_k} \ldots u_2^{\nu_2} u_1^{\nu_1} \nabla_{\nu_k} \ldots \nabla_{\nu_2} \nabla_{\nu_1} R^\alpha{}_{\beta,\lambda\mu} u^\lambda \bar{u}^\mu, \ldots$$

définissent des éléments de $d\sigma^*_x$. A l'aide des identités de Ricci pour la connexion affine, on montre que l'espace vectoriel engendré par (9.1) définit une sous-algèbre de $d\sigma^*_x$. Il existe donc un sous-groupe connexe σ'_x de σ^*_x admettant l'algèbre ainsi définie $d\sigma'_x$. A ce groupe σ'_x, on donne le nom de *groupe d'holonomie infinitésimale* en x.

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[3]) Une section locale analogue peut être introduite sur $\mathscr{E}$ (V_n).

Pour chaque x, il existe un voisinage $V(x)$ tel que pour tout $y \in V(x)$ $\dim \sigma'_y \geqq \dim \sigma'_x$.

Par suite, si $\dim \sigma'_x = \dim \sigma^*_x$ pour tout $x \in V_n$, on déduit d'un raisonnement par ouvert-fermé que $\dim \sigma^*_x = \text{const.}$ sur V_n et ainsi que

$$\sigma^*_x = \sigma'_x = \sigma_x(V_n). \tag{9.2}$$

2) Selon Nijenhuis [15], on appelle *oscillation* de l'holonomie infinitésimale en x

$$O'(x) = \min_{U \ni x} \max_{y \in U} \dim \sigma'_y - \dim \sigma'_x.$$

Un point x est régulier pour l'holonomie infinitésimale si $O'(x) = 0$, singulier dans le cas contraire. L'ensemble E' des points singuliers pour l'holonomie infinitésimale est un ensemble *fermé sans point intérieur* [15].

Soit U un voisinage de points réguliers, a un champ local de tenseurs qui, pour tout $x \in U$, définit un élément de l'algèbre de Lie $d\sigma'_x$. Sur la définition de cette algèbre, on vérifie qu'il existe un voisinage $V \subset U$ tel que, pour tout $x \in V$, la valeur de la différentielle absolue ∇a, pour tout vecteur en x, définit encore un élément de $d\sigma'_x$.

Nous pouvons choisir le voisinage V adapté à des coordonnées géodésiques d'origine x et introduire au-dessus de V la section locale étudiée dans § 8.2. A l'aide de celle-ci, on voit que si $y \in V$ les éléments de $d\sigma^*_x$ déduits des éléments de $d\sigma'_y$ par transport le long de la géodésique reliant x à y appartiennent à $d\sigma'_x$. Ainsi *en tout point x régulier pour l'holonomie infinitésimale*

$$\sigma'_x = \sigma^*_x. \tag{9.3}$$

En introduisant les composants connexes V^i_n de $V_n - E'$, on voit que

$$\sigma'_x = \sigma^*_x = \sigma_x(V^i_n) \quad (x \in V^i_n). \tag{9.4}$$

3) *Si la variété V_n est analytique réelle et munie d'une* connexion analytique, on vérifie à l'aide de développement en séries selon les puissances de s que $d\sigma^*_x = d\sigma'_x$. Il en résulte que *dans ce cas le groupe d'holonomie $\sigma_x(V_n)$* coïncide *avec le groupe d'holonomie infinitésimale σ'_x en* tout point x [5]).

10. *Etude du groupe ϱ_x dans le cas où σ_x est inéductible.*

Un groupe linéaire réel est réductible ou irréductible selon qu'il laisse invariant ou non un sous-espace vectoriel réel non trivial.

1) Soit ϱ un groupe de Lie connexe de transformations affines d'un espace affine $\mathscr{E}_q$. Il existe un homomorphisme naturel de ϱ sur un groupe σ de trans-

[4]) La coubure et la torsion de la connexion affine définissent la „courbure" de la connexion infinitésimale introduite sur $\mathscr{E}$ (V_n) dans § 6.2.

[5]) Certains de ces résultats étaient connus dès 1952, de Chevalley et Marcel Berger. Ils figurent dans Nijenhuis [15] sous une forme très élaborée.

formations linéaires de l'espace vectoriel associé. *Si σ est irréductible,* on démontre que ou bien ϱ est isomorphe naturellement à σ, ou bien ϱ contient toutes les translations de $\mathscr{E}_q$.

2) Appliquant ce résultat aux groupes d'holonomie ϱ_x et σ_x d'une variété à connexion affine, il vient:

Si σ_x est irréductible, ou bien ϱ_x est isomorphe naturellement a σ_x, ou bien il contient toutes les translations [6]).

III. *Variétés riemanniennes et pseudokählleriennes.*

11. *Groupes d'holonomie d'une variété riemannienne.*

Supposons la variété V_n munie d'une structure de variété riemannienne de métrique ds^2. L'espace $\varepsilon(V_n)$ des bases orthonormées de V_n dans cette métrique admet une structure d'espace fibré principal dont le groupe structurel est le groupe orthogonal $0(n)$. Toute connexion infinitésimale sur $\varepsilon(V_n)$ définit d'une manière naturelle une connexion infinitésimale sur $E(V_n)$, c'est-à-dire une connexion affine.

Le théorème fondamental de la géométrie riemannienne affirme qu'il existe une connexion infinitésimale et une seule sur $\varepsilon(V_n)$ telle que la connexion affine associée soit sans torsion. Cette connexion est dit la connexion riemannienne de la variété envisagée.

Les groupes d'holonomie d'une variété riemannienne peuvent être réalisés comme groupes de déplacements (resp. rotations) de l'espace ponctuel (resp. vectoriel) euclidien tangent en x. Nous les désignerons par les mêmes notations que dans la partie II.

C est d'abord dans le cas des variétés riemanniennes que les principales propriétés des groupes d'holonomie ont été établies [3].

12. *Réductibilité d'une variété riemannienne et applications.*

Dans ce § nous raisonnerons sans modifier les notations sur le revêtement universel de la variété initiale munie de la métrique image réciproque; autrement dit, nous supposerons V_n simplement connexe.

1) Une variété riemannienne V_n simplement connexe est dite *réductible* si son groupe d'holonomie σ_x est réductible. S'il en est ainsi, l'espace vectoriel T_x peut être décomposé d'une manière et d'une seule à l'ordre près en une somme directe de sous-espaces orthogonaux T_x^a $(a = 0, 1, \ldots, k)$, invariants par σ_x, tels que σ_x induise l'identité sur T_x^0 et des représentations irréductibles sur T_x^a $(a \neq 0)$. Par transport des T_x^a, on obtient sur V_n des champs complètement intégrables et un feuilletage de V_n en $(k + 1)$ systèmes de feuilles. Tout point x de V_n admet un voisinage $U(x)$ qui est produit topologique de voisinages $U^a(x)$

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[6]) Ce résultat a été indiqué, sous des formes légèrement différentes les unes des autres par A. Borel, Nijenhuis et Sasaki.

de feuilles passant par x. Le ds^2 de V_n est somme des ds^2 induits sur les feuilles, de sorte que V_n est „décomposable" au sens de la géométrie différentielle locale. Les feuilles d'indice 0 sont localement euclidiennes.

2) On peut alors établir le théorème suivant [3] [12]

Théorème. Le groupe d'holonomie σ_x est produit direct

$$\sigma_x = \prod_{a=1}^{a=k} \sigma_x^a$$

ou σ_x^a induit sur T_x^b $(b \neq a)$ la représentation triviale.

En effet si $r \in \sigma_x$ et si s^a est la rotation de T_x^a induite par r, σ_x contient la rotation r^a égale à s^a sur T_x^a et à l'identité sur T_x^b $(b \neq a)$. Car si r est induite par le lasso

$$l(x, y)^{-1}\, l_y(x, y) \quad [l_y \in U(y)]$$

r^a le sera par le lasso

$$l(x, y)^{-1}\, l_y^a(x, y)$$

où l_y^a est la projection de l_y sur $U^a(y)$. Le lemme de factorisation établit alors la propriété.

Un sous-groupe irréductible connexe de $SO(q)$ étant fermé dans $SO(q)$, on déduit aisément du théorème précédent que *σ_x est fermé dans le groupe des rotations propres de T_x*. Ainsi σ_x est compact.

3) En ce qui concerne le groupe ϱ_x, on obtient immédiatement un théorème de produit direct analogue au précédent. En appliquant le résultat de § 10.1 à chacun des groupes ϱ_x^a obtenus, réalisés comme sous-groupe du groupe des déplacements de T_x^a, on voit que ϱ_x^a est fermé dans ce groupe. Il en résulte que *le groupe ϱ_x est fermé dans le groupe des déplacements propres de l'espace ponctuel euclidien tangent en x.*

Si ϱ_x est naturellement isomorphe à σ_x (§ 10) et par suite compact, on voit à l'aide de la mesure de Haar sur ϱ_x que *ϱ_x laisse invariant un point η de l'espace tangent en x* (Sasaki).

4) Dans [2], Marcel Berger a étudié l'algèbre de Lie de *σ_x pour une variété riemannienne V_n irréductible non symétrique.* Sa méthode repose sur les travaux d'Elie Cartan concernant les groupes linéaires irréductibles et consiste à étudier le tenseur dérivé du tenseur de courbure (qui définit des éléments de $d\sigma_x$) pour les différentes structures d'algèbre de Lie qui peuvent se présenter. A l'aide des identités de Bianchi, il montre pour beaucoup de ces structures que ce tenseur est nécessairement nul, ce qui contredit l'hypothèse de non symétrie. Il obtient ainsi, selon la dimension n, une liste exhaustive des σ possibles pour une variété riemannienne irréductible non symétrique. Sa méthode s'étend sans difficulté aux variétés à métriques non définies. Il obtient aussi sur l'existence de formes invariantes par le groupe d'holonomie des résultats précis.

13. *Cas où la variété riemannienne est complète.*

1) Si la variété riemannienne V_n complète, simplement connexe, est réductible, il résulte d'un beau théorème de Georges de Rham [19] qu'il existe une isométrie globale de V_n sur le produit riemannien des feuilles issues d'un point de la variété. Un résultat semblable a été donné par A. G. Walker.

2) Considérons une variété riemannienne V_n complète, simplement connexe, irréductible et supposons que ϱ_x soit naturellement isomorphe à σ_x et par suite laisse invariant un point η de l'espace tangent en x. On peut alors construire un point x_0 de V_n tel que ϱ_{x_0} laisse invariant le point x_0 de l'espace tangent en x_0. Il existe une géodésique et une seule joignant x_0 à un point x arbitraire de V_n. En utilisant des coordonnées normales d'origine x_0, on établit que V_n est identique à l'espace euclidien. Par suite:

Théorème (Sasaki). Le groupe d'holonomie ϱ_x d'une variété riemannienne V_n complète, simplement connexe, irréductible contient toutes les translations, a moins que V_n ne soit identique à l'espace euclidien.

14. *Variétés pseudokählériennes.*

Une variété *pseudokählerienne* est une variété de dimension paire ($n = 2p$) munie d'une 2-forme de rang $2p$ échangeable avec la métrique et à *différentielle absolue nulle* dans la connexion riemannienne. Une telle variété admet une structure presque complexe intégrable (pseudocomplexe). Si la variété et la structure presque complexe sont analytiques réelles, la variété est kählerienne.

Pour qu'une variété riemannienne V_{2p} soit pseudokählerienne, il faut et il suffit que son groupe d'holonomie Ψ soit sous-groupe de la représentation réelle de $U(p)$. Si la variété pseudokählerienne V_{2p}, simplement connexe est réductible, ses feuilles non triviales sont nécessairement pseudokähleriennes [12].

Pour que σ soit sous-groupe de la représentation réelle de $SU(p)$, il faut et il suffit que la courbure de Ricci de la variété pseudokählerienne V_{2p} soit nulle. Supposons cette courbure différente de 0 : σ admet alors un centre non discret. Si en outre σ est irréductible, il en résulte que le centralisateur connexe $C(\sigma)$ de σ dans le groupe orthogonal $SO(n)$ est contenu dans σ [10].

Certains de ces résultats peuvent être étendus de manière convenable aux groupes d'holonomie des connexions hermitiennes [14] des variétés pseudohermitiennes.

IV. *Espaces homogènes.*

15. *Espaces homogènes riemanniens.*

Soit $V_n = G/H$ un espace homogène de Lie (G connexe, H compact sans sous-groupe $\neq \{e\}$ invariant dans G) $\tilde{H}^0_x$ la composante connexe de e du groupe linéaire d'isotropie en $x \in V_n$. Prenons sur cet espace une métrique riemannienne invariante par G; $\tilde{H}^0_x$ peut être identifié à un groupe de rotations de l'espace vectoriel tangent en x.

Si σ_x est irréductible, on peut affirmer que $\tilde{H}^0_x$ est sous-groupe de σ_x dans les cas suivants:

a) *si V_n n'est pas pseudokählerienne.*

b) *si V_n étant pseudokählerienne, la courbure de Ricci de la variété est différente de* 0.

Ces résultats se déduisent du §14. Le cas où $\tilde{H}^0_x$ coïncide avec σ_x est celui où V_n est espace symétrique irréductible. On notera que si G est compact, la courbure de Ricci n'est nulle que pour un espace localement euclidien [11].

Nomizu a étudié la réductibilité des espaces homogènes riemanniens (en particulier G simple entraînerait σ irréductible). De plus il a indiqué, en utilisant un résultat inédit de Chevalley, une définition purement algébrique de l'algèbre d'holonomie d'un espace homogène riemannien à $\tilde{H}^0$ irréductible [17].

Dans un travail consacré aux connexion affines invariants sur un espace homogène G/H (H réductif mais non nécessairement compact) il a généralisé, sous une forme élégante, aux ,,espaces affines symétriques'' les principaux théorèmes de la théorie des espaces riemanniens symétriques d'Elie Cartan [18].

16. *Espaces homogènes hermitiens.*

Supposons que la variété $V_{2p} = G/H$ admette une structure complexe invariante par G. En introduisant sur V_{2p} une métrique hermitienne invariante par G et une connexion hermitienne invariante déduite de la connexion riemannienne par ,,tronquage'', j'ai pu établir le théorème suivant [14]:

Si un espace homogène complexe G/H (H compact) admet un groupe linéaire connexe d'isotropie irréductible (dans le réel), ou bien il admet une structure d'espace hermitien symétrique irréductible (en particulier il est kählerien), ou bien sa classe de Chern de degré 2 *est nulle. Si G est compact, la nullité de la classe de Chern n'est réalisée que pour un espace localement unitaire (euclidien).*

On en déduit en particulier que tout domaine borné homogène de C^n pour lequel le groupe linéaire connexe d'isotropie peut être choisi irréductible est symétrique [10]. Les raisonnements font intervenir le groupe d'holonomie σ de la connexion hermitienne introduite.

Les espaces homogènes kähleriens ont été étudiés par Lichnerowicz [11, 13] et par Armand Borel et André Weil (résultats encore inédits). Dans le cas où G est compact on peut se ramener au cas où G est simple de centre réduit à $\{e\}$, et il existe un tore T de G tel que H soit le centralisateur de T dans G. Les variétés correspondantes sont toutes algébriques (Borel-Weil) et jouissent de propriétés variées.

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UNSOLVED PROBLEMS IN MATHEMATICS

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No manuscript of this lecture was available

CURRENT PROBLEMS OF MATHEMATICAL STATISTICS [1]

JERZY NEYMAN

1. *Introduction.* The purpose of the present paper is to give a general characterization of mathematical statistics understood as an independent mathematical discipline, to illustrate its contacts with experimental sciences and to indicate briefly several important directions of current research.

2. *Two broad categories of statistical problems: problems of stochastic models and problems of induction.* An early effort at a precise delineation of the domain of mathematical statistics is due to Emile Borel. In his book [1, p. 161] [2], first published in 1909, Borel gives the following definition of mathematical statistics.

> "Le problème général de la statistique mathématique est le suivant:
>
> "Déterminer un système de tirages effectués dans des urnes de composition fixe, de telle manière que les résultats d'une série de tirages, interprétés à l'aide de coefficients fixes convenablement choisis, puissent avec une très grande vraisemblance conduire à un tableau identique au tableau des observations."

In 1921, the following classification of problems of statistics was given by R. A. Fisher [2, p. 313].

> "(1) Problems of Specification. These arise in the choice of the mathematical form of the population.
>
> (2) Problems of Estimation. These involve the choice of methods of calculating from a sample statistical derivatives, or as we shall call them statistics, which are designed to estimate the values of the parameters of the hypothetical population.
>
> (3) Problems of Distribution. These include discussions of the distribution of statistics derived from samples, or in general any functions whose distribution is known."

At first glance, the two definitions quoted appear to be very different. However, closer examination, particularly taking into account the relevant contexts, reveals that the two scholars must have had essentially the same phases of statistical work in mind. Furthermore, although mathematical statistics has

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1) This paper was prepared with the partial support of the Office of Naval Research.

2) Figures in square brackets refer to the list of references given at the end of this paper.

undergone a very substantial evolution during the decades elapsed since the above passages were first published, the subdivision of the problems indicated still persists. However, the distribution of emphasis has changed.

Borel's definition emphasizes the invention of a system of draws from urns which is most likely to reproduce the observational data. This corresponds, roughly, to what Fisher termed the problem of specification. Currently, the same kind of problem is designated by the term, "problem of a stochastic model of a given class of phenomena." Under the powerful stimulus of the indeterministic approach to problems of experimental sciences such as physics, general biology, genetics, astronomy, and social sciences, the problem of stochastic models has recently attracted very considerable attention.

The two other types of problems mentioned by Fisher, problems of estimation and of distribution, correspond to Borel's passage referring to the "fixed coefficients" and to the "great likelihood" of duplicating the observations by the constructed system of sampling balls from urns. While, in the passage quoted from Borel, the emphasis on this particular aspect of statistical work is only slight, the relevant problems are set out clearly in the context. In fact, it is to Borel that we owe the first modern theoretical considerations of the possibility of testing a statistical hypothesis using some function of the observations (statistics) which are "en quelque sorte remarquables."

In modern literature, the problems in question are designated variously as problems of induction, or problems of inductive behavior or, following Abraham Wald, problems of statistical decisions. Briefly, the problems are as follows.

(i) *Problem of estimation.* Given a stochastic model M of a class of phenomena, specified up to the values of certain parameters θ and concerned with a random vector X, to determine a function $\theta(X)$ which could be used, "most advantageously" to represent the value of θ after the observations have determined the value of X.

(ii) *Problem of testing statistical hypotheses.* We are given a class Ω of different stochastic models all concerned with the same random vector X but ascribing to it different distributions. In all cases X is capable of assuming values contained in a set $\mathscr{X}$ (the sample space). Let ω stand for a subset of Ω and H the hypothesis that the actual distribution of X is an element of ω. The problem consists in determining a subset w of $\mathscr{X}$, termed the critical region for testing H against Ω, to be used in the following manner: the hypothesis tested H is "rejected" if and only if the observations determine $X \in w$. Here again the selection of the critical region w is subject to a requirement that it be "most advantageous".

(iii) *General problem of statistical decisions.* The above two problems have the following elements in common: following the observation on X we intend to select a particular "action", which is a function of X. In the case of the problem of estimation, the "action" contemplated consists in ascribing to the parameter θ the value of the function $\theta(x)$ corresponding to the observed value x of the random variable X. In the case of testing the hypothesis H, the contemplated action consists in either rejecting H or in abstaining from such rejection, according to whether or not the observation determines $X \epsilon w$.

Generalizing the situation, we contemplate two spaces, $\mathscr{X}$ the sample space and $\mathscr{A}$ the space of possible decisions a. The problem consists in defining, in the most "advantageous" manner, a function $a(x)$ from $\mathscr{X}$ to $\mathscr{A}$ to be used in the following manner: whenever the observations determine $X = x$, then the action taken is $a(x)$.

All the above problems have no mathematical meaning until the exact meaning of the word "advantageous" is clearly defined.

It will be noticed that the problem of statistical decisions involves interesting conceptual aspects. The process of solution of any particular problem of this kind involves the evaluation of various distribution functions. This particular phase of work was described by Fisher as the distribution problem. Frequently, the distribution problems are of great technical difficulty and their solutions are of primary importance for the applications. It is in this domain that the present development of mathematical statistics owes so much to Fisher. In order to appreciate all he has done, it is sufficient to consult the excellent books by Cramér [3] and by M. G. Kendall [4] in which the distribution problems solved by Fisher are carefully presented in a systematized way. However, the distribution problems do not present any independent conceptual interest and, therefore, the proposed classification of problems of statistics involves only two categories: the problem of stochastic models and the problem of statistical decisions. Some examples given in the following sections illustrate the general character of these problems.

3. *Stochastic model of clustering — two illustrations.* In the late 1930's, cooperation with Geofrey Beall [5], then an entomologist interested in the distribution of larvae over a field, attracted my attention to a particular stochastic machinery, the model of clustering, which appears to have a considerable field of application.

Let S be a Euclidean space of n dimensions. We consider the distribution of "particles" P in S. The machinery of this distribution requires the consideration of another kind of particles, which are termed "cluster centers" and denoted by C. The following "structural" postulates are adopted. (i) To every Borel subset s of S there corresponds a random variable $\gamma(s)$ representing the number of cluster centers in s. Thus $\gamma(s)$ is nonnegative and integer valued. (ii) The random variable $\gamma(s)$ considered as a set function is completely additive.

(iii) To each cluster center there corresponds a random variable ν, nonnegative and integer valued, which represents the number of particles P attached to the cluster center at C. The random variables ν corresponding to different cluster centers are mutually independent and identically distributed. Also they are independent of the positions of the cluster centers. (iv) Given the coordinates of a cluster center C, the coordinates of a particle P belonging to the cluster center at C are random variables following a fixed conditional distribution, the same for all clusters and for all particles. Also, given the positions of all cluster centers, the positions of all particles are mutually independent.

The above four postulates determine what might be called the general structural model of clustering. Various specializations make it applicable to a great variety of phenomena. In each case it is necessary to specify not only the dimensionality of S (three-space, plane or a line) but also sufficiently flexible distributions which are likely to approximate the actual distributions of $\gamma(s)$, ν and of the coordinates of particles given those of the cluster centers. In the case of Beall, the "particles" P were larvae distributed over a field and the "cluster centers" C the masses of eggs deposited by moths. Currently two other classes of phenomena of clustering are being studies.

(1) *Distribution of galaxies in space.* The published literature on the subject of the distribuion of galaxies in space [6, 7, 8, 9, 10] refers specifically to the case of a nonexpanding universe. However, further work [11, 12] postulates the possibility that the universe is expanding, with the rate of expansion depending on a constant h (Hubble's constant). If $h = 0$, then the universe is static. If $h > 0$, then the universe is expanding. It is hoped that by using counts of images of galaxies visible on the photographic plates it will be possible to test the hypothesis that the universe is static.

(2) *Ionization in the cloud chamber produced by cosmic rays.* Let T stand for the track of a cosmic particle within the ionization chamber. T is divided into m "cells" L_i, for $i = 1, 2, \ldots, m$ of equal length. Following its track, the cosmic particle encounters an atom from time to time and splits into two ions. Every such event, the "primary event", determines a "cluster center". It is assumed that to every Borel set s on T of measure $m(s)$, there corresponds a Poisson variable $\gamma(s)$ with expectation $\lambda m(s)$, representing the number of primary events occurring on s and that, if $s_1, s_2, \ldots$ are disjoint, then $\gamma(s_1), \gamma(s_2), \ldots$ are completely independent. Here λ is an adjustable parameter, the expected number of primary events per unit length of the track. The negative ion produced by a primary event may, but need not, have enough energy to strike other atoms in the vicinity of the primary event and produce more pairs of ions. Thus, to each primary event there corresponds an integer valued random variable $\nu \geqq 1$, representing the number of pairs of ions produced at that event.

Each positive ion creates a droplet which is observable. The random variables ν corresponding to different primary events are mutually independent and identically distributed. Also they are independent of the location of the primary events. To each negative ion there corresponds a fixed probability θ of producing an observable droplet. It is assumed that the occurrence of a droplet on a particular negative ion is independent of all other events considered. After the droplets on particular ions are formed, they take part in the Brownian motion and $p_i(u)$ represents the conditional probability, given the coordinate u of the primary event on the track T, that a droplet generated by this primary event will be found in the cell L_i.

The observable random variables are the numbers $X_1, X_2, \ldots, X_m$, of droplets in the consecutive cells of track T. Because of the Brownian motion, these variables are dependent and their joint probability generating function is

$$(1) \qquad G_{X_1, X_2, \ldots, X_m}(t_1, t_2, \ldots, t_m) = E\{\prod_{n=1}^{m} t_1^{X_i}\}$$
$$= \exp\{-\lambda \int_{-\infty}^{+\infty} [1 - G_\nu((1+\theta)S(u) + \theta S^2(u))]du\}$$

where $G_\nu(t)$ stands for the probability generating function of ν and

$$(2) \qquad S(u) = \sum_{n=1}^{m} p_i(u)(1 - t_i).$$

Except for a specific assumption regarding the Poisson property of the primary events, formula (1) represents a general model of the phenomenon of ionization. In order to obtain numerical results it is necessary to specialize $G_\nu(t)$ and the probabilities $p_i(u)$ corresponding to the scheme of Brownian motion contemplated. Table I gives a comparison between the actual counts of droplets in particular cells and the expectations based on one particular specialization of formula (1).

Table I. *Frequency distribution of number of droplets in cells of a track of a cosmic particle*

No. of drops	0	1	2	3	4	5	6	7	8	9
Obs.	0	0	0	4	6	5	8	11	16	14
Frequency Comp.	.4	.4	2.3	2.2	5.9	5.3	10.2	8.6	13.2	10.5
No. of drops	10	11	12	13	14	15	16	17	18	19
Obs.	11	15	7	10	7	3	2	6	2	1
Frequency Comp.	13.5	10.3	11.5	8.4	8.3	5.8	5.3	3.6	3.0	1.9
No. of drops	20	21	22	23	24	25	26	Totals		
Obs.	3	2	1	0	0	0	1	135.0		
Frequency Comp.	1.5	.9	.7	.4	.3	.2	.1	134.7		

The counts of droplets along the track are due to Professor William B. Fretter of the University of California, to whom I am deeply indebted. The interesting point about the distribution reproduced is that the probability of an even number of droplets in a cell is always of the same order of magnitude as, but somewhat greater than, that of the next odd number.

It appears plausible that, with an appropriate specialization of the function $G_\nu(t)$, formula (1) will represent satisfactorily the phenomenon of ionization in all cases. The purpose of building the model is to deduce appropriate procedures for estimating the constants that are of particular importance in the study of cosmic rays. One such constant is the product $I = \lambda E(\nu)$ representing the expected number of pairs of ions generated by the cosmic particle per unit of track. In this particular case, just as in any other case where a stochastic model of a phenomenon is studied, there immediately appear various problems of statistical decision: (a) should one act on the assumption that the given model is adequate to represent the phenomena studied? If so then (b) what arithmetical procedures should one apply in order to obtain reliable estimates of the parameters involved? (c) What is the accuracy of these estimates? etc.

4. *Problem of identifiability*. A very interesting aspect of the problem of models is the question of identifiability. As Borel [1] expected half a century ago, it is quite possible that two or more different stochastic models imply identical distributions of the same observable random variable X. Naturally in a case of this kind, no amount of observation on X can provide any means for deciding which of the models concerned is closer to the actual machinery governing the variability of X. In certain cases, all one is interested in is the distribution of X and the identity of the machinery producing this distribution is of no consequence. In cases of this kind, one follows Borel's advice and adopts the particular model that is the simplest. An example of this kind occurred in the study of the distribution of galaxies in which an infinity of different distributions of the random variables $\gamma(s)$ combined with an appropriate distribution of ν can produce the same spatial distribution of galaxies. The simplest assumption regarding $\gamma(s)$, which falls within the particular class considered, is that $\gamma(s)$ is a Poisson variable. Thus, without loss of generality this particular assumption was adopted.

In other cases, however, the identity of the machinery governing the variation of X is of prime practical importance. Then the statistician and the experimenter are faced with the problem of additional observations of some other random variables $Y, Z, \ldots$, with respect to which the model is identifiable.

An interesting example of this kind refers to the number X of accidents assumed nonfatal, incurred by an individual in a unit period of time. Two essentially different models lead to identical distributions of X. Let $P_{m,n}(t_1, t_2)$ denote the conditional probability of incurring exactly $X = n$ accidents in the time interval $[t_1, t_2)$ given that prior to t_1 the individual concerned incurred exactly m accidents. According to one of the models, due to Greenwood, Yule [13] and Newbold [14], for any given individual

$$(3) \qquad P_{m,n}(t_1, t_2) = e^{-\lambda(t_2-t_1)} [\lambda(t_2 - t_1)]^n / n!.$$

where λ is a measure of the individual's own accident proneness and is treated as a particular value of a random variable Λ following a Pearson Type III law. Since (3) is independent of m, the accidents already incurred have no effect on probablities of more accidents in the future. On the other hand, the whole population of individuals is a mixture of those with high and those with low accident proneness. As is well known, the Greenwood-Yule-Newbold scheme implies that, for a randomly selected individual, X follows a negative binomial distribution.

The alternative scheme, due to Pólya [15], postulates that the accidents are "contagious" and subject to "time effect". The basic postulate is that

$$(4) \qquad \left.\frac{\partial P_{m,1}(t_1, t_2)}{\partial t_2}\right|_{t_2=t_1} = -\left.\frac{\partial P_{m,0}(t_1, t_2)}{\partial t_2}\right|_{t_2=t_1} = \lambda \frac{1 + \mu m}{1 + \nu t_1}$$

$$(5) \qquad \left.\frac{\partial P_{m,n}(t_1, t_2)}{\partial t_2}\right|_{t_2=t_1} = 0$$

for $n = 2, 3, \ldots$ Here λ, μ and ν are positive numbers, the same for all individuals of the group studied. The curious thing is that Pólya's assumptions, different as they are from those of the preceding model, imply identical properties of the random variable X. In other words, with respect to X the two models are unidentifiable. Should the same conclusion apply not only to the random variable X but also to all other variables that the observation of accidents might provide, then the difference between the models mentioned would be no more than of a purely academic interest. However, as one feels intuitively, the difference between the presence and absence of contagion in accidents must be relevant to the policy of personnel management and that, therefore, with respect to some observable variables the two models must be identifiable.

Recently it was shown [16] that the difference between the two models becomes apparent when the observations include something more than the variable X, perhaps the numbers X_1 and $X_2 = X - X_1$ of accidents incurred in two successive subintervals, the union of which constitutes the original period of observation. More particularly, a test for the presence of the Pólya contagion [17] was developed using the exact times of occurrence of all the accidents incurred.

Another interesting case of nonidentifiability was discovered [18] quite some time ago and is still a subject of study. In its simplest form it is as follows.

Let ξ be a random variable connected with another random variable η by the linear equation

$$\eta = \alpha\xi + \beta \tag{6}$$

where α and β are unknown constants. The variables ξ and η are unobservable. Instead we may observe the variables X and Y representing the measurements of ξ and η, respectively,

$$X = \xi + u, \qquad Y = \eta + v \tag{7}$$

where the "errors of measurements" u, v are assumed to be independent of ξ and η. Frequently it is also assumed that u and v can be split into two components, possibly degenerate,

$$u = u_1 + u_2, \qquad v = v_1 + v_2 \tag{8}$$

of which u_1 and v_1 are mutually independent and arbitrarily distributed and u_2 and v_2 are normal variables, possibly correlated.

It is obvious that the model described has a tremendous field of application and in each case the problem consists in estimating either the slope α in equation (6) above or both α and β.

The interesting point is that, if $u_1 = u_2 = 0$ and ξ is a normal variable then the constants α and β are unidentifiable. The ease with which this case of nonidentifiability presented itself suggested that cases of nonidentifiability are very common and that, therefore, it is more or less useless to attempt the problem of estimation. Many papers followed giving solutions of various modifications of the original problem. Of these we shall mention one by Abraham Wald [19] containing a rich bibliography, another by Hemelrijk [20] and a third by Scott [21]. The very ingenious solution of Hemelrijk is concerned with the situation where the observed values of X and Y correspond to a sequence of not all equal nonrandom values $\xi_1, \xi_2, \ldots$ of ξ and where the assumptions regarding the errors (u, v) are very broad indeed. Unfortunately, in many cases the randomness of variation of ξ is an essential part of the model of the phenomena studied [22].

In 1942 there appeared an article by R. C. Geary [23] which was overlooked by many authors, including the present. From this article it follows that the originally established case of nonidentifiability of α and β is an exception rather than a rule. In addition to the linear case as in (6), Geary treated the case where η is a polynomial of an arbitrary degree in ξ. The conditions that are necessary and sufficient for the identifiability of α and β were established with all precision by Reiersol [24]. Essentially, these conditions require only that the distribution of ξ be nondegenerate and nonnormal. This discovery of Reiersol constituted a challenge to other statisticians, to construct such estimates of the parameters α and β which would be consistent in the most general case of identifiability. The first problem of this kind, referring to cases of identifiability of the directional parameter α alone, was solved [25] in 1951. The linear multivariate case was subsequently treated by Jeeves [26]. Finally, a new approach to the same problem, based on the method of minimizing the conveniently defined distance between two distribution functions, was published in a series of papers by Wolfowitz. The first of these papers [27] appeared in 1952. The last has just been published [28]. These papers treat the problem of estimation of both parameters α and β. Unfortunately, none of these publications treat the problem of precision of the various estimates, a problem which involves very considerable difficulty. A novel approach is due to Charles M. Stein whose results were partly reported in a colloquium in 1953.

Wolfowitz' method of minimum distance represents a very interesting extension of the same method applied previously [29, pp. 252—254, 30] in a much simpler situation.

5. *Trends of research on statistical decision problems — General remarks.* Previously, we remarked that the problems of statistical decisions have no mathematical meaning until one specifies the meaning of the phrase "most advantageous" taken in quotation marks. As frequently is the case with fruitful mathematical concepts, those of the modern theory of statistical decisions can be traced to Gauss and before him, to Laplace. Essentially, there are two distinct ideas underlying the theory of statistics. One is that every method of selecting a course of action on the ground of observations of some random variables causes the action chosen to become a random variable. The second idea is that for a rational choice of a method of adjusting our actions to the available sets of observations, a definition of what the economists like to call a utility function is unavoidable. Both these ideas were familiar to Laplace [31] who at the time was concerned with the problem of estimation.

Consider the situation in which a stochastic model M of some phenomena defines a sample space $\mathscr{X}$ of some random variable X. Also, with every real number θ the model M associates a distribution function $F_\theta(x)$ and a datum of the problem is that there exists a value θ^* of θ such that $F_{\theta^*}(x)$ coincides with the distribution function of X. However, the identity of θ^* is unknown and the quaesitum of the problem is a function $\theta(X)$ from $\mathscr{X}$ to the real line, to be termed the estimate or the estimator of the parameter θ. The contemplated use of this function is as follows: when the observations on X yield the value x, then we shall act on the assumption that $\theta^* = \theta(x)$.

In order to select the estimate $\theta(X)$ rationally, Laplace proposed to consider the process of estimation as a game of chance in which the statistician can never gain anything but can incur losses depending on the error in the estimate. Laplace was aware of the fact that the assessment of these losses must be subjective and selected for himself the absolute value $|\theta^* - \theta(X)|$ of the difference between the true value of the parameter and the estimate as the measure of loss or of "detriment" resulting from the error of estimation. The expected value of this loss function

$$R_{\theta(X)}(\theta^*) = \int |\theta^* - \theta(x)| \, dF_{\theta^*} \tag{9}$$

is called the risk function corresponding to the estimate $\theta(X)$ and serves as a basis for assessing the goodness of this estimate in relation to the particular value θ^* of the parameter θ.

Gauss [32] followed the steps of Laplace. However, he noticed that in many problems the formulae gain in simplicity if Laplace's loss function is replaced by its square, $[\theta^* - \theta(X)]^2$. This change took root and caused the subsequent preoccupation with the method of least squares. Unfortunately, while the machinery of the least square method became very popular, many of its users seem to have lost sight of the basic ideas behind the method.

Early in the present century the concept of detriment function was the subject of studies by Edgeworth [33]. Thereafter it was forgotten until 1933 when the idea was rediscovered and published in a joint paper [34] by E. S. Pearson and the present writer. However, after mentioning the idea, we did nothing with it. It stayed dormant until the appearance on the statistical scene of Abraham Wald who, in a series of remarkable papers and books [35, 36] used the idea of loss function and of risk very extensively and created out of them a firm foundation for modern statistical theory. Since Wald's untimely death in 1950, his work has been continued by a number of followers of whom we shall mention Blackwell and Girshick [37] and Wolfowitz. In addition to obtaining new results referring to particular loss functions, modern research is concerned with conclusions applicable to various classes of such functions.

In general, the present stage of development of mathematical statistics may be compared with that of analysis in the epoch of Weierstrass. During the preceding decades, a very considerable number of special problems were solved without particular care for generality and rigor. The workers in the field were in too great a hurry to broaden the domain of research to bother with what they considered troublesome details. At the present moment, we observe the inevitable reaction. The basic concepts of the theory are being revised, the commonly accepted assertions are being examined in order to determine the exact domain of their validity and, just as in the times of Weierstrass and immediately thereafter, counterexamples are being constructed in order to show the insufficiency of old proofs. It is difficult to determine exactly when this revisionary epoch began. The first attempts at a clearer and more precise grounding of the theory were published shortly before the Second World War [38–41]. In fact, the concept of power of a test, and those of most powerful and unbiased tests and of similar regions introduced then still continue to be subjects of many studies. The same applies to the theory of confidence regions. However, the really general trend towards firmer foundations developed after the war and here, I feel, the book by Cramér [3] appears to have been instrumental. Although mostly concerned with the probabilistic background rather than with the theory of statistics itself, this book assembled and made easily available the results of many schools of thought which hitherto were not generally known to American and British statisticians, including Cramér's own results and those of Kolmogorov, Khintchine and Slutsky. Above all, the book set an unprecedented standard of mathematical precision.

In order to illustrate the revisionary trend in recent work, one or two examples may be helpful.

6. *Maximum likelihood estimates.* The first example is concerned with the problem of excellence of the so-called maximum likelihood estimates. This problem originated from the writings of Edgeworth [33] in 1908/9. Fascinated by the results of Laplace, Edgeworth tried to prove that the *a priori* variance of the *a posteriori* "most probable" value of a parameter must be less than the variance of any other estimate of the same parameter. In this Edgeworth had the help of a Professor Love whom I have not been able to trace. The *a posteriori* "most probable" value was computed assuming uniformity of the *a priori* probability density of the parameter. Thus, this most probable value was that value of the estimated parameter which maximized the probability density of the observable random variables, evaluated at the actually observed values of these variables.

Some thirteen years later [2, 42], the same idea occurred to Fisher. However, Fisher detached himself from considerations of the *a posteriori* probabilities and preferred to introduce a measure of confidence in any hypothetical values of the unknown parameters, termed the mathematical likelihood. Thus, Edgeworth's most probable value obtained a new name, the maximum likelihood estimate. Its general use was emphatically advocated both as a matter of principle, because it maximizes the likelihood, and because of its remarkable properties of having, for large samples at least, the minimum variance. By a remarkable coincidence, the early examples used by Fhisher in order to illustrate the excellence of the maximum likelihood estimates were those originally discussed by Edgeworth. Both authors were concerned with the estimation of the parameters involved in Pearson's distributions and both contrasted the precision oi the maximum likelihood estimates with that of estimates based on the method of moments.

In spite of the fact that the proofs offered were manifestly inaccurate, the illustrative examples appeared convincing and the Edgeworth-Fisher assertions regarding the maximum likelihood estimates became generally accepted and are repeated in a great many books and in countless articles. In this connection it may be regretted that the original papers by Edgeworth were overlooked for quite some time and that his name is mentioned only rarely.

The two most important properties ascribed to maximum likelihood estimates are conveniently expressed in terms of the following concepts due to Fisher. A sequence of functions $\{\theta_n(X)\}$ with $\theta_n(X)$ depending upon the first n random variables of a sequence $\{X_n\}$, is called a consistent estimate of a parameter θ, if as $n \to \infty$, the value of $\theta_n(X)$ tends in probability to θ. A consistent estimate $\{\theta_n(X)\}$ is called asymptotically efficient if, as $n \to \infty$, it tends to be normally distributed with mean equal to θ and with the minimum asymptotic variance.

With this terminology, the two asserted properties of the maximum likelihood estimates are that, in conditions specified only very vaguely, they are consistent and asymptotically efficient.

There are on record a number of attempts to prove the above assertions. Steadily improved proofs of the property of consistency were given in succession by Hotelling [43], Doob [44, 45] and Wald [46]. The first rigorous attempt to prove efficiency is due to Wilks [47]. However, the proof given refers to what may be called restricted efficiency. The asymptotic variance of the maximum likelihood estimate was proved to be not necessarily an absolute minimum but only a minimum with respect to a certain class of estimates and under certain specified conditions to be satisfied by the distributions of the observable random variables. Subsequently, [29], a whole class of estimates, termed regular best asymptotically normal estimates, (RBAN estimates), was discovered referring to general multinomial distributions, having the same asymptotic properties as the maximum likelihood estimates. In particular, compared with a certain category of "regular" estimates, the asymptotic variances of the RBAN estimates are minimum. These results were later improved by Barankin and Gurland [48]. For quite some time there were no examples known where the maximum likelihood estimates failed to possess the properties of consistency and/or asymptotic efficiency, and therefore uncertainty prevailed whether or not the restrictions underlying the proofs of consistency and/or efficiency of the maximum likelihood estimates were really necessary. The first counterexamples were published in 1948 [49]. However, these examples refer to a rather special case where the random variables of the sequence $\{X_n\}$ considered each follow a different distribution involving so-called "incidental parameters". It was shown that in this case the maximum likelihood estimates of the structural parameters may be inconsistent and may converge in probability to values different from the value of the parameters estimated. Also, it was shown that, even though consistent, a maximum likelihood estimate may be asymptotically normal and have an asymptotic variance uniformly greater than that of an alternative estimate.

The really surprising counterexamples, referring to the most classical case of estimating the mean ξ of normal distribution with unit variance, are due to J. L. Hodges, Jr. Let $\overline{X}_n$ stand for the arithmetic mean of n independent observations on a normal variable with expectation ξ and unit variance. As is well known, $\overline{X}_n$ is the maximum likelihood estimate of ξ, its distribution is normal about ξ and its variance is equal to $1/n$. Let T_n be an alternative estimate of ξ defined as follows. If $|\overline{X}_n| > n^{-1/4}$ then $T_n = \overline{X}_n$. Otherwise $T_n = \alpha\overline{X}_n$ where α is an arbitrary positive number less than unity. It is easy to see that T_n has the following properties (i) T_n is always asymptotically normal and, as $n \to \infty$, tends to ξ in probability. (ii) If $\xi \neq 0$ then the asymptotic variance of T_n coincides with that of $\overline{X}_n$ and, thus, is equal to $1/n$. (iii) If $\xi = 0$, then the asymptotic variance of T_n is equal to α^2/n and, thus, is less than that of $\overline{X}_n$. This example shows not only that the maximum likelihood estimate $\overline{X}_n$ of the parameter ξ is not asymptotically "efficient" in the sense of the original definition, but also, by a slight modification of the argument, that no efficient estimate of ξ exists at all. If no restriction of regularity is placed on the class of asymptotically normal estimates, then at each value of ξ, the greatest lower bound of the asymptotic variances is zero and cannot be reached.

In the above example the "efficiency" of T_n exceeded that of $\overline{X}_n$ at just one point $\xi = 0$. This point may be called the point of superefficiency of T_n. The method of condensation of singularities then allows us to define estimates with richer and richer sets of points of superefficiency. The question arises as to how rich the sets of superefficiency may be. Also, whether or not if one gains in efficiency at some values of the estimated parameter, one must lose something at others. Questions of this kind are treated in an important memoir [50] by LeCam. Among other things it is shown that the set of points of superefficiency must be of Lebesgue measure zero. The general conclusion of this study is that, although failing to possess the various properties of excellence originally claimed for them, maximum likelihood estimates frequently are quite respectable.

7. *Lower bound of the variance of an estimate.* As long ago as 1908, Edgeworth [33] realized that, at least for a category of estimates, and for a given distribution of the observable random variables, there must exist a positive

lower bound of the variances of estimates of a given parameter. Edgeworth was not clear about the exact nature of the category of estimates concerned and described them vaguely as "symmetric functions of the observations having the properties of an average." Nevertheless, by somewhat misty operations, he obtained an expression for the lower bound sought. The same lower bound was later discovered by Fisher [42]. Both Edgeworth and Fisher were primarily concerned with large sample theory. Immediately after the war, exact proofs of the inequality giving the lower bound of the mean square error appeared simultaneously in publications by Cramér [3] and by C. R. Rao [51]. Dr. Scott and the present writer [49] claim priority in recognizing the importance of these results, referring to samples of fixed size. Also we bear the responsibility for labeling the formula the Cramér-Rao inequality. The label took root and the current literature contains a considerable number of contributions concerned with the Cramér-Rao inequality and with various extensions. In this respect we wish to express our regrets for having overlooked the previous publications of Fréchet [52] and of Darmois [53] who discovered and proved the same inequality somewhat earlier and published it during the war. Because of the difficulties in following the literature in conditions of a world war, there is no doubt that the discoveries were entirely independent and it appears advisable to relabel the result the Frechet-Darmois-Cramér-Rao inequality.

The inequality results from the remarkable properties of the logarithmic derivatives of the probability density with respect to the estimated parameter. Of the subsequent work on the subject we shall mention those of Bhattacharyya [54], Barankin [55], and Chapman and Robbins [56]. Quite recently, the very interesting studies by Berkson [57, 58] brought to light the significance of the derivative of the bias appearing in the Fréchet-Darmois-Cramér-Rao inequality illustrated on certain minimum χ^2 estimates with the mean square error less than that of the maximum likelihood estimate. Further important results in the same direction are obtained by J. L. Hodges, Jr. [59].

8. *Sufficient statistics.* One of the most interesting concepts of statistical theory, that of sufficient statistics, was introduced by Fisher [42]. Originally somewhat vague, the concept gradually gained in rigor [60—62]. The latest results are due to Halmos and Savage [63]. The original idea of Fisher was that sufficient statistics must play an unusual role in the problem of point estimation. It happened, however, that the first extensive use of sufficient statistics by authors other than Fisher, was made in testing hypotheses and in the theory of confidence intervals, where they provide an easy method of constructing similar regions [40]. Important results in this direction are due to Lehmann and Scheffé [64, 65]. An important theorem concerned with point estimation was discovered first by Rao in 1945 [51] and then, independently, by Blackwell in

1947 [66]. The very simple theorem is to the general effect that if T_1 is an unbiased estimate of θ and T_2 is a sufficient statistic for θ, then the conditional expectation of T_1 given T_2 is also an unbiased estimate of θ with a variance that cannot exceed that of T_1. An interesting use of this theorem appears in the paper of N.L. Johnson [67]. The theorem discussed is occasionally called the Blackwell theorem. A more appropriate label seems to be the Rao-Blackwell theorem.

9. *Testing of statistical hypotheses*. The basic concepts of the theory were published in the late 1920's and in the 1930's and constitute what some colleagues are kind enough to call the Neyman-Pearson theory. First came [68] the definition of the term "statistical hypothesis" meant to denote any assumption regarding the distribution of the observable random variables. This definition was accompanied by the concept of a test as a rule of rejecting the hypothesis tested in some specified cases and of abstaining from such rejection in other cases. The same paper contains the concept of the two kinds of error that one may commit in testing a hypothesis. The next step in the development was the realization of the necessity of considering not only the hypothesis H tested but the set Ω of all simple hypotheses, which in a given problem are deemed admissible. The original proof published [69] in 1929 is reproduced in the more recent publication [70]. From the perspective of a quarter of a century, further progress should have been almost immediate. However, the routine of thought established by the literature of the preceding decades was weighing heavily on our minds and it took us the whole of five years to devise the concept of power and that of the most powerful test [38]. In this struggle the discussion regarding the possibility of a rational theory of testing hypotheses conducted some years before by the pessimistic Joseph Bertrand [71] and by the optimistic Emile Borel [72] was very inspiring. The most relevant remarks, due to Borel, concerned the functions of the observations "en quelque sorte remarquables" which, Borel believed, could serve as test criteria. However, at the time the distinction between statistical and other hypotheses was not clear and this, perhaps, was the reason why the idea of the functions *en quelque sorte remarquables* took quite some time to crystallize. The rediscovery of the concept of loss function in 1933 was accompanied by the first result referring to a class of loss functions as contrasted with the results referring to particular loss functions of Laplace or of Gauss. The result in question [34] was the concept of the uniformly most powerful test. At the same time [38], considerations of composite hypotheses brought the necessity of considering the fascinating problem of similar and then of bisimilar regions. In spite of a considerable number of important concontributions, this problem is still far from completely solved. Because of its somewhat unusual mathematical character it appears desirable to give a brief description.

Let Φ be a family of distribution functions F, defined over n-dimensional Euclidean space S. Let α be a given number, $0 < \alpha < 1$, and $f(x)$ a system of k given functions from S to the real line, satisfying the condition

$$\int_S f_i(x)\, dF = 0$$

for all $F \in \Phi$ and for $i = 1, 2, \ldots, k$.

Definition 1. If a measurable subset s of S has the property that for every $F \in \Phi$

$$\int_s dF = \alpha,$$

then s is called similar to S with respect to Φ and of size α.

Definition 2. If a measurable subset s of S satisfies Definition 1 and, in addition, is such that

$$\int_s f_i(x) dF = 0$$

for all $F \in \Phi$ and for $i = 1, 2, \ldots, k$, then s is called bisimilar to S with respect to the family Φ of distributions and with respect to the functions $f_i(x)$.

The problems of similar and bisimilar regions consist (i) in determining whether or not, for given Φ and f_i, such regions (or sets) exist at all and (ii) in finding a general method of constructing all the subsets of the kind described. The present knowledge seems to be limited to the following.

(a) If the family Φ is finite and if every function $F \in \Phi$ is continuous then, for every α, these exists an infinity of similar regions. However, no general method of constructing all such sets is available. This result in the present wording was first published in 1946 [73]. However, I am indebted to Dr. Ester Seiden for calling my attention to the fact that it is an easy consequence of results in the topology of distributions, previously published by Liapounoff [74].

(b) It was shown by Feller [75] that for infinite families Φ there may exist no similar regions.

(c) For special families Φ the conditions necessary and sufficient for similarity of a subset s were given in 1933 [38]. This result was somewhat generalized [40] in 1937. The concept of bisimilar regions and the method of constructing them for certain restricted families of distributions were published in 1935 [41].

The whole theory is connected in various ways with the problem of moments, with the theory of orthogonal expansions and with the theory of integral equations. Subsequent important contributions are due to Scheffé [76], Lehmann and Scheffé [64, 65], Hoel [77], Ghosh [78] and Nandi [79]. Also, one should mention a recent paper [80] by Sverdrup which appears to have been largely inspired by the writings of the authors just mentioned, particularly by those of Lehmann and Scheffé. The difficulty of the general problem may be illustrated by that experienced in the attempts to solve a particular problem, pestering statisticians since about 1935. This problem, known as the Behrens-Fisher problem and very important in applications, consists in testing the hypothesis that the means of two normal distributions coincide while nothing is known or assumed about the variances. A lucky guess by Romanovsky [81] subsequently rediscovered by Bartlett [82], inspired Scheffé [83] who found a family of similar regions and determined a test of the hypothesis which is most powerful with respect to this family. Scheffé's results were later extended by Barankin [84]. Also Welch [85] has invented an alternative test procedure based on a certain expansion. Numerically this expansion seems to work. However, a proof of its convergence is still lacking. Also, the structure of the most general similar region is still unknown.

Whenever uniformly most powerful tests exist, they constitute a most desirable solution. Unfortunately, uniformly most powerful tests very rarely exist and then the solution of a practical problem of testing a given hypothesis requires the formulation of a new mathematical problem and of a new concept. Such new concepts, those of unbiased tests and of locally unbiased and locally most powerful tests were introduced in 1936 and 1938 [86]. Although at first sight appealing, occasionally these tests have unexpectedly unpleasant properties as discovered in the forthcoming publication by Lehmann [87]. Thus, the simplest problem of statistical decisions, that of testing a hypothesis is still far from being solved. Current work is still concerned with finding a principle, applicable in cases of sufficient generality and leading to a satisfactory solution. However, quite a few significant results have been achieved and further progress may be confidently expected from the new generation of statisticians. It is hoped that a recent book [88], summarizing the basic ideas of the statistical decision problem on an elementary level, will prove helpful in educating this new generation.

10. *Concluding remarks.* A paper meant for presentation at the plenum of an International Congress of Mathematicians may be expected to summarize the recent progress achieved in a given domain with its various ramifications. This has been attempted in the above pages. However, the recent advancement in the theory of statistics has been so great and its various ramifications so numerous that, after exceeding the space alloted to the present paper, I find myself with the embarrassing feeling of inadequacy. Time and space do not allow more than a mention of the remarkable results of Stein [89] indicating most interesting possibilities of sequential procedures. Neither can I do justice

to the works on the power of non parametric tests by Hoeffding, Lehmann and the Dutch School, nor to what may be called the "Young English School", headed by Barnard, David, Hammersley and David Kendall. Also, I regret to have to omit a detailed discussion of the brilliant results of T. W. Anderson, Hotelling and Miss Sitgreaves concerned with multivariate analysis and those of Robbins on the one hand and of Grenander and Rosenblatt on the other referring to the different aspects of decision problems concerned with stochastic processes. Finally, no more than a brief mention can be given to the new form of the multidecision problem solved by Tukey and Scheffé. Each of these directions of thought deserves quite a few pages and to present them all adequately a substantial volume would be necessary. It is hoped that some such detailed summary of the recent progress in a number of directions of statistical research will be published in the *Proceedings* of the Third Berkeley Symposium to be held within the next 12 months.

Before concluding I wish to thank very heartily the Organizing Committee of this Congress for the honor of being invited to give the present paper.

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EINIGE FRAGEN DER APPROXIMATION VON FUNKTIONEN DURCH POLYNOME

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1. In der Geschichte der theoretischen Untersuchungen, die der Approximation von Funktionen durch Polynome gewidmet sind, bemerkt man zwei Perioden.

Im vorigen Jahrhundert, im Laufe der ersten Periode, waren die Bemühungen der Mathematiker auf die Ausarbeitung der Approximationsmethoden einzelner individueller Funktionen gerichtet. Jedoch wurden zu der Zeit ebenfalls die Grundlagen zur gegenwärtigen Approximationstheorie von Funktionen gelegt.

P. L. Tschebyscheff führte den Begriff der besten Approximation ein und entdeckte den Satz, mit dessen Hilfe wir erkennen, ob die gegebene stetige Funktion auf dem Segment durch das Polynom auf die beste Weise oder nicht approximiert wird.

Tschebyscheff, sowie seine Schüler und Nachfolger, untersuchten ausserdem die speziellen Eigenschaften der Polynome, die sogenannten Extremaleigenschaften, die als Beweisinstrument eine Rolle in der modernen Approximationstheorie von Funktionen spielen. Die bekannten Markoffsche Ungleichungen für algebraische Polynome und die Bernsteinsche Ungleichung für trigonometrische Polynome sind Beispiele dieser Eigenschaften.

Die zweite Periode fällt schon in unsere Zeit. Sie formierte sich im Anfang unseres Jahrhunderts, nachdem Weierstrass seinen Satz bewies, der die prinzipielle Möglichkeit der Approximation auf dem Segment einer beliebigen stetigen Funktion durch Polynome feststellte.

Während im vorigen Jahrhundert die Approximation einzelner Funktionen den Untersuchungsgegenstand bildete, wird in der gegenwärtigen Zeit die Hauptaufmerksamkeit Fragen der Approximation von Funktionenklassen gewidmet.

Gegeben wird die Klasse der Funktionen, die (auf irgendeine Weise normierbar) analytisch, differenzierbar, der Lipschitzbedingung genügend usw. sind. Gegeben wird ausserdem die Approximationsmethode der Funktionen dieser Klasse durch Polynome. Gefordert ist, dass alles nur mögliche über die Grösse oder den Änderungscharakter der Approximationen der Funktionen dieser Klasse ausgesagt wird.

Dieser Vortrag setzt sich zum Ziel, einen kurzen Überblick über einige Ergebnisse, die sich auf dieses Thema beziehen, und in den letzten 20 Jahren erhalten wurden, zu geben. Dabei wird vielmehr die Aufmerkzamkeit nur auf solche Fragen gerichtet, mit denen ich selbst in meinen persönlichen Untersuchungen nahe Berührung hatte.

2. Vor allen Dingen wird uns folgende Aufgabe interessieren.

Gegeben sei die Funktionsklasse $\mathfrak{M}$ und gegeben sei die Approximationsmethode der Funktion $u_n(f, x)$ durch Polynome. Solche Methoden können zum Beispiel Fouriersche Summen, Fejérsche Summen n-ter Ordnung, beste oder algebraische Interpolationspolynome und trigonometrische Polynome n-ten Grades sein.

Gefragt ist, wie die obere Grenze

$$\sup_{f\in\mathfrak{M}} | f(x) - u_n(f, x) | = \mathscr{E}_{u_n}(\mathfrak{M}) \tag{1}$$

der erwähnten Approximation unter den Funktionen der Klasse $\mathfrak{M}$ ist.

In den Arbeiten von Lebesgue, Bernstein, Jackson, Vallée-Poussin, Borel und anderer Mathematiker wurden die Grössenordnungen der erwähnten oberen Grenzen in einer Reihe von Approximationsfällen untersucht, die für die Mathematische Analysis wichting sind.

Später, schon in den dreissiger Jahren, kam die Tendenz auf, genaue und asymptotische Ausdrücke für diese Konstanten zu suchen. Wir wenden uns ausführlicher gerade dieser Seite des Problems zu.

Angenommen, r sei eine natürliche Zahl und $\mathfrak{M}$ eine positive Zahl.

