Conceptions for Relating the Evolution of Mathematical concepts to Mathematics Learning - Epistemology, History, and Semiotics Interacting

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ABSTRACT

Despite all the intense and international efforts of research into the teaching-learning processes of mathematics, Euclid's famous dictum is still valid according to which there is no royal way to mathematics. A growing number of approaches has as its focus the nature of mathematics and investigates whether, by taking into account this nature, the teaching-learning processes might be improved. A common pattern of these approaches can be called to be a "genetic" one, i.e., to establish a relation between the historical evolution of mathematics and the learning of mathematics.

The paper ten discusses how interactions between epistemology and history of mathematics can contribute to better qualify teachers to cope with the conceptual problems inherent to the nature of mathematics. An outlook to the importance of semiotics within the history of mathematics is given for reflection within mathematics education.

INTRODUCTION

The integration of historical elements is a longstanding issue in mathematics education. The ICMI Study of 2000, History in Mathematics Education, represents its most elaborate state of the art (Fauvel&Maanen 2000). Yet, the mainstream of approaches and proposals for the use of mathematics history in teaching mathematics takes history of mathematics as a ready-made collection of facts, easily transposable to the aims of teaching.

In fact, the main justification usually given for the direct use resides in practical methods of classroom teaching: historical elements are claimed to increase the motivation of the pupils, by showing them that the seemingly abstract mathematical system is a living system, that it was developed by human beings and that it is related to the cultural history of mankind - or of a particular nation. Even if not explicitly reflected, the underlying epistemological assumption about the nature of mathematics is that of a continuistic growth.

I should like to refer here to a still not sufficiently known but seminal paper by Antonio Miguel of 1997 where besides the positive effects the problematic issues of the use of history in classrooms are reflected (Miguel 1997). He was only followed by Man-Keung Siu in 2004 with his 16 thought-provocative arguments for not using history in classrooms (Siu 2004).

Actually, all the approaches concerning a use of history are based on certain epistemological views about the nature of mathematics, but in general they remain implicit, and use underlying assumptions. And in order to make the approaches productive, these views should not only be made explicit, but also be reflected within the frames of theoretical discussions in historiography and sociology of science as well as in mathematics education.

What I am interested in, is, whether there exists - beyond the merely accidental contribution of the motivational function - a productive function of the history of mathematics for the actual mathematical practice and for research in the learning process. If one wants to tackle such a question one has to challenge a view of mathematics which is deeply grounded in the common-day philosophy of many mathematicians: I do mean the view of an essentially cumulative nature of the development and growth of mathematical knowledge. According to this common-day philosophy (or epistemology), modern mathematics contains already all fruitful achievements of earlier periods, in an abridged and rationalized manner - so that one could say that contemporaneous mathematics presents in a condensed form the "logic" of history. Consequently, there would be no inherent reason for analyzing the processes of development of mathematical concepts. Likewise, no intrinsic moment would exist for a use of history in teaching – other than to constitute an exhibition of remarkable facts and dates. There would be left just one dimension for historical research: the dimension of factual data like those of priority - who invented first Lemma X, who invented first theorem Y? - and those on ordering and connection of the propositions and of regional/geographical distribution of mathematical knowledge. I confess that such a restricted view or epistemology is too unsatisfactory for me.

Gladly enough, there are recent conceptual developments in historiography of mathematics and in didactics of mathematics, which allow to question the traditional cumulative view and which allow new insights in the relations between history, teaching and learning. The common feature in these developments resides in new approaches to consider the subjectivity in the development of knowledge - as regards the learning person as well as the researching person.

THE GENETIC PRINCIPLE: KEY APPROACHES

Let us begin to look at some prominent genetic approaches and how they conceptualized the role of mathematics history.

In fact, it was an outstanding mathematician and a mathematician who probably was the one who did the utmost for a productive relation between mathematics and mathematics education and who decisively promoted the genetic principle: this person was Felix Klein, at the turn from the 19th to the 20th centuries. Klein was deeply convinced of the pedagogical superiority of the genetic principle – yet he never gave concrete suggestions for practising it. Nevertheless, from his assertions, one can deduce some of the intended characteristic features.

Firstly, he expressed, in 1907, the conviction that this didactical principle had won the dominance within mathematics education:

> "While a systematic manner of exposing mathematics instruction dominated earlier on, which overemphasized the formal aspects of knowledge, this did change more and more over the last years. Today, in German schools, this methodology is overcome. You can remark this victory of the genetic methodology, in the most impressive way, by the establishment of the already mentioned propedeutic geometry teaching" (Klein 1907, 24; my transl.).

A first concrete hint is, hence, that Klein understood a "genetic ordering of the teaching subjects" as opposed to the traditional "systematic" teaching. A next hint is that he recommended the so called biogenetic law as the basis for establishing a good syllabus:

> "This basic law should apply mathematics instruction, too, like any instruction, at least in general: teaching should, by tieing to the natural disposition of the youth, lead them slowly to higher things and eventually even to abstract formulations, by following that same path on which the entire mankind struggled to climb from its naïve primitive state upwards to more developed insight. [...] A decisive obstacle for a dissemination of such a natural and truly scientific teaching methodology seems to be the lack of historical knowledge, which becomes so often evident" (Klein 1911, 590 f.; my transl.).

