Amongst Mathematicians and conversations on the teaching and learning of mathematics at university level: the case of visualisation

Elena Nardi, University of East Anglia, Norwich, UK, e.nardi@uea.ac.uk

How do students in the beginning of their undergraduate studies in mathematics cope with the requirement for rigour? Why do they so often resort to the familiarity of number? Why do they have problems with constructing examples and with identifying and accepting counterexamples? How do they manage to express in symbols their thoughts about the convergence of a sequence? Why is reference to the domain of a function so conspicuously absent in their writing? How do their teachers at university help them acquire the 'genre speech' of university mathematics and the mathematician's 'toolbox' of useful images, theorems and techniques? Do these teachers pursue the help of mathematics education researchers in these complex tasks? If at all, how? If not, why not?…

The above questions provide a flavour of the issues that the research I am reporting in this paper aimed to explore. To this purpose I am drawing on the data and analyses presented in Amongst Mathematicians: Teaching and Learning Mathematics at University Level, a 2008 Springer monograph (Nardi, 2008) that was based on this research. The study offers a perspective on how mathematicians: perceive student learning; describe and reflect on their own teaching practices; and, perceive their relationship with mathematics educators. Its evidence base is a series of focused group interviews with mathematicians from across the UK. Its analyses were presented in the format of a dialogue between two fictional, yet entirely data-grounded characters, M and RME, mathematician and researcher in mathematics education. (See the Appendix for a typical page from *Amongst Mathematicians*: each piece of dialogue between M and RME sets out from a discussion of a sample of student work, typically a piece of writing. The samples of student work exemplify topical learning and teaching issues – as highlighted in the literature and in previous research conducted by myself and colleagues at the universities of East Anglia and Oxford. Examples of relevant bibliography are cited in the footnotes accompanying the dialogue between M and RME.)

In what follows I first outline the study's background, aims and methods. I then discuss three samples of findings focusing, respectively, on: an issue of student learning (*Sample I*, the role of visualisation in mathematical reasoning and argumentation, highlighted in the literature as key to the students' early experiences of university mathematics); related pedagogical issues (*Sample II*); and, in closing, issues regarding the relationship between the respective communities of M and RME (*Sample III*).

TALUM, A NEW AND RAPIDLY DEVELOPING FIELD OF RESEARCH

TALUM, the *Teaching and Learning of Undergraduate Mathematics*, is a relatively new and rapidly developing field of mathematics education research Holton, 2001). As, particularly in the 1990s, mathematics departments started to respond to the decline in the number of students who opt for mathematical studies at university level (Hillel, 2001), the realisation that, beyond syllabus change, there is also the need to reflect upon tertiary pedagogical practice began to grow (McCallum, 2003). The research programme I am reporting here was conceived and carried out with the aim to address this need in a systematic and original way.

The study I am focusing on in this paper is underlain by a rationale for a certain type of TALUM research. The study draws on several traditions of educational research reflected in the five, essential characteristics listed below: it is collaborative, context-specific and data-grounded and, through being non-prescriptive and non-deficit, it aims to address the often difficult relationship between the communities of mathematics and mathematics education. A fundamental underlying belief of this work is that development in the practice of university-level mathematics teaching is manageable, and sustainable, if driven and owned by the mathematicians who are expected to implement it.

This rationale for collaborative, practitioner-engaging and context-specific research draws heavily on Barbara Jaworski's *Co-Learning Partnerships* (2003) and John Mason's *Inner Research* (1998). In these types of research practitioners of mathematics teaching engage with research and they, along with the researchers, become co-producers of knowledge about learning and teaching; they become educational co-researchers (Wagner, 1997). In this sense the study is a first step towards engaging with Developmental Research (van den Akker, 1999), a much needed type of research in undergraduate mathematics education. Furthermore the study has aimed to steer clear of a tendency (that sometimes studies of teaching suffer from) towards a 'deficit' and 'prescriptive' discourse on pedagogy, where the emphasis is on the identification of what it is thought teachers ought to be doing and are not doing, and on appropriate remedial action (Dawson, 1999). The work I am reporting here is located explicitly within a non-deficit and non-prescriptive discourse.

The study (Nardi, 2008) is the latest in a series of studies aiming to:

- explore students' learning in the first, and sometimes, second year of their undergraduate studies – mostly in Analysis, Linear Algebra and Group Theory and mostly through observing them in tutorials (Nardi, 1996; Nardi, 2000) and analysing their written work (e.g.: Nardi & Iannone 2001); and,
- engage their lecturers in reflection upon learning issues and pedagogical practice – mostly in individual (Nardi, Jaworski and Hegedus, 2005) and group interviews (Iannone and Nardi, 2005).