Wenn die Funktion f mit der Periode 2π eine Ableitung r-ter Ordnung $f^{(r)}(x)$ hat, die der Ungleichung $| f^{(r)}(x) | \leqq \mathfrak{M}$ genügt, so wollen wir sagen, dass sie zur Klasse $W^{(r)} = W^{(r)}(\mathfrak{M})$ gehört.

Es sei ferner $0 < \alpha \leqq 1$. Wenn die Ableitung r-ter Ordnung der Funktion mit der Periode 2π der Lipschitzbedingung mit der Konstante $\mathfrak{M}$ genügt

$$| f^{(r)}(x + h) - f^{(r)}(x) | \leqq \mathfrak{M} | h |^{\alpha},$$

so wollen wir sagen, dass sie zur Klasse $W^{(r)}H^{(\alpha)} = W^{(r)}H^{(\alpha)}(\mathfrak{M})$ gehört.

Angenommen, die Funktion $f(x)$ gehört jetzt zur Klasse $W^{(r)}(\mathfrak{M})$ und $S_n(f, x)$ ist ihre Fouriersche Summe n-ter Ordnung, dann gilt die Ungleichung

$$| f(x) - S_n(f, x) | < C_r \mathfrak{M} \frac{\ln n}{n^r}.$$

Diese Ungleichung wurde schon in den frühen Arbeiten von Lebesgue [15] und S. N. Bernstein [2] betrachtet. Damals sorgten die Autoren nicht dafür, dass die Konstante C_r möglichst klein sei. Sie sorgten allerhöchstens dafür, die Ungleichung im Sinne der Grössenordnung möglichst klein zu halten.

A. N. Kolmogoroff machte im Jahre 1934 den nächsten Schritt vorwärts zur Verstärkung dieses Resultates [11]. Er zeigte, dass folgende asymptotische Gleichheit existiert:

$$\mathscr{E}_{s_n}(W_r) \approx \frac{4M}{\pi^2} \frac{\ln n}{n^r} \quad (n \to \infty).$$

In Anschluss an Kolmogoroff bekamen späterhin ich [20], B.-Sz.-Nagy [18], S. Stečkin [31], I. Natanson [19] und andere, A. Timan [33], eine Reihe asymptotischer Abschätzungen ähnlicher Art für verschiedene in der Analysis wichtige Approximationsmethoden und Funktionsklassen.

Bemerkt sei, dass Stečkin die Aufgabe Kolmogoroffs löste für Potenzreihen, welche auf der Grenze des Kreises der Klasse $B^{(r)}$ gehören, die ähnlich der Klasse $W^{(r)}$ bestimmt ist.

3. Ein anderer wichtiger Untersuchungszyklus bezieht sich auf die Abschätzungen der besten Approximationen von Funktionen. Dieser Zyklus wurde in den Arbeiten von J. Favard begonnen, der die Ungleichung von G. Bohr und S. N. Bernstein verallgemeinerte. Weiter wurde der Zyklus fortgesetzt anfangs in den Arbeiten von N. I. Achieser, M. G. Krein und B. Sz. Nagy, und später in meinen und W. K. Dsjadyks Arbeiten.

Ich werde mich bemühen, dies zu Erklären, indem ich das Schema der Banachschen Räume benutze.

Wir betrachten im Banachschen Raum einen Körper $\mathfrak{M}$, der konvex, abgeschlossen und symmetrisch bezüglich des Nullelements ist. Diesen Körper wollen wir mit Klasse $\mathfrak{M}$ bezeichnen.

Es sei in unserem Banachschen Raum noch ein linearer n-dimensionaler Unterraum L gegeben, durch dessen Elemente (Polynome) wir die Elemente der Klasse $\mathfrak{M}$ approximieren wollen.

Es sei $E_n(x)$ der Abstand des Elementes x vom Unterraum L. Die kleinste Approximationskonstante der Klasse $\mathfrak{M}$ mit Hilfe von Polynomen aus L ist gleich

$$\mathscr{E}_n = \sup_{x \in \mathfrak{M}} E_n(x).$$

Wenn wir einen Hilbertschen Raum haben und $u_n(x)$ die Projektion des Elementes x auf L ist, so gilt $E_n(x) = || x - u_n(x) ||$ und folglich ist unsere Konstante $\mathscr{E}_n$ gleich der oberen Grenze

$$\mathscr{E}_n = \sup_{x \in \mathfrak{M}} || x - u_n(x) ||, \tag{2}$$

wobei $u_n(x)$ der lineare Operator ist.

In anderen, nicht Hilbertschen Räumen, ist die Sache komplizierter, da das x am nächsten liegende Element des Unterraums L mit Hilfe einer nichtlinearen Operation gefunden wird.

Und hier taucht das Problem auf: Ist es nicht trotzdem möglich, einen solchen linearen Operator $u_n(x)$ zu finden, der die Klasse $\mathfrak{M}$ im Unterraum L abbildet, und eben solchen der der Gleichheit (2) genügt?

In solcher abstrakten Form ist das Problem nicht in einem konkreten Banachschen Raum, ausser dem Hilbertschen, gelöst. Hier ist die Rede vom Existenzbeweis des linearen Operators $u_n(x)$, den wie die beste lineare Methode zur Betrachtung der Extremalaufgabe nennen werden.

Jedoch sind die Untersuchungen der obigen Autoren gerade Lösungen dieses Problems in einer Reihe für die Analysis konkret wichtiger partieller Fälle in der Metrik (C) (stetiger Funktionen) und der Metrik (L) summierbarer Funktionen.

J. Favard [6] löste diese Aufgabe in der Metrik (C) für die Klasse $W^{(r)}(\mathfrak{M})$.

Es sei $E_n(f)$ die beste Approximation einer Funktion mit der Periode 2π mit Hilfe trigonometrischer Polynome n-ter Ordnung, das heisst der Abstand in der Metrik (C) der Funktion f zum Unterraum der genannten Polynome. Dann gilt die Gleichheit

$$\sup_{f \in W^{(r)}(\mathfrak{M})} E_n(f) = \frac{A_r \mathfrak{M}}{n^r} \quad (r, n = 1, 2, \ldots),$$

wobei die Konstante A_r effektiv ausrechenbar ist.

J. Favard erhielt auch die beste lineare Methode für die Extremalaufgabe. Sie sieht folgendermassen aus:

$$u_n(f, x) = \frac{a_0}{2} + \sum_{1}^{n-1} \lambda_\varkappa^{(n)} (a_\varkappa \cos \varkappa x + b_\varkappa \sin \varkappa x), \tag{3}$$

wobei $a_\varkappa$, $b_\varkappa$ Fourierkoeffizienten der Funktion f sind, und die Zahlen $\lambda_\varkappa^{(n)}$ durch die Klasse $W^{(r)}$ bestimmt werden und effektiv ausgerechnet werden können.

Eben dieses Resultat wurde später von N. I. Achieser und M. G. Krein [1] erhalten, die parallel den Fall der Klasse $\overline{W}_r$ betrachteten, der trigonometrisch der Klasse W_r konjugiert ist, und die dieses Ergebnis allgemeiner Klassen von periodischen, differenzierbaren oder analytischen Funktionen verallgemeinerten.

Die erwähnte Theorie lässt sich völlig auf die Klasse $\mathfrak{M}$ der periodischen Funktionen erstrecken, die in Form des Integrals

$$f(x) = \int_0^{2\pi} \varkappa(x - t) \varphi(t) dt \quad (|\varphi| \leqq \mathfrak{M})$$

darstellbar sind, wenn nur der Kern $\varkappa(t)$ den speziellen Bedingungen genügt. Diese Bedingungen werden auf jeden Fall für die von B. Nagy untersuchten Kerne erfüllt.

Es sei bemerkt, dass vor Kurzem der sowjetische Mathematiker W. K.

Dsjadyk [5] diese Aufgabe für die Klasse $W^{(r)}(\mathfrak{M})$ löste, wenn $r < 1$ und die Ableitungen im Sinne von Weyl verstanden werden.

4. Indem ich zu den Summen (3) zurückkehre, möchte ich bemerken, dass sie als Sonderfall die meisten bekannten Methoden der Summierung von Fourierschen Reihen bei passenden Koeffizienten $\lambda_\varkappa^{(n)}$ enthalten (Fejérsche Summen, Summen von Vallée-Poussin, Cesaro, Rogosinski-Bernstein usw.).

Gewöhnlich wurde jede solcher Summierungsmethoden für sich betrachtet. Jedoch lässt sich in der letzten Zeit die Tendenz feststellen, diese Summen im allgemeinen Falle zu untersuchen. Hierbei werden vorher an die Koeffizienten $\lambda_\varkappa^{(n)}$ nur allgemeine Bedingungen gestellt: Monotonie, Konvexität, Quasikonvexität, Darstellbarkeit in Form von $\lambda_\varkappa^{(n)} = \varphi\left(\frac{\varkappa}{n}\right)$ usw. Schon bei diesen Bedingungen gelingt es, weitgehende notwendige und hinreichende Konvergenzkriterien dieser Methoden zu bekommen, die eine grosse Anzahl früher erhaltener Teilfälle umfasst. Darüberhinaus gelingt es sogar, für die oben bestimmten oberen Grenzen nicht nur die Grössenordnungen, sondern auch die asymptotischen Ausdrücke zu erhalten.

Im Zusammenhang damit sind die Arbeiten von E. Hille und I. D. Tamarkin, J. Favard, G. Alexits, S. Losinksy und S. Nikolsky zu erwähnen, die sich auf Fragen der Konvergenz beziehen, und die späteren Arbeiten von B. Sz.-Nagy und A. F. Timan, die sich durch grosse Allgemeinheid und Genauigkeit der erzielten Ergebnisse auszeichnen.

5. Ich möchte noch eine Bemerkung machen. Wenn die Funktion f der Periode 2π zur Klasse $W^{(r)}H^{(\alpha)}(\mathfrak{M})$ gehört, so genügt, gemäss der Ungleichung von Jackson, ihre beste Approximation der Ungleichung $E_n(f) \leqq C\mathfrak{M}n^{-r-\alpha}$ $(n = 1, 2, \ldots)$.

Die umgekehrte Behauptung bei α kleiner als eins ist auch richtig. Diese wurde von Vallée-Poussin festgestellt der die Umkehrung des Satzes Bernstein endgültig präzisierte. Jedoch ist bei α gleich eins die umgekehrte Behauptung im allgemeinen nicht richtig.

Der Ausweg aus dieser Lage fand Zygmund. Er hat etwas andere Klassen vorgeschlagen, die wir mit $H^{(r)}$ bezeichnen.

Es sei $r = r' + \alpha$, wo r' ganz und $0 < \alpha \leqq 1$. Auf Definition $H^{(r)} \doteq W^{(r')}H^{(\alpha)}$, wenn $0 < \alpha < 1$; aber für $\alpha = 1$, $f \in H^{(r)}$, wenn $f_{(x)}^{(r')}$ stetig ist und

$$|f^{(r')}(x+h) - 2f^{(r')}(x) + f^{(r')}(x-h)| \leqq \mathfrak{M}\,|h|. \tag{4}$$

Es gilt folgender Satz.

Damit die Funktion f zur Klasse $H^{(r)}$ gehört, ist es notwendig und hinreichend dass die Ungleichung $E_n(f) < \frac{c}{n^r}$ $(n = 1, 2, \ldots)$ erfüllt ist.

6. Übrigens ist zu bemerken, dass bei ganzem r Untersuchungen, die sich das Ziel setzen, genaue Konstanten für die Klassen $H^{(r)}$ zu erhalten, sehr schwer ausfallen.

Nehmen wir ein Beispiel: Wenn eine Funktion an den Enden des Segments $[-1, 1]$ gleich Null ist und der Lipschitzbedingung mit der Konstante $\mathfrak{M}$ gleich eins genügt, so ist $\max |f(x)| = 1$. Uns ist die genaue Lösung dieser Aufgabe für Funktionen, die der Zygmundbedingung mit der Konstanten eins genügen, unbekannt. A. F. Timan gab die nichttriviale Abschätzung $|f(x)| < \frac{4}{3}$. Sie ist jedoch nicht endgültig.

Hier sei noch eine Aufgabe, deren Lösung ich nicht kenne: Die genaue obere Grenze des Integrals $\int_0^{2\pi} f(x) \sin x dx$ ist unter den Funktionen f, die der klasse $H^{(r)}(\mathfrak{M})$ angehören, zu finden.

Im Übrigen gibt es für die Klassen $W^{(r)}$ und $W^{(r)}H^{(\alpha)}$ auch Aufgaben, die wir bisher nicht lösen konnten.

Uns ist zum Beispiel der genaue asymptotische Ausdruck für die Konstanten

$$\sup_{f \in W(0) H(\alpha)(\mathfrak{M})} E_n(f) = \mathscr{E}_n \quad (0 < \alpha < 1)$$

und $\sup\limits_{f \in W(1)(\mathfrak{M})} |f(x) - \bar{\sigma}_n(f, x)|$ unbekannt, wobei $\bar{\sigma}_n$ die in der Fejérschen Summe konjugierte Summe ist.

Für die Summen vieler Veränderlicher gelang es bisher nur in Einzelfällen solche asymptotischen Ausdrücke zu erhalten (siehe [4]).

7. Zuletzt ist noch zu erwähnen, dass die Jacksonschen Ungleichungen für die beste Approximation von Funktionen eine Abschätzung von oben geben. Vor kurzem erhielt S. B. Stečkin [31] eine Abschätzung von unten für Funktionen aus den Klassen $H^{(r)}$ und sogar allgemeinerer.

8. Gehen wir zum nicht periodischen Fall über. Angenommen, dass $W^{(r)}(\mathfrak{M})$ Klassen nicht periodischer Funktionen sind, gegeben auf der Strecke $[-1, +1]$. Des Weiteren werden die Funktionen dieser Klassen genau so bestimmt, wie vorher. Jetzt ist schon die Rede von der Approximation dieser Funktionen durch algebraische Polynome: $P_n(x) = a_0 + a_1 x + \ldots + a_n x^n$. Angenommen, $E_n(f)$ bezeichnet die beste Approximation der Funktion f auf der Strecke $[-1, +1]$ in der Tschebyscheffmetrik durch Polynome $P_n(x)$.

Es gelang, zu zeigen, dass die obere Grenze der besten Approximationen $E_n(f)$ unter den Funktionen der nicht periodischen Klasse $W^{(r)}(\mathfrak{M})$ asymptotisch gleich der oberen Grenze im periodischen Falle ist:

$$\sup_{f \in W^{(r)}(\mathfrak{M})} E_n(f) \approx \frac{A_r}{n^r} \quad (n \to \infty,\ r = 1, 2, \ldots). \tag{5}$$

Der Fall $r = 1$ wurde von mir [21] bewiesen, den allgemeinen Fall bewies S. H. Bernstein [2].

Die dieser Aufgabe entsprechende asymptotisch beste Linearmethode sieht so aus [37]:

$$u_n(f, x) = a_0 + \sum_{1}^{n} x^{\varkappa} \int_{-1}^{t} \alpha_{\varkappa}^{(n)}(t) f^{(r)}(t) dt,$$

wobei $\alpha_{\varkappa}^{(n)}(t)$ die durch die Klasse $W^{(r)}$ zu bestimmenden Funktionen sind. Gleichzeitig wurde diese Aufgabe in der Metrik (L) gelöst [23].

9. Ich möchte noch bei einigen Ergebnissen verweilen, die ich nur im Falle $r = 1$ formulieren werde. Es wurde gezeigt, dass die aus (5) folgende Ungleichung

$$\limsup_{n \to \infty} n E_n(f) \leqq \frac{\pi}{2} \mathfrak{M}$$

die auch für die Funktionen der Klasse $W^{(r)}(\mathfrak{M})$ gilt, genau ist.

Interessant ist, dass im Falle einer Beschränkung auf solche Funktionen der Klasse $W^{(r)}(\mathfrak{M})$, die eine Ableitung mit Unstetigkeiten 1-ter Art haben, für sie schon eine andere Ungleichung gilt, nämlich [22]:

$$\limsup_{n \to \infty} n E_n(f) \leqq \limsup_{n \to \infty} n E_n(|x|) = \mu \approx 0{,}282$$

wobei μ die Bernsteinsche Konstante, die kleiner als $\frac{\pi}{2}$ ist.

Aus der Jacksonschen Ungleichung folgt, dass die allergrösste Ordnung,die die beste Approximation einer individuellen Funktion der Klasse $W^{(1)}$ annehmen kann, gleich $O\left(\frac{1}{n}\right)$ ist. Seinerzeit bewies S. Bernstein, dass diese Grössenordnung für die Funktion $|x|$ erreicht wird. Wir können jetzt sagen, dass nicht vom Standpunkt der Grössenordnung, sondern von dem des asymptotischen Verhaltens der besten Approximation, es in der Klasse $W^{(1)}$ noch schlechtere Funktionen gibt, als $|x|$.

10. Ich möchte noch die Aufmerksamkeit darauf richten, das die Konstante A_r naturgemäss in anderen präzisen Extremalaufgaben der Analysis auftaucht. Im Zusammenhang damit sei folgendes Resultat von Kolmogoroff angeführt [12].

Es seien $\mathfrak{M}_0, \mathfrak{M}_{\varkappa}, \mathfrak{M}_n$ $(0 < \varkappa < n)$ drei positive Zahlen. Damit es auf der ganzen reellen Achse eine Funktion $f(x)$ existiert, für die

$$\mathfrak{M}_i = \sup_{-\infty < x < \infty} |f^{(i)}(x)| \quad (i = 0, \varkappa, n)$$

gilt ist notwendig und hinreichend, dass die Ungleichung

$$\mathfrak{M}_\varkappa \leqq \frac{A_{n-\varkappa}}{(A_n)^{\frac{n-\varkappa}{n}}} \mathfrak{M}_0^{\frac{n-\varkappa}{n}} \mathfrak{M}_n^{\frac{\varkappa}{n}}$$

stattfindet.

11. Ich möchte auch bemerken, dass die Theorie der Approximationen von Funktionen in der letzten Zeit um eine neue Seite bereichert wurde. Ich denke an ganze Funktionen vom Exponentialtypus oder, wie man sie noch nennt, ganze Funktionen endlichen Grades.

An der Schaffung der Approximationstheorie von Funktionen auf der reellen Achse durch ganze Funktionen endlichen Grades hat S. Bernstein grossen Anteil. Unter den mir bekannten anderen Arbeiten, die sich auf dieses Problem beziehen, bemerke ich die Arbeiten von N. Achieser, M. Krein, Boas, Kober, B. Sz.-Nagy, S. Nikolsky.

In einer dem periodischen Fall analogen Weise lassen sich auch, für die auf der geraden definierten Funktionen, die Klassen $H^{(r)}$ und — für die Metrik L_p die Klassen $H_{L_p}^{(r)}$ definieren, welche in bezug auf ganze Funktionen endlichen Grades sich ebenso verhalten, wie die betreffenden periodischen Klassen in bezug auf die trigonometrischen Polynome.

Der enge Zusammenhang mit der Theorie der Approximationen durch trigonometrischen Polynome ist hier so gross, dass sogar die genauen Approximationskonstanten in analogen Fällen völlig übereinstimmen [1, 17, 14]. Hier formuliere ich nur folgendes Resultat von S. Bernstein:

Bei allgemeinen Voraussetzungen, die an die Funktion gestellt werden, gilt die Gleichheit

$$\lim_{n\to\infty} E_n\left(f, \frac{n}{p}\right) = A_p(f),$$

wobei $E_n(f, a)$ die beste Approximation der Funktion f auf dem Segment $[-a, a]$ mit Hilfe eines algebraischen Polynoms n-ten Grades ist und $A_p(f)$ beste Approximation f durch eine ganze Funktion p-ten Grades.

12. Der letzte Teil meines Vortrages bezieht sich auf Fragen der Approximation von Funktionen mit mehreren Veränderlichen.

Die Klassen $H^{(r_1, \ldots, r_n)}$ [2] und $H_{L_p}^{(r_1, \ldots, r_n)}$ [38], [1], [25] von Funktionen mehrer Verändlicher sind nach dem Muster der Klassen $H^{(r)}$ und $H_{L_p}^{(r)}$ von Funktionen mit einer Veränderlichen aufgebaut, ebenso die gewöhnliche Theorie der Approximationen mit direkten und inversen Sätzen.

Im Falle der Metrik L_p ist es erforderlich, um die direkten und inversen Sätze zu vollkommenen „Abschliessen" zu bringen, dass die partiellen Ableitungen der Funktionen der Klassen $H_{L_p}^{(r_1, \ldots, r_n)}$ im verallgemeinerten Sinne zu verstehen sind, so wie es in der modernen Mathematischen Physik üblich ist.

Überhaupt muss konstatiert werden, dass einige Ideen die in der Mathematischen Physik schon in den dreissiger Jahren auftauchten sich in den letzten Jahren berührten mit der Theorie der Approximationen durch Polynome, was wiederum zu ihrer weiteren Entwicklung führte. Ich denke dabei an die in der mathematischen Physik bekannten Einbettungsatz von Soboleff [29].

Auf Grund der auf entsprechender Weise entwickelten Approximationstheorie von Funktionen durch ganze Funktionen endlichen Grades, wurde es ermöglicht, die Sätze von S. Soboleff und seines Schülers W. Kondrascheff [1]) [13] zu präzisieren und eine Klasse dieser Sätze voll umzukehren.

Es gibt zwei Typus von Soboleffschen Sätzen. Zum ersten Typus gehören die Theoreme, mit deren Hilfe man nach gegebener Differentialeigenschaft der Funktionen, ausgedrückt in der Metrik L_p, ihre Differentialeigenschaften in der Metrik $L_{p'}$ erkennen kann, wobei $1 \leqq p < p' \leqq \infty$.

Wenn zum Beispiel das Quadrat der Funktion $f(x, y)$ und die Quadrate ihrer ersten und zweiten verallgemeinerten Ableitungen in der Ebene integrierbar sind, so folgt daraus, dass die Funktion in der Ebene stetig ist. Sie gehört darüberhinaus zur Klasse $H^{(1,1)}$, das heisst sie genügt der (4) mit der Variablen x.

Zum zweiten Typus gehören die Sätze, mit deren Hilfe man nach gegebener Differentialeigenschaft, ausgedrückt bezüglich des Gebietes $\mathscr{G}$ die Differentialeigenschaft dieser Funktion erkennen kann, wenn man sie in der Mannigfaltigkeit betrachtet, die zu $\mathscr{G}$ gehört, und eine kleinere Zahl von Dimensionen hat als $\mathscr{G}$.

Wenn zum Beispiel die Funktion $f(x, y)$, gegeben in der Ebene, zur Klasse $H_{L_2}^{(1,1)}$ gehört, so gehört sie, wie die Funktion von x bei fixiertem y zur Klasse $H_{L_2}^{(1/2)}$.

Es zeigt sich jedoch, dass die Sätze des zweiten Typus völlig reversibel sind [27]. Aus dem Grunde kann man, wenn auf der x-Achse eine beliebige Funktion $\varphi(x)$ gegeben ist, die zur Klasse $H_{L_2}^{(1/2)}$ gehört, sie auf der ganzem Ebene fortsetzen so, dass die fortgesetzte Funktion aufs Neue zur Klasse $H_{L_2}^{(1,1)}$ gehört.

Für solche Untersuchungen von Funktionen mehrerer Veränderlicher mit Hilfe von Approximationsmethoden durch ganze Funktionen endlichen Grades oder durch trigonometrische Polynome wurde, neben der Anwendung der Bernsteinschen Ungleichung, die Anwendung anderer Extremalungleichungen [25] notwendig. Sie sehen so aus:

$$\|\mathscr{G}\|_{L_p}^{(m)} \leqq C \left(\sum_{m}^{n} \nu_\varkappa\right)^{\frac{1}{p}} \|\mathscr{G}\|_{L_p}^{(n)} \quad (1 \leqq m < n)$$

$$\|\mathscr{G}\|_{L_{p'}}^{n} \leqq C \left(\sum_{0}^{n} \nu_\varkappa\right)^{\frac{1}{p}-\frac{1}{p'}} \|\mathscr{G}\|_{L_p}^{(n)} \quad (1 \leqq p < p' \leqq \infty).$$

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[1]) Es sei noch die zu dieser Frage gehörende Arbeit von I. Petrowsky und K. Smirnoff [28] erwähnt.

Hier ist $\mathscr{G}$ eine ganze Funktion mit mehreren Veränderlichen des Grades $\nu_1 \ldots \nu_n$ und C eine Konstante, die nur von n abhängt.

Die erste Ungleichung gibt die Möglichkeit, die Norm der Funktion $\overline{\overline{\mathscr{G}}}$ im m-dimensionalen Unterraum durch die Norm abzuschätzen, die im n-dimensionalen Unterraum berechnet ist. Die zweite Ungleichung gibt die Möglichkeit die Norm $\overline{\overline{\mathscr{G}}}$ in der Metrik $L_{p'}$ zu berechnen, wenn die Norm $\overline{\overline{\mathscr{G}}}$ in der Metrik L_p bekannt ist. Die zweite Ungleichung war Jackson bei $n = 1$ und $p' = \infty$ früher bekannt.

Die Resultate, von denen hier die Rede ist, finden ihre Anwendung in der Theorie der direkten Methoden der Variationsrechnung [26, 29]. Als Beispiel betrachte ich die einfachste Dirichletsche Aufgabe.

Bekannt ist, dass die in der Ebene $\mathscr{G}$ harmonische Funktion $u(x, y)$ die an ihrer Grenze Γ gegebene Werte $u|_\Gamma = \varphi(s)$ annimmt, als Funktion zu bekommen ist, für welche das Integral

$$\mathscr{D}[F] = \iint\limits_{\mathscr{G}} \left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right\} dxdy$$

ein Minimum erreicht unter allen möglichen Funktionen F, die der Randbedingung $F|_\Gamma = \varphi(s)$ genügen.

Jedoch ist die Dirichletsche Aufgabe längst nicht bei allen, sogar stetigen, Randwertfunktionen $\varphi(s)$ lösbar. Schon Weierstrass brachte ein Beispiel einer stetigen Randwertfunktion, für welche die Dirichletsche Aufgabe nicht durch die Variationsmethode lösbar ist, da die ihr entsprechende harmonische Funktion ein unendliches Dirichletsches Integral $\mathscr{D}[u] = \infty$ hat.

In den modernen Handbüchern der Mathematischen Physik werden bei der Begründung der Variationsmethode unmittelbar die Eigenschaften der Randwertfunktion nicht gegeben. Um diese Schwierigkeit zu umgehen, wird angenommen, dass zum Mindesten eine zulässige Funktion existiert, die die gegebenen Randeigenschaften besitzt. Mit anderen Worten: Es wird angenommen, dass die Klasse der zulässigen Funktionen nicht leer ist, und das ist schon hinreichend, um die Existenz von Funktionen in dieser Klasse zu begründen, die das Dirichletsche Integral zum Minimum werden lassen, und die Tatsache zu begründen, dass die zum Minimum führende Funktion die einzige ist und harmonisch ist.

Trotzdem taucht die Frage auf: Welche Eigenschaften sollen die Randbedingungen besitzen, damit eine zulässige Funktion existiert? Solche Bedingungen kann man in den allgemeinsten Fällen von Randwertaufgaben elliptischen Typs geben, wenn die Grenze des Gebietes, für welches die Aufgabe gelöst wird, genügend glatt ist.

Notwendige Bedingungen wurden in den Arbeiten von S. Soboleff und W. Kondrascheff erhalten. Indem man die Approximationsmethoden durch Polynome anwendet, kann man sie in einigen Fällen präzisieren, jedoch kann man auch ihnen nahe (in den Bezeichnungen der Klassen $H^{(r)}$ mit der Genauigkeit bis auf ε) hinreichende Bedingungen geben.

Zum Beispiel gibt es im Falle der von uns betrachteten Dirichletschen Aufgabe mit hinreichend glattem Rand folgende Behauptung:

Damit den Bedingungen $\mathscr{D}[F] < \infty$, $F|_{\Gamma} = \varphi(s)$ genügende zulässige Funktion existiert, ist es notwendig, dass die Randwertfunktion zur Klasse $H_2^{(\frac{1}{2})}$ gehört, und hinreichend, dass die Funktion φ zur Klasse $H_2^{(\frac{1}{2}+\varepsilon)}$ gehört.

Zum Schluss möchte ich bemerken, dass gerade die Methoden der Approximation von Funktionen durch Polynome bei der Erfassung des Verhaltens von Funktionen in linearen Unterräumen des n-dimensionalen Raumes benutzt wurden. Der weitere Übergang von den Unterräumen zu kurvigen Rändern der Gebiete erforderte schon Betrachtungen, die für die Funktionstheorie überhaupt charakteristisch sind. Hierbei waren uns einige Ergebnisse von M. K. Hestenes [8] und H. Whitney [35] nützlich.

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GEOMETRY UPON AN ALGEBRAIC VARIETY

B. SEGRE

See Volume III under Symposium B.
Algebraic Geometry

RECENT DEVELOPMENTS IN RELAXATION TECHNIQUES

E. STIEFEL

I. One of the usual procedures for the solution of a system of real linear equations

$$Ax = k \tag{1}$$

is iteration. Beginning with a trial point x_0, a sequence of points x_i is to be constructed which approaches the point of exact solution $A^{-1}k$. For the sake of simplicity we shall take x_0 as the origin. In order to check the goodness of any approximation, we set x_i into the equations and form the *residual*-vector

$$r_i = k - Ax_i. \tag{2}$$

The iteration procedure consists of following a set of rules, which determine how to compute x_{i+1} from the preceding approximations. We shall call this set of rules the iteration algorithm and we limit ourselves to so-called *linear iterations* of the type

$$x_{i+1} = \sum_{j=0}^{m-1} C_i^{(j)} x_{i-j} + v_i.$$

The $C_i^{(j)}$ are matrices, which may depend upon A; the v_i are vectors. m is known as the *order* of the iteration, for exactly m preceding approximations appear on the right hand side. In particular we shall investigate second order procedures. They may be put in the form

$$x_{i+1} = B_i x_i - C_i x_{i-1} + v_i. \tag{3}$$

As Forsythe [1]) has pointed out in his thorough report on the solution of linear equations, it should be required that the solution be a fixed point of this transformation; that is, the point $x_{i-1} = x_i = x_{i+1} = A^{-1}k$ must satisfy equation (3). This permits the elimination of the non-homogeneous terms v_i. After some simple juggling of the terms, (3) may be written in the form

$$\Delta x_{i+1} = A_i r_i + C_i \Delta x_i, \quad A_i = (1 - B_i + C_i)A^{-1}. \tag{4}$$

Here Δx_{i+1} is the correction $(x_{i+1} - x_i)$ and the matrices A_i and C_i are arbitrary as the B_i and C_i were.

As you see in the general second-order procedure we may compute the correction from the present residual together with the preceding correction. A first order algorithm does not contain the term with Δx_i; and therefore does not use the history of the iteration. In an algorithm of higher order than the second, additional terms arise:

$$D_i \Delta x_{i-1} + E_i \Delta x_{i-2} + \ldots \tag{5}$$

II. The classical methods of iteration are all of first order:

$$\Delta x_{i+1} = A_i r_i \tag{6}$$

They differ from each other only in the choice of the matrices A_i. The more important of these methods are the following.

1) In the "Cyclic single" step procedure (*Gauss, Seidel, Nekrasov, Liebmann*) the A_i are equal to each other and are chosen as the inverse of the lower triangle of the given matrix A.

2) Less familiar is the fact that the *elimination algorithm of Gauss* is also a first order iteration procedure. Here the A_i are the reciprocals of the triangular matrices which are generated during the reduction of A.

3) In the so-called gradient methods, the A_i are scalars a_i:

$$\Delta x_{i+1} = a_i r_i. \tag{7}$$

The correction arising at each step is thus proportional to the residual. (*Richardson, Temple, Hestenes, Stein.*)

The convergence of these first-order procedures has been thoroughly investigated (*v. Mises, Collatz, Ostrowski, Weissinger*).

III. Although the convergence of a given algorithm certainly suggests its use for practical computation, it is most desirable that the convergence be monotone. No one likes to walk around in circles or in spirals. This is the point at which the principle of *relaxation* enters. The fundamental idea is to reduce the value of an appropriate measure of error at each step. In the following we shall assume the given matrix A to be *symmetrical* and *positive* definite. A suitable error measure is then the length φ of the error vector $y_i = A^{-1}k - x_i$, measured not in the cartesian sense but in the sense of the A-metric:

$$\varphi_i = (Ay_i, y_i) = (A^{-1}r_i, r_i). \tag{8}$$

The comma means the ordinary cartesian scalar product. Although the following theory could be developed with a number of other error measures, the φ-measure has two advantages:

1) It is invariant under coordinate transformations.
2) If the given system (1) is the system of normal equations for a problem in the calculus of observations, then the error measure φ is nothing else than the sum of the squares of the errors (plus an additive constant).

The three examples of first order algorithms given above are indeed relaxation methods in this sense; that is, the value of φ actually diminishes with each step. With the gradient method however, the a_i must be positive and smaller than certain bounds α_i.

Using the terminology of the theory of economic behaviour, we may say that the basic idea of relaxation is a *tactic*, in that one attempts to reduce φ by as much as possible in each single step. What we really desire, though, is a *strategy*. Of all the algorithms which have a given number n of steps, we want to choose the one yielding x_n closest to the true solution. The determination of such a strategy is a difficult mathematical problem and there is still much work to be done in this direction. Perhaps a closer study of the relationship to the theory of games would be useful. Certain partial results have already been found, however, and it appears to me that these represent the major advances in the theory of relaxation since the last international congress at Harvard. In particular the rules for *over-relaxation* have been investigated by *Ostrowski* [2]) and *Young* [3]).

Today I should like to outline a complete solution of the strategy problem in the special case of "*scalar*" iteration schemes. By scalar iteration is meant a linear algorithm in which the matrices involved in (4) and (5) are scalars. In the case of a second order procedure, the iteration algorithm becomes

$$\Delta x_{i+1} = a_i r_i + c_i \Delta x_i. \tag{9}$$

The method resulting from the choice of the a_i and c_i as independent of i, that is $a_i = a$, $c_i = c$, was first proposed by *Frankel* [4]) and further investigated by *Hochstrasser* [5]).

IV. Now it follows from (9) that any approximation x_i must be a linear combination of the iterated vectors

$$k, Ak, A^2k, \ldots, A^{i-1}k, \tag{10}$$

assuming $x_0 = 0$. In fact, this is true for scalar linear processes of any order and characterises these processes.

We may write

$$x_i = F_{i-1}(A)k, \tag{11}$$

where F_{i-1} is a polynomial with real coefficients of degree $(i - 1)$. The residual of x_i is

$$r_i = k - Ax_i = [1 - AF_{i-1}(A)]k. \tag{12}$$

Thus we shall have to do not only with the polynomials $F_{i-1}(A)$ but also with the *residual polynomials*

$$R_i(A) = 1 - AF_{i-1}(A). \tag{13}$$

Equation (12) becomes

$$r_i = R_i(A)k. \tag{14}$$

Let us consider the polynomials F_{i-1} and R_i for a real variable λ as argument instead of the matrix A. It follows from (13)

$$R_i(\lambda) = 1 - \lambda F_{i-1}(\lambda). \tag{15}$$

Thus for $\lambda = 0$

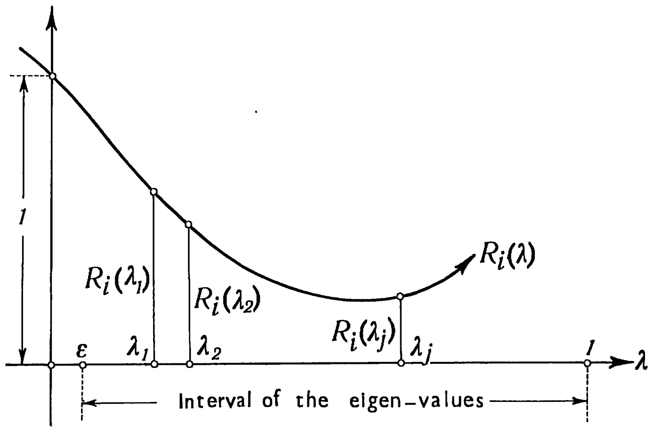
$$R_i(0) = 1. \tag{16}$$

It will turn out that this is the essential property of the residual polynomials. The residual polynomials have a simple and direct meaning in connection with the progressive approximations during the relaxation. This becomes clear if we write our equations in the coordinate system of the principle axes of A.

Since A was assumed symmetric and positive definite, its eigen-values λ_j are positive. Writing k_j for the components of k and r_{ij} for the *j-th* component of r_i we have in consequence of (14)

$$r_{ij} = R_i(\lambda_j)k_j. \tag{17}$$

Since $k = r_0$, the values $R_i(\lambda_j)$ of the residual polynomial $R_i(\lambda)$ tell by what percentage the components of the original residual have been reduced after the *i-th* iteration step (note the figure).



Our problem of finding the best strategy may now be stated somewhat inexactly as follows. A polynomial $R_n(\lambda)$ of given degree n must be found with the properties

a) $R_n(0)$ must be equal to one.

b) The values of the polynomial must be small over the interval of the λ-axis containing the eigen-values.

To state this precisely we use the error measure φ in the second form of equation (8).

The *j-th* component of $A^{-1}r_i$ is

$$\frac{R_i(\lambda_j)}{\lambda_j} k_j$$

and hence

$$\varphi_i = \sum_{(j)} \frac{R_i^2(\lambda_j)}{\lambda_j} k_j^2. \tag{18}$$

It will be convenient to use a more general measure of error ψ called the "*continuous*" error measure. In order to define it, we first need an *upper bound for the eigenvalues*, which without loss of generality may be set equal to 1. ψ is then given by

$$\psi_i = \int_0^1 \frac{R_i^2(\lambda)}{\lambda} \varrho(\lambda) d\lambda, \tag{19}$$

where $\varrho(\lambda)$ is an arbitrary density function defined over the interval $o \leqq \lambda \leqq 1$. The old *"discrete"* error measure φ is a special case of ψ corresponding to the density function

$$\varrho(\lambda) = \sum_{(j)} k_j^2 \cdot \delta(\lambda - \lambda_j), \tag{20}$$

where δ is the *Dirac*-function.

The following theorem can now be proved.

In the family of polynomials of *n-th* degree $R_n(\lambda)$ which satisfy $R_n(0) = 1$, the error measure ψ takes on its minimum value for the $(n + 1)$-*th* polynomial of the orthogonal set belonging to the density function $\varrho(\lambda)$.

V. This solves the strategy problem completely, but only if we are able to construct a scalar iteration algorithm yielding this minimal value of ψ after n steps.

Let $R_0(\lambda), R_1(\lambda), \ldots, R_n(\lambda)$ be the orthogonal set belonging to $\varrho(\lambda)$ such that each of the R_i satisfies (16). Note that the R_i can not be normalized in the usual sense. It is well known that three successive orthogonal polynomials are related by a recursion formula of the form

$$R_{i+1}(\lambda) = (d_i - a_i\lambda)R_i(\lambda) - c_iR_{i-1}(\lambda).$$

In consequence of (16) it follows that $d_i - c_i = 1$ and thus

$$R_{i+1}(\lambda) = (1 + c_i - a_i\lambda)R_i(\lambda) - c_iR_{i-1}(\lambda). \tag{21}$$

As given by (13) and (11) there are a sequence of polynomials $F_{i-1}(\lambda)$ and a sequence of approximations x_i associated with the $R_i(\lambda)$. Taking (14) and (21) into account, the corresponding residuals satisfy the recursion formula

$$r_{i+1} = (1 + c_i - a_iA)r_i - c_ir_{i-1} \tag{22}$$

and after proper substitutions it follows that

$$\Delta x_{i+1} = a_ir_i + c_i\Delta x_i. \tag{23}$$

By the definition (9) this is a scalar second-order process. In other words we must simply *carry out an iteration process of second order using coefficients taken from the recursion formula of the orthogonal set determined by the chosen density function.* Since this is the best possible scalar process, *it is not necessary to consider scalar procedures of higher order.*

As *Young* [6]) has remarked for a special case, the end result after n steps of our procedure may be reached by n steps of a *first order* scalar algorithm, that is, of a gradient method. Although the end values x_n coincide, the intermediate points do not. The second order procedure seems to be preferable in the sense of the adopted error-measure, since each of the intermediate points

x_i represents the better approximation after i steps. Furthermore the gradient algorithm requires the computation of the roots of $R_n(\lambda)$ but it needs less numerical work during the iteration.

VI. The selection of a density function $\varrho(\lambda)$ in the interval $0 \leqq \lambda \leqq 1$ must still be made. I wish to discuss three possibilities.

1) If no other information about the eigen-values is known except an upper bound, then we must take $\varrho(\lambda) > 0$ for $0 < \lambda < 1$ in order to cover the entire spectrum of A. Furthermore $\varrho(0)$ must be equal to 0 to insure the existence in the integral (19) and $\varrho(1)$ may vanish. These conditions suggest the form

$$\varrho(\lambda) = \lambda^\alpha (1-\lambda)^\beta f(\lambda), \ \alpha > 0, \ \beta > -1, \tag{24}$$

where we assume $f(\lambda)$ to have a continuous second derivative over $0 \leqq \lambda \leqq 1$ and to be bounded:

$$0 < c_1 \leqq f(\lambda) \leqq c_2, \quad 0 \leqq \lambda \leqq 1. \tag{25}$$

It can be shown that then the relaxation converges; that is

$$R_n(\lambda) \to 0 \text{ for } n \to \infty \text{ and } 0 < \lambda < 1. \tag{26}$$

The simplest possibility is $f(\lambda) \equiv 1$ yielding the *hypergeometric* or *Jacobian polynomials*.

$$R_n(\lambda) = F(-n, n+\alpha+\beta+1, \alpha+1; \lambda), \tag{27}$$

where F is the hypergeometric function of Gauss. α and β must still be chosen in consideration of the problem at hand. Some while ago *Lanczos* [7]) proposed the special case $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$. The behavior of the hypergeometric relaxation for large n is described asymptotically by the formulas;

a) for small λ we have

$$R_n(\lambda) \backsim \Lambda_\alpha(2n\sqrt{\lambda}) \tag{28}$$

where Λ_α is *Lommel's* function.

$$\Lambda_\alpha(x) = \frac{2^\alpha \alpha!}{x^\alpha} J_\alpha(x) \tag{29}$$

tabulated in *Jahnke-Emde*.

b) In the interior of the interval, $R_n(\lambda)$ oscillates with an amplitude given by

$$\frac{\alpha!}{\sqrt{\pi}} (n\sqrt{\lambda})^{-(\alpha+\frac{1}{2})} (1-\lambda)^{-\frac{\beta}{2}-\frac{1}{4}} \tag{30}$$

β may be called the *parameter of over-relaxation* for the following reason. Suppose β to be large, than the density

$$\varrho(\lambda) = \lambda^\alpha (1-\lambda)^\beta$$

diminishes rapidly as λ approaches 1. This means that the components of the residuals corresponding to the small eigen-values are more heavily weighted than the higher ones and therefore will be more rapidly eliminated. This is highly desirable in the application of relaxation techniques to partial differential equations.

For example, assume that the given system (1) of equations has been roughly solved using the sum of μ terms of the *Neumann-series*

$$A^{-1} = \sum_{(\nu)} (1 - A)^{\nu}. \tag{31}$$

The residual given by this rough solution is

$$r_{\mu} = (1 - A)^{\mu} k \tag{32}$$

corresponding to the residual polynomial

$$R_{\mu}(\lambda) = (1 - \lambda)^{\mu}. \tag{33}$$

In order to improve this rough solution we continue with hypergeometric relaxation using $\beta = 2\mu - \frac{1}{2}$. From (30) it follows that during this procedure (33) will be multiplied asymptotically by

$$\frac{\alpha!}{\sqrt{\pi}} (n\sqrt{\lambda})^{-\left(\alpha+\frac{1}{2}\right)} (1 - \lambda)^{-\mu}.$$

Hence the final amplitude of residuals is

$$\frac{\alpha!}{\sqrt{\pi}} (n\sqrt{\lambda})^{-\left(\alpha+\frac{1}{2}\right)}.$$

The tendency of Neumann's series to neglect the lower eigen-values is thus removed.

Another method to improve the convergence of the Neumann's series has been proposed by *Rutishauser* [8]) using the transformation of a power series into a continued fraction by his so-called QD-algorithm.

We have had satisfactory results with hypergeometric relaxation in Zürich.

2) If a lower bound $\varepsilon > 0$ for the eigen-values is also known, then $\varrho(\lambda)$ should vanish in the interval $0 \leqq \lambda \leqq \varepsilon$, since there are no residuals there to be liquidated. This leads to the method of *Shortley* and *Flanders* [9]), who use Tschebyscheff-polynomials in the remaining interval $\varepsilon \leqq \lambda \leqq 1$ and arrive at the best strategy in the sense of Tschebyscheff-approximation.

3) Obviously we are particularly interested in choosing the density-function (20) such that the ψ-measure is identically to the old discrete φ-measure. Since the eigen-values must be considered unknown, the explicit construction of the iteration-algorithm given in section V must be modified. Using (17) we find

$$\int_0^1 R_p(\lambda) R_q(\lambda) \varrho(\lambda) d\lambda = \sum_{(j)} k_j^2 \int_0^1 R_p(\lambda) R_q(\lambda) \delta(\lambda - \lambda_j) d\lambda$$

$$= \sum_{(j)} R_p(\lambda_j) R_q(\lambda_j) k_j^2 = \sum_{(j)} r_{pj} \cdot r_{qj} = (r_p, r_q).$$

The left side is zero for $p \neq q$ because of the orthogonality of our polynomials. Thus this relaxation process has the special property that the residual vectors form an orthogonal set. The coefficients a_i, c_i in (23) must therefore be computed in such a way that the residual r_{i+1} defined by (22) is orthogonal to r_i and r_{i-1}.

After finite many steps, the residual vector must vanish since there can only be a finite number of orthogonal vectors. The iteration thus reaches the exact solution after a finite number of steps, as is the case with the elimination method of Gauss. As *Hestenes* [10]) proved recently, every finite iteration is equivalent to the process of *conjugate gradients* developped by him and the author. In conclusion we may thus state:

Among all scalar iteration algorithms the method of conjugate gradients as given in the original papers [11]) *is the best strategy in the sense of the φ-measure of error.*

In particular for a problem of the calculus of observations this method gives the smallest sum of the squared errors which can be achieved in a given number of iteration steps by a scalar process [12]).

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MATHEMATICS AND METAMATHEMATICS

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No manuscript of this lecture was available

EIGENFUNCTION PROBLEMS ARISING FROM DIFFERENTIAL EQUATIONS

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1. Recent research on eigenfunction problems arising from differential equations has been largely influenced by the importance of Schrödinger's equation in quantum mechanics. This equation can be put into the form

$$\Delta\psi + (\lambda - q)\psi = 0, \tag{1.1}$$

where, in n dimensions

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2},$$

$q = q(x_1, x_2, \dots, x_n)$, λ is the eigenvalue parameter, and ψ is the 'wave function'. The region involved is usually the whole space.

In one-dimension, for example, we consider the ordinary differential equation

$$\psi''(x) + \{\lambda - q(x)\}\psi(x) = 0. \tag{1.2}$$

The primary object of the theory is to determine the 'eigenvalues', i.e. those values λ_n of λ for which the equation has a solution $\psi_n(x)$ of integrable square, called an eigenfunction. Such solutions corresponding to different values of λ are orthogonal, in the sense that

$$\int_{-\infty}^{\infty} \psi_m(x)\psi_n(x)dx = 0 \quad (m \neq n), \tag{1.3}$$

and we can suppose that this integral is equal to 1 if $m = n$. The functions $\psi_n(x)$ then form a normal orthogonal set. It is known that, under certain conditions, an arbitrary function $f(x)$ can be expanded in terms of the eigenfunctions, in the form

$$f(x) = \sum_{n=0}^{\infty} c_n\psi_n(x), \tag{1.4}$$

where

$$c_n = \int_{-\infty}^{\infty} f(t)\psi_n(t)dt. \tag{1.5}$$

In the case where the interval concerned is finite, this is the classical Sturm-Liouville theory. The corresponding theory for the case of an infinite interval is due to Weyl, who deduced it from the theory of integral equations.

2. The object of my book *Eigenfunction Expansions associated with Second-order Differential Equations* (Oxford 1946) was to give a theory of these formulae depending only on the calculus of residues and related ideas, and not on the theory of integral equations or on any general theory of linear operators The procedure adopted was as follows. If λ is any non-real number, there is, by a fundamental theorem of Weyl, a solution $\psi_1(x, \lambda)$ of (1.2) whose squared modulus is integrable over $(-\infty, 0)$. Similarly there is a solution $\psi_2(x, \lambda)$ whose squared modulus is integrable over $(0, \infty)$. Let $\omega(\lambda)$ denote the Wronskian of ψ_2 and ψ_1. Let

$$(2.1) \qquad G(x, y, \lambda) = \frac{\psi_1(x, \lambda)\psi_2(y, \lambda)}{\omega(\lambda)} \quad (x \leqq y), \quad \frac{\psi_2(x, \lambda)\psi_1(y, \lambda)}{\omega(\lambda)} \quad (x > y).$$

This is called the Green's function. In the simplest cases, G is a meromorphic function of λ, its poles being the eigenvalues. Let $f(x)$ be the function to be expressed in terms of the eigenfunctions, and let

$$(2.2) \qquad \Phi(x, \lambda) = -\int_{-\infty}^{\infty} G(x, y, \lambda) f(y) dy.$$

We integrate Φ round a large contour in the λ-plane. The residues at the poles are found to be the terms of the series (1.4). Also it is found that, when λ is large, $\Phi(x, \lambda)$ is approximately equal to $\lambda^{-1}f(x)$, so that $\int \Phi(x, \lambda)d\lambda$ is approximately equal to $2\pi i f(x)$. A proof of the expansion formula (1.4), under suitable conditions, is thus obtained.

A simple example is that in which $q(x) = x^2$. Then $\lambda_n = 2n + 1$, and

$$(2.3) \qquad \psi_n(x) = 2^{-\frac{1}{2}n} (n!)^{-\frac{1}{2}} \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2} H_n(x),$$

where $H_n(x)$ denotes the Hermite polynomial of degree n.

The function G is, however, not necessarily meromorphic. Consider, for example, the case in which $q(x) = 0$. Then $\psi_1(x) = e^{-ix\sqrt{\lambda}}$, $\psi_2(x) = e^{ix\sqrt{\lambda}}$, and $\omega(\lambda) = -2i\sqrt{\lambda}$. In this case G has no poles, but a branch-point at $\lambda = 0$. In order to apply Cauchy's theorem, the integral of Φ round a large circle of radius R has to be completed by a loop starting from R, going round the origin, and returning to R again. We thus obtain, instead of a sum of residues, an integral with respect to λ over $(0, R)$. The formula obtained on making $R \to \infty$ is

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos x\sqrt{\lambda}}{2\sqrt{\lambda}} d\lambda \int_{-\infty}^{\infty} \cos y\sqrt{\lambda} f(y) dy + \frac{1}{\pi} \int_0^{\infty} \frac{\sin x\sqrt{\lambda}}{2\sqrt{\lambda}} d\lambda \int_{-\infty}^{\infty} \sin y\sqrt{\lambda} f(y) dy,$$

a form of Fourier's integral formula.

The 'spectrum' is the set of values of λ which contribute to the expansion formula. In the former case, it consists of the eigenvalues λ_n, and is said to be discrete. In the latter case, it consists of the whole interval $(0, \infty)$, and is said to be continuous. Details of the theory are given in my book.

3. I have recently constructed a similar theory of partial differential equations — the case $n > 1$ of (1.1). Naturally this is not so simple as the case $n = 1$. However, the results are, to a certain extent, the same. Consider, for example, the problem in two dimensions. Let $f(x, y)$ be the function to be expanded. We want to construct the function $\Phi(x, y, \lambda)$ which is the solution of

$$\Delta\Phi + \{\lambda - q(x, y)\}\Phi = f(x, y).$$

Suppose first that $q = 0$, and that the region is the square $(0, \pi; 0, \pi)$, with $\Phi = 0$ on the boundary. The expansion formula is then an ordinary double Fourier series. We obtain

$$\Phi(x, y, \lambda) = -\int_0^\pi\int_0^\pi G(x, y, \xi, \eta, \lambda) f(\xi, \eta) d\xi d\eta,$$

where

$$G(x, y, \xi, \eta, \lambda) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin mx \sin ny \sin m\xi \sin n\eta}{m^2 + n^2 - \lambda}$$

is the Green's function.

We can next, by a process of iteration, solve the corresponding problem in which q is any continuous function. By a change of variable, the formulae for a square $(-b, b; -b, b)$ are then obtained. Finally, on making $b \to \infty$, a solution of the problem for the whole space is obtained. We prove the existence of a Green's function $G(x, y, \xi, \eta, \lambda)$, and

$$\Phi(x, y, \lambda) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, \xi, \eta, \lambda) f(\xi, \eta) d\xi d\eta.$$

The expansion theorem is obtained by integrating Φ round a large contour, as before.

If the Green's function is a meromorphic function of λ, its poles are the eigenvalues, and the expansion formula is of the series type, as in (1.4) and (1.5). For example, let $q(x) = x^2 + y^2$. The eigenfunctions are $\psi_m(x)\psi_n(y)$, where $\psi_n(x)$ is defined by (2.3). The corresponding eigenvalues are $\lambda = 2m + 2n + 2$. The Green's function is

$$G(x, y, \xi, \eta, \lambda) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_m(x)\psi_n(y)\psi_m(\xi)\psi_n(\eta)}{2m + 2n + 2 - \lambda}.$$

If $q(x, y) = 0$, the Green's function is found to be $\frac{1}{4}i\, H_0^{(1)}(\varrho\sqrt{\lambda})$, in the ordinary notation of Bessel functions, where $\varrho = \sqrt{\{(x-\xi)^2 + (y-\eta)^2\}}$. This leads to the expansion formula

$$f(x, y) = \frac{1}{4\pi} \int_0^\infty d\lambda \int_{-\infty}^\infty \int_{-\infty}^\infty J_0(\varrho\sqrt{\lambda}) f(\xi, \eta) d\xi d\eta,$$

the spectrum being continuous over $(0, \infty)$.

The general formulae which include both these types involve Stieltjes integrals. There are similar results in any number of dimensions.

In the following sections I consider as examples of this general theory some well known problems in quantum mechanics.

4. *The hydrogen atom.* The problem of the hydrogen atom in quantum mechanics is related to the three-dimensional equation

$$\Delta\psi + \left(\lambda + \frac{a}{r}\right)\psi = 0, \tag{4.1}$$

where $r = \sqrt{(x^2 + y^2 + z^2)}$ and a is a positive constant. It is easy to construct eigenfunctions of this equation since, in polar coordinates, it is separable. Let (r, θ, φ) denote the usual spherical polar coordinates. Then the functions

$$\psi_{l,m,n} = Cr^{-1}\chi_{l,n}(r)\, P_l^m(\cos\theta) \frac{\cos}{\sin} m\varphi \tag{4.2}$$

are eigenfunctions of (4.1) if $\chi_{l,n}(r)$ is an eigenfunction of the equation

$$\chi''(r) + \left(\lambda + \frac{a}{r} - \frac{l^2 + l}{r^2}\right)\chi(r) = 0 \quad (0 < r < \infty), \tag{4.3}$$

C being a normalization constant. The equation (4.3) has a series of discrete eigenvalues

$$\lambda_{l,n} = -\frac{a^2}{4(l + n + 1)^2}, \quad (n = 0, 1, \ldots), \tag{4.4}$$

the corresponding eigenfunctions being expressible in terms of generalized Laguerre polynomials. Hence the numbers (4.4) are also eigenvalues of (4.1), the eigenfunctions being given by (4.2). Each equation (4.3) also has a continuous spectrum over $(0, \infty)$, and hence so has (4.1).

What is less obvious is that all eigenfunctions of (4.1) arise in this way. However, it can be shown that, if $G[r, \theta, \varphi, r', \theta', \varphi', \lambda]$ is the Green's function of (4.1), expressed in polar coordinates, and $G_l(r, r, \lambda)$, where $l = 0, 1, \ldots$, are the Green's functions of (4.3), then

$$G[r, \theta, \varphi, r', \theta', \varphi', \lambda] = \frac{1}{2\pi rr'} \sum_{l=0}^\infty (l + \tfrac{1}{2})\, G_l(r, r', \lambda)\, P_l\{\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\varphi - \varphi')\}.$$

This formula suggests at once, and can be used to prove, that the left-hand side is regular wherever all the functions $G_l(r, r', \lambda)$ are regular, and so has no singularities other than those of the functions $G_l(r, r', \lambda)$. Hence there are no eigenvalues of (4.1) except those given by (4.4).

5. *The helium atom.* The theory of the helium atom in quantum mechanics (in a slightly simplified form) involves the six-dimensional equation

$$\Delta_1\psi + \Delta_2\psi + (\lambda - q)\psi = 0 \tag{5.1}$$

over the whole space, where ψ and q are functions of x_1, y_1, z_1, x_2, y_2 and z_2,

$$\Delta_\nu = \frac{\partial^2}{\partial x_\nu^2} + \frac{\partial^2}{\partial y_\nu^2} + \frac{\partial^2}{\partial z_\nu^2} \quad (\nu = 1, 2),$$

$$r_\nu = \sqrt{(x_\nu^2 + y_\nu^2 + z_\nu^2)} \quad (\nu = 1, 2),$$

$$r_{12} = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}},$$

and

$$q = \frac{a}{2r_{12}} - \frac{a}{r_1} - \frac{a}{r_2},$$

where a is a positive constant. This equation is not separable, and no explicit formulae for the eigenvalues are known. Much research has been devoted to the problem of finding approximate numerical values of the eigenvalues, but it is only recently that a Japanese writer, Tosio Kato, has proved that discrete eigenvalues actually exist. His method uses the general theory of linear operators, but its essential features can be taken over into the theory considered here. I shall show briefly how this is done.