Klein mentioned here a factual restriction regarding a general application of this teaching method, which he had characterized as being simultaneously natural and truly scientific: the lack of sufficient historical knowledge – apparently he meant the teachers of mathematics. Another hint how Klein conceived of the genetic curriculum is that he postulated mathematics instruction should begin with the continuous, i.e. with geometry, like mathematics itself he claimed, and only after that proceed to the discontinuous, i.e. to the number concept and to algebra (Klein 1899, 136).

It is highly revealing that the genetic principle became prominent again in almost the same wording in the 1960s, as a reaction against the so-called modern mathematics, against a one-sided orientation of school mathematics at the structure of mathematical science. It was in the famous memorandum of 65 mathematicians from Canada and the USA – among them Birkhoff, Courant, Kline, Polya, André Weil, and Wittenberg, published in 1962, which argued for the genetic principle:

> "in order to explain an idea (one should) refer to its genesis and retrace the historical formation of the idea. This may suggest a general principle: The best way to guide the mental development of the individual is to let him retrace the mental development of the race – retrace its great lines, of course, and not the thousand errors of detail. [...] On the whole, we may expect greater success by following suggestions from the genetic principle than from the purely formal approach to mathematics" (Memorandum 1962).

As you will note, both in Felix Klein's view as in that of these North-American mathematicians, the biogenetic law featured prominently. I will discuss this issue soon. But first let me discuss some works, which have been esteemed as realizations of the genetic principle sensu Felix Klein.

Alexis-Claude Clairaut (1713-1765) was an important French mathematician and physicist. He wrote tow textbooks, one on geometry in 1741 and the other one on algebra in 1746. They have often been claimed to be realizations of the genetic principle. This characterization is misleading, however: it is better to attest them a problem-oriented or heuristic approach (see Schubring 1983a, Glaeser 1983, Schubring 2003, 54ff.).

The geometry textbook intends to develop geometry step by step, always motivated by practical questions like measuring quantities in fields, in the landscape, in farming, and generally in land surveying. At a first glance, the geometry textbook realizes Felix Klein's demands to develop the geometrical notions – beginning from natural, "primitive" questions.

A closer analysis shows, however, that Clairaut did not succeed in a "natural" evolution of the conceptual field, according to an unfolding of "original" problems and of their consequences. Rather, he imposes what should be the next, seemingly practical question to be solved. Moreover, Clairaut's approach does not realize the claim to lead from simple notions to abstract knowledge. Rather, he refrains from all abstraction and theorization. And his claim to follow the historical evolution of geometry is not realized, neither: Clairaut postulates, in fact, *how it might have been*, how the "inventors" did proceed – his historical-genetic claim can hence at best be appreciated as a "rational reconstruction" – in the sense of Lakatos.

The lack of abstraction was consciously intended: The book was produced for a mundane public, not for use in schools and systematic teaching. Actually, it was written for a marquise who desired to be instructed in some leisure mathematics. This explains Clairaut's main methodological concern: ne pas rebuter les commençants – not to scare off the beginners. For the algebra textbook, the problem-oriented approach was even more difficult to realize.
In the famous *Encyclopédie* by Diderot and d'Alembert, in the key entry about textbooks, Clairaut's textbooks were sharply criticized for omitting essential proofs and hence for lack of rigor. Moreover, they were criticized for providing nothing but a sample of propositions instead of a methodically constructed architecture (d'Alembert 1755, 497 r).

A much more elaborated and theoretically reflected conception has been presented by Otto Toeplitz (1881-1940) - a German mathematician whose main book is translated at least into English and who was quite active for improving the teaching of mathematics in schools and in universities between the two world wars. Toeplitz pleaded for using history as a pivotal didactical means - he called this the "genetic methodology" and introduced the distinction between a "direct" genetic methodology and an "indirect" genetic one.

In a key paper of 1927, Toeplitz proposed to return to the "roots" of the concepts and to present them thus as living beings. As Toeplitz said, one could pursue two different ways to realize this goal in the teaching practice:

> "One can either present the discoveries to the students with all its dramatic circumstances and let thus grow for them the questions, concepts and facts - I would call this the direct genetic methodology - or one can learn oneself from such an historical analysis what is the real meaning, the true essence of each concept, and one can draw conclusions from such an analysis for the teaching of this concept which are no longer tied to the historical development - I am calling this second approach the indirect genetic methodology" (Toeplitz 1927, 92f.).

While the direct genetic methodology corresponds to the already discussed direct use of history in teaching, the second, indirect approach is interesting since it takes into consideration the role of the teacher and understands the teacher as actively reflecting the historical processes and as transmitting their essence by his teaching. Toeplitz's indirect approach looks not so much on knowledge, but on meta-knowledge and his main focus is on how to provide teacher-students in their training with such a meta-knowledge about mathematics.

Toeplitz has used this methodology in his own courses at the university, in particular on the infinitesimal calculus. This course has been published as a book: "The development of the infinitesimal calculus, exposed according to the genetic methodology" (Toeplitz 1972/1963). Unfortunately, despite its promising approach, this book cannot really serve as a model for the proposed methodological use of history, since Toeplitz's program to reveal the decisive turning points and ruptures in the historical processes is hardly realized: Toeplitz discerned mainly three fundamental concepts, which determined, by their development, the emergence of the infinitesimal calculus. For two of them, the "infinite process" and the number concept, Toeplitz tries to show that the ancient Greeks did already achieve all essential steps and that later developments were but an unfolding and a change of exterior form of these first achievements. For instance, in the famous dispute between Dedekind and Lipschitz, whether Dedekind's concept of real numbers was new or identical with the notions of the Greek Eudoxos, Toeplitz took the part of Lipschitz in claiming that Eudoxos already operated with the concept of real numbers while Dedekind had insisted that the notion of completeness was missing entirely in Greek mathematics and could not be derived, not even implicitly from geometrical ideas. Toeplitz admitted for the function concept only that it emerged as a new concept in modern times, but even here he tried to show that Ptolemy was already aware in Hellenist time of this concept (see Schubring 1978).