The studies were conducted at the Universities of Oxford and East Anglia in the UK between 1992 and 2004. Further studies, that aim to refine the themes emerging from the earlier studies, as well as take steps towards collaborative implementation of innovative pedagogical practice, are currently in progress.

STUDENT DATA, THE DATA THAT 'BECAME' M AND THE *RE-STORYING* APPROACH

The dialogues between M and RME that I exemplify in the following pages originate in eleven lengthy (approximately four-hour / half-day) focused group interviews with 20 mathematicians of varying experience and backgrounds from across the UK. In the interviews discussion was triggered by Datasets consisting of students' written work, interview transcripts and observation protocols collected during (overall typical in the UK) Year 1 introductory courses in Analysis / Calculus, Linear Algebra and Group Theory – see background studies listed in the previous page. Datasets had been distributed to the interviewees at least a week prior to the interview and were about a dozen pages long, split in four to six sections. A typical section of the Dataset typically consisted of:

- a mathematical problem (including its formulation as well as the suggested solution distributed to the students once they had submitted their written responses to their tutor)
- two typical student responses, often reflecting learning issues highlighted in relevant mathematics education literature

The interviews were conducted according to the principles of Focused Group Interviews (Madriz, 2001). Below I explain the narrative approach of *re-storying*

(Clandinin and Connelly, 2000) adopted in this work and the composition process through which the dialogues between M and RME came to be. In short the process of re-storying involves reading the raw transcripts, identifying and highlighting experiences to be told across this raw material and then constructing a new story that reflects these experiences. In this sense, while fictional, the new story is entirely data-grounded. In addition to the work of narrative researchers such as Clandinin and Connelly cited above a particularly helpful way of seeing the brand of re-storying I have used is Jerome Bruner's account of how the mind constructs a sense of reality through 'cultural products, like language and other symbolic systems' (1991, p3). The dialogues between M and RME in (Nardi, 2008) were constructed entirely out of the raw transcripts of the interviews with the mathematicians and then thematically arranged in *Episodes*. (For an example of the construction process see p27-28 in (Nardi, 2008)).

(Subsequently in (Nardi, 2008) chapters were constructed as series of *Episodes*, sometimes also broken in *Scenes*. Each *Episode* starts with a mathematical problem and usually two student responses. A dialogue between M and RME on issues exemplified by the student responses follows. Other examples of relevant student work are interspersed in the dialogue and links with relevant mathematics education research literature are made in the footnotes. *Special Episodes* are episodes that supplement the discussion in the main *Episodes* and *Out-Takes* are slightly peculiar or too specific incidents that stand alone and outside the more 'paradigmatic' material of the main *Episodes* but somehow address the wider theme of a chapter.)

Below I outline briefly a rationale for the dialogic format employed in the study that goes a little beyond a conventional methodological account. It may look like a digression but the brief text that follows is deeply ingrained into the study's, and the book's, *raison-d'-être*.

A BRIEF DIGRESSION REGARDING THE DIALOGIC FORMAT

> '…all you can do, if you really want to be truthful, is to tell a story'
>
> Paul Feyerabend (1991), quoted in Mason (1998, p367)

The idea for the character of M of course is not new – neither is the idea of a conversation between a researcher in mathematics education and a mathematician (Sfard, 1998a). Sfard's Typical Mathematician (1998b, p495) and Davis &

Hersh's Ideal Mathematician (1981) pre-date this study's M. Dialogue as a form for communicating and debating ideas is a format most quintessentially used by philosophers such as Plato, Galileo, Berkeley, Feyerabend and, crucially for mathematics educators, Lakatos in Proof and Refutations (1978). In theatre as well, authors such as Tom Stoppard (Arcadia) and Michael Frayn (Copenhagen, Democracy) have deployed the dialogic format in admirable attempts to help the subtle meet the artful effectively. In this sense the ultimate aim for using the dialogic format as a way of representing processed data is to employ storytelling as a different kind of science:

> Vanbrugh: [...] The plot already exists... in real life. The play and all its scenes.
> Cibber: A drama documenting facts? [...] Will you allow yourself the same liberties as Shakespeare? Taking liberties with facts converts facts into plays.
>
> Vanbrugh: No liberties... just facts in this play.
>
> *Calculus*, Scene I (Djerassi & Pinner, 2003)

SIX THEMES ON THE TEACHING AND LEARNING OF UNIVERSITY MATHEMATICS

As mentioned earlier, the dialogues between M and RME were thematically arranged in *Episodes*. Then clusters of Episodes around each one of the following six themes constituted the six chapters of data analyses presented in (Nardi, 2008):

- students' mathematical reasoning; in particular their conceptualisation of the necessity for proof and their enactment of various proving techniques;
- students' mathematical expression and their attempts to mediate mathematical meaning through words, symbols and diagrams;
- students' encounter with fundamental concepts of advanced mathematics –
- Functions (across the domains of Analysis, Linear Algebra and Group Theory) and Limits;
- pedagogical practices at university level; and,
- the often fragile relationship between M and RME as well as the necessary and sufficient conditions for their collaboration.