We consider, at the same time as the above problem, three other problems: that of

$$\Delta_1\psi + \Delta_2\psi + (\lambda - Q)\psi = 0 \tag{5.2}$$

over the whole space, where

$$Q = -\frac{a}{r_1} - \frac{a}{r_2};$$

and the problems of the equations (5.1) and (5.2) in the finite regions $r_1 \leqq R$, $r_2 \leqq R$, with $\psi = 0$ on the boundary. In the problems with finite regions, the spectra are necessarily discrete. Let the eigenvalues be denoted by $\lambda_{N,R}$ and $\mu_{N,R}$ respectively. Since $Q \leqq q$ everywhere, it follows from a general principle in the theory of eigenvalues [1] that $\mu_{N,R} \leqq \lambda_{N,R}$ for all values of N. The proof depends on the formulae

[1]) See Courant and Hilbert, *Methoden der math. Physik*, I, (zw. Aufl.), 357.

$$D_{R,q}(f) = \int_{r_1 \leqq R,} \cdots \int_{r_2 \leqq R} \left\{ \left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial z_2}\right)^2 + q f^2 \right\} \boldsymbol{dx}_1 \boldsymbol{dx}_2 = \sum_{n=0}^{\infty} \lambda_{N,R} c_{N,R}^2,$$

where $\boldsymbol{dx}_1$ stands for $dx_1\, dy_1\, dz_1$, etc., and the $c_{n,R}$ are the 'Fourier coefficients' of the function f in the expansion related to (5.1); and

$$D_{R,Q}(f) = \sum_{n=0}^{\infty} \mu_{n,R}\, d_{n,R}^2,$$

where the $d_{n,R}$ are the 'Fourier coefficients' in the expansion related to (5.2). Clearly $D_{R,Q}(f) \leqq D_{R,q}(f)$ for any f. Now let $f = \psi_{0,R}$, the eigenfunction related to $\lambda_{0,R}$. Then

$$\lambda_{0,R} = D_{R,q}(\psi_{0,R}) \geqq D_{R,Q}(\psi_{0,R}) \geqq \mu_{0,R} \sum_{n=0}^{\infty} d_{n,R}^2 = \mu_{0,R},$$

the first case of the theorem. The general theorem can be proved similarly.

Now the problem of (5.2) can be solved, since the equation is separable. In fact it has eigenfunctions

$$\psi_N = \psi_m(\boldsymbol{x}_1)\, \psi_n(\boldsymbol{x}_2)$$

and corresponding eigenvalues $\mu_N = \alpha_m + \alpha_n$, where $\psi_m(\boldsymbol{x})$ and α_m denote eigenfunctions and eigenvalues of (4.1). Thus

$$\mu_0 = -\tfrac{1}{2} a^2,\ \mu_1 = \mu_2 = \mu_3 = \mu_4 = -\frac{5a^2}{16}, \ldots,$$

the spectrum being discrete over the range $\lambda < -\frac{1}{4}a^2$. To the right of this point, the spectrum of (5.2) is continuous, since to $\alpha_0 = -\frac{1}{4}a^2$ has to be added a continuous range of positive values of α_n.

By another general principle stated by Courant and Hilbert, $\mu_N \leqq \mu_{N,R}$ for each N. Hence $\mu_N \leqq \lambda_{N,R}$ for each N. It follows that the number of numbers $\lambda_{N,R}$ in any interval $\lambda \leqq -\frac{1}{4}a^2 - \delta$, where $\delta > 0$, is bounded independently of R, being at most equal to the number of numbers μ_N in this interval. Since discrete eigenvalues λ_N of (5.1) can only arise as limits of $\lambda_{N,R}$ as $R \to \infty$, it follows that there are at most a finite number of eigenvalues λ_N to the left of $-\frac{1}{4}a^2 - \delta$. Thus, if the spectrum of (5.1) extends to the left of $-\frac{1}{4}a^2$, it can consist in this range only of discrete eigenvalues.

It remains to show that there are actually eigenvalues of (5.1) to the left of $-\frac{1}{4}a^2$. For this purpose we consider the equation

$$\Delta_1\psi + \Delta_2\psi + (\lambda - Q')\psi = 0, \tag{5.3}$$

where

$$-Q' = \frac{a}{r_1} - \frac{a}{2r_2}.$$

This has eigenfunctions of the form $\psi_m(\mathbf{x}_1)\omega_n(\mathbf{x}_2)$, where ψ_m denotes the same function as before, and ω_n the same function with a replaced by $\frac{1}{2}a$. The corresponding eigenvalues are $\nu_N = \alpha_m + \beta_n$, where β_n is obtained from α_n by replacing a by $\frac{1}{2}a$. Thus

$$\beta_0 = -\frac{a^2}{16}, \ \beta_1 = \beta_2 = -\frac{a^2}{64}, \ldots,$$

and

$$\nu_0 = -\frac{5}{16}a^2, \ \nu_1 = \nu_2 = -\frac{17}{64}a^2, \ldots.$$

Since the inequality $q \leqq Q'$ does not hold for all values of the variables, we cannot conclude that

$$D_q(f) \leqq D_{Q'}(f)$$

for all functions f. However, this inequality does hold if $f = \psi_0(\mathbf{x}_1)\omega_n(\mathbf{x}_2)$. For it is true if

$$\int \cdots \int (q - Q') f^2 \mathbf{dx}_1 \mathbf{dx}_2 \leqq 0,$$

and so if

$$\iiint (q - Q') \psi_0^2 \mathbf{dx}_1 \leqq 0$$

for all $\mathbf{x}_2$. Since ψ_0 depends on r_1 only, and not on the angle variables θ_1 and φ_1, this is true if

$$\int_0^{\pi} \int_0^{2\pi} (q - Q') \sin \theta_1 \, d\theta_1 \, d\varphi_1 \leqq 0,$$

which can be verified without difficulty. It follows that λ_N does not exceed the N^{th} eigenvalue of (5.3) corresponding to an eigenfunction of the above special form. Since the ν's are all less than $-\frac{1}{4}a^2$, it follows that there are an infinity of numbers λ_N less than $-\frac{1}{4}a^2$, i.e. that the original problem has an infinity of discrete eigenvalues less than $-\frac{1}{4}a^2$.

The above argument actually shows that

$$-\frac{1}{2}a^2 \leqq \lambda_0 \leqq -\frac{5}{16}a^2, \ -\frac{5}{16}a^2 \leqq \lambda_1 \leqq -\frac{17}{64}a^2, \ldots.$$

Much closer approximations to these numbers have been obtained by other methods.

6. *Hydrogen atom in an electric field.* The last example which I shall consider is that of a hydrogen atom in a weak electric field. The formalities of this problem are due to Schrödinger, and are well known. The equation which determines the energy levels can be put in the form

$$(6.1) \qquad \Delta\psi + \left(\lambda + \frac{a}{r} - \varepsilon z\right)\psi = 0$$

(three dimensions), where ε is proportional to the strength of the electric field, and is supposed small.

If $\varepsilon = 0$, the equation reduces to (4.1), and so has discrete eigenvalues given by (4.4). If we could apply the ordinary methods of perturbation theory, we should find that there were perturbed eigenvalues Λ_N given approximately by

$$(6.2) \qquad \Lambda_N = \lambda_N + \varepsilon\lambda_N^{(1)} + \dots$$

where

$$(6.3) \qquad \lambda_N^{(1)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\psi_N^2\,dx\,dy\,dz,$$

ψ_N being the eigenfunction of (4.1) corresponding to λ_N. This is equivalent to the formula used by Schrödinger, and the numbers $\lambda_N^{(1)}$ defined by (6.3) seem to have some importance in physical theory. Nevertheless it can be shown that the perturbed spectrum, i.e. the spectrum of (6.1) with $\varepsilon > 0$, is continuous over the whole range $(-\infty, \infty)$, there being no discrete eigenvalues Λ_N. To meet the requirements of both physics and mathematics, the problem has to be formulated in a different way.

Consider first the one-dimensional problem of the equation (1.2), with interval $0 \leqq x < \infty$ and boundary condition $\psi(0) = 0$. Let $\varphi(x, \lambda)$ be the solution of (1.2) such that $\varphi(0, \lambda) = 0$, $\varphi'(0, \lambda) = 1$, let $f(x)$ be any function of integrable square, and let

$$g(\lambda) = \int_0^{\infty} \varphi(t, \lambda) f(t)\,dt.$$

Then the Parseval formula is of the form

$$(6.4) \qquad \int_0^{\infty} f^2(x)\,dx = \int_{-\infty}^{\infty} g^2(\lambda)\,d\varrho(\lambda),$$

where $\varrho(\lambda)$ is a non-decreasing function of λ. We may consider this formula as defining the way in which the representation of $f(x)$ in terms of the functions $\varphi(x, \lambda)$ is allocated among the points of the spectrum. If the spectrum is discrete, $\varrho(\lambda)$ is constant except at the eigenvalues, and we can omit from the right-hand side all except the integral taken over a set E of intervals of which the eigenvalues are interior points. Now suppose that, in (1.2), $q(x)$ is replaced by $q(x) + \varepsilon a(x)$, where $a(x)$ is another given function. If the perturbed spectrum were discrete, the perturbed eigenvalues Λ_N would be related to the unperturbed eigenvalues λ_n by the formulae (6.2), where

$$\lambda_n^{(1)} = \int_0^\infty a(x)\,\psi_n^2(x)\,dx. \tag{6.5}$$

The Parseval formula for the perturbed problem is of the form

$$\int_0^\infty f^2(x)\,dx = \int_{-\infty}^\infty g^2(\lambda, \varepsilon)d\varrho(\lambda, \varepsilon). \tag{6.6}$$

It can be shown, in simple cases at any rate, that, if the perturbed spectrum is continuous, then

$$\int_0^\infty f^2(x)\,dx = \int_E g^2(\lambda, \varepsilon)d\varrho(\lambda, \varepsilon) + o(1), \tag{6.7}$$

where $f(x)$ is fixed, $\varepsilon \to 0$, and E is now a set depending on ε, which, as $\varepsilon \to 0$, converges on the set of numbers Λ_n satisfying the above formulae. It may, for example, be a set of intervals of the form

$$(\lambda_n + \varepsilon\lambda_n^{(1)} - \varepsilon^{3/2},\ \lambda_n + \varepsilon\lambda_n^{(1)} + \varepsilon^{3/2}), \tag{6.8}$$

where $\lambda_n^{(1)}$ is defined by (6.5). The numbers $\lambda_n^{(1)}$ thus regain their place in the theory. The result is that the perturbed spectrum, though not confined to discrete points, is, in a sense, highly concentrated in the neighbourhood of certain points, which thus play much the same part as perturbed eigenvalues.

In three (or more) dimensions the same ideas can be expressed as follows. Let $G(\mathbf{x}, \boldsymbol{\xi}, \lambda, \varepsilon)$ be the Green's function related to the perturbed problem, $\mathbf{x}$ standing for the three variables x, y and z, and $\boldsymbol{\xi}$ for ξ, η and ζ. Let

$$H(\mathbf{x}, \boldsymbol{\xi}, \lambda, \varepsilon) = \lim_{\nu \to 0} \int_0^\lambda \operatorname{im} G(\mathbf{x}, \boldsymbol{\xi}, \mu + i\nu, \varepsilon)d\mu,$$

where λ is now real. Let $f(\mathbf{x})$ be any function of integrable square, and let

$$J(f, \lambda, \varepsilon) = \frac{1}{\pi} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty H(\mathbf{x}, \boldsymbol{\xi}, \lambda, \varepsilon) f(\mathbf{x}) f(\boldsymbol{\xi})\, d\mathbf{x}\, d\boldsymbol{\xi}.$$

This a bounded non-decreasing function of λ, and the Parseval formula is

$$\iiint f^2(\mathbf{x})\,d\mathbf{x} = J(f, \infty, \varepsilon) - J(f, -\infty, \varepsilon). \tag{6.9}$$

Our object is now to prove that, as $\varepsilon \to 0$,

$$\iiint f^2(\mathbf{x})\,d\mathbf{x} = \Sigma \left\{J(f, \lambda_n'', \varepsilon) - J(f, \lambda_n', \varepsilon)\right\} + o\,(1) \tag{6.10}$$

where the summation is over a set of intervals $(\lambda_n', \lambda_n'')$ of the same type as the above set E.

7. In order to apply these ideas to the problem of (6.1), we make the transformation (used by Schrödinger)

$$x = uv \cos \varphi, \; y = uv \sin \varphi, \; z = \tfrac{1}{2}(u^2 - v^2),$$

where u and v vary over $(0, \infty)$, and φ varies over $(0, 2\pi)$. Let $\Psi = u^{1/2} v^{1/2} \psi$. Then (6.1) transforms into

$$(7.1) \qquad \frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2} + \left(\frac{1}{u^2} + \frac{1}{v^2}\right) \frac{\partial^2 \Psi}{\partial \varphi^2} + \Big\{ \lambda(u^2 + v^2) + 2a - \tfrac{1}{2}\varepsilon\,(u^4 - v^4) + \tfrac{1}{4}\left(\frac{1}{u^2} + \frac{1}{v^2}\right) \Big\} \Psi = 0.$$

In this form, the equation is separable. Let $G(\mathbf{x}, \boldsymbol{\xi}, \lambda, \varepsilon) = G[u, v, \varphi, s, t, \varphi', \lambda, \varepsilon]$ be the Green's function of (6.1), and let

$$H[u, v, \varphi, s, t, \varphi', \lambda, \varepsilon] = \lim_{\nu \to 0} \int_0^\lambda \operatorname{im} G[u, v, \varphi, s, t, \varphi', \mu + i\nu, \varepsilon] d\mu.$$

Let $G_n(u, v, s, t, \lambda, \varepsilon)$ be the Green's functions associated with the equations

$$(7.2) \quad \frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2} + \Big\{ \lambda(u^2 + v^2) + 2a - \tfrac{1}{2}\varepsilon(u^4 - v^4) - (n^2 - \tfrac{1}{4}) \left(\frac{1}{u^2} + \frac{1}{v^2}\right) \Big\} \Psi = 0,$$

and let

$$H_n(u, v, s, t, \lambda, \varepsilon) = \lim_{\nu \to 0} \int_0^\lambda \operatorname{im} G_n(u, v, s, t, \mu + i\nu, \varepsilon) d\mu,$$

where λ is real.

It can be shown that

$$(7.3) \quad G[u, v, \varphi, s, t, \varphi', \lambda, \varepsilon] = \frac{1}{\pi (uvst)^{1/2}} \sum_{n=0}^{\infty} e_n G_n(u, v, s, t, \lambda, \varepsilon) \cos n(\varphi - \varphi'),$$

where $e_0 = \frac{1}{2}$, $e_1 = e_2 = \ldots = 1$; and hence

$$(7.4) \quad H[u, v, \varphi, s, t, \varphi', \lambda, \varepsilon] = \frac{1}{\pi (uvst)^{1/2}} \sum_{n=0}^{\infty} e_n H_n(u, v, s, t, \lambda, \varepsilon) \cos n(\varphi - \varphi').$$

Also it can be shown that $H_n(u, v, s, t, \lambda, \varepsilon)$ is the integral over $(0, \lambda)$ of the function

$$(7.5) \quad h_n(u, v, s, t, \lambda, \varepsilon) = \sum_{m=0}^{\infty} \chi_{m,n}(u, \lambda, \varepsilon) \chi_{m,n}(s, \lambda, \varepsilon) \operatorname{im} \gamma_n(v, t - \varkappa_{m,n} \lambda, \varepsilon).$$

Here $\varkappa_{m,n} = \varkappa_{m,n}(\lambda, \varepsilon)$ and $\chi_{m,n}(u, \lambda, \varepsilon)$ are the eigenvalues and eigenfunctions associated with the equation

$$(7.6)\quad \chi''(u) + \left\{ \varkappa + \lambda u^2 - \tfrac{1}{2}\varepsilon u^4 - (n^2 - \tfrac{1}{4})\frac{1}{u^2} \right\} \chi(u) = 0, \quad (0 < u < \infty),$$

in which $\varkappa$ is now the eigenvalue parameter; and $\gamma_n(v, t, \varkappa, \lambda, \varepsilon)$ is the Green's function associated with the equation

$$(7.7)\qquad \omega''(v) + \left\{ \varkappa + 2a + \lambda v^2 + \tfrac{1}{2}\varepsilon v^4 - (n^2 - \tfrac{1}{4})\frac{1}{v^2} \right\} \omega(v) = 0.$$

The problem is thus reduced to the study of the ordinary differential equations (7.6) and (7.7). By means of some rather complicated analysis, of which the details are given in paper V of my series of papers on partial differential equations, a result of the form (6.10) can be proved. A rigorous form of the perturbation theory of the equation (6.1) is thus obtained.

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ABSTRACT VERSUS CLASSICAL ALGEBRAIC GEOMETRY

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SEMI-GROUP THEORY AND THE INTEGRATION PROBLEM OF DIFFUSION EQUATIONS

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0. *Introduction.* The analytical theory of one-parameter semi-groups deals with the exponential function in infinite dimensional function spaces. It is a natural generalization of the theorem of M. H. Stone [1] (Cf. I. Gelfand [1] and M. Fukamiya [1]) on one-parameter group of unitary operators in Hilbert space. Since the publication of E. Hille's book „Functional Analysis and Semi-groups" (Cf. K. Yosida [1]), many contributions were added to amplify the theory and its applications. It would be impossible to give a complete report of the theory in a one-hour talk. Fortunately, Professor Hille's talk „Abstract Cauchy's Problem" and mine will supplement each other on the subject.

1. *The semi-group theory.* Let $\{T_t\}$, $0 \leqq t < \infty$, be a one-parameter family of bounded linear operators on a Banach space X to X such that

(1.1) $T_t T_s = T_{t+s}$,

(1.2) $T_0 = I$ (the identity operator),

(1.3) $\sup_t \| T_t \| \leqq 1$,

(1.4) $\text{strong} \lim_{t \downarrow 0} T_t x = x$.

Then we have the following three theorems (E. Hille [1] and [5]; K. Yosida [1]).

The Differentiability Theorem. The set $D = D(A)$ of x for which

(1.5) $\text{strong} \lim_{h \downarrow 0} h^{-1} (T_h - I)x = Ax$

exists is strongly dense in X. A is a closed linear operator on $D(A)$ to X such that, when $n > 0$, the resolvents

(1.6) $I^{(n)} = (I - n^{-1}A)^{-1}$

exist as bounded linear operators on X to X satisfying

(1.7) $I^{(n)}x = n \int_0^\infty \exp(-nt)\, T_t x \, dt,$

(1.8) $\| I^{(n)} \| \leqq 1$,

(1.9) $D_t T_t x = \text{strong} \lim_{h \to 0} h^{-1} (T_{t+h} - T_t)x = A T_t x = T_t A x$ for $x \in D(A)$.

The Representation Theorem. We have

(1.10) $T_t x = \text{strong} \lim_{n \to \infty} (I - n^{-1} t A)^{-n} x$ (E. Hille),

(1.10)′ $Tx = \text{strong} \lim_{n \to \infty} \exp (t A I^{(n)}) x = \lim_{n \to \infty} \sum_{m=0}^{\infty} (m!)^{-1} (nt(I^{(n)} - I))^m x$

(K. Yosida)

uniformly in t in any bounded interval of t. In this sense A is the „infinitesimal generator" of the semi-group $\{T_t\}$, and we may write

(1.10)″ $T_t x = \exp (tA) x$.

The Converse Theorem. Let a linear operator A on a dense domain $D(A) \subseteqq X$ to X be such that, for $n > 0$, the resolvents (1.6) exist as bounded linear operator on X to X satisfaying (1.8). Then A is the infinitesimal generator of the semi-group (1.10) (and equivalently (1.10)′) satisfying (1.1)—(1.4).

In order to extend the range of the applicability of the theory, E. Hille, W. Feller and R. S. Philips gave various generalizations of the above result. Suggested by N. Dunford [1] and E. Hille [1], R. S. Phillips [1] proved that the strong measurability in t of T_t satisfying (1.1) implies the strong continuity in t of T_t for $t > 0$. E. Hille [1] proved that (1.1) and the strong continuity for $t > 0$ imply

(1.11) $\alpha = \inf_{t > 0} t^{-1} \log \| T_t \| = \lim_{t \to \infty} t^{-1} \log \| T_t \|$.

By making use of (1.11), R. S. Phillips [2] proved the result:

A closed linear operator A with dense domain is the infinitesimal generator of a semi-group satisfying (1.1), (1.2) and (1.4) if and only if there exist two constants ω and M such that the resolvents $I^{(n)}$ exist for $n > \omega$ with the estimates

(1.12) $\| (I^{(n)})^m \| \leqq M (n/(n - \omega))^m, \quad (m = 0, 1, \ldots)$.

W. Feller [8] (Cf. E. Hille [1]) discussed the semi-group satisfying (1.1) and the strong continuity for $t > 0$. By (1.11) we may assume, without losing the generality,

(1.3)′ $\limsup_{t \to \infty} \| T_t \| \leqq M < \infty$,

(1.13) the range $\{T_t x;\ x \in X,\ t > 0\}$ is strongly dense in X.

He introduced the „continuity set" $\Sigma \subseteqq X$ by the condition

(1.14) strong $\lim_{t \downarrow 0} T_t x = x$ whenever $x \in \Sigma$, and proved that Σ, which is strongly dense in X, may be considered to be a Banach space by the new norm

$$(1.15)\quad |||\, x \,||| = \sup_{t>0} \| T_t x \|.$$

Thus $\{T_t\}$ in Σ constitutes a semi-group satisfying (1.1)—(1.4). Moreover, he proved that, for the corresponding infinitesimal generator A in Σ,

$$(1.16)\quad |||\, x \,||| = \sup_{t>0} \| T_t x \| = \sup_{m\geqq 0} \sup_{n>0} \| (I - n^{-1}A)^{-m} x \|.$$

In this way, Feller extended Phillips' result (1.12) to the case of the semi-group unbounded near $t = 0$.

The formula (1.9) suggests us to apply the semi-group theory to the Cauchy's problem for the „equation of evolution":

$$(1.17)\quad D_t f(t) = Af(t), \text{ strong} \lim_{t\downarrow 0} f(t) = f \in X.$$

Professor Hille will give a talk on such „Abstract Cauchy's Problem" in its full generality. The present speaker wishes to talk about a special but important class of diffusion equations in a connected domain R of an m-dimensional, orientable, C^∞ Riemannian space with the metric $ds^2 = g_{ij}(x)dx^i dx^j$.

2. *The forward and the backward diffusion equations.* We are thus concerned with the forward and the backward diffusion equations in R: [1])

$$(2.1)\quad E_{tx} f = \frac{\partial f(t,x)}{\partial t} - A_{tx} f(t,x) = 0,\ t > s,$$

$$A_{tx} f(t,x) = g(x)^{-1/2} \frac{\partial^2}{\partial x^i\, \partial x^j} \left(g(x)^{1/2} a^{ij}(t,x) f(t,x)\right)$$

$$- g(x)^{-1/2} \frac{\partial}{\partial x^i} \left(g(x)^{1/2} b^i(t,x)\right) + c(t,x) f(t,x),$$

$$\text{strong} \lim_{t\downarrow s} f(t,x) = f(x) \in L_1(R),$$

$$(2.2)\quad E^*_{sy} h = -\frac{\partial h(s,y)}{\partial s} - A^*_{sy} h(s,y) = 0,\ s < t,$$

$$A^*_{sy} h(s,y) = a^{ij}(s,y) \frac{\partial^2 h(s,y)}{\partial y^i \partial y^j} + b^i(s,y) \frac{\partial h(s,y)}{\partial y^i} + c\,(s,y) h(s,y),$$

$$\text{strong} \lim_{s\uparrow t} h(s,y) = h(y) \in C(R).$$

In diffusion theory, (2.1) and (2.2) are respectively satisfied by the probability density and the transition probability of a simple Markoff process.

[1]) The forward diffusion equation is often called as the Fokker-Planck's equation.

This is the reason why we consider (2.1) and (2.2) respectively in the Banach space $L_1(R)$ and $C(R)$; $C(R)$ is the Banach space of bounded uniformly continuous functions in R and $L_1(R)$ is the Banach space of Borel measurable functions integrable over R with respect to the measure $dx = g(x)^{1/2}\, dx^1 \dots dx^m$, $g(x) = \det (g_{ij}(x))$.

Probabilistic meaning of the coefficients $a^{ij}(t, x)$ and $b^i(t, x)$ are as follows. Let, in a simple Markoff process in R, $P(s, t, x, E)$ denote the transition probability that a mass, located at $x \in R$ at the time moment s, is transferred in the Borel set $E \subseteqq R$ at a later time moment t. Let the density $p(s, t, x, y)$ of $P(s, t, x, E)$ exist and is C^3 in (s, t, x, y), and let the „continuity condition"

$$(2.3)\quad \lim_{\Delta \downarrow 0} \int_R \operatorname{dis}(x, y)^3\, P(t, t + \Delta, x, dy) \Big/ \int_R \operatorname{dis}(x, y)^2\, P(t, t + \Delta, x, dy) = 0$$

be satisfied. Then the limits

$$(2.4)\quad a^{ij}(t, x) = \lim_{\Delta \downarrow 0} \Delta^{-1} \int_{U(x)} (y^i - x^i)(y^j - x^j)\, P(t, t + \Delta, x, dy),$$

$$b^i(t, x) = \lim_{\Delta \downarrow 0} \Delta^{-1} \int_{U(x)} (y^i - x^i)\, P(t, t + \Delta, x, dy)$$

exist independently of the choice of the coordinate neighbourhood $U(x)$ of x (A. Kolmogoroff [1], W. Feller [1]). Thus the quadratic form $a^{ij}(t, x)$ must be $\geqq 0$ in R. And while $a^{ij}(t, x)$ is a contravariant symmetric tensor, $b^i(t, x)$ must obey, by the coordinate change $x \to \bar{x}$, to the transformation rule

$$(2.5)\quad \bar{b}^i(t, \bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} b^k(t, x) + \frac{\partial^2 \bar{x}^i}{\partial x^k\, \partial x^s} a^{ks}(t, x).$$

The Markoff porcess is called „temporally homogeneous" when

$$(2.6)\quad P(s, t, x, E) = P(s + u,\ t + u, x, E) \text{ for every } u.$$

In such case we may put

$$(2.7)\quad P(t, x, E) = P(s, s + t, x, E).$$

and the coefficients $a^{ij}(t, x)$ and $b^i(t, x)$ may be taken as independent of t. For the temporally homogeneous Markoff process, the semi-group theory enables us to prove the existence of the limits (2.4) under milder hypotheses than (2.3). Let, for example, R be a homogeneous space such that the group G of isometries S of R constitutes a Lie group transitive on R, and let us assume that the Lie subgroup of those S which leave a fixed point $x \in R$ invariant be compact. Let, moreover, the process be „spatially homogeneous" in the sense that

(2.8) $P(t, x, E) = P(t, Sx, SE)$ for every $S \in G$.

Then (K. Yosida [9]) the condition

(2.9) $\lim_{h \downarrow 0} h^{-1} \int_{\text{dis}(x,y)>\varepsilon} P(h, x, dy) = 0$ for every $\varepsilon > 0$

implies the existence of (2.4). The proof is obtained by applying the Differentiability Theorem to the semi-group in $C(R)$:

(2.10) $(T_t f)(x) = \int_R f(y) P(t, x, dy).$

W. Feller [6] proved the following result. Let $\{T_t\}$ be a positivity preserving semi-group of bounded linear operators on $C[r_1, r_2]$, $-\infty \leqq r_1 < r_2 \leqq \infty$, satisfying (1.1)—(1.4). Let the infinitesimal generator A of $\{T_t\}$ map any $x(s) \in C[r_1, r_2]$ vanishing in some interval $|s - s_0| < \delta$ onto a function vanishing at $s = s_0$. Let, moreover, for each $x(s) \in D(A)$, the domain of A, the derivatives $x'(s)$ and $x''(s)$ exist. Then A is of the form

(2.11) $A = a(s) \dfrac{\partial^2}{\partial s^2} + b(s) \dfrac{\partial}{\partial s} + c(s)$ with $a(s) \geqq 0$ and $c(s) \leqq 0$,

at every point s_0 where there exists no linear homogeneous relation among $x(s_0)$, $x'(s_0)$ and $x''(s_0)$ for $x \in D(A)$.

3. *Integration of the diffusion equations.* For the sake of simplicity, we assume that the coefficients $a^{ij}(t, x)$, $b^i(t, x)$ and $c(t, x)$ are C^∞ functions of the local coordinates $(x^1, \ldots, x^m)$. We also assume that the quadratic form $a^{ij}(t, x)\xi_i\xi_j$ is > 0 in R for $\sum_i (\xi_i)^2 > 0$.

The formula (1.10)′ in the Representation Theorem shows that we may integrate (1.9) by letting n tend to ∞ in the approximate equation

(1.9)′ $D_t T_t^{(n)} x = A^{(n)} T_t^{(n)} x,\ t \geqq 0,\ A^{(n)} = AI^{(n)} = n(I^{(n)} - I),$
$T_0^{(n)} x = x \in X.$

This procedure, combined with the Converse Theorem, suggests us a method of integration of (2.1) as follows (K. Yosida [12]).

Let D be a linear set of C^∞ functions $f(x)$ with compact carriers such that D is $L_1(R)$-dense in $L_1(R)$. We regard $A_t = A_{tx}$ as a linear unbounded operator on $D(A_t) = D \subseteqq L_1(R)$ to $L_1(R)$, and let $\bar{A}_t$ be the smallest closed extension of this A_t. We assume that D and the coefficients $a^{ij}(t, x)$, $b^i(t, x)$ and $c(t, x)$ are such that the following Hypothesis is satisfied:

Hypothesis. Let, for all sufficiently large integer (independently of t), the resolvents

(3.1) $I_t^{(n)} = (I - n^{-1}\bar{A}_t)^{-1}$

exist as bounded linear operators on $L_1(R)$ to $L_1(R)$ such that

(3.2) $I_t^{(n)} f(x)$ is non-negatieve and $\int_R I_t^{(n)} f(x)dx = \int_R f(x)dx$ if $f \in L_1(R)$ is non-negative,

(3.3) $I_t^{(n)} f$ is strongly continuous in t.

A bounded linear operator on $L_1(R)$ to $L_1(R)$ is called a „transition operator" if it satisfies (3.2).

We then consider the approximate equations

(3.4) $D_t f_{st}^{(n)} = A_t^{(n)} f_{st}^{(n)},\ t \geqq s,\ A_t^{(n)} = \bar{A}_t I_t^{(n)} = n^{-1}(I_t^{(n)} - I),$
$f_{ss}^{(n)} = f \in L_1(R).$

Since $A_t^{(n)}$ is a bounded operator, we may integrate (3.4) by the method of successive approximation. The solution $f^{(n)}(s, t, x) = f_{st}^{(n)}(x)$ satisfies, by (3.2),

(3.5) $f^{(n)}(s, t, x)$ is non-negative and $\int_R f^{(n)}(s, t, x)dx = \int_R f(x)dx$ if $f \in L_1(R)$ is non-negative.

We introduce the „distribution" (L. Schwartz [1])

$$(3.6)\quad (f_{st}^{(n)}, \varphi) = \int_R f^{(n)}(s, t, x)\varphi(x)dx$$

at the testing function $\varphi(x)$ which is a C^∞ function whose carrier is compact and contained in an open domain of R. We have, from (3.4),

$$(3.7)\quad (f_{st}^{(n)}, \varphi) - (f, \varphi) = \int_s^t (A_\tau^{(n)} f_{s\tau}^{(n)}, \varphi)d\tau = \int_s^t (I_\tau^{(n)} f_{s\tau}^{(n)}, A_\tau^* \varphi)d\tau.$$

Thus, by (3.5) and (3.7), the distributions $(f_{st}^{(n)}, \varphi)$ are equi-bounded and equi-continuous in t. Hence, by the separability of $L_1(R)$, we may choose a subsequence $\{n'\}$ of $\{n\}$ such that, for every $t \geqq s$, for every φ and for every $f \in L_1(R)$ simultaneously, we have

$$(3.8)\quad \lim_{n' \to \infty} (f_{st}^{(n')}, \varphi) = (T_{st} f, \varphi),$$

where the distribution $T_{st} f$ satisfies

$$(3.9)\quad \frac{\partial}{\partial t}(T_{st} f, \varphi) = (T_{st} f, A_t^* \varphi),\ t \geqq s,\ (T_{ss} f, \varphi) = (f, \varphi).$$

Because of (3.5), the distribution $T_{st} f$ is defined by a measure $\varrho_{stf}(dx)$:

$$(3.10)\quad (T_{st} f, \varphi) = \int_R \varphi(x)\varrho_{stf}(dx).$$

On the other hand, we may construct a reasonable parametrix for E_{sx}^* (K.

Yosida [11] which was suggested by S. Minakshsunderaram [1] (with A. Pleijel). Cf. S. Minakshsundaram [2] and S. Ito [1]) as follows [1]).

For any point $x_0 \in R$ and for sufficiently small neighbourhood $U(x_0)$ of x_0, we may choose a compact neighbourhood $V(x_0) \subseteqq U(X_0)$ and a parametrix $H(s, t, x, y)$ with the properties

(3.11) $H(s, t, x, y)$ is, for $t > s$, C^∞ in (s, t, x, y);
$E^*_{sx}H(s, t, x, y) = K(s, t, x, y)$ is C^2 in (s, t, x, y) even when $t = s$;
$H(s, t, x, y) \equiv 0$ if x or y is outside of $U(x_0)$;
for any $f \in L_1(R)$, we have $f(x) = \lim_{t \downarrow s} \int_R f(y)H(s, t, x, y)dy$
$= \lim_{s \uparrow t} \int_R f(y)H(s, t, y, x)dy$ almost everywhere in $V(x_0)$.

Hence, taking the limit, firstly letting $n' \to \infty$ and then letting $\varepsilon \downarrow 0$ in the formula

$$(3.12) \quad \int_R f^{(n)}(s, t, y)H(t, t+\varepsilon, x, y)dy = \int_R f^{(n)}(s, s, y)H(s,t+\varepsilon, x, y)dy$$
$$+ \int_s^t d\tau \left\{ \frac{\partial}{\partial \tau} \int_R f^{(n)}(\tau, t, y)H(\tau, t+\varepsilon, x, y)dy \right\},$$

---

1) $H(s, t, x, y)$ may be defined as follows. We denote by $\Gamma(\tau, x, y)$ the square of the geodesic distance of x and y according to the metric $dr(\tau)^2 = a_{ij}(\tau, x)dx^i dx^j$, where the matrix $(a_{ij}(\tau, x)) = (a^{ij}(\tau, x))^{-1}$, and let $S(x, y)$ be the geodesic distance of x and y according to the original metric $ds^2 = g_{ij}(x)dx^j dx^i$. Then $H(s, t, x, y)$ is given by $H(s, t, x, y) = \pi^{-m/2} (a(t, x)/g(x))^{1/2} \cdot (t-s)^{k-m/2} \exp(-\Gamma(s, x, y)/4(t-s))$

$$\cdot \sum_{i=0}^{k} u_i(s, x, y)(t-s)^i \delta(S(x, y)).$$

Here the integer k is $> (2 + m/2)$ and u_i are defined recursively by

$$\frac{1}{2}\left(a^{pj}\frac{\partial \Gamma}{\partial x^j}\right)\frac{\partial u_i}{\partial x^p} + \left(-\frac{m}{2} + i + \frac{M}{4} + \frac{1}{4}\frac{\partial \Gamma}{\partial s}\right) = N(u_{i-1}) + \frac{\partial u_{i-1}}{\partial s}$$

with the initial condition

$$u_{-1} \equiv 0, \quad u_0(s, x, x) \equiv 1;$$

the coefficients M and $N(F)$ being defined by

$$M = a^{ij}\frac{\partial^2 \Gamma}{\partial x^i \partial x^j} + b^i \frac{\partial \Gamma}{\partial x^i},$$

$$N(F) = a^{ij}\frac{\partial^2 F}{\partial x^i \partial x^j} + b^i \frac{\partial F}{\partial x^i} + cF,$$

and the function $\delta(S)$ is infinitely differentiable and $\geqq 0$ for $S \geqq 0$ such that $\delta(S) = 1$ for $0 \leqq S \leqq \eta$ and $\delta(S) = 0$ for $S \geqq 2\eta$ with a suitably chosen positive constant η.

we obtain, remembering (3.9)—(3.11),

$$(3.13)\quad \int_R \varphi(x)\varrho_{stf}(dx) = \int_R f(x)dx\left\{\int_R \varphi(y)\,H\,(s,t,x,y)dy\right\}$$

$$+\int_s^t d\tau\left\{\int_R \varrho_{s\tau f}(dx)\left[\int_R \varphi(y)\,K\,(\tau,\ t,\ x,\ y)\ dy\right]\right\}.$$

Thus the measure $\varrho_{stf}(dx)$ is absolutely continuous with respect to dx, and the density $f(s, t, x)$ satisfies, almost everywhere in $V(x_0)$,

$$(3.14)\quad f(s,t,x) = \int_R f(y)\,H\,(s,t,x,y)dy + \int_s^t d\tau\left\{\int_R f(s,\tau,y)K(\tau,t,y,x)dy\right\}.$$

The right hand side is, for $t > s$, continuously differentiable once in t and twice in x. Thus we have obtained a solution of

$$(2.1)'\quad \frac{\partial f(s,t,x)}{\partial t} = A_t f(s,t,x),\ \ t > s,$$

$$\lim_{t\downarrow s} f(s,t,x) = f(x) \in L_1(R) \text{ for almost every } x.$$

We see, by (3.14), that the value $(T_{st}f)(x) = f_{st}(x) = f(s, t, x)$ for fixed (s, t, x) may be considered as a bounded linear functional of $f \in L_1(R)$. Hence we obtain the kernel representation

$$(3.15)\quad f(s,t,x) = \int_R P(s,\ t,\ x,\ y)f(y)dy \text{ for every } f \in L_1(R).$$

Since the conjugate space $L_1(R)^*$ of $L_1(R)$ is the totality of bounded Borel-measurable functions on R, the kernel $P(s, t, x, y)$ is, for fixed (s, t, x), bounded Borel measurable in y. In this way we have obtained the fundamental solution of (2.1)′.

If we modify and apply the above method of intergation to (2.2), we also obtain a fundamental solution $Q(t, s, y, x)$ of (2.2). By taking Q instead of H in (3.12), we obtain

$$(3.16)\quad f(s,t,x) = \int_R Q(t,s,y,x)f(y)dy,$$

and hence the fundamental symmetry

$$(3.17)\quad P(s,t,x,y) = Q(t,s,y,x).$$

The above argument shows that, in case $Q(t, s, y, x)$ too exists, the uniqueness of the solution of (2.1)′ is guaranteed so that the original sequence $\{f_{st}^{(n)}\}$ itself converges to f_{st}:

$$(3.8)'\quad \lim_{n\to\infty}\ (f_{st}^{(n)},\ \varphi) = (f_{st},\ \varphi).$$

4. *Results concerning the hypothesis in* 3. The integration of the diffusion equation (2.1)′ is thus reduced to the verification of the Hypothesis (3.1)—(3.3), and also to the verification of the similar Hypothesis concerning (2.2) if the uniqueness of the solution of (2.1)′ is to be guaranteed. If we disregard the condition (3.3), we have only to verify (3.1)—(3.2) for each fixed t. Thus the results from the „Abstract Cauchy's Problem" pertaining to the temporally homogeneous diffusion equations may immediately be applied to the temporally inhomogeneous diffusion equations. This is one of the advantages of our method of integration. It is also to be noted, as was shown above, that the corresponding transition probability is given by the density $P(s, t, x, y)$.

Various conditions may be obtained under which the Hypothesis is satisfied. Assuming $c(t, x) \equiv 0$, such conditions may be derived as follows. (Cf. K. Yosida [6]). We start with the Green's integral theorem

$$(4.1)\quad \int_G \big(h(x)(A_t f)(x) - f(x)(A_t^* h)(x)\big)dx = \int_{\partial G} B(f, h)dS$$
$$= \int_{\partial G} g(x)^{1/2} a^{ij}(t, x)\left(h(x)\frac{\partial f}{\partial x^j} - f(x)\frac{\partial h}{\partial x^j}\right)\cos(n, x^i)dS$$
$$+ \int_{\partial G}\left\{\frac{\partial g(x)^{1/2}a^{ij}(t, x)}{\partial x^j} - g(x)^{1/2} b^i(t, x)\right\} f(x)h(x)\cos(n, x^i)dS,$$

where n denotes the outer normal at the point x of the boundary ∂G of the connected domain $G \subseteqq R$, and dS is the hypersurface element of ∂G. Let $D(A_t)$ denote a linear set of C^∞ functions $f(x)$ on R with compact carriers satisfying certain boundary condition on ∂R. The boundary condition should be chosen in such a way that the following lemma is satisfied.

Lemma. Let $f(x) \in D(A_t)$ be positive (negative) in a connected domain $G \subseteqq R$ such that $f(x)$ vanishes on $\partial G - \partial R$. Then, we have, for any positive number n, the inequality

$$(4.2)\quad \int_G \Big(f(x) - n^{-1}(A_t f)(x)\Big)dx \geqq \int_G f(x)dx \geqq 0 \left(\leqq \int_G f(x)dx \leqq 0\right).$$

Moreover, it is assumed that $D(A_t)$ is so chosen that

$$(4.3)\quad \int_R (A_t f)(x)dx = 0 \text{ for } f \in D(A_t).$$

We assume that $a^{ij}(t, x)\cos(n, x^i)\cos(n, x^j) > 0$ on ∂R so that we may define the outer „transversal" direction $d\nu$ at $x \in \partial R$ by

$$(4.4)\quad d\nu = dx^i / g(x)^{1/2} a^{ij}(t, x)\cos(n, x^j) \quad (i = 1, 2, \ldots, m).$$

Then we see, by (4.1), that each of the following boundary conditions satisfies (4.2)—(4.3):

(4.5) $f = 0$ on ∂R and $\int_{\partial R} B(f, 1) dS = 0$;

(4.6) $B(f, 1) = 0$ on ∂R;

(4.7) $g(x)^{1/2} a^{ij}(t, x) \dfrac{\partial f}{\partial x^j} \cos(n, x^i) = k(x) f(x)$ on ∂R and $\int_{\partial R} B(f, 1) dS = 0$

for a function $k(x)$ satisfying $\dfrac{\partial g(x)^{1/2} a^{ij}(t, x)}{\partial x^j} -$

$g(x)^{1/2} b^i(t, x) \cos(n, x^i) + k(x) \leqq 0$ on ∂R.

We obtain, from (4.2),

(4.8) $\| f - n^{-1} A_t f \| \geqq \| f \|$ for $n > 0$ and $f \in D(A_t)$.

Hence the bounded linear inverse $(I - n^{-1}\bar{A}_t)^{-1}$ exists for the smallest closed extension $\bar{A}_t$ of the operator A_t with the domain $D(A_t)$. Therefore the resolvent $I_t^{(n)} = (I - n^{-1}\bar{A}_t)^{-1}$ exists if and only if

(4.9) the range $\{(I - n^{-1}A_t)f;\ f \in D(A_t)\}$ is strongly dense in $L_1(R)$.

Because of the condition (4.3), we may prove, in case the resolvent $I_t^{(n)}$ exists, that $I_t^{(n)}$ is a transition operator. Hence the existence of the transition resolvent is equivalent, under (4.5) or (4.6) or (4.7), to the condition (4.9). This condition, in turn, is equivalent to the non-existence of the bounded Borel measurable function $\tilde{h}(x)$ not equivalent to zero and satisfying

(4.10) $\int_R \tilde{h}(x)(f(x) - n^{-1}(A_t f)(x)) dx = 0$ whenever $f \in D(A_t)$.

By the „ellipticity" of the operator A_t, we see (L. Schwartz [1]) that such function $\tilde{h}(x)$ is equal almost everywhere to a c^∞ function $h(x)$ satisfying

(4.11) $(A_t^*) h(x) = n h(x)$.

Moreover, from (4.1) and (4.10)—(4.11), we see that $h(x)$ must satisfy the following boundary condition: Let $\{R_i\}$ be a monotone increasing sequence of connected domains $\subseteqq R$ such that the boundary ∂R_i tends, as $i \to \infty$, to the boundary ∂R very smoothly. Then

(4.12) $\lim_{i \to \infty} \int_{\partial R_i} B(f, h) dS = 0$ for every $f \in D(A_t)$.

Hence, under the boundary condition (4.5) or (4.6) or (4.7), the conditions (3.1)—(3.2) in the Hypothesis are surely satisfied if and only if there does not exist bounded solution $h(x) \not\equiv 0$ of (4.11) satisfying (4.12).

Therefore, when $c(t, x) \equiv 0$, the Hypothesis and the Hypothesis concerning (2.2) are satisfied in the case of compact Riemannian space R (K. Yosida [4] and [8]). In fact, in such case, the solution of (4.11) cannot have positive maximum nor negative minimum on R. This result assures, in particular, the existence and the uniqueness of the Brownian motion on the surface of the 3-sphere. This result was obtained in other ways and independently by F. Perrin [1] and K. Yosida [3]. We may also prove, corresponding to the boundary condition (4.6), the existence and the uniqueness of the Brownian motion with reflecting barrier condition in a domain R of the m-dimensional Euclidean space, under the condition that the boundary ∂R does not extend to ∞ (K. Yosida [7] and F. V. Atkinson [1]).

Finally, it is to be noted that K. Ito [1]—[3] devised, by virtue of his "stochastic integral", another method of integration of the diffusion equations.

5. *Researches by E. Hille, W. Feller and T. Kato.* For the case of one dimension, E. Hille and W. Feller gave thorough investigations, details of which will be discussed in the talk by Professor Hille. The present speaker will mention here of some salliant features of their results. E. Hille [2] and [3] gave a fairly complete condition for the existence and the uniqueness of the solution, satisfying the transition property similar to (3.5), of the equations

$$(5.1)\quad \frac{\partial f(t,x)}{\partial t} = \frac{\partial^2 a(x)f(t,x)}{\partial x^2} - \frac{\partial b(x)f(t,x)}{\partial x},\ t>0,$$

$$\text{strong}\lim_{t\downarrow 0} f(t,x) = f(x) \in L_1(r_1, r_2),$$

$$(5.2)\quad \frac{\partial h(t,x)}{\partial t} = a(x)\frac{\partial^2 h(t,x)}{\partial x^2} + b(x)\frac{\partial h(t,x)}{\partial x},\ t>0,$$

$$\text{strong}\lim_{t\downarrow 0} h(t,x) = h(x) \in C[r_1, r_2],\ -\infty \leqq r_1 < r_2 \leqq \infty.$$

One of his results says that the Hypothesis for (5.1) with $r_1 = -\infty$, $r_2 = \infty$ is surely satisfied if

$$(5.3)\quad \sup_x\,(a''(x) - b'(x)) < \infty \text{ and}$$

$$\sup_x\,(|a'(x)| + |b(x)|)/(|x|+1) < \infty.$$

E. Hille [4] also discovered an „explosive solution" whose norm tends to ∞ as $t \uparrow t_0 < \infty$. W. Feller [4] gave an exhaustive study of the interplay between (5.1) and (5.2). The remarkable fact is his discovery that, from the view point of the semi-group theory, the corresponding adjoint of (5.2) is no longer the differential equation (5.1). It is, for example, of the form

$$(5.1)'\quad f_t(t,x) = ((a(x)f(t,x))_x - b(x)f(t,x))_x + \tau\sigma^{-1}V_2(t)p'(t), \text{ where}$$

$$V_1'(t) = p_1\sigma^{-1}V_2(t),\ V_2'(t) = -p_2\sigma^{-1}V_1(t) - \lim_{x\to r_2}\{(a(x)f(t,x))_x - b(x)f(t,x)\}$$

Thus an essentially new boundary condition is introduced: We interprete $f(t, x)$ as the mass density at the time moment t at the point x in the interior of (r_1, r_2), and $V_j(t)$ as the mass at r_j at the time moment t. Then the mass flows out of the interior at r_2 but not at r_1. The mass concentrates at r_2 from where it flows out at the constant ratio p_2/σ. The outflowing mass goes in part to r_1, in part to the interior of (r_1, r_2) and in part it disappears; the three parts being in the ratio $p_1 : \tau : (p_2 - p_1 - \tau)$ and the part going to r_1 being inactive forever.

In the case when R degenerates to an enumerable set, the temporally homogeneous diffusion equations reduce to a pair of equations (A. Kolmogoroff [1]).

$$(5.4)\quad p'_{ij}(t) = \sum_{k=1}^{\infty} a_{ik} p_{kj}(t),\ p'_{ij}(t) = \sum_{k=1}^{\infty} p_{ik}(t)\, a_{kj}\ \ (i, j = 1, 2, \ldots),$$

where the matrix $A = (a_{ij})$ is subject to the condition

$$(5.5)\quad a_{ij} \geqq 0\ \ (i \neq j),\ a_{ii} \leqq 0 \text{ and } \sum_{j=1}^{\infty} a_{ij} = 0.$$

The equations (5.4) have been integrated by J. L. Doob, W. Feller and P. Lévy probabilistically. The integration by the semi-group theory was recently taken up by E. Hille and T. Kato.

E. Hille [10] discussed the special case of triangular matrices $A = (a_{ij})$: $a_{ij} = 0$ for $j > i$. He integrated the equations in the Markoff algebra M of matrices $B = (b_{ij})$ with the norm $\| B \| = \sup_i \sum_{j=1}^{\infty} | b_{ij} | < \infty$. Let $D(A)$ be the subspace of M for which $AB \in M$, and let M_A be the strong closure of $D(A)$. The semi-group of transition operators

$$(5.6)\quad P(t)B = \text{strong} \lim_{n\to\infty} \exp\,(tA(I - n^{-1}A)^{-1})B$$

is strongly continuous for $t \geqq 0$ when acting on M_A. The infinitesimal generator A_0 of $P(t)$ is the contraction of A to the subset of $D(A)$ in which $AB \in M_A$. While $P(t)B$ is the only solution of

$$(5.7)\quad Y'(t) = AY(t),\ Y(t) \in D(A),\ t > 0,\ \text{strong} \lim_{t\downarrow 0} Y(t) = B,$$

it is shown, by an example, that $P(t)$ is not the only solution of $Z'(t) = Z(t)A$, $\lim_{t\downarrow 0} z_{ij}(t) = \delta_{ij}$.

T. Kato [2] integrated the equation (5.4) in the Banach space (l_1). Let $D(A)$ consist of those $x \in (l_1)$ such that $Ax \in (l_1)$, and let D_0 be the linear subspace spanned by the vectors $x_j = (0, \ldots, 0, 1, 0, \ldots)$, viz. x_j is the vector whose components are nought except the j-th component 1. Let H

be the diagonal part of A and put $K = A - H$. Let H_0 be the contraction of H to D_0, and let H_1 be the smallest closed extension of H_0. It is proved that the operator

(5.8) $A_r = H_1 + rK \ (0 < r < 1)$ with the domain $D(H_1)$

is the infinitesimal generator of a semi-group $P_r(t)$, and that the

(5.9) $P(t) = \text{strong} \lim_{r \to 1} P_r(t)$

constitutes a semi-group whose infinitesimal generator G satisfies

(5.9) $Gx_j = Ax_j \ (j = 1, 2, \ldots)$.

$P(t)$ is a non-negative operator such that $\| P(t) \| \leqq 1$. Moreover, it is shown that

(5.10) $\lim_{t \to \infty} \sum_j p_{ij}(t) = 1 - \lim_{\lambda \downarrow 0} b_i(\lambda)$, where

$b_i(\lambda) = \lim_{n \to \infty} \sum_j b_{ij}^{(n)}(\lambda), \ b_{ij}^{(n+1)}(\lambda) = \sum_k b_{ik}^{(n)}(\lambda) b_{kj}^{(1)}(\lambda),$

$b_{ij}^{(1)}(\lambda) = 0$ (for $i = j$) and $= a_{ij}/(\lambda + a_{ij})$ (for $i \neq j$).

6. *Perturbation theory of semi-groups by T. Kato and R. S. Phillips.* The method of the integration in § 3 may be modified so as to be applicable to other function spaces than $L_1(R)$; it may, for example, be modified to be applied to the integration of the time dependent Schrödinger equations in Hilbert space. However, more abstract methods, both aiming at direct generalizations of semi-group theory were devised by T. Kato and R. S. Phillips in connection with the equation of evolution

(6.1) $D_t U(t, s)x = A_t U(t, s)x, \ t > s,$

$\text{strong} \lim_{t \downarrow s} U(t, s)x = x \in X.$

T. Kato [1] assumed the following conditions:

(i) The domain D of the closed linear operator A_t is independent of t and is strongly dense in X in such a way that, for $\alpha > 0$, the resolvents $(I - \alpha A_t)^{-1}$ exist as bounded linear operator on X to X with the norms $\leqq 1$. (ii) The operator $B(t, s) = (I - A_t)(I - A_s)^{-1}$is uniformly bounded in norm for every $t \geqq s$. (iii) $B(t, s)$ is, at least for some s, of bounded variation in t in norm, viz.

$$\sum_j \| B(t_{j-1}, s) - B(t_j, s) \| \leqq N(a, b) < \infty$$

for every partition $a = t_0 < t_1 \ldots < t_n = b$ of the interval (a, b). (iv) $B(t, s)$ is, at leat for some s, weakly differentiable in t and the differential quotient $\partial B(t, s)/\partial t$ is strongly continuous in t.

He proved that, under the above conditions, the limit

$$(6.2)\quad U(t, s)x = \text{strong} \lim_{\max|t_j - t_{j-1}| \to 0} \prod_{k=n}^{1} \exp\left((t_k - t_{k-1}) A_{t_{k-1}}\right) x, \text{ where}$$

$$s = t_0 < t_1 \ldots < t_n = t,$$

exists for every $x \in X$ and gives a unique solution of (6.1) at least for $x \in D$.

R. S. Phillips' result [2] corresponds to the special case

$$(6.3)\quad A_t = A + B_t.$$

Herein A is the infinitesimal generator of a semi-group in X and B_t is a strongly continuously differentiable function of t with values in the space of the bounded linear operators on X to X. In this sense, Phillips called the problem of the integration of (6.1) as the „perturbation theory for the semi-groups". His method of integration is entirely different from Kato's. R. S. Phillips [3], published quite recently, gives a discussion of the perturbation theory under milder conditions of the continuity of $U(t, s)$ at $t = s$.

Their results are general and elegant. However, it would not be easy to apply their results to the concrete equations such as (2.1) and (2.2), since they assumed that the domain of A_t is independent of t. It is highly desirable, at least to the speaker, that their methods should be modified so as to be adapted to the integration of the diffusion equations (2.1) and (2.2).

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SHORT LECTURES

Abstracts of those communications in sections which did not reach the Editorial Committee in time and therefore could not be printed in Volume II, which was issued before the Congress took place.

Section I

ALGEBRA AND THEORY OF NUMBERS

SHORT LECTURES

DIE KUBISCHE FERMAT-GLEICHUNG IN QUADRATISCHEN KÖRPERN

ALEXANDER AIGNER

Die Möglichkeit von $x^3 + y^3 = z^3$ in $K(\sqrt{m})$ hängt weitgehend von der Faktorenzerlegung der Zahl m ab. Es gibt gewisse Produkte von Primzahlen der Art $3n + 1$ mit 2 als kubischem Rest oder der Art $3n + 2$, für welche auf Grund gegenseitiger quadratischer Rest- oder Nichtrest-Beziehungen ihrer Teiler in $K(\sqrt[3]{2})$ — etwa durch einen Graphen zu veranschaulichen — die Gleichung unmöglich ist und es auch stets bleibt, wenn zu diesem Produkt noch allfällige Primzahlen mit 2 als kubischem Nichtrest hinzutreten. Diese Einschränkungen gibt es im Falle $|m| \equiv 1(3)$, während sie bei $|m| \equiv 2(3)$ nicht erscheinen, wo also die Möglichkeit der Gleichung immer offen bleibt.

GRAZ (AUSTRIA), HUMBOLDTSTRASSE 17.

THEORY OF REMAINDERS IN THE ALGEBRA OF GRASSMANN

GOVERDHAN LAL. BAKHSHI

The paper develops some results in connection with the theory of Prof Lepage about the Remainders with respect to a non-singular quadratic form in the algebra of Grassmann. It is known that this theory leads to an analogous theory in the associative algebra of α's, which is generated by the linearly independent elements $1, \alpha_1, \alpha_2, \ldots, \alpha_n$, with the rules

$$1 \,.\, \alpha_i = \alpha_i \,.\, 1 = \alpha_i$$
$$\alpha_i^2 = 0$$
$$\alpha_i \wedge \alpha_j = \alpha_j \wedge \alpha_i \quad (i \neq j).$$

The fundamental form $H = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and a form ϱ_k of degree k in the α's is a remainder, if and only if $\varrho_k \wedge H^{n-2k+1} = 0$.

The set R_k of remainders of degree k is a vectorial space, of which the dimensionality is known, and a system of its generators is constituted by the product of differences $(\alpha_{i_1} - \alpha_{i_2})(\alpha_{i_3} - \alpha_{i_4}) \ldots (\alpha_{i_{2k-1}} - \alpha_{i_{2k}})$.

One of the essential results of this paper is to construct effectively in two manners a basis of this space.

The first consists in determining in the set of generators indicated above, a system of linearly independent elements, such that each generator can be shown as a linear combination of these elements.

Besides, two formulae are obtained, which enable us to calculate directly the remainder of the monomial $\alpha_i \wedge \alpha_2 \wedge \ldots \wedge \alpha_k = H_k^k/k!$.

The first of these formulae gives the remainder as a linear combination of the generators indicated above and finally as linear combination of the basis constructed.

The second formula gives the remainder $r(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k)$ of $\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_k = H_k^k/k!$, as a polynomial in H and H_k.

The set of elements $r(\alpha_{i_1} \wedge \alpha_{i_2} \wedge \ldots \wedge \alpha_{i_k})$ constitutes a system of generators for the space R_k, and from this a second basis is extracted.

It is known that any form can be expanded as a linear combination of the powers of H, the coefficients being the remainders. In this paper are also found two formulae which enable us to calculate effectively these coefficients.

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QUESTIONS D'ANALYSE DIOPHANTIENNE MULTIGRADE

ALBERT GLODEN

Rappelais qu'un système multigrade normal est un système diophantien de la forme

$$\sum_{i=1}^{k+1} A_i^k = \sum_{i=1}^{k+1} B_i^k, \quad (k = 1, 2, \ldots, n),$$

l'un au plus des A_i ou des B_i pouvant s'annuler.