We can see therefore that Toeplitz remained attached to the traditional view of a continuous, cumulative development in the history of mathematics so that his own notion of an indirect approach could not become fruitful. His underlying conception seems, too, to be effected by that notion, which is commonly called the "biogenetic law": Toeplitz claimed that the development of mathematical concepts uses in general to follow "the easy ascent from the more simple to the more complex" and that this historical ascent might be used didactically (Toeplitz 1927, 95).

The example of Toeplitz's conception therefore again shows that the main problem for a revealing use of history resides in an adequate conception of historical development. While most of the other scientific disciplines are discussing - since Thomas Kuhn's famous book on scientific revolutions - revolutions in their field and ruptures in the conceptual development, mathematics seems to close its mind to realize an analogous epistemological change. The traditional epistemology stressing the uniform, continuous and cumulative character of this "queen of the sciences" is, apparently, too strong. A telling example for this exceptional position of mathematics has been formulated by the French philosopher Gaston Bachelard who has convincingly analyzed epistemological ruptures in the exact sciences, but who has consciously excepted mathematics from these analyses:

> "The history of mathematics is a miracle of regularity. There are periods of standstill, but it knows no periods of errors" (Bachelard 1975, 25; my transl.).

Actually, the notion of error will provide a key to challenge this epistemological view.

RELATION BETWEEN RESEARCH AND TEACHING

In order to tackle this question let me present you some of the mentioned new approaches in historiography of mathematics. Their main feature is constituted by studying the interrelationship between the system of production of new mathematical knowledge and the systems encompassing and supporting mathematics. These new types of historical research, which have evolved over the last decades, focus in particular on one specifically related social sub-system: on the education system, since the dissemination of mathematical knowledge is essentially bound to the education system and since teaching positions were for a long period the only relevant professional careers for mathematicians. The analysis of the relationship between mathematics seen as a social system and its surrounding systems has progressed much beyond the fruitless dichotomy of internal versus external determination of mathematical ideas and has particularly contributed to better understanding the circumstances of mathematical production.

A primordial element in these analyses is given by a re-evaluation of the relation between teaching and research. The traditional view of this relation has been that the scientific part exclusively plays the active, productive role and that the didactical side always is the passively receiving partner, which transposes the received into the instruction system (a view, still perpetuated by Chevallard's concept of transposition didactique). The relation between scientific knowledge and school knowledge was therefore understood as operating only in one direction. This one-directional view has been denounced in 1978 by Willem Kuyk – the author of "Complementarity in mathematics" (1977) - by comparing it with the relation between stalactites and stalagmites (Kuyk 1978, 5):

"Mathematics is not a stalactite hanging over a stalagmite", thus denying the view that mathematics education grows but by receiving some drops from above, from the supreme instance. The instructional system cannot be understood in the simplistic way of a stalagmite, which receives some drops from the stalactite while it is growing. My intention is to show that the re-evaluation of the relation between research and teaching allows at arriving at another understanding of historical development.

An important publication on this way has been the article by Judith Grabiner of 1974: "Is mathematical truth time dependent?" At the same time, Hans Wußing had remarked that the new system of teaching higher mathematics, emerging in France since the end of the 18th century, contributed decisively to establishing new standards of rigor, to promote research on the foundations of mathematics and to falsify propositions, which had been thought to be true (Wußing 1974, XVIII).

My own research on the development of mathematics in Prussia (a leading state in Northern Germany) in the 19th century done in the early 1980s, has shown that the profession of mathematics teachers at secondary schools constituted the social basis which enabled the establishment of mathematics as an autonomous discipline within the university system. Moreover, the type of interest of these teachers in mathematics decisively moulded the production of pure mathematics for which Prussian and later German mathematics has become so well known: Actually, the interest of these teachers - themselves regarded as "scholars" - in rigor and in a consequent architecture of mathematics yielded important achievements in foundational questions and in clarifying basic notions (see Schubring 1983b, 158 ff.).

Resuming these briefly outlined researches and results on the history of mathematics in its context, one can say:

- firstly, the teaching of mathematics has influenced the development of mathematical research. The dimension of instruction and teaching has therefore to be considered for an adequate notion of historical understanding of mathematics (see Schubring 2001);
- secondly, ruptures and emergence of novel directions in history of mathematics are largely due to epistemological changes, which are connected to changes in the systems related to the system of scientific activities;
- and thirdly, didactical research on learning processes can reveal means and categories which are usefully applicable to analyze also processes of scientific development.

The last two propositions aim at including the subjectivity of the student and of the scientist into the theoretical framework. In order to explain and to apply these propositions I want to discuss two aspects on which much didactical research has been done over the last decades in order to study the subjective element in the learning process. These two aspects are the errors and the obstacles.

Errors

The investigation of pupils' errors in the learning process constitutes a major field of didactical research since several decades – actually, as one of the main features of the emergence of mathematics education as a scientific discipline. Didactics of mathematics has increasingly established more refined experimental instruments to analyze pupils' errors and discusses theoretical models for interpreting errors.