In the rest of this paper I collate samples of data and findings from across the above themes. Sample I reports manifestations of student perceptions of the role of visualisation as evident in their mathematical writing. Sample II reports their lecturers' reactions, mathematical and pedagogical, to these manifestations and outlines a pedagogical role for the mathematician in fostering a fluent interplay between rigour and visual insight. Finally, Sample III collates elements of the discussion between M and RME which focuses on the benefits for pedagogical practice ensuing from engagement with educational research.

Notes

All quotations that follow, except otherwise noted, are utterances of the character M – page numbers indicate pages in (Nardi, 2008). Also: the data and analyses reported in these samples have appeared partly also in (Nardi , 2009a and b).

Sample I: Students' perceptions on the role of visualisation

Students often have a turbulent relationship with visual means of mathematical expression. When they find difficulty in connecting different representations (for instance: formal definitions and visual representations), they often abandon visual representations - which tend to be personal and idiosyncratic - for ones they perceive as mathematically acceptable (Presmeg, 2006). Here we take a look at M's perspective on students' attitudes towards visualisation and on the ways in which these attitudes – and ensuing behaviour – can be influenced by teaching. The discussion eventually becomes about the importance of building bridges between the formal and the informal in constant negotiation with the students.

First and foremost M describes pictures as efficient carriers of meaning – in the case of $| \; |$ as distance, for example:

> 'What the students really need to be thinking about is what $| \; |$ means on the number line and as a distance. But they so often get stuck to the algorithmic habit of solving this without knowing what it means. And that stubbornness can be a nightmare.
>
> What I mean by *what it means* is, for example, seeing, what an equality or inequality involving $|x\text{-}1|$ means pictorially on the real line. Once you have seen it on the line, the answer to your question is obvious. That is why I am

> a huge fan of them using all sorts of visual representation: because the ones who do, almost invariably are the ones who end up writing down proper proofs.', *p238*

Instead students often feel ambivalence towards 'picture', even wondering 'are pictures mathematics?!'

> 'Students often mistrust pictures as *not mathematics* – they see mathematics as being about writing down long sequences of symbols, not drawing pictures – and they also seem to have developed limited geometric intuition perhaps since their school years. I assume that, because intuition is very difficult to examine in a written paper, in a way it is written out of the teaching experience, sadly. And, by implication, out of the students' experience. It is stupefying sometimes to see their numb response to requests such as imagining facts about lines in space or what certain equations in Complex Analysis mean as loci on the plane.', *p139*

This ambivalence can lead to a narrow, inflexible, even mutually exclusive adherence to informal or formal modes of thinking:

> '… students somehow end up believing that they need to belong exclusively to one of the two camps, the informal or the formal, and they do not understand that they need to learn how to move comfortably between them', *p140*

Now let's delve into the above general statements about student tendencies in the context of a specific mathematical problem and see how they pan out.

The premise for the discussion is the following mathematical problem (typically given to Year 1 mathematics undergraduates in a Semester 1 Calculus / Analysis course):

Write down a careful proof of the following useful lemma sketched in the lectures. If $\{b_n\}$ is a positive sequence (for each n, $b_n > 0$) that converges to a number $s > 0$, then the sequence is bounded away from 0: there exists a number $r > 0$ such that $b_n > r$ for all n. (Hint on how to start: Since $s > 0$, you might take $\frac{1}{2}s = \epsilon > 0$ in the definition of convergence.)

One acceptable approach to this is described in the notes below (written by the lecturer of the course the problem originally comes from):

Let $\epsilon = \frac{s}{2}$ in the definition of convergence. Then there is an N such that $n > N \Rightarrow |b_n - s| < \frac{s}{2} \Rightarrow b_n \geq \frac{s}{2}$. Then, for any n, $b_n \geq r = \min\{b_1, \ldots, b_N, \frac{s}{2}\}$ which is the minimum of finitely many positive quantities, hence is positive.

Questions / issues touched in the discussion included: what responses would you expect from the students to this problem; what difficulties may they face; if you were to discuss this problem with a student how would you do so?
One of the issues that emerged in the course of discussing this problem concerns the fact that, in the second line of the lecturer's notes, it is a perfectly acceptable part of the argument to 'leave out' of the inequality the terms b1, b2, …bN. Why this is helpful can also be visible in a simple picture that portrays the 'boundedness away from zero' of the significant majority of the sequence's terms. Students treated the contingency of such a picture variably. See Table 1 which shows three typical Year 1 student responses and the comments made on them – with regard to the presence, absence and quality of such a picture in the students' scripts – by M.