A. *Système trigrade normal et quadrilatère inscrit à cotés et diagonales en nombres entiers.*

Nous démontrons que le problème consistant à déterminer tous les quadrilatères inscrits dans un cercle ayant pour cotés et diagonales des nombres entiers est équivalent à la résolution du système trigrade normal. Par des transformations linéaires nous ramenons le système trigrade normal au système suivant:

$$\begin{cases} \sum_{i=1}^{3} a_i^2 = \sum_{i=1}^{3} b_i^2 \\ a_1 a_2 a_3 = b_1 b_2 b_3, \end{cases}$$

dont la solution complète est donnée par des formules du 6e degré à 6 paramètres arbitraires. Nous donnons ensuite des formules pour des quadrilatères inscrits héroniens.

B. *Un procédé de résolution du système sextigrade normal.*

Nous ramenons la résolution de ce système à celle du système trigrade normal

$$\sum_{i=1}^{3} A_i^k = \sum_{i=1}^{4} B_i^k, \qquad (k = 1, 2, 3).$$

Nous explicitons les solutions dans les deux cas suivants:

1) La différence entre deux termes d'un membre de la trigrade de départ égale celle de deux termes de l'autre membre.

2) La différence entre deux termes d'un membre de la sextigrade à construire égale la différence entre deux termes de l'autre membre.

C. *Systèmes multigrades dans le corps des nombres complexes.*

Dans l'anneau des entiers on a établi des systèmes multigrades normaux jusqu à l'ordre 9. La question de savoir s'il en existe d'un ordre supérieur, et plus généralement, de tous ordres > 9, reste ouverte.

Nous démontrons que, par contre, dans le corps des nombres complexes il existe des systèmes multigrades normaux de tous ordres.

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ERHALTUNGSSÄTZE DER IDEALTHEORIE UND IHRE FOLGERUNGEN

Heinrich Grell

Auf Grund der Untersuchungen von Akizuki, Cohen, Grell und Krull läßt sich folgender allgemeiner idealtheoretischer Erhaltungssatz formulieren:

Ist $\mathfrak{o}$ ein nullteilerfreier Ring mit Einselement und Gültigkeit der eingeschränkten Minimalbedingung für die Ideale und $\mathfrak{O}$ ein $\mathfrak{o}$ umfassender nullteilerfreier Ring, dessen Quotientenkörper $Q(\mathfrak{O}) = K$ von endlichem Rang über dem Quotientenkörper $Q(\mathfrak{o}) = k$ von $\mathfrak{o}$ ist, so ist auch $\mathfrak{O}$ ein Ring mit Gültigkeit der eingeschränkten Minimalbedingung für die Ideale. Dabei wird weder eine Voraussetzung über die Charakteristik der Körper noch eine solche über die Separabilität von K bez. k gemacht; auch braucht $\mathfrak{O}$ nicht aus Elementen von K zu bestehen, die bez. $\mathfrak{o}$ ganz-algebraisch sind. Die eingeschränkte

Minimalbedingung ist für einen Ring mit der z.B. in allgemeinen Zahlkörperordnungen bestehenden Idealtheorie äquivalent; sie besagt neben einer gewissen Irreduzibilitätsbedingung die Teilerfremdheit je zweier verschiedener Primideale und die eindeutige Produktdarstellung der Ideale durch Primärideale.

Dieser Satz erlaubt es u.a., bei einem beliebigen endlichen algebraischen Zahlkörper K eine vollständige Übersicht über alle in K gelegenen, ein Einselement enthaltenden Ringe $\mathfrak{r}$ mit $Q(\mathfrak{r}) = K$ und Gültigkeit der eingeschränkten Minimalbedingung für die Ideale mittels gewisser einfachster Normaltypen zu gewinnen. Insbesondere lassen sich alle Dedekindschen Ringe, d.h. die mit Gültigkeit der gewöhnlichen Idealtheorie, erschöpfend charakterisieren.

Eine ausführliche Darstellung wird in den „Mathematischen Nachrichten", Berlin, erscheinen.

BERLIN C2, UNTER DEN LINDEN 6.

ON THE DISTRIBUTION OF STRONGLY ADDITIVE NUMBER THEORETIC FUNCTIONS

HEINI HALBERSTAM

A function $f(m)$ is called *strongly additive* if

$$f(m_1 m_2) = f(m_1) + f(m_2), \quad (m_1, m_2) = 1$$

and

$$f(p^\alpha) = f(p) \quad , \quad p \text{ prime}, \ \alpha = 2, 3, \ldots$$

We define $A_n = \sum_{p<n} f(p)/p, \quad B_n = \sum_{p<n} f^2(p)/p,$

and assume only that

$$f(p) = O(1), \quad B_n \to \infty \text{ as } n \to \infty.$$

An outline will be given of the proof, by elementary methods, of the following:

Theorem.

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} (f(m) - A_n)^k}{n B_n^{k/2}} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega^k e^{-\frac{1}{2}\omega^2} d\omega, \quad k = 1, 2, 3, \ldots$$

Let $R_n(\omega)$ be the number of m, $1 \leqq m \leqq n$, such that

$$f(m) < A_n + \omega B_n^{1/2},$$

and put $\sigma_n(\omega) = R_n(\omega)/n$. Then $\sigma_n(\omega)$ is a distribution function and the theorem may be restated as

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \omega^R d\sigma_n(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega^R e^{-1/2\omega^2} d\omega, \quad R = 1, 2, \ldots$$

This means that *the moments of the distribution characterised by* $\sigma_n(\omega)$ *tend to the moments of the Gaussian distribution*. It may be deduced that

$$\lim_{n\to\infty} (\sigma_n(\omega)) = (2\pi)^{-1/2} \int_{-\infty}^{\omega} e^{-1/2 u^2}\, du,$$

which is a theorem of Erdös and Kac (American J. Math. 62 (1940).

By similar methods asymptotic formulae may be obtained for sums such as

$$\sum_{m=1}^{n} (f(m+1) - f(m))^R, \text{ and } \sum_{m=1}^{n} \{(f(m+1) - A_n)(f(m) - A_n)\}^R.$$

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SUR LA LOI GÉNÉRALE D'ASSOCIATIVITÉ

SHOKICHI IYANAGA

Une loi de composition écrite sous forme multiplicative entre les deux éléments d'une structure algébrique $A = \{a, b, c, \ldots\}$ est dite associative, comme on sait, si on a l'identité $(ab)c = a(bc)$ pour toute combinaison de trois éléments a, b, c de A. On sait que l'on a, dans toute structure algébrique avec une loi de composition associative, une loi plus générale d'associativité, d'ordinaire formulée vaguement en disant que „le produit de plusieurs éléments de A est indépendant de manière de mettre les parenthèses entre ces éléments", par exemple: $((ab)c)d = a((bc)d) = (ab)(cd)$, mais il parait qu'une formulation exacte de cette loi n'a pas encore été donnée. On se propose d'en donner une.

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APPROXIMATIONS DIOPHANTIENNES LINÉAIRES ET HOMOGÈNES

V. Jarník

Soit $\Theta = (\theta_{ji})$ $(j = 1, \ldots, s;\ i = 1, \ldots, r)$ une matrice de rs nombres réels. Pour $t \geqq 1$ posons

$$\psi(t) = \psi(\Theta, t) = \operatorname{Min} \left(\max_{1 \leqq j \leqq s} | \theta_{j1}x_1 + \ldots + \theta_{jr}x_r + x_{r+j}| \right),$$

où l'on prend le minimum pour tous les systèmes de nombres entiers $x_1, x_2, \ldots, x_{r+s}$ tels que $0 < \operatorname{Max} \left(| x_1 |, \ldots, | x_r | \right) \leqq t$. Soit $\alpha = \alpha(\Theta)$ resp. $\beta = \beta(\Theta)$ la borne supérieure de tous les nombres γ tels que

$$\limsup_{t \to +\infty} t^\gamma \psi(t) < +\infty \text{ resp. } \liminf_{t \to +\infty} t^\gamma \psi(t) < +\infty.$$

On a donc $r/s \leqq \alpha \leqq \beta \leqq +\infty$. Laissons de côté le cas banal, où $\psi(t) = 0$ à partir d'un certain t. Alors on sait encore que pour $r = 1$ on a $\alpha \leqq 1$.

On peut démontrer les inégalités suivantes:

I. Pour $r = 2$, $s \geqq 1$ on a $\beta \geqq \alpha(\alpha - 1)$.

II. Pour $r > 2$, $s \geqq 1$, $\alpha > (5r^2)^{r-1}$ on a $\beta \geqq \alpha^{\frac{r}{r-1}} - 3\alpha$.

III. Dans le cas $r = 1$, $s \geqq 2$ on doit supposer que, parmi les nombres de Θ, il y ait au moins deux nombres linéairement indépendants. Dans ce cas, on a $\beta \geqq \dfrac{\alpha^2}{1 - \alpha}$ pour $\alpha < 1$, $\beta = +\infty$ pour $\alpha = 1$.

Le terme principal (pour $\alpha \to +\infty$) au second membre de I, II est $\alpha^{\frac{r}{r-1}}$ et ce terme est définitif, car, à chaque $a > 2^{r-1}$ on peut faire correspondre une matrice Θ telle que

$$\alpha = a, \ \beta = \alpha^{\frac{r}{r-1}}.$$

On a un résultat analogue dans le cas III.

On peut remplacer quelques-uns de ces résultats par des résultats plus précis.

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CARMICHAEL-ZAHLEN UND NEUERE HOLLÄNDISCHE ARBEITEN

WALTER KNÖDEL

Jene natürlichen Zahlen c, die der Kongruenz

$$a^{c-1} \equiv 1 \pmod c$$

für jedes a mit $(a, c) = 1$ genügen, ohne Primzahlen zu sein, nennt man Carmichaelsche Zahlen. Für die Anzahl $z(x)$ der $c \leqq x$ läßt sich zeigen, daß

$$z \leqq x \cdot e^{-k\sqrt{\log x \cdot \log\log x}}$$

für $K = 2^{-\frac{1}{2}} - \varepsilon$. Der Beweis besteht aus zwei Teilen. Der erste fußt auf Sätzen über Carmichael-Zahlen von N. G. W. H. Beeger (On composite numbers n for which $a^{n-1} \equiv 1 \pmod n$ for every a prime to n, Scripta math. *16*, 133—135 (1950)) und H. J. A. Duparc (On Carmichael numbers, Simon Stevin *29*, 21—24 (1951/52)), der zweite beruht auf einer Veröffentlichung von N. G. de Bruijn (On the number of positive integers $\leqq x$ and free of prime factors $> y$, Indagationes math. *13*, 50—60 (1951)).

WIEN, KRICHUBERG. 12.

THE CLASS-NUMBER OF THE SYMMETRIC SEMIGROUP

ALEXANDER MURRAY MACBEATH

By analogy with the definition of the symmetric group as the group of one-one mappings of a set on itself, we define the *symmetric semigroup* Σ_n of order n as the semigroup consisting of all one-valued mappings into itself of a set consisting of n objects.

If $f, g \in \Sigma_n$, we say that f is *equivalent* to g if there is a one-one map $s \in \Sigma_n$ such that $f = sgs^{-1}$. Our object is to study the number σ_n of equivalence-classes in Σ_n by means of the generating function

$$\sigma(x) = 1 + \sum_{n=1}^{\infty} \sigma_n x^n.$$

With each $f \in \Sigma_n$ is associated a 'diagram' consisting of n points representing the set of n elements on which f acts, each point being joined by a line-segment to its image under the map f. For a one-valued map the diagram consists of a number of cycles, and from each vertex of a cycle springs a 'root-tree' in the sense of Cayley. The diagrams of equivalent mappings are the same, apart from the numbering of the points.

It is possible to deduce from this that the desired generating function $\sigma(x)$ is closely related to the generating function $t(x) = \sum_{1}^{\infty} T_n x^n$, where T_n is the number of root-trees with n nodes (including the root). In fact we have the identity:

$$\sigma(x) = \prod_{n=1}^{\infty} (1 - t(x^n))^{-1}. \qquad (1)$$

The function $t(x)$ has been studied by Cayley, Polya, and Otter; it is known that its power series has radius of convergence $\tau = .33832...$, and that its only singularity on the circle of convergence is at $x = \tau$, in the neighbourhood of which $t(x)$ has an expansion:

$$t(x) = 1 + \sum_{n=1}^{\infty} a_n \left(1 - \frac{x}{\tau}\right)^{n/2} \quad \begin{matrix} a_1 \neq 0 \\ a_2 \neq 0 \end{matrix} \qquad (2)$$

where the coefficients a_n are complex numbers. From (1), (2) it follows that

$$\sigma(x) = b\left(1 - \frac{x}{\tau}\right)^{-\frac{1}{2}} + P\left(\left(1 - \frac{x}{\tau}\right)^{\frac{1}{2}}\right)$$

where P is a power series with non-zero radius of convergence. Hence, for some value of the constant k, we find the following asymptotic expression for σ_n:

$$\sigma_n \sim k\tau^{-n} n^{-\frac{1}{2}}. \qquad (3)$$

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INTEGER SOLUTIONS OF THE EQUATION $ax^3 + ay^3 + bz^3 = bc^3$

Louis Joel Mordell

It is shown that this equation has an infinity of integer solutions apart from the trivial ones given by $x + y = 0$, $z = c$. This is a particular case of the more general result that the equation

$$xf(x, y, z) = yg(x, y, z),$$

where f and g are polynomials of the second degree in x, y, z with integer coefficients, has in general an infinity of integer solutions.

The proof depends upon an application of the Pellian equation and will be published in the Journal of the London Mathematical Society.

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SUR UNE NOUVELLE GÉNÉRALISATION DE LA NOTION DE NILPOTENCE D'UN GROUPE

G. Pick

On définit habituellement la nilpotence d'un groupe à l'aide d'une chaîne de centres ascendentes ou à l'aide d'une chaîne de centres descendentes, les deux définitions étant équivalentes.

Pour donner une nouvelle définition nous utilisons le radical nilpotent des groupes défini pour la première fois par H. Fitting. Ce radical est le plus grand sousgroupe normal nilpotent qui, comme l'a montré Fitting, contient tous les sousgroupes normaux nilpotents du groupe G.

Nous introduirons maintenant la notion de hypercentre $Z_i(G)$ que l'on définit par ce que $Z_i(G)/Z_{i-1}(G)$ est le radical nilpotent du groupe $G/Z_{i-1}(G)$. L'hypercentre ainsi défini est caractéristique.

En même temps on introduit la notion de i-ième puissance hypercommutatorielle. On l'a défini par ce qu'elle est l'intersection de tous les sousgroupes normaux N de G pour lesquels $R_{i-1}(G)/N$ est le radical nilpotent de G/N. La puissance hypercommutatorielle est à son tour caractéristique.

Pour la première puissance hypercommutatorielle on a

$$R_1(G/N) = \frac{R_1(G) \cup N}{N} \cong \frac{R_1(G)}{(R_1G) \cap N}.$$

En outre on a en général

$$R_1 R_{i-1}(G) \subset R_i(G).$$

Partant de ces notions on définit la hypernilpotence. On dira qu'un groupe est hypernilpotent s'il existe un nombre naturel k tel que $R_k(G) = e$. Pour l'hypernilpotence il y a des propriétés analogues à la nilpotence. Par exemple on démontre que si la chaîne des puissances hypercommutatorielles est terminée par e alors la chaîne des hypercentres se termine par G et réciproquement.

Enfin on peut montrer que les sousgroupes normaux hypernilpotents d'un groupe G forment un treilli.. En effet, il est presque évident que l'intersection de deux sousgroupes normaux hypernilpotents l'est de même. Il reste, pour démontrer la proposition, de montrer que la réunion de deux sousgroupes A et B normaux hypernilpotents l'est de même. On le déduit du fait que R_i $(A \cup B)$ est contenu dans

$$R_i(A \cup B) \subseteqq R_i(A) \cup [R_{i-1}(A) \cap B] \cup [R_{i-2}(A) \cap R_1(B)] \dots$$
$$\dots \cup [A \cap R_{i-1}(B)] \cup R_i(B).$$

On déduit d'ici que dans le cas des groupes infinis un ensemble de sousgroupes normaux, constituant un treilli, contenant l'identité et avec limitation maximale pour les sousgroupes normaux contient toujours un radical hypernilpotent U. On peut montrer dans ce cas qu'il existe toujours un nombre naturel k tel que $Z_k(G) = U$ soit le radical nilpotent du groupe.

Une conséquence importante de ces théorèmes est que tout sousgroupe résoluble est hypernilpotent.

Tenant compte du fait qu'il y a comme l'a montré Tchernicov ou Tobvin, des p-groupes qui n'ont pas de centre, il résulte que la définition de la résolubilité à l'aide de l'hypernilpotence nous peut donner la possibilité de généraliser la notion de résolubilité.

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SUR UN ENSEMBLE FERMÉ D'ENTIERS ALGÉBRIQUES

CHARLES PISOT

On sait depuis les travaux de M. Salem que l'ensemble S des entiers algébriques $\theta > 1$, dont tous les conjugués autres que θ sont en module strictement inférieurs à un, est un ensemble fermé. M. Siegel avait déterminé les nombres de S inférieurs à $\sqrt{2}$ (il y en a deux, ce sont les zéros des polynomes $z^3 - z - 1$ et $z^4 - z^3 - 1$) et il a suggéré que $\theta_0 = (1 + \sqrt{5})/2$ était le plus petit point limite de S. Avec M. Dufresnoy, nous avons établi des conditions nécessaires et suffisantes que doivent remplir les coefficients a_n d'un développement $1 + a_1 z + \ldots + a_n z^n + \ldots$ pour que la somme représente une fonction méromorphe, bornée en module par un sur $|z| = 1$ et ayant dans $|z| < 1$ un seul pôle simple. Ces conditions nous ont permis de déterminer tous les nombres de S, inférieurs à une valeur légèrement supérieure à θ_0. Ces nombres θ se rangent, sans omission, dans l'ordre suivant:

Soient θ'_n zéro de $(z^2 - z - 1)z^n + z^2 - 1$, θ''_n zéro de $(z^2 - z - 1)z^n + 1$, θ'^*_n zéro de $(z^2 - z - 1)z^n - z^2 + 1$, θ''^*_n zéro de $(z^2 - z - 1)z^n - 1$, enfin θ^{**} zéro de $z^6 - 2z^5 + z^4 - z^2 + z - 1$, alors on a: $1 < \theta'_1 < \theta'_2 < \theta'_3 < \theta''_3 < < \theta'_4 < \theta''_4 < \theta'_5 < \theta^{**} < \theta''_5 < \ldots < \theta''_{n-1} < \theta'_n < \theta''_n < \ldots < \theta_0 < \ldots < < \theta''^*_n < \theta'^*_{n+1} < \theta''^*_{n-1} < \ldots < \theta''^*_{16}$. Il y a une infinité de ces nombres, et θ_0 en est leur unique point d'accumulation.

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VERALLGEMEINERTE ASYMPTOTISCHE DICHTEN

HANS ROHRBACH und BODO VOLKMANN

Der Dichtesatz von M. Kneser für die asymptotische Dichte (Math. Zeitschr. *58*, 1953) lässt sich durch Einführung von Gewichten verallgemeinern, wie sie J. G. van der Corput (Proc. Ak. Wet. Amsterdam *50*, 1947) bei seiner Erweiterung des Mannschen Dichtesatzes für die finite Dichte betrachtet hat. Es sei φ_x eine für alle ganzzahligen x definierte reelle monoton nicht abnehmende Funktion, $\varphi_x = 0$ für $x < 0$ und $\varphi_x > 0$ für $\varphi(x) = \varphi_{-[-x]}$ die zugehörige Treppenfunktion (x reell), insbesondere also $\varphi_i = \varphi(i)$ für $i = 0, 1, 2, \ldots$ Ferner sei bei ganzzahligem x für eine Menge $\mathfrak{A} = \{a_1, a_2, \ldots\}$ nichtnegativer ganzer Zahlen mit e_i als charakteristischer Funktion

$$ {}_{\varphi}A(x) = \sum_{i=1}^{x} e_i \varphi_i, \quad \Phi(x) = \sum_{t=1}^{x} \varphi(t). $$

Ist dann $\mathfrak{Z}$ die Menge der natürlichen Zahlen, so heisst

$$ D_\varphi(\mathfrak{A}) = \varliminf_{x\to\infty} \frac{{}_{\varphi}A(x)}{{}_{\varphi}Z(x)} = \varliminf_{x\to\infty} \frac{1}{\Phi(x)} \sum_{\substack{a \leq x \\ a \in \mathfrak{A}}} \varphi(a) $$

eine (durch die Gewichte φ_i) *verallgemeinerte asymptotische Dichte* von $\mathfrak{A}$. Für $\varphi(x) \equiv 1$ ergibt sich die gewöhnliche Anzahlfunktion $A(x)$ und asymptotische Dichte $D(\mathfrak{A}) = \varliminf_{x\to\infty} \frac{1}{x} A(x)$ von $\mathfrak{A}$.

Wir beschränken uns auf Funktionen $\varphi(x)$, die den folgenden Bedingungen genügen:

(1) Für jede reelle Konstante c sei $\lim_{x\to\infty} \varphi(x + c)/\varphi(x) = 1$.

(2) Für jede Menge der Form $\mathfrak{A}h = \{a_1 h, a_2 h, \ldots\}$ sei

$$ D_\varphi(\mathfrak{A}h) = 1/hD_\varphi(\mathfrak{A}) \qquad (h \text{ ganz}). $$

(3) Für alle ganzzahligen x sei $\varphi^2(x + 1) \geqq \varphi(x) \cdot \varphi(x + 2)$.

Dann gilt zwar, dass für pseudorationale Mengen $\mathfrak{A}$ im Sinne von B. Volkmann (Journ. f. Math. *190*, 1952) stets $D_\varphi(\mathfrak{A}) = D(\mathfrak{A})$ ist, aber es lässt sich durch Beispiele (etwa $\varphi(x) = x^\alpha$, $\alpha > 0$) zeigen, dass $D_\varphi(\mathfrak{A}) \neq D(\mathfrak{A})$ ist bei passendem $\mathfrak{A}$. Bezeichnet man nun für $n \geqq 1$ Mengen $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ und jedes $x \geqq 0$ mit $a_i(x)$ das kleinste Element $a \geqq x$ von $\mathfrak{A}_i$, ist ferner $\tilde{x} = \max(a_1(x), \ldots, a_n(x))$ und

$$ \psi(\mathfrak{A}_1, \ldots, \mathfrak{A}_n; \varphi) = \varlimsup_{x\to\infty} \frac{\varphi(\tilde{x})}{\varphi(x)}, $$

so gilt der folgende verallgemeinerte Dichtesatz:

Sind $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ $(n \geqq 1)$ Mengen nichtnegativer ganzer Zahlen mit der Summe $\mathfrak{A} = \sum_{i=0}^{n} \mathfrak{A}_i$, für die bei geeigneter Numerierung $\psi(\mathfrak{A}_1, \ldots, \mathfrak{A}_n; \varphi) < \infty$ ist, so ist entweder

$$D_\varphi(\mathfrak{A}) \geqq \varliminf_{x\to\infty} \frac{1}{\Phi(x)} \sum_{i=0}^{n} {}_\varphi A_i(x)$$

oder es gibt eine ganze Zahl g derart, dass $\mathfrak{A} \sim \mathfrak{A}^G = \sum_{i=0}^{n} \mathfrak{A}_i^g$ und

$$D_\varphi(\mathfrak{A}) \geqq \varliminf_{x\to\infty} \frac{1}{\Phi(x)} \sum_{i=0}^{n} {}_\varphi A_i^g(x) - \frac{n}{g}$$

ist. Dabei bedeutet $\mathfrak{A}_i^g$ die Menge aller m, für die es ein $a_i \in \mathfrak{A}_i$ mit $m \equiv a_i$ (mod g) gibt, ferner die Äquivalenz $\mathfrak{A} \sim \mathfrak{B}$, dass $\mathfrak{A}$ und $\mathfrak{B}$ bis auf höchstens endlich viele Elemente übereinstimmen.

Der Beweis lässt sich im wesentlichen wie der des Kneserschen Dichtesatzes erbringen. Eine ausführliche Darstellung erscheint im Journal für die reine und angewandte Mathematik *194* (1954/55).

MAINZ.

SUR LE PREMIER CAS DU (DERNIER) THÉORÈME DE FERMAT

B. SPYRIDON SARANTOPOULOS

En m'occupant avec le premier cas du théorème de Fermat j'ai obtenu plusieurs résultats, dont nous donnons ici les suivants:

1. Si le nombre μ est premier et impair $(\mu > 2)$ et il existe des entiers positifs x, y, z, premiers deux à deux, non divisible par μ, satisfaisant à l'équation du Fermat

$$x^\mu + y^\mu = z^\mu, \tag{1}$$

ou, plus généralement, à la congruence

$$x^\mu + y^\mu \equiv z^\mu(\mu^2) \tag{2}$$

et si $\varepsilon, \eta, \vartheta$ sont les restes correspondants de la division de x, y, z par μ, alors les congruences

$$\varphi(\alpha) = (\alpha + 1)^\mu - \alpha^\mu - 1 \equiv 0(\mu^2), \ \eta \equiv \alpha\varepsilon(\mu), \ \vartheta \equiv (\alpha + 1)\varepsilon(\mu)$$

seront satisfaites à même temps par une seule valeur de α appartenant à la suite $1, 2, 3, \ldots, \mu - 2$. Et *inversement*, ε étant un nombre entier positif quelconque inférieur à $\mu - 1$.

2. Excepté les valeurs $\omega = \mu - 1$ et $\omega = 0$, on voit que les congruences

$$\varphi(\omega) \equiv 0 \quad \text{et} \quad \sigma(\omega) = \frac{\varphi(\omega)}{\mu\omega(\omega+1)} \equiv 0(\mu)$$

ont les mêmes racines. La recherche donc des racines de $\sigma(\omega) \equiv 0(\mu)$ est un sujet intéressant, puisque la solution de la congruence (2) dépend de ces racines. Cette recherche nous a conduit à montrer qu'il existe les identités suivantes:

$$(3) \quad \begin{aligned} &(\omega^{\mu-1} - 1)F(\omega) + \sigma(\omega)\varphi(\omega) = \frac{2^{\mu-1}-1}{\mu}\varphi(1) \quad (\text{si } \mu = 6k-1) \\ &(\omega^{\mu-1} - 1)F(\omega) + \sigma(\omega)\varphi_1(\omega) = \frac{2^{\mu-1}-1}{\mu}\varphi_1(1)(\omega^2+\omega+1), \ (\text{si } \mu = 6k+ \end{aligned}$$

$F(\omega)$, $\varphi(\omega)$, ainsi que $F_1(\omega)$ et $\varphi_1(\omega)$ désignant des polynomes fixes à coefficients entiers et des degrés $\mu - 4$, $\mu - 2$, $\mu - 6$, $\mu - 4$ respectivement.

3. Si $\varphi(1) \not\equiv 0(\mu)$, $(\mu = 6k - 1)$, ou $\varphi_1(1) \not\equiv 0(\mu)$ $(\mu = 6k + 1)$ et $\sigma(\omega) \equiv 0(\mu)$ a une racine quelconque $\alpha (1 \leqq \alpha \leqq \mu)$, on en conclut tout de suite à l'aide des identités (3) que $\frac{2^{\mu-1}-1}{\mu}$ est divisible par μ. Ainsi on a une expression du théorème de Wieferich plus générale.

4. Si $\varphi(1) = \mu$, $(\mu = 6k - 1)$, ou $\varphi_1(1) = \mu$ $(\mu = 6k + 1)$ la congruence $\sigma(\omega) \equiv 0(\mu)$ aura nécessairement une racine au moins. Plus précisément:

5. Si μ est de la forme $\mu = 6k - 1$ et $\frac{2^{\mu-1}-1}{\mu}$ est divisible par μ, la congruence $\sigma(\omega) \equiv 0(\mu)$ aura les racines $1, \frac{\mu-1}{2}, \mu - 1$. Toutes les autres racines, si elles existent, constituent des groupes des six éléments chacun. Si α désigne une racine d'un de ces groupes, le groupe contiendra six racines de la forme

$$\alpha, \mu - 1 - \alpha, \frac{\mu\lambda - 1}{1+\alpha}, \frac{\mu(1+\alpha-\lambda)}{1+\alpha}, \frac{\mu\eta - (1+\alpha)}{\alpha}, \frac{\mu(\alpha - \eta)+1}{\alpha}$$

λ et η étant des entiers positifs, les plus petits, de sorte que les fractions soient réduites aux entiers. Le nombre total des racines de $\sigma(\omega) \equiv 0(\mu)$ ne peut pas dépasser $\mu - 8$.

6. Si $\mu = 6k + 1$ les conclusions du numéro précédent se trouvent en vigueur, mais $\sigma(\omega) \equiv 0(\mu)$ a de plus comme racines les deux racines de la congruence

$$\omega^2 + \omega + 1 \equiv 0(\mu).$$

7. En admettant que les théorèmes de Wieferich et de Marimanoff sont en vigueur pour la relation (1), on voit qu'on doit avoir $\mu \geqq 23$ si $\mu = 6k - 1$ et $\mu \geqq 31$ si $\mu = 2k + 1$.

Il en résulte tout de suite que pour $\mu = 5, 7, 11, 13, 19$ un au moins des théorèmes de Wieferich et Marimanoff, et par suite la relation (1), ne peut être en vigueur.

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CHARACTERS OF COMMUTATIVE SEMI-GROUPS

Štefan Schwarz

By a character of a semi-group S we mean a complex-valued function $\chi(x)$, satisfying the relation $\chi(a) \cdot \chi(b) = \chi(ab)$ for every $a, b \in S$. The problem is to find the structure of the semi-group S^* of all characters of S.

Suppose first that S is finite. Then S can be written as a sum of disjoint sets $S = \sum_e P_e$, where P_e is a semi-group containing one and only one idempotent e. Let G_e be the maximal subgroup of S with e as unity element.

The semi-group S^* can be written as a sum of disjoint groups $S = \sum_J G_J$. The group G_J (with the idempotent ε_J) is the set of all characters vanishing on an arbitrarily chosen prime ideal J of S.

The idempotents of S and those of S^* can be partially ordered in an obvious manner. The „gross structure" of S^* is as follows. The partially ordered set of non-zero idempotents of S^* and the semi-lattice of idempotents $\in S$ are anti-isomorphic. If in this correspondence $e \longleftrightarrow \varepsilon_J$ holds, then $G_J^* \cong G_e$. The „fine structure" is (with obvious notations) essentially given by means of the following statement: if $e_2 < e_1$ (hence $\varepsilon_{J_1} < \varepsilon_{J_2}$), then $G_{J_2} \cdot \varepsilon_{J_1} \cong G_1 e_2$.

Some results can (under suitable restrictions) be extended to some types of infinite semi-groups (for instance bicompact semi-groups).

There exist interesting Galois connexions between various subsets of S and S^*.

A REFINED CLASSIFICATION OF SEMI-GROUPS LEADING TO GENERALIZED POLYNOMIAL RINGS WITH A GENERALIZED DEGREE CONCEPT

DOV TAMARI

In the Bull. Amer. Math. Soc. 54 (1948), 153 ([1]), we gave a certain classification of rings and semi-groups. Here, a more refined classification of semi-groups, motivated by the immersion problem of semi-group-rings is based on the theorem on systems of linear equations and the enumeration principle applied in our paper on Birkhoff-Witt rings, Proc. Amer. Math. Soc. 4 (1953), 197 ([2]). Certain order properties of the semi-group permit a closer generalization of polynomial rings by the use of a generalized degree concept.

Let $\mathfrak{R}(\mathfrak{S})$ be a semi-group-ring („algèbre de monoïde" of Bourbaki, Algèbre I), $\mathfrak{R}$ a ring regular on the right (say) and $\mathfrak{S}$ a (cancellation) semi-group. The necessary and sufficient condition for the immersibility of $\mathfrak{R}(\mathfrak{S})$ into a field of *right quotients* A/B $(= AB^{-1}, A, B \in \mathfrak{R}(\mathfrak{S}))$ is the transfer of right regularity from $\mathfrak{R}$ to $\mathfrak{R}(\mathfrak{S})$, i.e.: (1) $\mathfrak{R}(\mathfrak{S})$ has no divisors of zero; (2) any two $A, B \in \mathfrak{R}(\mathfrak{S})$ possess a (non-trivial common right multiple (C.R.M.).

We suppose (1) satisfied and ask: how must $\mathfrak{S}$ be in order to satisfy (2)? The found property of $\mathfrak{S}$ is independent of $\mathfrak{R}$ and the limiting case (conjunction) of an infinity of properties $\{\mathfrak{P}^r_j\}$ $j = 2, 3, \ldots$, restricting more and more the (considered class of semi-groups. Denote $|\mathfrak{a}|$ the cardinal number of the (finite) complex $\mathfrak{a} \subset \mathfrak{S}$ (only finite complexes of $\mathfrak{S}$ will be considered); $\mathfrak{S}$ *has the property* $\mathfrak{P}^r_\infty$, *resp.* $\mathfrak{P}^r_j$, *if, and only if, to every* $\mathfrak{a}$, *resp. to every* $\mathfrak{a}$ *with* $|\mathfrak{a}| \leqq j$, *one can find a complex* $\mathfrak{x}$ *such that* $|\mathfrak{a}\mathfrak{x}| < 2|\mathfrak{x}|$ ($|\mathfrak{a}\mathfrak{x}| \leqq |\mathfrak{a}||\mathfrak{x}|$ is trivial in every well defined multiplicative system; $\mathfrak{P}^r_2$ expressing existence of C.R.M.'s, only such semi-groups of [1] are considered). Exemples show that for $j < k \leqq \infty$ $\mathfrak{P}^r_j$ does not imply $\mathfrak{P}^r_k$. Similarly we define a left classification principle $\{\mathfrak{P}^l_i\}$, $i = 2, 3, \ldots$ by $|\mathfrak{x}\mathfrak{a}| < 2|\mathfrak{x}|$. Left and right properties being independent, a semi-group is *exactly of type* (i, j) if it satisfies $\mathfrak{P}^l_i$ and $\mathfrak{P}^r_j$, but neither $\mathfrak{P}^l_{i+1}$ (if $i < \infty$) nor $\mathfrak{P}^r_{j+1}$ (if $j < \infty$).

Any ascending sequence of complexes $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots \subset \mathfrak{a}_n \subset \ldots$ with $\cup \mathfrak{a}_n = \mathfrak{S}$ is a *graduation* of $\mathfrak{S}$: whatever i and j, there exist k with $\mathfrak{a}_i\mathfrak{a}_j \subset \mathfrak{a}_k$. Denote $K(i, j)$ the smallest such k. The graduation *exhibits* the property $\mathfrak{P}^r_\infty$ (say) of $\mathfrak{S}$ if, whatever i, there exist j with $|\mathfrak{a}_{k(i,j)}| < 2|\mathfrak{a}_j|$. If, in particular,

$$\lim_{j=\infty} \gamma_j \left(\equiv \frac{|\mathfrak{a}_{j+1}|}{|\mathfrak{a}_j|}\right) = 1 \text{ it suffices to choose a } j \text{ with } \gamma_{j+p} < 2^{1/(k(i,j)-j)}$$

whatever $p = 1, 2, 3, \ldots$ Choose, e.g., a generating set $\mathfrak{g}$ of $\mathfrak{S}$ and put

$\mathfrak{a}_i = \bigcup_{\nu=}^{i} \mathfrak{g}^{\nu}$ ($\mathfrak{g}^0 = \varnothing$ or $\{e\}$). $\mathfrak{S}$ need not be finitely generated: all finite complexes are in finitely generated subsemi-groups with corresponding graduations, all of which might exhibit $\mathfrak{P}^r_\infty$. $k(i, j)$ becomes a subadditive function and $\gamma_{j+p} < 2^{1/i}$ will do.

These methods can succeed for more general algebras $\mathfrak{R}(\mathfrak{S})/\mathfrak{N}$, $\mathfrak{N}$ a two-sided ideal in $\mathfrak{R}(\mathfrak{S})$ generated by the left sides of the defining relations $\Sigma \varrho_i s_i \doteq 0$ ($\varrho_i \in \mathfrak{R}$, $s_i \in \mathfrak{S}$), as shown by the special case of Lie algebras treated in [2].

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CONSIDERAZIONI SUGLI ELEMENTI MODULARI IN UN p-GRUPPO

GIOVANNI ZACHER

Sia L un reticolo (lattice). Un elemento di L si dirà, secondo Zappa, un elemento modulare di L se gode della seguente proprietà: Detto R un qualsiasi sottoreticolo modulare di L, il reticolo $\{m, R\}$ generato da m ed R risulta ancora modulare.

Se L é un reticolo in cui ogni sottoreticolo modulare è contenuto in un reticolo modulare massimo, allora gli elementi modulari formano un sottoreticolo modulare che coincide coll'intersezione di tutti i sottoreticoli modulari massimi di L.

Applicando il concetto di elemento modulare al reticolo $L(G)$ di un p-gruppo G d'ordine p^{α}, si dimostra che ogni elemento modulare di $L(G)$ dev'essere un sottogruppo permutabile con ogni sottogruppo di G; la condizione evidentemente non é sufficiente.

Se G è un p-gruppo (d'ordine finito) di esponente p, se H è un sottogruppo proprio di G, ed è un elemento modulare in $L(G)$, il gruppo G ha il reticolo $L(G)$ necessariamente modulare, vale a dire G è un p-gruppo abeliano elementare. Se G è un p-gruppo di esponente p^{ν}, con $\nu > 1$, l'esistenza di un elemento modulare in $L(G)$ che sia sottogruppo proprio di G non comporta che $L(G)$ sia modulare. Si da un esempio.

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Section II

ANALYSIS

SHORT LECTURES

SUR LA CARACTÉRISATION D'UNE CLASSE DE FONCTIONS DÉRIVABLES PAR LEURS SÉRIES DE FOURIER

GEORGES ALEXITS

Soit $f(x)$ une fonction continue périodique définie dans l'intervalle $(0, 2\pi)$ et $\tilde{f}(x)$ sa fonction conjuguée. Désignons par $\sigma_n^\alpha(x)$ la n-ième moyenne (C, α) de la série de Fourier de $f(x)$ et par $\tilde{\sigma}_n^\alpha(x)$ la moyenne correspondante de la série conjuguée.

En ce qui concerne l'ordre de grandeur de l'approximation d'une fonction dérivable, nous pouvons démontrer le théorème suivant:

I. *Si la dérivée $f'(x_0)$ existe à un point x_0, alors*

$$\tilde{f}(x_0) - \tilde{\sigma}_n^\alpha(x_0) = O\left(\frac{1}{n}\right)$$

pour tout $\alpha > 1$.

Parmi les fonctions dérivables en certains points,nous pouvons caractériser une classe assez étendue des fonctions presque partout dérivables:

II. *Pour que la fonction continue $f(x)$ ait presque partout une dérivée L^p-intégrable $(p > 1)$, il faut et il suffit que, pour tout $\alpha > 0$ on ait*

$$| f(x) - \sigma_n^\alpha(x) | \leqq \frac{F(x)}{n}$$

où $F(x)$ est une fonction L^p-intégrable.

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LA CONVERGENCE PRESQUE PARTOUT DES DÉVELOPPEMENTS DE FONCTIONS CONTINUES EN SÉRIES DE POLYNOMES ORTHOGONAUX

G. ALEXITS

Il y a peu de temps que nous avons démontré que la série de Fourier d'une fonction $f \in L^2$ est presque partout convergente dans un intervalle (a, b), lorsque le module de continuité quadratique

$$\omega_2(\delta, f; a, b) = \sup_{|h| \leq \delta} \left\{ \int_a^b [f(x+h) - f(x)]^2 dx \right\}^{1/2}$$

est de l'ordre de grandeur $O(1/\sqrt{\lambda(1/\delta)})$ où $\lambda(x)$ désigne une fonction monotone croissante satisfaisant à la condition

$$\int_1^\infty \frac{dx}{x\lambda(x)} < \infty. \tag{1}$$

En même temps, S. B. Stečkin a également démontré ce théorème, indépendamment de nous.

En ce qui concerne les développements en séries de polynomes orthogonaux, nous pouvons démontrer le théorème suivant: Soit $w(x)$ une fonction intégrable dans l'intervalle $(-1, 1)$ satisfaisant à la condition $0 < w(x) = O((1 - x^2)^{-1/2})$ et $\{p_n(x)\}$ le système complet de polynomes orthogonaux déterminé par la fonction de poid $w(x)$. Soit

$$\omega(\delta, f; a, b) = \sup_{\substack{|h| \leq \delta \\ a \leq x \leq b}} | f(x+h) - f(x)|$$

le module de continuité de $f(x)$ dans l'intervalle (a, b).

Si les polynomes $\{p_n(x)\}$ *sont bornés dans leur ensemble dans l'intervalle* (a, b) *situé à l'intérieur de* $(-1, 1)$ *et*

$$\omega(\delta, f; a, b) = O\left(\frac{1}{\sqrt{\lambda(1/\delta)}}\right)$$

où $\lambda(x)$ *satisfait à la condition* (1), *alors le développement*

$$f(x) \sim \sum_{n=0}^{\infty} c_n p_n(x)$$

converge presque partout dans (a, b).

Les polynomes orthogonaux classiques sont bornés dans leur ensemble dans tout intervalle (a, b) intérieur à $(-1,1)$. En prenant donc $\lambda(x) = (\log x)^{2\alpha}$ avec $\alpha > 1/2$, on obtient le corollaire suivant:

Les développements classiques d'une fonction satisfaisant à une condition de Dini-Lipschitz

$$| f(x+h) - f(x) | = O\left((\log 1/h)^{-\alpha}\right)$$

avec $\alpha > 1/2$ *convergent en* $(-1, 1)$ *presque partout.*

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GEOMETRICAL NOTIONS IN BANACH ALGEBRAS

H. F. Bohnenblust and Samuel Karlin

In the Banach algebras which are considered, only such norms are admitted, which satisfy the conditions $||u|| = 1$ and $||xy|| \leqq ||x|| \, ||y||$. A study of the geometry of Banach algebras as it is determined by the norm reveals certain distinctive features of the unit sphere, which in some cases allow to characterize algebraical notions of the algebra purely in terms of geometrical properties, and lead to direct intuitive proofs of properties of mappings of algebras into algebras. It is shown that the unit element of a Banach algebra is always a vertex of the unit sphere, where the notion of vertex must be modified, of course, since the vector space is complex rather than real. Commutative algebras can always be renormed, the union of all admissible unit spheres fill out the full "cylinder" $|M(x)| = 1$ in the direction of the radical which is determined by the maximal ideals M. These results are used, in particular, to show that a commutative star algebra is symmetric if and only if the convex spanned by the maximal ideals coincide with the set of positive functionals normalized to be equal to one at the unit element. Further applications to C-star-algebras show that the involution $x \to x^*$ is uniquely determined by the norm and the more important result that the normalized positive functionals coincide with the planes of support at the unit element. In turn this result makes it evident that a linear mapping of norm one of a C-star-algebra into another one preserves positivity if it transforms the unit element into the unit element.

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SUR L'ALLURE DES FONCTIONS HARMONIQUES À LA FRONTIÈRE

M. Brelot

On sait qu'une fonction surharmonique $u > 0$ sur un domaine euclidien Ω admet en un point-frontière irrégulier Q une pseudo-limite. Dans le cas de u harmonique, on peut préciser, au moins si u est bornée. Introduisons, pour une suite $M_n \to Q$ et appartenant à Ω pour n assez grand, la condition d'être „maximale", c'est-a-dire que pour une fonction de Green G_{p_0}, $G_{p_0}(M_n)$ tende vers la lim. sup. de G_{p_0} en Q. Cela est indépendant de $p_0 \in \Omega$, et d'une altération du domaine hors d'un voisinage de Q. Alors une fonction harmonique bornée u

au voisinage de Q sur Ω admet sur toute suite maximale une même limite (égale à la pseudo-limite). Cela suggère en particulier une étude dans les espaces de Green au voisinage d'un point de la frontière idéale de Martin où la fonction de Green aurait une lim. sup. > 0.

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SUBHARMONIC FUNCTIONS IN SEVERAL COMPLEX VARIABLES

Hans Joachim Bremermann

A series of the form $\sum_{\nu=0}^{\infty} a_\nu(z)w^\nu$, where w, z are complex variables and the $a_\nu(z)$ are functions holomorphic in a domain D of the z-plane converges uniformly in a domain $\{(z, w)|z \in D \Lambda | w | < R(z)\}$. The function -log $R(z)$ has the property to be subharmonic in D. This result was discovered by F. Hartogs in 1906 (Math. Ann., vol. 62). Later subharmonic functions were studied for their own sake and the origin was more or less forgotten. Applications to several complex variables were made again after a subclass of the general subharmonic functions was introduced for n complex $= 2n$ real variables by K. Oka (Tohôku Math. J., vol. 49 (1942)) who called them "pseudoconvex functions" and independently by P. Lelong (Compt. Rend. 215 (1942) and 216 (1943), also Ann. Scient. Ec. Norm. Sup., vol. 62 (1945)) who called them "plurisubharmonic functions". (The "Hartogs functions" (Bocher-Martin, Several Complex Variables) are closely related). Let $\mathfrak{z} = (z_1 \ldots z_n)$ be n complex variables. Then $V(\mathfrak{z})$ is called "plurisubharmonic" (plsh.) if $-\infty \leqq V(\mathfrak{z}) < +\infty$ and if $V(\mathfrak{z})$ is upper semicontinuous and if the Hermitean differential form $\sum_{\mu,\nu=1}^{n} \frac{\partial^2 V}{\partial z_\mu \partial \bar{z}_\nu} dz_\mu d\bar{z}_\nu$ (the derivatives in the sense of L. Schwartz) is positive semi-definite. A series $\sum_{\nu=1}^{\infty} a_\nu(\mathfrak{z})w^\nu$ (the $a_\nu(\mathfrak{z})$ holomorphic in a domain $D \subset C^n$) converges uniformly in a domain $\{\mathfrak{z}|\mathfrak{z} \in D \Lambda |w| < R(\mathfrak{z})\}$. The function -log $R(\mathfrak{z})$ has the property to be plsh. in D. (Bremermann, Thesis Muenster 1951).

Let $\delta_D(\mathfrak{z})$ denote the distance of the point $\mathfrak{z}$, $\mathfrak{z} \in D$, from the boundary $\overline{D} - D$, then -log $\delta_D(\mathfrak{z})$ is a plsh. function if and only if D is a domain of holomorphy. (K. Oka. l.c., Bremermann, l.c. and Math. Ann., vol. 128 (1954) and "Complex Convexity", Technical Report, Stanford 1954). (For domains of holomorphy see Behnke-Thullen, Ergebn. d. Math., vol. 3 no. 3 (1934)).

The domains for which -log $\delta_D(\mathfrak{z})$ is plsh. are called "pseudoconvex domains". There is a close correspondence between convex domains and pseudo-convex domains and convex functions and plsh. functions.

1) Formal: A function f of n real variables is convex if and only if $\sum_{\mu,\nu=1}^{A} \frac{\partial^2 f}{\partial x_\mu \partial x_\nu} dx_\mu dx_\nu$ is positive semidefinite; and a domain in real space is convex if and only if -log $\delta(x_1 \ldots x_n)$ is a convex function. 2) Every theorem that is valid for all pseudo-convex domains and plsh. functions (and not involving "existence" quantification) is true in a corresponding form for convex domains and functions. (Bremermann, Complex Convexity, l.c. Compare also P. Lelong, Journ. de Math., vol. 31 (1952). This result is obtained by the aid of tube domains (introduced by Bochner-Martin, l.c.). The Bergman kernel function is a special plsh. function. Plsh. functions can succesfully be used as a new "extended class" to solve boundary value problems with boundary values prescribed on the "distinguished boundary surface" of an "analytic polyhedron" (unpublished). Further applications see: P. Lelong, Colloque sur les fonctions de plusieurs variables, Bruxelles 1953 pp. 21-40.

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SOPRA UNA FORMA PIÙ AMPIA DEL PROBLEMA DI CAUCHY PER I SISTEMI DI EQUAZIONI A DERIVATE PARZIALI DEL PRIMO ORDINE

SILVIO CINQUINI

In due Memorie in collaborazione con altro autore (1) ci siamo occupati di una forma più ampia del problema di Cauchy relativo all'equazione a derivate parziali

$$p(x, y) = f(x, y, z(x, y), q(x, y)), \tag{a}$$

$\left(\text{ove } p(x, y) = \frac{\partial z}{\partial x}, \ q(x, y) = \frac{\partial z}{\partial y}\right)$ e abbiamo stabilito sia teoremi di esistenza sia teoremi di unicità, i quali forniscono un'estensione, all'equazione (a), di quell'ordine di idee che C. Carathéodory ha sviluppato per le equazioni differenziali ordinarie.

Si tratta ora di introdurre tale ordine di idee nello studio dei sistemi di equazioni a derivate parziali del tipo

(b) $p_j(x,y)=f_j(x,y,z_1(x,y),\ldots,z_r(x,y),q_1(x,y),\ldots,q_r(x,y)),\ (j=1,2,\ldots,r),$

cominciando a occuparci dell'unicità delle soluzioni del problema di Cauchy relativo a tale sistema.

Tale questione sembra presentare difficoltà piuttosto serie, perchè, anche se ci limitiamo alla teoria classica e prendiamo in considerazione quelle ricerche di A. Haar (2) che a noi paiono particolarmente significative, troviamo un risultato relativo a un sistema lineare (vale a dire di primo grado nel complesso delle derivate parziali), il quale è stato raggiunto sotto condizioni di derivabilità che sono ben lontane da quella ampiezza di ipotesi che è già stata realizzata per l'equazione (a).o

Ciò premesso, per proseguire second l'ordine di idee che abbia mo già sviluppato nei riguardi dell'equazione (a), è da supporre per le funzioni $f_j(x, y, z_1, \ldots, z_r, q_1, \ldots, q_r)$, che figurano nei secondi membri delle equazioni del sistema (b), soltanto una condizione di Carathéodory, relativa alle variabili $z_1, \ldots, z_r, q_1, \ldots, q_r$ con coefficienti che sono funzioni della variabile x quasi continue e integrabili secondo Lebesgue, mentre come soluzioni del sistema (b) si considerano r-ple di funzioni $z_1(x, y), \ldots, z_r(x, y)$, ognuna delle quali: 1⁰) è continua assieme alla propria derivata parziale del primo ordine rispetto a y; 2⁰) per ogni y fissato è funzione assolutamente continua della sola x; e si intende che su ogni parallela all'asse x le r-ple considerate soddisfano al sistema (b) per quasi tutti i valori di x.

Nella presente comunicazione ci limitiamo a considerare il caso particolare, in cui nel secondo membro di ciascuna delle equazioni (b) compare una sola derivata parziale rispetto alla variabile y, e precisamente quella della stessa funzione, di cui figura nel rispettivo primo membro la derivata parziale rispetto a x, vale a dire i sistemi della forma

$$p_j(x, y) = f_j(x, y, z_1(x, y), \ldots, z_r(x, y), q_j(x, x)),\ (j = 1, 2, \ldots, r);$$

per ragioni di spazio l'enunciato del teorema di unicità, che abbiamo rilevato, viene dato in altra pubblicazione (3) assieme alla sua dimostrazione. Questa è basata su un sistema di disuguaglianze, il quale viene studiato, facendo appello soltanto a delicate considerazioni di teoria delle funzioni di variabile reale, mediante un procedimento approssimativo che abbiamo ripetutamente sviluppato nel caso dell'equazione (a) e che viene ulteriormente raffinato.

[1] M. Cinquini Cibrario e S. Ginquini. Annali di Matematica T. XXXII (1951), pp. 121—155.

S. Cinquini. Rend. Accademia naz. dei Lincei vol. XI (1951), pp. 255—259.

M. Cinquini Cibrario e S. Cinquini. Annali Scuola Normale superiore di Pisa vol. VI (1952), pp. 187—243.

[2] A. HAAR. Atti del Congresso internazionale dei matematici di Bologna (1928), T. III, pp. 5—10.
[3] S. CINQUINI. In corso di stampa nei Rend. Accademia naz. dei Lincei.

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UNA ESTENSIONE NELLO STUDIO DEI SISTEMI DI EQUAZIONI A DERIVATE PARZIALI

MARIA CINQUINI CIBRARIO

In una serie di lavori ho studiato a fondo i sistemi di equazioni del primo ordine a derivate parziali

$$p_i = f_i(x, y; z_1, z_2, \ldots z_m; q_1, q_2, \ldots, q_m) \quad (i = 1, 2, \ldots, m) \tag{1}$$

imponendo alle f_i opportune condizioni di derivabilità.

In due Memorie recenti, scritte in collaborazione (cfr. [1] e [2]), abbiamo cercato di estendere ad una equazione a derivate parziali

$$p = f(x, y, z, q) \tag{a}$$

l'ordine di idee introdotto da Carathéodory nello studio delle equazioni differenziali ordinarie.

Volendo estendere i risultati di tali Memorie ai sistemi (1), si incontrano notevoli difficoltà; in un primo tentativo ho considerato il caso in cui il sistema sia

$$p_i+\varrho_i(x, y, z_1, z_2, \ldots, z_m)q_i=f_i(x, y, z_1, z_2, \ldots, z_m), \quad (i=1, 2, \ldots, m), \tag{2}$$

dove le ϱ_i possono essere distinte o no. *E'* noto (cfr. [3]) che, con opportune ipotesi di derivabilità, il sistema (1) si può ricondurre al sistema (2).

Per il sistema (2) ho dimostrato un teorema di esistenza simile a quello contenuto nella Memoria [1] e relativo alle (a).

Le funzioni $\varrho_i(x, y, z_1, z_2, \ldots, z_m)$, $f_i(x, y, z_1, z_2, \ldots, z_m)$ sono supposte definite assieme alle derivate $\frac{\partial \varrho_i}{\partial y}, \frac{\partial \varrho_i}{\partial z_j}, \frac{\partial f_i}{\partial y}, \frac{\partial f_i}{\partial z_j}$ $(i, j = 1, 2, \ldots, m)$, per x variabile in un intervallo $(0, a_0)$ e per tutti i valori reali di $y, z_1, z_2, \ldots z_m$; per ogni $\bar{x}$ fissato di $(0, a_0)$ sono continue in $y, z_1, z_2, \ldots, z_m$, e per ogni $(m + 1)$ — pla fissata $(\bar{y}, \bar{z}_1, \bar{z}_2, \ldots \bar{z}_m)$ sono quasi continue in x; esistono una costante positiva

H e tre funzioni $K(x)$, $L(x)$, $\Lambda(x)$, definite in $(0, a_0)$, ivi sommabili e $\geqq 0$, tali che sia

$$(3)\quad \begin{cases} |\varrho_i(x, y, z_1, z_2, \ldots, z_m)| \leqq H, \quad |f_i(x, y, z_1, z_2, \ldots, z_m)| \leqq K(x), \\ \left|\dfrac{\partial \varrho_i}{\partial y}\right| \leqq L(x), \left|\dfrac{\partial \varrho_i}{\partial z_j}\right| \leqq L(x), \left|\dfrac{\partial f_i}{\partial y}\right| \leqq L(x), \left|\dfrac{\partial f_i}{\partial z_j}\right| \leqq L(x) \; (i, j = 1, 2, \ldots, m) \end{cases}$$

per quasi tutti gli x di $(0, a_0)$ e per ogni $(m+1)$—pla reale $(y, z_1, z_2, \ldots, z_m)$,

$$(4)\quad \left| \frac{\partial \varrho_i(x, y^{(1)}, z_1^{(1)}, z_2^{(1)}, \ldots, z_m^{(1)})}{\partial y} - \frac{\partial \varrho_i(x, y^{(2)}, z_2^{(2)}, z_2^{(2)}, \ldots, z_m^{(2)})}{\partial y} \right|$$

$$\leqq \Lambda(x) \left\{ |y^{(1)} - y^{(2)}| + \sum_{j=1}^{m} |z_j^{(1)} - z_j^{(2)}| \right\},$$

per quasi tutti gli x di $(0, a_0)$ e per tutte le coppie di $(m+1)$—ple $(y^{(1)}, z_1^{(1)}, z_2^{(1)}, \ldots, z_m^{(1)})$ e $(y^{(2)}, z_1^{(2)}, z_2^{(2)}, \ldots, z_m^{(2)})$; relazioni analoghe alle (4) valgano per le derivate $\dfrac{\partial \varrho_i}{\partial z_j}$, $\dfrac{\partial f_i}{\partial y}$, $\dfrac{\partial f_i}{\partial z_j}$ $(i, j = 1, 2, \ldots, m)$.

Siano $\Phi_i(y)$, $(i = 1, 2, \ldots, m)$ m funzioni definite per ogni y reale e aventi derivate prime lipschitziane; in queste ipotesi:

Esiste un numero positivo a $(\leqq a_0)$, e almeno un sistema di funzioni $z_i(x,y)$, $z_2(x, y), \ldots, z_m(x, y)$, le quali sono definite e continue assieme alle derivate $\dfrac{\partial z_i}{\partial y}$ nella striscia $0 \leqq x \leqq a$, per ogni y sono assolutamente continue in x, per ogni x sono lipschitziane in y (con costante indipendente da x), soddisfano le

$$z_i(0, y) = \Phi_i(y), \quad (i = 1, 2, \ldots, m),$$

e, inoltre, fissato y, per quasi tutti gli x soddisfano le (2).

Una Memoria recente di P. Lax (cfr. [4]) introduce il concetto di integrale generalizzato di un sistema di equazioni a derivate parziali in un indirizzo diverso dal nostro.

[1] M. Cinquini Cibrario e. S. Cinquini, Ann. di Mat. (4), **32** (1951), 121—155.
[2] M. Cinquini Cibrario e S. Cinquini, Ann. Scuola Normale Superiore di Pisa (3), **6** (1952), 187—243.
[3] R. Courant e P. Lax, Commun. pure appl. Math. **2** (1949), 255—273.
[4] P. Lax, Commun. pure appl. Math. **6** (1953), 231—258.