As major results of these researches I need here to mention only briefly: errors are not merely expressions of an individual's "defects", of missing attention, or the consequence of missing knowledge or due to an accidental specific situation. Errors can therefore not be simply remedied by increasing discipline, attention and diligence of the pupils.

Empirical research has shown that errors are rather causally determined and often of a systematic nature. Errors can be analyzed and described as resulting from patterns and notions, which can be internally consistent but which do not coincide with the notions and operations as intended by the teacher. A first consequence of these researches has been to identify as causes of the errors either difficulties of the pupils in grasping the new information in teaching or problems in the interaction of the variables influencing mathematics instruction (teacher, curriculum, pupil, context of the school). But even in this research, errors of students were understood as indicators for individual difficulties (Radatz 1979). Further research has, however, increasingly questioned that these specific patterns are signs for merely individual difficulties.

A radical research program developed in this field is that of social interactionism, initiated and developed by the Bielefeld group: Bauersfeld, Krummheuer, and Voigt, since the 1980s:[1] in this program the status of errors is challenged. The basis for this program is the philosophy of constructivism as developed in particular by Glasersfeld: There exists no objective meaning of notions and concepts. Each individual constructs his own meanings given his experience and background. It is only by the social interaction between the individuals that communication takes place and that the individual constructions can gain a certain convergence. It is by the process of social interaction, that a specific construction becomes acknowledged as common knowledge, as "objective" (Bauersfeld 1983).

[1] Paul Cobb, in his speech at ICME 11, after having received the ICMI Freudenthal medal, remembered the formative significance of his cooperation with this group.

Mathematics teaching is particularly suited for studying the processes of establishing a common knowledge shared by the participants of the communication in a class since there is no direct exterior reality, which would allow testing the validity of the individual constructions. The teaching process can be described as a negotiation between teacher and students and where the teacher tries to establish working procedures, which may be more or less stable. The original Bielefeld group has used particular experimental instrumentations like video-recording of the teacher-student interactions and developed methods for transcribing the interactions in order to make them analyzable and reproducible to other researchers. This research program has yielded very remarkable results and shown that what is usually seen, by the teacher, as errors are in fact misunderstandings: the students use to "see" other notions in the material presented by the teacher than the teacher had in mind.

For instance, in the teaching materials used for introducing the notions of the first natural numbers several objects from the real world are shown. The student should "abstract" from the real world features and just retain the cardinal number. The analysis of the interaction process shows, however, that the students direct their attention to other elements in the pictures and effect therefore other "abstractions". It takes a long time until the students can divine what the teacher wants to hear and that conventions become routinized upon signals given by the teacher. This learning "success" can be a merely superficial one and the working procedure can break down when the teacher uses a different symbolization (see Voigt 1985).

This concept of social interactionism need not remain restricted to school teaching and didactics. It can equally well be applied to research in mathematics and therefore to history, too. How does it happen that a new theory is adopted in mathematics, that a concept is regarded as rigorous or rejected as not rigorous, that a proposition is regarded as false? This is neither by the decision of an individual nor by the universal insight of an eternal truth, rather we find, here too, negotiating processes in the mathematical community, interactions in this social community, which determine about acknowledgement or refutation. Before I discuss consequences of this view for the growth of mathematical knowledge, there is to mention yet another dimension relevant for history in the didactical research on students' errors.

In fact, in the didactical research on errors one does not locate all problems in the modes of interaction and in the communication process, but one also emphasizes possible causes in the mathematical content of the communication: One analyzes the teaching material or the exposition of the teacher if they might not be correct from the mathematical point of view or if they contain missing links which could have caused that a student did not grasp a mathematical notion and its operations. This is surely a legitimate approach in didactics of mathematics but it is not a sufficient one: In almost all didactical theories, the mathematical knowledge is taken as objective or absolute precondition for learning which will not be questioned. This starting point of didactics is, however, insofar not sufficient, as there exists no *a priori* evidence that the mathematical knowledge used for teaching really is complete, organized consequently and coherently and without missing links. Didactical research should be aware of inherent problems in mathematics itself: unsolved or even undetected problems in the logic or in the epistemology of mathematics, ambiguous or even misleading notations. The teacher who has been initiated to the language of mathematics and its peculiar operating procedures will not be able to remark such inconsistencies, but the student as naive, as non-initiated, might be hindered by such problems inherent to mathematics - what the teacher marks as error can be an indication for deficiencies within the mathematical knowledge.

It is particularly this dimension of unsolved internal or epistemological problems in mathematics by which the teaching process can effect an impetus for progress in mathematics or can even effect ruptures within the established system of mathematics.

Since school mathematics represents to a greater degree the condensed essence of the historical development than the actual research knowledge we did arrive at a first productive use of mathematics history for didactical research, namely by supplying the means for analyzing those conceptual, notational or epistemological problems of mathematics which are due to certain stages of the historical development and which effect errors or misunderstandings by the side of the students.