Overall M's insights into students', and M's own, perceptions regarding the role of visualisation revolved around the following four axes:

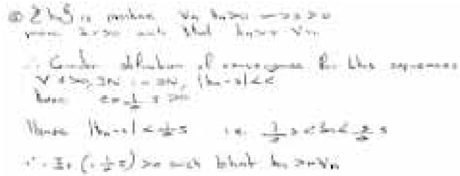
- Usefulness of visual representations: firm and unequivocal ('Graphs are good ways to communicate mathematical thought', p. 143);
- Usefulness of educational technology, e.g. graphic calculators: caution and concern ('Calculators are nothing more than a useful source of quick illustrations', p. 143);
- Students' varying degrees of reliance on graphs (both in terms of frequency and quality); and,
- The potentially creative fuzziness of the 'didactical contract' at university level with regard to the role of visualization.

We will now focus on the last two. In a nutshell, M's views are largely put forward in the light of how mathematicians employ visualisation in their own mathematical practice. The emergent perspective is of the need for a clarified didactical contract (Brousseau, 1997), in which students are encouraged to emulate the flexible ways in which mathematicians to-and-fro between analytical rigour and often visually-based intuition.

Table 1. Three ways of relating to 'pictures'.

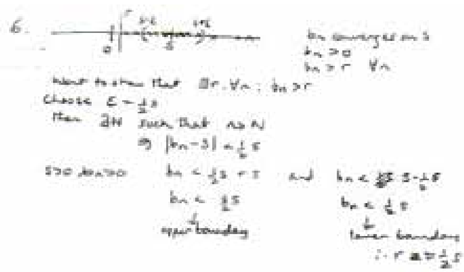
Student N, no picture

Student N has not left out of his argument a small but significant number of terms in the sequence he is working on. 'Had the student drawn a picture, he would have seen he had left them out'.



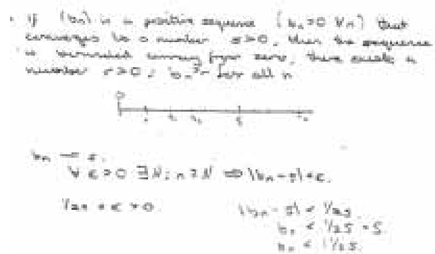
Student H, unhelpful picture

Student H, emulates 'the type of picture drawing seen in lectures'. She however 'needed a more helpful picture'. It is encouraging though that both Students N and H pinned down an understanding of | | as a 'distance between things'.



Student E, not benefiting from picture

Student E has not 'used this diagram as a source of inspiration for answering the question'. Instead 'she drew this, on cue from recommendations that are probably on frequent offer during the lectures, and then returned to the symbol mode unaffected'. So 'there is no real connection between the picture and the writing'.



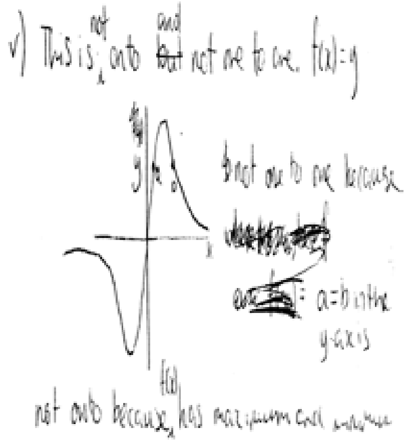
Examples and M utterances from p. 140 and pp. 195–199 in (Nardi, 2008)

The premise of the discussion is a question in which students were invited to explore whether certain functions from R to R were one-to-one and onto. In the two examples of student responses below M identifies two distinct ways in which students typically appear to rely on graphical evidence – see Table 2.

Table 2. Two ways of relying on graphs.

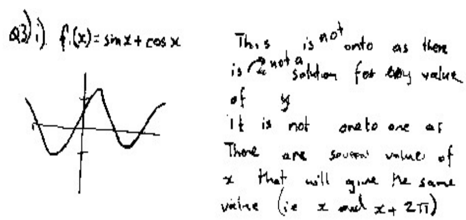
Student WD, absence of transition from picture to wording

'I am concerned about the answer being provided before the graph is produced but I also observe that the answer has been modified on the way – which may mean the graph did play some part after all in the student's decision making. If the student had drawn a line through points a and b, I would be a bit more convinced that the student is actually building the argument from what they see in the graph. I am also disappointed by the absence of a transition from the picture to some appropriate words and with the use of a=b to denote that points a and b on the curve have same y. What a use of the equals sign! In this sense…'



Student LW, no construction evidence

'… I am more sympathetic to Student LW …who may need the Intermediate Value Theorem to complete the argument in part (i) – the IVT is true after all –, the picture is almost perfect, all the shifting etc. is there, but this is still an incomplete answer. Still there is no construction evidence.'



