History and Epistemology in Mathematics Education

Proceedings of the Seventh European Summer University

ESU 7

Edited by:
Evelyne Barbin
Uffe Thomas Jankvist
Tinne Hoff Kjeldsen

HPM

AARHUS UNIVERSITY
DANISH SCHOOL OF EDUCATION

History and Epistemology in Mathematics Education

Proceedings of the
Seventh European Summer University
ESU 7

Copenhagen, Denmark
14 - 18 July 2015

Edited by:
Evelyne Barbin, University of Nantes
Uffe Thomas Jankvist, Danish School of Education, Aarhus University
Tinne Hoff Kjeldsen, University of Copenhagen

IMPRESSUM:

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Layout: Leif Glud Holm

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Printed in Denmark by Danish School of Education, Aarhus University
1st edition 2015

978-87-7684-737-1 (Printed version)
978-87-7684-736-4 (Electronic version)

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ESU 7 (2014) Proceedings Presentation

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University of Nantes, Aarhus University, University of Copenhagen

This ESU was the seventh *European Summer University on the History and Epistemology in Mathematics Education*. The Summer University was held from 14th to 18th of July 2014 in the Danish School of Education, Aarhus University, Campus Emdrup in Copenhagen (Denmark). The initiative of organizing a *Summer University on the History and Epistemology* in Mathematics Education belongs to the French Mathematics Education community of the IREM in the early 1980s. From these meetings emerged the organization of a Summer University on a European scale, as *the European Summer University* (ESU) on the History and Epistemology in Mathematics Education, starting in 1993. Since then, ESU was organized in 1996, 1999, 2004, 2007, 2010 and 2014 in different places in Europe: Montpellier (France), Braga (Portugal), Louvain-la-Neuve and Leuven (Belgium), Uppsala (Sweden), Prague (Czech Republic), Vienna (Austria) and Copenhagen (Denmark). By now, it has been established into one of the main international activities of the HPM Group (the *International Study Group on the Relations between the History and Pedagogy of Mathematics*), which is an affiliate of the International Commission on Mathematical Instruction. From 2010 onwards – the Summer University is organized every four years, so that every two years at least one major international meeting of the HPM Group will take place; namely, ESU and the HPM Satellite Meeting of ICME.

The purpose of ESU is not only to stress the use of history and epistemology in the teaching and learning of mathematics, in the sense of a technical tool for instruction, but also to reveal that mathematics should be conceived as a living science with a long history and a vivid present. The main idea of the Summer University is threefold: i) to provide a school for working on a historical, epistemological and cultural approach to mathematics and its teaching, with emphasis on actual implementation, ii) to give the opportunity to mathematics teachers, educators and researchers to share their teaching ideas and classroom experiences related to a historical perspective in teaching, and iii) to motivate further collaboration along these lines, among teachers of mathematics and researchers on history and education of mathematics in Europe and beyond. In accordance with this, the ESU is more a collection of intensive courses than a conference for researchers. More specifically, it is a place where teachers and researchers meet and work together. It is also a place where beginners, more experienced researchers and teachers present their teaching experiences to the benefit of the participants and get constructive feedback from them. It refers to all levels of education – from primary school, to tertiary education – including in-service teachers' training. The focus is preferably on work and conclusions based on actual classroom experiments and/or produced teaching and learning materials.

These Proceedings collect papers or abstracts corresponding to all types of activities included in the scientific programme of ESU 7: plenary lectures, panel discussions, workshops based on didactical, pedagogical, historical and/or epistemological material, oral presentations and posters. This volume is divided into eight sections, seven sections corresponding to each of the seven main themes of ESU 7, and a section for posters:

1. Tools of history and epistemology, theoretical and/or conceptual frameworks for integrating history in mathematics education;
2. Classroom experiments & teaching materials, considered from either the cognitive or/and affective points of view; surveys of curricula and textbooks;
3. Original sources in the classroom, and their educational effects;
4. History and epistemology as tools for an interdisciplinary approach in the teaching and learning of mathematics and the sciences;
5. Cultures and mathematics;
6. Topics in the history of mathematics education;
7. History of mathematics in the Nordic countries;
8. Posters

The reader of these Proceedings will find papers on most of the plenaries, workshops and presentations given at ESU 7. There were two panel sessions: one on *Technics and technology in mathematics and mathematics education* and one on *The question of evaluation and assessment of experiences with introducing history of mathematics in the classroom*. The reports from the panel discussions are found in the section for theme 1.

The number of participants to ESU 7 (2014) was 130 – a list of participants is found at the end of the Proceedings. It is important to remark that they came from many parts of the world, with many persons participating for the first time in a meeting organised by the HPM group.

We thank the Danish School of Education and their staff for financial support and help. A special thank is extended to Pernille Ussing-Nielsen and her staff of student helpers for extremely professional organization before, during and after the meeting. We also thank Taylor & Francis and Springer for supporting the conference.

Finally, we thank the members of the HPM Scientific Committee for the quality of the programme and of the reviewing process of these Proceedings.

THEME 1:

TOOLS OF HISTORY AND EPISTEMOLOGY, THEORETICAL AND/OR CONCEPTUAL FRAMEWORKS FOR INTEGRATING HISTORY IN MATHEMATICS EDUCATION

Plenary Lecture

PHILOSOPHIES AND THEORIES BEHIND HISTORY AND EDUCATION: THIRTY YEARS AFTER HANS FREUDENTHAL

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This paper goes back to a paper of the Dutch mathematician and philosopher Hans Freudenthal. We analyse and develop two purposes of this paper of 1983: the idea to not separate history and education in the reflection on mathematical education and the notion of anti-didactical inversion where this idea is active. We will examine four situations (1) Philosophy or theory behind History and Education (2) Didactics and History of Mathematics (3) Philosophy and Theory behind using of History in classrooms (4) Curricula, Didactics and History. We will continue with the notion of anti-didactical inversion to examine two orders of knowledge: historical and didactical orders. From this, we question the role of history of mathematics in the reflection on the curricula in mathematics.

INTRODUCTION: THE PAPER OF HANS FREUNDENTHAL (1983)

In the ICM Conference of 1983, Freudenthal presented a paper, titled « The Implicit Philosophy of Mathematics, History and Education ». He called "philosophy of history" "what we can learn from the history of old mathematics for the sake of teaching people […], one philosophy behind both history and education, or if they are two, that one is common to both" (Freudenthal, 1984, 1695). He stressed the relations between history and education, but more, he did not want to separate them in his reflection. He considered that the historical course could be used in teaching, but "people who teach mathematics as a ready-made system prefer anti-didactical inversion". He also noticed about the use of history of mathematics for teaching: "In fact we have not yet understood the past well enough to really give them [young learners] this chance to recapitulate it [the historical learning process]" (Freudenthal, 1984, 1696). Indeed, the history of mathematics is not an easy subject if we want to use it as a tool for teaching. In 1937, the historian of mathematics Gino Loria wrote: "always I did my best to prove to my students [future teachers] that history of mathematics is a very serious subject; which has to be studied very seriously" (Loria, 1937, 275). In a recent paper on the historical dimension in teaching, Niels Jahnke stressed: "History of maths is difficult!" (Jahnke, 1994, 141).

Freudenthal asked the question of the existence of either a philosophy or a theory behind history. For Imre Lakatos, "history without some theoretical 'bias' is impossible" (Lakatos, 1970, 107), while for the historian Paul Veyne "history has neither structure nor method and in advance it is certain that any theory in this domain is still-born" (Veyne, 1971, 144). Lakatos and Veyne represent two opposite conceptions, which do not lead to the same kind of history. In the first case, it is "a rational reconstruction of history" (Lakatos 1970), as Lakatos wrote, which explains features or reinforces a theory. In the second case, the history tells "an intrigue" to understand facts. Veyne criticized the introduction of theories or ready-made frameworks to write history. I introduced the idea of an *histoire dépaysante* in a paper[i] of 1991 (Barbin, 1991), where I quoted Veyne who wrote that "the event is difference and the characteristic effort of the historian's profession and what gives it its flavor are well known: astonishment at the obvious" (Veyne, 1971, 7). A "rational history" can be written with several kinds of theories: mathematics, didactics, sociology, psychology, etc. In this paper we will meet some didactical theories: theory of conceptions, realistic mathematics education, theory of beliefs and radical constructivism.

In his paper, Freundenthal discussed the notion of anti-didactical inversion, which he had written about in a book edited ten years before, *Mathematics as an educational task* (1973), and later in *Didactical phenomenology of mathematical structures* (1983). In this last book, opposing the mental objects to the mathematical concepts, he wrote:

> Children learn what is number, what are circles, what are adding, what is plotting a graph. They grasp them as *mental objects* and carry them out as *mental activities*. It is a fact that the concepts of number and circle, of adding and graphing are susceptible to more precision and clarity than those of chair, food and health. Is this the reason why the protagonists of concept attainment prefer to teach the number concept rather than number, and, in general, concepts rather than mental objects and activities? Whatever the reason may be, it is an example of what I called the *anti-didactical inversion*. (Freudenthal, 1999, x)

Teaching a concept rather than a mental object is an anti-didactical inversion. Here, this inversion reverses the convenient didactical order, which is the phenomenological one. The question of the order of knowledge in general had been a constant concern in mathematical teaching from the 17th century to the Reform of modern mathematics.

To examine philosophies or theories behind history and education, in each part of this paper, we will compare two authors – historians, philosophers, teachers or researchers in didactics – about history and didactics, use of history in classrooms, curricula and history. These authors had been chosen, to focus on the teaching of curve, tangent and function and the order of their knowledge. Many of them wrote on the methods of tangents of the 17th century, so we begin by presenting original texts written by Pierre de Fermat, René Descartes, Gilles de Roberval and Isaac Barrow.

METHODS ON TANGENTS IN THE 17TH CENTURY

There exist propositions on tangents in Greek Antiquity, but the authors didn't explain how they found the result and they prove them by reductio ad absurdum. The geometer of the 17th century researched direct methods to find the tangents: these are called methods of invention.

Fermat's method

Fermat's method of tangents appeared in a text of 1636, entitled "Method for maximum and minimum", and is an application of this last method.

> Let us consider, for example, the parabola with vertex *D* and diameter *DC*; let *B* be a point on it which the line *BE* is to be drawn tangent to the parabola and intersecting the diameter at *E*. We choose on the segment *BE* a point *O* where we draw the ordinate *OI*; we also construct the ordinate *BC* of the point *B*. We have then $CD / CI > BC^2 / OI^2$, since the point *O* is exterior to the parabola. But $BC^2 / OI^2 = CE^2 / IE^2$, in view of the similarity of the triangles. Hence $CD / CI > CE^2 / IE^2$.

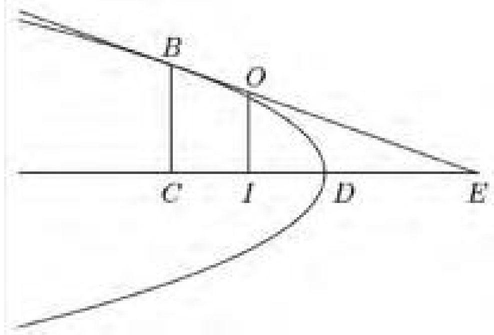


Figure 1. Fermat's tangent of a parabola

> Now the point *B* is given, consequently the ordinate *BC*, consequently the point *C*, hence also *CD*. Let $CD = d$ be this given quantity. Put $CE = a$ and $CI = e$; we obtain: $d / (d - e) > a^2 / (a^2 + e^2 - 2ae)$. Removing the fraction: $da^2 + de^2 - 2\,dae > da^2 - a^2e$. Let us then adequate, following the precedent method; by taking out the common terms we find: $de^2 - 2\,dae \approx - a^2e$," or, which the same, $de^2 + a^2e \approx 2\,dae$. Let us divide all terms by *e*: $de + a^2 \approx 2\,da$. On deleting *de*, there remains $a^2 \approx 2\,da$, consequently $a = 2d$ (Fermat, 1891, 122-123).

Fermat considered a point *B* on a parabola, the tangent *BE* and he choose a point *O* on this tangent. He knew the relations established by Apollonius to characterize the points of a parabola. The point *O* is exterior to the parabola, so by similarity of triangles *BCE* and *OIE*, he obtained the first inequality. He introduced letters and he transformed the previous inequality by another one between algebraic expressions. Then he applied the rules of his method of maximum and minimum. Now the inequality became what he called an *adequation*, He divided the two members by *e* and then deleted *e*. So he obtained an a*dequation* without *e* and transformed it in an equation that gives the result: *CD* is equal to *DE*.

Descartes'method

In his *Geometry* of 1637, Descartes gave a method to find a normal *CP* to a curve. The normal is the perpendicular to the tangent.

> Let *CE* be the given curve, and let it be required to draw through *C* a straight line making right angles with *CP*. Suppose the problem solved, and let the required line be *CP*.

Produce CP to meet the straight line GA, to those points the points of CE are to be related. Then, let $MA = CB = y$; and $CM = BA = x$. An equation can be found expressing the relation between x and y. I let $PC = s$, PA = v, whence PM = $v - y$. Since PMC is a right angle, we see that s^2, the square of the hypotenuse, is equal to $s^2 = x^2 + v^2 - 2vy + y^2$, the sum of the two squares. […]

For example, if CE is an ellipse, we have $x^2 = ry - (r/q)\, y^2$. By means of these last two equations, I can eliminate one of the two quantities x and y from the equation expressing the relation between the points of the curve and those of the straight line GA. Eliminating x^2 the resulting equation is $y^2 + (qry - 2qvy + qv^2 - qs^2) / (q - r)$. […]

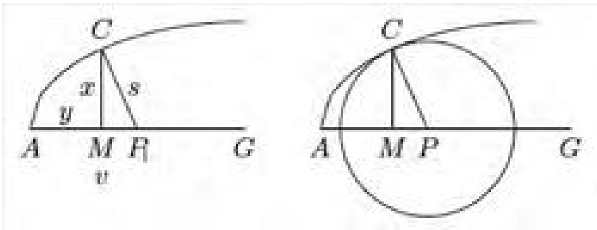


Figure 2. Descartes' method of tangents

Observe that if the point P fulfils the required equations, the circle about P as centre and passing through the point C will touch but not cut the curve CE [...]. It follows that the value of x, and y, or any other such quantity, that is, will be two-fold in this equation, that is the equation will have two equal roots. Furthermore, it is to be observed that when an equation has two equal roots, it must be similar in form to the expression obtained by multiplying by itself the difference between the supposed unknown quantity and a known quantity equal to it [...]. This last step makes the two expressions correspond term by term. For example, I say that the first equation found in the present discussion, […] must be of the same form as the expression obtained by making $e = y$ and multiplying $y - e$ by itself, that is $y^2 - 2ey + e^2 = 0$. We may then compare the two expressions term by term (Descartes, 1925, 342-348).

Descartes introduced letters for the coordinates of the point C, for CP and PA. With Pythagoras, he obtained a first equation. He took the example of an ellipse, for which the equation has two parameters r and q. He eliminated x from the two equations and obtained a new equation. Then he examined the situation where CP is the normal to the curve. In this situation, the circle about P as centre and passing through the point C will touch but not cut the curve CE. Thus, the last equation must have two equal roots. Indeed this equation is satisfied for points both belonging to the curve and to the circle. Then Descartes observed that when an equation has two equal roots, it must be similar to certain expression. For the example, the equation has to be similar to an equation that has two roots equal to e. By comparing the two equations term by term, Descartes obtained the unknowns v and s, and so the position of the normal CP.

Roberval's method

Roberval invented his method around 1635, but his "Observations on the composition of movements and on the means to find the tangents to curves" were edited in 1693.

> Axiom or principle of invention. The direction of a movement of a point, which describes a curve, is the tangent of the curve in each position of this point.
>
> General rule. From the specific properties of the curved line (which you will be given) examine the different movements, which the point describes where you wish to draw a tangent: from all these movements compose one single movement, draw the direction of that movement, and you will have the tangent to the curved line.
>
> First example of the tangent to the parabola. It is clear by the above description that the movement of *E* which describes the parabola is composed of the movements of two equal straight lines, the one is the line *AE*, the other is the line *HE* on which it moves with the same velocity than the point *I* in the line *BA*, which is the same than the one of the line *AE* by construction, since always *AE* is equal to *BI*. Accordingly, since the direction of the equal movements is known, that is along the given straight lines *AE*, *HE*, if you divide the angle *AEH* in two [equal] parts by the line *CE*, [...] the line *EC* is the tangent (Roberval, 1693, 80-81).

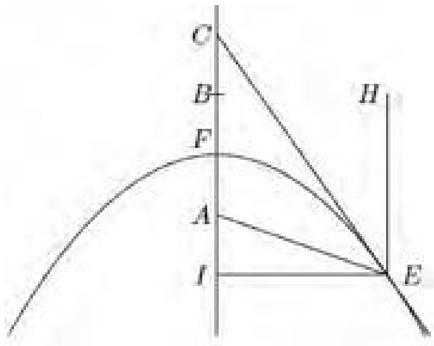


Figure 3. Roberval's tangent to the parabola

Roberval's general rule to find tangents to a curve contains three steps: to examine the different movements of the point describing the curve, to compose them in one single movement and to draw the direction of this movement. Roberval's first example is a parabola. He knew the characteristic of the point of a parabola given by the equality of the distances *EA*, of E to the focal A, and *EH*, of *E* to the perpendicular at the axis passing by B. He concluded that the movement, which describes a parabola, is composed of two movements, one in the direction of *EA* and the other one in the direction of *EH*. Since the segments are equals, the bisector is the tangent.

Barrow's method

Barrow introduced "indefinitely small" parts of tangent and curve in his *Lectiones geometricae* of 1670:

> Let *AP*, *PM* be two straight lines give, in position of which *PM* cuts a given curve in *M*, and let *MT* be supposed to touch the curve at *M*, and to cut the straight line at *T*. "In order

to find the length of the straight line *PT*, I set off an indefinitely small arc, *MN* of the curve; then I draw *NQ*, *NR* parallel to *MP*, *AP*. I call $MP = m$, $PT = t$, $MR = a$, $NR = e$, and other straight lines, determined by the special nature of the curve, useful for the matter in hand, I also designate by name; also I compare *MR*, *NR* (and through them, *MP*, *PT*) with one another by means of an equation obtained by calculation; meantime observing the following rules.

I omit all terms containing a power of *a* and *e*. I reject all terms which do not contain *a* and *e*. I substitute *m* for *a* and *t* for *e*. So *PT* is found and the tangent is obtained.

1. In the calculation, I omit all terms containing a power of *a* and *e*, or products of these (for these terms have no value).

2. After the equation has been formed, I reject all terms consisting of letters denoting known or determined quantities or terms which do not contain *a* or *e* (for these terms, brought over to one side of the equation, will always be equal to zero).

3. I substitute *m* (or *MP*) for *a*, and *t* (or *PT*) for *e*. Hence at length the quantity of *PT* is found (Barrow, 1670, 80-81).[ii]

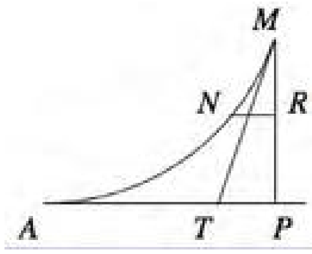


Figure 4. Barrow's method

Barrow considered an indefinitely small arc *MN* of the curve. He associated letters to the segments of the figure: *NR* is called *e*. He compared the sides *MR* and *NR* of the triangle *MNR* to the sides *MP*, *PT* of the triangle *MPT*. Like in Fermat's method we have to observe rules. As *NM* is indefinitely small, he considered it as a straight line and used similar triangles of the figure.

PHILOSOPHY OR THEORY BEHIND HISTORY AND EDUCATION

We begin by comparing two authors concerning the philosophy or theory behind history. We then continue with two authors often quoted in research in didactics.

Histoire dépaysante against rational history

Léon Brunschvicg was a French philosopher who wrote a book on "the steps of the mathematical philosophy" in 1912. Derek Whiteside was an English historian who wrote a paper on the "patterns of mathematical thought" in 17th century in 1962.

Brunschvicg used the experience of history against a "pedagogical tradition of the philosophy" and the "dogmatic systems", he wanted to write the history of a "collective acquisition of knowledge between incidents of the invention and forms of the discourse" (Brunschvicg, 1912, 459). Whereas Whiteside wrote "a study of the particular mathematical forms which developed in the 17th century with emphasis on their interconnections rather than on their philosophical aspects", and wanted "to isolate significant trends of development" (Whiteside, 1962, 179). The results are very different, but we will compare them on two points only. Brunschvicg wrote a *histoire dépaysante,* where he gave the words exactly used by the authors and long quotations for Fermat and Descartes. It is also a history oriented on the research of differences. Brunschvicg compared the mathematical materials used and the intentions of the geometers, examined the disputes between them about the value of the methods.

Whiteside didn't research differences, but similarities. Thus, he pointed to the "slight differences of treatment required in the two approaches" of Fermat and Descartes. To obtain this result, he translated the texts into the modern language of limits, which leads to a none disorienting reading of the texts. He concluded his paper with a continuous, recurrent and limited view of history: " In fact – and in summary – what was done in 17th century mathematics [...] was sufficient to provide rich pickings for 18th century mathematicians seeking a lead into the unknown" (Whiteside, 1962, 384).

Philosophy behind History and Education

Raymond Louis Wilder was an American mathematician interested by philosophy, he wrote in 1972 a paper "History in the Mathematics Curriculum: Its Status, Quality, and Function", also *Evolution of mathematical concepts* (1969) and *Mathematics as a cultural system* (1981). Gaston Bachelard was a French philosopher, he wrote many books, and two has been translated into English: *The formation of the scientific mind* (1938) and *The new scientific spirit* (1934).

In his paper of 1972, Wilder researched the necessary conditions to introduce history of mathematics in curriculum and he wrote:

> Actually, the standpoint from which I believe we should present the history of mathematics is at an even higher level than mathematics. By this I mean, to take a broad view of mathematics as a living, growing organism, which is continually undergoing evolution; in short we should study it as a culture (Wilder, 1972, 483).