Obstacles

We can deepen the discussion of the use of history for teaching by the means of didactical categories if we regard the specific contributions by French didacticians. The emphasis on the knowledge itself - what one can call the epistemic dimension -, which is largely missing in German and North-American didactical research on errors, constitutes in French research one of the main issues. One uses in France a category for didactics of mathematics, which has originally been established for studies in history of science. I mean the category of *obstacles épistémologiques*, of epistemological obstacles, put forward in 1936 by the already mentioned French philosopher Gaston Bachelard. It gained particular influence after a re-edition of his works in 1975. Bachelard's conceptions have been transposed by Guy Brousseau to didactics of mathematics, who has developed a didactical theory of obstacles. Its main aim is to overcome to attribute errors only to subjective causes in the students. Brousseau discerns in particular the following types:

- didactical or didacto-genetic obstacles: by this he means learning difficulties or bareers which originate from the conception or structure of the curriculum, from the particular teaching concept, from didactical concepts,
- and, secondly, epistemological obstacles. According to Brousseau, these obstacles to learning are rooted in the nature of mathematical knowledge and can therefore not be avoided. They are constitutive for the respective knowledge, they become visible in some stage of the historical development and can be identified by historical analysis.

According to Brousseau's theory, where a model of stages is applied, there are inherent contradictions within the types of knowledge tied to the lower stages: the knowledge shows itself effective as long as applied within these restricted areas, but reveals to be an obstacle when it becomes applied to situations of a higher stage. Some knowledge can therefore, due to inherent reasons, function as an obstacle against progress on the next stage (Brousseau 1997, 84).

One can therefore understand his theory as a "transposition" of Bachelardian ideas to didactics. Both theories on whom Brousseau relies, by Bachelard and by Piaget, imply a teleological vision: the certainty to be able to achieve the most "mature", the most elevated level of science, of human thinking.

A number of studies has been carried through on the basis of this research program, for instance on the difficulties of students with the limit concept in calculus, with the notion of infinite, and on students' notions of basic geometric concepts. A particularly profound study of the limit concept, both for the historical and for the didactical side is the collective work published in 2005 by a group of Italian researchers: *Oltre ogni limite* – Beyond any limit.

The quality of such research relies to an important part on the reliability of the historical analysis: otherwise, the empirical findings on students' difficulties are interpreted according to prejudices or to a common-day understanding about the nature of breaks, ruptures and problems in the historical development. The demand for detailed and qualified historical research is the more imperative as historiography of mathematics traditionally tended to restrict itself to the ideas of the "great men", the "heroes" - an emphasis by which the real difficulties experienced by the larger contemporaneous mathematical community can hardly be taken into account.

We arrive thus at a second, "indirect", use of history for didactical research: In order to fill the enormous gaps of knowledge about the mathematical thinking and practice in the larger group of mathematical practitioners, it constitutes a challenging task for the historiography of mathematics to study debates and controversies about the status and nature of relevant mathematical concepts.

This second use is not thought of in the way of deriving recipes for teaching, but as elements for the didactical research on epistemological obstacles and to enrich the meta-knowledge of teacher students and of teachers.

Starting from such a conception, I have done extensive research on the history of negative numbers. The results were significant contributions for history and for didactics, namely on the role of errors for mathematicians and for the teaching process and, likewise, on the notion of epistemological obstacles in history and in teaching (see Schubring 2005a; 2005 b; 2007).

The function of history in this French conception

Understanding Brousseau's theory is facilitated by comparing its two versions, of 1976 and of 1983, which is easy, since many of his publications were translated in the volume Theory of Didactical Situations (1997). He uses to emphasize that obstacles are unavoidable, but also that one should not reinforce them explicitly:

Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative rôle in the knowledge being sought. (Brousseau 1997, 87).

In 1983, after his controversy with Georges Glaeser about the meaning of the term "obstacle" and basing himself now on the study by Duroux –

the first concrete investigation in French didactique to identify epistemological obstacles, this time regarding the notion of absolute value – Brousseau relied much more on history of mathematics and attributed it a decisive function:

But it can prove itself to be fruitful for teaching insofar as:

- the obstacles in question are truly identified in the history of mathematics;
- they have been traced in students' spontaneous models;
- the pedagogical conditions of their "defeat" or their rejection are studied with precision in such a way that a precise didactical project can be proposed to teachers,
- the assessment of such a project can be considered positive. (Brousseau 1997, p. 93-94)

This strengthened function of history reveals, however, a weakness of the conception: history has to serve as source for errors committed by mathematicians. Thus, history has no productive function; it serves as an element of a recipe for research:

From the outset, therefore, researchers should

a. find recurrent errors, and show that they are grouped around conceptions;
b. find obstacles in the history of mathematics;
c. compare historical obstacles with obstacles to learning and establish their epistemological character. (Brousseau 1997, p. 99)

Here, one finds no active role for history. This seems to be related to the fact that Brousseau did no integrate a key element of Bachelard's conception: the notion of a rupture between empirical knowledge and scientific knowledge, which is of enormous importance expressly for didactical research. Furthermore, in the 1983 conception, there is not the supposed symmetry between the side of history and the side of the learner: Since obstacles were declared to be insurmountable - "incontournables" and "insurmontables" (Brousseau 1989, quoted in Brousseau 1998, 154) -, while students' errors should be surmountable, there is a drastic asymmetry. And scientific progress would be impossible when obstacles could not be overcome. Glaeser's understanding of obstacles as "difficulties", hence without a normative character, lent itself better for historical investigations.

The general weakness of the conception of epistemological obstacle resides in the problem that the history of mathematics is regarded as a fixed collection no longer open to questions and research. Traditional historiography, onto which this didactical conception would base itself, is not adapted for answering to these new questions, for a use in didactics and learning: they did not look for the "normal" mathematicians who would better reveal the obstacles sought for than the traditional heroes.