Examples and M utterances from p. 144 in (Nardi, 2008)

M is particularly keen to stress what he calls the 'irony in using the graph to produce evidence that a function is one to one or onto' (p144) as the ability to construct this graph would in itself require this knowledge:

> '…I find this evidence compelling but still this is not a complete answer. This picture is potent and I see a certain danger in its sophistication: the fact, for example, that, if a function has a maximum, it cannot be onto is immediately graspable from the graph. However some unpacking is still necessary in order to provide a full justification of the claim.
>
> I am a proponent of starting with a diagram but I do not wish to see this placing value on starting with a diagram giving the students a false sense of obligation to do so, another hurdle to get over. I want them to think of doing so as a totally natural procedure to follow but also do it correctly.', p144

From this quotation and the one below begin to emerge some of the terms of the renewed, clarified didactical contract mentioned earlier:

> 'I would be far less frustrated if I could find evidence in the students' writing that the diagram is used almost as a third type of language, where the other two are words and symbols, as an extension of their power to understand: just drawing a diagram bigger, or, for example in the first picture, putting in a horizontal line that goes through the points *a* and *b*. I am afraid students do not use pictures to their full potential. Of course I see that relying on their power therein lies a danger but I would like to see students make a sophisticated use of this power and be alert to their potential to be misleading too.', p145

First of all students need to be alerted to what I term here 'the creative fuzziness of the 'didactical contract'. 'Fuzziness' is used here to denote the necessary acknowledgement that a clear-cut distinction between their obligation to engage with mathematics formally *or* informally, in a mutually exclusive way, is too simplistic. It is also ineffective; in fact, hence the use of 'creative', it is exactly this to-ing and fro-ing between the formal and the informal modes of engaging with mathematics that will ultimately turn out to be the most effective. M outlines two significant phases in imparting this new type of didactical contract:

Allow the use of visual insight, acknowledging that the students, by the nature of introductory university courses, are already using unproven facts:

> 'Students should be allowed at this stage to use the graphs for something more than simply identifying the answer because after all they allowed to use all sorts of other facts – the uniqueness of cubic roots is one of those facts – that have not been formally established yet.
>
> So if the Intermediate Value Theorem is implicit in their finding the answer by looking at the graph, then let that be! Of course one needs to check: an actual value of a and b there would be very reassuring. At this stage I feel sympathy for them and want to let them say this function is onto because of the uniqueness of the cubic root. Because at this stage, well, I don't want to tell you what the cube root of two is … I want to tell you the cube root of eight is. I am not sure I even know how to exhibit the cube root of two without resorting to some quite sophisticated ideas.', p145

Eventually prove, conveying that ultimately mathematics is mainly about establishing facts via proof:

> 'I am happy with using the ingredients for proving a claim and then, at some later stage, spending some time on establishing those ingredients formally. So prove that e^x is injective via the IVT and then later on prove the IVT. This to me is fine as long as I know that all along I have been leaving some business-to-be-finished on the side. That kind of rigour is fine with me.', p146

Below M concludes with two pertinent observations on this matter.

SAMPLE II: PEDAGOGICAL PRACTICE WITH REGARD TO VISUALISATION

M describes three key elements to a teacher's response to the student perceptions outlined in Sample I: acknowledgement of the innately human need for visual insight, raising students' awareness and celebration of this typically very personal need, assist them in pursuing the construction of such insights:

> '…they need to learn how to move comfortably between [the formal and the informal]. Because in fact this is how mathematicians work! I still remember acutely my own teachers' explanations of some Group Theory concepts via

> their very own, very personal pictures. I am a total believer in the Aristotelian *no soul thinks without mental images.* In our teaching we ought to communicate this aspect of our thinking and inculcate it in the students. Bring these pictures, these informal toolboxes to the overt conscious, make students aware of them and help them build their own.
>
> And I cannot stress the last point strongly enough: we need to maintain that these pictures are of a strictly personal nature and that students should develop their own. All I can do is describe vividly and precisely my own pictures and, in turn, you pick and mix and accommodate them according to your own needs.', p237

At the heart of this three-step plan of support is the frank acknowledgement that this approach to visualisation reflects the ways mathematics is understood and created by mathematicians themselves. Further elaborating the 'this is how mathematicians work' statement above M adds:

> 'Lest we forget some very clever people regarded [IVT] not needing a proof either! People like Newton. […] there is an irony in the fact that validating the truth of the statement in IVT means that all the pictures that students have been drawing are retrospectively true – like drawing the solutions of an equation. This irony in fact is nothing other than another piece of evidence of a constant tension within pure mathematics: that you want to use these methods and occasionally you need a theory to come along and make them valid. And you need these means, diagrams etc., so badly. Yes, they are not proofs but they do help students acquire first impressions, start inventing some suitable notation.', p238

M proceeds with the presentation of examples from mathematics where the above is the case. I omit these due to limitations of space but they are available in: p238 (geometric problems in the complex plane), p240 (exponentials) and p 241 (powers) in (Nardi, 2008); and (Nardi, 2009b).