He described this evolution by giving the stages in evolution of geometry, of real number system, aspects of reality, etc. He described the "forces of mathematical evolution" like "environmental stress", "hereditary stress", etc. and he explained the evolution inside these frameworks.

The purpose of Bachelard was not to establish a Curriculum, but he thought that history of sciences could help students "to learn to invent":

> Teaching about the discoveries that have been made throughout the history of science is an excellent way of combating the intellectual sloth that will slowly stifle our sense of mental newness. If children are to learn to invent, it is desirable that they should be given the feeling that they themselves could have made discoveries (Bachelard, 1991, 10).

It is also meant to disorientate (*dépayser*) the teachers: "we must also disrupt the habits of objective knowledge and make reason uneasy. This is indeed part of normal pedagogical practice"(Bachelard, 1991, p.245). Bachelard stressed on the polemical character of knowledge. For him, "scientific operation is always polemical; it either confirms or denies a prior thesis, a pre-existing model, an observation protocol; [...] it reconstructs first its own models and then reality" (Bachelard, 1984, 12-13).

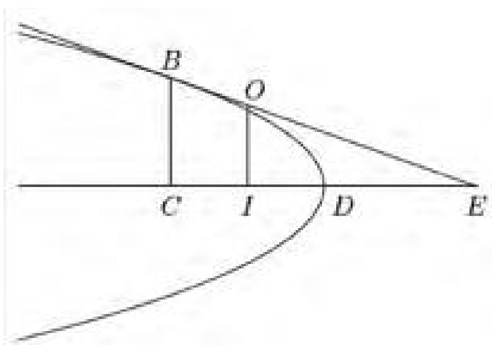
His epistemology is inscribed in a negative philosophy, an open philosophy which struggles against the tendency to systems, against positivism and empiricism, like we read in his book *Philosophy of no*. It is an epistemology of the difference and of rupture: "Specifying, rectifying, diversifying: these are dynamic ways of thinking that escape from certainty and unity, and for which homogeneous systems present obstacles rather than imparting momentum"(Bachelard, 1991, 27). It is both a constructivist and historical epistemology, where Bachelard introduced the notions of epistemological obstacle and rectification of knowledge, and stressed the role of problems in the historical construction of the sciences.

DIDACTICS AND HISTORY OF MATHEMATICS

Maggy Schneider is a Belgium researcher in didactics. In her thesis of 1988, she examined the difficulties of students to find the tangent from the calculus of the slope. Michèle Artigue is a French researcher in didactics, who wrote in 1990 a paper on relations between epistemology and didactics.

A question of order: comparing Fermat's and Barrow's methods for tangents

To explain the difficulties of students to obtain tangent from the calculus of the slope, Schneider explained that for the students, the tangent is a "mental object" linked with the idea of slope, while the infinitesimal calculus begins with the derivate number (Schneider, 1988, 291-292). Thus, the phenomenological order goes from the notion of tangent to the notion of slope, while the anti-didactical order (which is the scholarly order) goes from the calculus of the slope to the tangent. It is a case of an anti-didactical inversion. For Schneider, history helps to understand the difficulties of the students by comparing Fermat's and Barrow's methods for tangents.



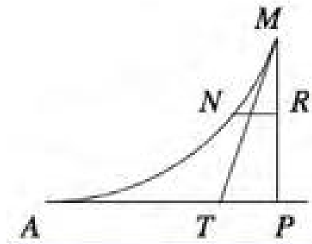


Figure 5. Fermat's and Barrow figures of tangents

She explained the difficulties by the fact that “the pupils seem nearer to Fermat”. Indeed, Fermat doesn’t use the slope, while in the procedure of Barrow, the triangle MNR gives the slope. Schneider didn’t use the Barrow’paper but a simple and short explanation given by the historian Morris Kline (1972). So, she didn’t mention that the geometer introduced “indefinitely small” parts of tangent and curve (see above). Thus, she did not examine the relative questions: what is a curve for Fermat and Barrow? Or for the students?

Nine conceptions on tangent of a curve

In her paper, Artigue re-situated “the trajectory of the notion of conception” in the French didactical community in ten pages, with the purpose of grouping “in relevant class for didactical analysis” the multitude of conceptions on a given object (Artigue, 1990, 265). For her, a historical analysis can show the diversity of the “points of view” on the “object” of tangent. In consequence, she gave a catalogue of nine conceptions on tangent and the names of the mathematicians associated which them.

For instance she wrote that for Euclid a straight line is tangent to a curve when having a common point with the curve, we cannot lead any straight line between the curve and the tangent at this point. Here, we recognize a result proven in the *Elements* for the tangent of the circle, but it is not the definition of the tangent. She wrote also that for Descartes, a straight line is tangent to a curve if it has a common point with the curve and is perpendicular to the normal in this point. Here the question became to know what is a normal for the geometer. She added that “this generalizes the notion of tangent to a circle via the osculatory circle” and so Descartes finds the tangent of a cycloid in Book II of his Geometry (Artigue, 1990, 275). It is a very modern reading of the original text, which causes confusion, since the Book II treats only the algebraic curves and so the cycloid cannot be there. For Roberval, Artigue wrote: “the tangent to a curve in a point M is the vector velocity in M of a moving point describing a curve”. The notion of vector velocity arrived only in the end of the 19th century, so the purpose is not to render a comprehensive or disorienting history. The purpose is situated in the field of the theory of didactics, and the researcher concluded that the notion of “conception” corresponds to an “intermediary level in the operational effectiveness of the didactical analysis”.

Nicolas Rouche, the director of the thesis of Schneider, followed Freudenthal when he asked: what can we learn from educating the youth for understanding the past of mankind? This idea is the contrary of the usual one, which is that we can learn from the past for education. Bachelard is close to this when he remarked: “the idea of the epistemological obstacle can be examined in the historical development of scientific thought and also in educational practice” (Bachelard, 1991, 27). For them, the purpose was not to separate history and education. On the contrary, by referring to the theory of Yves Chevallard, Artigue separated epistemology and didactics:

> The student cannot be reduced to an epistemic subject or to a cognitive subject. His behaviour is also and almost determined in priority by his status of didactical subject. [The epistemological analysis] shows all that separates these two fields: the epistemological one and the didactical one. This is this fact, which is at the centre of the theory of Y. Chevallard already quoted (Artigue, 1990, 278).

HISTORY OF MATHEMATICS IN CLASSROOMS

Laurent Vivier is a French teacher, who wrote a paper in 2010, on "a theoretic background on the notion of tangent in the secondary teaching" (Vivier, 2010), he proposed to solve a problem on the teaching of tangent by the Descartes' method on tangents. Evgenios Avgerinos and Alexandra Skoufi are teachers in Rhodes. In a paper of 2010 "On teaching and learning calculus using history of mathematics: a historical approach of calculus", they used the Fermat's method on tangents.

Descartes' method in classroom: an adaptation

Laurent Vivier tried to solve one problem of teaching, which is how to introduce the notion of tangent before the notion of derivative? It is a question on an anti-didactical inversion since in the French Curriculum, the derivate calculus is presented before the tangent and used to find tangents. He considered that a historical light would permit to define an alternative teaching: thus, history is used against an anti-didactical inversion. For this purpose, he compared Descartes and Fermat's methods from the point of view of a teaching approach. For him, Descartes' method has the advantages to correspond to a properly defined class of curves, to be an entirely algebraic method and to be easily understood. He remarked that it could be adapted to find a straight line and not a circle which is tangent to a curve. While, Fermat's method permits us to find tangents to algebraic curves easily, it has the disadvantages to be difficult to explain and "it is already in analysis". Moreover, Fermat didn't give a class of curves for which the method works.

Vivier adapted the Descartes' method by intersecting the curve by a straight line, here a parabola. We can note that Descartes used also this method in his correspondence. He proposed a problem to his students where he considered a parabola $y = x^2$, a point A with coordinates (a, a^2) and the secants passing through A whose the equations are $y = k\,(x - a) + a^2$. The question is to find the tangent among the secants. Vivier concluded his paper by this question: what is a curve?

Fermat' s method in classroom: a rational re-construction

Avgerinos and Skoufi introduced the teaching of differential calculus inspired by the principles of the theory of Realistic Mathematics Education of Koeno Gravemeijer, which promotes real situations in teaching. They wrote:

> Fermat discover how applies the [method of maximum and minimum] before in extrema process of neighbouring points, using the mysterious E, for finding tangent line of a curve

> $y = f(x)$. Let P (a, b) is a point of parabola and P' a neighbouring point in curve with coordinates $(a + E, f(a + E))$. If the P' be found too much near the P then could one say that the secant PP' coincides with the tangent in the P (Avgerinos & Skoufi, 2010, 94)

The authors proposed a rational re-construction of Fermat's method of tangent, where they used coordinates and function symbolisms. Further, they used the slope of the tangent, trigonometry and finally the notion of limit. It is not a *histoire dépaysante*, in despite or because they had been disoriented by the "mysterious E" of the method. They guided students to apply the method to the function $f(x) = -x^3$. They considered a point P of the curve with coordinates (x, y) and a neighbouring point P' with coordinates $(x + E, f(x + E))$, T the section of tangent with x-axis and $TQ = c$. The students are asked to calculate the ratio $f(x) / c$, then "to set inside" $E = 0$ to find the result. In this re-construction, the difference between a curve and a function is not examined, nor the history of the concept of function.

In contrast to Avgerinos and Skoufi, Schneider used Fermat's method to understand the students (see above), because there is no slope in the procedure of Fermat. We have two completely different readings of Fermat's method. In the historical reading of Schneider, the method is linked with the "mental object" tangent of the students. So the students are nearer to Fermat, because for them the notion of slope is not associated with the notion of tangent. In her study, the history comes against an anti-didactical inversion because it permits to understand the difficulties of the pupils with the order of the Curricula, which goes from the analysis to the slope. While the modern interpretation of Avgerinos and Skoufi obeys and reinforces the anti-didactical inversion.

CURRICULA, DIDACTICS AND HISTORY

Anna Sierpinska is a researcher in didactics who works on understanding. In one of her first papers "On understanding the notion of function" of 1992, she wrote on the relations between history and didactics. David Dennis is a researcher in mathematics and science education, he wrote in 1995 a thesis on historical perspective for the curriculum titled "Historical perspective for the Reform of Mathematics curriculum geometric curves drawing devices and their role in the transition to an algebraic description of functions".

From « epistemological obstacles » to a theory of « beliefs »

The purpose of Sierpinska' paper concerns the evaluation of teaching: "any evaluation of a teaching design [...] has to be based on a framework that is external to it. We must have some theory about understanding and about understanding functions against which to construct or to evaluate our projects" (Sierpinska, 1992, 25). It became a theoretical problem in scientific didactics. In her paper, she introduced a notion of epistemological obstacle: "If once, we know in a new way, we contemplate our old ways of knowing and what we see are things that prevented us from knowing in a new way. Some of these things may be qualified as epistemological obstacles" (Sierpinska, 1992, 27). Her notion of obstacle is not the same as that of Bachelard, because it is something to avoid, while for Bachelard the obstacles are normal components in the process of knowing. She distinguished three levels to explain obstacles: "attitudes, beliefs and convictions", together with "schemes of thought" and "technical levels". Then she stressed the role of beliefs and schemes of thought, since, as she explained, an obstacle will be overcome if we are able to stand back from our beliefs or scheme of thought, if we see their consequences and are able to consider other points of view.

To develop and reinforce her theory, Siepinska employed the history of mathematics. She wrote that the first definitions of the concept of function presented it as an algebraic expression. Below, we will see that the history is more complicated. Then she gave some definitions of the concept of function, those of Johan Bernoulli, Leonhard Euler (in his *Introductio*), Louis Lagrange and Augustin Louis Cauchy, to conclude that mathematicians have always researched to describe relationships. For her, curves are not interesting by themselves in history but they provided a context in which analytic tools for describing relationships could be developed. She added that Leibniz introduced his calculus and the first definition of function in the context of analytical geometry and that it is in this context that he and Bernoulli coined the term « Function » and came to formulate its first definition, but it is not exact as we will see, since the context was geometrical only.

Sierpinska saw the geometric diagram of a function as an epistemological obstacle. As she explained, students happen to identify functions with the geometric diagrams sometimes used to represent them, some students view the diagrams in "synthetic and concrete way", other students have "a more analytic view of analytical representations of functions" but "the line does not represent the relation" and "rather the line is represented by the relation". For her, the didactical order, which goes from function to curve, is not questioned. Moreover, she thought that it is the historical order from some definitions of the 18th century. The idea that this order would be an anti-didactical inversion does not emerge.

On curves and functions: epistemological versus historical studies of concepts

In a part of the paper of 1992, titled "epistemological studies versus historical studies of concepts", Sierpinska wrote:

> An epistemological study of a concept differs from its history. Histories of a mathematical concept are usually presented as if the concept's development followed a smooth curve with positive gradient. Learning cannot be thus modelled. At greater cognitive depths catastrophe occurs (Sierpinska, 1992, 58).

By these words, she separated history and education, contrary to Freudenthal. The issue is that she did not criticize the few historical works that she read. Yet, as we already saw, the history of mathematics depends on the historian. Probably, she read authors like Wilder but not others, like Brunschvicg. From this point of view, to come back to the Leibniz's texts themselves is interesting, as we have already seen.

Leibniz gave a first definition of function in a paper of 1694 "[On] constructions of a curve from a property of its tangents", but he used the word "function" in 1673 and in 1692 with the same meaning and about the same problem, the inverse problem of tangents. The inverse problem of tangents is a geometrical problem, which consists to find a curve when the tangents in each point are known.

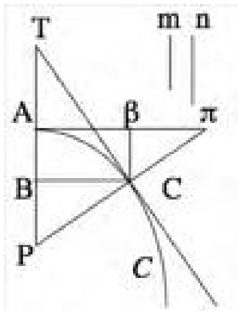


Figure 6. Leibniz's geometrical figure for the definition of function

In 1694, Leibniz wrote: "I call function a finite straight line exclusively determined from straight lines drawn from a fixed point to a given point of the curve. Coordinates *CB*, *Cβ*, tangent *CT*, sub-tangent *BT*, normal *CP* are functions of the point" (Leibniz, 1989, 271). That means, that the context is not algebraic but geometrical. The calculus was invented to solve problems on curves and not problems on functions. More precisely, Leibniz introduced the notion of function to solve a difficult problem, which is to find a curve tangent to a family of circles. For this purpose he used his calculus, and two ways to characterize a curve, which are a differential equation or a series. Twenty years later, Bernoulli gave another definition in a paper "On the isoperimetrics" of 1718: "Definition. We called function a variable magnitude or quantity composed in any manner of this variable magnitude x and of constants F x". He did not indicate the manner, but of course he did not only consider algebraic polynomial equations. In his *Introduction to Analysis of the Infinite* of 1748, Euler called function an analytical expression but he changed, after the controversy on the vibrating strings, to define a function as a dependence between variables.

History against anti-didactical inversion

David Dennis applied the ideas of the "radical constructivism" of Jere Confrey, who used epistemological arguments to critique standard historical descriptions of mathematics and used history to describe, examine and legitimise students' conceptions. Dennis saw history as a source of contexts and activities, his intention is to use mathematical history "to create a broad and flexible notion of how language evolved in response to activities and experience". He knew and criticized the use of history of mathematics by Sierpinska because she briefly describes a variety of historical conceptions of functions and gives some details from original sources, but as he wrote, "her overall theory of history and its relations to education, remain progressive-absolutist". He added:

> History is not seen by her as a source of conceptual diversity, but almost as a set of pitfalls to be avoided or overcome. She suggests that some sense of history can be useful in helping students to overcome these possible obstacles, but there is no indication that student investigations within a given historical conception might offer valuable insights that are obscured by modern conventions (Dennis, 1995, 27).

For him, the purpose of historical and investigations is quite different, in particular:

> Historical discussions of the social and technological history of the scientific revolution would connect such mathematical investigations directly with larger cultural issues, but most importantly these investigations would provide students with more appropriate, dynamic, geometric experience (Dennis, 1995, 200).

Dennis questioned the teaching of the concept of function: "a fundamental goal of mathematics education is for students to develop an understanding of the concept of a function. In mathematics classrooms curves are usually created from algebraic equations or numerical data, and only rarely by physical or geometric actions" (Dennis, 1995, p.198). Like Schneider, he asked that a pedagogical problem should be linked with the curricula. He remarked that, even before algebraic equations are found, one can often determine tangent lines, areas between curves, and arc lengths of curves, all from an analysis of the actions which produced the curves. History shows that, as we saw with the methods of tangents. Thus, his question concerns an anti-didactical inversion and opposes an historical order of knowledge and to a didactical order, which goes from function to curve. Thus, he was interested by curves themselves. Here, history is used against an anti-didactical inversion and as a tool to criticize Curricula:

> What is governing our choice of curriculum? It would seem to be regulated by algebraic convenience. Students are asked to consider many curves that I have never seen in daily life, simply because their equations are tractable.
>
> The role of functions as conceptual tools for the analysis of curve drawing actions reverses the usual epistemic role that they play in current curriculum where functions are used to create curves (Dennis, 1995, 175).

That means that, contrary to Chevallard's theory, the cognitive subject can be more important than the didactic subject. In his thesis, Dennis gave numerous and various examples of construction of curves in history: curves are the heart of the learning of analysis.

ANTI-DIDACTICAL INVERSION: ON HISTORICAL AND DIDACTICAL ORDERS

We come back to the anti-didactical inversions met until here. We saw that Schneider considered the order from slope to tangent, Vivier the order from calculus to tangent and Dennis the order from function to curve. All these inversions concern the order between notion of function and notion of curve. The historical order goes from the notion of curve to the notion of function, but between them there are constructions of concepts of curve in the years 1630, which are strongly linked with the methods of tangents (Barbin, 1996). Here I distinguish notion and concept in this manner: a notion takes is meaning in relation with problems (to solve them) and a concept takes its meaning in relation with concepts into a theory. For the curve, for instance, we can speak about a notion of parabola as a way to solve the problem of the duplication of a cube, but it appears as a concept in the Apollonius' *Conics*.

Fermat and Barrow proposed a notion of curve in their works on tangents. Following the dispute on tangents between Descartes and Fermat, the latter one felt obliged to give " a foundation" to his method (Barbin, 2015). He wrote in a paper titled "On the same method": "we suppose the tangent already found at a given point on the curve, and we consider by adequality the specific property of the curve, not only on the curve itself, but on the tangent to be found (Fermat, 1981, 141). As we saw, Barrow consider that "an indefinitely small part of the tangent can be substituted for an indefinitely small arc of the curve". That means that in these two methods, a curve can be considered as composed by parts of tangents. This notion of curve permits them to give an account for the procedures of their methods.

Descartes gave two definitions of what he called a "geometric curve". In the second Book of his *Geometry* he characterized them by this way: "they can be conceived as described by a continuous motion or by several successive motions, each motion being completely determined by those which precede; for in this way an exact knowledge of the magnitude of each is always obtainable" (Descartes, 1925, 316). But some lines later, he added: "all points of those curves which we may call 'geometric' that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation". He did not prove that these two definitions, one in terms of motions and the other in terms of equations, are equivalent. But he gave some examples, where he defined a curve by motions and obtained an equation for the points of the curve.

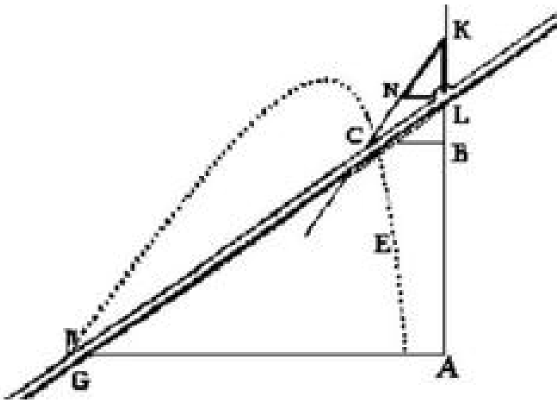


Figure 7. A Descartes' curve defined by motions

In the first paper "Nova methodus" of 1684, where Leibniz introduced his calculus, he explained that his method does not only concern curves associated to algebraic equations, but also the others, what he called "transcendental curves". He wrote:

> It is clear that our method also covers transcendental curves – those that cannot be reduced by algebraic computation, or have no particular degree – and thus holds in a most general way without any particular conditions.
>
> In its principle, to find a tangent consists of drawing a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinitive number of angles, which for me is equivalent to the curve.[...] We can always obtain the value of *dx* : *dy*, the ratio of *dx* to *dy*, or the ratio of the required *DX* to the given *XY* [*dx* : *dy* :: *DX* : *XY*] (Leibniz, 1989, 110-111).

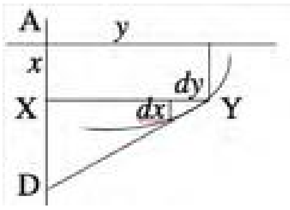


Figure 8. The infinitesimal triangle in the Leibniz's method

A curve can be considered as a polygon with an infinite number of infinitely small sides. Leibniz used the similarity of the infinitesimal triangle, with slides *dx* and *dy*, and the triangle *XDY* to establish the fundamental proportion of his calculus.

Descartes distinguished two kind of curves: the "geometric curves" described with motions, where each motion is completely determined by the others, and the "mechanic curves", which are described by independent motions – like the spiral. Moreover, he announced that the geometrical curves are expressed by algebraic equations. Leibniz also distinguished two kinds of curves: the curves associated with algebraic equations called "algebraic curves", and the others, called "transcendental curves". Thus, the mechanic curves of Descartes are not the transcendental curves of Leibniz, because Descartes always considered the curves as produced by motions, while for Leibniz, all the curves had a "regular rule": algebraic equation, differential equation or series. This distinction is important in a pedagogical context, but also in a historical context, since Leibniz and Newton researched different ways to construct curves, despite of their calculus (Bos, 1986, Knobloch, 2006). The concrete production of curves enlarges the teaching to the studies of motions and optics, to physic problems where the unknown is a curve (Barbin, 2006).