Actually, it had never been investigated whether one of the normative pillars of the concept of epistemological obstacles is really justified, namely whether an historical obstacle necessarily shows up as a learning obstacle. I have therefore undertaken a case study to test this issue: it concerned the multiplication of quantities, which proved to constitute over various centuries a genuine conceptual obstacle in arithmetic and which characteristically no longer constitutes an obstacle in learning – essentially due to an epistemological switch which had happened in the meantime: quantities no longer constituting the conceptual fundament of mathematics (Schubring 2005 b).

Critique of the biogenetic law

In almost all the genetic approaches, which I have presented to you so far, almost inevitably the so-called biogenetic law showed up – either implicitly or explicitly. This is true since Felix Klein's first pleas. Even in Brousseau's conception it shows up implicitly. He uses to relate to "spontaneous" reactions of students, i.e. to answers before teaching the respective concept (Brousseau 1997, 93). According to him, these spontaneous answers reveal epistemological obstacles and correspond at the same time to the naïve hypotheses of the first scientists. This implies not only the implicit acceptation of the biogenetic law, but negates at the same time the profound social and cultural changes, which effect that children of today start at decisively different conditions than earlier generations.

The recapitulation hypothesis originated from a transfer of biologism to cognitive development. It was in particular Haeckel's famous law for biological development of the. species which was grafted to psychology. The graft from biology on psychology and education was effected, among others, by the philosopher Herbert Spencer

> "the education of the child must accord, both in mode and arrangement, with the education of mankind, considered historically. In other words, the genesis of knowledge in the individual must follow the same course as the genesis of knowledge in the race." (quoted from Branford 1908, 326).

This grafted biogenetical principle, or principle of parallelism, had become a largely shared topic in education by the end of the 19th and the early 20th centuries and, remarkably enough, in particular in mathematics education. In fact, it would seem that mathematics was, and still is, the only school discipline where this principle has become so prominent. I cannot remember anybody to have claimed it being applicable, say, to physics or to chemistry. Strangely enough, the biogenetic law, no longer prominent in the first half of the 20th century, made a more or less explicit return to mathematics education in its second half, and in particular in approaches for using mathematics history in teaching (see Schubring 2004).

An instructive and concise introduction to the entire problematic of parallelism and of the biogenetic law is the excellent paper of 2002 by Luis Radford and Fulvia Furinghetti. They elaborate not only Piaget's and Garcia's deficits in conceiving of cultural and social impacts on cognitive formation, but they also present L. Vygotski's alternative approach as that of one of the few psychologists to have profoundly investigated socio-cultural influences on cognitive processes. As they put it, "the merging of the natural and the socio-cultural lines of development in the intellectual development of the child definitely precludes any recapitulation" (Radford/Furinghetti 2002, pp. 634 – 642; here: 637).

The major flaw in all the approaches based on parallelism is that they presuppose history of mathematics as a definitely established corpus of knowledge, which is beyond controversy. This is, however, far from being true. The historiography of mathematics has hitherto concentrated on the "peaks", on the "heroes" of mathematics, and it has practiced a resultatist view, searching for forerunners of the results of present mathematics, and thus ever and again reproducing the continuist view of development we always find in how didacticians assess the history of mathematics.

For uses in education, another type of historiography and of research has to be attained, however, a view which unravels the contributions of scientific communities at large, identifying and assessing conceptual ruptures, and in this way documenting conceptual developments in different relations of subsystems to their encompassing systems (cf. Schubring 2002). This will make it possible to better establish the social and cultural contexts and their impact on scientific development – an approach hitherto only postulated, but never really elaborated.

Resuming our discussion of the conceptions of epistemological obstacles and of the biogenetic law (or parallelism) we have to state that both are not adapted for a productive use of the history of mathematics. Both are normative approaches and do thus hamper experimental research in both domains, in history and in mathematics education – they are prejudicial for open-ended research.

Furthermore, al the discussed genetic approaches and these last two in particular presuppose a universally homogeneous conceptual development over time. However, there does not exist a "Gesamt-Intellektueller", an all-comprising intellectual. Conceptual developments occur within determinate and specific groups, the so-called scientific communities which have as primary references for their conceptual frames the values and norms of their particular cultural environment, their directly surrounding systems – which one may shortly call "context". Therefore, there does likewise not exist an absolute simultaneousness or parallelism of conceptual developments in different cultures.

Errors in mathematics

I can now come back to my proposed approach to start form the subjectivity of the person and its group: I spoke already of this approach for the learner, within the conception of social constructivism. I should now turn to the other side, which is relevant here, to the scientist – and now not limited by *a priori* assumptions about a *Naiveté* of early scientists etc., but based on a productive role of interaction between research and learning. In such a sense, one is able to investigate more freely possible errors of scientists, and in particular of mathematicians.

In present day convictions it seems to be unthinkable to acknowledge the possibility of serious errors in the history of mathematics, as exemplified by Bachelard's exclusion of errors in mathematics. Earlier generations seem to have had less problems with such a possibility. A telling example is provided by Martin Gebhardt, the author of the first ICMI Study on the role of mathematics history for mathematics instruction in 1912. He assured:

> "With the proof by history that error and controversy play their role and are important in mathematics, too, the abysm, which separates it from other sciences, in particular also from the natural sciences, will disappear to a considerable degree" (Gebhardt 1912, 83).