In the course of the interviews M stressed repeatedly how much of the pedagogical awareness and the potentially effective pedagogical practices evidenced above became available through participation in this study.

SAMPLE III: BENEFITS FROM ENGAGEMENT WITH EDUCATIONAL RESEARCH

M often juxtaposed the accusation for 'indecipherability', futility and irrelevance of mathematics education research often mounted by the mathematics community (Ralston, 2004) to the potent experience of participating in these interviews. Often M cited improved access to understanding students as a primary benefit of this participation:

> '… it is in these discussions exactly that these sessions have proved enormously valuable already. There are things I will teach differently. There are things that I feel like I understand better of mathematics students than I did before. And I appreciate the questioning aspects of the discussion and I realise how one should be liaising with the other lecturers simultaneously lecturing the students and discussing what things we are doing that confuse them.', p260

A substantial part of this understanding consisted of realising the extent of student difficulty:

> '…these discussions are already beginning to influence the way I think about my teaching. I think discussing the examples is a very good starting point, and a well-structured one. By seeing these often terrifying pieces of writing I am faced with the harsh reality of the extent of the students' difficulties. Too often I see colleagues who are in denial and opportunities like this are poignant reality checks! […] I am therefore grateful for this opportunity to face the music, so to speak.', p261

A significant outcome of this understanding is fostering an appetite, and capacity, for change, pedagogical innovation, even reform, away from conventional views of mathematics and how it is learnt and taught:

> 'There is substance in this; it is important.
>
> Suppose you have a schoolteacher. So, here is someone who has to run classes and, for some reason or another, their view of mathematics is no other than

> an instrumental one: you apply this rule, you put this in and you get this out. Suppose that such a person one day meets *Concept Image* and all that. All of a sudden he learns that these things are all out there and that changes that person's professional view entirely. It can change the whole classroom, it can change the whole mathematical process. That is precisely what we want.
>
> A lot of the problems you have to deal with when you meet our students is that they have a very singular view of mathematics, a rather poor view of mathematics. So, I mean, that sort of debate that is happening here is on some of the building blocks around which, it seems to me, if made available at the school level for practitioners, would be hugely interesting. To get away from this sort of mathematics which is quite poor in a way.', p262

M often concluded the discussion emphasising the gaining of awareness, and a renewed appreciation of openness regarding questions of pedagogy:

> 'I think now I don't have any more answers than when I started but certainly I don't take things for granted anymore, from colleagues or from students.
>
> I think I am much more open-minded on what might be going on inside other people brains. The material that you have got here has given the evidence that sure, it is fascinating glancing in other people's heads.
>
> And I have become much more conscious about the spoken word. What I say can have an impact, saying the right thing at the right time when you get one opportunity to introduce the students for the first time to how mathematics works and not fluff the line. That I think has made a big influence on the way I lecture.', p263

Often the discussion between M and RME signalled direct parallels with the educational literature Sample 3: M and RME – benefits, change EXAMPLE of parallel with literature – M's comment below on the importance of substantial feedback to students' written work echoes Mason's recommended tactics on this matter (*Focusing on what is mathematical; Developing a language; Finding something positive to say; Selecting what to mark; Summarising your observations; and, Providing a list of common errors or a 'corrected' sample of student argument*, Mason, 2002, Chapter 5):

> […] examining these pieces of data was something of a reminder, if not a revelation, of the devastating importance of detailed responses to written work. In some sense every not totally perfect piece of written work has an interesting important story to tell that needs to be engaged with and responded to.', p263

> Therefore it will be far from a surprise to say that the entire study incarnates rather aptly the much needed synergy between mathematicians and mathematics educations often discussed by Michèle Artigue:

> '...we, mathematicians as well as didacticians […] have to act energetically in order to create the positive synergy between our respective competences which is necessary for a real improvement of mathematical education, both at secondary and at tertiary levels. Obviously such a positive synergy is not easy to create and is strongly dependent on the quality of the relationship between mathematicians and didacticians'. (Artigue, 1998, p482/3)

CONCLUDING REMARKS: WHAT HAS NOT BEEN AND …A FUTURE FOR M/RME?