History shows the role of the methods of tangent in the construction of a concept of curve, linked with a concept of tangent, into a theory. In Roberval, a curve is the trajectory of a point in motion, the tangent is the direction of motion and the method is cinematic. The concept of curve takes it's meaning into a theory of the cinematic. In Descartes, a curve is described by an equation, the tangent is obtained thanks to the equation of a circle and the method is algebraic. The concept of curve takes it's place in algebra. In Leibniz' infinitesimal method, a curve has to be conceived as a polygon with infinitively small sides, the tangent is one of its sides and the method use infinitesimal magnitudes. The new theory is the calculus of differences.

Conclusion: Curriculum and anti-didactical inversions

The historical order is not the order proposed in the Curricula, and we observe that anti-didactical inversions is a subject of many works – some of them are examined in this paper. But, accordingly with Freudenthal, a historical course would be used for teaching. Vivier examined this possibility locally by adapting Descartes' method, and Dennis did more radically. But is it possible to use history of mathematics without changing the Curricula? The answer given by Avgerinos and Skoufi consists in a re-construction where the result is a hybridization, not necessarily comprehensive by students and not more efficient than the classical calculus. As we saw also, history of mathematics is used in didactics research, more often to evaluate or to reinforce didactical theories than to construct a teaching method. In this kind of research, history and education are separated, contrary to the Freudenthal' s philosophy.

The question can be also asked in another manner: how teachers and researchers have to advance in face of Curricula, which are producers of anti-didactical inversions? Can history be used and adapted in any Curriculum? What will be the meaning of these changes? What will be the results? Luis Radford is a researcher in science of education who examined these questions in a paper of 1997, he wrote:

> The way in which an ancient idea was forged may help us to find old meanings that, through an adaptive didactic work, may probably be redesigned and made compatible with modern curricula in the context of elaboration of teaching sequences [...] in order to reconstruct accessible presentations [of history] for our students (Radford, 1997, 32)

The proposal would be to reconstruct history of mathematics, to render it compatible with curriculum thanks to didactic works. With the examples given in this paper and others, we can imagine the danger of a terrible anti-didactical inversion, which would be the "didactical transposition" of history of mathematics. As Freudenthal wrote in 1986, Chevallard's didactical transposition is "the expression of an anti-didactical conception" (Freudenthal, 1986, 327). But, another axis would be to consider history of mathematics as a source to construct new Curriculum, introducing a most important interdisciplinary and cultural part in teaching. Dennis asked what is governing our choice of curriculum. We can add why should we prefer to see a student as a "didactical subject" rather than as a "cognitive or epistemological subject". Indeed, the question of the order of knowledge is an important one in teaching, linked with epistemological ideas of simplicity and generality, which concerns the comprehension of students inside mathematics, but also in relations with other scientific fields. Freudenthal's paper has the virtue of stressing the role of history of mathematics to examine the anti-didactical inversions but also to propose a reflection on the order of knowledge in Curricula.

Acknowledgment. I thank Leo Rogers very much for his comments and his help to write the paper in English.

NOTES

[i] It was an answer to the researcher in didactics Yves Chevallard, about two manners to make history with "bare hands" or with "hands full", full of didactical concepts.

[ii] translated in Struik, D. J. (1969), p.259.

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Panel Discussion 1

TECHNICS AND TECHNOLOGY IN MATHEMATICS AND MATHEMATICS EDUCATION

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The use of computer technology for teaching and learning of mathematics has several consequences and does sometimes give rise to both controversies and misunderstandings. We address these problems by both a philosophical and a historical approach, investigating what it actually is that goes on when new technologies enter mathematics as a discipline and mathematics education as a societal practice. Our analysis suggests a focus on continuities in time and place in the sense that it is necessary to understand the history of "tool use" in mathematics and the various ways that scholastic and non-scholastic mathematical practices adopt such tools. Furthermore we point to the strong interrelation between mathematics as a body of knowledge, mathematical activity and the technologies used for mathematical work. Finally we discuss how different theoretical lenses and epistemological outsets give rise to different guidelines and conclusions regarding the use of computer technology in mathematics education.

TECHNOLOGY IN MATHEMATICS EDUCATION; BEYOND PRO AND CON

Despite 30 years of use in mathematics education and substantial research and development activities, computer technology has not brought the positive changes originally envisioned (Artigue, 2010, Hoyles, 2014). In this plenary panel discussion we have allowed ourselves to take a helicopter view on the understandings of the uses of technology and ask some of the big questions that become apparent. In a sense we wish to understand why the use of technology in mathematics education can give rise to such hopes and at the same time be considered as a major disappointment. The panel should bring us further in an understanding of how to conceptualize the use of computer technology in the teaching of mathematics, and illuminate the debate pro and con the use of such technologies for teaching mathematics. In the panel we address the following questions:

- How, and to what extent, does the use of computer technology in mathematical activities change *mathematical work processes*, what *mathematics is* and how it is *understood and learned*? More specifically:

  - **1a.** How does the use of computer technology in mathematical activities change *mathematical work processes?*
  - **1b.** How does the use of computer technology in mathematical activities change what *mathematics is*?
  - **1c.** How does the use of computer technology in mathematical activities change how mathematics is *understood and learned*?
- Is the use of computer technology in mathematics and mathematics education best viewed as *in continuity* with *or* as a break away from the use of non-computer technology*?*
- How can different theories describe doing and learning mathematics with computer technology?

We have struggled to negotiate a version of the questions that can be embraced by all of us. And we do suggest that any attempt to answer these questions will at least allow a more fine-grained discussion of the reasons for bringing computer technology to the mathematics classroom as well as an increased understanding of the resulting changes to classroom practice.

UNDERSTANDING THE QUESTIONS

When we tried to answer the questions, we realized that all of the question could be answered both from the perspectives of activities in education and from the perspective of activities in mathematics (such as we have asked the second question). However this leads to another unclarity – what is meant by *in education* and *in mathematics?*

This unclarity invites us to consider mathematical practices in various settings. For simplicity we will talk about *educational settings* and *research settings.* Furthermore the educational setting refers both to students at different levels and to teachers of mathematics. Of course aspects of *vocational/work life*, *citizenship*, and *private life* also involves mathematics, but for the sake of simplicity we will address the questions from three perspectives: *researchers of mathematics*, *the mathematics student*, and *the mathematics teacher*. And hence our discussion speaks into the organization suggested by table 1.

	Mathematics Student	Mathematics Teacher	Mathematician
How the use of computer technology in mathematical activities changes *mathematical work processes*	Addressed in the section "technology and mathematical work processes"		
How the use of computer technology in mathematical activities changes what	Addressed in the section "technology and the nature of mathematics"		

mathematics is	
How the use of computer technology in mathematical activities changes how mathematicsis *understood and learned?*	Addressed in the section: "How is the use of computer technology in mathematical activities changing how mathematics is understood and learned"
Is the use of computer technology in mathematics and mathematics education best viewed as *in continuity with* or as a *break away* from the use of non-computer technology?	Addressed in the section: "computers as continuity or rupture in the development of mathematics"
How can different theories describe doing and learning mathematics with computer technology?	Addressed in the section: "how do different theories describe doing and learning mathematics with computer technology?"

Table 1: Matrix showing different approaches to the question of technology and mathematics learning.

Finally the perspective that we take also affects our possible answers. The questions mean different things if addressed from specific theoretical perspectives, and they hence have different answers. In our panel debate we have addressed the questions from cognitive, didactical and disciplinary perspectives. We will not fill out the entire matrix from each perspective. Rather we will use the matrix as a guide to navigate when several approaches address the same question. In the following we shall address the questions one by one.

TECHNOLOGY AND MATHEMATICAL WORK PROCESSES

We address the question of technology for mathematical work process from the perspective of students, teachers and researchers work processes.

Students' mathematical work processes

In general, the use of computer technology promotes the emergence of new solving techniques, which can facilitate many calculations to the students (Lagrange, 2005).

These instrumented techniques allow students to try many individual cases eventually reaching generalizations; Trouche et al. (1998) (cited in Lagrange, 2005) showed for example how some students obtained an expression of the n^{th} order derivative of $(x^2 + x + 1)e^x$, by reflecting on several particular cases using a calculator with CAS (Computer Algebra System) capabilities.

The distribution of algebraic and arithmetic work to computer technologies does give rise to some problems, certain tasks and topics (for instance trigonometric triangle calculations) cannot be worked with by students in meaningful ways, and do not train the algebraic skills they did in previous technological situations (Misfeldt, 2014).

Computer technologies provide users with different representational resources and new possibilities for using familiar forms of representation (Morgan, Mariotti & Maffei, 2009). For example, students can quickly draw graphical representations of mathematical objects, but they can also manipulate and explore these representations dynamically. While the benefits of such resources seem intuitively clear, it should be pointed out that there still lacks a proper understanding of external cognition and how graphical representations work (Scafie & Rogers, 1996).

Finally computer technology can also modify mathematics students' study processes. Due to the omnipresence of the Internet and mobile devices, students can have immediate and unlimited access to various sources of mathematical information. Contemporary mathematics students rely on non-traditional sources of mathematics. For instance, the study of van de Sande (2011) shows how mathematics students from different regions of the world are turning to Internet-based open forums looking for advice that could help them to solve their doubts related to their mathematical tasks.

Mathematics teachers' work processes

The availability of computer technology affects the teachers' work in many respects. For instance, mathematical tasks that the teacher can offer to her/his students could become obsolete when the use of computer technology is allowed in the classroom (Lagrange, 2005). A task that could be considered a challenging problem in a setting where computer technology is not available can become a trivial exercise in a technological-aided setting, in the sense that only applying a command or pressing a button on the calculator could solve it. Thus the need for redesign of mathematical tasks arises. It is necessary to rethink the mathematical activities in order to make them more meaningful and challenging in a technology-aided environment.

Computer technology offers the possibility to enrich teachers' instructional techniques. The use of technology may promote the emergence of new teaching techniques; for example, the work of Drijvers, Doorman, Boon, Reed & Gravemeijer (2010) provides a taxonomy of various forms of work that can arise when teachers teach mathematics with the aid of computational tools. Internet resources such as YouTube can make the mathematical problems given to students more interesting by providing them with realistic contexts in which these problems could be embedded (Stohlmann, 2012).

Technology can also help to expand teachers' instructional spaces, i.e., teachers can provide their students with mathematics instruction beyond the walls of the classroom. There are for example mathematics teachers who video record their mathematics lessons and make them available to their students so they can review the lesson in the privacy of their home (see for example the concept of *flipped classroom*, Talbert, 2014; Tucker, 2012). On the other hand, the use of mobile technology can help teachers to organize mathematical activities outside the classroom where students can use elements of the real world to study mathematical objects and their properties, for instance (Wijers, Jonker & Drijvers, 2010) report the use of a mathematical game based on the use of mobile devices and GPS technology, in which students draw geometric shapes and explore their properties.

Teachers' work has been described within the documental approach focusing on the interplay between various resources including computer technologies, that the teachers use in preparation, conduction, and documentation of their teaching, and their actual practice (Gueudet, Buteau, Mesa, & Misfeldt, 2014; Gueudet, Pepin, & Trouche, 2012). Despite the undeniable potential, integrating technology in the mathematics classroom also raises several difficulties, and increased the complexity of teaching mathematics (Tabach, 2013).

Mathematicians' work processes

It is undeniable that the work processes of professional mathematicians benefit from the calculation capabilities of computational tools to the point that it can be argued that the introduction of computer technology in mathematics has changed mathematics in several different ways. Four main points can be mentioned:

- Computers have made it easier to search, store and share information.
- Computers have opened the possibility of more powerful explorative experimentation.
- Computers have made certain types of computationally heavy proofs possible.
- Computers, and associated complex and large data sets from various fields, have changed what problems are considered interesting.

Hence computational tools support already existing work processes (such as communicating, searching information etc.), allow mathematicians to conduct experiments that could lead to the formulation of conjectures and new theorems that can subsequently be demonstrated in a more formal way.

TECHNOLOGY AND THE NATURE OF MATHEMATICS

The use of tools has accompanied mathematical work processes throughout the history of mathematics: ruler and compass, abacus, curve-drawers, perspectographs, planimeters are examples of historical mathematical tools.

Such tools were used to support mathematical activities and at the same time they contributed to and influenced the progress of mathematical knowledge.

As one example of this we can consider the abacus (this example is thoroughly discussed in Bartolini-Bussi & Mariotti, 2008). The abacus can "easily" evoke to experts the place-value notation of integer numbers, and indeed it is often used in primary schools as a didactical aid, and it is still used in some countries in everyday life.

The first appearance of the Sumerian abacus dates back to the period 2700–2300 BC (Selwyn, 2001). Anyway it took centuries to pass from the computation practice based on the use of the abacus to the development of a “new” way to represent written numbers (the place-value notation was originally developed by Indians and introduced in Europe in the XIII century by Fibonacci; and it took centuries before it was widely accepted).

> “From an historical perspective, the positional system is not “embedded” but rather an important yet unexpected “by-product” (and even a late one) of the century use of abaci in computation”. (Bartolini-Bussi & Mariotti, 2008, p. 761).

This example illuminates the role that tools played and still play in the historical development of mathematics. Tools help represent mathematical actions and objects, create new representations, develop new forms of treatment of representations, and give birth to new mathematical objects and new ways of thinking of mathematical objects. The example also shows how complex this process can be and how unexpected the results may be. The potential of representational, communicative, data-storing and data-processing affordances of todays computer technology are strong and hence we will describe below how computer technology is destined to impact the development of mathematics in unforeseeable ways. Drawing on evolutionary approaches to cognition, Kaput and Shaffer argue that “computational media are in the process of creating a new, virtual culture based on the externalization of highly general algorithmic processing that will in turn lead to profoundly new means of embodying, enriching and organizing all aspects of human experience” (2002, p. 288), that is a new stage of human cognitive development. In the next sections we will zoom in on the effects that tools has on mathematics as a discipline, and see how it changes for *researchers* and for *teachers* and *students*.

The researcher perspective

None of the changes in work processes of mathematicians described above are philosophically innocent, since such changes in the work practice might lead to more fundamental changes in the field of mathematics.

The fact that computers have made it easier to search, store and share information has not only made the day-to-day work of mathematicians easier, but has also introduced qualitatively new ways of conducting mathematical research. An illustrative example is the On-Line Encyclopedia of Integer Sequences (OEIS.org) that by June 1, 2013, had been cited in 2399 papers (according to https://oeis.org/wiki/Works_Citing_OEIS). Thus, computer based tools for sharing and searching information has provided a new ways for finding and exploring mathematical theorems.

Explorative experiments are certainly not something new to mathematics. Gauss’ discovery of the prime number theorem which gives an estimate for the total number of primes less than a given number[1], could serve as a historical example (Goldstein 1973). However, the introduction of computers has given us new, powerful tools for explorative experimentation (see e.g. Sørensen 2010) and has led to a new recognition of the experimental aspects of mathematical research, most notably with the birth of the journal *Experimental Mathematics*, which is specifically devoted to increase the awareness of the role played by experiments in mathematical discoveries (Epstein et. al., 1992; current statement of the journal's philosophy: http://www.emis.de/journals/EM/expmath/philosophy.html). Consequently, it is fair to say that the introduction of computers has led to an increase in both the awareness and power of explorative experiments as a method for mathematical discovery.

The advent of computer assisted proofs such as the Appel and Haken's 1976 proof of the four colour theorem (Appel & Haken 1977a & b) has not only opened the possibility of using computation heavy proofs, but has also led to the recognition that mathematics can no longer be viewed as a priori knowledge (for discussion, see Johansen & Misfeldt, n.d.). Other mathematicians have suggested more radical reforms. Most notably, Doron Zeilberger has argued that mathematicians should not invest energy in actually proving mathematical theorems. Instead they should focus their work on transforming mathematical problems into a form, where computers can attack them (e.g. Zeilberger 1999a, 1999b). Zeilberger furthermore has argued that the introduction of computers should lead to a fundamental change in the mathematical epistemology, where we accept a class of 'almost-true' theorems (Zeilberger, 1993). These observations suggest that the introduction of computers in the mathematical practice has led to pragmatic changes in the day-to-day work of the mathematicians, as well as in the methodology and epistemology of mathematics.

The student and teacher perspective

The teaching of mathematics requires a shared conceptualisation of what is being taught. Hence discussing what mathematics "is" in a technological society becomes important in order to develop learning goals and curriculum. As we saw above, the change in researchers' practice caused by the use of computer technology has affected mathematics as a discipline, and in the same way students' and teachers' use of technology in the classroom affects what mathematics is for them.

Hence two types of change can be observed; development that results from the practice of teaching and learning of mathematics in classroom settings and development from the way mathematics is done in research and professional life, affecting the target knowledge for teaching mathematics. As described in the previous

[1] If we let $P(x)$ designate the total number of primes less than or equal to a given positive real number x, the theorem more precisely states

$$\lim_{x\to\infty} \frac{P(x)}{x/\ln(x)} = 1$$

chapter digital technology allows students and teachers to distribute calculations to computational technology and to communicate more, and in different modalities. Different educational systems address these changes and possibilities differently, but potentially these technologies tone down the value of computational skills, and tone up the ability to communicate and make meaning from diverse digital representations.

The effects from outside of the mathematics classroom come from many sources. We have here discussed mathematical research, which is one obvious source for conceptualising what mathematics is. However more such sources exist. The way mathematics is used in professional life is affected by technology, and so is the relevance of studying mathematics both in order to cope with various aspects of life and in order to understand and participate in the democratic debate. Hence, all reasons for studying mathematics (Niss, 1996) are somehow affected by technology. The emerging goals for mathematics teaching as a result of technology is described in the next section.

HOW IS THE USE OF COMPUTER TECHNOLOGY IN MATHEMATICAL ACTIVITIES CHANGING HOW MATHEMATICS IS UNDERSTOOD AND LEARNED

In the previous sections we have described how educational research around computer technology, for example CAS and DGS (Dynamic Geometry Systems), have studied the micro processes of learning mathematics with computers. These tools change students' mathematical work processes and hence affect their learning. We suggest that the resulting changes can be described as questions of new goals, new didactical problems, and new didactical potentials.

Since a number of mathematical work processes outside school is affected by computational technology, it is natural to reconsider goals for schooling**.** Currently the role of programming in the mathematics and science curriculum is discussed (Caspersen & Nowack, 2011; Rushkoff, 2011; Wolfram, 2010), because of the increased importance of programming in society. As a contrast long division is often described as a mathematical process that, due to the widespread use of calculators, is not necessary for lower secondary school pupils to master anymore. The discussion however is more difficult than this. Although we might all agree that it is not really important to be able to calculate the quotient of two 7-10 digit numbers fast and efficient, it does not mean that it is not important to know how it is done. And if students never do the actual process, then there is a risk that they might never learn how to do it right (the same goes for the solution of equations, algebraic simplifications and several other mathematical processes). Learning problems as a result of blackboxing is well documented (Guin, Ruthven, & Trouche, 2005; Nabb, 2010)**.** If students are consistently using a CAS to perform algebraic reductions and solutions of equations, then it is less likely that they are able to perform such calculations without the tool. This can affect learning because the student lose track of the processes that is hidden by the tool (Jankvist & Misfeldt, 2015).

The tools also offer a number of new potentials in terms of construction, inductive reasoning and experimentation. The increased potential in diagrammatic reasoning has for example been investigated in relation to DGS (Laborde, 2005; Mariotti, 2000).

COMPUTERS AS CONTINUITY OR RUPTURE IN THE PRACTICE OF MATHEMATICS, AND MATHEMATICS EDUCATION

We will address the question of continuity vs. rupture through two different frameworks: distributed cognition and the theory of semiotic mediation. The two frameworks stress quite strongly the aspects of continuity between the use of computer and non-computer technology in mathematics and mathematics education rather than the aspects of rupture. Both perspectives consider computer and non-computer technology as particular 'artefacts' designed by humans in order to produce intended effects (Rabardel 1995, p. 49).

From the point of view of distributed cognition computers can be seen as epistemic artefacts that allow cognitive tasks to be distributed and completed by epistemic actions. Although the introduction of computers in mathematics has led to qualitative changes in mathematical research, the use of epistemic artefacts is not at all new to mathematics. On the contrary, throughout its history mathematics has been intimately connected with the use of cognitive artefacts; we have always strived to create tools, algorithms and representational systems that allow us to reduce the demands mathematics poses on human cognition. The use of such artefacts can be traced back to at least the Upper Paleolithic period where carved bones were allegedly used as tallying sticks. Furthermore, studies of animals, human infants and isolated tribes have shown that our ability to do mathematics without the aid of cognitive tools is very limited. To put it roughly, we have the ability to do basic arithmetic with sets containing less than five elements, and we are able to judge the size of large sets with approximation (Feigenson, Dehaene & Spelke, 2004; see also Johansen 2010, p. 49 for discussion). We are however not able to judge, say, whether there is 10 or 11 elements in a set without the aid of a cognitive tool, such as a sequence of counting words.

The theory of semiotic mediation considers the role of computer technology in fostering mathematics learning process focusing on the commonalities between computer and non-computer technology, stressing how they contribute not only to the accomplishment of mathematical tasks, but also to the individuals' construction of mathematical knowledge. Computer, ruler and compass, abacus, and curve-drawers (just to mention some materials often used in schools) are artefacts conceived and designed to be used according to certain modalities in order to solve tasks. In this sense, artefacts embody people's collective experiences, and modes of acting, thinking, and communicating; i.e. they embody collective social knowledge and experience (Stetsenko, 2004) which "assures" the correct functioning of the artefact. And for this very reason artefacts can be viewed as "bearers of historically deposited knowledge from the cognitive activity of previous generations" (Radford, 2008, p. 224).

Through the use of an artefact for accomplishing a task, the individual has in a sense access to the historically and culturally established knowledge embodied in it. In fact the process of using an artefact for accomplishing a task involves two components having opposite orientations. On the one hand, the process is oriented towards the objects of the action: the artefact is a means to transform the object; on the other hand it is oriented towards the individual, it permits the individual's consciousness-raising of the object itself of the artefact-mediated action (Rabardel, 1995). The use of an artefact even structures the individual's action and thinking, drives his attention and perception. This means that artefacts not only serve to facilitate already existing mental processes, they also transform them (Cole & Wertsch, http://www.massey.ac.nz/~alock/virtual/colevyg.htm).