And by errors he meant, as he emphasized, not those which can happen to each mathematician, but those which are characteristic for an entire epoch – like the conviction of convergence of the series 1-1+1-1+1- … having as limit ½, defended by Grandi, Leibniz, and Euler, among others. And in 1904, E. Maillet, a French mathematician had called to collect remarkable errors of mathematicians, as an instance of self-reflection. The resulting collection was published in 1935 by Maurice Lecat, a specialist in variational calculus. It is not well known, neither in historiography nor in mathematics education. The collection documents about 500 errors, attributed to 330 mathematicians – among them many minor figures, but also famous mathematicians. Lecat stated that there was only one famous mathematician who never committed an error: Evariste Galois. Thus, Lecat dedicated to him an honorary page, i.e. an empty page (Lecat 1935, 39).

Given this dimension and extension of committing errors in mathematical research, on the one hand, and the acceptance of "errors" as good mathematics over extended periods, I am now able to formulate my main hypothesis/research guideline/proposition:

It is a consequence of the program of social constructivism resp. social interactionism that so-called students' errors can no longer be called "errors" if they follow a definite strategy, jointly shared by that entire social group. Analogously, this applies to communities of scientists, too, and in particular to mathematicians. Regarding chemistry, I should like to recall the phlogiston theory, which was accepted by chemists over centuries (see Kuhn 1962).

This specific claim of such a constructivism has to face the objection: where remains the objectivity of mathematics, which has always been maintained to be the major characteristic of this science?

In fact, the consequence of my conception is that there exists no objectivity, at least no overall objectivity. Not only in learning, meanings of concepts are subject to negotiation processes, so that differences in meanings established by various groups will disappear as result of interactions when these groups get into communication and achieve shared meanings, but also in science a common understanding will at first be restricted to social communities, which are tied together by certain conditions to form a basic unit of communication, say by sharing a common culture and language. Let me call this basic unit a scientific community of first order. In general, one can assume that they will share, too, the epistemological view of their subject. While there might co-exist different epistemological and conceptual views of mathematics in separated mathematical communities, there should begin processes of interaction at the moment when such separated communities come in contact with each other. Consequently, either the values and conceptions remain mutually alien so that – if there are no other pressures for establishing shared conceptions – the communities will continue to be separated, or a negotiating about the differences will begin with the effect of certain compromises or dominations.

This hypothesis about a relative objectivity as result of negotiation processes between originally separated mathematical communities can be tested by investigating – not a "clash" of cultures – but the effect when two cultures with different conceptions of knowledge are colliding.

A first such test is presented by the transmission of number signs and of decimal fractions from India to the Arab civilization, studied by Mahdi Abdeljaouad. As is well known our so-called Arab number signs are in reality Indian signs, as well as the establishment of zero and of the decadal number system. The Arabs used, like the Greeks, the Phoenician manner of designing numbers by letters of the alphabet. And for fractions, they either used Babylonian sexagesimal fractions or Egyptian unit fractions. In the main period of Islamic culture, from the 8th to the tenth centuries, the Indian numbers and the decimal fractions had not found acceptance. Al-Uqlidisi who had tried to introduce them, by a significant textbook in 952, had no success and his book was forgotten, until a re-edition by Saidan in 1966. The resistance against the Indian way of mathematics is clearly documented by a polemic appreciation uttered by Al-Biruni in the 11th century, in his introduction to the book "History of India":

> "The Indians to not dispose of philosophers like the Greeks who have exposed their subjects in their texts entirely scientifically. They have produced almost no book, which is not a downright collection of rubbish and where get mixed all varieties of popular beliefs. The spirit of authority dominates in them. As far as I am concerned, I can assure that their books of arithmetic and mathematics are comparable with nothing else than stone cairns containing some fragments of ceramics or with pearls hidden in the dung/manure of camels." (quoted from Abdeljaouad 1978, 14; my transl.)

I have published a more recent example of mutually exclusive visions of mathematics last year: the case of Edmund Külp, the teacher of Georg Cantor who in his youth was educated according to the values of French mathematics – and that meant of physico-mathématique: a vision of applied and applicable mathematics. Becoming transferred to Germany, Külp had to suffer a purely formal, inapplicable mathematics – the mathematics of permutations and transpositions of the German combinatorial school. Due to the incompatible meanings of that French and that German mathematics, Külp failed with his project to pursue an academic career at a German university and had to serve for decades in primary teacher education to make his living – until he managed to become a teacher at a trade school where some French mathematics was admissible (Schubring 2007).

Role of semiotics: the development of signs

A particularly illuminating quotation by Destutt de Tracy, a French philosopher, of 1801 underlines the productive role of teaching for research, for obtaining new knowledge, which I am emphasizing in my approach to the use of history of mathematics. This quotation presents an evaluation of the historically first experience to disseminate scientific knowledge, to elementarize science and making it accessible to a general public. Reflecting the ambitious projects of the French Revolution to produce such truly elementary textbooks, Destutt de Tracy resumed:

> "When one is about to expose a scientific fact, one often remarks that it necessitates to undertake before new observations, and – better investigated – it presents itself by a quite different point of view. At other occasions, it proves that it is the principles of science itself, which need to be revised, or one has to fill numerous gaps to connect them mutually. Briefly, the matter is not to disseminate the truth, rather one has to detect it" (vol. 1, p. 4 f., of his *Projet d'Éléments d'Idéologie*; quoted from Schubring 1982, 114 (my translation).

A particularly important dimension of the challenge to research by teaching as explained by Destutt de Tracy is presented by the representation of mathematical objects, by the sign function of concepts where essential elements use to be hidden and where it is in particular the effort to teach them which effects an explication of implicit and hidden assumptions and conceptual moments.