While the study reported in this paper focused on matters of learning and teaching that could be broadly described as 'cognitive only' (the discussion rarely turned to topical socio-affective matters such as gender, affect, equity etc.) its aspirations to meet at least two objectives were nonetheless rather wide: obviously, to listen to what M, experienced learner, doer and teacher of mathematics, has to say about learning and teaching; and, less obviously, to allow a certain image of M to emerge (characterised by pedagogical awareness, perceptiveness and sensitivity) which would be in contrast to widespread pedagogical stereotypes of university mathematicians. And, to do so through its distinctive characteristics (context-specific, example-centred, mathematically-focused samples of data, discussed in a relaxed yet focused, unthreatening and mutually respectful research ambience).

In resonance with its non-prescriptive character the study refrained from direct recommendations for practice. However, soon after its completion, a brief guide with a focus on the teaching of proof was published following a request by the UK's Higher Education Academy (Nardi and Iannone, 2006). Alongside several studies that aim to refine some of this study's findings (e.g. Ioannou and Nardi, 2009), in the (hopefully near!) future we aim to continue with more directly developmental work, namely: the construction, implementation and evaluation of innovative practice. We are currently in the process of designing a series of such interventions in collaboration with colleagues from mathematics departments in the UK.

ACKNOWLEDGEMENTS

I would like to thank the British Academy, the London Mathematical Society and the University of East Anglia for the financial support that allowed me to accept the invitation to this lecture at ICME11 in Mexico. I would also like to extend my warmest thanks to Paola Iannone, invaluable Research Associate to the UEA-based studies on which *Amongst Mathematicians* is based.

REFERENCES

Artigue, M. (1998). Research in mathematics education through the eyes of mathematicians. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics Education as a research domain: A search for identity* (pp. 477 - 490). Dordrecht / Boston / London: Kluwer Academic Publishers.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht/Boston/London: Kluwer Academic Publishers.

Bruner, J. (1991). The Narrative construction of reality. *Critical Inquiry, 18*(1), 1-21.

Clandinin, D.J., & Connelly, F.M. (2000). *Narrative inquiry: experience in story in qualitative research*. San Francisco, CA: Jossey-Bass.

Davis, P., & Hersh, R. (1981). *The mathematical experience* Harmondsworth, UK: Penguin Books.

Dawson, S. (1999). The enactive perspective on teacher development: "A path laid while walking" In B. Jaworski, T. Wood & S. Dawson (Eds.), *Mathematics teacher education: International critical perspectives* (pp. 148-162). London: Falmer.

Djerassi, C., & Pinner, D. (2003). *Newton's darkness: Two dramatic views*. London: Imperial College Press.

Feyerabend, P. (1991). *Three dialogues on knowledge*. Oxford, UK: Blackwell.

Hillel, J. (2001). Trends in curriculum: A working group report. In D. A. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 59 – 70). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Holton, D. A. (Ed.). (2001). *The teaching and learning of mathematics at university level: An ICMI study*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Iannone, P., & Nardi, E. (2005). On the pedagogical insight of mathematicians: 'interaction' and 'transition from the concrete to the abstract'. *Journal of Mathematical Behavior, 24*(2), 191-215.

Ioannou, M. & Nardi, E. (2009). Engagement, abstraction and visualisation: Cognitive and emotional aspects of Year 2 mathematics undergraduates' learning experience in Abstract Algebra. *Proceedings of the Conference of the British Society for Research into the Learning of Mathematics, 29*(2), 35-40.

Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: Towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics, 54*(2/3), 249-282.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, UK: Cambridge University Press.

MacCallum, W. G. (2003). Promoting work on education in mathematics departments. *Notices of the American Mathematical Society, 50*(9), 1093–1098.

Madriz, E. (2001). Focus groups in feminist research In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2 ed.). Thousand Oaks, CA: Sage Publications.

Mason, J. (1998). Researching from the inside in mathematics education. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics Education as a research domain: A search for identity* (pp. 357 - 377). Dordrecht / Boston / London: Kluwer Academic Publishers.

Mason, J. (2002). *Mathematics teaching practice. A guide for university and college lecturers*. Chichester, UK: Horwood Publishing.

Nardi, E. (1996). *The novice mathematician's encounter with mathematical abstraction: Tensions in concept-image construction and formalisation* Unpublished doctoral thesis, University of Oxford, UK (available at http://www.uea.ac.uk/~m011).

Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in Group Theory. *Educational Studies in Mathematics,* 43(2), 169-189.

Nardi, E. (2008). *Amongst Mathematicians: Teaching and learning mathematics at university level*. USA: Springer.

Nardi, E. (2009a) 'Because this is how mathematicians work!': Pictures' and the creative fuzziness of the didactical contract at university level. In F. Lin, F. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proceedings of ICMI Study 19: Proof and Proving in Mathematics Education* (Vol 1, pp. 112-117). Taipei, Taiwan.