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ON AN EQUATION OF T. UNO AND R. YOKOMI

ROBERTO CONTI and GIOVANNI SANSONE

T. Uno and R. Yokomi have studied by graphical methods the equation

$$\frac{dy}{dx} = \frac{-xy}{y^2 - (x+1)\left[(x-1)^2 + \lambda\right]}$$

containing the real parameter λ, to the effect of describing one mode of appearance of limit cycles corresponding to the value $\lambda = 0$.

The same equation, or rather the system

$$\begin{cases} \dfrac{dx}{dt} = y^2 - (x+1)\left[(x-1)^2 + \lambda\right] \\[2ex] \dfrac{dy}{dt} = -xy, \end{cases}$$

is studied here at some length and the following results are obtained:

i) if $\lambda \leqq 0$ or if $\lambda \geqq 1$ there is no limit cycle;

ii) if $0 < \lambda < 1$ there are just two limit cycles, symmetrical with respect to the x-axis;

iii) when λ is sufficiently near to 0 there are limit cycles which have an arbitrarily small distance from a graph formed by two singular points and by two separatrices joining them;

iv) when λ is sufficiently near to 1 there are limit cycles entirely contained in an arbitrarily small circle surrounding the point $(0, \sqrt{2})$.

Proofs are based upon standard methods and applications are made to the differential equation

$$\frac{d^2x}{dt^2} + \left[3x^2 - (1-\lambda)\right] \frac{dx}{dt} \left[+ + 2x(x\ \ 1)\, x - 1)^2 + \lambda\right] = 0$$

of Liénard type.

VIA CAIROLI 39,
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ÜBER FUNKTIONALGLEICHUNGEN ZWISCHEN EPSTEINSCHEN ZETAFUNKTIONEN MIT VERSCHIEDENEN WERTEN DER „UNTEREN" PARAMETER

Otto Emersleben

Das Nachfolgende beschränkt sich auf solche Epsteinschen Zeta-funktionen, deren „obere" Parameter verschwinden und deren quadratische Form die Kugelform ist, definiert in der Halbebene $\sigma > p$ der $s = \sigma + i\tau$-Ebene durch die Reihe:

$$^{(p)}Z\begin{vmatrix} 0 \\ h \end{vmatrix}(s) \equiv Z\begin{vmatrix} 0 & \dots & 0 \\ h_1 & \dots & h_p \end{vmatrix}(s) = \sum_{k_1=-\infty}^{\infty} \dots \sum_{k_p=-\infty}^{\infty}{}' \frac{e^{2\pi i \sum_{j=1}^{p} k_j h_j}}{\left(\sum_{j=1}^{p} k_j^2\right)^{\frac{s}{2}}},$$

darüber hinaus durch analytische Fortsetzung.

Wie bei der Riemannschen Zetafunktion $\iota(s) = \frac{1}{2}.{}^{(1)}Z\begin{vmatrix} 0 \\ 0 \end{vmatrix}(s)$ gibt es auch bei den Epsteinschen Verallgemeinerungen Funktionalgleichungen zwischen Funktionswerten für die Argumente s und $p - s$, also Spiegelungen an der kritischen Geraden $\sigma = \frac{p}{2}$. Hierbei erfolgt eine Vertauschung der unteren und oberen Parameter verbunden mit einem Vorzeichenwechsel bei diesen.

Für einzelne Werte der Parameter sind auch bereits Funktionalgleichungen zwischen solchen Zetafunktionen bei gleichem Wert des Arguments bekannt. Z.B. kennt man in der Funktionalgleichung

$$^{(p)}Z\begin{vmatrix} 0 \\ \frac{1}{2} \end{vmatrix}(s) = {}^{(p)}f(s).\,{}^{(p)}Z\begin{vmatrix} 0 \\ 0 \end{vmatrix}(s)$$

die Funktion $^{(p)}f(s)$ explizit für $s = 1, 2, 4$ und 8. (Ber. Math. Tagung Berlin 1953, 233-250).

Von besonderem Interesse für die Anwendungen ist jedoch eine Funktionalgleichung, die für jede Ordnung p der Zetafunktionen aufgestellt werden kann. Wenn n eine natürliche Zahl bedeutet, lautet sie:

$$\sum_{k_1=1}^{n} \dots \sum_{k_p=1}^{n} Z\begin{vmatrix} 0 & \dots & 0 \\ h_1 + \frac{\varkappa_1}{n} & \dots & h_p + \frac{\varkappa_p}{n} \end{vmatrix}(s) = n^{p-s} Z\begin{vmatrix} 0 & \dots & 0 \\ nh_1 & \dots & nh_p \end{vmatrix}(s).$$

Der vorgetragene analytische Beweis benutzt in der Halbebene $\sigma > p$ die bekannte zahlentheoretische Identität (k eine ganze Zahl)

$$\sum_{l=1}^{n} e^{2\pi i k \frac{l}{n}} = n \text{ , wenn } n|k,$$
$$= 0 \text{ , sonst.}$$

Für $p = 2$ ist daneben eine weitere Funktionalgleichung zu erwähnen:

$$Z\begin{vmatrix} 0 & 0 \\ x & y \end{vmatrix}(s) + Z\begin{vmatrix} 0 & 0 \\ \frac{1}{2} - x & \frac{1}{2} - y \end{vmatrix}(s) = 2^{1-\frac{s}{2}} Z\begin{vmatrix} 0 & 0 \\ x + y & x - y \end{vmatrix}(s).$$

Eine ausführlichere Veröffentlichung erscheint in den Mathematischen Nachrichten.

Wenn man auf diese Funktionalgleichung die vorher erwähnte Funktionalgleichung mit der Spiegelung $s \longleftrightarrow p - s$ anwendet, ergeben sich Summen von Epsteinschen Zetafunktionen mit verschiedenen Werten der *oberen* Parameter.

BERLIN-ZEHLENDORF, TREIBJAGDWEG 16.

TRANSFINITE DIAMETER AND FOURIER SERIES

MICHAEL FEKETE

The aim of this paper is to contribute to the study of the exceptional sets E of all points x in $(-\pi, \pi)$ at wich the Fourier series F: $a_0/2 + \Sigma\ (a_n \cos nx + b_n \sin nx)$ of a function $f(x)$ of the L^2-class diverges provided the coefficients satisfy a condition of the type $\Sigma(a_n^2 + b_n^2)\omega(n) < \infty$ with $\omega(n) \to \infty$. Measured by the different set-functions: length (if $\omega(n) = \log n$), logarithmic capacity $(\omega(n) = n)$, $(1 - \alpha)$-capacity ($\omega(n) = n^\alpha$ with $0 < \alpha < 1$), the extent of E was found, in all cases, to be zero. (Cf. Plessner; Beurling; Salem and Zygmund)

Noticing the identity of the above concepts of capacity with those of the transfinite diameters as defined by Fekete and Pólya-Szegö, the present author applied, in studying E, his notion of the generalized transfinite diameter $\delta(C, g)$ of compact (linear) sets C, in case of an (even) generator function $g(x)$, positive and continuous for $0 < x \leqq \pi$, strictly increasing to $+\infty$ as $x \downarrow 0$ but, nevertheless, fulfilling $\int_0^\pi g(t)\, dt < \infty$ — a restriction, necessary (H. R. Ursell) and sufficient (Fekete) that $\delta\ (C, g)$ does not vanish identically. Thus he proved the following proposition, including those quoted above: Let the Fourier constants $\lambda_n = \frac{2}{\pi}\int_0^\pi g(t) \cos nt\, dt$ of $g\ (x)$ be positive for $n \geqq 1$ and satisfy the conditions $\Sigma\lambda_n = \infty$; $\left|\sum_{\nu=1}^{n} \lambda_\nu \cos \nu\, x\right| \leqq A\, g(x) + B$ with $0 < A, B < \infty$ whenever $0 < x \leqq \pi$. Then the closure $\overline{E}$ of the set E of points x of divergence in $(-\pi, \pi)$ of F has a vanishing transfinite diameter $\delta(\overline{E}, g)$ w.r. to $g(x)$ provided $\Sigma(a_n^2 + b_n^2)/\lambda_n$ converges. The proof uses the generalization by Salem and Zygmund of a known lemma of Kolmogoroff and Seliverstoff, combined with a recent result by the author: Under the "law of repulsion" $g(t)$, the minimum of the energy-integral $I(\mu) = \int_0^\pi \int_0^\pi g(|x-y|)d\mu(x)d\mu(y)$ for variable distribution $\mu(x)$ of the unitmass spread on $(0, \pi)$ concentrated on a subset E — i.e. $\int_0^\pi d\mu(x) = \int_E d\mu(x)$ — equals $g(\delta(\overline{E}, g))$, $\overline{E}$ denoting the closure of E.

The latter auxiliary theorem, applied in case $\mu(x) = m(x)/m(\pi)$ where $m(x)$ means the length of $E \cap (0, x)$, implies that $m(\pi)$ — i.e. the the length of E itself — satisfies the inequality $g(\delta(\overline{E}, g))m(\pi) \leqq 2\int_0^\pi g(t)dt$ and, therefore, vanishes whenever $\delta(\overline{E}, g) = 0$. Hence, under the assumptions of the foregoing proposition, F converges almost everywhere in $(0, \pi)$.

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UNE MÉTHODE GÉNÉRALE POUR OBTENIR DES FORMULES DE QUADRATURE

ALDO GHIZZETTI

On montre que toutes les formules de quadrature connues peuvent être déduites de la formule classique de Green, d'une manière systématique qui donne en même temps l'expression de l'erreur. On peut ainsi se débarrasser complètement de la théorie de l'interpolation.

VIA PAVIA 86 - ROMA.

FOURIER TRANSFORMS OF THE CLASS L_p

EDWIN HEWITT

Let G be a locally compact Abelian group and let G^* denote the character group of G. For every number p, $1 \leqq p \leqq 2$, the Fourier transform of the space $L_p(G)$ exists, carrying $L_p(G)$ into or onto $L_{p'}(G^*)$, where $p' = p/(p-1)$. The theorem of Plancherel-Weil asserts that this transform, $f \to f^*$, is one-to-one onto for $p = 2$ and is linear and norm-preserving. The transformation is one-to-one into for $p < 2$ and G infinite. The Hausdorff-Young inequality asserts that $\| f^* \|_{p'} \leqq \| f \|_p$, for all p. For $1 < p < 2$, equality can occur in the Hausdorff-Young inequality if and only if f on G has the form

$$f(x) = \alpha\chi(x)$$

for x in some compact open subgroup A of G and $f(x) = 0$ for x not in A, where χ is a character, or if f is a translate of such a function. For compact G (all p) and for non-compact G, $1 < p \leqq 4/3$, the theorem is due to Hewitt, for non-compact G and $4/3 < p < 2$, to I. I. Hirschman, Jr.

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SULL' INTEGRAZIONE DELLE FORME DIFFERENZIALI

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Siano F_{p-1} e G_{n-p+3} due forme differenziali rispettivamente di grado $p-1$ ed $n-p+3$, definite in un dominio T di un S_n euclideo. Se è $3 \leqq p \leqq n+1$ e se F_{p-1} e G_{n-p+3} hanno integrale nullo rispettivamente su tutti i $(p-1)$—cicli e su tutti gli $(n-p+3)$—cicli contenuti in T, esiste una forma U_{p-2} che verifica simultaneamente le equazioni

$$(1) \qquad dU_{p-2} = F_{p-1}, \quad dU^*_{p-2} = G_{n-p+3},$$

nella seconda delle quali U^*_{p-2} è la forma aggiunta di U_{p-2}. La U_{p-2} è univocamente determinata dalla conoscenza dei suoi valori al contorno e di certi integrali della U^*_{p-2}. Questo risultato sussiste nella sola ipotesi che F_{p-1} e G_{n-p+3} abbiano coefficienti hölderiani e in tale ipotesi i coefficienti della U_{p-2} risultano dotati di derivate prime hölderiane, per modo che la U_{p-2} riesce soluzione effettiva delle (1) e non soltanto soluzione generalizzata nel senso di Gillis.

Per giungere ai risultati più sopra enunciati occorre preventivamente completare le ricerche di W. V. D. Hodge sulla rappresentazione mediante potenziali di frontiera delle soluzioni dei problemi di Dirichlet e Neumann per le forme armoniche in un dominio T dello spazio euclideo, abbandonando l'ipotesi, considerata da Hodge, che T sia una n-cella.

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GRENZEIGENSCHAFTEN VON FUNKTIONEN MEHRERER VERÄNDERLICHEN UND IHRE ANWENDUNG IN VARIATIONSAUFGABEN

S. M. Nikolsky

Für je drei positive Zahlen p, r, M, wobei $1 \leqq p \leqq \infty$ betrachten wir eine Klasse von Funktionen, $H_p^{(r)}(M)$, die im n-dimensionalen reellen Raum R_n der Punkte $(x_1, \ldots, x_n)$ definiert sind.

Die Klassen werden auf folgende Weise bestimmt:

Angenommen $r = \bar{r} + \alpha$, wobei $\bar{r}$ eine ganze Zahl ist und α der Ungleichung $0 < \alpha \leqq 1$ genügt.

Es seien

$$\Delta_{hx} \varphi = \varphi(x + h) - \varphi(x),$$
$$\Delta_{hx}^2 \varphi = \varphi(x + h) - 2\varphi(x) + \varphi(x - h).$$

Nach der Definition gehört die Funktion f zur Klasse $H_p^{(r)}(M)$, wenn sie und ihre verallgemeinerten partiellen Ableitungen bis einschliesslich $\bar{r}$-ter Ordnung

$$f_{x_i}^{(\bar{r})} = \frac{\partial^{\bar{r}} f}{\partial x_i^{\bar{r}}}$$

p-ter Potenz über R_n integrierbar sind, und ausserdem die verallgemeinerte Lipschitzbedingung erfüllt ist:

$$\left(\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} | \Delta_{hx_i} f_{x_i}^{(\bar{r})} |^p \, dx_1 \ldots dx_n \right)^{1/p} \leqq M \, | h |^\alpha,$$

wenn $0 < \alpha < 1$. Oder, wenn $\alpha = 1$, die verallgemeinerte Zygmundbedingung erfüllt ist

$$\left(\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} | \Delta_{hx_i}^2 f_{x_i}^{(\bar{r})} |^p \, dx_1 \ldots dx_n \right)^{1/p} \leqq M \, | h |.$$

Im Raum R_n werden wir die $(\bar{r} + 2)$ mal differenzierbare Mannigfaltigkeit S der m-ten Dimension betrachten $(1 \leqq m < n)$.

Analog dem Falle $H_p^{(r)}(M)$ kann man die Klassen $H_p^{(r)}(S, M)$ der Funktionen φ, gegeben in der Mannigfaltigkeit S bestimmen. Einzelheiten der Bestimmung lassen wir weg. Das grundlegende, die entsprechenden Sätze von Soboleff und Kondrascheff verallgemeinernde Theorem, lautet:

Es sei $f \in H_p^{(r)}(M)$ und

$$\varrho_\lambda = r - \lambda - \frac{n - m}{p} \qquad (1)$$

wobei $\lambda \geqq 0$ eine ganze Zahl ist. Dann gehört jede partielle Ableitung λ-ter Ordnung

$$f^{(\lambda)} = \frac{\partial^\lambda f}{\partial x_1^{\lambda_1} \ldots \partial x_n^{\lambda_n}} \quad \left(\sum_1^n \lambda_k = \lambda\right)$$

betrachtet in der Mannigfaltigkeit S zur Klasse $H^{(\varrho\lambda)}(S, M')$, wo M' eine entsprechende Konstante ist.

Der formulierte Satz kann umgekehrt werden. Und zwar gilt folgender inverser Satz.

Unsere Mannigfaltigkeit S bestehe aus einer endlichen Zahl einander überdeckender Stücke σ.

In jedem Punkt p des Stückes σ sind, normal zu σ $n - m$ untereinander orthogonale $\bar{r} + 1$ mal differenzierbare Einheitsvektoren

$$\overline{N}_{1,\sigma}, \ldots, \overline{N}_{n-m,\sigma}$$ aufgerichtet.

Das System (λ) aus $n - m$ nichtnegativen ganzen Zahlen $\lambda_1, \ldots, \lambda_{n-m}$ nennen wir zulässig, wenn die Bedingung $\varrho_\lambda = r - \lambda - \frac{n-m}{2} > 0$, wobei $\lambda = \lambda_1 + \ldots + \lambda_{n-m}$ erfüllt ist.

Angenommen, zu jedem Stück σ und jedem zulässigen System (λ) existiert eine in σ bestimmte Funktion $\varphi_{(\lambda)\sigma}$, die zur Klasse $H_p^{(\varrho_\lambda)}(\sigma, M)$ gehört.

Die Funktionen $\varphi_{(\lambda)\sigma}$ die verschiedenen einander überdeckenden Stücken σ entsprechen, sollen untereinander so abgestimmt sein, dass für eine und dieselbe Funktion $f(x_1, \ldots, x_n)$ die Gleichheiten

$$\left.\frac{\partial^\lambda f}{\partial N_{1,\sigma}^{\lambda_1} \ldots \partial N_{n-m,\sigma}^{\lambda_{n-m}}}\right|_\sigma = \varphi_{(\lambda),\sigma} \tag{2}$$

gelten können.

Dann kann man in R_n tatsächlich die Funktion f erhalten, die zur Klasse $H_p^{(r)}(\overline{M})$ gehört, und zwar in solcher Weise, dass für alle Stücke σ und alle zulässigen Systeme (λ) gleichzeitig die Gleichheiten (2) erfüllt sind. Dann gilt die Ungleichung

$$\overline{M} < c\,(\sum_{\substack{(\lambda)\\ \sigma}} \|\varphi_{(\lambda)\sigma}\|_{L_p} + M),$$

wobei $\|\varphi_{(\lambda)\sigma}\|_{L_p}$ die Norm der Funktion $\varphi_{(\lambda)\sigma}$ in bezug auf L_p ist. (Für die Anwendung siehe meinen Übersichtsvortrag).

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UEBER DIE RIEMANN'SCHE PERIODENRELATION FÜR OFFENE FLÄCHEN

ALBERT PFLUGER

Kann die von Riemann bewiesene bilineare Periodenrelation für Abel'sche Integrale erster Gattung auf null-berandete Flächen (Flächenklasse 0_G) übertragen werden? Für spezielle zweiblättrige Flächen ist diese Frage von P. J. Myrberg [1]) und K. I. Virtanen [2]) und allgemein für die Flächenklasse 0_G von Letzterem [2]) und L. V. Ahlfors [3]) untersucht worden. Hier wird diese Frage für spezielle zweiblättrige Flächen wieder aufgegriffen.

Zur Konstruktion der Riemann'schen Fläche R bringen wir mit P. J. Myrberg auf der positiven reellen Achse der z-Ebene unendlich viele Schlitze I_n, $n = 0, 1, 2, \ldots$ an, die sich nur im Unendlichen häufen und verheften kreuzweise zwei Exemplare dieses Schlitzgebietes. Diese zweiblättrige Fläche R, oberes Blatt π_+ unteres Blatt π_- gehört zur Flächenklasse 0_G.

Es werden auf R zwei kanonische Schnittsysteme betrachtet, die Wege A_n^1, B_n^1, $n = 1, 2, \ldots$ des ersten und die Wege A_n^2, B_n^2, $n = 1, 2, \ldots$ des zweiten Systems. B_n^1 ist ein im positiven Sinn durchlaufener, zwischen I_{n-1} und I_n hindurchgehender Kreis $|z| = r$ in π_+. Ein Weg in π_+ von I_{n-1} nach I_n und zurück in π_- nach I_{n-1} liefert A_n^1. B_n^2 ist der im positiven Sinn durchlaufene Schlitz I_n in π_+; ein Weg von I_0 in der obern Halbebene von π_+ nach I_n und von dort in der untern Halbebene von π_- zurück nach I_0 liefert den Schnitt A_n^2. Die Gesamtheit der auf R harmonischen Differentiale ω von endlicher Norm, mit $(\omega_1, \omega_2) = \iint_R (a_1a_2 + b_1b_2)dxdy$, $\omega_i = a_idx + b_idy$, $i = 1, 2$, als innerem Produkt, ist ein Hilbertraum H. Für einen geschlossenen Weg c setzten wir $\int_c \omega = \omega(c)$ d.i. die Periode von ω längs c. Wir haben die Aufgabe das innere Produkt (ω_1, ω_2) durch die A- und B-Perioden der ω_i und deren konjugierter Differentiale ω_i^*, $i = 1, 2$ darzustellen. Je nach der metrischen Struktur von R ist hierfür das erste oder zweite Schnittsystem geeignet. Um dies zu präzisieren betrachten wir die Jordankurven der z-Ebene, welche einen festen Kreis $|z| = r_0$ vom unendlich fernen Punkt trennen und mit der positiven reellen Achse genau einen Schnittpunkt haben. Diese Kurven zerfallen in zwei Klassen;

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1) P. J. MYRBERG. Ueber transzendente hyperelliptische Integrale erster Gattung. Ann. Acad. Sci. Fenn. Ser. A I Nr. 14 (1943).

2) K. I. VIRTANEN. Ueber Abelsche Integrale auf nullberandeten Riemannschen Flächen von unendlichem Geschlecht. Ebenda Nr. 56 (1949).

3) L. V. AHLFORS. Normalintegrale auf offenen Riemannschen Flächen. Ebenda Nr. 35 (1947)

die erste Klasse C_1 besteht aus den Kurven, die keinen der Schlitze I_n treffen, die zweite Klasse C_2 aus jenen, die genau einen Schlitz treffen. λ_{C_i} sei die Extremallänge der Kurvenmenge C_i, $i = 1, 2$. λ_{C_1} und λ_{C_2} können nicht gleichzeitig positiv sein. Es gilt der Satz: Für beliebige ω_1 und ω_2 aus H und eine zugehörige Teilfolge n_ν der natürlichen Zahlen ist

$$(\omega_1, \omega_2) = \lim_{\nu \to \infty} \sum_{k=1}^{n_\nu} (\omega_1 (A_k^i) \omega_2^* (B_k^i) - \omega_1^* (A_k^i) \omega_2 (B_k^i))$$

im Falle $\lambda_{C_i} = 0$, $i = 1, 2$.

Die verwendete Methode lässt sich auf allgemeinere zweiblättrige Flächen übertragen.

BÜCHNERSTRASSE 7, ZÜRICH 6.

LES POLYNOMES DE S. N. BERNSTEIN ET LE PROBLÈME DE L'INTERPOLATION

TIBERIU POPOVICIU

1. Les problèmes d'interpolation les plus simples reviennent à substituer à la fonction $f(x)$, définie dans l'intervalle fermé $[a, b]$, la fonction

$$F(x; f) = \sum_{i=0}^{n} \varphi_i(x) f(x_i), \tag{1}$$

où les points (les noeuds) distincts $x_i \in [a, b]$ et les fonctions $\varphi_i(x)$, définies dans $[a, b]$ sont indépendants de la fonction $f(x)$.

On peut poser le problème de la détermination et de l'étude des procédés (1), qui non seulement restent non-négatifs pour tout $f(x)$, non-négatif, mais qui en outre conservent aussi certaines propriétés relatives à l'allure de la fonction $f(x)$. Par ex., qui conservent la monotonie, la convexité, etc., de la fonction $f(x)$.

2. Si la fonction (1), non seulement reste non-négative pour tout $f(x)$ non-négatif, dans tout l'intervalle $[a, b]$, mais conserve dans $[a, b]$ aussi toute propriété de convexité de tout ordre de la fonction $f(x)$, les $\varphi_i(x)$ sont des polynomes du degré $\leqq n$ et, plus exactement, $F(x; f)$ est un polynome du degré $\leqq k$ pour tout polynome $f(x)$ du degré k, pour $k = 0, 1, \ldots$ En outre il faut et il suffit que les inégalités

$$\sum_{i=0}^{n} (| x_i - \lambda | + x_i - \lambda)^s \varphi_i^{(s+1)}(x) \geqq 0, \text{ pour } x \in [a, b], s = 0, 1, \ldots$$

soient vérifiées [3].

Remarquons encore que si $\{F_n(x; f)\}$ est une suite infinie de fonctions de la forme (1), qui jouissent de la propriété de conservation de l'allure précisée plus haut, et si cette suite tend dans $[a, b]$ vers $f(x)$, pour $f(x) = 1, x, x^2$, alors la suite converge absolument et uniformément vers $f(x)$ dans $[a, b]$, lorsque $f(x)$ est continue dans cet intervalle.

3. Le procédé le plus simple de la forme (1) est constitué par les polynomes de S. N. Bernstein [1]

$$B_n(x; f) = \sum_{i=0}^{n} \binom{n}{i} f\left(\frac{i}{n}\right) x^i(1-x)^{n-i} \quad ([a, b] = [0, 1]). \tag{2}$$

J'ai démontré autrefois [2], que ces polynomes conservent toute propriété de convexité de la fonction $f(x)$. Mais il y a des propriétés plus complètes. La formule qui donne la dérivée du polynome (2),

$$B_n'(x; f) = n \sum_{i=0}^{n-1} \Delta_{\frac{1}{n}} f\left(\frac{i}{n}\right) \binom{n-1}{i} x^i(1-x)^{n-1-i}$$

nous montre que si $f(x)$ est formé par m morceaux monotones (est monotone par segments), le polynome (2) jouit de la même propriété. En général, s étant un nombre naturel ou 0, la dérivée d'ordre $s+1$,

$$B_n^{(s+1)}(x; f) = n(n-1)\ldots(n-s) \sum_{i=0}^{n-s-1} \Delta_{\frac{1}{n}}^{s+1} f\left(\frac{i}{n}\right) \binom{n-s-1}{i} x^i(1-x)^{n-s-1-i}$$

nous montre que si $f(x)$ est formé par m morceaux de fonctions non-concaves ou non-convexes d'ordre s, le polynome (2) est formé par au plus $(m-1)(s+2) + 1$ morceaux analogues.

A l'aide des propriétés des fonctions d'ordre n par segments [3] on peut encore compléter ces propriétés.

4. Les procédés d'interpolation qui conservent les propriétés de convexité et, en général certaines allures de la fonction $f(x)$, ont aussi un intérêt pratique. Il est important que dans la représentation graphique d'une fonction provenant d'un problème pratique, les points discrets, obtenus par observations, puissent servir à représenter approximativement la variation du phénomène étudié par une courbe ayant une allure théoriquement prévue.

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ON THE THEOREMS OF CONSISTENCY FOR RIESZ SUMMABILITY

Badri Nath Prasad

1. In connection with the Riesz process of summability of infinite series, involving 'type' and 'order', and using the usual notations, the theorem which asserts that $(R, \lambda_n, \varkappa) \subset (R, \lambda_n, \varkappa')$, $\varkappa' > \varkappa \geqq 0$, is termed the "first theorem of consistency". The corresponding theorem for absolute summability states that $| R, \lambda_n, \varkappa | \subset | R, \lambda_n, \varkappa' |$, $\varkappa' > \varkappa \geqq 0$.

Questions have been raised to consider the relative effectiveness of any two Riesz methods of which the types are different while the orders are identical. The first answer to a question of this kind was embodied in the classical "second theorem of consistency". In 1916 Hardy obtained a generalisation of this theorem, replacing the 'type' by a special kind of its "logarithmico-exponential" function. In 1932 Hirst further extended the scope of this theorem by proving that under certain sufficient conditions on φ, $(R, \lambda_n, \varkappa) \subset (R, \varphi(\lambda_n), \varkappa)$. Recent work (1951—52) by Kuttner shows that, if $\varkappa$ is an integer, Hirst's conditions are also necessary.

2. Following the idea of Hardy's paper of 1916, Chandrasekharan proved in 1942 the direct analogue of Hardy's theorem for absolute summability. Very recently Pati obtained the following generalised second theorem of consistency for absolute Riesz summability.

If $\varphi(t)$ is a non-negative monotonic increasing function of t, $t \geqq 0$, steadily tending to infinity as t tends to infinity, such that, for positive integral $\varkappa$, $\varphi(t)$ is a $(\varkappa + 1)$-th indefinite integral for sufficiently large t, and

$$t^r \varphi^{(r)}(t)/\varphi(t) \in BV(h, \infty) \quad (r = 1, 2, \ldots, \varkappa),$$

then the summability $| R, \lambda_n, \varkappa |$ of an infinite series implies its summability $| R, \varphi(\lambda_n), \varkappa |$.

Lately Prasad and Pati have extended the scope of applicability of the second theorem of consistency by establishing the following theorem, which supplements Pati's theorem for the case in which $\varkappa$ is non-integral and $\varphi^{(1)}(t)$ is a monotonic non-decreasing function of t.

If $\varphi(t)$ is a non-negative monotonic increasing function of t, $t \geqq 0$, steadily tending to infinity as t tends to infinity, such that $\varphi^{(1)}(t)$ is monotonic non-decreasing, $\varphi(t)$ is a $(k + 2)$-th indefinite integral for sufficiently large t, $(k = [\varkappa])$, and

$$t^r \varphi^{(r)}(t)/\varphi(t) \in BV(h, \infty) \quad (r = 1, 2, \ldots, k + 1),$$

then the summability $| R, \lambda_n, \varkappa |$ of an infinite series implies its summability $| R, \varphi(\lambda_n), \varkappa |$.

3. Denoting by (*) the hypotheses of these theorems, we observe that, by virtue of the first theorem of consistency, $| R, \lambda_n, \varkappa | \subset | R, \varphi(\lambda_n), \varkappa' |$, $\varkappa' > \varkappa$, under (*). It should, however, be possible to establish this result under conditions less stringent than those in (*). Work is being done to establish what may be termed a "unified theorem of consistency" which may possibly lead to the conclusion that $| R, \lambda_n, \varkappa | \subset | R, \varphi(\lambda_n), \varkappa' |$, $\varkappa' \geqq \varkappa \geqq 0$, under suitable conditions without any appeal to the first theorem of consistency.

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SOME REMARKS ON SCHLICHT FUNCTIONS

MAXWELL O. READE

In this paper we summarise several results on certain special classes of schlicht functions.

W. Kaplan introduced a class of schlicht mappings which he called "almost convex". These functions $f(z)$ defined in the unit disc, are characterized by an inequality of the form $Re \dfrac{f'(z)}{\varphi'(z)} > 0$ where $\varphi(z)$ is a convex schlicht mapping of the unit disc. By using the standard Carathéodory inequality on the coefficients of analytic functions with positive real part, we show that the Bieberbach conjecture holds for all normalized "almost convex" schlicht mappings. This result includes that for the star mappings as a special case.

The corresponding exterior mapping problem is introduced and studied both from the geometric point of view and the "coefficient" point of view.

A generalization of the classic Noshiro-Warschawski sufficient condition that a function be schlicht is given; this generalization consists in replacing the usual convex domain of definition and the corresponding condition $Re\, f'(z) > 0$ by a domain whose boundary has a tangent vector having a total variation that differs little from 2π and the condition that $Re\, e^{i\alpha} f'(z) > 0$ for all sufficiently small α. From this last condition we obtain a new class of schlicht mappings, using the technique of W. Kaplan, who used the Noshiro-Warschawski criterion to obtain his "almost convex" class. For our new class, we obtain an integral condition that must be satisfied by members of the class and we obtain the usual coefficient bounds for the normalized members of the new class.

Study-type theorems are obtained for some of the interior and exterior mapping problems noted above.

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L^1-SPACES ON LOCALLY COMPACT GROUPS

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Let G be a locally compact group and g a closed normal subgroup of G. Let the Haar measures be so normalized that $\int_G f(x)dx = \int_{G/g} dx' \int_g f(x\xi)\, d\xi$, for $f(x) \in L^1(G)$.

Then

(I) $L^1(G/g)$ is a homomorphic image of $L^1(G)$, i.e., $L^1(G/g)$ is isomorphic and isometric with the quotient-space $L^1(G)/K$, where K is a certain closed linear subspace of $L^1(G)$.

(II) If the subgroup g is, in addition, abelian (or compact), then for all $f(x)$ in $L^1(G)$

$$\inf \int_G \left| f(x) - \sum_{n=1}^{N} b_n f(\sigma_n x) \right| dx = \int_{G/g} dx' \left| \int_g f(x\xi) d\xi \right| ,$$

where N ranges over all positive integers, the coefficients b_n range over all complex numbers satisfying the condition

$$\sum_{n=1}^{N} b_n = 0,$$

and the group elements σ_n over the *subgroup* g. This implies a relation similar to (I) for closed linear subspaces of $L^1(G)$ which are "invariant under translations" with respect to the subgroup g, and the subspaces of $L^1(G/g)$.

This is a generalisation of theorems contained in the author's thesis (Investigations in Harmonic Analysis, Trans. Amer. Math. Soc., vol. 73 (1952), pp. 401-427). A full account will appear under the title "Über L^1-Räume auf Gruppen" I, II in Monatshefte für Mathematik, vol. 58 (1954); the results are given there in a somewhat more general form.

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VERALLGEMEINERUNG DER PARTIELLEN DIFFERENTIALGLEICHUNGEN DER PHYSIK.

JULIO REY PASTOR

Das Problem der allgemeinen Integrale der Gleichungen $u_{xy} = 0$ und $u_{xx} - u_{yy} = 0$ ist bisher nicht in befriedigender Weise gelöst worden. Die Lösungsansätze $u = F(x) + G(y)$ bzw. $u = F(x+y) + G(x-y)$ führen zur

Bedingung der zweimaligen Differenzierbarkeit von F und G; in den verschiedenen physikalischen Problemen, die zu den genannten Gleichungen führen, ist diese Bedingung jedoch unwesentlich. Die Methoden von Leray, Bochner u. a. durch Verallgemeinerung der Gleichungen die zu starke Bedingung zu vermeiden, sind für die Physik nicht befriedigend, und der Vorschlag von Schwartz, die beliebige Summe $F(x) + G(y)$ als Integral von $u_{xy} = 0$ zu betrachten, auch wenn beide Funktionen nicht differenzierbar sind, ist nicht ohne weiteres sinnvoll.

Definieren wir als verallgemeinerte Ableitung

$$u^*_{xy}(a, b) = \lim_{h \to 0;\, k \to 0} d_u(a, b; h, k)/hk$$

wobei $d_u(a, b; h, k) = u(a + h, b + k) + u(a, b) - u(a + h, b) - u(a, b + k)$, so erhalten wir folgenden Satz:

Ist $u^*_{xy} = 0$ in jedem Punkte des Gebietes V so ist $d_u(a, b; h, k) = 0$ für jedes achsenparallele Rechteck, das samt seinem Innern in V liegt, und $u = F(x) + G(y)$ ist das allgemeine Integral der Gleichung $u^*_{xy} = 0$. (F und G endlichwertig in V).

Ebenso erhält man $F(x + y) + G(x - y)$ als allgemeines Integral der verallgemeinerten d'Alembert'schen Gleichung $A^*u = 0$ die anstatt des Operators $Au = u_{xx} - u_{yy}$ den verallgemeinerten Operator:

$$A^*u = \lim\, [u(a + h, b) + u(a - h, b) - u(a, b + h) - u(a, b - h)]/h^2$$

enthält.

In ähnlicher Weise definieren wir für jeden Punkt P:

$$\Delta^*u(P) = \lim_{r \to 0} \frac{u_m(P) - u(P)}{r^2/4}$$

wobei $u_m(P)$ der Mittelwert der Werte von u in den Eckpunkten eines achsenparallelen Quadrates von Mittelpunkt P und mit Diagonallänge $2r$ bezeichnet (Übertragung der Definition der Schwarz'schen Ableitung auf Funktionen von zwei Variablen). Der Hauptsatz der Laplace'schen Gleichung $\Delta u = 0$ (Eindeutigkeit der Lösung unter bestimmten Bedingungen) lässt sich auf die Gleichung $\Delta^*u = 0$ übertragen, ebenso wie die wichtigsten übrigen Eigenschaften von $\Delta u = 0$.

Ohne Schwierigkeit lassen sich verschiedene weitere Verallgemeinerungen der betrachteten Differentialgleichungen und von $\Delta\Delta u = 0$, auch für beliebig viele Dimensionen, einführen.

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RECHERCHE DES INTÉGRALES DE S. LIE D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES DU PREMIER ORDRE

NICOLAS SALTYKOW

Considérons l'équation *semi-linéaire*

$$p_1 + H(x_1, x_2, \ldots, x_n, p_2, p_3, \ldots, p_n) = 0.$$

La condition qu'elle admette une intégrale complète de classe q s'exprime au moyen des *polynômes caractéristiques*, de l'équation donnée. L'intégrale requise est définie par le système d'équations aux différentielles totales equivalent au système d'équations linéaires aux dérivées partielles du premier ordre

$$p_1 + \sum_{j=1}^{q} \varphi_{0j} p_{n-q+j} = \theta_1(x_1, x_2, \ldots, x_n, C_1, C_2, \ldots, C_{n-q-1}),$$

$$p_{k+1} + \sum_{j=1}^{q} \psi_{kj} p_{n-q+j} = \theta_{k+1}(x_1, x_2, \ldots, x_n, C_1, C_2, \ldots, C_{n-q-1}),$$

les premiers membres de ces dernières équations représentant les polynômes caractéristiques mentionnés et $C_1, C_2, \ldots, C_{n-q-1}$ désignant les constantes arbitraires des intégrales des caractéristiques. Les conditions d'involution de ces dernières équations sont suffisantes pour obtenir l'intégrale cherchée.

Comparaison avec les autres méthodes et applications.

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SULLA VALIDITÀ DI FORMULE INTEGRALI NELLE ALGEBRE

MICHELE SCE

Le funzioni monogene (o regolari nel senso di Fueter) in un'algebra reale d'ordine n dotata di unità soddisfano, con le loro componenti, ad un sistema di n equazioni a derivate parziali lineari del primo ordine a coefficienti costanti. Si mostra anzitutto che la forma caratteristica del sistema è il determinante di una matrice appartenente alla prima od alla seconda rappresentazione regolare dell'algebra, secondo che si tratti di monogeneità sinistra o destra; si trova quindi una relazione tra i divisori dello zero dell'algebra e gli spazi lineari caratteristici dell'equazione a derivate parziali d'ordine n associata alla forma caratteristica. La soluzione del problema di Cauchy per tale equazione permette infine di dare espressioni integrali per le componenti delle funzioni monogene; ne conseguono formule integrali in vari tipi di algebre.

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SUR CERTAINS ESPACES VECTORIELS LOCALEMENT CONVEXES

JOSÉ SEBASTIÃO E SILVA

Nous disons qu'une suite (E_n) d'espaces localement convexes est ***régulière*** si les conditions suivantes sont vérifiées: 1) quel que soit n, $E_n \subset E_{n+1}$ et la topologie de E_{n+1} induit dans E_n une topologie moins fine que celle de E_n; 2) la boule de E_n est relativement compacte dans E_{n+1}. Nous nommons *espace* (LN^*) tout espace localement convexe exprimable comme limite inductive d'une suite régulière d'espaces normés.

Soit E un espace localement convexe, limite inductive d'une suite régulière (E_n) d'espaces normés. Alors on a les propositions suivantes:

I. Pour qu'un ensemble A soit fermé dans E il faut et il suffit que, pour tout n, $A \cap E_n$ soit fermé par rapport à la topologie de E_n.

II. Pour qu'un ensemble A soit borné dans E, il faut et il suffit qu'il existe un n tel que $A \subset E_n$ et A soit borné par rapport à la topologie de E_n.

On démontre en outre les théorèmes suivants:

III. Dans un espace (LN^*) tout ensemble borné est relativement compact.

IV. Tout espace (LN^*) est réflexif. Son dual fort est un espace (M).

V. Pour qu'un espace localement convexe E soit le dual fort d'un espace (LN^*), il faut et il suffit que E soit la limite projective d'une suite d'espaces normés par rapport à des applications linéaires continues $\varrho_{m,n}$ de E_n dans $E_m (m \leqq n)$ telles que $\varrho_{n,n+1}$ transforme la boule de E_{n+1} dans une partie relativement compacte de E_n.

Les espaces (LN^*) sont très importants pour les applications. Ils jouent un rôle essentiel dans la théorie des distributions et dans la théorie des fonctionnelles analytiques.

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UEBER EINE ART FAKTORENFOLGEN IN DER THEORIE DER ALGEBRAISCHEN GLEICHUNGEN

L. TCHAKALOFF

In Zusammenhang mit den Untersuchungen über die Präzisierung der Mittelwertsätze, angewandt auf algebraische Polynome, hat Herr J. Favard folgendes Problem gestellt und gelöst:

Welches sind die notwendigen und hinreichenden Bedingungen dafür, dass die unendliche Folge reeller Zahlen

$$c_0, c_1, c_2, \ldots \tag{1}$$

folgende Eigenschaft besitzt: ist

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \tag{2}$$

ein reelles Polynom, dessen Koeffizienten der Bedingung

$$c_0 a_0 + c_1 a_1 + \ldots + c_n a_n = 0 \tag{3}$$

genügen, so hat $f(x)$ mindestens eine reelle Wurzel. Die Lösung dieses Problems hat Herr Favard auf das Hamburgersche Momentenproblem zurückgeführt.

In dieser Mitteilung soll das entsprechende Problem für endliche Folgen

$$c_0, c_1, \ldots, c_n \qquad (n \geqq 1) \tag{4}$$

reeller Zahlen gelöst werden, und zwar mit rein algebraischen Hilfsmitteln, d.h. ohne die Ergebnisse des Momentenproblems zu benutzen. Es wird ausserdem die Frage nach der Existenz und effektiven Bestimmung einer Minimalmenge reeller Zahlen untersucht, wo jedes der Bedingung (3) genügende Polynom (2) mindestens eine reelle Wurzel besitzt.

Wir betrachten die Klasse K_n der nicht identisch verschwindenden reellen Polynome n-ten oder niederen Grades in x. Die endliche Folge reeller Zahlen (4) möge eine *ausgezeichnete Faktorenfolge* heissen, wenn jedes Polynom (2) der Klasse K_n, dessen Koeffizienten der Gleichung (3) genügen, sein Vorzeichen im Intervall $(-\infty, +\infty)$ wechselt, d.h. mindestens eine reelle Wurzel ungerader Ordnung besitzt. Ein einfaches Beispiel einer $(n + 1)$-gliedrigen ausgezeichneten Faktorenfolge bilden die Zahlen $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}$, was man leicht erkennt, wenn man berücksichtigt, dass in diesem Fall der Bedingung (3) die Gestalt $\int_0^1 f(x)dx = 0$ gegeben werden kann. Es gilt nun folgender

Satz I. *Die reelle Zahlenfolge (4) stellt dann und nur dann eine ausgezeichnete Faktorenfolge dar, wenn die quadratische Form*

$$F(u_0, u_1, \ldots, u_K) = \sum_{p,q=0}^{K} c_{pq} u_p u_q \qquad \left(K = \left[\frac{n}{2}\right]\right)$$

definit ist.

Es sei (4) eine feste ausgezeichnete Faktorenfolge. C_n bedeute die Klasse der nicht identisch verschwindenden Polynome (2) vom Grade $\leqq n$ mit reellen Koeffizienten a_r, die der linearen Gleichung (3) genügen. Unter einer *Minimalmenge* E_n inbezug auf die Klasse C_n verstehen wir eine Menge reeller Zahlen mit folgender Beschaffenheit: 1) ein beliebiges Polynom der Klasse C_n hat mindestens eine Wurzel ungerader Ordnung, die der Menge E_n angehört; 2) keiner echten Teilmenge von E_n kommt die Eigenschaft 1) zu. Die von uns angewandte Methode gestattet nun, sämtliche Minimalmengen inbezug auf C_n effektiv zu bestimmen. Ist z.B. $n = 2K - 1 > 3$, so stellt das offene Intervall (α, β) die einzige Minimalmenge dar inbezug auf die Klasse C_n, wobei α und β die extremen Wurzeln des Polynoms

$$P_K(x) = \begin{vmatrix} c_0 & c_1 \dots c_K \\ c_1 & c_2 \dots c_{K+1} \\ \dots & \dots \\ c_{K-1} & c_K \dots c_{2K-1} \\ 1 & x \dots x^K \end{vmatrix}$$

bedeuten. Die Beweise dieser Ergebnisse werden demnächst in die Acta Mathematica der Ungarischen Akademie der Wissenschaften veröffentlicht werden.

LITERATUR.

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ÜBER EIN ABGESCHWÄCHTES AUSWAHLPOSTULAT [1])

KLAUS WAGNER

M^* bezeichne die Potenzmenge einer Menge M.

Def. 1. Wir sagen, eine Menge M erfüllt das „abgeschwächte" Auswahl postulat (P), wenn es eine eindeutige Abbildung φ von M^* in sich gibt, sodass für jedes aus mindestens zwei Elementen bestehende $A \subseteqq M$ gilt: $0 \subset \varphi(A) \subset A$

Satz 1. Erfüllt eine Menge M unser (P), so lässt sie sich ordnen (i.S. von vollständig ordnen).

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[1]) Die gemeinsam mit W. Kinna verfasste Arbeit ist inzwischen zur Veröffentlichung in die Fundamenta Mathematicae (Warschau) angenommen und soll im nächsten Band dieser Zeitschrift gedruckt werden.

Korollar: Wenn φ jedem nicht leeren $A \subseteqq M$ eine aus nur einem Element bestehende Menge zuordnet, so ist die im Beweis von Satz 1 definierte Ordnung eine Wohlordnung.

Def. 2. Eine Menge M heisse spaltbar geordnet, wenn sie geordnet ist und es ausserdem ein solches φ gibt, das jedem aus mindestens zwei Elementen bestehenden $A \subseteqq M$ ein echtes, nicht leeres Anfangsstück von A zuordnet.

Satz 2. Eine Menge M erfüllt dann und nur dann (P), wenn es eine spaltbare Ordnung von M gibt.

Satz 3. Eine Menge M erfüllt dann und nur dann (P), wenn M mit einer Teilmenge der Potenzmenge einer wohlgeordneten Menge äquivalent ist.

Beispiel: Hat eine geordnete Menge M die Eigenschaft, dass es darin eine relativ zu ihr dichte, wohlordnungsfähige Teilmenge gibt, so erfüllt sie (P). Also auch noch das Kontinuum erfüllt (P).

(Def. 1 und Satz 1 stammen m.E. vom Ref., Def. 2 und Sätze 2 und 3 von W. Kinna in Solingen).

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ANALYTISCHE THEORIE DER DIFFERENTIALGLEICHUNGEN AUF DER GRUNDLAGE VON ENTWICKLUNGEN NACH BESSELSCHEN FUNKTIONEN

RUDOLF WEYRICH

Ausgehend von der von Carl Neumann entwickelten und von Gegenbauer erweiterten Theorie der Neumannschen Reihen lässt sich die Fuchssche Theorie der Differentialgleichungen auch auf der Grundlage von Entwicklungen nach Besselfunktionen aufbauen. Die entsprechenden Reihen ergeben sich durch Umordnung aus den betreffenden Potenzreihen bei sich deckenden Konvergenzbereichen. Beide Arten von Reihen weisen für kleine Argumentwerte etwa gleich gute Konvergenz auf, doch zeigen sich bei grossen Parameterwerten die Neumannschen Reihen, die hinsichtlich der Parameter asymptotischen, aber konvergenten Charakter haben, den Potenzreihen überlegen. Es ist dies eine unmittelbare Folge der wegen ihrer Oszillationseigenschaften grösseren Schmiegsamkeit der Besselschen Funktionen. Durch Eintragen der wohlbekannten asymptotischen Entwicklungen der Besselfunktionen in solche

Neumannsche Reihen erhält man sehr einfach entsprechende asymptotische Ausdrücke. Auch sind die Neumannschen Reihen für numerische Rechnungen gut brauchbar, da für die Besselfunktionen ausgezeichnete Tafeln vorliegen. Als Beispiele werden die entsprechenden Entwicklungen der Laméschen und der Sphäroïd-Funktionen herangezogen. Das Verfahren lässt sich auf viele andere wie etwa die Whittakerschen Funktionensysteme ausdehnen.

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Section III

GEOMETRY AND TOPOLOGY

SHORT LECTURES

ÜBER DIE HOMÖOMORPHIE VON PUNKTMENGEN

P. Alexandroff

Es sei zunächst A ein in einer Simplizialzerlegung τ gegebenes Polyeder (in einem R^n). Dann sind auch die Unterteilungen $\alpha, \beta, \ldots$ des Komplexes τ gegeben und diese bilden ein gerichtetes System, wenn man $\beta \succ \alpha$ setzt, sobald jedes (offene) Simplex von β auf einem (offenbar, einzigen) Simplex von α (seinem Träger) liegt: Ist $\beta \succ \alpha$ und lässt man jedem Simplex von β seinen Träger in α entsprechen, so entsteht eine Abbildung π_α^β des Komplexes β in (sogar auf) den Komplex α, die Projektion heisst, und die allein durch die Bilder von Eckpunkten von β eindeutig bestimmt ist. Identifiziert man jedes Simplex mit seinem Eckpunktgerüst, so lässt sich die Projektion π_α^β als mehrdeutige Abbildung der Menge aller Eckpunkte von α in die Menge aller Eckpunkte von β betrachten, und zwar als eine unmittelbare Verallgemeinerung der gewöhnlichen simplizialen Abbildungen.

Das gerichtete System der Komplexe $\alpha, \beta, \ldots$ mit den zugehörigen Projektionen π_α^β heisst das zur Triangulation τ gehörende kombinatorische Spektrum des Polyeders A.

Ist jetzt $A \subset R^n$ eine beliebige Punktmenge des R^n, so betrachtet man die gerichtete Folge sämtlicher Simplizialzerlegungen sämtlicher Polyeder, die im R^n liegen und die Menge enthalten. Ordnungsrelation und Projektion — wörtlich wie soeben.

Dieses gerichtete System mit den zugehörigen Projektionen nennt man das geometrische Spektrum der Punktmenge A.

Sind jetzt die $\alpha, \beta, \ldots$ irgendwelche (stern-endliche) abstrakte Komplexe, die ein gerichtetes System bilden, und ist für $\beta \succ \alpha$ eine Projektion π_α^β definiert, so dass bei $\gamma \succ \beta \succ \alpha$ die „abgeschwächte Transitivitätsbedingung $\pi_\alpha^\beta \pi e_\gamma \subseteqq \pi_\alpha^\gamma e_\gamma$ für jeden Eckpunkt g von γ gilt, so erhält man ein abstraktes Spektrum; offenbar ist jedes geometrische Spektrum Spezialfall eines abstrakten (für geometrische Spektra gilt sogar die nicht abgeschwächte Transitivität $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$).

Unsere Frage lautet: Wie erkennt man an den geometrischen (bzw. kombinatorischen) Spektren zweier Punktmengen (bzw. zweier Polyeder) $A \subseteqq R^n$, $A' \subseteqq R^{n'}$, dass diese Punktmengen untereinander homöomorph sind?

Die Beantwortung dieser Frage geschieht dadurch, dass auf eine gewisse

Weise der Begriff eines konfinalen Teilspektrums eines gegeben Spektrums eingeführt und sodann der folgende Satz bewiesen wird:

Satz. Es seien $A \subseteqq R^n$, $A' \subseteqq R^{n'}$ gegeben, S und S' die geometrischen (bzw. S_τ und $S_{\tau'}$) die kombinatorischen Spektra von A und A'. Dann lautet die notwendige und hinreichende Bedingung für die Homöomorphie von A und A' folgendermassen:

> Es soll in S ein konfinales Teilspektrum s und in S' ein konfinales Teilspektrum s' derart existieren, dass diese Spektra s und s' konfinale Teilspektra eines und desselben abstrakten Spektrums Σ sind.

Bleibt übrig die Definition eines konfinalen Teilspektrums Σ' des gegebenen Spektrums Σ. Zunächst soll Σ', als gerichtetes System betrachtet, Teil des Systems Σ sein und dann — falls $\pi_{\alpha'}^{\beta'}$ die Projektionen in Σ' und π_α^β diejenigen in Σ sind — für jeden Eckpunt $e_{\beta'} \in \beta' \in \Sigma'$ bei $\beta' > \alpha'$ in Σ' die Inklusion $\pi_{\alpha'}^{'\beta'} e_{\beta'} \subseteqq \pi_{\alpha'}^{\beta'} e_\beta$ gelten.

Um jetzt die Konfinalität zu definieren, nennen wir zuerst *eine Projektionsmenge* des Spektrums Σ jede Menge $\xi = \{e_\alpha\}$ von Eckpunkten verschiedener Komplexe $\alpha, \beta, \ldots$ usw. des Spektrums Σ, die mindestens einen und immer nur endlichviele Eckpunkte eines jeden $\alpha \in \Sigma$ enthält und der folgenden Bedingung genügt: zu je endlichvielen Elementen $e_{\alpha_1}, \ldots, e_{\alpha_s}$ lässt sich ein solches $\alpha = \alpha\,(e_{\alpha_1}, \ldots, e_{\alpha_s})$ bestimmen, das für $\beta > \alpha$ und $e_\beta \in \xi$ stets

$$e_{\alpha_1} \in \pi_{\alpha'}^\beta e_\beta, \ldots, e_{\alpha_s} \in \pi_{\alpha_s}^\beta e_\beta$$

gilt.

Schliesslich heisst das Teilspektrum Σ' des Spektrums Σ ein konfinaler Teil dieses Spektrums, falls folgende beiden Bedingungen erfüllt sind: ergänzt man irgendein Projektionssystem ξ' des Spektrums Σ' durch sämtliche Projektionen (in Σ) der Elemente von ξ', so erhält man ein Projektionssystem des Spektrums Σ; umgekehrt, behält man in einem Projektionssystem des Spektrums Σ nur die zu dem Spektrum Σ' gehörenden Eckpunkte, so entsteht ein Projektionssystem des Spektrums Σ'.

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THÉORIE DES JEUX ET STRUCTURES TOPOLOGIQUES

CLAUDE BERGE

Sous sa forme extensive la plus générale, on peut assimiler un *n-personnes jeu*, sur un espace abstrait $X = \sum_1^n X_i$, à une structure (au sens de Bourbaki), composée:

1. de la règle Γ, qui est une application multivalente de X dans lui-même (cf. notre Thèse);

2. de la préférence R_i de chaque joueur (i), qui est une relation de quasi-ordre dans X.

On aura un *jeu topologique* pour le joueur (i) si:

1. $X = \sum_1^n X_i$ est la somme topologique d'espaces X_i topologiques.

2. Γ est une application continue de X dans lui-même.

3. R_i est une relation de quasi-ordre continue dans X.

On peut donc parler de jeux „topologiques" au même titre, par exemple, que pour les groupes „topologiques".

Les jeux topologiques sont d'ailleurs extrêmement fréquents, si l'on songe à tous les jeux, comme les jeux de poursuites, où l'on retient la notion intuitive de positions „voisines".

Pour un jeu $(\Gamma, R_1, R_2, \ldots, R_n)$, topologique pour le joueur (i), nos principaux résultats sont les suivants:

1. L'ensemble des positions dans lesquells (i) peut garantir une position strictement plus favorable que y est ouvert.

2. Le meilleur gain $\varphi_i(x_0)$ auquel (i) peut prétendre dans la position x_0 est représenté par une fonction réelle φ_i semi-continue inférieurement sur X. Si la durée du jeu est limitée, φ_i est continue.

3. X étant un espace topologique uniforme, les „stratégies de (i)", qui sont des opérateurs dans X, forment un espace Σ^i topologique uniforme (au sens de la topologie faible des opérateurs).

L'ensemble des bonnes stratégies de (i) est fermé dans Σ^i.

4. Si X est un espace localement compact, si l'ensemble Γx est fermé quel que soit x, et si la durée du jeu est limitée, il existe pour (i) une bonne stratégie effective.

D'autres théorèmes généraliseront très exactement des résultats connus sur les jeux des stratégies composées (Von Neumann-Morgenstern), qui est un jeu topologique — d'ailleurs trivial en ce qui concerne la règle.

8, RUE MATRAT, ISSY-LES-MOULINEAUX, (F.).

LE RETI O DI GUICHARD E IL TEOREMA DI PERMUTABILITÀ DI BIANCHI

RENATO CALAPSO

Dalla più generale rete O di Guichard di uno spazio a sei dimensioni si deduce, mediante la trasformazione di Lie composta, introdotta dall'autore, la più generale configurazione relativa al „teorema di permutabilità" di Bianchi, per le trasformazioni asintotiche delle superficie dello spazio ordinario, e viceversa.

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ON SIEGEL-HUA SPACES: PLANE ELEMENTS OF ZERO SECTIONAL CURVATURE

VITTORIO DALLA VOLTA

I name SIEGEL-HUA Space the well known Riemannian space, with $n = p(p+1)$ real dimensions (p integer > 1), whose linear element is $ds^2 = tr(Y^{-1}dZY^{-1}d\overline{Z})$, where $Z = X + iY = (z^{ij})$ is a symmetric complex matrix with p rows, and det $Y \neq 0$. — I study here the problem of finding the plane elements with zero sectional curvature; the question — first attacked by the late Prof. F. Conforto — was solved for $p = 2$ by B. Segre, while I succeeded in giving a geometrical construction of the concerned elements for any p. Before presenting any result, however, let us recall the main properties of our space: a) (1) admits a continuous, transitive, group of motions, $G_{p(2p+1)} \equiv G$, whose transformations are analytic (in the $\frac{1}{2}p(p+1)$ complex variables z^{ij}), and of the form : $Z' = (ZC + D)^{-1}(ZA + B)$ (with $T = \begin{pmatrix} D & B \\ C & A \end{pmatrix}$ symplectic: $TMT^{-1} = kM$; $M = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$; k real scalar $\neq 0$); b) (1) is both symmetric and kählerian; c) G is the *complete* continuous group of motions for (1); this I proved elementarily, considering Killing's equations; the computations were greatly simplified by b); c) G leaves invariant the eigen-values of $Y^{-1}dZY^{-1}d\overline{Z}$, which appear as (infinitesimal) invariants with respect to G; as far as I know, the first geometrical meaning for these is given here, later. — Due to the transitivity of G, it will suffice to determine the concerned elements through one point P of the space; the question may be then studied in the projective space $S_{p(p+1)-1} \equiv S^*$, of the directions through P; in this interpretation, F. Conforto had already found that the "lines of zero curvature" describe an algebraic system ∞^{p2}, Γ, of S_{p-1}, so that for a generic point A in S^* goes one and only one space of Γ; for it, I found the following construction, invariant with respect to the linear group G' induced in S^* by the isotropy group of P: for c), G' leaves invariant p algebraic forms, $F^{(i)}$ ($i = 1, \ldots, p$) (of degrees $2, \ldots, 2p$); consider then A in S^*, and its polar hyperplanes, $\tau^{(i)}$ with respect to the $F^{(i)}$'s; for A generic, the $\tau^{(i)}$ meet in a space, whose polar space with respect to the quadric $F^{(1)}$ is just the space of Γ through A. — The above process does not apply if and only if A belongs to the Jacobian variety of the $F^{(i)}$'s, Φ; Φ is therefore the locus of the *foci* of Γ. It turns out that Φ is a reducible algebraic variety, with two components, Φ_1 and Φ_2, both of $p(p + 1) - 3$ dimensions; the points of Φ_1 (Φ_2) *correspond to the directions through P for which one at least* (*two at least*) *of the invariants quoted under c*) *vanishes* (*are equal*). The "lines of zero curvature" through a point of Φ belong also to a linear space, but of dimension $> p - 1$; and in this case too I succeeded in finding a geometrical construction.