The didactical potential of the artefact is related to the mediation oriented towards the individual. The use of an artefact for accomplishing a task may trigger the students' development of personal meanings concerning the object of the artefact-mediated action, that are potentially coherent with historically established mathematical meanings. In educational settings this process is not spontaneous but mediated by the teacher (Bartolini-Bussi & Mariotti 2008, Maracci & Mariotti, 2013).

Summing up, this general perspective contributes to understand the role that artefacts may play in the mathematical research, teaching and learning process, illuminating the aspects of continuity between computer and non-computer technology. Even if the use of computer technology is bringing undeniable shifts in work processes of students, teachers and researchers of mathematics, there are still aspects of continuity between computer technology and non-computer technology and between their use and roles in mathematics education. The use of computers in mathematics is an extension of a practice that goes back a long time.

HOW DO DIFFERENT THEORIES DESCRIBE DOING AND LEARNING MATHEMATICS WITH COMPUTER TECHNOLOGY?

So far we have mainly focused on interactional theories such as distributed cognition and the theory of semiotic mediation. Such theories provide an important starting point for developing our understanding of the use of computers in *mathematical practice*. However, theoretical constructs have different centres of gravity proposing different issues and problems. Being aware that the complexity of the issue at stake requires us to view the problem from different angles, we are left with the question of how to approach the issue of comparing theoretical perspectives.

Mediating concepts and questions

We can seek inspiration in two European research projects TELMA and ReMath. In these projects one of the aims was to investigate the role of theoretical frameworks in the design and in the analysis of the educational use of computers for mathematics education. With this focus these projects investigated how different theories drive the design and analysis in different ways. Hence theoretical constructs studying mathematics education can be compared through specific attention on three interrelated poles (Cerulli et al. 2006):

1. a set of features/characteristics of the tool;
2. a specific educational goal; and
3. a set of modalities of employing the tool in a teaching/learning process with respect to the chosen educational goal

Different theories contribute differently to analyse these poles and their relationship, some theories are more sensitive to issues related to one pole and leave the others in the shadow. For instance, when considering the educational goals that can be pursued through the use of artefacts, one can (or not) focus on epistemological issues concerning specific mathematical contents or practices, express the educational goals in terms of cognitive processes possibly considering specific cognitive difficulties, address the process of construction of knowledge as a social or an individual process, be concerned about institutional expectations, and so on.

Let us examine three theoretical approaches in that respect. The instrumental approach (Rabardel 1995, Guin & Trouche 1999) raises the crucial importance of considering the process through which students develop the “utilization schema” of an “instrument”. That draws the attention on the pragmatic/operational side of the knowledge developed by students, involving both knowledge of the artefact and mathematical knowledge. The theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) explicitly raise the epistemological issue of the relationship between the meanings which individuals autonomously develop when using an artefact and the culturally established mathematical meanings, and addresses it through a semiotic lens. The anthropological theory of didactics (Chevallard, 1992), on its side, explicitly address the question of the institutional expectations and of the compatibility of the forms and contents of the activity mediated by the artefact and those valued by the educational institutions.

The above summary is not meant to compare or evaluate the three mentioned theories, but simply to point out that different theories offer specific theoretical tools, which inevitably can address only part of the complexity of mathematics teaching and learning with artefacts. Analogously, we could attach several dimensions even to the other poles of the construct of didactical functionality: the features of an artefact, and their modalities of use.

What we have sketched above is in fact the so-called Concern Methodological Tool (elaborated within the TELMA project, Artigue et al, 2009, and refined in the ReMath project, Artigue et al.2006, Mariotti et al. 2007) which is meant to express (some of) the main different dimensions and sensitivities through which different theories contribute to conceptualize the *features of the tool*, the *educational goal* which can be pursued through the use of these feature, and the *modalities of employing the tool* in a teaching/learning process with respect to the chosen educational goal (Artigue et al., 2009).

INTERCONNECTED CHANGING PRACTICES

To conclude this discussion of how we should conceptualize the use of computer technologies in mathematics education, we have suggested that the influence should be studied with different theoretical lenses (interactional, cognitive, curricular) and different focus points. One important focus point is the actual artefacts (e.g. a computer algebra system) used by students, teachers and mathematicians, as well as the direct influence that such artefacts has on practices. And as we have shown, different practices are influenced in different ways. If we return to our initial questions, we have addressed how the use of computer technology change mathematical work and learning. We have done so by looking at mathematics as an essentially tool-driven practice. This has given us the insights that the use of tools is a necessary part of the mathematical practice and that the introduction of new tools is a common event both in mathematics research and in education. New tools act as drivers for the development of mathematical research. From this perspective the introduction of computers is not a special event but is in continuity with the development and practice of mathematics. The introduction of new cognitive tools however, change the cognitive landscape and consequently force us to reconsider what mathematical tasks we consider important and worth learning and what problems and learning situations we should design in order to teach these tasks in a meaningful way. The problem of blackboxing described above, illustrates this process well. If a new cognitive tool, such as a CAS-system, effectively hides the intermediate steps in a task and turns the task into the use of a simple solve function we should ask whether the task is worth teaching anymore, and if it is, we should also ask how to do that in a meaningful way.

The question of whether we should view computer technology in mathematics (and mathematics education) as in continuity with or as a break away from the use of non-computer technology is almost answered by our approach to the first question. Considering mathematics as essentially a tool driven practice, puts the tool in the centre of the activity and almost forces the continuity perspective. If we say that the tools that people use have always significantly affected mathematics, and that these tools always have changed over time, then computational tools are just a natural and continuous development. However, we are able to see some accelerated changes in mathematical practices as a consequence of computer technology. These changes relate to the practices of both teachers, students and researchers of mathematics, as described in the paper.

The observation that our view of mathematics as a tool driven practice, at least to some extent, forces a view of computer technology and mathematics that are in continuity with other tool uses in mathematics, does give some insights to the last question. A different conception of mathematics, for instance a realist one, considering tools as mere means to obtain pure mathematical insights, could legitimate other answers to our questions, and hence prescribe other reasonable views and practices on the use of computer technology in mathematics education. We have seen that different theoretical lenses construct the use of tools in mathematics education differently, and that these theoretical lenses can be compared by how they construct the tool, the learning goal and the modes of using the tool (Cerulli et al. 2006). However, we should also be aware that philosophical construction of what mathematics is, what technology is, and what education is, can play a role for how the questions put up in this panel will be answered.

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Panel Discussion 2

THE QUESTION OF EVALUATION AND ASSESSMENT OF EXPERIENCES WITH INTRODUCING HISTORY OF MATHEMATICS IN THE CLASSROOM

Leo Rogers[a], Ysette Weiss-Pidstrygach[b], Janet Heine Barnett[c], David Guillemette[d] & Frederic Metin[e]

[a]University of Oxford, [b]University of Mainz, [c]Colorado State University – Pueblo, [d]Université du Québec à Montréal, [e]Université de Bourgogne

INTRODUCTION

The question of Evaluation and Assessment that we were asked to consider contains many different aspects of general beliefs and principles, of personal didactic and pedagogic decisions, and of internal freedoms and external constraints. The use of history of mathematics in education and teaching of mathematics also concerns the broad cultural aspects of our subject surveyed by Alan Bishop (1988). Areas not considered explicitly in this short report are questions of equity and social justice, of race and gender, which are the concern of all sensitive educators.

This report is intended first to survey the contexts, options, possibilities, and situations surrounding the problems of assessment and evaluation, and to offer a number of questions that we all have to consider when we plan a course, and before we make an assessment of our students' work.

These aspects were considered by the panel members, and since each of us work in different contexts, our work situations and observations can be found in the statements at the end of this report.

Leo Rogers

EVALUATION AND ASSESSMENT: CLARIFYING THE TERMS

It is clear that the words *Evaluation* and *Assessment* are found in different contexts and have slightly different meanings in different languages, and these meanings are often confused.

The word *Test* is also used to mean some kind of assessment, and has its own particular contexts and intentions.

In UK English we use all three words:

a) Evaluation is about objects, ideas, entities, and beliefs. It indicates what it is about the subject matter, namely the history of mathematics, its use in education, and often about some mathematics, that we **value**.

Our values derive from our own philosophy of mathematics and of history of mathematics, and are inherent in any beliefs, principles or practices we hold when teaching students, or designing their assessment.

b) Assessment is about estimating quantity, or agreeing a 'measure', or finding out what students 'know' in some way and is generally qualified as:

i) Formative Assessment (sometimes called *Continuous* Assessment) which is about observing changes over time in relatively short-term periodic checks on a student's progress, like in-class discussions, weekly tests of facts, or short essays or projects.

These are often used to explore students' understanding of a concept or to *check on whether our own teaching has been effective*.

ii) Summative Assessment is the traditional assessment taken at the end of a student's course, like a test at the end of a semester or end of year examinations.

c) Test: usually means a single short summative assessment.

It could be argued that if we have used formative assessments during a course or semester, then we have enough information, and we do not need to ask the student to perform a summative assessment at the end of the period.

There is also the possibility of applying these aspects of evaluation and assessment in the context of assisting an individual to reflect upon their own progress as 'self-assessment' as noted below (Ipsative assessment).

BASIC QUESTIONS: BACKGROUND CONTEXTS AND PRINCIPLES TO CONSIDER

Value judgements govern answers to all these questions that can be considered and debated with colleagues.

- **Why** Assess? *Deciding* on effects or outcomes we expect or seek.
- **What** to Assess? *Becoming aware* of and deciding on what we are looking for.
- **How** to Assess? *Selecting* the method we regard as being more 'truthful' or 'fair' for different kinds of valued knowledge.
- **How** to Interpret? *Making sense* of observations, measurements and impressions gathered by whatever means we employ and explaining, appreciating and *attaching meaning* to 'raw' data.
- **How** to Manage the data? As expressed in words, numbers, statements, student profiles, personal interviews, etc.
- **How** to Respond? *Expressing and communicating* appropriate response sensitively to individuals and communities.
- In all contexts ***Feedback*** for those being assessed is important.

NORM REFERENCING AND CRITERION REFERENCING

Humanities and Social sciences generally use qualitative assessment methods whereas the 'Exact' sciences, like Mathematics and Physics tend to use quantitative assessment methods. However, with History of Mathematics we have to select what is appropriate; our work involves Essays and Projects as well as solving Mathematical Problems and following Calculations, so both Qualitative and Quantitative methods depend on the kinds of questions asked about the material being studied.

Norm Referencing

The essential characteristic of **norm-referencing** is that students are awarded their grades on the basis of their ranking within a particular group. This involves fitting a ranked list of students' 'raw scores' to a pre-determined distribution for awarding grades. Usually, grades are spread to fit a 'bell curve' (a normal distribution), either by qualitative judgements or by statistical techniques of varying complexity.

Norm-referencing is based on the assumption that an approximately similar range of performance can be expected for any student group.

Criterion-referencing, as the name implies, involves determining a student's grade by comparing their achievements with clearly stated criteria for learning outcomes and clearly stated standards for particular levels of performance.

Unlike norm-referencing, there is no pre-determined grade distribution and a student's grade is not influenced by the performance of other students. Theoretically, all students within a particular group could receive very high (or very low) grades depending solely on the levels of individuals' performances against the established criteria. The goal of criterion-referencing is to report student achievement against objective reference points that are independent of the group being assessed.

Ipsative assessment. (Self - assessment)

In this mode of assessment, a person's performance is **compared with their own earlier performance**, to determine whether any improvement has been made, or any 'added value' brought about. Such assessment might involve setting a learner pre-course, or post-course assessment or keeping track of how a student's average percentage mark or overall grade changes as they progress through an entire course. In all cases, however, the benchmark against which any change in performance is measured is the person's own previous performance - not the performance of other people. (Andrade & Valtcheva 2009)

Small Groups or Individual Students, and Peer-Assessment

Clearly, small groups of students will not fit into a formal pattern as implied above, and judgements on individuals may be made on the experience (often over a number of years) of the assessor. However, in such cases, more than one examiner, or an external moderator is usually involved. Topping (1998) shows that Peer Assessment is "of adequate reliability and validity in a wide variety of applications."

Once a decision about the type of assessment has been made, the actual tools - the test, the essay, the project or report (be it quantitative or qualitative) may be applied.

THE VALIDITY - RELIABILITY SPECTRUM

Validity (or truthfulness)

A valid assessment is one that measures what it is intended to measure. The assessment tools must be appropriate – for example a practical skill cannot be measured solely by a written test.

On the other hand, for a statement to be valid, it depends on personal, idiosyncratic, discursive, cultural, individual, and affective factors. Hence judgements about valid statements are very difficult, often subjective and raise questions about extension over time and space:

- Can assessment be extended over time and in different situations?
- Can predicting future results and behaviours become more robust?
- Can an individual retain a particular ability whilst maintaining the disposition to act in the same way over time?

Reliability (or consistency)

For a result to be reliable it needs to be objectively measureable, testable, appropriate, and repeatable. A reliable result is necessarily restricted to a narrow range of results. This is the scientific ideal. Even so, reliability involves the expectations of students and teachers, individual predispositions and attitudes, experiences and personalities, qualities of experience, and conceptions of abstract entities.

It is well known that two people can witness the same thing (a result, a process, or entity) but disagree about its meaning or significance. In our own experience we can find a wide variation in marks in test papers or essays, over time, and between individual assessors.

Our Problem is to *seek a path between these two concepts*, to balance the *nomothetic* demands of the (quantitative) mark scheme against the *idiographic* uniqueness of the (qualitative) student response.

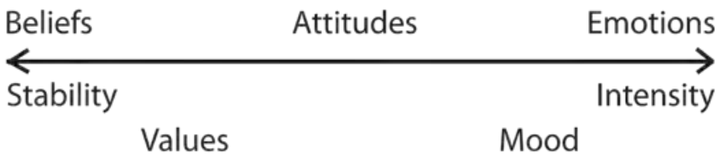
(Educational Studies in Mathematics, 2001; Smith et. al. 1996).

FRAMEWORKS, TAXONOMIES AND TEMPLATES FOR ASSESS-MENT

Bloom *et al.* (1956) published a taxonomy developed for educational assessment. It was originally designed for application to all school subjects, and provided definitions for each of the six major categories in the **cognitive** domain. The categories were *Knowledge, Comprehension, Application, Analysis, Synthesis,* and *Evaluation.*

At the time there were a number of mathematics educators who adapted it to their own views of assessment of mathematics, and many in mathematics education have shown it particularly ill-fitting for use in mathematics (Kilpatrick 1993). Later, a taxonomy of objectives for the **affective** domain was published (Kratwohl, 1964) which dealt with *beliefs, attitudes and emotions* as representing increased levels of affective involvement, with consequent decreased levels of cognitive involvement, increasing levels of intensity of response, and decreasing levels of stability of response. (Krathwohl, 2002, Evans, et.al. 2006, Hannula 2012.) Recent versions of Bloom’s Taxonomy offer Characterising, Organising, Valueing, Responding, and Receiving as the main affective domain categories.

The Range of Affective Aspects



(Krathwohl 1964, 2002)

Richard Skemp’s (1976) paper on *Instrumental* and *Relational* understanding had a significant impact on teachers’ views about learning mathematics and influenced much research on assessing *mathematical thinking processes* rather than the production of results.

Generally, variations of Bloom’s Taxonomy fail to identify *levels of learning* as opposed to designing different types of question (Freeman and Lewis 1998), and that its hierarchical nature is flawed, as certain levels in it may be considered interdependent (Anderson and Sosniak 1994, Kadijević, 2002). However, the most important difficulty with using taxonomies relates to the classification process itself, specifically:

(a) *It is difficult to put certain questions into just one category.* More involved questions can include routine aspects and procedural calculations as part of the solution process.

(b) *It is difficult to know what skills and thinking are employed by individual students to answer a question.* For example, when asked to prove a theorem, a student may learn a proof by rote and reproduce it from memory; or understand the principles and associated concepts and definitions, and use these to independently develop a proof when assessed.

(Darlington, 2013).

Competency-Based Learning and Assessment

In this category system we define a set of competencies (criteria), about what students should know and be able to do, and develop valid, reliable assessments for them. Similar to the taxonomy above, it defines a set of objectives that, however well intentioned, is open to the same problems, and we still have to make choices about what criteria are important and what ideas or principles we value in a given context. Some versions of this approach allow students to take examinations more than once and obtain feedback, so that they finally qualify when they have met all the criteria. A typical competency based situation is the assessment of teacher training which relies on cognitive skills, effective performance, affective rapport, and qualitative judgments.

ABILITY THINKING AND 'LEVELS' OF ABILITY

Ability thinking

- is ingrained in our educational systems.
- is an entity that determines 'how much' or 'how fast' an individual can learn
- describes similar levels of attainment; hence students with assumed similar ability are taught together
- leads to the common practice of grouping / setting / streaming in school environments
- influences interactions between teachers and learners and between learners themselves
- is the dominant discourse for teachers, pupils, parents, policy makers, curriculum planners, test writers. etc.
- we all use *levels of ability* – so it is 'obviously true'!

Ability it is never consistently defined and only understood in the sense that A is 'better' (in some particular skill, or group of skills) than B.

THE SCHOOL, COLLEGE, CURRICULUM, AND THE EDUCATIONAL SYSTEM

We find ourselves working inside a local educational system, in a particular institutional social context and choose to abide by its rules of governance. The situation imposes *constraints* which may limit our choices of teaching material and methods, but it could also offer *affordances* (Gibson 1977) namely, possibilities of choice, development, and action.

Guidelines, Constraints and the Curriculum itself are often politically motivated to some degree or other. We may follow the guidelines and test the constraints of the system, and explore the nature of the school or college curriculum; its opportunities and affordances which include regulations about assessment and evaluation methods. (Gresalfi, Barnes & Cross, 2012)

Our Pupils' or Students' age or position in the learning programme will determine the approach we have to the situation.

8 - 10 (Primary School)

10 - 18 (Middle and Secondary Schools)

16 - 20 $^{+}$ (High School and College)

20+ University students and *Teachers' Professional Development*

Curriculum Control

What we are able to achieve is subject to different kinds of control

(a) Professional control (what is valued) this concerns the content and nature of the subject matter; of children, pupils and students; of teaching and learning and understanding.

(b) Political control (what is ideologically desirable) justification of content – and methodology – system constraints and affordances ...

These controls depend on the philosophy (ideology) of both Administrators. Educators and Teachers. (Ernest, 1989, 1991; Furinghetti & Pehkonen 2002; Andrews, Paul 2007; Drew & Hannafin 2011).

Clarifying our Objectives and Alternatives

- For including the use of History of Mathematics as an element of mathematics courses at different levels
- For teaching the history of mathematics as a separate course.
- For choosing appropriate assessment methods
- For evaluating the process of 'use and assessment'

Range, Suitability and Significance of Historical Materials

The materials we use with our students can create a diversity of mathematical experiences**,** including cultural contexts and historical awareness. However, we have to be aware of the different possibilities of assessment modes available, and make our own judgments about their use.

(Ball, et. al. 1998)

- **Physical materials:** Original texts, documents, engravings, memorials, manuscripts, letters, diagrams,
- **Historical 'events':** Births, deaths, social-economic events, publications, and similar well-substantiated dates. How important is it to remember the date of an event?

- **Historical 'facts' and 'problems':** Theorems, propositions, conjectures, arguments, calculations, explanations, and improvements or refutations of these.
- **Interpretations, Influences**, and comparison of different accounts (past and present) – primary sources and secondary sources; 'history' books and translations of original texts.
- **Cultural contexts: e**xplores links between the cultural-historical dimension of mathematical practices and an individual's likely mathematical thinking.

All of these aspects have different values, qualities and affordances when used with different groups of students. Choosing an appropriate system for assessment will not only allow us to encourage the well-grounded and vigorous development of our students but also,

"…. investigating the process of how knowledge grows through researching historical materials themselves, and through evidence of the growth of mathematical ideas, and using this material either directly or indirectly in the classroom, reaches for similar understandings and operationally valid results as 'main-stream' educational theory." (Rogers (2014: 120).

STATEMENTS FROM PANEL MEMBERS

Janet Heine Barnett

Colorado State University, Pueblo.

I teach at a Mid-size Regional State University and I teach Undergraduate mathematics majors; mostly upper division courses, and Prospective teachers - mostly at both lower and upper secondary school. Guidelines for history of mathematics CBMS (2012) suggests:

For middle grades: *A history of mathematics course can provide middle grades teachers with an understanding of the background and historical development of many topics in middle grades.*

For high school: *The history of mathematics can either be woven into existing mathematics courses or be presented in a mathematics course of its own. ……*

"It is particularly useful for prospective high school teachers to work with primary sources. Working with primary sources gives practice in listening to ``wrong'' ideas. Primary documents show how hard some ideas have been, for example, the difficulties that Victorian mathematicians had with negative and complex numbers helps prospective teachers appreciate how hard these ideas can be for students who encounter them for the first time. Finally, primary documents exhibit older techniques, and so give an appreciation of how mathematics was done and how mathematical ideas could have developed."

I use history of mathematics in my teaching with Guided Reading Modules based on Original Historical Sources: (Barnett et.al. 2014)

"*Learning Mathematics and Computer Science via Primary Historical Sources*"

This involves joint work with colleagues at New Mexico State University Funded by US National Science Foundation with 33 existing modules available at www.cs.nmsu.edu/historical-projects

The typical Structure of a Primary Source Project (PSP) contains:

- Historical and biographical background
- Excerpt(s) from original source(s)
- A project narrative to guide student reading of excerpt(s)
- Student tasks based on excerpt(s)
- Concluding Comments / Epilogue

The primary goal is to support student learning of core material in contemporary undergraduate courses using classroom assessment of student learning of mathematics using:

Reading and Study Guides (Including classroom Preparation and Reading exercises); Written Homework Sets; Observations of Class Group Work and Contributions to Whole Class Discussions; Written Exams, including Comprehensive Final Student

Interviews.

Additional Goals: Motivate and support development of a deeper level understanding that reaches beyond basic content objectives

ASSESSMENT will include student comments on benefits of learning from original sources, and a theoretical perspective uses (Sfard (2008/2010)

There are Plans for a new project: "Evaluation with research" a component of a new NSF grant proposal (pending). Project Evaluator: Kathy Clark, Florida State University.