Nardi, E. (2009b) Gaining insight into teaching and learning mathematics at university level through Mason's inner research. In S. Lerman & B. Davis (eds.), *Mathematical action and structures of noticing* (pp. 111-120). NL: Sense Publishers.

Nardi, E., & Iannone, P. (2001). On convergence of a series: The unbearable inconclusiveness of the Limit Comparison Test. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 399-406). Utrecht, The Netherlands.

Nardi, E., & Iannone, P. (2006). *How to prove it: A brief guide for teaching proof to mathematics undergraduates*. Norwich, UK: Commissioned by the Higher Education Academy (Mathematics, Statistics and Operational Research branch). ISBN 978-0-9539983-8-8. Available at http://mathstore.ac.uk/publications/Proof%20Guide.pdf.

Nardi, E., Jaworski, B., & Hegedus, S. (2005). A spectrum of pedagogical awareness for undergraduate mathematics: From 'tricks' to 'techniques'. *Journal for Research in Mathematics Education,* 36(4), 284 - 316.

Presmeg, N.C. (2006). Research on visualization in learning and teaching mathematics: emergence from psychology. In A.Gutierrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205–235). Dordrecht, NL: Sense Publishers.

Ralston, A. (2004). Research mathematicians and mathematics education: A critique. *Notices of the American Mathematical Society,* 51, 403 - 411.

Sfard, A. (1998a). A mathematician's view of research in mathematics education: An interview with Shimshon A. Amitsur. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics Education as a research domain: A search for identity* (pp. 445 - 458). Dordrecht / Boston / London: Kluwer Academic Publishers.

Sfard, A. (1998b). The many faces of mathematics: Do mathematicians and researchers in mathematics education speak about the same thing? In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics Education as a research domain: A search for identity* (pp. 491 - 512). Dordrecht / Boston / London: Kluwer Academic Publishers.

van den Akker, J. (1999). Principles and methods of developmental research. In J. van den Akker, R. Maribe Branch, K. Gustafson, K. Nieveen & P. T. (Eds.), *Design approaches and tools in education and training* (pp. 1-14). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Wagner, J. (1997). The unavoidable intervention of educational research: A framework of reconsidering researcher-practitioner cooperation. *Educational Researcher,* 26(7), 13-22.

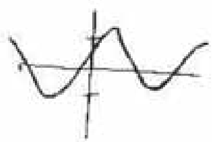
APPENDIX

A typical page from *Amongst Mathematicians* (Nardi, 2008). An example of student work at the top of the page becomes the trigger for the dialogue between M and RME in the middle. In the footnotes the reader is referred to relevant bibliography.

142

CHAPTER 4: MEDIATING MATHEMATICAL MEANING

Q3) i) $f(x) = \sin x + \cos x$



This is not onto as there is not a solution for any value of y
It is not one to one as There are several values of x that will give the same value (i.e. x and x + 2π)

Student LW

M: Of course if one uses the calculator then it makes no difference: all parts of the question are equally simple or complex. But that's a big issue in itself, the use of the graphic calculator!

RME: I sense you fervently want to raise it!

M: Well, a calculator could give you a good picture in part (i), for example. The students here seem to have access to a calculator and it seems to me they have missed a terrific amount of practice in the Analysis which is how these questions ought to be tackled. Generally students lack graphing skills and awareness of what to look for when asked to produce a graph[24].

RME: But surely the technology should help them see more examples and develop such expertise.

M: Yes, they have access to more pictures, and more easily, but have no feel for what to look out for.

[24] Intertwined here are at least two issues: constructing graphs and extracting mathematical meanings from interpreting graphs – for a review on graph comprehension and definitions of what constitutes good graph sense see (Friel et al, 2001) and the reference to (Roth & Bowen, 2001) in E7.4, Scene III. The two are linked as, particularly in the absence of a graphic calculator, for example, resorting to an understanding of a function's properties is a necessary step towards the construction of its graph. Researchers have used graphing tasks to investigate graphing skills and students' developing comprehension. E.g. drawing on the theoretical construct of APOS (Action, Process, Object, Schema) as well as Piaget & Garcia's triad of levels for schema development (intra, inter, trans) Baker et al (2000) studied undergraduates' comprehension of a non-routine graphing problem in Calculus. The triad was applied for properties (condition-property schema) and for intervals (domain, interval schema). Several student difficulties were observed: with cusp point, vertical tangent, removal of the continuity condition and second derivative. Overall co-ordinating information about properties and intervals was a problem – as was the resilience of incorrect images and overly emphasis on first derivative. Generally the two-schema interplay (property, interval) was difficult for the students