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LA SUITE SPECTRALE D'UN SYSTÈME DIFFÉRENTIEL EXTÉRIEUR

Paul Dedecker

On connaît le rôle de la suite spectrale dans la théorie des espaces fibrés. On se propose de montrer qu'une situation apparentée se présente lorsque l'on se donne un système différentiel extérieur sur une variété différentiable (paracompacte) V. Aux chaînes *singulières* différentiables à coefficients réels on substitue les chaînes *régulières* (cubiques) de V; elles se définissent comme les chaînes habituelles mais en supposant que les applications caractérisant les simplexes sont différentiables et de rang égal à la dimension du simplexe; soit $C(V)$ le complexe correspondant. Un système différentiel extérieur sur V s'obtient en considérant un idéal homogène et différentiel I de l'algèbre $G(V)$ des formes extérieures de V. On définit alors le sous-complexe $R(I) \subset C(V)$ des chaînes intégrales de I, puis une filtration de $C(V)$ par des espaces $A^0 = R(I) \subset A^1 \subset \ldots \subset A^p \subset \ldots \subset C(V)$. En gros une chaîne appartient à A^p si elle est engendrée par une famille à p paramètres de chaînes intégrales. On en déduit par dualité une filtration de l'algèbre $G(V)$ par des espaces $A^{*0} = G(V) \supset A^{*1} \supset \ldots \supset A^{*p} \supset \ldots$ où A^{*p} est l'espace orthogonal à A^{p-1}. On a notamment

$I \subset A^{*1}$. Dans le cas d'une variété fibrée V, A^{*p} est l'espace des formes de degré $\geqq p$ en les différentielles des coordonnées locales de la base. Plus généralement on peut considérer le cas d'une variété feuilletée, ce qui conduit à une généralisation de la théorie des invariants intégraux de E. Cartan. On définit également une suite spectrale généralisée à deux indices $(E_{r,s}, d_{r,s})$ avec $H(E_{r,s}, d_{r,s}) = E_{r+s,s}$ (la suite classique correspond à $s = 1$).

Une application intéressante est fournie par le calcul des variations, un problème variationnel conduisant à un système différentiel extérieur d'une variété appropriée. Dans le cas des intégrales simples, la méthode d'intégration des équations canoniques par la méthode d'Hamilton-Jacobi s'explique par la trivialité du terme $E^{*2,0}_{2}$ d'une suite spectrale *locale*. L'extension de cette méthode aux intégrales multiples d'ordre $q + 1$ est subordonnée à la question suivante: un certain cocycle de $E^{*2,q}_{1}$ est-il un cobord?

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SUR LES PSEUDOGROUPES DE TRANSFORMATIONS DE LIE

Charles Ehresmann

Voir la définition d'un pseudogroupe de transformations dans: „Sur la théorie des espaces fibrés" (Colloque Topologie algébrique, Paris 1947) et „Structures locales" (Annali di Mat. 1954).

La notion de pseudogroupe de transformations est fondamentale en géométrie. Elle est équivalente à celle de pseudogroupe d'automorphismes locaux d'une *structure locale*.

Soient V_n et V_m deux variétés de classe $\geqq r$, $I^r(V_n, V_m)$ l'espace des r-jets de V_n dans V_m et $\Pi^r(V_n)$ le groupoïde des r-jets inversibles de V_n dans V_n. Un *système différentiel* Φ_r est une variété extraite de $I^r(V_n, V_m)$. On définit l'espace $\Phi_{r,\lambda}$ des germes de solutions et les prolongements successifs de Φ_r: Φ_{r+1}, Φ_{r+2}, ..., Φ_∞, où Φ_∞ est la limite projective de la suite $\Phi_r \leftarrow \Phi_{r+1} \leftarrow \Phi_{r+2} \leftarrow \ldots$ Une solution (généralement multiforme) correspond à un ouvert de $\Phi_{r,\lambda}$. Le système Φ_r est complètement intégrable si $\Phi_{r,\lambda}$ se projette *sur* Φ_r. *Système de Mayer-Lie*: Φ_r est une *variété* telle que son application canonique dans $I^{r-1}(V_n, V_m)$ soit localement biunivoque. Il est complètement intégrable si Φ_{r+1} se projette *sur* Φ_r. Dans ce cas Φ_{r+1} correspond à un champ complètement intégrable d'éléments de contact dans Φ_r, dont les variétés intégrales correspondent aux solutions de Φ_r.

Un *pseudogroupe de Lie* Γ est l'ensemble des solutions uniformes d'un sous-groupoïde $I^r(\Gamma)$ de $\Pi^r(V_n)$, muni d'une structure de sous-variété. Γ sera dit de *type fini* r si $I^r(\Gamma)$ est un système de Mayer-Lie. Le théorème de stabilité de Reeb entraîne: *Supposons* V_n *compact et simplement connexe,* Γ *de type fini* r, $I^r(\Gamma)$ *connexe ou admettant une composante connexe transitive dans* V_n. *Alors* Γ *se déduit par localisation d'un groupe de transformations de Lie.* On définit une notion d'*espace complet* permettant de généraliser ce théorème.

Sur un pseudogroupe Γ on définit diverses topologies:

I. Les ensembles de solutions des ouverts de $I^r(\Gamma)$ forment une base d'une topologie sur Γ.

II. Soit K un compact de V_n, W un ouvert de $I^r(\Gamma)$. Soit $\Omega(K, W)$ l'ensemble des $\varphi \in \Gamma$ tels que: 1) la source U de φ contient K; 2) $j^r\varphi$ relève U dans W. Les ensembles $\Omega(K, W)$ forment une base d'une topologie $\mathcal{T}_r$. L'ensemble des $\mathcal{T}_r$ engendre une topologie $\mathcal{T}$.

III. Dans II remplaçons 2) par 2'): $j^r\varphi$ relève K dans W. Les ensembles $\Omega(K, W)$ engendrent une topologie généralisant celle de la convergence compacte.

Ces topologies servent à définir les notions de *noyau* et de *germe* de pseudogroupe.

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SUR LES STRUCTURES INFINITÉSIMALES RÉGULIÈRES

Charles Ehresmann

Utilisons les notions et notations introduites dans une série de publications récentes (Voir références dans Colloque International Géométrie différentielle, Strasbourg 1953, p. 97).

Les structures qui font l'object de la Géométrie différentielle peuvent être définies en partant de la notion de *prolongement d'une variété différentiable* V_n. Supposons V_n de classe r et posons $r = k + l$. Soit Φ un sous-groupoïde de $\Pi^k(V_n)$, espace des k-jets inversibles de V_n dans V_n. Un prolongement d'ordre k et de classe l de V_n, relativement à Φ, est une variété E munie d'une projection p sur V_n et admettant Φ comme *groupoïde d'opérateurs*: Le composé θz de $\theta \in \Phi$ et $z \in E$ est défini si $\alpha(\theta) = p(z)$ et on a $p(\theta z) = \beta(\theta)$, où $\alpha(\theta) =$ source de θ, $\beta(\theta) =$ but de θ; les applications p et $(\theta, z) \to \theta z$ sont de classe l. Soit Ψ l'ensemble des jets $j^l_z(z \to \theta_{p(z)}z)$, où $x \to \theta_x$ est un relèvement local de classe l de V_n dans Φ; c'est-à-dire $\alpha(\theta_x) = x$. *Transitivité des prolongements*: Tout prolongement d'ordre l de E, relativement à Ψ, est un prolongement d'ordre r de V_n, relativement à Φ^l (prolongement d'ordre l de Φ).

Une *structure infinitésimale pure* d'ordre k est une section σ d'un prolongement E d'ordre k de V_n. Si $E = H^k(V_n)/G$, où $H^k(V_n)$ = espace des repères d'ordre k, G = sous-groupe fermé de L_n^k, la structure σ est appelée *régulière* ou G-structure. Il lui correspond dans $H^k(V_n)$ un sous-espace H', espace des repères distingués, qui est un espace fibré principal à groupe G. On définit les notions de *prolongement d'une structure infinitésimale* et de *covariant différentiel.*

Soit G un sous-groupe de L_n^1. A toute G-structure σ on peut associer au moins une connexion affine. Si σ' est le prolongement du 1er ordre de σ, on peut définir la *torsion* d'un élément de σ'. Celle-ci est nulle si l'élément correspond par un repère du 2e ordre à la G-structure triviale sur R^n. Si la torsion est identiquement nulle, on peut associer à σ une connexion affine sans torsion. Au Congrès de Géométrie différentielle (Italie 1953) j'ai donné une caractérisation des groupes G tels que la torsion soit un covariant représenté par le tenseur de torsion d'une connexion affine associée, plus particulièrement tels qu'à toute G-structure on puisse associer d'une manière covariante une connexion affine déterminée.

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ON THE GENERAL THEORY OF ENVELOPES

Mario O. González

The author considers the m-parameter family of hypersurfaces

$$f(u_i; \alpha_j) = 0, \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, m. \tag{1}$$

By adjoining the partial derivatives $f_{\alpha_j} = 0$ the following system is obtained:

$$\begin{cases} f(u_i; \alpha_j) = 0 \\ f_{\alpha_j}(u_i; \alpha_j) = 0 \end{cases} \tag{2}$$

which contains $m + 1$ equations. It is assumed that the conditions of the implicit function theorem are satisfied for $u_i = u_i^0$, $\alpha_j = \alpha_j^0$, so that for these values $\dfrac{\partial(f, f_{\alpha_1}, \ldots, f_{\alpha_m})}{\partial(u_n, \alpha_1, \ldots, \alpha_m)} \neq 0$. Then the system (2) defines $u_n, \alpha_1, \ldots, \alpha_m$ implicitly as functions of $u_1, u_2, \ldots, u_{n-1}$, and, in particular, u_n as a function of

$u_1, u_2, \ldots, u_{n-1}$, in a certain neighborhood of (u_i^0) $(i = 1, 2, \ldots, n-1)$. From (1) it follows

$$\frac{\partial u_n}{\partial u_i} = -\frac{f_{u_i}}{f_{u_n}} \quad (i = 1, 2, \ldots, n-1). \tag{3}$$

These derivatives as obtained from the system are

$$\frac{\partial u_n}{\partial u_i} = -\frac{\dfrac{\partial(f, f_{\alpha_1}, \ldots, f_{\alpha_m})}{\partial(u_i, \alpha_1, \ldots, \alpha_m)}}{\dfrac{\partial(f, f_{\alpha_1}, \ldots, f_{\alpha_m})}{\partial(u_n, \alpha_1, \ldots, \alpha_m)}} = -\frac{f_{u_i} \cdot \dfrac{\partial(f_{\alpha_1}, f_{\alpha_2}, \ldots, f_{\alpha_m})}{\partial(\alpha_1, \alpha_2, \ldots, \alpha_m)}}{f_{u_n} \cdot \dfrac{\partial(f_{\alpha_1}, f_{\alpha_2}, \ldots, f_{\alpha_m})}{\partial(\alpha_1, \alpha_2, \ldots, \alpha_m)}}. \tag{4}$$

Thus (3) and (4) have the same values if $\dfrac{\partial(f_{\alpha_j})}{\partial(\alpha_j)} \neq 0$ and if $f_{u_i}^2 + f_{u_n}^2 \neq 0$. Therefore if all these conditions are satisfied, (2) defines implicity a hypersurface to which each member of the family in the neighborhood of (u_i^0) is tangent.

In the theorem above, the coordinate system (u_i) is not specified; also it is to be noted that no parametric representation of the envelope need to be introduced.

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ÜBER FLÄCHEN IN 3-DIMENSIONALEN MANNIGFALTIGKEITEN LÖSUNG DES ISOTOPIEPROBLEMS FÜR DEN KREISKNOTEN

Wolfgang Haken

M^3 sei eine (gegebenenfalls berandete) 3-dimensionale Mannigfaltigkeit. Eine geschlossene in M^3 eingebettete 2-dimensionale Mannigfaltigkeit M^2 wird als irreduzibel in M^3 bezeichnet, wenn es keine geschlossene Kurve in M^2 gibt, die in M^2 nicht homotop 1 ist, und in $M^3 - M^2$ ein singularitätenfreies Flächenstück berandet. Ist eine beliebige Zerlegung von M^3 gegeben, so werden deren Eckpunkte mit kleinen Kugeln umgeben, die Kanten mit Vollzylinderstücken und die Flächenstücke mit schmalen Scheiben. Eine 2-dimensionale Mannigfaltigkeit in M^3 nennen wir nun eine Normalmannigfaltigkeit (bezüglich der gegebenen Zerlegung von M^3), wenn sie diese Kugeln, Zylinderstücke und Scheiben in einer bestimmten, verhältnismäßig einfachen Weise durchsetzt. Es läßt sich leicht zeigen, daß jede in M^3 irreduzible 2-dimensionale Mannigfaltigkeit durch isotope Deformationen in eine Normalmannigfaltigkeit überführt werden kann. Ferner ergibt sich, daß alle überhaupt möglichen Normalmannigfaltigkeiten (bezüglich einer fest gegebenen Zerlegung von M^3) ein-

eindeutig den nicht-negativ-ganzzahligen Lösungen eines gewissen homogenen linearen Gleichungssystems entsprechen, dessen Koeffizienten sich von der gegebenen Zerlegung ablesen lassen. — Die vorstehend angedeuteten Betrachtungen lassen sich ohne Weiteres auch auf solche berandeten 2-dimensionalen Mannigfaltigkeiten ausdehnen, deren Ränder ganz im Rande von M^3 liegen. — Die nicht-negativ-ganzzahligen Lösungen des Gleichungssystems lassen sich aus endlich vielen nicht-negativ-ganzzahligen „Fundamentallösungen" als Linearkombinationen mit nicht-negativ-ganzzahligen Koeffizienten darstellen. Die diesen Fundamentallösungen eindeutig entsprechenden Normalmannigfaltigkeiten werden die Fundamentalmannigfaltigkeiten (bezüglich der gegebenen Zerlegung) genannt.

Es läßt sich eine einfache Anwendung dieser Betrachtungen auf die Knotentheorie geben. K sei ein Knoten in M^3, $'M^3$ gehe aus M^3 durch Ausbohren von K hervor; eine beliebige Zerlegung von $'M^3$ sei gegeben. Gibt es nun in $'M^3$ eine zusammenhängende orientierbare 2-dimensionale Mannigfaltigkeit, deren Rand eine zu K parallele Linie in dem Randtorus von $'M^3$ ist, so läßt sich zeigen, daß sich notwendig eine derartige 2-dimensionale Mannigfaltigkeit von minimalem Geschlecht bereits unter den endlich vielen Fundamentalmannigfaltigkeiten befinden muß. Daraus ergibt sich ein Verfahren, um das Geschlecht eines Knotens in einer beliebigen 3-dimensionalen Mannigfaltigkeit M^3 in endlich vielen Schritten zu berechnen, insbesondere also, zu entscheiden, ob sich der betreffende Knoten in einen Kreis deformieren lasse. In ganz ähnlicher Weise ergibt sich ein Verfahren, nach dem man in endlich vielen Schritten entscheiden kann, ob zwei Knoten K_1, K_2 in M^3 miteinander verkettet sind, oder nicht, das heißt, ob es in M^3 eine 2-dimensionale Sphäre gibt, die M^3 so in zwei Teile zerlegt, daß deren einer K_1, deren anderer K_2 enthält.

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ÜBER DIE BAHNKURVEN DER DREHUNGSGRUPPEN VOM FALTUNGSTYPUS IM HILBERTSCHEN RAUM

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In einem reellen Hilbertschen Raum H wird die Gruppe G der Drehungen vom Faltungstypus betrachtet, deren Transformationen dadurch gekennzeichnet sind, daß sie die Koordinatenachse Π_0 sowie das System der Koordinatenebenen Π_r ($r = 1, 2, \ldots$) invariant lassen. In den Koordinatenebenen Π_r erfahren somit die Komponenten eines Vektors X Drehungen um den

Koordinatenursprung O. Jede Drehung aus G erzeugt sodann eine einparametrige kontinuierliche Gruppe, deren Bahnkurven $X(t)$ eingehend untersucht werden. Aus dem System der Ableitungsvektoren $X^{(n)}$ $(n = 0, 1, 2, \ldots)$ der Bahnkurve $X(t)$ werden durch Orthogonalisierung und Normierung die Vektoren ξ_m $(m = 0, 1, 2, \ldots)$ des begleitenden Vielbeins gewonnen, für deren Ableitungen nach der Bogenlänge s die Frenetschen Formeln

$$\frac{d\xi_m}{ds} = - k_{m-1}\, \xi_{m-1} + k_m\, \xi_{m+1} \quad (m = 0, 1, \ldots)$$

gelten. Aus den Krümmungen k_m $(m = 0, 1, 2, \ldots)$ lassen sich sodann für die Schmiegsphären der Dimensionen 1, 2, 3, ... die Koordinaten der Mittenvektoren und die Radien bestimmen. Die Bahnkurve $X(t)$ ist durch ihre Krümmungen bis auf die Drehungen vom Faltungstypus bestimmt, doch reichen hiezu die Radien der Schmiegsphären allein nicht aus. Die Radien der Schmiegsphären der Bahnkurve $X(t)$ bestimmen jedoch im Verein mit den Radien der Schmiegsphären des Tangentenbildes $X'(t)$ die Krümmungen der Bahnkurve $X(t)$ und damit die Gestalt derselben. Der dargelegte Problemkreis steht im engsten Zusammenhang mit der Theorie der orthogonalen Polynome und dem Stieltjes'schen Momentenproblem. Es zeigt sich, daß die Frenetschen Formeln als Rekursionsformeln eines Systems orthogonaler Polynome gedeutet werden können, das mit dem System der Ableitungsvektoren verknüpft ist. Die Bedingungen, denen die Krümmungen der Bahnkurve $X(t)$ genügen, ergeben sich aus den Voraussetzungen für die Lösbarkeit des Stieltjes'schen Momentenproblems.

WIEN IV., KARLSPLATZ 13.

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К некоторым вопросам теории размерности.

Некоторые теоремы теории размерности, известные для пространств счетного веса, доказываются для любых метрических /иногда даже любых нормальных/ пространств.

В частности, приводятся теоремы о размерности образа нормального пространства при непрерывном отображении в пространство Банаха и теорема о замкнутых покрытиях метрического пространства P обладающих тем свойством, что пересечение любых m+1 множеств покрытия или пусто или имеет размерность = dim P − m.

SUR LES FONCTIONS RATIONNELLES HOMOTOPES À DES HOMÉOMORPHIES

KAZIMIERZ KURATOWSKI

Le résultat principal s'exprime comme suit:

Soient A un sous-ensemble localement connexe du plan des nombres complexes (plan de Gauss) et $p_0, \ldots, p_n$ un système de points situés en dehors de A et tels que A coupe le plan entre tout couple de ces points. Soit

$$r(z) = c(z - p_0)^{k_0} \ldots (z - p_n)^{k_n}$$

où $k_0 + \ldots + k_n = 0$ et $k_j \neq 0$ pour $0 \leqq j \leqq n$.

Admettons que la fonction r soit déformable sur A en une homéomorphie de manière qu'au cours de cette déformation aucune fonction intermédiaire n'admet la valeur 0. On a alors

$$| k_0 | + \ldots + | k_n | \leqq 2n.$$

WARSZAWA 12, KIELECKA 42.

SUR LES VARIÉTÉS LOCALEMENT DÉFORMABLES D' UN ESPACE COMPLET

SIMONE LEMOINE

Considérons un espace riemannien complet [1] V_n dans lequel est plongée une variété V_{n-1} complète pour la métrique induite, V_{n-1} et V_n étant de classe C^6, en général. D'après les propriétés des géodésiques des espaces complets [2] la notion de ligne de partage [3] relative à un point d'une surface convexe se généralise: à tout point $x \in V_n$ correspond le domaine $\Delta(x)$ constitué par: — les points à l'infini, s'il en existe, de toute géodésique passant par x, — les points ξ tels que, g étant une géodésique passant par x et ξ, ce point ξ soit le premier pour lequel la *longueur* de l'arc $x\xi$ de g cesse d'être égale à la *distance* de ξ à x dans V_n.

Quel que soit $x \in V_n$, toute géodésique issue de x a au moins un point dans $\Delta(x)$

$$\Delta(x) \neq \varnothing, \qquad V_n = \Delta(x) \cup C_n;$$

C_n est une cellule ouverte, $\Delta(x)$ est de dimension $n - 1$, au plus.

Considérons celles des V_{n-1} plongées dans V_n qui sont localement déformables [4]. Les V_n contenant de telles V_{n-1} sont exceptionnelles pour $n > 3$; alors existe nécessairement, en tout point $x \in V_{n-1}$, une sous-variété $V_{n-3}(x)$ totalement géodésique dans V_n, donc complète et $V_{n-3}(x) \cap \Delta(x) \neq \varnothing$.

Les conditions de déformabilité locale sont satisfaites pour certaines V_{n-1} de tout V_n localement euclidien et pour certaines V_3 des V_4 à courbure constante.

Théorème I. — Dans la sphère à 4 dimensions, toute V_3 localement déformable, complète pout la métrique induite par son plongement, contient nécessairement l'antipode de chacun de ses points.

Théorème II. — Dans un espace V_n à métrique définie positive, simplement connexe et soit localement euclidien ($n > 3$), soit localement hyperbolique ($n = 4$) toute V_{n-1} localement déformable et complète pour la métrique induite par son plongement, possède nécessairement un domaine à l'infini.

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SUR LES SOUS-PSEUDOGROUPES ET PROLONGEMENTS D'UN PSEUDOGROUPE DE LIE

PAULETTE LIBERMANN

Cet exposé résume en partie un travail consacré aux pseudo-groupes de Lie [4], où par l'utilisation de la théorie des jets de C. Ehresmann [2], sont étudiées certaines questions relatives à la théorie des "groupes infinis" de E. Cartan [1].

Soit Γ un pseudogroupe de transformations analytiques opérant sur une variété V_n (on ne considère que de tels pseudogroupes); Γ est un *pseudogroupe de Lie* s'il est *complet d'ordre* q (c'est-à-dire si Γ est l'ensemble des solutions du groupoïde $J^q(\Gamma)$ associé à Γ, engendré par tous les q-jets $j^q_x f$ où $f \in \Gamma$) et si $J^q(\Gamma)$ est une *sous-variété analytique complètement régulière* de l'ensemble $J^q(V_n, V_n)$ de tous les q-jets dont la source et le but appartiennent à V_n : $J^q(\Gamma)$ est défini localement par un système d'équations aux dérivées partielles vérifiant certaines conditions de régularité.

Le *prolongement canonique* d'ordre s de Γ est le pseudogroupe Γ^s (opérant sur $J^{s-1}(\Gamma)$), ensemble de toutes les applications f^s: $X^{s-1} \to X^{s-1}\,(j^{s-1}_x f)^{-1}$ où $X^{s-1} \in J^{s-1}(\Gamma)$ et $f \in \Gamma$. Les prolongements canoniques successifs de Γ sont des pseudogroupes de Lie et si $s \geq q$, Γ^s est du premier ordre.

Un *sous-pseudogroupe* Γ' de Γ (c'est-à-dire un pseudogroupe Γ' tel que $\Gamma' \subset \Gamma$) est dit *complet par rapport à* Γ, *d'ordre relatif* h si Γ' est l'ensemble des applications de Γ, solutions du groupoide $J^h(\Gamma')$ associé à Γ', h étant l'ordre minimum des groupoïdes associés jouissant de cette propriété. On démontre que le sous-pseudogroupe $\Gamma'^{(h+1)}$ du prolongement canonique Γ^{h+1} de Γ, engendré par toutes les applications se projetant sur V_n suivant des applications de Γ' est d'ordre relatif 0. Si, de plus, Γ' est un pseudogroupe de Lie, on démontre que les classes d'intransitivité suivant $\Gamma'^{(h+1)}$ déterminent sur $J^h(\Gamma)$ un *feuilletage complètement régulier*. Inversement à un tel feuilletage, correspond moyennant certaines conditions d'intégrabilité, un pseudogroupe de Lie Γ', sous-pseudogroupe de Γ, d'où la détermination de tous les pseudogroupes de Lie, sous-pseudogroupes de Γ.

Un pseudogroupe $\tilde{\Gamma}$, opérant sur une variété V_m est le *prolongement* d'un pseudogroupe Γ, opérant sur une variété V_n s'il existe une application p de V_m *sur* V_n telle que tout $f \in \Gamma$ soit la projection d'au moins une application $\tilde{f} \in \tilde{\Gamma}$ (c'est-à-dire si $fp = p\tilde{f}$). Si à tout $f \in \Gamma$ correspond une seule application $\tilde{f} \in \tilde{\Gamma}$ le prolongement est dit *holoédrique* (exemple: prolongements canoniques d'un pseudogroupe de Lie); il est *mériédrique* dans le cas contraire.

En introduisant les notions de *feuilletage invariant par un pseudogroupe* et de *sous-pseudogroupe associé à un feuilletage* on définit les pseudogroupes *primitifs, holoédriquement et mériédriquement imprimitifs*. Un pseudogroupe Γ est dit *simple* si tout prolongement holoédrique de Γ est holoédriquement imprimitif ou primitif; on en déduit la notion de sous-pseudogroupe invariant.

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52 RUE DE LA TOUR D'AUVERGNE.
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SUR CERTAINES PROJECTIONS DES VARIÉTÉS ALGÉBRIQUES

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Soit V^r une variété algébrique (absolument irréductible) à r dimensions dans un espace projectif de dimension quelconque, définie sur un corps k. Soit $m = \max(r + d - 1, 2r + 1)$, où d est le maximum des dimensions des espaces tangents de Zariski à la variété V. Alors on démontre le théorème suivant:

Théorème. Il existe une correspondance birationnelle et birégulière (engendrée par une projection), définie sur k, entre la variété V et une variété V' d'un espace projectif de dimension m. Si le corps k est fini, la correspondance est définie sur une extension finie k' de k.

Corollaire. Si la variété V est sans singularités il en est de même de V' et V' se trouve dans un espace de dimension $2r + 1$.

Les demonstrations vont paraitre dans un autre recueil.

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SIMPLEX-INHALTE IM ELLIPTISCHEN R_3

W. MAIER

Im dreidimensionalen Raum fester Krümmung $R > 0$ bestimmen 4 nicht komplanare Punkte ein Simplex, dessen Herstellung durch 6 Kantenlängen oder Vorgabe der entsprechenden 6 Keilwinkel bewirkt werden kann. Durch Lotkonstruktion von einer Ecke auf die Gegenfläche und aus dem Fußpunkt auf die drei Kanten seiner Fläche zerfällt das beliebige Simplex in 6 besondere, die je von nur 3 Parametern abhängen und seit L. Schläfli [1]) als Orthoscheme bezeichnet werden. Jedes Orthoschem [$= Os$] hat neben seinen 6 Kanten nur noch 9 nicht auf $\frac{\pi}{2}$ entartete Winkel, und es sollen diese $15 = \binom{6}{2}$ Bestimmungselemente mit $0 \leqq h < k \leqq 5$ bezeichnet werden durch (h, k). Bildet man weiter die Ergänzungen

[1]) Schläfli, Werke I, Basel (1950), p. 243.

$$\frac{\pi}{2} - (h, k) = (k, h)$$

und den vierfachen Inhalt des aus den Keilwinkeln 01, 05, 45 aufgebauten *Os'*

$$S \begin{pmatrix} 10 \\ 05 \\ 54 \end{pmatrix} = S,$$

so berechnet sich dessen Inhaltszuwachs bekanntlich aus

(1) $$\frac{1}{2} dS = (12)\, d\,(45) - (23)\, d\,(50) + (34)\, d\,(01).$$

Die Ausscheidung der Kanten zugunsten der Winkel in (1) gelingt, wenn eine Invariante ι bestimmt wird aus

(2) $$tg(01)\, tg(34) = tg(12)\, tg(45) = i\, tg\, \iota.$$

Es folgt

$$0 < i\,.\,\iota$$

und mit Ansätzen der imaginären Geometrie Lobatschewskys [2])

(1′) $$i\, d\, S \begin{pmatrix} 10 \\ 05 \\ 54 \end{pmatrix} = d(10)\, ln\, \frac{\cos(\iota-10)}{\cos(\iota+10)} - d(05)\, ln\, \frac{\cos(\iota-05)}{\cos(\iota+05)} + \\ + d(54)\, ln\, \frac{\cos(\iota-54)}{\cos(\iota+54)}.$$

In der für ganze g längs

$$i(x + \frac{\pi}{2} + \pi\, g) < 0$$

geschlitzten Ebene erklären wir die Transzendente

$$\int_{x}^{0} d\xi\, ln \cos \xi = \Lambda(x)$$

und gelangen wie Coxeter [3]) zu

(3) $$i\, S = \left\{ \begin{array}{l} \Lambda(\iota - 10) - \Lambda(\iota - 05) + \Lambda(\iota - 54) \\ + \Lambda(\iota + 10) - \Lambda(\iota + 05) + \Lambda(\iota + 54) \end{array} \right. - 2\,\Lambda(\iota).$$

Wird im elliptischen Raum der Krümmung 1 an einem Ende der längsten *Os*-Kante ein Ergänzungs-*Os* angebaut, so daß jeweils dort (h, k) verlängert wird um (k, h), dann geht dadurch

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[2]) N. J. Lobatschewsky, Imaginäre Geometrie. Leipzig (1904), p. 100 ff.

[3]) H. S. M. Coxeter, Quart. J. Math., Oxford, Serie 6, (1936), p. 23 ff.

$$S\begin{pmatrix}10\\05\\54\end{pmatrix} \to S^*\begin{pmatrix}34\\45\\50\end{pmatrix} \text{ und } \iota \to \iota^* = \frac{\pi}{2} - \iota.$$

Führt die Wiederholung des Prozesses

$$S^* \to S^{**},$$

so gibt (3)

$$i(S - S^{**}) = \begin{cases} \Lambda(\iota - 10) - \Lambda(\iota - 05) + \Lambda(\iota - 43) - \Lambda(\iota + 32) \\ + \Lambda(\iota + 10) - \Lambda(\iota + 05) + \Lambda(\iota + 43) - \Lambda(\iota - 32) \end{cases}$$

während (1) andererseits

$$i(S - S^{**}) = 2\,i[(01)(34) - (23)(50)]$$

liefert.

Nach Ausscheidung der geometrischen Hilfsbegriffe bleibt eine lineare Funktionalgleichung zwischen 8 Λ-Funktionen mit drei unabhängigen Argumenten. Aus

$$0 < ix_0;\ \nu = 1, \ldots, 4;\ |\,Re\,(x_0 \mp x_\nu)\,| < \frac{\pi}{2};\ tg\, x_1\, tg\, x_3 = tg\, x_2\, tg\, x_4 = -\, ictg\, x_0$$

folgt somit

$$(4) \qquad x_1 x_3 - x_2 x_4 + \frac{1}{2} \sum_{h=1}^{4} h\, i^{2h-1} \left[\Lambda\,(x_0 - x_h) + i\pi x_h + \Lambda\,(x_0 + x_h)\right] = 0.$$

Während bei Lobatschewsky [2]) erst durch 19, bei Rogers [4]) durch 10 kombinierte Λ-Funktionen der Rückgang auf elementare Funktionen vollzogen wurde, ergibt das Eingreifen der elliptischen Geometrie durch (4) eine Verknüpfung von nur 8 Λ-Funktionen, deren Argumente trigonometrisch auf 3 unabhängige reduzierbar sind.

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UN THÉORÈME D'UNICITÉ BIRATIONNELLE POUR LES FONCTIONS ALGÉBRIQUES DE PLUSIEURS VARIABLES

Ermanno Marchionna

Considérons dans un espace projectif complexe $S_r(x_1, x_2, \ldots, x_{r-1}, u)$, une V_{r-2} multiple, F_u, d'ordre μ, définie par une fonction algébrique, à μ valeurs,

$$u = u(x_1, x_2, \ldots, x_{r-1}),$$

sur une V_{r-2} algébrique

[4]) L. J. Rogers, Proc. Lond. Math. Soc., London, Serie 2, Bd. 4, (1907), p. 174 ff.

$$F(x_1, x_2, \ldots, x_{r-1}) = 0.$$

Soit

$$A(x_1, x_2, \ldots, x_{r-1}, u) = 0$$

la V_{r-1} qui définit la fonction u.

Nous dirons que F_u est une variété multiple *générale* lorsque F et A sont des variétés générales dans leurs espaces.

Pour une telle variété multiple (si l'on suppose $\mu > 4$) il existe ce théorème d'unicité birationnelle:

,,*Deux fonctions algébriques des points de F, ayant sur F la même variété de diramation de F_u, sont birationnellement identiques*''.

La démonstration est assez complexe et on la obtient surtout au moyen de considérations de Topologie et de Théorie des Groupes: ici nous pouvons seulement en donner un bref résumé:

On établit d'abord un critère d'identité birationnelle:

,,*Soient F_u et $F_{u'}$ deux variétés multiples, définies sur une même V_{r-2}:*

$$F(x_1, x_2, \ldots, x_{r-1}) = 0$$

par deux fonctions u et u', ayant la même variété de diramation.

Supposons que les courbes f_u et $f_{u'}$, sections de F_u et $F_{u'}$ par un même espace S_3 (parallèle à l'axe u et d'ailleurs générique) soient birationnellement identiques. Alors F_u et $F_{u'}$ sont elles aussi birationnellement identiques''.

En vertu de ce critère notre question se réduit à démontrer l'identité birationnelle de deux courbes multiples f_u et $f_{u'}$, qui soient les sections de deux surfaces multiples F_u et $F_{u'}$ satisfaisant aux conditions suivantes.

a) F_u et $F_{u'}$ soient définies sur une surface générale

$$F(xyz) = 0$$

de l'espace ordinaire, par deux fonctions

$$u = u(xyz); \qquad u' = u'(xyz)$$

ayant sur F la même courbe de diramation D.

b) F_u soit une surface multiple générale d'ordre $\mu > 4$.

Le problème est résolu en deux pas.

I) On construit d'abord une forme canonique de la surface de Riemann de la courbe multiple générale f_u, en donnant le système des substitutions fondamentales du groupe de monodromie (qui est imprimitif).

Cette construction qui donne un instrument essentiel pour notre recherche, est faite profitant de la forme limite (de M. Chisini) des courbes de diramation des plans multiples généraux.

II) Ensuite, on montre l'identité birationnelle des deux courbes multiples f_u et $f_{u'}$ (et à ce point-là le problème est résolu).

A ce but on prouve l'isomorphisme des groupes de monodromie de deux fonctions associées à f_u et $f_{u'}$; et pour cela il faut conduire une analyse de topologie et de théorie des groupes plutôt difficile qui exige aussi l'extension d'une condition d'invariance d'Enriques, bien connue.

(VIALEX ABRUZZI 44) MILANO, ITALIA.

GESCHLOSSENE MINIMALFLÄCHEN II [1]

M. PINL

Nach einem Satz von Kommerell und Struik [2]) kann unter hinreichenden Differenzierbarkeitsvoraussetzungen jede Fläche als Minimalfläche angesehen werden, wenn man sie in einem geeigneten Riemannschen Raum einbettet. Eine solche Einbettung kann man eine *extremale* oder auch ein *Minimaleinbettung* nennen. Dabei zeigt sich, daß Eigenschaften, die auf „gewöhnlichen" Minimalflächen (d.h. bei Einbettung im gewöhnlichen euklidischen Raum) simultan nicht bestehen können, im geeeignet abgeänderten Einbettungsraum verträglich werden. So ist die geschlossene und abwickelbare Fläche

$$x_1 = \cos u,\ x_2 = \sin u,\ x_3 = \cos v,\ x_4 = \sin v$$

sicherlich keine Minimalfläche des euklidischen R_4 der kartesischen Koordinaten x_1, x_2, x_3, x_4 wohl aber eine solche im dreidimensionalen sphärischen Einbettungsraum

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2.$$ [3])

Ebenso ist die Schnittkugel der beiden Räume

$$(1) \qquad x_1 = u_1,\ x_2 = u_2,\ x_3 = u_3,\ x_4 = 0$$

und

$$(2) \qquad x_1 = \frac{v_1 + v_2}{1 + v_1 v_2},\ x_2 = \frac{i(v_2 - v_1)}{1 + v_1 v_2},\ x_3 = \frac{v_1 v_2 - 1}{v_1 v_2 + 1},\ x_4 = v_3$$

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[1]) vgl. M. Pinl, Geschlossene Minimalflächen I, Compositio Mathematica 12 (im druck)

[2]) vgl. J. A. Schouten und D. J. Struik, Einführung in die neueren Methoden der Differentialgeometrie II, § 11, S. 94; Groningen 1938.

[3]) Ausführliche Darstellung in [1].

sicherlich keine Minimalfläche des euklidischen Koordinatenraumes (1) wohl aber eine solche des Zylinderraumes (2) mit positiver konstanter Gaußscher Krümmung und geschlossenem Verlauf. Desgleichen bilden die Kugelerzeugenden kein Imprimitivitätssystem in Bezug auf die Translationsgruppe des euklidischen Koordinatenraumes (1) wohl aber eine solche in Bezug auf die Translationsgruppe im nichteuklidischen Zylinderraum (2).

2 FULLER ROAD,
RAMNA-DACCA, EAST-BENGAL, PAKISTAN.

SUR LES QUADRIQUES GÉNÉRALISÉES

BELLINO ANTONIO ROSINA

Je présente ici une famille de surfaces algébriques que j'appelle *quadriques généralisées*, jouissant de propriétés analogues aux propriétés diamétrales des quadriques. Ce sont des surfaces d'ordre $2n$ $(n > 1)$

$$F^{2n} \equiv \varphi^n + \varphi^{n-1} f_1 + \sum_{s=0}^{2n-2} U_s(x, y, z) = 0,$$

où φ, f_1, U_s sont, des formes respectivement quadratiques, linéaires, de degré s dans les variables (x, y, z). Le plan à l'infini rencontre F^{2n} suivant une conique K de multiplicité 2 (au moins) comptée n fois. Pour $n = 2$ on a les F^4 à conique double, surfaces étudiées par Clebsch, Cremona, Lemonnier etc., dont les plus remarquables sont les cyclides de Darboux, objet de nombreuses recherches. Nouvelles propriétés de ces surfaces F^4 seront déduites des propriétés diamétrales que je trouve pour les quadriques généralisées F^{2n} en suivant la méthode de M. Piazzolla-Beloch dans la théorie diamétrale des surfaces [1]).

Après avoir donné la définition du discriminant (Δ) je fais une classification des surfaces F^{2n} parfaitement analogue à celle des quadriques.

Je distingue ainsi les surfaces F^{2n} en *quadriques généralisées propres* et *spécialisées* selon $\Delta \neq 0$ ou $\Delta = 0$; en outre d'après la nature de la conique K (imaginaire ou réelle, irréductible ou réductible) où je trouve des cas que j'appelle *elliptique, hyperbolique, parabolique*. Je trouve que les quadriques généralisées (propres ou spécialisées) elliptiques ou hyperboliques ont leurs plans diamétraux passant par un même point (*point principal*) situé à une distance finie; dans le cas parabolique général ils passent par un même point situé à une distance infinie (*point principal impropre*); dans le cas parabolique spé-

[1]) V. M. Piazzolla-Beloch, Teoria diametrale delle superficie algebriche, U.M.I., 1951

cialisé les plans diamétraux passent, en général, par une même ligne droite, lieu du point principal indéterminé; en particulier la F^{2n} peut avoir un seul plan diamétral conjugué à toutes les directions de l'espace et lieu du point principal indéterminé.

Enfin la F^{2n} (propre ou spécialisée) dans le cas elliptique ou hyperbolique présente trois plans principaux deux à deux perpendiculaires.

S'il y en a une infinité, ils passent:

1) par une même ligne droite; mais il existe alors un autre plan principal qui passe par le point principal de la surface où il rencontre perpendiculairement la ligne droite précédente.

2) par un même point, à une distance finie (*point principal*); on a alors la sphère généralisée:

$$(x^2 + y^2 + z^2)^n + (x^2 + y^2 + z^2)^{n-1}f_1 + \sum_{s=0}^{2n-2} U_s(x, y, z) = 0.$$

Dans le cas parabolique (propre ou spécialisé) la F^{2n} admet, en général, deux plans principaux rectangulaires. En particulier on peut avoir:

1) une infinité de plans principaux qui passent par une même ligne droite, lieu du point principal indéterminé.

2) un seul plan principal, lieu du point principal indéterminé.

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EINE KLASSE VON HUYGENS-SCHEN DIFFERENTIALGLEICHUNGEN

Karl L. Stellmacher

Sei für eine lineare hyperbolische Differentialgleichung 2. Ordnung $\mathfrak{L}u = 0$ das Cauchysche Anfangswert-Problem gestellt: auf einer raumartigen Anfangshyperfläche $\mathfrak{A}$ soll $u(x)$ nebst Ableitungen 1. Ordnung vorgegebene Werte annehmen. In einem gewissen Nachbargebiet $\mathfrak{G}(x)$ von $\mathfrak{A}$ existiere genau eine Lösung. Man nennt die Differentialgleichung in $\mathfrak{G}(x)$ Huygenssch, wenn die Lösung in jedem Punkte $P \subset \mathfrak{G}$ nur von den Anfangsdaten auf dem Durchschnitt der Punkt-Menge $\mathfrak{A}$ dem charakteristischen Konoid mit der Spitze in P abhängt. Bekannte Beispiele („triviale") für solche „Huygensschen" Differentialgleichungen sind $\square\, u = u_{x_0 x_0} - \sum_{i=1}^{m} u_{x_i x_i} = 0$ falls $m + 1 \equiv 0 \pmod{2}$ Besonders Hadamard hat wiederholt auf das Problem hingewiesen, alle solche

Huygensschen Differentialgleichungen zu bestimmen. (Insbs. Lectures on Cauchy's problem; New Haven 1921). Ich möchte hier einige Beispiele Huygensscher Differentialgleichungen mitteilen mit einer Variablen-Zahl $m + 1 > 4$:

$$(1)\qquad \left(u_{x_0x_0} - \frac{v_0(v_0+1)}{x_0^2}\right) - \sum_{i=1}^{m}\left(u_{x_ix_i} - \frac{v_i(v_i+1)}{x_i^2}\right) = 0.$$

Sie sind Huygenssch dann und nur dann, wenn 1.) $m + 1 \equiv 0 \pmod 2$ 2.) $v_j = 0, 1, 2, \ldots$ $(j = 0, 1, \ldots m)$ und 3.) $\sum_{j=0}^{m} v_j \leqq \frac{m-3}{2}$. Die Gleichungen (1) sind von der Gestalt $\Box u + \tau(x)\, u = 0$ mit $\tau(x) \not\equiv 0$. Sie lassen sich daher nicht durch Koordinatentransformation, lineare Transformation der Unbekannten $u = \varrho(x)\bar{u}$ und Multiplikation der Differentialgleichung mit einem Skalar $\delta(x)$ $(\varrho(x),\ \delta(x) > 0$ und $\epsilon\, D_2$ in $\mathfrak{G})$ auf die triviale Form $\Box u = 0$ bringen. — Der Beweis für das Huygenssche Verhalten der genannten Gleichungen (1) wird am besten in der Gestalt

$$(\bar{1})\ \bar{\mathfrak{L}}\bar{u} \equiv \bar{u}_{x_0x_0} + \frac{2v_0+2}{x_0}\bar{u}_{x_0} - \sum_{i=1}^{m}\left(\bar{u}_{x_ix_i} + \frac{2v_i+2}{x_i}\bar{u}_{x_i}\right) = 0$$

erbracht, in die (1) durch die Transformation $u = \prod_{i=0}^{m} x_i^{v_i+1}\,\bar{u}$ übergeht. Sind speziell die v_i ganze Zahlen und ist etwa $m \sum_{i=1}^{m} (2v_i + 3) = f \geqq 2v_0 + 3$, so sind alle Lösungen von $(\bar{1})$ auch Lösungen der ultrahyperbolischen Differentialgleichung (2) $\sum_{j=1}^{f} \bar{u}_{x_{0j}} = \sum_{i=1}^{m}\sum_{j=1}^{2v_i+3} u_{ij}$, wofern $x_i = \sqrt{\sum_{j=1}^{2v_i+3} x_{ij}^2}$ $(i = 0, 1, \ldots m)$ gesetzt wird. — Hängt umgekehrt eine Lösung von (2) nur von diesen x_j ab, aber $u_{x_{0j}} \equiv 0$ (für $j = 2v_0 + 4, 2v_0 + 5, \ldots, f$) so ist sie auch Lösung von $(\bar{1})$. Daher ist der Integralsatz von Asgeirsson anwendbar (Math. Ann. 1936, S. 321 *f.*). *) Nach Ausführung gewisser Quadraturen in diesem Integralsatz erhält man dann eine Volterrasche Integralgleichung, die sich durch eine endliche Anzahl passender Differentiationen lösen lässt. Die gewonnene Lösungsformel gestattet den Huygensschen Charakter unter den angegebenen Bedingungen abzulesen. — Die Hadamardsche Grundlösung von (1) lässt sich explicite angeben. In dieser Grundlösung verschwindet der logarithmische Anteil genau unter den angegebenen Bedingungen 1.), 2.), 3.). Die Konstruktion dieser Grundlösung interessiert vielleicht auch deshalb, weil man dabei auf eine Appellsche Verallgemeinerung der Gaussschen hyper-

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*) Auch Courant-Hilbert II, 417 ff.

geometrischen Reihe mit mehreren unabhängigen Variablen geführt wird, insbesondere auf die in dieser Theorie benutzten überbestimmten Systeme von partiellen Differentialgleichungen. (Siehe P. Appell u. J. Kampé de Fériet: Fonctions Hypergéométriques et . . ., Paris 1926, insbesondere S. 114, Gl. (2) für F_B und S. 117.).

MATHEM. INST., GÖTTINGEN.

ESPACES HOMOGÈNES ET GROUPES DE LIE EXCEPTIONNELS

JACQUES LÉON TITS

L'objet de cette communication est l'étude d'une classe d'espaces homogènes, que nous nommerons ici R-espaces, à laquelle appartiennent notamment les espaces projectifs complexes, les hyperquadriques projectives complexes (intrinsèques) et plus généralement tous les espaces homogènes de dimension minimum des groupes de Lie complexes simples, les grassmanniennes des espaces projectifs complexes, les grassmanniennes des hyperquadriques (en appelant ainsi l'espace des variétés linéaires de dimension donnée d'une hyperquadrique projective). Nous indiquons ici quelques théorèmes généraux relatifs aux R-espaces; l'application de ces théorèmes permet de donner une description géométrique assez détaillée de n'importe quel R-espace particulier, arbitrairement choisi. En envisageant plus spécialement le cas des R-espaces des groupes exceptionnels, on peut obtenir ainsi diverses interprétations géométriques de ces groupes.

La définition des R-espaces est la suivante. Soient G un groupe semi-simple complexe, C une sous-algèbre de Cartan de l'algèbre de Lie de G. Σ un système de racines simples de G par rapport à $\overline{C}$, Σ' une partie de Σ, $G(\Sigma')$ le sous-groupe (connexe) de G engendré par $\overline{C}$ et par les vecteurs propres et correspondant à toutes les racines négatives et aux racines positives appartenant à Σ', et enfin $G[\Sigma'] = G/G(\Sigma')$. *Les R-espaces sont les espaces $G[\Sigma']$ ainsi obtenus.*

Dans l'espace projectif E_n à n dimensions sont définies des variétés linéaires à $-1, 0, 1, 2, \ldots, n$ dimensions, jouissant des propriétés suivantes: toute v.l. à i dimensions de E_n est, au point de vue intrinsèque, un espace projectif à i dimensions dont les v.l. sont les v.l. de E_n qui y sont contenues; l'intersection de deux v.l. de E_n est une v.l.; l'espace des v.l. de dimension i donnée de E_n contenant une v.l. donnée, de dimension $j \leqq i$, est isomorphe à la grassmannienne des v.l. de dimension $i-j-1$ de E_{n-j-1}. De même, on peut définir dans tout R-espace E diverses espèces de sous variétés, généralisant les v.l. des diverses dimensions de E_n, que nous nommerons l-variétés et qui jouissent des propriétés suivantes: toute l-variété de E est, au point de vue intrinsèque, un R-espace dont les l-variétés sont les l-variétés de E qui y sont contenues; l'intersection de deux l-variétés est une l-variété; l'espace des l-variétés d'espèce donnée de E contenant une l-variété donnée est un R-espace; l'ensemble vide est une l-variété. De plus, si nous associons au R-espace $E = G[\Sigma']$ (v. plus haut) le schéma obtenu en marquant d'un signe distinctif les sommets de la figure de Schläfli du système Σ qui appartiennent à Σ', des règles simples permettent de déduire immédiatement de ce schéma quelles sont les diverses espèces de l-variétés de E et leurs relations d'inclusion, la structure intrinsèque des l-variétés, la structure des l-variétés d'espèce donnée contenant une l-variété donnée, . . .

Toute R-espace $E = G[\Sigma']$ peut être plongé d'une infinité de manières dans un espace projectif de telle façon que les éléments de G considérés comme des transformations de E s'étendent en des projectivités; un tel plongement est déterminé, à un isomorphisme près, par un système d'entiers positifs associés aux racines de Σ'. Tout plongement φ de E dans un espace projectif P détermine par restriction, des plongements des l-variétés de E dans des v.l. de P; les systèmes d'entiers caractéristiques de ces plongements se déduisent aisément du système d'entiers caractéristiques de φ.

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PROBLEMI INTEGRALI SULLE TRASFORMAZIONI PUNTUALI

Mario Villa

1. Sono state svolte recentemente ricerche riguardanti la determinazione di trasformazioni puntuali tra spazi lineari che, nell'intorno del 2° ordine di una coppia generica di punti corrispondenti, si comportano in modo particolare.

Sorgono ora ricerche analoghe di caratterizzazione di trasformazioni puntuali che si comportano in modo particolare nell'intorno del 3° ordine di una coppia generica di punti corrispondenti. E ciò offre notevole interesse per le trasformazioni puntuali fra due piani in quanto per queste si hanno invarianti (proiettivi) solo a partire appunto dall'intorno del 3° ordine.

Nelle ricerche svolte col possente metodo del riferimento mobile intervengono in modo essenziale le omografie tangenti, e con esse le corrispondenze linearizzanti di ČECH, collegate a loro volta a certe due forme differenziali quadratiche Ω_1, Ω_2.

Da tempo, nelle ricerche locali sulle trasformazioni puntuali, io ho studiato le trasformazioni quadratiche osculatrici, che svolgono nell'intorno del 2° ordine un ufficio analogo a quello delle omografie tangenti per l'intorno del 1° ordine. Mi parve quindi assai probabile che nelle nuove ricerche integrali, relative all'intorno del 3° ordine, col metodo del riferimento mobile dovessero pure svolgere un'azione importante tali trasformazioni quadratiche.

Ciò si verifica appieno come viene provato dalla presente comunicazione. Apparirà come le omografie tangenti, la corrispondenza linearizzante di ČECH, le forme Ω_1, Ω_2 trovano perfetto riscontro nelle trasformazioni quadratiche osculatrici, in una nuova corrispondenza (analoga a quella di ČECH) relativa a quest'ultime, in certe forme differenziali cubiche Θ_1, Θ_2. E i nuovi enti si presentano, nel metodo del riferimento mobile, in modo analogo e con la stessa spontaneità dei primi.

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Section IV

PROBABILITY AND STATISTICS

SHORT LECTURES

ON THE SUM OF INDEPENDENT k-DIMENSIONAL RECTANGULAR VARIATES.

FRANCISCO AZORIN

Given n independent k-dimensional variates

$$\xi_{(i)} = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{ik}), \ i = 1, 2, \ldots, n$$

the ξ_{ij} being one-dimensional components of the $\xi_{(i)}$ and independent rectangular variates (0,1), we shall now define the variate $\xi = \xi_{(1)} + \xi_{(2)} + \ldots + \xi_{(n)}$ as a k-dimensional variate or random vector which has as one-dimensional components the k-variates: $\xi_j = \xi_{1j} + \xi_{2j} + \ldots + \xi_{nj}$, $j = 1, 2, \ldots, k$ and has therefore, as its distribution function

$$F(x_1, \ldots, x_k) = P(\sum_{i=1}^{n} \xi_{i1} \leqq x_1, \ldots, \sum_{i=1}^{n} \xi_{ik} \leqq x_k)$$

and as its density function

$$\frac{1}{[(n-1)!]^k} \prod_{i=1}^{k} \left[x_i^{n-1} - \binom{n}{1} (x_i - 1)^{n-1} + \binom{n}{2} (x_i - 1)^{n-2} + \ldots \right]$$

with range

$$0 \leqq x_i \leqq n \quad (i = 1, 2, \ldots, n).$$

All the marginal ξ_{ij} variates being rectangular (0,1), its characteristic function is

$$\left(\frac{e^{it_1} - 1}{it_1}\right)^n \ldots \left(\frac{e^{it_k} - 1}{it_k}\right)^n.$$

If $n \to \infty$ the characteristic function of the standardized variates converges to $e^{-\frac{t^2}{2}}$ and it follows that the k-dimensional variate has as its c.f. $e^{-\frac{1}{2}(t_1^2 + \ldots + t_k^2)}$ which is the c.f. of the k-dimensional normal distribution

$$\frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}(x_1^2 + \ldots + x_k^2)}.$$

For $k = 2$, $n = 2$, we have the two-dimensional generalization of the so called *triangular distribution*. Its marginal distributions are in fact triangular distributions with range (0,2), and the density function has the expressions

0	0	1	2
1	xy	$x(2-y)$	
2	$(2-x)y$	$(2-x)(2-y)$	

The first five moments are

$$\alpha_{10} = \alpha_{01} = 1, \ \sigma_1^2 = \sigma_2^2 = \tfrac{1}{6}, \ \mu_{11} = 0.$$

The tangent planes at the cuspidal point (1,1,1) are

$$x + y - z = 1, \quad x - y - z = -1$$
$$x - y + z = 1, \quad x + y + z = 3.$$

The surface wich represents the density function is composed of four regions, onesheet hyperboloids, the range of the distribution being the square with principal vertices (0,0) and (2,2).

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UEBER BESONDERE GLEICHUNGSSYSTEME DER WIRTSCHAFSMATHEMATIK UND DER STATISTIK

FELIX BURKHARDT

Fuer die oekonomischen Groessen
P_i = Produktionswert des Produktionszweiges i,
M_{ik} = Wert der Lieferung von i an K,
B_i = Wert der Lieferung von i an die nicht produzierenden Bedarfstraeger,

$$\frac{M_{ik}}{P_k} = m_{ik} \quad (i, k = 1, 2, \ldots, n)$$

besteht die Matrizengleichung

(1) $(\mathfrak{E} - \mathfrak{M})\,\mathfrak{P} = \mathfrak{B}$

$$\mathfrak{M} = \{m_{ik}\}, \ \mathfrak{P} = (P_1, \ldots, P_n), \ \mathfrak{B} = (B_1, \ldots, B_n).$$

Die Koeffizientenmatrix hat die Eigenschaft $\lim\limits_{\nu\to\infty} \mathfrak{M}^\nu = 0$.

Auf Grund von $\lim\limits_{\nu\to\infty} \mathfrak{M}^\nu = 0$ gelten die Beziehungen

(2) $\mathfrak{N}(\mathfrak{E} - \mathfrak{M}) = \mathfrak{E}$,

$\mathfrak{N} = \mathfrak{E} + \mathfrak{M} + \mathfrak{M}^2 + \ldots$ (Neumann'sche Reihe).

(3) $\mathfrak{N}(\mathfrak{E} - \mathfrak{M})\mathfrak{P} = \mathfrak{P}$,

(4) $\mathfrak{N}\mathfrak{B} = \mathfrak{P}$.

Die Formel (4) ermoeglicht, aus den Werten m_{ik} und B_i die Werte P_i auf iterativem Wege mittels des Lochkartensverfahrens zu bestimmen.

Iteration:

$\mathfrak{P}^{(0)} = \mathfrak{B}$

$\mathfrak{P}^{(1)} = \mathfrak{B} + \mathfrak{M}\mathfrak{P}^{(0)}$

$\mathfrak{P}^{(2)} = \mathfrak{B} + \mathfrak{M}\mathfrak{P}^{(1)}$.

Lochkartenverfahren:

System I: Elemente der Matrix $\mathfrak{M}$,

System II: Elemente der Matrix $\mathfrak{B}$.

Analoge Gleichungssysteme ergeben sich bei anderen praktischen Aufgaben, z.B. bei der Kostenumlegung.

Hinreichende Kriterien fuer $\lim_{\nu \to \infty} \mathfrak{M}^\nu = 0$ sind:

$$\max \sum_{k=1}^{n} | m_{ik} | < 1, \quad \max \sum_{i=1}^{n} | m_{ik} | < 1, \quad \sum_{i,k=1}^{n} m_{ik}^2 < 1.$$

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ÉTUDE DE LA CORRÉLATION DE DEUX ESPACES VECTORIELS ALÉATOIRES

JÉRÔME CHASTENET DE GÉRY [1]) ET JEAN-MARIE SOURIAU [2])

Le but de cette étude est le suivant:

Une relation linéaire étant présumée entre les éléments de deux espaces vectoriels $\mathscr{E}$ et $\mathscr{F}$ à nombre fini de dimensions, on observe simultanément et indépendamment une suite de couples de vecteurs, et on cherche:

1) à apprécier la validité de l'hypothèse formulée;

2) éventuellement, à calculer „au mieux" la relation.