Ysette Weiss-Pidstrygach

Mathematical Institute, Johannes Gutenberg-University of Mainz. DE.

A Community of Practice

I teach courses in mathematics education for Mathematics student teachers for the gymnasium at the university of Mainz (Germany). There exist different forms of assessment and evaluations, like tests, oral examination, essays, coursework, seminar papers, presentations and homework assignments. But it seems that the biggest impact on self-concept and self-esteem of the Mathematics student teachers as future mathematics teachers are the written maths examinations. Today's students were brought up in a school system with normative approaches to human development.

They had to produce a required output in situations that are created and determined by others. A different approach is taken in the process of value creation in a community of practice (Wenger et al, 2002). The development of a community of practice starts with a university course in mathematics education. The use of historical and cultural perspectives in university mathematics education can support the development of self-esteem and maturity. It can bring together students with similar interests. In (Weiss-Pidstrygach & Kaenders, 2015) we present the concept of a seminar on the analysis of mathematical school textbooks and of learning contexts based on the consideration of historical excerpts. Such a seminar can become a starting point for a community of practice of student teachers, mathematics educators, historians, mathematicians, mathematics teachers and school textbook authors with the potential to develop social recognition and personal appreciation of the individual interests and talents of its members and their joint activities. We choose to work on historical excerpts in mathematical school textbooks, because for teacher students this topic is strongly related to their future practice: In Germany, there are a handful of schoolbook series that are used extensively in school. At present, most of them have historical insertions. Since the historical references that we deal with in the seminar stem from books that teachers use in their daily teaching, they constitute a link of this activity with the practice.

In countries where textbooks are not used in the classroom, the concept of the seminar can be adapted to other learning aids with historical references.

Weiss-Pidstrygach, Y., & Kaenders, R. Using historical School book excerpts for the education of mature mathematics teachers. In Proceedings of **CERME** 9, 4th-8th February 2015.

Wenger, E., McDermott, R.A. & Snyder, W. (2002). *Cultivating communities of practice: A guide to managing knowledge.* Harvard Business Press.

Frederic Metin

École Supérieure du Professorat et de l'Éducation, Université de Bourgogne**.**

I am a mathematics teacher trainer at the School of Education of the University of Burgundy in Dijon, and my major tasks are:

- to give literary students the opportunity of improving their skills in basic mathematics;
- to prepare students to take the competitive exam which will make them civil servants;
- then to train them into the construction of their own professional style

But in the various classes I teach, History of Mathematics is a minor subject, but can be the core of some courses, with for instance a special training session on how to use original texts in the classroom at middle school and high school levels.

Nevertheless I use History of Mathematics for both enlightening student's knowledge in mathematics and putting some distance between them and this knowledge, plus linking this knowledge to other disciplines. Of course the question of assessment is a difficult one: why? How? And does it even make sense to assess the historical aspects of a course on mathematics?

For example, when you try to make sense of recreational problems contained in a manuscript course of geometry from a 17th century Jesuit college, what kind of the way do you have to make sure the students understood the contents and methods? A simple answer will be: give them items 1 and 2 as exercises and hide item 3, that you will keep for the special moment of assessment. The ideal original texts are the ones where the methods are obscurely described, not well explained or even not mentioned. The natural assessment will be the simple explanation of the mathematics in the text.

Take practical geometry: studying the usual theorems in their unusual but 'useful uses' of the past will provide a kind of *depaysement* which makes assessment obvious, or obviously irrelevant: you just check the understanding of the underlying mathematical thinking, but you might rather reconstruct it. To avoid the trap, you can ask unusual (for me) question as 'is that approximation accurate?' or 'what is your opinion about the notations?' or even 'how do you feel about the text?' The problem then is that there is no unique and impersonal answer, and you thus have to accept different points of view, which is quite unfamiliar in assessing mathematics.

David Guillemette

Université du Québec à Montréal.

From my part, I'll concentrate on experiences lived with my students that are pre-service secondary school teachers. Aiming at "disorientation" with the reading of original texts, I'll try to explain our account, to underline our perspective of disorientation argument and to say few words about the problematic of assessment in this context.

In my thesis, I manage to describe the experience of disorientation of my students involved in the reading of original texts. When adopting a phenomenological stance, major themes emerge from the analysis. Two of them are the experience of *otherness* and *empathy*.

Students are saying that they are trying very hard to understand the mathematics depicted in original texts. They show great difficulties concerning language, notation, implicit argument, style, definitions, interpretations, typography, etc. Literally, they "suffer the texts". For now, in this context, I see the reading of original sources as a 'hermeneutic extreme sport' … and without helmet. The experience of *otherness* seems brutal, from a cognitive and affective point of view, it sometimes includes shocks and violence.

From Levinas, I learned that violence is a "thematisation of the Other", a reification of the Other, a way to make the Other a Mine, and that to understand something is to control it, make violence to it. I saw few acts of violence during my experimentation, for instance, someone said: "Fermat was doing this or that".

That's why *otherness* is linked with *empathy*. Again with Levinas, and also with Bakhtine, *empathy* could be heard as an effort of a non-violent relation with the Other, in this case, a way of keeping alive the subjectivity of the authors, keeping it fragile and mysterious. The question is how to accompany the students in this ordeal, in this hardship experience of *otherness*? How to maintain an *empathic* relation with the authors? I try to address these questions from a fundamental pedagogical point of view.

From these bases, the question of assessment in this context is for me a question of affectivity and a question of being-with-others. If assessment should support students, what are the actions that could support *empathy*? (Bakhtin, 1981, Levinas, 1985, 2010, 2011)

Discussion among the audience and the panel

There was a general discussion between the audience and the panel members clarifying points of view and contexts. Most of the audience concentrated on the university training of teachers and history in the context of teaching mathematics at this level.

However, the situation in the secondary school and some secondary teachers were present, and the following points were made by Ewa Lakoma on behalf of the situation in secondary education

Concering Secondary Education.

Ewa Lakoma

Institute of Mathematics, Military University of Technology.

In Poland there is a system of education: 6+3+3, starting with children at 7 years old. After each step of education there is outer examination, the same for the whole population of students at this level in Poland, leaded by the Central Examination Board [1, 2].

In fact students, when learning, are also preparing to sit these examinations. After the second stage (gymnasium) the examination opens the door to the Lycee. After the third stage the 'matura examination' is the entrance examination for the universities.

When we look at the textbooks that were presented in 2000 in the ICMI book of Fauvel and van Maanen (eds.), we can notice that many of these examples still exist in current textbooks but now they are treated rather as additional material for students. Sets of exercises preparing students for the exams are the most important.

Currently, in school practice, the history of mathematics is often present as mathematical projects, that students develop individually or in a small team and present to a classroom audience. They have to find some information on the Internet and then usually they prepare a multi-media presentation. What is most important from the point of view of assessment in this activity is the level of invention of students, their social ·competences, the level of using IT, the history of mathematics is an illustration of these activities. But it is also often evident that students are really interested in old historical materials, mathematical examples, and their solutions. They sometimes really learn something new in mathematics.

In the education system, the result of the final examination after a given level decides on a position for the student at the next higher level. Current school practice is that the history of mathematics is placed mainly in the context of better preparation of students for these examinations and has its value when it appears among questions and tasks in the examination.

An example of such a situation was found in the test after the first phase of education for 12 years old pupils in 2011.

Example 1 - the 'historical' context in the examination

The text to consider was the famous anecdote about the young pupil Karl Gauss whose teacher gave pupils the task of adding all the numbers from 1 to 40.

In the text there is also a presentation of the reasoning of Gauss.

Just below the text there is some short information:

Karl Gauss (1777-1855) - *German scholar, mathematician, astronomer, physicist; obtained the title of Doctor at the age of* 22. *In 1807 was* a *Professor. One of the greatest mathematicians of the world.*

And after that text there were eight multiple-choice exam questions (to select one correct answer among four statements): six questions consider the situation in the classroom from the point of view of the teacher, for example:

After checking the notebook the teacher realized that they needed to:

A. *move Karl to the next class;* B. *call his parents ;*C. *to develop his talent.*

D. teach him 'a *lesson '.* - and the last two questions were supposed to be 'mathematical':

When was this lesson? (i.e. How old was Gauss?)

A At the turn of the seventeenth and eighteenth centuries.

B. In the second half of the eighteenth century.

C. In the late eighteenth and early nineteenth century.

D. In the first half of the nineteenth century.

and

How old was Gauss, when he became a professor? A. 22, B. 30, C. 48, D. 78

We can pose many important questions such as:

Is this really considering historical material in a way that we prefer?

Was it really about the history of mathematics?

What is the conclusion from results of these 'mathematical' questions?

Example 2:

The History of mathematics as a theme for lessons in the Polish language

Surprisingly, the oldest book for geometry written in Polish by Stanislaw Solski, was titled: *The Polish Geometrist,* (1683), and has been known by using it at historical lessons in the Baroque Palace of the King Jan III Sobieski in Vilanov, Warsaw.

Students are able to attend at the historical lesson (real or virtual) and are able to read some pages of this book. The intention is linguistic - to recognise some old Polish words, but from the point of mathematics this is very important book, because in it we can find the creation of Polish names for the most fundamental mathematical notions.

CONCLUSIONS

As long as the history of mathematics is absent from the examination tasks, the status of considering historical materials will be still seen as 'an appendix' to the main stream of 'common' examination tasks.

For both teachers and students the use and consideration of historical materials must be clearly justified (otherwise teachers will claim there is *'no time'* to consider it)

It is a good idea to integrate areas of the history, language, culture and mathematics in order to place some historical original materials to consider, but it needs considerable cooperation between teachers of different subject areas. '

In any case, the history of mathematics is interesting for students and valuable from the point of view of their cognitive development.
The most important problem is how to profit from the short time between the examinations whose results decide the future career of young people.

This contribution points to the importance of the history of mathematics in our cultural education, and the problems about raising the awareness of history without trivialising the subject within the traditional structure of a formal examination.

This also reflects on the problems raised by the members of the panel above who, in university contexts have much more freedom to choose their mode of assessment. Clearly, what can be done in school depends upon the significant external constraints of the system, and wherever possible, the mode of assessment needs to be appropriate for the level of sophistication of the students.

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CURVES IN HISTORY AND IN TEACHING OF MATHEMATICS: PROBLEMS, MEANINGS, CLASSIFICATIONS

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In mathematical teaching nowadays curves appear as graphs of functions. They have more or less become an exercise of applications of calculus, and their place is less and less important. So the interest for curves disappeared for students and for teachers in secondary schools and also in universities.

But in history of mathematics, from Greek geometry to mathematics of today, curves play an important role. We can learn many things from history. The first point is that curves are not only a pedagogical object to judge the competencies of students: they were invented to solve problems of geometry, optics, etc. Another point was to examine how the curves can be drawn, produced or constructed, and we find many possibilities given by mathematicians of the past. Last point but not the least was to classify the curves. The purpose of the workshop is to examine some historical steps in the long history of curves, and the goal is to reintroduce curves in teaching as a rich, interesting, open subject. We will examine the possibility to create an European Team of teachers and researchers working together to progress on this subject.

Texts taken from Geminus of Rhodes (1st century), the Mathematical Collection of Pappus of Alexandria (3rd century), the Commentaries of Eutocius of Escalon (6th century), the Geometry of Descartes (1637), papers of Van Schooten (1654) and Leibniz (1693), the Introduction to the Infinitesimal Analysis of Euler (1748), papers of Peaucellier (1868), and (Kempe (1877), the Mechanisms for the Generation of curves of Artobolevsky (1964).

MATHEMATICS AS A TOOL-DRIVEN PRACTICE: THE USE OF MATERIAL AND CONCEPTUAL ARTEFACTS IN MATHEMATICS

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In this paper we introduce the concept of cognitive artefact and show how such artefacts are used in mathematical activities. By analysing different instances of artefact use we argue that our use of cognitive artefacts can lead to (at least) three different types of qualitative shifts in our mathematical capacity. Cognitive artefacts may allow: 1) expansions of practices in otherwise impossible ways, 2) extensions of mathematical domain, and 3) creative mediation of different mathematical areas. We argue that the use of cognitive artefacts – and in 2) and 3) – the choice in artefacts influence the development and content matter of mathematics. Our analysis of the role played by cognitive artefacts shows that mathematics is essentially a tool driven practice. We close the paper by discussing consequences of this realization for the choices we face concerning the introduction of CAS-tools in mathematics education.

INTRODUCTION

The computer has made its entry into mathematics teaching and learning – which has created heated debates with very strong opinions for and against. This paper is not a part of this debate, at least not directly. We are not addressing the advantages and disadvantages of computer assisted teaching and learning of mathematics – as a matter of fact, we are not discussing computers at all. Rather, we take this debate as an opportunity to shift the focus from the computer as such to the use of tools in mathematics in general, to move beyond the "good"/"bad" discussion of computers and instead ask: What can we learn about mathematics if we view mathematics as a tool-driven practice in research and in every-day (or practical) mathematics?

The concept of cognitive artefacts has drawn a lot of attention in contemporary cognitive science (see Heersmink 2013 for an overview and Hoyles & Noss 2009 for some educational implications), and in this paper we are using this concept to explore how tools affect the development of mathematics. In the following we introduce the concept of cognitive artefacts, and we use it to analyse four concrete pieces or episodes in ancient and modern mathematics in order to explore and pinpoint different ways in which mathematics can be viewed as driven by tools. We identify three kinds of qualitative shifts in these pieces of mathematics that are due to the use of such tools. We will close the paper by discussing what implications the perspective offered by the concept of cognitive artefacts could have on mathematics education.

WHAT ARE COGNITIVE ARTEFACTS?

A cognitive artefact is a human made object that is used to aid, improve or enhance human cognition (cf. Hutchins, 2001, p. 126). Typical examples of cognitive artefacts include shopping lists, calendars, address books and GPS navigation devices. Such tools allow us to think better, more reliably or with less effort (cf. Kirsh & Maglio, 1994). They do so not by enhancing our mental capacity, but rather by changing the cognitive landscape and offer new and cognitively less expensive ways of solving a given task.

In parts of the literature cognitive artefacts are exclusively associated with physical objects (e.g. Hutchins, 2001), while other theorists operate with a more inclusive definition where conceptual artefacts such as procedures, rules and certain concepts are also accepted as cognitive artefacts (e.g. Norman, 1993, p. 4). In this paper we will use the concept in this last more inclusive sense. This choice is motivated by several observations. Firstly, as also noticed by Norman, algorithms and rules of thump are clearly human creations, they are artefacts, and they can in some cases play the same role in human cognition as physical cognitive artefacts, i.e. they aid, improve or enhance our thinking. Secondly, in many – if not most – cases the physical device taken in isolation is not enough to accomplish the given cognitive task. You will also need to know certain algorithms or rules for operating the device. Thus it is natural to include the conceptual artefacts in the totality of resources needed in order to accomplish the task. Lastly, in some cases the physical part of the artefact can even be internalised. The alphabet for instance can be seen as a cognitive artefact that is used to reduce the cost of search operations; if the books in the library were not alphabetized it would be much harder to find the one you need. However, whether you carry a piece of paper with the alphabet written down or have memorized the alphabet is not important. In both cases you use the same artefact.

In the following we will describe how cognitive artefacts are used in mathematics and identify three different ways in which artefacts have led to qualitative shifts in our ability to perform mathematical cognition.

EXPANDING THE GIVEN

The first claim we wish to make in this paper is that mathematics is essentially a tool-driven activity. Over the last two decades cognitive science has shown that humans and several other species of animals have an inborn ability to solve tasks we would describe as mathematics. In short, we can do basic arithmetic on sets with less than four elements, and we can judge the approximate size of larger sets (Feigenson et.al. 2004; see also Johansen 2010, pp. 49 for discussion). Our inborn abilities however do not allow us to do anything more than that. So if we want to find out what 5+6 is or judge whether a set contains 9 or 10 elements, we have to use qualitatively different cognitive abilities and strategies (Núñez 2009).

The limits of our inborn abilities were effectively demonstrated in a study on members of the Amazonian Pirahã tribe (Frank, Everett, Fedorenko, & Gibson 2008). This tribe is especially interesting in this context because their language does not contain number words, and consequently the Pirahã does not have access to the technology of counting. In the study a subject was shown a small number of objects and was asked to match the sample by placing a similar number of objects on a table. In test conditions where the sample was hidden the performance of the subjects decreased as the size of the sample increased; with a sample size of four objects most subjects were able to match the sample correctly, but with a sample size of ten objects most subjects would fail the test. In a follow-up study similar results were obtained with participants from Boston who were deprived the ability to count (Frank, Fedorenko & Gibson 2008).

Tests such as these show that normal adult humans cannot perform simple tasks such as matching a hidden sample of ten objects without cognitive support. We simply have to use some kind of tool in order to solve this task. One of the tools that can be used in this respect is counting. Counting involves a large amount of highly complex cognitive mechanisms, such as the ability to group objects in certain ways, but first and foremost it involves a counting sequence, such as the sequence of words “one”, “two”, “three” etc. In our analysis a counting sequence is a clear example of a conceptual cognitive artefact.

From a mathematical point of view the example might be banal, but there is a more general lesson to be learned from it. Our ability to think – also mathematically – is determined by the cognitive context we are positioned in, that is: by the cognitive artefacts and other cognitive support available to us. An Amazonian Indian cannot suddenly begin to count, even if she wants to and even though she has the cognitive hardware (so to speak) needed in order to do so. It is simply not within her cognitive reach. The introduction of counting thus constitutes a radical change in our cognitive landscape. With access to counting we can perform tasks that are impossible for us to do without. Counting allow us to expand our inborn ability to handle the size of sets with digital precision. Without counting (or similar techniques) we can handle sets with 1 to 4 elements, but with counting we can handle larger sets with the same degree of precision.

A similar story can be told about basic arithmetic. We seem to have an inborn ability to do addition and subtraction, but only on small sets. With the introduction of the proper cognitive artefacts these abilities can be expanded so as to be applicable to sets of arbitrary size. In this case the proper artefacts could be conceptual artefacts such as rules and algorithms or tables of basic products, but also physical artefacts such as the abacus, counting boards or representational systems that allow basic calculations to be performed (see e.g. Menninger 1992, pp. 299 and Johansen & Misfeldt 2015 for examples and analysis). It is not our ambition at this place to provide historical analysis or account for the genealogy of counting or arithmetic. The fact that we use tools is not due to historical contingencies. It is due to the cognitive conditions we face as human beings; without cognitive support our mathematical abilities are extremely limited. The kind of tools and cognitive artefacts we use is however a result of historical development and below we will provide historical case studies illustrating the importance of such developments.

CHOICE MATTERS

In this section we will expand our analysis by showing some of the roles cognitive artefacts play in academic mathematics and by illustrating why the choice of cognitive artefact matters.

We will begin by looking at Proposition 18 from Book V in Euclid's *The Elements*. The proposition is stated and explained in the following way:

> **Proposition 18**
>
> *If magnitudes be proportional* separando*, they will also be proportional* componendo.
>
> Let *AE*, *EB*, *CF*, *FD* be magnitudes proportional *separando*, so that, as *AE* is to *EB*, so is *CF* to *FD*; I say that they will also be proportional *componendo*, that is, as *AB* is to *BE*, so is *CD* to *FD* (Heath, 2006, p. 427).

Even with this explanation it might be difficult to understand the exact content of the theorem. In Heath's translation the reader is offered cognitive support in form of the following diagram (here, slightly simplified):

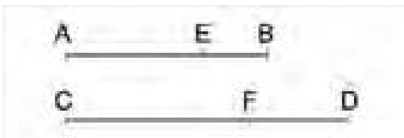

Figure 1: Diagram representing Euclid V.18

In fact, there are diagrams (or rather: figures) like this on almost every page of Heath's translation. This is puzzling in the sense that Euclid carefully describes all of the needed constructions in the text. So why has Heath included the figures in the book? They add nothing to the content of the text and thus seem completely superfluous.

In order to answer this we must turn to the cognitive role such visual representations play. Of course we could read the text and imagine the appropriate figure in our mind's eye. It would however take a considerable effort – even in simple cases such as the above. Our short-term memory is very limited and not completely reliable, so from a cognitive point of view it makes sense to off-load some of the cognitive work to a material object, in this case: a figure drawn on paper. The figure is in other words a highly specialised cognitive artefact. To introduce a more precise concept, we can say that in this case the artefact has an *anchoring* role for our cognition (Hutchins 2005).

The conceptual structure we need to build in order to understand the content of Euclid's theorem is anchored in the physical drawing. The anchor keeps the general structure stable and allow us to focus on and manipulate local parts of the structure; we can for instance imagine what would happen if we moved the point *E* or we could add new elements to the drawing (as Euclid actually does in the proof of the theorem). In this case the anchor seems to be a fairly natural depiction of the content it anchors; it simply represents magnitudes as line segments.

We will not go through the details of the proof here and the reader does not need to understand it in details, but we will nevertheless include the proof in full in order for the reader to form an impression of the cognitive workload it would take to actually understand and read the proof. In other words, we want to prove a point, not a theorem. This being said, the proof goes like this:

> For, if *CD* be not to *DF* as *AB* to *BE*, then, as *AB* is to *BE*, so will *CD* be either to some magnitude less than *DF* or to a greater. First, let it be in that ratio to a less magnitude *DG*. Then, since, as *AB* is to *BE,* so is *CD* to *DG*, they are magnitudes proportional *componendo*, so that they will also be proportional *separando*. Therefore, as *AE* is to *EB*, so is *CF* to *FD*. But also, by hypothesis, as *AE* is to *EB*, so is *CF* to *FD*. Therefore also, as *CG* is to *GD, so is CF* to *FD*. But the first *CG* is greater than the third *CF*; therefore the second *GD* is also greater than the fourth *FD*. But it is also less: which is impossible. Therefore as *AB* is to *BE* so is not *CD* to a less magnitude than *FD*. Similarly we can prove that neither is it in that ratio to a greater: it is therefore in that ratio to *FD* itself. Therefore, etc. (Heath 2006, p. 427).