The last part of the lecture will be devoted to briefly expose the role of semiotics for such a new approach to the relation between history and didactics. In fact, there is a forerunner for the present approaches to introduce semiotics into mathematics education: It is Karl Menger (1902-1985), the important philosopher of the Vienna circle, logician, mathematician and economist. He had to flee the Nazis and emigrated to the United States. Since the late 1940s, Menger has published several papers and even a seminal book, which give excellent descriptions and analyses of inconsistencies in mathematics and notational ambiguities, most of them remain even today to be solved and pupils and students are left with the obstacles to get through the misleading paths.

His publications are not only well instructed in history, in semiotics, and in teaching, but they are written with such a deep humour that it is a real pleasure to read his profound analyses.

It is highly remarkable that a review of his seminal book refounding the teaching of the calculus emphasizes the same points as Destutt de Tracy:

> "It becomes clear after reading the book that the invention of the new notation was an essential step toward the clarification of the basic ideas and their applications and is thus amply justified" (Review of: Karl Menger, *Calculus, A Modern Approach*, by H. E. Bray, in: *American Mathematical Monthly*, vol. 61, Sept. 1954, 483-492, on p. 483).

A key starting point for Menger are notational ambiguities in mathematics, which use to be ever again transmitted to the next generation as time honoured and therefore not questionable. A particularly striking example are the twelve different meanings of the seemingly so innocent letters x and y. In fact, the meanings range from numerical variables, over indicators for the identity function, indeterminates, specific fluents, function variables, to "dummies" (Menger 1956a).

Menger has sharply criticized the negation of notational and conceptual problems arising from the weight of unchallenged history. Summarizing the mainstream thinking of mathematicians at least of his time, he lets them say:

> "Since for the past two hundred years and to this day, all mathematicians and scientists have achieved complete mastery of mathematics with its time honored procedures and in its traditional presentation, and since furthermore,

> the difficulties here discussed do not disturb any accomplished mathematician in the least, youngsters who cannot cope with them must be mathematically utterly incompetent. To revise procedures or symbols for their sake is not worth anyone's while since their study of mathematics cannot, under any circumstances, be profitable either to those mathematical morons themselves or to anyone else" (Menger 1956b, 584).

In a perfectly satirical manner, Menger has denounced the sticking to historical traditions in his series of papers on Gulliver, in particular in the first one entitled: "Gulliver in the land without one, two, three".

His starting point is the juxtaposition – historically to be often found – of the first numbers being treated as quantities (or "named numbers") and the greater ones as numbers. And he ridicules a didactical retrogression by which all numbers are treated as quantities or named numbers. Here, the mainstream mathematicians defending this anti-didactic transformation are called the IMMORTALS, abbreviation of: The Island's Major Mathematicians of Real Talent and Learning (Menger 1959).

This invented example of retrogression, of a use of history where one needs to get liberated from historical dust, serves as an introduction for what is a key element in Menger's theories: the establishment of an algebra of functions. For this, he first criticizes a notational ambiguity, which causes many learning problems: the often missing distinction between a function and a value of this function - both being usually designated by f(x) (or cumbersome formulations like: "the function which is expressed by f(x)"). Rather, one has to designate a function by its name; one is thus able to distinguish the function from its value at a certain point.

More generally, however, his conclusion is that one does not need variables in calculus, that they constitute but dummies, and that one has rather to reflect on naming functions to be able to operate with functions. In this sense, he calls variables "dummies" and shows that these are elements of historical tradition, from which teaching has to be liberated.

On the other hand, he develops his algebra of functions by the introduction of a notational innovation: the basic element of this algebra is the identity, the neutral element. He calls it the function I, namely:

$$I : x \to x, \quad I^2 : x \to x^2$$

And this otherwise neglected neutral element enables him to introduce an operational calculus with functions. Therefore, this new approach of semiotics implies a double role of history of mathematics for teaching: on the one hand, it reveals outdated mathematical practices, which need to be deleted to improve the teaching-learning process. On the other hand, it reveals forgotten or marginalized conceptions which had been established in some mathematical community in an earlier period and which need to be valourised and updated for present day teaching purposes. In fact, Menger's operational calculus is a direct continuation of the Derivation Calculus established by Arbogast in the wake of the French Revolution – exactly as a realization of the *méthode analytique* of the Enlightenment, which should contribute to disseminating the scientific knowledge. It is not by accident that in these analytic approaches the role of symbols is decisive for clarifying the meaning of the concepts and for enhancing their teaching and learning.

In fact, Menger's algebra of functions confirms again the systematic relation between processes of algebraization and reflection on the use of symbols. Semiotics promises fruitful impacts on the use of history for teaching!

Menger's ideas have had some impact and influence in the great Curriculum-Projects in the USA, during the 1960s, in particular within CSMP, but with their end they remain rather neglected. His legacy constitutes treasures, which still remain to be excavated and brought to light and to use!

CONCLUSION

Although conceptually attractive, approaches to use history of mathematics for mathematics teaching show theoretical shortcomings as well as problematics in the experimental designs. As a major reason, continuistic visions of mathematical development proved to be underlying so that history of mathematics was not able to exert a productive function. The most promising conception, the indirect genetic method of Toeplitz, suffered in his realization from his peculiar teleological view of development: all the essence being already contained as a germ in Greek mathematics. But the kernel of his vision, to unravel the conceptual depth and meaning from turning points in the history, provides a precious approach at least for teacher training. Yet, historiography of mathematics still has to broaden its research areas to comply with such a vision. Semiotics provides promising contributions.

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