Dans ce but on définit et on construit un opérateur qu'on appelle: *opérateur de corrélation de l'espace* $\mathscr{E}$ *avec l'espace* $\mathscr{F}$, et qui est apte à fournir tous les renseignements demandés.

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[2]) Professeur, Institut des Hautes Études — Tunis.

La qualité de la corrélation sera figurée par un *spectre de corrélation*; on indiquera de plus le scalaire qui, si l'on désire s'en tenir à un seul nombre, nous paraît caractériser le mieux la corrélation dans son ensemble.

La „meilleure" relation sera figurée par un opérateur linéaire, dont on donne une expression, et représentée dans un système de référence par une matrice de nombres.

Cette étude trouve des applications pratiques en physique chaque fois que l'on détermine un opérateur linéaire, non par la mesure de „coefficients d'influence" mais par la mesure indépendante de couples de vecteurs; ceci est en particulier la cas quand on observe des phénomènes naturels, champ des courants telluriques en deux points du sol, par exemple.

THEOREMS ON CONFIDENCE REGIONS

L. M. Court

Theorem I: Let $\delta_i(\boldsymbol{E})$, $i = 1, . . ., s$ be a confidence region with confidence coefficient α_i for the subset $\tilde{\theta}_i = (\theta_{i1}, \ldots, \theta_{im_i})$ of the parameter set $(\theta_1, \ldots, \theta_k)$, identifying the probability distribution $F(E \mid \theta_1, \ldots, \theta_k)$, where the parameter subsets may or may not be pairwise disjunct. Then

$$P\{\tilde{\theta}_i \in \delta_i(\boldsymbol{E});\ i = 1, \ldots, s \text{ simultaneously} \mid \theta_1, \ldots, \theta_k\} \geqq \sum_{i=1}^{s} \alpha_i - s + 1.$$

That is, the s regions $\delta_i(\boldsymbol{E})$ can be used to define a subset of the parameter space which is a confidence region for the parameterset $U_{i=1}^{s} \tilde{\theta}_i$ with an assured confidence coefficient of $\sum_{i=1}^{s} \alpha_i - s + 1$.

Theorem II: Let $P(S \mid \theta, \varphi)$ be a probability function defined for any set S of a Borel field over the sample space $\mathscr{S}$, the parameter $\theta \in$ the space Ω_1 and $\varphi \in \Omega_2$. Let $\boldsymbol{E}$ be any point of $\mathscr{S}$ and $T(\boldsymbol{E})$ a statistic, i.e. a mapping of $\mathscr{S}$ onto the space $\mathscr{I} \in \boldsymbol{T}$. Suppose further that there exists a family of regions (sets) $I_{\mathscr{I}}[\theta, \varphi]$ in $\mathscr{I}$, one for each point of $\Omega_1 \times \Omega_2$ (the topological space product of Ω_1 and Ω_2), such that

1) $$I_{\mathscr{I}}[\theta, \varphi] \subseteqq I_{\mathscr{I}}[\theta, \varphi']$$

whenever the relation $\Re(\varphi', \varphi)$ subsists between the primed and unprimed φ parameters. Suppose also that

2) $$P\{T(\boldsymbol{E}) \in I_{\mathscr{I}}[\theta, \varphi] \mid \theta, \varphi\} \geqq \alpha$$

and that there exists a mapping $\hat{\varphi}(\boldsymbol{E}, \theta)$ of the space $\mathscr{S} \times \Omega_1$ into the space Ω_2 such that

3) $$P\{\mathfrak{R}(\hat{\varphi}, \varphi) \mid \theta, \varphi\} \geqq \beta.$$

Then

4) $$P\{T(\boldsymbol{E}) \in I_{\mathscr{I}}[\theta, \hat{\varphi}(\boldsymbol{E}, \theta)] \mid \theta, \varphi\} \geqq \alpha + \beta - 1.$$

That is, $T(\boldsymbol{E})$ and $I_{\mathscr{I}}[\theta\, \varphi(\boldsymbol{E}, \theta)]$ can be used to define a confidence region $J_{\Omega_1}[\boldsymbol{E}] \subseteqq \Omega_1$ with an assured confidence coefficient of $\alpha + \beta - 1$.

In the usual applications $\mathscr{S}$ and $\mathscr{I}$ will be euclidean vector spaces of dimensions n and m, and Ω_1 and Ω_2 analogous spaces of dimensions k_1 and k_2. The $I_{\mathscr{I}}[\theta, \varphi]$ will be subsets of $\mathscr{I}$, each of intrinsic dimensionality m, and $\hat{\varphi}(\boldsymbol{E}, \theta)$ a vector function with k_2 components.

Theorem II being somewhat complicated, it is profitable to illustrate it with a few examples. Let $\overline{\boldsymbol{x}}$ be the mean of N observations, each distributed normally with common mean μ and standard deviation σ. Take $I_{\mathscr{I}}[\theta, \varphi] = I_{\mathscr{I}}[\mu, \sigma] = (\mu - 2\sigma_{\overline{X}} \leqq x \leqq \mu + 2\sigma_{\overline{X}})$ where $\sigma_{\overline{X}}$ is the population variance of $\overline{\boldsymbol{x}}$. Then obviously $I_{\mathscr{I}}[\mu, \sigma] \subseteqq I_{\mathscr{I}}[\mu, \sigma')$ if $\sigma' \geqq \sigma$ — — this is the $\mathfrak{R}(\varphi', \varphi)$ for the present situation. We take $T(\boldsymbol{E}) = \overline{\boldsymbol{x}}$ so that

$$P\{\overline{\boldsymbol{x}} \in (\mu - 2\sigma_{\overline{X}} \leqq x \leqq \mu + 2\sigma_{\overline{X}}) \mid \mu, \sigma\} \geqq .95.$$

Suppose for our first example that it is *known* that $\sigma \leqq$ some constant c; we can then take $\hat{\varphi}(\boldsymbol{E}, \theta) = \varphi(\boldsymbol{E}, \mu) = c$ and it is clear that

$$P\{\sigma \leqq c\} = P\left\{\sigma_{\overline{X}} \leqq \frac{c}{N}\right\} = 1.$$

It follows then from the theorem that

$$P\left\{\overline{\boldsymbol{x}} \in \left(\mu - \frac{2c}{N} \leqq x \leqq \mu + \frac{2c}{N}\right) \middle| \mu, \sigma\right\} \geqq .95 + 1 - 1 = .95.$$

The x-interval within "( )" is $I_{\mathscr{I}}[\theta, \hat{\varphi}(\boldsymbol{E}, \theta)]$ for the present example.

$$J_{\Omega_1}[\boldsymbol{E}] = J_\mu[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n] = \left(\overline{\boldsymbol{x}} - \frac{2c}{n} \leqq \mu \leqq \overline{\boldsymbol{x}} + \frac{2c}{n}\right).$$

In the second example, let P, T, $I_{\mathscr{I}}$ and $\mathfrak{R}$ remain as before but take $\hat{\varphi}(\boldsymbol{E}, \theta) = h\,|\,\overline{\boldsymbol{x}} - \mu\,|$ where h is a constant such that

$$P\{\sigma \leqq h\,|\,\overline{\boldsymbol{x}} - \mu\,|\ \mu, \sigma\} = \beta.$$

It is clear that no matter how large $\beta < 1$ is, it is possible to choose h so that this equation is satisfied. Theorem II then says that

$$P\left\{\overline{\boldsymbol{x}} \in \left(\mu - \frac{2h|\overline{\boldsymbol{x}} - \mu|}{N} \middle| \leqq x \leqq \mu + \frac{2h\,|\,\overline{\boldsymbol{x}} - \mu\,|}{N}\right) \middle| \mu, \sigma\right\} \geqq .95 + \beta - 1 = \beta - .05$$

Again, $I_{\mathscr{J}}\ [\theta, \hat{\varphi}\ (\boldsymbol{E}, \theta)]$ is the point set defined by the inequalities in the "( )". $J_{\Omega_1}\ [\boldsymbol{E}] = J_{\mu}\ [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ is found by taking for any particular value of $\overline{\boldsymbol{x}}$, the values of μ satisfying these inequalities. The confidence coefficient associated with $J_{\mu}[\boldsymbol{E}]$ is $\geqq \beta - .05$.

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ON A NEW AXIOMATIC FOUNDATION OF THE THEORY OF PROBABILITY

ALFRÉD RÉNYI

The author has given a short account of a new axiomatic foundation of the theory of probability, in which the basic concept is that of conditional probability, and absolute probabilities may not exist. The new theory is a generalization of that of A. N. Kolmogorov*, and includes such schemes which could not be fitted in any of the existing theories, and which have important applications especially in physics.

Let H denote a set, T_1 a Borel field of subsets $A, B, C, \ldots$ of H, called events. Let further a non-void subset T_2 of T_1 be given, and a set function of two variables $P\ (A|B)$ defined for $A \in T_1$ and $B \in T_2$, which satisfies the following axioms:

I. $0 \leqq P\ (A|B) \leqq 1$ and $P\ (B|B) = 1$.

II. For any fixed $B \in T_2$ $P\ (A|B)$ is a σ-additive set function on T_1.

III.** $P\ (A|BC).\ P\ (B|C) = P\ (AB|C)$.

We call $P\ (A|B)$ the conditional probability of A relative to B and $F = (H, T_1, T_2, P\ (A|B))$ a conditional probability field (c.p.f.). A c.p.f. is thus a set of probability fields (p.f.) in the sense of the theory of Kolmogorov which are connected by axiom III. If H is a set, T_1 a Borel field of subsets of H,

*) The author has been informed that in a lecture held some years ago Kolmogorov himself mentioned the possibility of such a generalization of his theory but he did not publish his ideas on this subject.

**) AB denotes the intersection of the sets A and B.

$\mu(A)$ a σ-additive measure on T_1, further if T_2 denotes the set of those $B \in T_2$ for which $0 < \mu(B) < +\infty$, then putting

$P(A|B) = \dfrac{\mu(AB)}{\mu(B)}$, clearly, $F = (H, T_1, T_2, P(A|B))$ is a c.p.f. If $H \in T_1$ and $\mu(H) = 1$, F is called the c.p.f. generated by the p.f. $(H, T_1, \mu(A))$. If $\mu(A)$ is not bounded, F can not be generated by a p.f.

Generalizations of the laws of large numbers as well of the central limit theorem for a c.p.f. have been found by the author. We mention here only the following consequence of the strong law of large numbers for a c.p.f., which is a generalization of a well known theorem of Borel on normal decimals:

Let us consider the development of a real number x $(0 < x < 1)$ into a Cantor series

$$x = \sum_{n=1}^{\infty} \frac{e_n(x)}{q_1 q_2 \cdots q_n}$$

where the „digits" $e_n(x)$ can take one of the values $0, 1, 2, \ldots q_n - 1$, and $q_n \geqq 2$ is a sequence of integers for which $\lim_{n\to\infty} q_n = +\infty$ and $\sum_{n=1}^{\infty} \frac{1}{q_n} = +\infty$. Let us denote by $f_n(x; k_1, k_2, \ldots, k_s)$ the number of those digits $e_r(x)$ $(r = 1, 2, \ldots, n)$ which are equal to one of the non-negative integers $k_1, k_2, \ldots, k_s$. Then we have for any $s \geqq 2$ and any choice of the non-negative integers $k_1, k_2 \ldots, k_s$ and for almost every x in (0.1) the relation

$$\lim_{n\to\infty} \frac{f_n(x; k_j)}{f_n(x; k_1, k_2, \ldots, k_s)} = \frac{1}{s} \quad (j = 1, 2, \ldots, s).$$

Thus the conditional relative frequencies of the digits are for almost every x in the limit equal for any finite collection of digits.

It can be shown further that for some types of Markov chains and other stochastic processes in cases when there does not exist a limiting distribution in the ordinary sense, there exists a conditional limiting distribution in the sense of the theory sketched above.

All these results will be published in a paper in print in the Acta Mathematica of the Hungarian Academy of Sciences.

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ON THE THEORY OF ORDER STATISTICS.

ALFRÉD RÉNYI

The author developed a new method, which reduces the problems concerning limiting distributions of order statistics to the study of sums of independent random variables. Many results of the theory of order statistics can be obtained by this method with surprising simplicity. The method is based on the fact, that if $\xi_1^* \leqq \xi_2^* \leqq \dots \leqq \xi_n^*$ are the order statistics from a sample taken from a population having the continuous cumulative distribution function (c.d.f.) $F(x)$, we have

$$(1) \qquad \xi_k^* = F^{-1}\left(\exp\left(-\sum_{j=1}^{n-k+1} \frac{\delta_j}{n-j+1}\right)\right) \quad (k = 1, 2, \dots, n)$$

where the random variabes δ_j $(j = 1, 2, \dots, n)$ are independent and have the same c.d.f. $1 - e^{-x}$ $(x \geqq 0)$ and $x = F^{-1}(y)$ denotes the inverse function of $y = F(x)$.

This fact is equivalent with the statement that the n random variables $\{F(\xi_k^*)/F(\xi_{k+1}^*)\}^k$ are independent and homogeneously distributed between 0 and 1, which is contained in the stencilled lectures (1947—1949) by D. van Dantzig [3], who kindly called the attention of the author to this fact.

One of the new results which have been obtained by the author by means of his method is as follows. Let $F_n(x)$ denote the c.d.f. of a sample of size n taken from a population having the continuous c.d.f. $F(x)$. A well known theorem of A. N. Kolmogorov gives a confidence band for the unknown c.d.f. $F(x)$ which band has the same breadth for all x. From a practical point of view a confidence band the breadth of which is proportional to the value of $F(x)$ has some advantages. Such a confidence band can be (approximately) given by means of the following theorem:
for any a $(0 < a < 1)$ we have for $y > 0$

$$(2) \quad \lim_{n\to\infty} P\left(\sqrt{n} \sup_{a \leqq F(x)} \left| \frac{F_n(x) - F(x)}{F(x)} \right| < y\right) = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k e^{-\frac{(2k+1)^2\pi^2(1-a)}{8ay^2}}.$$

The proof of (2) and of other similar results was published in [1].

In the second part of his lecture the author mentioned a new two sample test, of Wilcoxon's type, which is similar to a test proposed by E. Lehmann [2], but somewhat simpler. Let us denote by $(\xi_1, \xi_2, \dots, \xi_m)$ a sample taken from a population having the continuous c.d.f. $F(x)$ and $(\eta_1, \eta_2, \dots, \eta_n)$ a sample taken from an other population having the continuous c.d.f. $G(x)$. Let W_1 denote the number of valid pairs of inequalities $\eta_j < \xi_i$, $\eta_k < \xi_i$ $(i=1, 2, \dots, m;$

$j, k = 1, 2, \ldots, n; j \neq k$) and W_2 the number of valid pairs of inequalities $\xi_j < \eta_i, \xi_k < \eta_i$ ($i = 1, 2, \ldots, n; j, k = 1, 2, \ldots m, j \neq k$). The test proposed by the author is based on the statistic

$$W = \frac{W_1}{m\binom{n}{2}} + \frac{W_2}{n\binom{m}{2}}. \tag{3}$$

This test is consistent against any alternative hypothesis $G(x) \not\equiv F(x)$. This is the consequence of the fact that if $F(x)$ and $G(x)$ are continuous c.d.f. we have

$$\int_{-\infty}^{+\infty} F^2(x)dG(x) + \int_{-\infty}^{+\infty} G^2(x)dF(x) \geqq \tfrac{2}{3} \tag{4}$$

and equality stands in (4) if and only if $F(x) \equiv G(x)$.

Finally the author mentioned some unsolved problems, for instance the following one: which systems of curves on the (x, y)-plane can be a system of level curves $F(x, y) =$ const. of a two dimensional c.d.f. $F(x, y)$?

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SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

SHORT LECTURES

ON DIFFERENTIAL DIFFERENCE EQUATIONS AND CONTROL PROBLEMS

RICHARD BELLMAN and JOHN M. DANSKIN,

The paper discusses problems arising in the application of differential-difference equations to the theory of magnetism, fission processes, radder mechanisms and other processes involving a time lag.

RICHARD BELLMAN, SANTA MONICA, CALIFORNIA
JOHN M. DANSKIN, WASHINGTON, D.C.

SUR LES ÉQUATIONS DU MOUVEMENT D'UN GAZ

RATIP BERKER

La méthode indirecte qui consiste à faire certaines hypothèses (nature des trajectoires, forme analytique du champ des vitesses) et à essayer ensuite de déterminer complètement le mouvement, a pu être appliquée aux gaz avec succès après l'obtention des équations de Munk et Prim (ou de Hicks et de ses collaborateurs). Je me propose de montrer que l'on peut obtenir un autre système d'équations auquel la méthode indirecte s'applique d'une manière très avantageuse; il s'agit d'équations qui sont le résultat de l'élimination de toutes les grandeurs non-cinématiques des équations du mouvement y compris l'équation qui exprime la conservation de l'énergie.

Les équations obtenues sont générales et conviennent à tous les gaz quelle que soit la forme de l'équation d'état.

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SUR UN PROBLÈME MATHÉMATIQUE POSÉ PAR LA PHYSIQUE THÉORIQUE

Jean Louis Destouches

Le théorie dite „de la double solution" de M. Louis de Broglie présente un problème mathématique intéressant:

L'inconnue est une équation aux dérivées partielles non-linéaire; on connait une équation linéaire approchée de cette équation et les domaines où elle est pratiquement suffisante (équation des ondes de la mécanique ondulatoire usuelle). On connait aussi un certain nombre de conditions asymptotiques assez diverses pour les solutions de cette équation inconnue. Il faut d'abord montrer que ces conditions sont compatibles entre elles, puis essayer de parvenir à déterminer les formes d'équations qui répondent à la question. Actuellement ce problème n'est pas complètement résolu. On montre qu'il peut s'exprimer en termes de mécanique des fluides, et qu'il se ramène à la détermination des termes de sources, de potentiel des accélérations et de pression d'un fluide parfait en mouvement irrotationnel soumis au théorème de Lagrange. Ces termes doivent être tels que les conditions asymptotiques données soient remplies. On peut satisfaire aux conditions imposées.

BEMERKUNGEN ÜBER EINE KLASSE SYMMETRISIERBARER EIGENWERTAUFGABEN

Hans Karl Dettmar

Vorgelegt sei die Eigenwertaufgabe

$$M[y] = \sum_{k=0}^{m} f_k(x)y^{(k)}(x) = \lambda \left(\sum_{k=0}^{n} g_k(x)y^{(k)}(x)\right) = \lambda N[y] \tag{1}$$

$$U_\mu[y] = 0 \qquad (\mu = 1, 2, \ldots, m). \tag{2}$$

Die f_k und g_k sind im Grundintervall $a \leqq x \leqq b$ gegebene, reelle stetige Funktionen mit $f_m \neq 0$ und $g_n \neq 0$. Es ist $0 \leqq n < m$. Die Gleichungen (2) sind linear homogene, voneinander linear unabhängige Randbedingungen.

Es werde eine speziellere Problemklasse herausgegriffen. Eine reelle m-mal stetig differenzierbare, alle Randbedingungen erfüllende Funktion $u(x)$ heisse *Vergleichsfunktion*. Die Vergleichsfunktionen bilden einen reellen linearen Funktionenraum $\mathfrak{V}$. Unter Verzicht auf grösste Allgemeinheit sei eine Klasse von symmetrisierbaren Eigenwertaufgaben wie folgt definiert: Die Eigenwertaufgabe (1), (2) heisst *symmetrisierbar*, wenn ein gewisser linear homogener Funktionalausdruck $\Phi\ [y]$ existiert und wenn mit beliebigen $u, v \varepsilon \mathfrak{B}$ für die Formen

$$\{u, v\} = \int_a^b \Phi[u]M[v]dx \quad \text{und} \quad \langle u, v\rangle = \int_a^b \Phi[u]N[v]dx \tag{3}$$

die Symmetriebeziehungen

$$\{u, v\} = \{v, u\} \quad \text{und} \quad \langle u, v\} = \langle v, u\rangle$$

bestehen. Der Operator Φ heisse dann *Vergleichsoperator* der Eigenwertaufgabe. Sind überdies die beiden Formen positiv definit in $\mathfrak{B}$, so heisse die Aufgabe *volldefinit symmetrisierbar* und Φ ein *volldefiniter Vergleichsoperator*.

Jede selbstadjungierte Eigenwertaufgabe ist symmetrisierbar ($\Phi \equiv 1$). Darüber hinaus gibt es nicht selbstadjungierte Aufgaben, die symmetrisierbar sind. Für die Existenz eines Vergleichsoperators lassen sich unter Einschränkungen hinreichende Bedingungen angeben. Beschrąnkt man sich etwa auf reine Differentialausdrücke

$$\Phi[y] = \sum_{k=0}^{\pi} p_k(x)y^{(k)}(x) \qquad (0 \leqq \pi \leqq m), \tag{5}$$

dann ist für die Symmetrie von $\{u, v\}$ z.B. im Fall $m \leqq 4$ die Symmetrie det Differentialausdrücke $A_k[f, p]$ $(k = 1, \ldots, 4)$ in f und p und das Verschwinden gewisser Randausdrücke hinreichend (die auftretenden Ableitungen mögen existieren). Dabei sind die Operatoren A_k definiert durch

$$\begin{aligned} A_1\,[r, s] &= r_4s_3 \\ A_2\,[r, s] &= (r_4s_2)' + r_4s_1 + r_2s_3 \\ A_3\,[r, s] &= (r_4s_1)'' + 2(r_4s_0)' + (r_1s_3)' + r_2s_1 + r_0s_3 \\ A_4\,[r, s] &= (r_4s_0)''' + (r_0s_3)'' + (r_2s_0)' + r_0s_1. \end{aligned} \tag{6}$$

Überdies gilt, falls die Symmetrie der $A_k[f, p]$ in f und p erfüllt ist, dass

$$M^*\Phi \equiv \Phi^*M \tag{7}$$

ist, wo M^* bzw. Φ^* die zu M bezw. Φ adjungierten Operatoren bedeuten. Solche Bedingungen bieten eine Handhabe zum Auffinden von Vergleichsoperatoren.

Ist die Eigenwertaufgabe volldefinit symmetrisierbar, so gelten u.a. folgende Sätze: Die Eigenwertaufgabe besitzt abzählbar unendlich viele *Eigenwerte* ohne Häufungspunkt im Endlichen. Alle Eigenwerte sind positiv und lassen sich in einer Folge $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots$ anordnen (jeder Eigenwert ist entsprechend seiner Vielfachheit gezählt). Zu dieser Folge gibt es ein System von zugehörigen *Eigenfunktionen* $y_1, y_2, \ldots,$ das bezüglich der Metrik $\langle u ,v\rangle$

in $\mathfrak{V}$ *vollständig* ist. Die Eigenfunktionen können orthonormiert angenommen werden, d.h. $\langle y_1, y_k\rangle = \delta_{1k}$. *Ist* $R[u] = \{u, u\}/\langle u, u\rangle$ *der „mit* Φ *gebildete Rayleighsche Quotient" so gilt* $\lambda_k = \underset{u}{Min}\ R\ [u]$, *wo u alle Funktionen* $\neq 0$ *aus* $\mathfrak{V}$ *durchläuft, die zu* $y_1, \ldots, y_{k-1}$ *orthogonal sind.* y_k *ist Lösung dieser Aufgabe.* Ferner lassen sich Entwicklungssätze aussprechen.

Zur praktischen Lösung symmetrisierbarer Eigenwertaufgaben können die bekannten Verfahren (Ritzsches Verfahren, Iterationsverfahren u.a.) herangezogen werden. Diese lassen sich nach geeigneten Modifikationen auf die symmetrisierbaren Aufgaben anwenden. Die Verfahren hängen vom gewählten Vergleichsoperator ab. Zum Beispiel kann zur Durchführung eines Ritzschen Verfahrens die Extremaleigenschaft des mit Φ gebildeten Rayleighschen Quotienten $R[u]$ herangezogen werden. Ebenso lassen sich die bekannten Einschliessungssätze übertragen. Falls mehrere Vergleichsoperatoren angegeben werden können, besitzt man Freiheit in der Wahl des Operators für die numerische Rechnung. Diese Möglichkeit ist auch für die Behandlung selbstadjungierter Probleme fruchtbar.

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ENTROPY CHANGES IN THE EQUATIONS FOR RAREFACTION WAVES

R. F. DRESSLER

The role of entropy change in compressible, frictional flow of a gas is investigated analytically, based upon the three non-linear equations of one-dimensional unsteady flow applied to a centered rarefaction wave. Energy balance is maintained by the assumption that mechanical energy loss due to quadratic frictional forces reappears as thermal energy. The behavior of the specific entropy at the forward wavefront is discussed, using the non-isentropic equation of state for a polytropic gas as the basic model. Then the equations defining first order frictional effects are derived for particle velocity, local sound speed, and specific entropy. This is a system of three linear partial differential equations in the x, t plane containing variable coefficients and non-homogeneous terms. By utilizing the geometrical similarity of the three families of Mach lines for the undisturbed non-frictional wave, the functional form of the first order quantities can be ascertained. This then permits solution of the boundary value problem for these three unknowns in the x, t wedge defining the flow zone. The expansion procedure used is singular at the wavefront, and results are therefore not applicable in the region of the wavefront. The correction functions are found to be seventh degree polynomials plus first and second order pole terms which dominate near the wavefront. These expressions are compared with simpler results obtained when the heat generation and entropy change due to frictional forces are neglected and only the forces are retained. Velocities and densities obtained from the two models agree closely for the back half of the wave, but local sound speeds and entropy changes for these models differ widely at all flow points.

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DEDUZIONE VARIAZIONALE DELLE EQUAZIONI DEL CAMPO RELATIVISTICO UNITARIO

BRUNO FINZI

Le equazioni di campo, esternamente alla materia, sono tratte da Einstein, nella sua recente teoria relativistica unitaria, dal seguente principio variazionale: (1) $\delta\int R^{**}_{i\,k} g^{ik} w dx = 0$, dove: g^{ik} è il tensore fondamentale non simmetrico, $w = \sqrt{-\,\|\, g_{ik} \,\|}$, dx il prodotto dei 4 differenziali delle variabili spazio-temporali, $R^{**}_{i\,k} = R^{*}_{i\,k} - \left(\frac{1}{w}\, w;\, i\right); k$, $R^{*}_{i\,k} = R_{ik} + \Gamma_i; k$, R_{ik} il tensore di Riemann contratto e Γ_i il vettore di torsione. Einstein suppone però che il tensore fondamentale verifichi la relazione: (2) $(w g^{\underset{\vee}{ik}}), k = 0$. Le equazioni di campo tratte dalla (1), sotto la condizione (2), sono la (2) stessa e le: (3) $g_{ik;r} = 0$, (4) $R_{ik} = 0$, (5) rot $R_{\underset{\vee}{ik}} = 0$. Dalla (3) discende poi che (3′) $w; i = 0$, e dalla (2) e dalla (3) che è nullo il vettore di torsione: (2′) $\Gamma_i = 0$. Il procedimento seguito da Einstein ha carattere ibrido, perchè fa appello al principio variazionale (1) e alla condizione posta a priori (2), che è proprio quella che permette di individuare il campo elettromagnetico con la parte emisimmetrica del tensore fondamentale.

In questa comunicazione mostro che si può istituire una teoria relativistica unitaria analoga a quella einsteiniana, traendone le equazioni di campo dal solo principio variazionale (1), senza porre a priori nè la (2) nè altra relazione che vincoli il modo col quale si calcola la variazione. Si perviene così ancora alla relazione fondamentale (3), che, affermando la costanza del tensore fondamentale, permette di dedurre dal campo fondamentale il campo dei coefficienti di connessione $\Gamma_{ik}{}^{l}$, e si è condotti alle seguenti equazioni di campo: (6) $R^*_{i\,k}=0$, (7) rot $R^*_{\underset{\smile}{i\,k}}=0$, (8) div $R^*_{\underset{\smile}{i\,k}}=0$. Le (6) (7) (8) sono $10+4+4=18$ equazioni nelle 16 componenti di g_{ik}, e da queste equazioni si possono tratre, oltre alle 4 identità analoghe a quelle di Bianchi, che lasciano libertà nella scelta del riferimento, le due identità div rot $R^*_{\underset{\smile}{i\,k}}\equiv 0$, div div $R^*_{\underset{\smile}{i\,k}}\equiv 0$. Con la (3) si mantiene ancora valida la (3′), ma cadono le (2) e (2′), cioè non risulta nullo il vettore di torsione, nè costante il tensore di Ricci. Le relazioni (7) (8) permettono di interpretare il tensore $R^*_{\underset{\smile}{i\,k}}$ come tensore elettromagnetico di un campo neutro. Si può però interpretare come tensore elettromagnetico (9) $F_{ik}=R^*_{\underset{\smile}{i\,k}}-\Gamma^*_{\underset{\smile}{i,k}}=R_{\underset{\smile}{ik}}-\Gamma_{\underset{\smile}{ik}}{}^{l}\Gamma_{l}$, perchè risulta rot $F_{ik}=0$; mentre, posto (10) $w\dot{j}^{r}=(wF^{rs})_{,s}$, dalla (8) e dalla (9) si deduce: (11) $w\dot{j}^{r}=-(w\Gamma_{\underset{\smile}{i,k}}g^{ir}g^{ks})_{,s}$. Il vettore j^{r}, dipendente dalle derivate del vettore di torsione e annullantesi con questo, è solenoidale perchè (12) $(wj^{r})_{,r}\equiv 0$: esso può quindi interpretarsí come distribuzione elettrica nel campo non neutro F_{ik}, e l'identità (10) traduce la conservazione dell'elettricità.

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LES FORMES EXTÉRIEURES EN MÉCANIQUE

Francois Gallissot

1. *Point matériel.* La recherche d'une forme génératrice des équations différentielles du mouvement d'un point matériel, invariante dans les transformations du groupe Galiléen conduit à une forme extérieure unique ω de degré deux, définie sur la variété V_7 ayant pour base le produit topologique de l'espace Euclidien E_3 et de la droite numérique T, pour fibre l'espace numérique R^3.

Méthode analogue pour les équations de la dynamique en relativité restreinte.

2. *Système matériel.* Pour un système matériel formé de points et de solides dépendant de n paramètres q^2, l'extension de ω sur le groupe des déplacements conduit à la forme extérieure Ω de degré deux

$$\Omega = dp_2 \wedge dq^2 - dH \wedge dt + Q_2 dq^2 \wedge dt$$

définie sur une variété V_{2n+1}. La variété espace-temps de configuration V_{n+1} est fibrée, a pour base la droite numérique T, pour fibre V_n; V_{2n+1} est l'espace des vecteurs tangents aux fibres de V_{n+1}. Les équations différentielles du mouvement sont les caractéristiques de Ω, les trajectoires sur V_{2n+1} sont tangentes au champ E caractéristique définie par $i(E)\Omega = 0$ [1]).

3. *Liaisons*. Une liaison est dite imposée à un système holonome S:

1) si l'image de S est une sous-variété de V_{2n+1}, $a(M) = 0$

2) si on ajoute au champ E un champ de liaison E_l, champ du aux forces nécessaires à la réalisation de cette liaison.

Champ E_l et forme da sont liés par la relation $i(E + E_l)da = 0$.

Les types classiques de liaisons sont de la forme $E_l = \lambda e$, e champ de direction connu à priori, λ fonction numérique sur V_{2n+1} déterminée par

$$\lambda[i(e)da] + i(E)da = 0 \quad \text{si} \quad [i(e)da] \neq 0.$$

Cette hypothèse $i(e)da \neq 0$ est liée à la condition physique de non déformation des systèmes rigides.

Extension aux cas de p liaisons, aux liaisons unilatérales caractérisées par des signes imposées à priori à da et λ. Pour qu'à des conditions initiales données ne correspondent qu'un seul mouvement on démontre qu'il faut et il suffit que les mineurs diagonaux de la matrice $(i(e^h)da^k)$ soient tous positifs, ce qui a toujours lieu pour les liaisons holonomes et linéairement non holonomes classiques. Impossibilités et indéterminations bien connues depuis lex travaux de Painlevé sur le frottement ne sont donc dues qu'au concept même de liaison.

4. *Intégration*. Ω de rang $2n$ étant connue, on engendre le système Σ des équations différentielles du mouvement au moyen de $2n$ champs x^δ (δ 1 à $2n$) définis sur l'espace tangent à la variété V_{2n+1}: $L(x^\delta)\Omega = 0$.

En particulier x est générateur d'une transformation infinitésimale pour Σ, si l'opérateur $\theta(x)$ [2]) applique Ω sur le zéro de l'espace des formes.

Les opérateurs $\theta(\ )$, $i\ (\ )$, d étant liés par

$$\theta(\ ) = L(\ )\,.\,d + d\,.\,i(\ )$$

deux cas se présentent:

a) $d\Omega = 0$. A toute transformation infinitésimale correspond une intégrale première et réciproquement. L'intégration ne peut s'effectuer par quadratures que si l'on connait n champs en involution, ce qui revient à connaître df sur une variété $V_{4n+1} = V_{2n+1} \times S_{p_{2n}}$.

b) $d\Omega \neq 0$. r champs générateurs de transformations infinitésimales étant connus l'intégration se décompose en:

1) l'intégration d'un système de Pfaff complètement intégrable de rang $2n - r(P)$.

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[1]) i ( ) opérateur antidérivation de M. H. Cartan. Colloque Topologie Bruxelles 1950.

[2]) θ ( ) opérateur dérivation — idem.

2) l'intégration de r formes de Pfaff invarientes.

Dans les applications l'ordre du système complètement intégrable se trouve réduit de $(p + q)$ unités si l'on connait p intégrales premières et q liaisons. L'intégration s'achève par quadratures si $2n - r = p + q$ et si les r formes invarientes sont fermées modulo les intégrales de (P).

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VARIATIONAL THEOREMS AND THE EIGENVALUE PROBLEM FOR PLANE STRAIN IN PERFECT PLASTICITY

Herbert J. Greenberg

The author considers the problem of minimizing the functional $k \int_D \gamma \, dA$ where k is a positive constant, $\gamma = [(u_x - v_y)^2 + (u_y + v_x)^2]^{\frac{1}{2}}$, over the class of twice continuously differentiable functions $u(x, y)$, $v(x, y)$ defined over a plane finite domain D, bounded by a smooth closed curve C, subject to the subsidiary conditions $u_x + v_y = 0$ in D and $\int_C (X_u + Y_v) \, ds = 1$, where X and Y are given functions along C (the rectangular components of assigned surface tractions in equilibrium). Assuming the existence of a solution to this problem and denoting the minimum of the functional by $\lambda_0 > 0$, it is shown that as necessary conditions one obtains, wherever $\gamma \neq 0$, the equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \quad u_x + v_y = 0, \text{ in } D$$

$$\sigma_x n_x + \tau n_y = \lambda_0 X, \; \tau n_x + \sigma_y n_y = \lambda_0 Y, \text{ on } C,$$

where

$$\sigma_x = k\left(\frac{u_x - v_y}{\gamma} - p\right), \; \sigma_y = k\left(-\frac{u_x - v_y}{\gamma} - p\right), \; \tau = k\frac{u_y + v_x}{\gamma}, \; n_x \text{ and } n_y$$

are components of the unit outward normal, and $p = p(x,y)$ arises as a Lagrange multiplier. Interpreting u and v as rectangular components of the velocity strain and σ_x, σ_y, τ as stress components, p becomes the mean normal pressure and one sees that all of the equations of the Mises theory of plasticity for plain strain have been generated including equilibrium, stress-strain relations, yield condition, and boundary conditions. The minimum λ_0 of the functional becomes the multiple of the given surface tractions at which plastic flow occurs.

Next, the three equations for u, v, p, in D together with the boundary conditions with λ_0 replaced by λ, a parameter, are shown to constitute a non-linear eigenvalue problem with the eigenvalue λ_0 in the boundary conditions. Under proper additional restrictions covering the subdomains where $\gamma = 0$ in D, it is shown that λ_0 is the smallest and indeed only eigenvalue of the problem. The eigenvalue λ_0 can also be characterized by a known complementary maximum principle extended over a class of functions which can be interpreted as statically admissible stress distributions. Thus, upper and lower bounds can be found for λ_0.

These results supplement previous theorems of Drucker, Greenberg and Prager in plasticity and serve to clarify the mathematical relations between the variational problems and the eigenvalue problem. From the purely mathematical point of view, the equivalence of these unorthodox problems is interesting and seems to merit further study and generalization.

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ON THE NUMERICAL SOLUTION OF MIXED INITIAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

EUGENE ISAACSON

A number of finite difference methods have been proposed for the solution of the initial value problem. Two such numerical schemes are combined to obtain an efficient procedure for the solution of mixed boundary and initial value problems. The convergence of the process can be easily demonstrated for the case of two independent variables. Some remarks on possible generalizations for the case of three or more independent variables are made. Application of the method to flood-routing problems in rivers has been made in a report by J. J. Stoker, Andreas Troesch and the author.

By a minor modification the technique may be applied to special equations of mixed hyperbolic, elliptic type — an example from meteorology is described.

Reference is made to earlier work of J. Keller, P. Lax, R. Courant and M. Rees.

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CARTESISCHE DARSTELLUNGEN DES MINKOWSKISCHEN RAUMES

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Zwei Punktereignisse X, Y in einem Zeitraume P nennt man lichtkoinzident, wenn ein von der Stelle des Ereignisses X zu der Zeit, in welcher sich dieses Ereignis abspielt, ausgesendetes Lichtsignal die Stelle des Ereignisses Y in der Zeit, in der sich das Ereigniss Y abspielt, erreicht.

Die Menge aller lichtkoinzidenten Punktpaare wollen wir mit $\mathfrak{B}$ bezeichnen.

In der speziellen Relativitätstheorie macht man die Voraussetzung (A), dass es ein vierdimensionales System gibt (genauer: dass es eine schlichte Abbildung φ des Raumes P auf vierdimensionalen, reellen, cartesischen Raum Q gibt) so, dass der Umstand $X, Y \in \mathfrak{B}$ mit den Beziehungen

$$(\varphi (X) - \varphi (Y))^2 = 0, \; \varphi_4 (X) \leqq \varphi_4 (Y)$$

ausgedrückt werden kann. Dabei benützt man für die Summen und für die Vielfachen der Punkte $x = (x_1, x_2, x_3, x_4)$ des Raumes Q die gewöhnliche Symbolik und das Skalarprodukt der Punkte x, y definiert man durch die Form $x . y = x_1 y_1 + x_2 y_2 + x_3 y_3 - c^2 x_4 y_4$ wo c eine feste Zahl ist (also es ist $x^2 = x_1^2 + x_2^2 + x_3^2 - c^2 x_4^2$).
Es handelt sich nun darum, alle Abbildungen φ festzustellen, welche diese Eigenschaft besitzen.

Es gilt folgender Satz: Wenn φ und ψ zwei solche beliebige Abbildungen sind, so ist $\psi \varphi^{-1}$ eine lineare Abbildung des Raumes Q auf sich selbst.

Es ist also nicht notwendig (wie man es gewöhnlich tut) vorauszusetzen, dass der Raum P in gewissem Sinne metrisch ist, genauer, dass zu jedem Punktereignispaare eine Zahl $\varrho (X, Y)$ zugeordnet werden kann (das Quadrat des sogenannten Weltabstandes der Punkte X, Y), so dass

$$(\varphi (X) - \varphi (Y))^2 = \varrho (X, Y)$$

gilt.

Die Invarianz des Ausdruckes $(\varphi (X) - \varphi (Y))^2$ bei den Anderungen der Abbildung φ ist dagegen (bis auf einen Faktor, der von der Wahl X, Y unabhängig ist) eine Folge der schwächeren, oben schon gesagten Voraussetzung (A).

Diese Behauptung kann man verallgemeinern.

ALLGEMEINE BETRACHTUNGEN ÜBER DIE FORTPFLANZUNG DER ELEKTROMAGNETISCHEN WELLEN IN DEN BEWEGTEN KÖRPERN

GIOVANNI LAMPARIELLO

Die Maxwellschen Gleichungen der Elektrodynamik ruhender Körper in Bezug auf ein Inertialsystem S', bei Abwesenheit von Ladungen und Strömen, sind die Feldgleichungen

$$\text{(I')} \qquad \begin{cases} \dot{\mathfrak{B}}' = -\, c \,\mathrm{rot}'\, \mathfrak{E}' & \dot{\mathfrak{D}}' = c\,\mathrm{rot}'\,\mathfrak{H}' \\ \mathrm{div}'\,\mathfrak{B}' = 0 & \mathrm{div}'\,\mathfrak{D}' = 0 \end{cases} \qquad \cdot = \frac{\partial}{\partial t'}$$

und die Materiegleichungen, die für die isotropen homogenen Körper die Form

$$\text{(II')} \qquad \mathfrak{D}' = \varepsilon \mathfrak{E}', \qquad \mathfrak{B}' = \mu \mathfrak{H}'$$

annehmen, wo ε, μ zwei Zahlen sind, welche im Falle des Vakuums gleich eins sind.

Das System (I'), (II') kann zu einer einzigen Gleichung zweiter Ordnung, der Wellengleichung, zurückgeführt werden

$$\text{(III')} \qquad n^2 \frac{\partial^2 \varphi'}{\partial t'^2} - c^2 \Delta' \varphi' = 0 \qquad n^2 = \varepsilon\mu.$$

Jede kartesische Komponente der Grössen $\mathfrak{E}'$, $\mathfrak{H}'$ ist Lösung der (III'); für $n = 1$, wird die (III') die Wellengleichung des Vakuums

$$\text{(IV')} \qquad \frac{\partial^2 \varphi'}{\partial t'^2} - c^2 \Delta' \varphi' = 0.$$

1904 hat H. A. Lorentz entdeckt, dass das System (I'), (II'), im Falle des Vakuums, in Bezug auf die berühmten Transformationen, die seinen Namen tragen, Kovariant ist.

1908 hat H. Minkowski diese Eigenschaft angewandt, um die Gleichungen der Elektrodynamik bewegter Körper in Bezug auf ein Inertialsystem festzustellen.

Wenn $\vec{v}$ die Geschwindigkeit von S' in Bezug auf S ist, so sind die Feldgleichungen in Bezug aus S die Lorentzschen Transformierten der (I'), d.h.

$$\text{(I)} \qquad \begin{cases} \dot{\mathfrak{B}} = -\, c \,\mathrm{rot}\, \mathfrak{E} & \dot{\mathfrak{D}} = c\,\mathrm{rot}\,\mathfrak{H} \\ \mathrm{div}\,\mathfrak{B} = 0 & \mathrm{div}\,\mathfrak{D} = 0. \end{cases}$$

Die Materiegleichungen nehmen dagegen in Bezug auf S folgende Form an:

$$
\text{(II)} \qquad \begin{cases} \mathfrak{D} + \frac{1}{c} \vec{v} \wedge \mathfrak{H} = \varepsilon(\mathfrak{E} + \frac{1}{c} \vec{v} \wedge \mathfrak{B}) \\ \mathfrak{B} - \frac{1}{c} \vec{v} \wedge \mathfrak{E} = \mu(\mathfrak{H} - \frac{1}{c} \vec{v} \wedge \mathfrak{D}). \end{cases}
$$

Sie sind nicht die Lorentzschen Transformierten der (II′). Nun kann man fragen: Sind die Minkowskischen Gleichungen durch eine einzige Gleichung zweiter Ordnung ersetzbar? Ich beantworte diese Frage bejahend und beweise, dass die Gleichungen die Lorentzsche Transformierte der Wellengleichung (III′) ruhender Körper ist.

Im Falle einer einzigen räumlichen Variabel x, deren Richtung mit der Bewegungsrichtung zusammenfällt, ist diese Gleichung die folgende

$$
\text{(III)} \qquad (n^2 - \beta^2) \frac{\partial^2 \varphi}{\partial t^2} + 2(n^2 - 1) v \frac{\partial^2 \varphi}{\partial x \partial t} - c^2(1 - n^2\beta^2) \frac{\partial^2 \varphi}{\partial x^2} = 0
$$

wo $\beta = \frac{v}{c}$.

Für $\beta = 0$ wird diese Gleichung die Wellengleichung ruhender Körper (III′) und für $n = 1$, β beliebig, die (III) reduziert sich der Gleichung (IV′).

Das System (I), (II) hat Lösungen deren x-Komponenten null sind; die anderen Komponenten sind Lösungen der (III) und haben die Form

$$
\text{(1)} \qquad \varphi = A e^{i(kx - \omega t)}
$$

wo

$$
\text{(2)} \qquad c_1(\beta) = \frac{\omega}{k}
$$

eine Wurzel der algebraischen Gleichung

$$
\text{(3)} \qquad (n^2 - \beta^2) c_1^2 - 2(n^2 - 1) v c_1 - c^2(1 - n^2\beta^2) = 0
$$

ist. Die Existenz der ebenen Wellen, deren Phasengeschwindigkeit c_1 ist, wie eine direkte Folge der Minkowskischen Gleichungen, ist bewiesen.

Bekanntlich, hat von Laue die Phasengeschwindigkeit c_1 bei Anwendung des Einsteinschen Additionstheorems den Geschwindigkeiten v und

$$
\text{(4)} \qquad c_0 = c_1(0) = \frac{c}{\sqrt{\varepsilon\mu}} = \frac{c}{n}
$$

gefunden.

Im Falle einer geradlinig-polarisierten Welle, findet man, dass die Minkowskischen Gleichungen (I), (II) die Lösung

$$
\begin{cases}
\mathfrak{E}_x = \mathfrak{E}_z = 0,\ \mathfrak{H}_x = \mathfrak{H}_y = 0,\ \mathfrak{E}_y = A e^{i(kx-\omega t)},\ \mathfrak{H}_z = \sqrt{\dfrac{\varepsilon}{\mu}}\,\mathfrak{E}_y & (5) \\
\mathfrak{D} = \varepsilon^* \mathfrak{E} \quad , \quad \mathfrak{B} = \mu^* \mathfrak{H} & (6)
\end{cases}
$$

haben, wo

$$\varepsilon^* = \frac{c_0}{c_1}\varepsilon, \quad \mu^* = \frac{c_0}{c_1}\mu \tag{7}$$

ist.

Wir können ε, μ die dielektrische Konstante und magnetische Permeabilität der Ruhe, ε^*, μ^* die dielektrische Konstante und magnetische Permeabilität der Bewegung nennen. Abgesehen von Gliedern zweiter Ordnung in β, hat man

$$\varepsilon^* = \left(1 - \frac{n^2-1}{n}\beta\right)\varepsilon, \quad \mu^* = \left(1 - \frac{n^2-1}{n}\beta\right)\mu. \tag{7'}$$

Man hat also einen elektrischen und magnetischen Effekt erster Ordnung, entsprechend dem Fresnel-Fizeau Effekt, der bei der Erfahrung bestätigt werden wird.

Schliesslich bemerke ich, dass man die Formel

$$c_1 = \frac{c}{\sqrt{\varepsilon^* \mu^*}} \tag{8}$$

hat, die die Maxwellsche Beziehung der Elektrodynamik bewegter Körper erweitert.

Valdaora in Val Pusteria.

SUR L'ACCÉLÉRATION SÉCULAIRE DE LA LUNE

Cristóbal de Losada y Puga

L'accélération séculaire de la Lune présente un terme résiduel: la magnitude du phénomène est plus grande que celle dont on a pu trouver l'explication théorique. On n'a pas prêté une attention suffisante à la cause principale capable de produire une accélération séculaire: la possible existence d'un milieu résistant.

Dans l'impossibilité de faire une étude directe du problème, j'ai pensé qu'il serait intéressant d'établir une comparaison entre l'action de ce milieu sur le mouvement de la Lune et sur les mouvements des autres satellites du système solaire.

On ne peut pas déceler l'existence d'une accélération séculaire des satellites autres que la Lune; mais il n'est pas impossible de découvrir l'action du milieu résistant sur un autre élément caractéristique du mouvement planétaire, savoir, sur l'excentricité de l'orbite.

Or, on trouve que l'arrondissement de l'orbite d'un satellite, produit par un milieu résistant, est directement proportionnel à la racine carrée de μ (somme des masses de la planète principale et du satellite), et inversement proportionnel à la racine carrée de a (demi grand axe de l'orbite du satellite).

Et on peut voir que les satellites dont les orbites présentent une forte excentricité, ont effectivement une valeur petite de la relation μ/a.

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EINE HYDRODYNAMISCHE EXISTENZBETRACHTUNG

KARL MARUHN

Betrachtet wird eine homogene inkompressible reibungsfreie Flüssigkeit, die den ganzen Raum erfüllt. Ist zur Zeit t_0 das Geschwindigkeitsfeld bzw. das Wirbelfeld gegeben, so gibt es zu diesem als Anfangsbedingung nach bekannten Existenzsätzen für ein genügend kleines Zeitintervall genau eine Bewegung der Flüssigkeit. Wir fragen: Wie muß das Anfangsfeld beschaffen sein, damit sich eine (bezüglich eines festen oder in spezieller Weise mitbewegten Achsenkreuzes) stationäre Bewegung ergibt? Als Antwort wird eine naheliegende notwendige und hinreichende Bedingung angegeben, deren Herleitung die bekannten Cauchy'schen Relationen wesentlich berücksichtigt. Hierdurch ergeben sich die Grundlagen zu den Betrachtungen von Lichtenstein über gewisse quasistationäre Bewegungen (vgl. L. Lichtenstein, Grundlagen der Hydrodynamik, Berlin 1929).

DRESDEN A 20, PFAFFENSTEINSTR. 3.

A BASIC SET OF HOMOGENEOUS HARMONIC POLYNOMIALS IN 3 VARIABLES

E. P. MILES Jr (Speaker) AND E. WILLIAMS

Basic sets of solutions for $\Delta^2 [u (x, y, z)] = 0$ such as those obtained from the Whittaker integral (*Math. Ann.* Vol 57, 1903, p. 333) or from the infinitely many components of powers of the Ketchum Hyper variable (*Amer. Journ. of*

Math. Vol 51, 1929, p. 129) use trigonometric functions and must be individually computed. The authors present explicitly a polynomial basic set of degree N. Let n, a, b. c be non negative integers such that $a = 0$, or $a = 1$, and $a + b + c = n$. We define

$$\frac{M^n}{W_{a,b,c}}(x, y, z) = \sum (-1)^{\left[\frac{\alpha}{2}\right]} \frac{n!}{\alpha!\,\beta!\,\gamma!} \frac{\left[\frac{\alpha}{2}\right]!}{\left(\frac{b-\beta}{2}\right)!\left(\frac{c-\gamma}{2}\right)!} x^\alpha y^\beta z^\gamma$$

where the summation extends over all non negative α, β, γ such that (1) $\alpha \equiv a$, $\beta \equiv b$, $\gamma \equiv c$ mod. 2; (2) $\alpha + \beta + \gamma = n$; and (3) $\beta \leqq b$, $\gamma \leqq c$. These polynomials are by construction independent and $2n + 1$ in number. Since their laplacian vanishes also they form a basic set of solutions.

(Note added after presentation).

The authors are indebted to professors P. C. Rosenbloom and L. Bers who pointed out a related result of M. H. Protter (*Portugaliae mathematica* Vol. 10 Fasc 1, 1951). The $\frac{M}{W}$ polynomials represent a single explicit presentation of the four types obtained by Protter after an x, z interchange, but were derived independently by a different method. The authors results, generalized to k variables, to the wave equation, and to the equation $\sum_{j=1}^{k} \frac{\partial^s u}{\partial x_j^s} = 0$, will appear soon in the Proc. Am. Math. Soc.

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SUL TENSORE ELETTROMAGNETICO DELLA TEORIA UNITARIA DI EINSTEIN

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In una varietà quadridimensionale riemanniana o a connessione affine simmetrica (ed a tensore ε costante) ogni tensore doppio emisimmetrico può individuare il campo elettromagnetico quando gode di una delle seguenti proprietà:

a) è irrotazionale,

b) è solenoidale.

Nel caso a) può essere assunto come tensore elettromagnetico il tensore stesso, nel caso b) il suo coniugato.

Ciò è conseguenza di un'osservazione di carattere algebrico e di due ovvie proprietà di carattere differenziale. L'osservazione algebrica è che in ogni varietà quadridimensionale il coniugato di un tensore doppio emisimmetrico è pure un tensore doppio emisimmetrico; le due proprietà di carattere differenziale sono le seguenti: 1): la divergenza di un tensore emisimmetrico eguaglia è rotore del coniugato; 2): la divergenza del rotore di un tensore emisimmetrico è nulla (e quindi, passando al coniugato, è nulla la divergenza seconda di ogni tensore emisimmetrico).

Nella teoria di Einstein le considerazioni precedenti sono tutte valide purchè si costruiscano le divergenze con la sola parte simmetrica della connessione (divergenze neutre). Ma i tensori doppi emisimmetrici che più direttamente si presentano nella teoria, si ottengono o dal tensore fondamentale ($g_{\underset{\smile}{ik}}$ o $g^{\underset{\smile}{ik}}$) o dal tensore di curvatura contratto ($R_{\underset{\smile}{ik}}$ o $R^{\underset{\smile}{ik}}$).

Se si impone o risulta che sia nullo il rotore di $g_{\underset{\smile}{ik}}$, è tale tensore che fa da tensore elettromagnetico (recente proposta di Stephenson); se risulta nulla la divergenza di $g^{\underset{\smile}{ik}}$ è il suo coniugato che fa da tensore elettromagnetico (teoria di Einstein 1950 e 1953). Ma nella teoria di Einstein 1953 si annulla anche il rotore di $R_{\underset{\smile}{ik}}$. Potrebbe quindi essere assunto questo tensore come tensore elettromagnetico.

B. Finzi ha proposto nel 1953 una modificazione della teoria di Einstein in cui risultano nulla divergenza e rotore della parte emisimmetrica di $R'_{\underset{\smile}{ik}} = R_{ik} + \Gamma_{i,k}$ con Γ_i vettore di torsione; essa può fungere da tensore elettromagnetico in un campo neutro. Nella sua comunicazione a questo Congresso, aggiunge alla parte emisimmetrica considerata il termine $-(\Gamma_{i,k} - \Gamma_{k,i})$ (che lascia inalterato il rotore) per rendere valida la teoria anche per campi non neutri.

Osservo che lo stesso si potrebbe fare partendo dal coniugato del tensore considerato da Finzi. Le conseguenze ottenute potranno a posteriori indicare qual'è la scelta più opportuna.

VIA CORRIDONI 38, MILANO.

SUR LE SILLAGE D'UN OBSTACLE PERMÉABLE

RENÉ DE POSSEL AND JACQUES VALENSI

La théorie du „sillage d'Oseen" (Voir par ex. J. Pérès, Cours de mécanique des fluides, Paris 1936) d'un obstacle dans un fluide au repos à l'infini, revient à négliger $\vec{V} \Lambda \operatorname{rot} \vec{V}$ dans l'expression de l'accélération

$$\tfrac{1}{2} \operatorname{grad} \vec{V}^2 - \vec{V} \Lambda \operatorname{rot} \vec{V} + \frac{\partial \vec{V}}{\partial t}$$

évaluée dans le repère fixe. Des hypothèses relatives aux discontinuités permettaient d'évaluer les pressions et de choisir parmi les solutions.

1. Nous avons remarqué qu'il suffit d'admettre la continuité de la densité d'énergie $p + \frac{1}{2}\varrho V^2$ à la traversée du cylindre de génératrices parallèles à la vitesse à l'infini qui touche l'obstacle.

2. Nous avons étendu la théorie au cas d'un *obstacle perméable* infiniment mince, pensant que les résultats seraient plus voisins de l'expérience que dans cas d'un obstacle véritable. Pour cela, il était nécessaire de fixer des conditions sur l'obstacle simples et voisines de la réalité. Des résultats numériques publiés par divers auteurs [1]) et des photographies de filets colorés au passage à travers une toile, obtenues par l'un de nous, nous ont conduits aux lois [2])

$$k = -\Delta p/(\tfrac{1}{2}\varrho V^2) = \chi V^{\gamma-2} \cos i_1, \qquad n = \operatorname{tg} i_1/\operatorname{tg} i_2,$$

où V est la vitesse immédiatement avant la paroi, i_1, i_2 les angles de la vitesse avec la normale avant et après la paroi, χ, γ et n des constantes pour une paroi poreuse donnée.

3. Nous avons appliqué ces lois aux problèmes plans suivants: a) *obstacle de forme donnée,* b) *recherche du profil d'un obstacle souple en équilibre.* Elles conduisent à des *équations intégrales non linéaires à noyau de Cauchy*. L'étude de ces équations n'a pas été entreprise à notre connaissance et semble difficile.

4. Dans le cas de la bande plane, on obtient une solution approchée en admettant $n = 1$:

(1) $$w = A + B \log [(i - z)/(i + z)]$$

w étant la vitesse complexe, et l'obstacle s'étendant de $-i$ à $+i$.

L'accord avec l'expérience est parfait pour les lignes de courant avant l'obstacle [2]). Quant à la force subie par l'obstacle, (1) conduit à

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1) Taylor, G. I. and Davies, R. M. — Aeronautical research council, reports and memoranda, no 2237, 1944. — Simmons, L. F. G. and Cowdrey, C. F. id. no 2276, 1944.

2) Comptes Rendus Ac. Sc. Paris, 236 no. 2211, 1953.

3) — id. —, 238, p. 1966, 1954.

(2) $$C_x = -\Delta p/(\tfrac{1}{2}\varrho a^2) = 4(1-\sigma),$$

où a est la vitesse à l'infini et $\sigma = u/a$ la porosité aérodynamique (u étant la composante normale de la vitesse, supposée constante en tout point de l'obstacle. Des mesures [3]) faites par l'un de nous pour des toiles métalliques et des tôles perforées de diverses porosités, le nombre de Reynolds rapporté au diamètre des fils étant compris entre 40 et 400, nous ont conduits à la formule $C_x = 4(1-\sigma^2)$. Elle représente une parabole tangente à la droite (2) au point $\sigma = 1$, et l'accord est bon.

UNIVERSITÉ D'ALGER.