As we can see, even with the cognitive support offered by the diagram in figure 1, it would take a considerably effort to follow the proof. As it is, Heath gives us a hint to another way to attack the problem. He translates the problem to algebraic symbols. In this representation the theorem states that if $\frac{a}{b}=\frac{c}{d}$ then $\frac{a+b}{b}=\frac{c+d}{d}$. Once the theorem is stated in this way, its proof is no more than a simple calculation:

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a}{b}+1=\frac{c}{d}+1$$

$$\frac{a}{b}+\frac{b}{b}=\frac{c}{d}+\frac{d}{d}$$

$$\frac{a+b}{b}=\frac{c+d}{d}$$

Here, we use another cognitive artefact; abstract symbols. Contrary to the Euclidian proof we do not need to consider the content of the operations we perform. We just need to know a few fully formal rules that tell us how we are – and how we are not –

allowed to operate on the symbols. In other words, the artefact allows us to externalise the problem and solve it as a series of physical actions.

This example shows that different artefacts have different affordances. A diagram such as figure 1 offers a qualitatively different type of cognitive support than algebraic symbols, and tasks that might be difficult to perform using only the figure might be relatively easy to perform when using algebraic symbols (and vice versa). Thus, cognitive artefacts are not just cognitive artefacts. Different artefacts shape the cognitive landscape in different ways, and for that reason it matters what type of artefacts one have access to. What one can do – and maybe even what one can think – is determined by the cognitive artefacts one has access to.

ARTEFACTS AND THE DEVELOPMENT OF MATHEMATICS

We should keep in mind that cognitive artefacts are artefacts; they were not always around, but were developed by humans. Furthermore, as we argued in the second section, cognitive artefacts are necessary in order to do more than rudimentary mathematics. However, with the example analysed in the previous section it can also be asked whether the introduction of new cognitive artefacts into the mathematical practice can change the cognitive landscape in such a way that it not only allows us to expand our given abilities or to do something well-known more easily, but also allows us to perform qualitatively new tasks. In other words: Can the introduction of new cognitive artefacts lead to qualitative changes in the content matter of mathematics?

In this section we will discuss the possible connection between the development of new cognitive artefacts and developments of mathematics by analysing two cases: Cardano's introduction of complex numbers [1] and Minkowski's use of n-dimensional lattices. The first case involves relatively simple mathematics and is relatively distant in time, whereas the second case involves advanced mathematics and describes a relatively recent development.

Cardano and the complex numbers

In *Ars Magna* (1545) Cardano considered several problems of the type: Divide a given number into two parts such that the product of the parts is equal to another given number. In one of the cases he considered how to divide ten into two parts such that their product is 40 (Cardano 2007, p. 219). This type of problems has been known since antiquity and in Euclid's *The Elements* we have an algorithm that makes it possible to construct solutions geometrically in special cases (Proposition VI.28). The Euclidian algorithm however can only be applied if the square of half of the given number is greater than or equal to the given product (this is explicitly stated as a condition to the theorem (Heath, 2006, p. 518)). In this case the square of half the number is 25 and the given product is 40, so the condition is not fulfilled, and Cardano began his treatment by stating that "it is clear that this case is impossible" (Cardano 2007, p. 219). Nevertheless, Cardano pressed on and applies the Euclidian algorithm (or a version hereof). He constructs the square of half of the given line and represented the result geometrically, as seen in figure 2.

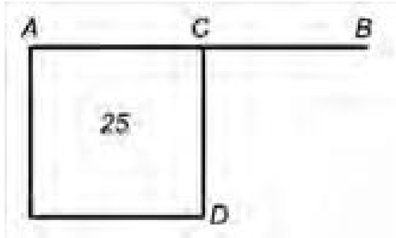

Figure 2: Drawing from Cardano (redrawn). The given line is represented as the line segment AB

As the next step the algorithm requires us to subtract the given area from the square of the given line and to find the square root of the result. In this case we will have to subtract 40 from 25 and construct the square root of the resultant area. This cannot be done geometrically – hence the condition in Euclid's proposition. Cardano responded to this problem by abandoning the geometric interpretation and representation of the situation. He simply replaced the geometric representation with abstract algebraic symbols, and then carried through with the rest of the steps in the algorithm interpreted not as geometric constructions, but as algebraic operations. This led him to the conclusion that the problem has the solutions $5+\sqrt{-15}$ and $5-\sqrt{-15}$, as the sum of these numbers are 10 while their product is 40 [2].

Solutions such as those found by Cardano cannot be found or even seen as long as one is using an algorithm based on a geometric interpretation of the situation. One cannot represent negative areas geometrically and hence from a geometrical point of view it does not make sense to subtract a larger area from a smaller one or to construct the square root of the resultant (negative) area. From an algebraic point of view the situation is different. With the proper representational system in place one can represent the square root of -15 just as well as one can represent the square root of 15 (although we might not be able to evaluate the former or understand it as a constructable geometric object, as Cardano was well aware. It was merely ink on paper, so to speak). In other words, the algebraic symbols used by Cardano allowed him to anchor and thus introduce and operate on a class of objects (square roots of negative numbers) that could not be anchored in the traditional geometrical representations. So in this case the development of a particular cognitive artefact (algebraic symbols) allowed a qualitative shift in the content of the mathematics Cardano was able to develop and work with (c.f. De Cruz & De Smedt 2013).

Minkowski lattice – An artefact in geometry of numbers?

Our final example is an episode in the history of modern mathematics regarding the German mathematician Hermann Minkowski's development of geometry of numbers and the concept of a general convex body. Before we enter into the mathematical details, we introduce a methodological triangle (figure 3) [3] that displays the relation between the historian, the materials/historical artefacts and the historical actors.

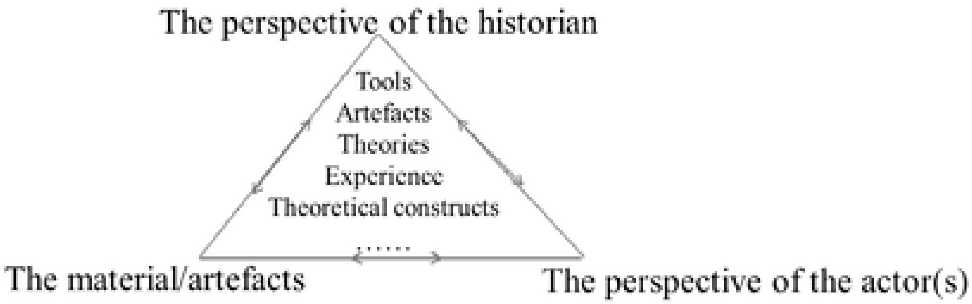

Figure 3: Methodological triangle

Reliability and validity of historical analyses depend on the relation between these three i.e. the relation between the perspective of the historian (from which perspective(s) is the historian writing his/her history?), the perspective of the historical actors (what were/was their intentions at the time?) and what material/artefacts does the historian have access to. In the following we will use Kjeldsen's (2008, 2009) historical analyses of Minkowski's development of the concept of a general convex body to pinpoint yet another way in which mathematics can be considered to be tool-driven. The relations in the methodological triangle and the validation of the historical analyses with respect to our agenda in this paper will be unfolded and discussed as we move along.

The idea of a general convex body was crystalized and constructed in the period 1887-1897. Two instances have been found: 1) Hermann Brunn's theses at Munich University from 1887 in which he introduced and investigated what we today will think of as general convex bodies in two and three dimensions. 2) Hermann Minkowski's work on positive definite quadratic forms that led to his development of geometry of numbers and the beginning of a theory for general convex bodies in the period 1887-1897. The short introductions to the history of convex analysis and geometry that can be found in textbooks and some historical accounts (see e.g. (Bonnesen and Fenchel 1934; Klee 1963; Gruber 1993)), are mostly written from the perspective of the present status and practice of the theory of convexity that is, from the conceptualization of modern mathematicians. In Kjeldsen's analysis there is a change of perspective from what we can call timeless "sameness" or from the universality of mathematics which the above mentioned historical accounts are written from, to the situatedness, to the local development of mathematics, to the practices of Brunn and Minkowski, a perspective where attention is paid to the tools and techniques they used, to their intensions and to unintended consequences of their work. She moves into Brunn's and Minkowski's "workshops" (with their tools, techniques, objects and their theories) through their manuscripts, their institutional affiliations and their mathematical cultures. She uses the historiographical tool of epistemic configuration (Rheinberger 1997; Epple 2004) in her historical analyses. Among other things, she argues that Minkowski's construction of the concept of a general convex body appeared as an unintended consequence of his work on positive definite quadratic forms.

Hermann Brunn introduced what he named "ovals" and "egg forms" in his thesis written in 1887 at the University of Munich. He defined an oval as a closed plane curve that has two and only two points in common with every intersecting straight line in the plane, and a full oval as an oval together with its inner points. Egg surfaces and egg bodies were defined as the corresponding objects in space. A mathematician of today will recognize these objects as convex sets in two and three dimensions. For Brunn they were what we could coin quasi-empirical objects whose mathematical properties such as curvature, area, volume and cross sections were unknown. The visual and intuitive, the quasi empirical status of Brunn's objects, were essential for his mathematical practice. He had very strong opinions about the methodology of geometry as he wrote in his thesis:

> I was not entirely satisfied with the geometry of that time which strongly stuck to laws that could be presented as equations quickly leading from simple to frizzy figures that have no connection to common human interests. I tried to treat plain geometrical forms in general definitions. In doing so I leaned primarily on the elementary geometry that Hermann Müller, an impressive character with outstanding teaching talent, had taught me in the Gymnasium, and I drew on Jakob Steiner for stimulation. (Brunn, 1887)

Brunn's mathematical objects can be seen as developing from artefacts from our material world – artefacts which Brunn turned into quasi empirical mathematical objects. In the discussion we will compare Brunn's objects with Minkowski's and discuss the role of their objects in the development of mathematics.

David Hilbert wrote in memory of his friend and colleague Hermann Minkowski (1864-1909) that Minkowski's geometrical proof of the so-called minimum theorem for positive definite quadratic forms was "a pearl of the Minkowskian art of invention" (Hilbert 1909). Besides being a very intuitive proof and providing a better upper bound for the minimum, Minkowski's work with the geometrical proof of the minimum theorem led to a new discipline in mathematics, geometry of numbers, it led to the idea of a general convex body hereby launching the beginning of the modern theory of convexity, and it led to the generalization of the concept of a straight line through Minkowski's introduction of what he called radial distance (which we today would call an abstract notion of a metric). A key object in these developments is the concept of a lattice which Minkowski used in his investigations of the minimum problem for positive definite quadratic forms in n variables. In the following we will explain the role of the lattice in Minkowski's work in order to discuss if and if so in what sense the lattice can be seen as a cognitive artefact and how these developments of Minkowski's in modern mathematics can be said to be driven by this tool.

A positive definite quadratic form f in n variables has the following form:

$$f(x) = \sum a_{hk}\, x_h\, x_k\ , \quad x = (x_1, x_2, \dots, x_n)\ , \quad a_{hk} = a_{kh}$$

where a_{hk} are real numbers.

The minimum problem for such forms is to: *Find the minimum value of the quadratic form for integer values of the variables – not all zero.*

Minkowski was inspired by Gauss and Dirichlet who had outlined and shown how positive definite quadratic forms in two and three variables, respectively, could be represented geometrically.

Following Gauss, we let

$$f: \quad axx + 2bxy + cyy$$

be a positive definite quadratic form in two variables. In a rectangular coordinate system, the level curves of such a form will form ellipses. Gauss (1863, p. 188–196) outlined how such a form can be associated with a lattice that is built up of congruent parallelograms through a coordinate transformation (see figure 4).

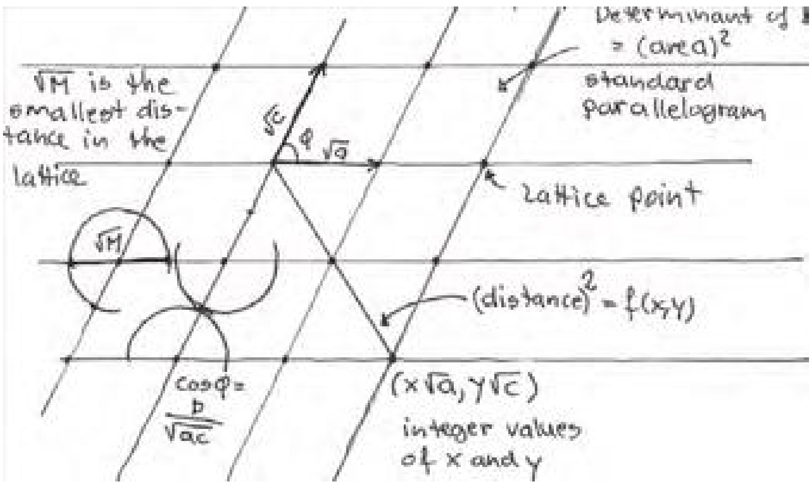

Figure 4: The lattice

The angle φ between the coordinate axes in the lattice is determined by cos $\varphi = b/\sqrt{ac}$. The points $(x\sqrt{a}\ , y\sqrt{c})$ for integral values of x and y are called lattice points. They form the vertices of the parallelograms. In this coordinate system the quadratic form measures the distance from lattice points to the origin for integral values of the variables:

$$f(x,y) = (\text{distance from the lattice point } (x\sqrt{a}\ , y\sqrt{c}) \text{ to the origin})^2$$

In this geometrical representation the minimum problem becomes the problem of finding the smallest distance between two points in the lattice. Minkowski reached an upper bound for the minimum for forms of three variables through geometrical reasoning in his probationary lecture for the habilitation in 1887. The technique he used was to place spheres with the smallest distance in the lattice as diameter around lattice points. Since the spheres will not overlap and they do not fill out the volume of the standard parallelotopes, he could deduce the following inequality:

$$V_{sp} < V_{par}$$
$$\left(\frac{4}{3}\right)\pi\left(\frac{\sqrt{M}}{2}\right)^3 < V_{par}$$
$$M < kD^{1/3}$$

Hereby he reached an upper bound for the minimum M of the quadratic form that depends solely on the determinant D of the form and the dimension. In 1891 he published a proof for the n-dimensional case.

Minkowski developed what he called Geometry of Numbers as a general theory of which positive definite quadratic forms could be treated geometrically. He realized that the essential property was not the ellipsoid shape of the level curves for positive definite quadratic forms but what we today will call the convexity property of these bodies. In a talk from 1891 Minkowski introduced the 3-dimensional lattice, not as a representation of a positive definite quadratic form in three variables, but as a collection of points with integer coordinates in space with orthogonal coordinates. In the lattice, he considered what he called a very general category of bodies that consists: “of all those bodies that have the origin as middle point, and whose boundary towards the outside is nowhere concave.” (Minkowski 1891). By then he had realized that it does not have to be a positive definite quadratic form that measures the distance in the lattice. It can be any body belonging to this category of bodies. The lattice had changed function from being a geometrical representation of a positive definite quadratic form to function as scaffolding for investigating the general categories of bodies mentioned above. A scaffolding which Minkowski began to investigate within the context of geometry of numbers that he was developing.

In a talk from 1893 he presented his ideas in more details. He introduced what he called the radial distance $S(ab)$ between two points, where S is positive if a and b are not equal to one another, otherwise S is zero. He also defined what he called the corresponding “Eichkörper” which consists of all the points u which radial distance to the origin is less than or equal to one: $S(ou) \leq 1$ (we would call this the unit ball). He emphasized that:

> If moreover $S(ac) \leq S(ab) + S(bc)$ for arbitrary points a, b, c the radial distance is called “einhellig”. Its “Eichkörper” then has the property that whenever two points u and v belong to the “Eichkörper” then the whole line segment uv will also belong to the

> "Eichkörper". On the other hand every nowhere concave body, which has the origin as an inner point, is the "Eichkörper" of a certain "einhellig" radial distance. (Minkowski 1911, vol I, p. 272-273)

Today we would recognize a radial distance that fulfills the triangular inequality and is reciprocal as a metric that also induces a norm.

Minkowski formulated his famous lattice point theorem in the talk: If $J \geq 2^3$, where J is the volume of the Eichkörper, then the Eichkörper contains additional lattice points. Minkowski's lattice point theorem connects the volume of a body with certain geometrical properties with points with integer coordinates. In his book *Geometry of Numbers*, he developed his theory for bodies in n-dimensional space.

In the course of Minkowski's research the lattice changed epistemic function from being a representation of positive definite quadratic forms, to become of interests in itself when Minkowski began to investigate the lattice and its corresponding bodies, to function as a tool – a scaffolding. Viewing the mathematical practice of Minkowski in this research episode from this particular perspective of mathematics as a tool driven enterprise, we can see that the lattice played a major role as a cognitive artefact, a tool that caused a qualitative shift in the research on the minimum problem for positive definite quadratic forms, in at least two ways:

- It provided the structure in which the "very general category of bodies" could be considered (Minkowski's talk from 1891).
- It functioned as a link between integer coordinates and the seize (volume) of the convex body.

We will finish this example by further exploring how the cognitive artefact of the lattice in this concrete episode of mathematical research enhanced our mathematical thinking, in what sense it led to a qualitative shift in our ability to perform mathematical cognition.

Brunn's egg-forms and ovals are quasi empirical mathematical objects which he investigated and proved theorems about by using the method and technique from synthetic geometry. In the preface or introduction to text books about convexity we can often read that general convex bodies were first investigated by Brunn and then further explored and extended by Minkowski (see e.g. (Bonnesen and Fenchel 1934; Klee 1963; Gruber 1993)). These short accounts of the development of the theory of convexity are written from the perspective of the modern theory, from the conceptualization of the writer, who focuses on the similarity of the bodies investigated by Brunn and Minkowski respectively. This is in the tradition of modern writings in mathematics where mathematical objects are presented as timeless entities (cf. Epple 2011).

If we change perspective from considering mathematical objects as timeless entities and instead focus on the situatedness in the actual production of mathematics, we have two trajectories of research emanating from each local context with a concrete mathematical practice. This is illustrated in figure 5.

Figure 5: The figure is adapted from Kjeldsen (2014)

There were two local contexts, Brunn's and Minkowski's, having each a concrete mathematical practice that was very different from one another. Minkowski and Brunn worked independently of each other and only became aware of each other's work after they both had developed and formulated their ideas. They met around 1893, and realized that they were both working on bodies with nowhere concave boundaries (see Kjeldsen 2009).

In order to explore this "sameness" or timelessness of mathematical entities from a historical perspective of mathematics we can play with the question whether Brunn, working with ovals and egg-bodies, within his mathematical workshop or "lab", could have reached the results of Minkowski, as illustrated by the stipulated trajectory in figure 6.

Figure 6: The figure is adapted from Kjeldsen (2014)

However, as our historical analysis of the concrete episodes of Brunn's and Minkowski's work with ovals and egg-bodies and positive definite quadratic forms, radial distance and "eichkörper", respectively, from the perspective of Brunn's and Minkowski's mathematical practices has shown, Minkowski's lattice point theorem could not have been developed within Brunn's mathematical workshop. There is nothing in Brunn's practice in this episode that connected the volume of his egg-bodies with points in space with integer coordinates. Brunn could not have asked the question of the lattice point theorem, as illustrated in figure 7.

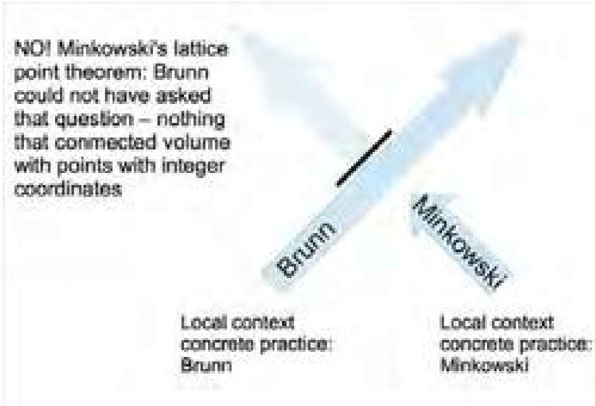

Figure 7: The figure is adapted from Kjeldsen (2014)

The content of concrete episodes of mathematical research, and the questions asked in such episodes depend on the objects and techniques (lattice, geometrical representation of quadratic forms vs quasi empirical egg forms, synthetic geometry) that are available and present for the mathematician in the particular research situation. Mathematicians' ability to think mathematically is determined by the cognitive context they are positioned in, that is: by the cognitive artefacts and other cognitive support available to them.

The lattice played a significant role in Minkowski's work. If we look at the dynamics of the knowledge production, we can see that in the beginning of the research episode, the lattice functioned as a representation for positive definite quadratic forms that made it possible for Minkowski to use the method of analytic geometry to work on the minimum problem. The lattice then became the object of investigation which led to Minkowski's introduction of the radial distance and the "Eichkörper". The lattice was the connecting link between the geometry of the nowhere concave bodies and arithmetic through the points with integer coordinates in the Euclidean coordinate system in n-dimensional space. In this sense, the lattice functioned as a cognitive artefact, a tool that drove the development of geometry of numbers. It caused a qualitative shift in the development of Minkowski's work on positive definite quadratic forms.

DISCUSSION

Through our four examples we have explored how cognitive artefacts are used in mathematics and we have identified three different ways in which artefacts have led to qualitative shifts in our ability to perform mathematical cognition: 1) as an expansion of the given (counting), 2) as an extension of what one can work and manipulate with (the square roots of negative numbers), and 3) as a scaffolding mediator between different mathematical areas (the lattice). The analyses show that cognitive artefacts are not just cognitive artefacts. Different artefacts shape the cognitive landscape in different ways, and for that reason it matters what type of artefacts we have access to. Our ability to think – also mathematically – is determined by the cognitive context we are positioned in, that is: by the cognitive artefacts and other cognitive support available to us.

In the introduction we alluded to the debate about the use of computers in the teaching of mathematics. We took this debate as an opportunity to shift the focus from the computer as such to the use of tools in mathematics in general, to move beyond the "good"/"bad" discussion of computers and instead ask: What can we learn about mathematics if we view mathematics as a tool-driven practice in research and in every-day (or practical) mathematics? We complete the loop by returning to the educational perspective. Today almost everybody in the Western world is intimately connected with smartphones, laptops, tablets and other devices that offer powerful computational support. This has radically changed the cognitive landscape we are situated within. We have to recognise this change and take informed decisions about what consequences it should have for our mathematical practice. At the outset, doing long division with smartphone is no less mathematical than doing it using an abacus or Hindu-Arabic numerals. In all the cases, students will be using cognitive artefacts. As we have shown in our analyses of our historical cases, the different artefacts have different affordances. They shape the cognitive landscape in different ways, and for that reason it matters what type of artefacts our students have access to. What they can do is determined by the cognitive artefacts they have access to. Our decisions concerning which artefacts to use and (more importantly) which to teach our students to use, should depend on an analysis of these affordances as compared to our need.

NOTES

1. We are indebted to Professor Jesper Lützen, University of Copenhagen, for bringing this case to our attention. Lützen presented his own treatment in a talk given at the Second Joint International Meeting of the Israel Mathematical Union and the American Mathematical Society, IMU-AMS in Tel Aviv, Israel, June 16-19, 2014.

2. It should be noted that Cardano used a slightly different representation. In his original manuscript the modern symbols + and – are represented as "p" and "m" respectively, and $\sqrt{\ }$ is represented as "R2". Thus in total his two solutions are stated as: 5 p:R2:m:15 and 5 m:R2:m:15 (Struik 1969, p.68).

3. Presented by Kjeldsen in the talk "Whose History? Minkowski's development of geometry of numbers and the concept of convex sets", held at at Second Joint International Meeting of the Israel Mathematical Union and the American Mathematical Society, IMU-AMS in Tel Aviv, Israel, June 16-19, 2014.

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Workshop

PERSONALISED LEARNING ENVIRONMENT AND THE HISTORY OF MATHEMATICS IN THE LEARNING OF MATHEMATICS

Caroline Kuhn & Snezana Lawrence

Bath Spa University

The proposed talk will describe the project, which aims to design and implement a personalized learning environment built around contextual historical material for the learning of mathematics. On the one hand, the project seeks to understand the principles that connect the personalized learning and digital technology, both as ways of providing individual input and collaborative learning at a distance; on the other hand it seeks to examine the role that history of mathematics may have in such a learning environment.

The talk will therefore concentrate on three aspects:

1. It will survey the existing and historical examples of personalized learning environments which use the history of mathematics as a contextual tool for the learning of mathematics
2. It will question the hows and whys on using the history of mathematics to underpin the epistemological aspect of mathematics education in digital environments
3. It will question whether the original sources, widely available on the Internet, can contribute to creating an authentic personalized learning environment, which rests on original research in mathematics.

The talk will be illustrated by the examples of personalized learning environments in mathematics that use some aspects of the history of mathematics already existing in the digital world. It will attempt to propose a brief explanations for creating a personalized learning environment, which has at its core the historical context of the development of mathematical sciences. Whilst the project is a recent collaboration between two authors, and empirical studies of the students' preferences in the learning of mathematics in digital environments is not abundant, we will aim to produce results of our initial data.

Workshop

TEACHING THE MATHEMATICAL SCIENCES IN FRANCE AND GERMANY DURING THE 18TH CENTURY: THE CASE STUDY OF NEGATIVE NUMBERS

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Bergische Universität Wuppertal

In the following, we provide the analysis of a particular case study in the mathematical teaching of the 18th century: how negative numbers were introduced to students, justified, and used in practice. We focus on a small selection of French and German textbooks, paying particular attention to their didactic approaches. Our main aim is to point out the similarities and differences between these presentations.

INTRODUCTION

During the 18th century, a question of concern in mathematical teaching was how negative numbers had to be interpreted, either within arithmetic or algebra (for further reading, see Schubring 2005). In the past, there had already been attempts for interpretations. Common explanations were possessions and debts, and quantities moving along in opposed directions. There are three German sources that give us an insight into the contemporary discussion and the associated problems with negative numbers. To these belong first *Gedanken über den gegenwärtigen Zustand der Mathematik* (1789) by Johann Andreas Christian Michelsen (1749-1797), second *Versuch das Studium der Mathematik durch Erläuterung einiger Grundbegriffe und durch zweckmäßigere Methoden zu erleichtern* (1805, published anonymously) by Franz Spaun (1753-1826), third the reaction on Spaun's writing, namely *Ueber Newtons, Eulers, Kästners und Konsorten Pfuschereien in der Mathematik* (1807) by Karl Christian von Langsdorf (1757-1834). Spaun criticized among other things that the meaning of the plus and minus operators have a double meaning; first representing the arithmetic operations of addition and subtraction, second as algebraic symbols for positive and negative numbers (cf. Spaun 1805, p. 7 and p. 18). This aspect concerning the opposed numbers caused difficulties for contemporaries. Spaun also spoke against the usage of the expression "negative" in order to denote negative numbers (cf. Spaun 1805, p. 7). In contrast, Langsdorf argued that this expression was a convention for mathematicians and could be used for negative numbers (cf. Langsdorf 1807, p. 12).

We consider a small selection of French and German textbooks from this period. In order to identify the selection, the following criteria had been taken into account. Firstly, we limited to textbooks written in a national language, namely French or German, during the 18th century. Secondly, we searched for textbooks that were written with a teaching purpose for higher education. Thirdly, we considered only the ones that were meant to provide a complete presentation of the mathematical sciences.[1] Afterwards, among this first raw selection, we chose some of the most renowned and used textbook, according to primary and secondary sources. In particular, for the French part, we deal with Bélidor's (1725), La Caille's (1741), Camus' (1749), Bézout's (1764, 1770), and Bossut's (1771). For the German part, we analyse Wolff's (1775) and Kästner's works. We also choose the textbook by Euler for the German case, but for the ease of the workshop in an English edition (1822). The first and original edition was written in German and published in 1770.

With our case study, we wanted to take a look at different approaches to negative numbers, and especially at their justifications, in the small selection of textbooks. After a brief presentation of the German and French circumstances (educational system, institutional conditions, position of mathematics, textbooks and their authors, …), we invited the participants to work in teams on the different sources. At the end of the workshop, every team presented their results. The aim was to show the differences among the various approaches to negative numbers at that time, also in comparison with the developments that lead to nowadays approaches. In order to make the study on the sources easier for the participants and to guarantee comparable results, we proposed the following questions for the analysis of the sources:

- Definition: Is there a definition of negative numbers? If yes, where is it in the textbook? Are there examples to explain the definition? If yes, what are they?
- Terminology: Which expressions are used?
- Are there interpretative models for negative numbers?
- Are there also non-mathematical remarks (philosophical, historical,…)? Is the difference between plus and minus once as arithmetic operators, once as algebraic symbols clear?
- Applications: How are negative numbers used in calculations? (subtraction, multiplication in algebra, quadratic equations)
- According to your experience, are there parallels or differences to nowadays approaches?
- Other impressions

This workshop was based on parts of the results that we got in the context of the project "*Traditionen der schriftlichen Mathematikvermittlung im 18. Jahrhundert in Deutschland und Frankreich*", financed by the Deutsche Forschungsgemeinschaft (DFG) at the Bergische Universität in Wuppertal. The final aim of this project is to establish a comparison between the German and French textbooks that were used during the 18th century to teach mathematical sciences in higher education. Therefore, we take into account many case studies, including the one at issue. Eventually, we hope to manage to analyze the emergence of traditions in teaching mathematics in this period, and also to retrace their possible origins in the textbooks written in Latin, especially by the Jesuits. To this purpose, we are moreover working on a comprehensive database, based on the software developed by another DFG project, the "*Personendaten-Repositorium*", at the Berlin-Brandenburgische Akademie der Wissenschaften.

FRANCE

All the French authors that we are taking into account delayed the treatment of negative numbers until the algebra part. Bélidor's approach is in this respect peculiar since he did not deal with elementary arithmetic at all, so that negative numbers are explained right at the beginning of his *Nouveau cours de mathématiques* (Book I). Indeed, he took for granted that his readers were acquainted with calculations with integer and fractional numbers and started, after having stated some basic geometrical definitions (without examples), with calculations with "algebraical quantities", that is with letters that are used as signs to point at non-defined numbers. Bélidor maintained that, when an algebraic quantity is preceded by no sign, that is, neither by + nor by – , he always supposed that it has the sign + and called it "positive quantity". On the other hand, the quantities that are preceded by the sign – are called "negative" (cf. Bélidor 1725, p. 11). He provided $+ab = ab$ and $–ab$ as examples, where he used to denote the algebraical quantity ab referring to the extremes a, b of a geometrical segment. Bélidor provided some interpretative models for negative numbers. First, he interpreted them as possessions and debts (cf. Bélidor 1725, p. 14). Later, he stressed that negative quantities are not "less real" than the positive ones. Indeed, they are opposite quantities, which means that they have contrary effects in calculations (cf. Bélidor 1725, p. 18 and p. 80). Bélidor never clearly stated the difference between plus and minus. Sometimes he used them as arithmetic operators, sometimes as algebraic signs. He suggested both viewpoints (cf. respectively Bélidor 1725, p. 8 and pp. 12-13 and Bélidor 1725, p. 14 and p. 18), but he never critically compared them. Negative quantities appear at first while dealing with the algebraical subtraction, where a "$–b$" stands alone. This means that there the "–" denotes the fact that "b" is negative, and it is not an operation. Many other examples, for instance the results of the multiplication $(–8abc)(–5bcd)$ and of the multiplication $(a–b)(a–b)$, can be found in the paragraph on algebraical multiplication. In this passage, Bélidor argued that, if the multiplicand has the sign + (respectively –), the multiplication is made by addition (respectively subtraction) of the same algebraical quantity. A classical example concerns the so-called rule of signs, namely when one or more negative multiplicands are involved. The most interesting examples, however, are to be searched for in the treatment of quadratic equations (cf. Bélidor 1725, pp. 158-166). Indeed, here he provided no general method for solving them, but rather a collection of solved examples. While commenting some of these, Bélidor affirmed that a negative root is to be considered a solution of the problem as well as and with the same degree of trustworthiness as a positive one. Again, he states that negative roots give a solution "in the sense that we intended", meaning that when one finds a negative solution he only has to adapt his interpretation, for instance in terms of debts. Finally, he remarked that the algebraic values are true and reasoned, even if sometimes it seems they don't have a meaning since they are far from what we had imagined.

In contrast, La Caille, as all the other authors that we are taking into account, firstly dealt with arithmetic, then with algebra in his *Leçons élémentaires de mathématiques*.

Again, as in all other works, only in the algebra part do the negative numbers occur. For La Caille algebra is a kind of arithmetic which is more general, faster, briefer, simpler, and that can be applied. Among its preliminary notions, he passed from the definition of "algebraic quantity" quite immediately to the one of "polynomial" (namely, an algebraic quantity that contains more than one term). Here we can find the only definition that can be assimilated to negative numbers: La Caille explained that there are two kinds of terms, the positive ones and the negative ones. These last are always preceded by the sign –, the other by the sign + (cf. La Caille 1741-1750, Vol. 1, p. 62). He only gave the example $+p-q-rr+x-y$, where no term stands alone with a "–". La Caille interpreted negative numbers as "opposite" quantities and he justified this term in the following way. Indeed, he explained that $-3a$ is the same quantity a taken three times, as for $+3a$, the only difference being that it is taken in the contrary direction. Apart from this and the usual signs rule for multiplication, it is hard to find some other concrete examples. But obviously La Caille is compelled to deal with negative numbers in solving quadratic equations. While giving the general solving method with the quadratic formula, La Caille repeatedly remarked that a solution can be negative (cf. La Caille 1741-1750, Vol. 1, pp. 130-135). He even mentioned that square roots of negative numbers can appear. To this purpose, he limited to explain that it is impossible to find a quantity that, being multiplied with itself, gives a negative product, but to this he added no judgment of value. When the problem that leads to an equation with a negative solution is interpreted in "real" life (for instance, when we search for the number of travelers), a negative solution only points to the fact that also this negative number (for instance, –6) satisfies the equation (cf. La Caille 1741-1750, Vol. 1, p. 135). La Caille also added that "of course" only the positive solution is the one that we were searching for. Further on, La Caille remarked that, when the result of a calculation gives a negative value for the unknown, this means that one has to take this unknown in the opposite direction compared to the one that they considered at the beginning (cf. La Caille 1741-1750, Vol. 1, p. 291).

In Camus' *Cours* negative numbers do not appear. Indeed, it only reaches an elementary level. As all the other French authors, Camus considered only positive numbers in the arithmetic, and algebra is not included at all in the table of contents.

Bézout's treatment of negative numbers in his *Cours de mathématiques* is highly detailed. We take into account the textbook for the navy since, concerning the topic of negative numbers, the differences from the textbook for the artillery are minor. Bézout gave the definition at the beginning of the algebra volume: as usual, the quantities which are preceded by the symbol + are positive, while the ones that are preceded by the symbol – are negative (cf. Bézout 1764-1769, Vol. 3, p. 9). No example is given at first, but then Bézout devotes a whole paragraph to the topic (cf. Bézout 1764-1769, Vol. 3, Réflexions sur les quantités positives et les quantités négatives, pp. 78-84). Among the French authors of our selection, he is the only one that explicitly discusses the distinction of + and – as operations and as properties of quantities. Bézout had already dealt in the usual way with + and – as addition and subtraction in the preceding paragraphs of the arithmetic and algebra parts dedicated to these topics. In this paragraph, he focused on + and – as “the way of being of quantities, one in regard to the others”. On the one hand, Bézout legitimated the negative quantities by means of the usual interpretative models, while, on the other hand, he weakened the ontological status of these quantities according to the following arguments. The discussion begins by observing that one quantity can be considered from two opposed viewpoints, and the analogies concerning possessions and debts, and opposite directions on a line are offered to the readers. In this theoretical part (since no examples are provided), Bézout stressed that the negative quantities are as much real as the positive ones except that they have a completely opposite “meaning” in calculations: indeed, negative quantities have properties opposite to the positive quantities, or they behave in an opposite way. On the other hand, when it comes to the applications (in particular in quadratic equations), Bézout provides a conceptual frame to let the students deal with negative numbers. His strategy is to weaken the rights of negative quantities to appear in the solution of a problem. Indeed, Bézout stated that each negative quantity points at a false assumption in the statement of the problem but, at the same time, it also points at its correction, since it would be enough to take the assumed quantity with the opposite symbol.

Finally, Bossut starts the discussion in the algebra volume of his *Cours de mathématiques* by defining, among others, the symbols + and – as operations. The first negative quantity $-b$ appears before the definition. According to Bossut, negative and positive quantities are of a same kind, but they are opposite regarding “their way of being” (cf. Bossut 1772-1775, Vol. 2, p. 10). He instantiated this definition with two examples from real life which provide as many interpretations. They boil down as usual to possessions and debts and to considering the opposite direction on a line. In the main, Bossut's textbook shows a lot of similarities with Bézout's one and, in the practice, for instance while dealing with quadratic equations, negative solutions are accepted without reserves. At this point, Bossut did not even need to extensively justify the negative solution. Referring to a numerical equation with a positive and a negative solution, Bossut briefly mentions that both solve the equation (cf. Bossut 1772-1775, Vol. 2, p. 189). His justification is the algebraic calculation in which he simply substituted the two solutions in the equation at issue. In the collection of examples that follows, when the equation derives from a problem with an interpretation in real life, the negative solution (if there is one) is also briefly interpreted as the opposite of the positive one (to gather or to loose water).

GERMANY

Wolff explained the negative numbers within the algebra chapter in the fourth volume of his *Anfangs=Gründe*, namely for solving equations. Wolff did not use the terms “positive”, “negative”, or “opposed” quantities, but described these quantities as money, debts, and lack (cf. Wolff 1775, Vol. 4, p. 1557).

Kästner is the first author who gave a concrete definition on opposed quantities within the arithmetic chapter at the beginning of his *Anfangsgründe*. Euler treated these numbers in his textbook on algebra. Kästner gave a definition of the opposed quantities:

> Opposed quantities are called quantities from the same kind, which are considered under such conditions that one of them reduces the other one. For instance assets and debts, moving forward and backward. One of these quantities, no matter which one, is called positive or affirmative; the opposed quantity negative or negating (Kästner 1800, p. 71).[2]

Euler defined negative numbers:

> All these numbers, whether positive or negative, have the known appellation of whole numbers, or integers, which consequently are either greater or less than nothing (Euler 1822, p. 5).

Euler proceeded the definition of negative numbers by attribution to a concrete number range, namely the integers. In Euler's definition, there is another aspect which is quite interesting. This concerns the expression "less than nothing". In the 18th century, an unanswered question was the interpretation of negative numbers. From a philosophical point of view, it is very difficult to label negative numbers as "less than nothing", because they are real objects, for instance debts. Therefore, Kästner saw the need to explain the expression "less than nothing" in his textbook (cf. Kästner 1800, pp. 72-74). He stated that one must distinguish between an "absolute nothing" and a "relative nothing". Concerning to the negative numbers, one must choose the meaning of the relative nothing, because a negative number or quantity can only exist because of its opposed (positive) quantity. It is wrong to denote a number negative in an absolute meaning. Euler equated "nothing" with the number "zero" and, with the help of a number line, showed the positive and negative numbers (cf. Euler 1822, p. 5).

During this time, Kästner's definition of the concept of negative numbers was well accepted to be precise. Kästner devoted a whole paragraph (§ 95) to the nature of negative numbers. He impressed with his remarks on the nature of negative numbers and the issue of "less than nothing" even the philosopher Immanuel Kant (1724-1804), who wrote about negative numbers in his work *Versuch den Begriff der negativen Größen in die Weltweisheit einzuführen* (1763).

Wolff explained the negative quantities in order to solve algebraic equations. But we cannot find a lot of examples with references to everyday life. Kästner introduced the negative numbers in a practical way. At the beginning, there are some examples with reference to everyday life. Then you can find "questions" which are used to explain the four basic operations with negative numbers. Kästner uses concrete numbers instead of letters as we can find in Wolff's algebra chapter. Also Euler uses concrete numbers for his explanation of negative numbers. While Wolff only mentions "quantities", Kästner once and Euler several times speak of "numbers". Another observation is that Wolff treats the negative numbers subordinated, while this topic is an independent one in the textbooks by Kästner and Euler.

For the German part, there was a common notion of the interpretation of negative numbers, namely as debts. This is the same as in earlier times (see introduction). Also nowadays this example is very popular and often used for the explanation of negative numbers.

In his book, Euler clearly points out the difference between arithmetic operations and algebraic symbols of plus and minus. First, he explains the arithmetic operations. After that, he introduces plus and minus as algebraic symbols serving as description of positive and negative numbers. Kästner and Wolff did not make this difference clear. This was a problem which Spaun criticized in his writing (see above).

By making reference to the German textbooks, we can see the development concerning the treatment of the negative numbers during the 18th century. This topic was detached from its treatment in the context of algebraic equations. Negative numbers became an independent part, either within the arithmetic or in the algebra chapter. This comes along with the fact that the authors gave a concrete definition of opposed quantities. In order to illustrate negative numbers, Kästner gave a lot of examples from everyday life (like assets and debts). Euler defined the negative numbers as part of the integers and illustrated them at the number line.

CONCLUSION AND SOME RESULTS

There are some interesting observations regarding the treatment of the negative quantities in the considered French and German textbooks. In French textbooks, negative numbers are treated within the algebra part, which is not always the case in Germany. While at the beginning of the 18th century Wolff treated the negative numbers within the algebra part, there is a shift in the course of the years. In the middle of the 18th century, Kästner explained negative numbers at the beginning of his *Anfangsgründe* within the arithmetic part. Although Euler treated the negative numbers in his textbook on algebra, he labeled negative numbers explicitly as "numbers". Also in Kästner's textbook we can find once the terminology "number" instead of quantities. On the contrary, the "number"-terminology is never used by the French authors.

There are commonly accepted interpretations of negative numbers, such as possessions and debts, and opposite directions, which are widely employed in both French and German textbooks. Overall, the difference between plus and minus sometimes as operations and sometimes as algebraic properties of quantities is not explicitly addressed; it is completely missing in La Caille since there negative numbers are only defined in the context of polynomials.

The development in France shows a clash of differing epistemological conceptions, which spanned from complete acceptance of negative numbers as solutions of problems (especially when those are originally formulated with no references to real life) to no acceptance (that is, the hypotheses of the problem should be reformulated), passing by a limited acceptance (that is, provided that one can stick to these negative numbers an interpretation to reconnect them with reality).

For the German part we can state that negative numbers were not only regarded as possible solutions in algebraic equations any more. Opposed numbers became an independent topic within arithmetic. This development shows that negative numbers were generally accepted.

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NOTES

1 Whatever "complete" means depends not only on each single author, but also on the time span. Indeed, there were some shifts in the 18th century in Germany concerning the framework of the mathematical sciences.

2 Translated by Desirée Kröger. Original quote in Kästner 1800, p. 71: "Entgegengesetzte Grössen heissen Grössen von einer Art, die unter solchen Bedingungen betrachtet werden, daß die eine die andere vermindert. Z.E. Vermögen und Schulden, Vorwärtsgehen und Rückwärtsgehen. Eine von diesen Grössen, welche man will, heisst man positiv oder bejahend, die ihr entgegengesetzte negativ oder verneinend".

Workshop

APPROACHING CONIC SECTIONS WITH MATHEMATICAL MACHINES AT SECONDARY SCHOOL

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In secondary and tertiary educations, students study conic sections mainly as algebraic objects with a graphical representation. They rarely meet conic sections from a synthetic point of view. In addition, the origin of conics - as curves obtained by cutting a cone with a plane - has become a sort of “simple story” to introduce them, but this is not always picked up by the teachers during the lessons. In Italy, the recent reform of secondary school education requests to develop a synthetic approach to geometry.

This workshop aims to discuss the main steps of a teaching experiment focusing on the introduction of conic sections at secondary school level (16-17 years-old students) following the methodology of mathematics laboratory with mathematical machines (Maschietto & Martignone 2008, Maschietto & Bartolini 2011). In this teaching experiment the historical dimension is very important, because each mathematical machine has a strong link with the history of mathematics (Bartolini Bussi, 2005). In particular, we have considered mathematical machines with tightened threads and articulated antiparallelograms (described in Van Schooten’s books), big models of cones cut by a plan representing Apollonius’s theory and big models showing Dandelin’s theorem (all are available at the Laboratory of Mathematical Machines in Modena, www.mmlab.unimore).