SECOND ORDER EFFECTS IN INFINITESIMAL ELASTICITY

MARKUS REINER

Infinitesimal elasticity is defined by the condition that the components of the displacement gradient are infinitesimal. In the classical theory of elasticity it is then assumed that higher powers of these components can be neglected in comparison with the first powers. The expressions for the strain components are thus made linear, and the stress-strain relation $p_{ij} = \lambda I_e \delta_{ij} + 2\mu e_{ij}$, which is linear in tensors, is also linear in displacement gradients. It is shown that this procedure gives too narrow results and masks the existence of second order effects which may, and ordinarily will, arise in infinitesimal elasticity. Second order terms cannot be neglected when first order terms are absent. In such cases the above stress-strain relation, while not loosing its tensorial linearity, becomes "quasi-linear". — As the most general measure of strain a linear combination of the "Lagrangean" or "material" Green-measure and the "Eulerian" or "spacial" Almansi-measure is assumed, and the quasi-linear stress-strain relation solved for cases of homogeneous pure strains resulting from isotropic pressure and uniaxial tension, and for the cases of simple shear and simple shearing stress in which the principal axes are rotated. It is shown that in the last two cases second order effects (namely the Kelvin and the Poynting effect) make their appearance, even if the displacement gradients are infinitesimal.

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THE USE OF POISSON'S FORMULA IN PATTERN SYNTHESIS

HERBERT ELLIS SALZER

In the employment of linear arrays in antenna design, one of the important problems of pattern synthesis is to obtain feeding coefficients which will produce very sharp beams. Although optimum patterns are obtained from the Dolph-Tschebyscheff distributions, there the numerical work in determining the feeding coefficients mounts as the number of sources increases. This present article indicates an entirely independent method of finding feeding coefficients which uses only one very special case of a general formula due to Poisson, to obtain extremely sharp patterns for broadside arrays. Although the resulting formulas are not as flexible as the Dolph-Tschebyscheff formulas and require a comparatively large number of sources, the amplitudes of the feeding coefficients are given at once by an extremely simple explicit expression which is just as easy to calculate for a formula with over a hundred terms as for a formula having just a few terms (and those amplitudes of the feeding coefficients for broadside arrays are invariably positive.)

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SUL PRINCIPIO DELL'AZIONE POTENZIALE

EDOARDO STORCHI

Il principio di Hamilton afferma, nel caso conservativo, che fra tutti i moti variati sincroni che rispettano le configurazioni estreme, il moto naturele e' quello per cui risulta stazionaria l'azione hamiltoniana:

$$A = \int_{t_0}^{t_1} (T + U)dt.$$

Introdotto il concetto di moto variato asincrono, dal principio di Hamilton si puo' desumere il principio di Hölder o dell'azione: „Fra tutti i moti variati asincroni isoenergetici (per i quali cioè è rispettata la condizione $\delta_a E = 0$ essendo E l'energia totale), il moto naturale è quello per cui risulta stazionaria l'azione:

$$A^* = \int_{t_0}^{t_1} 2T dt".$$

Orbene questi due importanti principi sono caso particolare di un unico principio che qui enunciamo:

,,Fra tutti i moti variati che rispettano le configurazioni estreme e tali che nel passaggio dal moto naturale al moto variato risulti: $\delta_a[E^{\lambda-1}dt^\lambda] = 0$ (essendo λ un parametro arbitrario), il moto naturale è quello per cui si mantiene stazionaria l'azione:

$$S = \int_{t_0}^{t_1} [(2 - \lambda)T + \lambda U]dt.''$$

Per $\lambda = 0$ i moti variati risultano isoenergetici ($\delta_a E^{-1} = 0$), l'azione (1) si riduce all'azione hölderiana e si ottiene il principio di Hölder; per $\lambda = 1$ i moti variati risultano sincroni ($\delta_a(dt) = 0$), l'azione (1) si riduce all'azione hamiltoniana e si ottiene il principio di Hamilton. Un caso pure notevole è quello corrispondente a $\lambda = 2$. Poichè nel passaggio dal moto naturale al moto variato rimane inalterata la quantità meccanica: $Q = Edt^2$ diremo isocronoenergetici i moti variati asincroni per i quali la condizione variazionale $\delta_a[Edt^2] = 0$ è rispettata. Si giunge allora alla formulazione del seguente principio variazionale dell'azione potenziale stazionaria: ,,Fra tutti i moti variati isocronoenergetici che respettano le configurazioni estreme, il moto naturale è quello per cui risulta stazionaria *l'azione potenziale*:

$$S = \int_{t_0}^{t_1} 2Udt$$

Nel caso di vincoli indipendenti dal tempo si puó dare al principio una ben diversa interpretazione. Dette infatti $q_1, q_2, \ldots, q_n$ le coordinate lagrangiane, risulta in tal caso:

$$2Edt^2 = \sum_1^n {}_{h,k} a_{hk} dq_h dq_k - 2U(q_1, q_2, \ldots, q_n)dt^2$$

e si vede allora come sia possibile adottare la quantità meccanica:

$$Q = 2Edt^2 = dS^2$$

quale quadrato dell'elemento lineare di una varità spazio-temporale ad $n + 1$ dimensioni. La validità del principio è dunque subordinata ad una variazione che non altera l'elemento lineare di un opportuno spacio-tempo: $\delta_a(dS) = 0$.

Nel caso del moto per inerzia il principio si riduce ad un principio del tempo stazionario. In particolare per il moto inerziale di un solo corpuscolo, la validità del principio del tempo stationario comporta una variazione che, dovendo rispettare la condizione:

$$\delta_a[dx^2 + dy^2 + dz^2 - \frac{2U}{m}dt^2] = 0$$

lascia inalterata la lunghezza della linea oraria in uno spazio-tempo pseudoeuclideo in cui la costante $c = \sqrt{\frac{2U}{m}}$ funge da velocità della luce.

VIA SEBINO 2 — MILANO (ITALIA).

SUCCESSIVA LINEARIZZAZIONE DELLE EQUAZIONI DEL CAMPO UNITARIO EINSTEINIANO

Paolo Udeschini

Riferendomi all'ultima versione data da Einstein alla teoria relativistica del campo unitario, considero le equazioni approssimate in successive linearizzazioni mettendo in evidenza le mutue azioni fra i due campi gravitazionale ed elettromagnetico.

Mentre in prima approssimazione le equazioni linearizzate separano il campo gravitazionale da quello elettromagnetico, già in seconda approssimazione tale separazione non ha luogo, e manifesta risulta l'interazione fra i due campi fondamentali, quello gravitazionale e quello elettromagnetico. E ciò vale nelle successive linearizzazioni e si possono stabilire delle equazioni di tipo ricorrente con un processo di successiva linearizzazione. I risultati ottenuti si accordano con quelli del caso generale, in cui non interviene approssimazione alcuna, come nello schema considerato da W. Hlavaty.

Via Ariosto 27, Milano (Italia).

Section VI

LOGIC AND FOUNDATIONS

SHORT LECTURES

ESQUEMAS REPRESENTATIVOS DE SISTEMAS REGIDOS POR UNA LÓGICA POLIVALENTE

BADILLO BARALLAT

Se empieza indicando la posibilidad de emplear el álgebra de la lógica polivalente en los sitemas, como los constituidos por ruedas dentadas electrónicas, que pueden ocupar sucesivamente in estados estables. A continuación se aplica dicha álgebra al establecimiento de los esquemas relativos a los mencionados sistemas. Seguidamente se ponen varios ejemplos de redes primitivas y sus correspondientes derivadas. Por último se insertan las tablas de valores en serie y en paralelo.

FUENCARRAL, NO 116, 3e DECHA, MADRID.

OBTENTION EN THÉORIE DES RELATIONS DE CERTAINES CLASSES D'ORIGINE LOGIQUE

ROLAND JEAN FRAÏSSÉ

I. Rappelons qu'une *classe arithmétique* est formée des relations (ou des systèmes de relations) vérifiant une formule Θ du calcul logique du premier ordre (*A. Tarski,* Proc. Intern. Congr. Math. 1950, p. 710); si Θ ne contient que des quantificateurs universels tous en tête de la formule, la classe est dite *universelle.* Considérons une relation $R(x_1, \ldots, x_n)$ comme une fonction définie sur un ensemble E et susceptible de prendre deux valeurs; la relation R' définie par $F \subseteq E$ par l'égalité: $R'(x_1, \ldots, x_n) = R(x_1, \ldots, x_n)$ lorsque $x_i \in F$ sera dite la *restriction* R/F. Une relation S isomorphe à une restriction de R sera *inférieure* à R ($S < R$). Nous écrirons $S \underset{p}{<} R$ (p entier naturel) si toute restriction de S définie sur p éléments au plus est inférieure à R.

II. On démontre que chacune des conditions suivantes équivaut à dire que K est universelle:

1. Il existe un ensemble fini de relations A définies sur un nombre fini d'éléments, tel que $R \in K$ équivaut à $A \not< R$ pour chaque A.

2. Il existe un entier p tel que, pour toutes relations R, S, les conditions $R \in K$ et $S \underset{p}{<} R$ entraînent $S \in K$.

3. Condition dûe à *R. Vaught*: Il existe un p tel que $R \in K$ si, et seulement si toute relation inférieure à R et définie sur p éléments au plus appartient à K.

III. Dans la condition II-2, remplaçons $S \underset{p}{\prec} R$ par: $S \underset{p}{\prec} R$ et $R \underset{p}{\prec} S$; on obtient alors pour K les classes arithmétiques particulières déduites des universelles par un nombre fini de complémentaires, réunions et intersections finies.

IV. Soient R, S définies sur les ensembles D, E. Un isomorphisme φ de R/F sur S/G (où $F \subseteq D$, $G \subseteq E$) sera dit un $(0, p)$-*isomorphisme de* R *vers* S si F et G ont au plus p éléments. Procédant par récurrence, nous dirons que φ est un (n, p)-*isomorphisme de* R *vers* S si, quel que soit $\overline{F}$ $(F \subseteq \overline{F} \subseteq D)$ ayant au plus p éléments, il existe un $(n-1, p)$-isomorphisme $\overline{\varphi}$ de R vers S défini sur $\overline{F}$ et se réduisant à φ sur F, et si l'on a la même condition en échangeant R, F, φ et S, G, φ^{-1}. Nous écrirons $R \overset{n}{\underset{p}{\sim}} S$ s'il existe un (n, p)-isomorphisme de R vers S, l'ensemble F étant vide.

On démontre que K est une classe arithmétique si, et seulement s'il existe deux entiers n, p tels que, pour toutes relations R, S, les conditions $R \in K$ et $S \overset{n}{\underset{p}{\sim}} R$ entraînent $S \in K$.

On retrouve les classes K du paragraphe III lorsque $n = 1$; en effet, $R \overset{1}{\underset{p}{\sim}} S$ équivaut à $R \underset{p}{\prec} S$ et $S \underset{p}{\prec} R$.

Berchères-sur-Vesgre (Eure et Loir) France.

ON THE INTUITIONISTIC AND FORMALISTIC THEORY OF REAL NUMBERS

Sigekatu Kuroda

In mathematics each of the logical symbols, $\wedge$, $\vee$, $\supset$, $\equiv$, $\exists$, and $\forall$, has two entirely different meanings, namely, intuitionistic and formalistic. To distinguish these two meanings, the suffix (+) or (—) should be attached to each of these symbols, as these symbols are used in the intuitionistic or formalistic sense respectively. These two kinds of symbols may well be considered to be related by the formulas

$$a\,\omega^- b \equiv^+ \neg\neg\, (a\,\omega^+ b), \quad \Omega^- x a(x) \equiv^+ \neg\neg\, \Omega^+ x a(x)$$

where ω stands for $\wedge$, $\vee$, $\supset$ or $\equiv$ and Ω for $\exists$ or $\forall$. The negation is the unique symbol, whose two meanings can not be distinguished in this way owing to the equivalence of triple negation to the mere simple one.

The concept of real numbers is defined by the fundamental sequence of rational numbers a_1, a_2, a_3, . . ., which satisfies the Cauchy condition

$$\forall^{\pm} \varepsilon (\text{rational} > 0),\ \exists^{\pm} n,\ \forall^{\pm} r,\ s > n,\ |a_r - a_s| < \varepsilon.$$

We obtain from this formula presumably 2^3 different concepts of real numbers according to the selection of the suffixes (+) or (—) of each symbol.

If we construct the real number theory, always using the words with (+) (or (—)) sign, we get the intuitionistic (or formalistic) theory of real numbers. There are also many intermediate theories, if we use symbols sometimes with (+) sign and sometimes with (—) sign. Thus the analysis of the modi (+) and (–) of logical symbols leads to different theories of real numbers. The intuitionistic and formalistic theories are characterised to be situated at the two extreme ends of the „spectra of modi".

On the other hand the two propositions $\neg \forall x a(x)$ and $\neg \exists x \neg a(x)$, which are in the formalistic logic contradictory to each other, can be intuitionistically at the same time valid for some appropriately constructed proposition $a(x)$. It is however very plausible that this can never occur for such a proposition $a(x)$, in which the range of the variable x and also of every variable, which may eventually occur in the inner construction of the proposition $a(x)$, are all enumerable sets. Assuming this be true, we obtain only two kinds of concepts of real numbers, which correspond to that in the intuitionistic and formalistic sense respectively.

Under this assumption the full analysis of modi of logical symbols is treated in detail concerning the real number theory.

CHIKUSA-KU,
NAGOYA JAPAN.

Section VII

PHILOSOPHY, HISTORY AND EDUCATION

SHORT LECTURES

SULL' INSEGNAMENTO MATEMATICO NELLE SCUOLE SECONDARIE E SUI TESTI SCOLASTICI

Vincenzo Amato

Ritengo mio dovere di uomo di scuola, già quasi estraneo alla matematica (diciamo cosi) militante, di rilevare le manchevolezze che, specialmente negli ultimi anni, sono apparte nell'insegnamento della matematica nelle scuole italiane.

Due guerre mondiali hanno aperto le porte a giovani impreparati, scientificamente e didatticamente, a coprire le cattedre nelle scuole medie e purtroppo non pochi di questi giovani hanno creducto di poter primeggiare scodellando libri di testo opachi e scorretti.

Non so se questo male della Scuola italiana sia comune con altri paesi; è però certo che occorre stroncarlo.

Secondo mei, i rimedi sono assai semplici se severamente applicati.

Il primo è quello di non supervalutare l'esame di laurea specie quando il docente sa bene che la lode che egli propone riguarda proprio quella parte della tesi dovuta a lui e non al candidate: saddisfazione ridicola quella di lodar sé e lesiva a colors che hanno faticato con le risorse del proprio ingegno!

Il secondo è quello di sottoporre, prima dell'adozione, i libri di testo ad una Commissione di stato con pieni poteri.

Quest'ultima richiesta potrebbe essere oggetto di un voto da parte delle Sottocommissioni per l'insegnamento della matematica nei paeti interessati.

Via Grotte Bianche 7,
Catania, Italia.

LE RÔLE DE LA MATHÉMATIQUE ET DU MATHÉMATICIEN DANS LA VIE CONTEMPORAINE

Guido Ascoli

Le caractère de la Mathématique dans une certaine époque est sans doute lié aux intérêts materiels et spirituels de cette époque. Les exemples abondent. Mais la Mathématique a aussi une logique intérieure qui trace en bonne partie son chemin. On ne pourrait y renoncer sans diminuer en même temps la valeur de notre culture et endommager aussi toute application.

Cette réserve faite, on peut demander quels sont les intérêts qui peuvent influer sur l'activité mathématique actuelle. Nous les reconnaissons dans la pénétration si imposante des Mathématiques dans toutes les sciences et techniques. J'ai questionné des personnes qualifiées de mon Pays sur les modalités de cette collaboration. En général les spécialistes travaillent pour leur compte; au surplus ils posent au mathématicien des questions bien encadrées pour avoir des formules résolutives ou des tabulations. C'est à quoi répond très bien notre „Istituto nazionale per le applicazioni del Calcolo", fondé et dirigé par M. Picone. Ce procédé est préféré; et partant on insiste pour une préparation mathématique plus profonde dans les Ecoles supérieures. Les demandes diffèrent en extension selon les spécialités; les plus élevées sont celles des électrotechniciens, en particulier pour les Télécommunications. On propose des cours complémentaires ou facultatifs, des diplomes speciaux; plus simplement, pour les chercheurs, la licence en Mathématique après les autres diplomes scientifiques ou techniques. On insiste sur le „bon usage" des Mathématiques: discussion du schéma choisi et des résultats, limitation de la précision à celle qui est permise par les données, etc.. On reconnait toutefois un rôle au mathématicien à cause de l'ampleur de ses vues et de sa mentalité critique; on envisage, dans l'avenir, son emploi dans l'industrie, actuellement nul, et qui, au surplus, pourrait intéresser seulement quelques grandes entreprises: élaboration de résultats expérimentaux, résumé de travaux théoriques etc. On recommande des expositions de théories élevées en forme accessible aux techniciens.

On ne peut pas considérer la Mathématique seulement comme connaissance ou comme outil; elle a aussi une valeur éducative, qui lui donne une place même dans une culture humaniste. Rien ne peut la remplacer comme exemple de langage rigoureux et univoque, de cohérence, de probité intellectuelle. Nous pensons que la diffusion de la mentalité mathématique dans notre temps doit avoir une influence bienfaisante sur le droit, sur le langage administratif, sur la conduite des discussions, sur l'esprit de tolérance. Il est juste, en somme, de révendiquer à notre science aussi un rôle moral et humain.

Corso Vittorio Emanuele 164 Torino (Italy).

REPORT ON MATHEMATICS INSTRUCTION IN DENMARK

S. Bundgaard

No manuscript of this lecture was available

REPORT OF THE DUTCH SUB-COMMISSION ON THE TEACHING OF MATHEMATICS IN THE NETHERLANDS TO STUDENTS OF 16-21 YEARS OF AGE

LUCAS NICOLAAS HENDRIK BUNT

The referee restricted himself to speaking about two points and to mention

1. certain general considerations which influenced the way in which the report is organized,
2. a number of recent trends in changing the curriculum of the secondary schools.

The restriction to the age-group 16-21 brought with it some difficulties. In the first place, the teaching of the typical secondary school mathematics starts when the pupils are about twelve years old, so that the students of the age of 16, to whom this report refers, have already had a considerable preliminary mathematical training.

In the second place, the university teaching of mathematics to students up to 21 years is only for small groups complete in itself, and usually a preparation to further study in mathematics.

For these two reasons it was here and there inevitable to give information on the instruction in mathematics to pupils and students outside the age-group 16-21.

A third difficulty resulting from the restriction mentioned above, is the consequence of the Netherlands system of getting and missing the remove. While the greater part of pupils on entering the secondary school are 12-13 years old, and so the students of 12-13 years of age nearly all occupy the first form of this school, it is not possible to indicate a definite form in which the age group 15-16, the pupils of the youngest year-group of this report, would be found. It can certainly not be said that form IV of the schools for secondary education is the one. Still, since somewhere a line of demarcation had to be drawn between that part of the mathematical study considered for description, and the part that was not, the report has been confined to the part which the legislator originally intended for the pupils 16 years of age and older. Its consequence is that, so far as the secondary school is concerned, the report is restricted to the forms IV, V and VI of the gymnasium and the forms IV and V of the modern secondary school.

The teaching in the two highest forms of both schooltypes is a more or less independent whole, which fact somewhat diminishes the unsatisfactory character of the selection of the lower limit for age.

The time to prepare this report in, was extremely short. Hence the more or less schematic character of some parts of it, moreover, there was hardly any opportunity to consult the other members of the subcommittee.

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Utrecht, Netherl.

REPORT OF THE BRITISH SUB-COMMISSION ON MATHEMATICAL INSTRUCTION FOR STUDENTS BETWEEN 16 AND 21 YEARS OF AGE

Mary Lucy Cartwright

The British Subcommission's replies to the Questionnaire only cover the educational system of England and Wales and most of the answers are over simplified generalizations because there is so much variety in practice. The Ministry of Education has no power to specify the subjects taught, nor the time allotted to them, nor does it examine. The Ministry has inspectors who report to the Minister and advise both State and Independent Schools. Compulsory education begins at 5 and ends at 15. The State system is a partnership between the Ministry of Education, the Local Education Authority (L.E.A.) and the Governing body and teachers of the schools. It consists of a primary stage from 5 to 11 and a secondary stage from 11 to further education at the University, Technical College, or in part time and evening classes. Both stages are free. There are three types of Secondary School, Grammar School with mainly academic basis, Technical School and Secondary Modern School. Admission to a Secondary School is by an examination arranged by the L.E.A. and there is much competition for Grammar School places. There are also Independent Schools; some are like Grammar Schools but the most famous are the so-called Public Schools for boys which admit pupils from the independent Preparatory Schools at the age of $13\frac{1}{2}$.

The syllabuses in Independent Schools and Grammar Schools are conditioned by the General Certificate of Education (G.C.E.) which is organized by examining boards established by the Universities. These boards are supervised by a committee containing representatives of the universities, the schools and the Ministry of Education. It is a written examination. Subjects may be taken at O = Ordinary, A = Advanced or S = Scholarship level in G.C.E. The O level is suitable for a pupil of 16 but may be taken later. The A level is taken 2 years later and fewer subjects are offered. The O and A levels in certain subjects are used by Training Colleges and Universities in admitting students and also by professional bodies. The S level papers are taken in one or two allied subjects and are used in awarding State Scholarships and Scholarships offered by the L.E.A. The Universities also offer scholarships and the examinations for these exert a strong influence on the teaching of mathematics in schools. About $\frac{3}{4}$ of the students at the universities have at least part of their fees paid by scholarships which may include a grant covering for maintenance. The usual first course at the University lasts 3 years and leads to a B.A. or B.Sc. degree. Most teachers in Grammar Schools and Public Schools have a degree. Teachers in Primary and Secondary Modern Schools do a 2 year course at a Training College. Specimen timetables and examination papers and other information have been submitted with the report.

Girton College,
Cambridge.

EDUCATION IN FRANCE

A. Châtelet

No manuscript of this lecture was available

THE TEACHING OF MATHEMATICS

Ramón Créspo Pereira

In teaching Mathematics we must consider three realities: 1) Mathematics, 2) The teacher, and 3) The pupil. But these three fundamental parts are in a functional relation of interdependence. So they have to be taken in the most concrete way because the teaching of Mathematics takes place always into a particular, real, cultural sphere. We must not tackle didactic problems with abstract methods. In particular cases too there are many possibilities of reticulation: elementary school, secondary school, University, etc. We have to take likewise into consideration the following fact. To learn what other persons have created or made, is not the same thing as to do original work. The learning process is quite different when we work in anything because it is vital for ourselves to know. If the learning action comes from external sectors to the pupil, and not from the very inside of the person who learns, the whole situation has dissimilar characteristics.

We must pay attention to the pupil. There are no pupils without a teacher and reciprocally. But let us look upon the following relational fact: The teacher is *for* the pupil, not *vice versa.*

I consider as a fundamental principle that what really matters in the teaching process is the spirit, not the litteral or formal statement. It is very convenient and opportune to know how to teach any particular mathematical lesson to a certain type of pupils. Yet, in my opinion, it is more important that the teacher should stimulate scientific interests and love for Mathematics in pupils. The most important thing to be attained is this: the pupil should regard Mathematics as a very wide field to work in, should believe that Mathematics is always open, i.e., not yet finished, and that in our science we can arrive at as beautiful heights as in the fine arts.

FERNANDO EL CATÓLICO, 28,
MADRID.

THE FUNCTION OF MATHEMATICS IN MODERN SOCIETY AND ITS CONSEQUENCES FOR THE TEACHING OF MATHEMATICS

D. VAN DANTZIG

This lecture appeared as report No 1 of the National Committee of the International Commission on Mathematical Instruction (ICMI) in the Netherlands

LE RÔLE DU MATHÉMATICIEN DANS LA VIE CONTEMPORAINE

GEORGES DARMOIS

Caractère exceptionnel du développement de ce rôle à notre époque. Vie et fécondité des disciplines mathématiques dans l'art de l'ingénieur au sens général, et dans la recherche technique.

Conséquences pour la structure nécessaire et pour le développement de l'enseignement et de la formation de la jeunesse et des adultes. Ne pas oublier que la source vive est la recherche fondamentale. Mais l'exposition nécessaire des théories modernes doit être simple et attrayante, ce qui pose depuis le début (et jusqu'à la fin) le problème pédagogique, qui exige un effort constant.

Types nouveaux des hommes nécessaires.

Les conseillers mathématiques des laboratoires, des grandes entreprises, des grands services de l'Etat.

Nouveautés particulières à la théorie des probabilités, la statistique mathématique.

Les mathématiques en biométrie, en économétrie.

Les mathématiques dans le calcul numérique.

Développement spécial des applications industrielles de la statistique. L'ingénieur statisticien sous ses différentes formes.

Conclusions.

7 Rue de l'Odéon - Paris 6e.

DETERMINANTEN IM UNTERRICHT ALLGEMEINBILDENDER SCHULEN

(Kurze Zusammenfassung).

R. Dolinsky

Die Behandlung der 2- und 3-reihigen Determinanten im Unterricht der höheren Schulen hat sich in Westdeutschland fast überall durchgesetzt und wird in einigen deutschen Ländern durch die Richtlinien verlangt. Man beschränkt sich dabei oft auf die Mitteilung und Anwendung der Cramerschen Regel.

Die Hauptbedeutung der Determinanten besteht aber nicht in einer Erleichterung des Rechnens, sondern darin, daß sie uns einen tieferen Einblick in die Zusammenhänge und Lösbarkeitsfragen erlauben.

Auf Grund langjähriger Erfahrung schlage ich folgendes Verfahren vor:

Man verzichte auf die Kombinatorik.

Man gehe von den zweireihigen und dreireihigen Determinanten aus und leite die Hauptregeln für die Determinanten und die Cramersche Regel ab.

Dann wird die Determinante beliebigen Grades auf folgende Weise definiert: Sie ist eine homogene Funktion der n-reihigen quadratischen Matrix, für welche alle Regeln gelten sollen, die für 2- und 3-reihige Determinanten abgeleitet wurden. Ferner soll das erste Glied gleich dem Produkt der Glieder der Hauptdiagonale sein.

(Diese Definition ist auf Grund des Weierstraßschen Theorems durchaus berechtigt).

Die praktische Berechnung einer n-reihigen Determinante erfolgt auf Grund der Entwicklung nach den Elementen einer Zeile. Man kann aber auch den ganzen Laplaceschen Entwicklungssatz im Unterricht durchnehmen.

Mit diesem Unterbau ist es möglich, den mathematischen Unterricht allgemeinbildender Schulen (entspr. deutschen Gymnasien) wesentlich zu vertiefen, und zwar können folgende Gebiete durchgenommen werden: allgemeine Theorie linearer Gleichungen (homogener und inhomogener), lineare Abhängigkeit von Zahlenfolgen,

Anwendungen der Determinanten auf die analytische Geometrie der Ebene und des Raumes,

Anwendungen auf die Auflösung algebraischer Gleichungen 3-ten und 4-ten Grades. (Hier benutze ich das von Prof. Dr. I. Schur in seinen Vorlesungen 1925-26 angegebene Verfahren, welches ich weiterentwickelt habe.)

Die Determinanten werden am besten bei Schülern im Alter von 14-15 Jahren (deutsche Obertertia) eingeführt und in den späteren Klassen immer wieder angewandt.

Der hier angegebene Umfang bezieht sich auf sprachliche Anstalten. Auf mathematisch-naturwissenschaftlichen Schulen kann der Unterricht wesentlich ausgebaut und vertieft werden.

STEPHANSTRASZE 6,
GREVENBROICH-ELSEN.

THE ADMINISTRATION OF MATHEMATICS EDUCATION IN THE UNITED STATES OF AMERICA

HOWARD FRANKLIN FEHR

Who determines the mathematics curriculum, who decides what mathematics must be studied by some or all of the students, and who supervises and executes the mathematics education of youth are important questions in all countries. In the United States of America, there are a variety of answers to each question.

There is no federal control of Education in the U.S.A. The control lies within each state, and most states delegate responsibility to the local city or town. The state laws usually make schooling a mandatory process up to age 16, and specify certain subjects that must be taken by all students. Almost all states make it mandatory to study mathematics through the first eight grades. This study includes arithmetic and intuitive geometry, with a very slight introduction to algebra. In seventeen of the states (including New York State), no mathematics study is compulsory after the eighth grade. In the other states, the usual requirement is one year of high school mathematics, the nature of which is very ambiguous. In a few states, over one-fourth of the high schools offer no mathematics beyond the ninth year for any of their pupils.

Outside agencies, however, cause mathematics to be offered and to be studied. For entrance to most liberal arts colleges and schools of engineering and technology, a minimum of three years credit in high school mathematics is required, many colleges demanding four years credit. Technical institutes demand a year of algebra and a year of geometry for entrance. Vocational schools, under a federal law which grants money to these schools, must teach one hour of mathematics, related to shop work, each day of the four high school years. The National Council of Teachers of Mathematics, with a membership of 10,000, through its committees, conventions, and publications, exerts strong pressure on local education bodies to offer and to require at least one year of mathematics for general students, three years of mathematics for college-bound students, and four years for scientific and engineering preparatory students.

The curriculum is made by state committees, by local committees, and by committees appointed by national organizations. These curriculums are, for the most part, guides which teachers can follow but must not adhere to rigidly. In New York State, the Regents is a controlling body which sets the syllabus for college preparatory mathematics and administers state-wide tests on the syllabus. This is the only state control of mathematics in the U.S.A. Even here, local option exists (but not usually exerted) for giving their own college preparatory courses and diploma. Iowa is the only other state with a state-wide testing program, which, however, is not mandatory on local bodies for granting credit. By far and large, the curriculum in most schools is set by the textbook which is used. The textbook writers thus become the important element in mathematics education.

Outside of several large cities, there is very little supervision. Each mathematics teacher is his own authority and he selects and teaches what he deems best. This freedom is good for the well-prepared teacher, bad for the poor teacher. The result is a free competition that has produced a fairly good program for average students, and (except for several outstanding schools) a gross neglect of capable and gifted students. This we are now trying to remedy. As long as colleges, business, industry, and leaders in educational thought continue to make demands for mathematics education, the schools will give and improve such education.

TEACHERS COLLEGE, COLUMBIA UNIVERSITY,
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SUMMARY OF A REPORT ON THE MATHEMATICAL INSTRUCTION IN SWEDEN FOR STUDENTS BETWEEN 16 AND 21 YEARS OF AGE

O. Frostman

The ordinary Swedish schools giving theoretical education to students between 16 and 19 years of age are called gymnasier. People who intend to continue their mathematical studies can do that either at a university (lowest and most common examination: filosofie kandidatexamen or filosofie ämbetsexamen) or at an institute of technology (civilingenjörsexamen). In both cases mathematics is studied during the first years, and many of the students pass their mathematical examination before 21 years of age.

The gymnasium has three sides. On the modern side, which has the largest course in mathematics, this subject is taught during about six hours a week. The course includes: *Arithmetics and algebra*: The real number concept. Polynomials in one variable, decomposition into linear factors, the connections between roots and coefficients, *n-th* roots, powers and logarithms. *Plane and solid geometry* approximately corresponding to the Elements of Euclid. Plane trigonometry. *Plane analytical geometry*: The straight line. The conic sections as quadratics (without xy-term). *Functions of one real variable*: The concepts of function, limit, continuity and derivative (unrigorous treatment). Maxima and minima. Integrals applied to calculation of areas and volumes. The class of functions considered is: Rational, trigonometric, exponential and logarithmic functions.

At the universities the mathematical material, which is included in fil. kand. — and fil.ämbetsexamen, is subdivided into three courses: The courses for one point (instruction during half a year), two points (another half years instruction) and three points. The course for three points is rather farreaching. The instruction is pursued as lectures and seminars (together 10—15 hours a week).

The course for *one point* includes: *Algebra*: Complex numbers, Combinatorial arithmetics and elementary number theory. *Calculus*: The use of real numbers is based on some postulate, e.g. the existence of the least upper bound. Limits of number sequences. Infinite series. Differential- and integral calculus (one variable), Rolle's theorem, the mean value theorem, Taylor's series, the Riemann integral. All elementary functions are studied. *Geometry*: Two dimensional vectors. Oblique affine coordinates and change of coordinates. The straight line. Conic sections in synthetic and analytic treatment.

The course for *two points* includes in addition to the course for one point: *Algebra*: Theory of determinants. Systems of linear equations. Symmetric functions. The fundamental theorem of algebra. Constructions with ruler and compass and in this connection elementary facts about fields of numbers. *Calculus*: Theory of irrational numbers. Extended course of infinite series. Improper integrals. Uniform convergence. Functions of several variables, Taylor's series, line integrals, multiple integrals. Ordinary differential equations. *Geometry*: Elementary facts about plane projective geometry. Three dimensional vectors, inner product, outer product. Oblique affine coordinates, change of coordinates. The straight line and the plane. A concise treatment of quadratics.

The course of mathematics at the institute of technology corresponds approximately to the university course for two points.

INSTITUT MITTAG-LEFFLER,
DJURSHOLM, SWEDEN.

DER MATHEMATISCHE UNTERRICHT IN ÖSTERREICH

FRITZ HOHENBERG

In Österreich ist das Realgymnasium die vorherrschende Gattung der allgemeinbildenden Schulen für die 10- bis 18-Jährigen. Daneben gibt es Realschulen und Gymnasien. Mathematik ist in allen drei Schulen Gegenstand der schriftlichen und mündlichen Reifeprüfung. In den letzten vier Klassen stehen dem Mathematikunterricht am Realgymnasium drei Wochenstunden zur Verfügung. Die Provisorischen Lehrpläne 1946 bezeichnen als Lehrziel des Mathematikunterrichts: „Kenntnis der elementaren Mathematik samt Erfassen und Anwenden des Funktionsbegriffs, Kenntnis der Grundbegriffe der Infinitesimalrechnung. Fähigkeit, das Mathematische in Form und Gesetzmäßigkeit an den Erscheinungen der Umwelt zu erfassen und die Bedeutung der Mathematik und ihrer Verfahren für Naturwissenschaften, Technik und öffentliches Leben zu erkennen. Schulung des räumlichen Vorstellungsvermögens." Verlangt wird eine sorgfältige, sparsame Stoffauswahl, eine gründliche, eindringliche Behandlung des Lehrguts, Vermeidung aller Beispiele, die weder mathematisch noch praktisch wertvoll sind. Bemerkungen zur Geschichte und Philosophie der Mathematik. An den Realschulen wird erhöhte Selbständigkeit im Lösen schwierigerer Aufgaben angestrebt. Darstellende Geometrie wird an Realschulen in den letzten vier, an Realgymnasien in den letzten zwei Schuljahren als selbständiges Unterrichtsfach mit je zwei Wochenstunden gelehrt.

Die Anforderungen im Mathematikunterricht sind in Österreich höher als in manchen anderen europäischen Staaten. Reformtendenzen weisen auf eine stärkere Querverbindung des Mathematikunterrichts zum Physikunterricht und auf eine stärkere Betonung der mathematischen und praktischen Anwendungen der darstellenden Geometrie. Als Unterrichtsmethode findet der Arbeitsunterricht wachsende Verbreitung.

An den drei österreichischen Universitäten wird Mathematik als Selbstzweck gelehrt, an den beiden Technischen Hochschulen und einigen anderen Hochschulen im Rahmen der Ausbildung für technische und andere Berufe. Das Studium der Mathematik kann mit dem philosophischen, u.U. mit dem technischen Doktorat abgeschlossen werden. Künftige Lehrer an Realgymnasien usw. legen die Lehramtsprüfung aus den Hauptfächern Mathematik und Physik oder aus den Hauptfächern Mathematik und Darstellende Geometrie ab. Die mathematische Ausbildung der Mathematiker und Ingenieure besitzt trotz mancher Hindernisse hohes Niveau. Günstig für die Ausbildung der Mathematiker ist die Möglichkeit des gleichzeitigen Studiums an Universität und Technischer Hochschule. So verbindet sich die Freizügigkeit, Selbstverantwortung und theoretische Schulung der Universität mit der straffen, mehr auf die Anwendungen gerichteten Schulung an der Technischen Hochschule.

KOPERNIKUSGASSE 24,
GRAZ, (AUSTRIA).

INTERMEDIATE MATHEMATICAL INSTRUCTION IN THE UNITED STATES

SAUNDERS MACLANE

This is a summary of the report of the United States subcommittee of the International Commission on Mathematical Education, covering the topic of educational methods for students between the ages of 16 and 21. In the United States, this age group covers the latter part of high school and the first part of college. Because of their strict separation, these two types of institutions show markedly different development. However, both are drastically influenced by the tremendous number of students to be educated. In the high schools, the standard curriculum has been fixed for generations, but the content has been reduced and new pedagogical devices have been introduced to make the mathematics appeal on practical terms to more students. New „General Mathematics“ curricula have also appeared. In the colleges, there are several standard types of courses available, depending on the type of institution. These have been altered some in recent years; for example, there is now less emphasis on analytic geometry, and more on statistics and other topics of use in the social sciences. Many improvements in instructional methods and types of courses are still necessary. The report closes with a summary of some of the current movements for reform.

5712 Dorchester Ave,
Chicago 37, Ill. USA.

PEDAGOGIA SCIENTIFICA INQUADRATA IN UNA TEORIA MATEMATICA

Elena Palazzo

Dopo aver studiato la teoria matematica dcl Prof. Fantappiè (sorta dallo studio delle equazioni fondamentali della fisica moderna), che è teoria unitaria del mondo fisico e biologico, ho trovato che in essa si puòinquadrare anche il campo educativo. Ricordo che in tale teoria si considerano fenomeni entropici, (retti da un principio di causalità, tendenti a passare, col tempo, dal differenziato all'omogeneo,) e fenomeni sintropici (retti da un principio di finalità, tendenti a passare, col tempo, dallo omogeneo al differenziato). La pedagogia che rientra in detta teoria ed è da essa illuminata, è la pedagogia scientifica (secondo quanto ho esposto in precedenti lavori approvati da Maria Montessori, che li ha sistematicamente usati nei suoi ultimi corsi.)

Cito qualche nuovo risultato.

I) Lc *nebule* scoperte dalla Montessori nel neonato, (cioè le energie potenziali di questo), si identificano nella teoria unitaria con l'*embrione* da cui dipendono i fenomeni del massimo interesse per la vita biologica.

II) I *periodi sensitivi* scoperti dal De Vries nel mondo biologico e quelli divinati dalla Montessori nello sviluppo psichico dell'uomo, si unificano in fenomeni sintropici della teoria unitaria.

III) Si ha una nuova analisi dell'acquisto della cultura matematica. L'attrazione del bambino, (e quindi dell'uomo,) alla aritmetica ed alla geometria [1]), è fenomeno sintropico diretto alla distinzione di forme e dimensioni. L'interesse per altro materiale, sintropicamente lo avvia alla numerazione, all'uso dei segni rappresentanti i numeri, al calcolo delle quattro operazioni. Nel susseguirsi dei periodi sensitivi, (dovuti tutti a determinari fenomeni sintropici,) lo spontaneo interesse per le matematiche di accende sempre più. Ad un tratto, sintropica-

[1]) Attraverso il materiale Montessori.

mente, il bambino si distacca dai sussidi materiali, vuole ragionare sull'astratto, come „obbedendo ad una spinta interiore diretta ha liberare l'anima da ogni legame". Verificatisi i fenomeni sintropici di maturazione interna, i fanciulli chiedono nuovo materiale, rispondente a nuove esigenze, materiale che poi, dopo nuova conquista sarà abbandonato. Nelle scuole elementari Montessoriane, attraverso il perfetto estrinsecarsi dei fenomeni sintropici di costruzione psichica, si conquistano concetti di uguaglianza, equivalenza, similitudine ecc., mentre nei ginnasi comuni gli allievi si dibattono più o meno faticosamente per impadronirsi di tali concetti. Per questi il fenomeno sintropico di autocostruzione è ostacolato, represso, da sistemi che si illudono di poterlo sostituire entropicamente.

Noto infine che il presente contributo può interessare l'inchiesta della *ciem* sulle funzioni delle matematiche e del matematico nella vita contemporanea, giacchè la teoria del Fantappiè, sorta dalla matematica, è tipico esempio dell'influsso di questa sulla pedagogia ed altre scienze (sociologia, diritto, biologia, ecc.)

ROMA, VIA PALESTRO 95.

QUELQUES REMARQUES RELATIVES À L'INFLUENCE DES TECHNIQUES SUR L'ÉVOLUTION DE LA GÉOMÉTRIE

RENÉ TATON

Le but de cet exposé est d'apporter quelques précisions relatives à une question encore très controversée: le problème des relations entre l'évolution des mathématiques et celle des autres branches des sciences pures ou appliquées, en particulier la physique et la technique.

Les tendances de plus en plus nettes des mathématiques actuelles vers l'abstraction et la formalisation font parfois oublier l'origine concrète de nombreuses notions essentielles, voire même de secteurs entiers des mathématiques. Sans nier la part prépondérante prise par la pensée logique dépouillée de tout souci d'ordre pratique dans l'édification du corps de doctrine de la mathématique actuelle, on ne doit pas pour autant négliger le rôle éminent joué en diverses phases essentielles de ce développement par des idées ou des problèmes d'origine concrète. Le choix des quelques exemples cités se limitera à la géométrie élémentaire et à l'analyse infinitésimale, mais une enquête analogue donne en d'autres secteurs des mathématiques des résultats similaires.

Il est indéniable que ce sont des préoccupations artistiques et des problèmes d'origine pratique qui ont présidé à la mise en lumière des premiers concepts d'ordre géométrique. Si en Grèce la géométrie prend un caractère de science théorique et déductive, cependant certains de ses progrès essentiels y tirent encore leur origine de préoccupations d'ordre concret. Nous citerons seulement les trois problèmes célèbres: duplication du cube, trisection de l'angle, quadrature du cercle, la part prise par la gnomonique dans l'éclosion de certains progrès de la théorie des coniques, l'intervention de divers résultats de géométrie pratique dans l'oeuvre d'Archimède et de Héron, et enfin le rôle éminent joué par la mécanique et l'astronomie dans la création de nouveaux secteurs des mathématiques: calcul infinitésimal, trigonométrie plane et sphérique, etc.

Mais l'influence de soucis d'ordre concret sur le développement de la géométrie est encore plus manifeste à partir de la Renaissance: création et développement de la perspective géométrique qui mène à la mise en lumière de la géométrie projective, influence parallèle de la gnomonique, de la théorie des ombres et de l'optique géométrique, etc, pour trouver son apogée dans l'oeuvre de Monge et de ses disciples qui mène au grand renouveau géométrique du XIXe siècle. A l'appui de cette thèse, nous rappelerons seulement l'origine concrété des divers problèmes relatifs aux différentes branches de la théorie des projections.

64 Rue Gay-Lussac Paris 5.

L'INSEGNAMENTO DELLA MATEMATICA IN ITALIA PER I GIOVANI DAI 16 AI 21 ANNI

Mario Villa

Viene fatta una relazione sull'insegnamento della matematica in Italia per i giovani dai 16 ai 21 anni.

Università,
Bologna.

THE ROLE OF MATHEMATICS AND MATHEMATICIAN AT PRESENT TIME

G. Kurepa

Halfhour lecture on invitation of the International Commission on Mathematical Instruction (ICMI)
See Volume III, Section VII

DOMAINE D'EXISTENCE DES INTÉGRALES D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES D'ORDRE SUPÉRIEUR AU PREMIER

NICOLAS SALTYKOW

Considérons l'équation aux dérivées du second ordre, aux désignations usuelles,

$$r + H(x, y, z, p, q, s, t) = 0.$$

L'intégrale générale du système de comparaison (Bull. de l'Acad. Serbe Sci. (N.S.) **5,** Cl. Sci. Math. Nat. Sci. Math. **1,** 149-156 (1952)) est définie par deux équations, d'après Lagrange, à une fonction arbitraire $f(C)$. On en tire, d'après les formules de Jacobi, pour avoir l'intégrale de Cauchy, deux équations

$$f(v) = \frac{1}{27}[3(y - b) - 4]\, e^{3/2v}; \; v = \frac{b\, e^{3/2v}}{b - y}.$$

La dernière relation démontre que v représente une fonction continue, monotone et croissante, sur l'intervalle $(0, b)$. Il en résulte que la fonction inverse y de v est continue et monotone dans le même domaine. Par conséquent, la première formule démontre que $f(v)$ est une fonction finie et continue de v, sous l'hypothèse $|y| < b$. On en conclut que la fonction de comparaison Z, ainsi que l'intégrale requise de Cauchy, est une fonction finie et continue dans tout le domaine d'holomorphie de l'équation étudiée.

Généralisant le théorème démontré sur l'équation aux dérivées partielles du second ordre

$$p_{11} + H(x_1, x_2, x_3, z, p_1, p_2, p_3, p_{12}, p_{13}, p_{23}, p_{22}, p_{33}) = 0,$$

on démontre l'existence de l'intégrale de Cauchy, généralisant la notion du domaine étoilé, dans le domaine étoilé d'holomorphie, dont les sommets se trouvent à l'intérieur du circle de rayon de l'holomorphie de l'équation étudiée.

IASCHE PRODANOVIĆ 38,
BELGRADE, JOUGOSLAVIE.

SHORT LECTURES

The Organizing Committee has requested the authors to indicate if and where further details about the subjects of their short lectures have been or will be published.

→ means: the author has stated that the paper either has been presented to or probably will be presented to the journal mentioned.

* means: still in preparation.

** means: no further publication is expected.

The title of a paper is only mentioned if it differs from the title of the short lecture concerned.

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HOLT, M.	II p. 353	*	
JONES, C. W.	II p. 355		Proc. Roy. Soc. Lond., A. 221, p. 257—267 (1954) and A. 228, p. 82—99 (1955).
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MCCREA, W. H.	II p. 361	*	
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NASH, W. A.	II p. 366	→	Proc. of the Amer. Soc. of Civil Engineers.
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KAMKE, E.	II p. 422		Jahresber. der deutschen Math. Ver., 57 p. 6—20 (1955).
MARKOVIĆ, Ž.	II p. 423	→	Rev. d'hist. des Sci. et de leurs appl. Paris.
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RÜPING, H.	II p. 425	*	
WOLFF, K. G.	II p. 428	*	

TABLE OF CONTENTS

ORGANIZATIONAL DATA

List of Committees 7
List of Delegates 13
List of Members 39
Scientific program 92
Secretary's Report 127
Address by the President of the Fields Medal Committee, Professor Hermann Weyl 161

ONE HOUR LECTURES

Alexandroff, P., Aus der mengentheoretische Topologie der letzten zwanzig Jahren 177
Borsuk, K., Sur l'élimination de phénomènes paradoxaux en topologie générale . 197
Brauer, R., On the structure of groups of finite order 209
Dantzig, D. van, Mathematical problems raised by the flood disaster 1953 218
Dieudonné, J., Le calcul différentiel dans les corps de caractéristique $p > 0$ . . . 240
Gelfand, I., Some aspects of functional analysis and algebra 253
Goldstein, S., On some methods of approximation in fluid mechanics 277
Harish-Chandra, Representations of semi-simple Lie groups 299
Jessen, B., Some aspects of the theory of almost periodic functions 305
Kolmogoroff, A. N., Obščaja teorija dinamičeskih sistem i klassičeskaja mehanika (Théorie générale des systèmes dynamiques et mécanique classique) 315
Lichnerowicz, A., Groupes d'holonomie 334
Neumann, J. von, Unsolved problems in mathematics 348
Neyman, J., Current problems of mathematical statistics 349
Nikolsky, S. M., Einige Fragen der Approximation von Funktionen durch Polynome 371
Segre, B., Geometry upon an algebraic variety 383
(See Volume III under symposium B, Algebraic Geometry, p. 497)
Stiefel, E., Recent developments in relaxation techniques 384
Tarski, A., Mathematics and metamathematics 392
Titchmarsh, E. C., Eigenfunction problems arising from differential equations . . 393
Weil, A., Abstract versus classical algebraic geometry 404
(See Volume III under symposium B, Algebraic Geometry, p. 550)
Yosida, K., Semi-group theory and the integration problem of diffusion equations 405

SHORT LECTURES

SECTION I: ALGEBRA AND THEORY OF NUMBERS

Aigner, A., Die kubische Fermat-Gleichung in quadratischen Körpern 425
Bakhshi, G. L., Theory of remainders in the algebra of Grassmann 425
Gloden, A., Questions d'analyse diophantienne multigrade 426
Grell, H., Erhaltungssätze der Idealtheorie und ihre Folgerungen 427
Halberstam, H., On the distribution of strongly additive number theoretic functions 428
Iyanaga, S., Sur la loi générale d'associativité 429
Jarník, V., Approximation diophantiennes linéaires et homogènes 430

Knödel, W. Carmichael-Zahlen und neuere holländische Arbeiten 431
Macbeath, A. M., The class-number of the symmetric semigroup 431
Mordell, L. J., Integer solutions of the equation $ax^3+ay^3+bz^3 = bc^3$ 432
Pick, G., Sur une nouvelle généralisation de la notion de nilpotence d'un groupe 433
Pisot, Ch., Sur un ensemble fermé d'entiers algébriques 434
Rohrbach, H. und Volkmann B., Verallgemeinerte asymptotische Dichten . . . 435
Sarantopoulos, B. S., Sur le premier cas du (dernier) théorème de Fermat . . . 436
Šchwarz, S., Characters of commutative semigroups 438
Tamari, D., A refined classification of semigroups leading to generalized polynomial rings with a generalized degree concept 439
Zacher, G., Considerazioni sugli elementi modulari in un p-gruppo 440

SECTION II: ANALYSIS

Alexits, G., Sur la caractérisation d'une classe de fonctions dérivables par leurs séries de Fourier 443
Alexits, G., La convergence presque partout des développements de fonctions continues en séries de polynomes orthogonaux 443
Bohnenblust, H. F., and Karlin, S., Geometrical notions in Banach algebras . . 445
Brelot, M., Sur l'allure des fonctions harmoniques à la frontière 445
Bremermann, H. J., Subharmonic functions in several complex variables 446
Cinquini, S., Sopra una forma più ampia del problema di Cauchy per i sistemi di equazioni a derivate parziali del primo ordine 447
Cinquini Cibrario, M., Una estensione nello studio dei sistemi di equazioni a derivate parziali 449
Conti, R. and Sansone, G., On an equation of T. Uno and R. Yokomi. 451
Emersleben, O., Über Funktionalgleichungen zwischen Epsteinschen Zetafunktionen mit verschiedenen Werten der „unteren" Parameter. 452
Fekete, M., Transfinite diameter and Fourier series 453
Ghizzetti, A., Une méthode générale pour obtenir des formules de quadrature . . 454
Hewitt, E., Fourier transforms of the class L_p 454
Miranda, C. Sull'integrazione delle forme differenziali 455
Nikolsky, S. M., Grenzeigenschaften von Funktionen mehrerer Veränderlichen und ihre Anwendung in Variationsaufgaben. 456
Pfluger, A., Ueber die Riemann'sche Periodenrelation für offene Flächen 458
Popoviciu, T., Les polynomes de S. N. Bernstein et le problème de l'interpolation 459
Prasad, B. N., On the theorems of consistency for Riesz summability 461
Reade, M. O., Some remarks on schlicht functions 462
Reiter, H., L^1-spaces on locally compact groups 463
Rey Pastor, J., Verallgemeinerung der partiellen Differentialgleichungen der Physik 463
Saltykow, N., Recherche des intégrales de S. Lie d'équations aux dérivées partielles du premier ordre 465
Sce, M., Sulla validità di formule integrali nelle algebre 465
Sebastião e Silva, J., Sur certains espaces vectoriels localement convexes 466
Tchakaloff, L., Ueber eine Art Faktorenfolgen in der Theorie der algebraischen Gleichungen 466
Wagner, K., Über ein abgeschwächtes Auswahlpostulat 468
Weyrich, R., Analytische Theorie der Differentialgleichungen auf der Grundlage von Entwicklungen nach besselschen Funktionen 469

SECTION III: GEOMETRY AND TOPOLOGY

Alexandroff, P., Über die Homöomorphie von Punktmengen 473
Berge, C., Théorie des jeux et structures topologiques. 475
Calapso, R., Le reti 0 di Guichard e il teorema di permutabilità di Bianchi 476
Dalla Volta, V., On Siegel-Hua spaces: plane elements of zero sectional curvature 476
Dedecker, P., La suite spectrale d'un système différentiel extérieur. 477
Ehresmann, Ch., Sur les pseudogroupes de transformations de Lie 478
Ehresmann, Ch., Sur les structures infinitésimales régulières 479
González, M. O., On the general theory of envelopes 480
Haken, W., Über Flächen in 3-dimensionalen Mannigfaltigkeiten; Lösung des Isotopieproblems für den Kreisknoten 481
Inzinger, R., Über die Bahnkurven der Drehungsgruppen vom Faltungstypus im Hilbertschen Raum. 482
Katetov, M., K nekotorym voprosam teorii razmernosti (Probleme der Dimensionstheorie) 483
Kuratowski, K., Sur les fonctions rationelles homotopes à des homéomorphies. . 484
Lemoine, S., Sur les variétés localement déformables d'un espace complet. . . . 484
Libermann, P., Sur les sous-pseudogroupes et prolongements d'un pseudogroupe de Lie 485
Lluis, E., Sur certaines projections des variétés algébriques 487
Maier, W., Simplex-Inhalte im elliptischen R_3 487
Marchionna, E., Un théorème d'unicité birationnelle pour les fonctions algébriques de plusieurs variables 489
Pinl, M., Geschlossene Minimalflächen II. 491
Rosina, B. A., Sur les quadriques généralisées 492
Stellmacher, K. L., Eine Klasse von Huygens-schen Differentialgleichungen . . . 493
Tits, J. L., Espaces homogènes et groupes de Lie exceptionnels 495
Villa, M., Problemi integrali sulle trasformazioni puntuali 496

SECTION IV: PROBABILITY AND STATISTICS

Azorin, F., On the sum of independent k-dimensional rectangular variates . . . 501
Burkhardt, F., Ueber besondere Gleichungssysteme der Wirtschaftsmathematik und der Statistik. 502
Chastenet de Géry, J. et Souriau, J. M., Étude de la corrélation de deux espaces vectoriels aléatoires 503
Court, L. M., Theorems on confidence regions 504
Rényi, A., On a new axiomatic foundation of the theory of probability 506
Rényi, A., On the theory of order statistics 508

SECTION V: MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

Bellman, R. and Danskin, J. M., On differential difference equations and control problems 513
Berker, R., Sur les équations du mouvement d'un gaz 513
Destouches, J. L., Sur un problème mathématique posé par la physique théorique 514
Dettmar, H. K., Bemerkungen über eine Klasse symmetrisierbarer Eigenwertaufgaben 514
Dressler, R. F., Entropy changes in the equations for rarefaction waves 516
Finzi, B., Deduzione variazionale delle equazioni del campo relativistico unitario 517
Gallisot, F., Les formes extérieures en mécanique. 518
Greenberg, H. J., Variational theorems and the eigenvalue problem for plane strain in perfect plasticity 520

Isaacson, E., On the numerical solution of mixed initial boundary value problems for hyperbolic equations 521
Knichal, Vl., Cartesische Darstellungen des Minkowskischen Raumes 522
Lampariello, G., Allgemeine Betrachtungen über die Fortpflanzung der elektromagnetischen Wellen in den bewegten Körpern 523
Losada y Puga, C. de, Sur l'accélération séculaire de la lune 525
Maruhn, K., Eine hydrodynamische Existenzbetrachtung 526
Miles Jr, E. P. and Williams, E., A basic set of homogeneous harmonic polynomials in 3 variables 526
Pastori, M., Sul tensore elettromagnetico della teoria unitaria di Einstein 527
Possel, R. de and Valensi, J., Sur le sillage d'un obstacle perméable 529
Reiner, M., Second order effects in infinitesimal elasticity 530
Salzer, H. E., The use of Poissons' formula in pattern synthesis 531
Storchi, E., Sul principio dell'azione potenziale 531
Udeschini, P., Successiva linearizzazione delle equazioni del campo unitario Einsteiniano 533

SECTION VI: LOGIC AND FOUNDATIONS

Barallat, B., Esquemas representativos de sistemas regidos por una lógica polivalente 537
Fraïssé, R. J., Obtention en théorie des relations de certaines classes d'origine logique 537
Kuroda, S., On the intuitionistic and formalistic theory of real numbers 538

SECTION VII: PHILOSOPHY, HISTORY AND EDUCATION

Amato, V., Sull'insegnamento matematico nelle scuole secondarie e sui testi scolastici 543
Ascoli, G., Le rôle de la mathématique et du mathématicien dans la vie contemporaine 543
Bundgaard, S., Report on mathematics instruction in Denmark 544
Bunt, L. N. H., Report of the Dutch sub-commission on the teaching of mathematics in the Netherlands to students of 16—21 years of age 545
Cartwright, M. L., Report of the British sub-commission on mathematical instruction for students between 16 and 21 years of age. 546
Châtelet, A., Education in France 547
Crespo Pereira, R., The teaching of mathematics 547
Dantzig, D. van, The function of mathematics in modern society and its consequences for the teaching of mathematics. 548
Darmois, G. Le rôle du mathématicien dans la vie contemporaine 548
Dolinsky, R., Determinanten im Unterricht allgemeinbildender Schulen 549
Fehr, H. F., The administration of mathematics education in the United States of America. 550
Frostman, O., Summary of a report on the mathematical instruction in Sweden for students between 16 and 21 years of age 552
Hohenberg, F., Der mathematische Unterricht in Osterreich 553
MacLane, S., Intermediate mathematical instruction in the United States . . . 554
Palazzo, E., Pedagogia scientifica inquadrata in una theoria matematica 555
Taton, R., Quelques remarques relatives à l'influence des techniques sur l'évolution de la géométrie. 556
Villa, M., L'insegnamento della matematica in Italia per i giovani dai 16 ai 21 anni 557
Kurepa, G. The rôle of mathematics and mathematician at present time 557
(See Volume III, p. 305)
Saltykow, N., Domaine d'existence des intégrales d'équations aux dérivées partielles d'ordre supérieur au premier 558
List of references concerning short lectures in Volumes I and II 559