The workshop is organised in steps as follow:

1. Introduction to mathematics laboratory with mathematical machines and to the context of the teaching experiment;
2. Working group on the analysis of worksheets for students concerning a first mathematical machine;
3. Collective discussion;
4. Working group on a second mathematical machine;
5. Collective discussion and historical perspective;
6. Working group on a third mathematical machine;
7. Presentation of the final step of the teaching experiment.

Participants: secondary school teacher, researchers.

Age of students involving in the teaching experiment: 16-17 years.

Materials for the participants: worksheets, historical texts, outline of the teaching experiments, mathematical machines.

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Workshop
TOOLS AND PROCEDURES FOR USING HISTORICAL MATERIALS IN THE CLASSROOM.

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This workshop was based on some ideas in my paper (Rogers 2011) where I developed the principles of using concept maps of ‘significant’ items pertaining to the *history* of mathematics and building a narrative of relevant *heritage* content (Grattan-Guinness 2004) from where we can develop particular orientations relevant to specific classroom contexts. (Rogers 2011: 7-13)

A number of examples were presented from workshops used with teachers and secondary pupils (ages 11-18) where problems adapted from historical contexts were offered for criticism to participants. The main objective of the workshop was to discuss the manner in which these problems or others like them may be introduced in the classroom *to foster the pupil’s own epistemological process* in building up their personal mathematical knowledge.

Colleagues attending this workshop who have used historical material with students were invited to bring their own examples of classroom problems for discussion.

A particular focus was the research-based evidence for attending to the ideas of Ratio and Proportion, Spatial and Geometrical Reasoning, Introduction to Functions and the Development of Algorithms and Algebraic notation. (Watson, Jones & Prat 2013)

Some questions to consider about affordances (Gibson 1997, Heft 2003) and constraints were offered when using historical materials as classroom problems:

- Can this material be used (or adapted) with pupils at any age
- What mathematics (if any) do pupils need to know in order to address the problem
- What kinds of problem-situations is this material designed to raise
- What is its potential for developing conceptual knowledge
- Does it have relevance for building a knowledge of mathematics as a science
- Do the ideas involved appear in different areas and at different levels of mathematics
- Does this material encourage mathematical communication
- Does this material encourage teachers’ own reflection processes

About 25 colleagues attended, and a useful and provocative discussion ensued. I thank those colleagues for their contributions and encouragement.

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Workshop

TEACHER TRAINING IN THE HISTORY OF MATHEMATICS

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The History of Mathematics could be a powerful tool for Mathematics teachers to improve their teaching, by offering to students a variety of ways to achieve mathematical concepts successfully. The Catalan Mathematics curriculum for secondary schools, published in June 2007, contains notions of the historical genesis of relevant mathematical subjects within the syllabus. However, there is no indication to develop the content associated with these subjects. We have designed a course for pre-service teachers of Mathematics with the aim of providing them with the knowledge needed to use historical materials in their classrooms. This contribution aims to analyze the implementation of these historical Mathematics activities.

INTRODUCTION [1]

By means of original sources and significant texts, it is possible to learn from the past and teach Mathematics through a historical and cultural approach (Fauvel & Maanen, 2000; Katz, & Tzanakis, 2011). Knowledge of the History of Mathematics provides teachers with an understanding of the foundations and the nature of Mathematics, and with a capacity for a better understanding of how and why the different branches of Mathematics have taken shape as well as their connection with other disciplines (Jankvist, 2009).

In fact, the History of Mathematics is a very useful tool to help in the comprehension of mathematical ideas and concepts (Demattè, 2010). It is also a very effective tool to help in the understanding of Mathematics as a useful, dynamic, humane, interdisciplinary and heuristic science.

On the one hand, the History of Mathematics can be used as an implicit resource in the design of activities to adapt some standard concepts to the teaching syllabus, to choose context, and to prepare problems and auxiliary sources.

On the other hand, the History of Mathematics can also be used in an explicit way to direct and propose research works at baccalaureate level using historical material, to design and impart elective subjects involving the History of Mathematics using ICT, to hold workshops, centenaries and conferences using historical subjects, and to use significant historical texts in order that students understand better mathematical concepts.

We have designed a course for teachers of Mathematics with the aim of providing them with the knowledge needed to use historical materials in their classrooms. In addition, the use of these historical materials allows the teacher to use a different approach in which at the start of the class the teacher sets the students a text to read and helps them to interpret it, by giving them some guidelines and questions to answer. The fact of having to locate the texts in their historical context also encourages an interdisciplinary approach and helps the students to understand Mathematics as a discipline which is linked to other disciplines. We supply the teachers with original sources on which the knowledge of mathematics in the past is based. They have to work with these sources, which consist of reading and interpreting a selection of classical mathematical texts as well as learning how to locate and use historical literature or historical online resources. The task of teachers also involves the recognition of the most significant changes in the discipline of Mathematics; those which have influenced its structure and classification; its methods; its fundamental concepts and its relation to other sciences. Some materials have also been chosen to emphasize the socio-cultural relations of mathematics with politics, religion, philosophy and culture in a given period, and most importantly to encourage teachers to reflect on the development of mathematical thought and the transformations of natural philosophy (Pestre, 1995). The final project is drawn up by teachers themselves and consists of designing an activity for the students based on the material with which they have worked throughout the course.

In this paper we present three of the activities carried out in the course:

1. Using Chinese problems and procedures from an ancient classical book for teaching Mathematics.
2. Introducing the quadratic equation using historical methods
3. Algebra and geometry in the Mathematics classroom

Most of these activities have been tried out in secondary schools and intended to inspire teachers to create their own activities. The criteria for selecting the specific texts consist of their relationship with the historical contexts in the Catalan curriculum (Catalunya. Decret 143/2007).

USING CHINESE PROBLEMS AND PROCEDURES FROM AN ANCIENT CLASSICAL BOOK FOR TEACHING MATHEMATICS

For the following activities we use Chinese problems and procedures from *The Nine Chapters* for teaching mathematics.

The Nine Chapters and the historical context

Ancient mathematical texts were compiled during the Qin dynasty (221–206 BC) and Han dynasty (206 BC–AD 220). The most influential of all Chinese mathematical books, *The Nine Chapters,* was probably compiled in early Han dynasty (Dauben, 2007, 227). The purpose of this practical manual of mathematics consisting of 246 problems was to provide methods to be used in solving everyday problems of engineering; surveying, trade, and also taxation (Lam, 1994). Scholars believe that *The Nine Chapters* has been the most important mathematical source in China for the past 2000 years, comparable in significance to Euclid's *Elements* in Western Culture. Along the centuries, some scholars made copious commentaries on the book to explain the implied mathematical concepts. Among those commentators are Liu Hui (ca. 220-280), one of the greatest mathematicians of ancient China and Li Chunfeng (602-670), an outstanding astronomer and mathematician.

As the name suggests, the book contains nine chapters, and we focus on Chapter 9 *"Gougu"* (or base and height), which deals with problems for solving right triangles, involving the Gougu procedure, the principle known in Western Culture as the Pythagorean Theorem.

Activities carried out by pupils of secondary education

We have proposed a sequence of activities carried out by pupils of secondary education, based on the problems in Chapter 9 of *The Nine Chapters.* The activities are designed according to the fundamental figures described by Liu Hui (263) and Li Chunfeng (656) in their commentaries on the classical text, analysed in the bilingual translation by Chemla & Shuchun (2005, 703-745) and following the suggestions about their pedagogic value by Siu Man-Keung (2000, 159-166).

In general, *The Nine Chapters* is organized as follows: first there is the classical text, dealing with the problem statement with specific numerical data; secondly, the questions; thirdly, the answers, then a brief description of the procedure to find the solution; then the commentaries by Liu Hui and Li Chunfeng, which provide the algorithms needed to solve the problems, and finally the explanations of how the algorithms work.

Going to the beginning of Chapter 9: The title *Gougu*, which means base (*gou)* and height *(gu)*, has a subtitle, "*Solving height and depth, width and length*". The chapter contains 24 problems on right triangles. Problems 1-12 (Chemla & Shuchun, 2005, 703-721) deal with the base and height procedure, and problems 13-24 deal with similar triangles (Chemla & Shuchun, 2005, 723-745). The following problems are related to situations in a real context where the initial geometric assumptions appear.

At the beginning of Chapter 9, the classical text states "*Base (gou) and height (gu) procedure*", but later Lui Hui adds the following:

> "The shorter side is called the base (gou), the longest side the height (gu) joining the corners with each another is called the hypotenuse (xian)" (Chemla & Shuchun, 2005, 705).

The classical text states the *Gougu* theorem like an algorithm:

> "If each is multiplied by itself and the results, once added, are divided by the square root extraction, the result is the hypotenuse" (Chemla & Shuchun, 2005, 705).

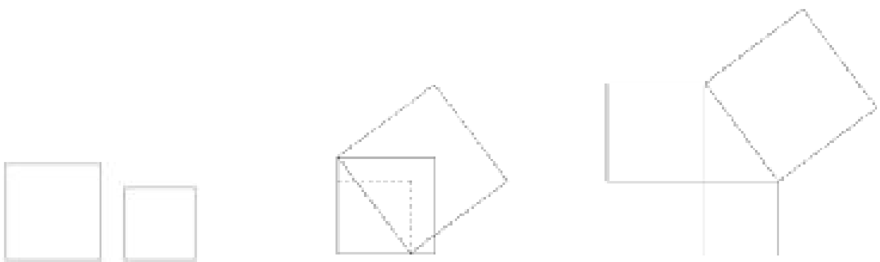
and later Lui Hui gives a geometrical proof :

> "The shorter leg multiplied by itself is the vermilion square, and the longer leg multiplied by itself is the blue-green square. Let them be moved about so as to patch each other, each according to its type. Because the differences are completed, there is no instability. Together they form the area of the square on the hypotenuse; extracting the square root gives the hypotenuse" (Chemla & Shuchun, 2005, 705).

Figure 1: *Gougu* theorem

We propose an activity for the students in which they try to obtain a similar proof for themselves. In order to prove that "the area of the square on side *c* is the sum of the areas of the squares on the sides *a* and *b",* they need to construct a square whose side is equal the hypotenuse from 2 squares of sides *a* and *b,* respectively. Then to prove the theorem, they have only to cut and paste figures.

The instructions for students could consist of the following: **a)** cut any two squares; **b)** place the small square inside the large square so that the two have a common vertex and base; **c)** draw the triangle on the side of the small square base and the height of the larger square; **d)** cut a third square of a side equal to the hypotenuse of the right triangle **e)** draw below the triangle and draw three squares obtained where appropriate (see Figure 2).



a b & c d

Figure 2: The steps of the student's instructions

Figure 3 below shows the work by one student:

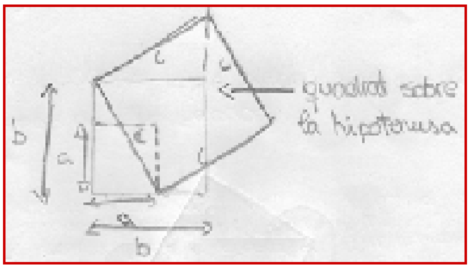


Figure 3: A student's production (14-15 years old)

Then, to prove the theorem (the largest area of the square is the sum of the areas of two other squares) it is only a matter of cutting and pasting (see Figure 4).

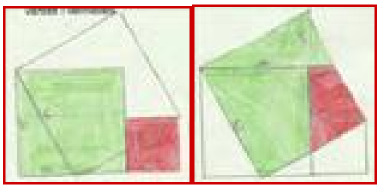


Figure 4: A student's production

Another activity we propose is based on problem 5, following the commentaries by Lui Hui. The wording is as follows:

> "Suppose we have a tree of 2 *zhang* as height and 3 *chi* as a perimeter. A climbing plant that grows from its base surrounds the tree seven times before reaching the top. One asks how long the climbing plant is" (Chemla & Shuchun, 2005, 709).

From the commentaries we can deduce that if we: a) roll up a sheet of paper forming a cylinder, simulating the trunk of the tree; b) draw the climbing plant around it; c) expand the sheet, we will obtain the solution, which is related to the *Gougu* theorem, as may be seen in Figure 5. Simply by adding seven times the hypotenuse, we will obtain the answer.

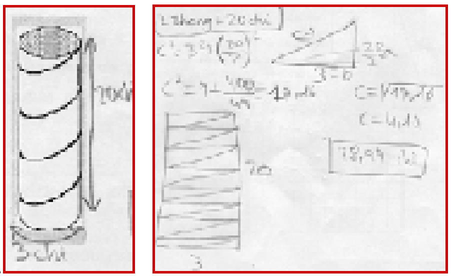
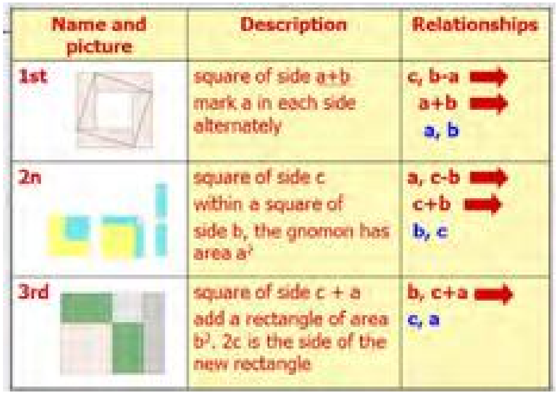


Figure 5: A student's production of problem 5

We also propose many interesting activities on the course by using special figures that the ancient Chinese employed to infer relationships between measures of the sides of the triangle, and sums and differences between them. They considered three geometrical figures, called “the fundamental figures”, which helped them to solve problems with right triangles in a geometrical way, that is, with “visual aids”.

The following table (Figure 6) shows these three fundamental figures and the relationships between their different measures to solve problems of right triangles:



Name and picture	Description	Relationships
1st	square of side a+b mark a in each side alternately	c, b-a → a+b → a, b
2n	square of side c within a square of side b, the gnomon has area a^2	a, c-b → c+b → b, c
3rd	square of side c + a add a rectangle of area b^2. 2c is the side of the new rectangle	b, c+a → c, a

Figure 6: The three fundamental figures and their relationships

These activities were conducted in the same way as the ancient Chinese, who in the absence of algebraic symbolism solved problems with reasoning based on geometry, and were very well accepted by the students. They were able to make sense of the rules of formal algebra, remarking that: "Now I understand it. These operations with letters are like the calculations we are doing with the figures!"

INTRODUCING THE QUADRATIC EQUATION USING HISTORICAL METHODS

In the following activities, we propose to solve equations using the al-Khwārizmī method (by completing squares); students can benefit from visual reasoning that combines algebra (in current notation) and geometry (Katz & Barton, 2007, 185-201).

Abu Ja'far Muhammad ibn Musa al-Khwārizmī (ca. 780-850)

His name indicates that he may have come from Khwarezm (Khiva), then in Greater Khorasan, which occupied the Eastern part of the Greater Iran, now the Xorazm Province in Uzbekistan.

He was a mathematician, astronomer and geographer during the Abbasid Empire, and a scholar at the House of Wisdom in Baghdad.

"The Compendious Book on Calculation by Completion and Balancing” (*Kitāb al-Mukhtasar fī hisāb al-jabr wa’l-muqābala* (ca. 813) المختصر حساب الجبر والمقابلة) was the most famous and important of all of al-Khwārizmī 's works (Djebbar, 2005, 211; Toomer, 2008).

In Renaissance Europe, he was considered one of the inventors of algebra, although it is now known that his work was based on older Indian or Greek sources.

The treatise Hisāb al-jabr wa'l-muqābala (ca. 813)

The book was translated into Latin by Robert of Chester (Segovia, 1145) as *Liber algebrae et almucabala,* hence "algebra", and also by Gerard of Cremona (ca. 1170). A unique Arabic copy of manuscript from 1342 is kept at the Bodleian Library in Oxford, and was translated into English in 1831 by Frederic Rosen.

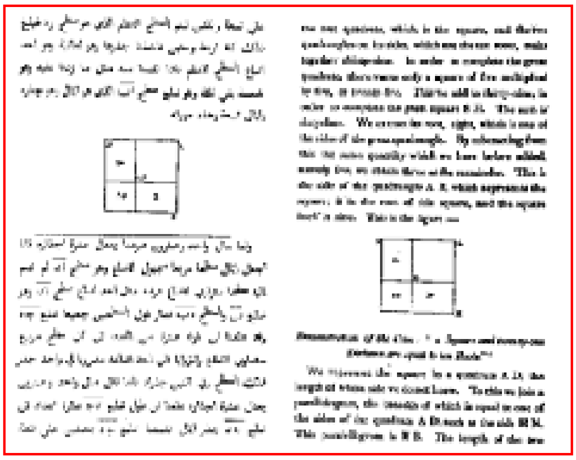


Figure 7. The text and diagrams in the Rosen edition (Rosen, 1831, 322/16)

We chose the text and the diagrams from the Rosen edition (see Figure 7) to design the activities for solving quadratic equations with visual reasoning (Rosen, 1831).

In the text, the author provided an exhaustive account of solving polynomial equations up to the second degree, and also discussed the fundamental methods of "reduction" and "balancing", which refers to the transposition of terms from one side of an equation to the other side, that is, the elimination of equal terms on both sides of the equation.

Al-Khwārizmī wanted to give his readers general rules for all kinds of equations and not just how to solve specific examples. His rules for solving linear and quadratic equations began by reducing the equation to one of six standard forms.

We will use the case: "a Square and ten Roots are equal to thirty-nine Dirhems", to design the activities. Al-Khwārizmī stated as follows (see Figure 8):

> "We proceed from the quadrate AB, which represents the square. It is our next business to add to it the ten roots of the same. We halve for this purpose the ten, so that it becomes five, and construct two quadrangles on two sides of the quadrate AB, namely, G and D, the length of each of them being five, as the moiety [half] of ten roots, whilst the breadth of each is equal to a side of the quadrate AB. Then a quadrate remains opposite the corner of the quadrate AB. This is equal to five multiplied by five: this five being half of the number of the roots, which we have

added to each of the two sides of the first quadrate. Thus we know that the first quadrate, which is the square, and the two quadrangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied by five, or twenty-five. This we add to thirty-nine in order to complete the great square SH. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrangle. By subtracting from this the same quantity, which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrangle AB, which represents the square; it is the root of this square, and the square itself is nine."(Rosen, 1831, 15-16).

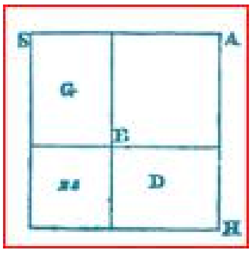


Figure 8. Geometrical justification by al-Khwārizmī (Rosen, 1831, 16)

We begin the activity by solving incomplete equations

a) The algebraic procedure

We use some sessions to solve incomplete equations with algebraic procedures, reduction and balancing. Students know how to solve linear equations and we apply this procedure to incomplete second-degree equations.

b) The geometrical procedure

We devote some sessions to the geometrical visualization of x^2 and, step by step, we introduce students to solving the second degree incomplete equations geometrically. They have to understand x and x^2 as the measure of the sides of the squares, and their areas, respectively (see Figure 9).

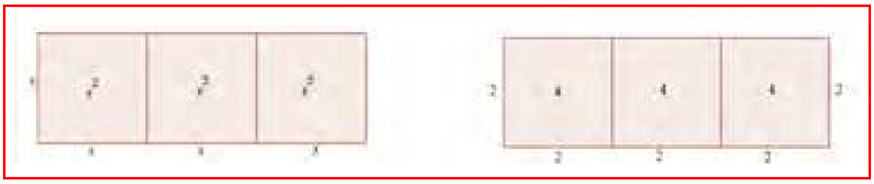


Figure 9: The geometrical interpretation of the incomplete equation $3x^2 = 12$

When students have discovered how to solve this kind of incomplete equation, we ask them to write equations knowing their solutions: For example:

$x = 3 \rightarrow x^2 = 9,\ 2x^2 = 18, \ldots$

$x = 0$ and $3 \rightarrow x^2 = 3x,\ 2x^2 = 6x, \ldots$

We solve equations like: $ax^2 = c$, and $ax^2 = bx$

Now we introduce the resolution of complete quadratic equations. The first example was the same as that by al-Khwārizmī: $x^2 + 10x = 39$ (Figure 10).

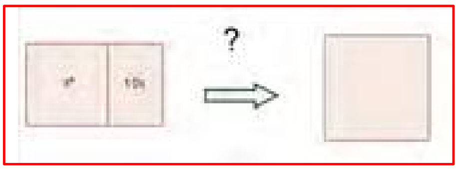


Figure 10: Is it possible to transform this rectangle of area 39 into a square?

As in the al-Khwārizmī procedure, we guide students to its solution with this idea (see Figure 11).

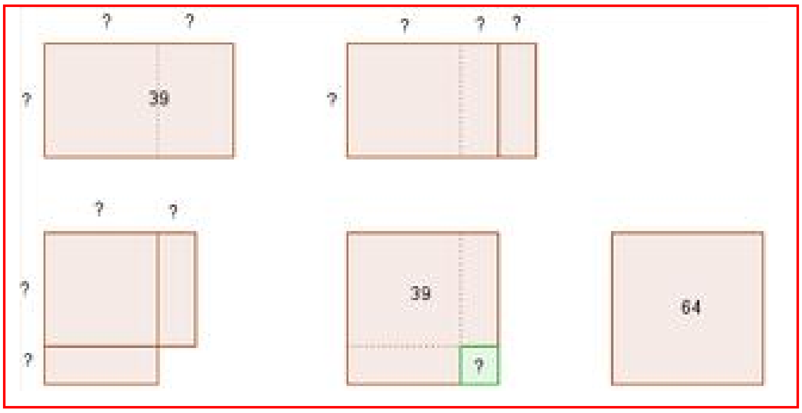


Figure 11: The diagrams in the sequence of activities

We also work with negative numbers, although al-Khwārizmī only worked with positive numbers.

Students know that 64 has two square roots, 8 and -8, and then we can obtain two solutions of the equation, the geometrical one $x = 3$ and $-8 = x + 5$ $x = -13$ (see Figure 12, when they solve $x^2+6x=40$)

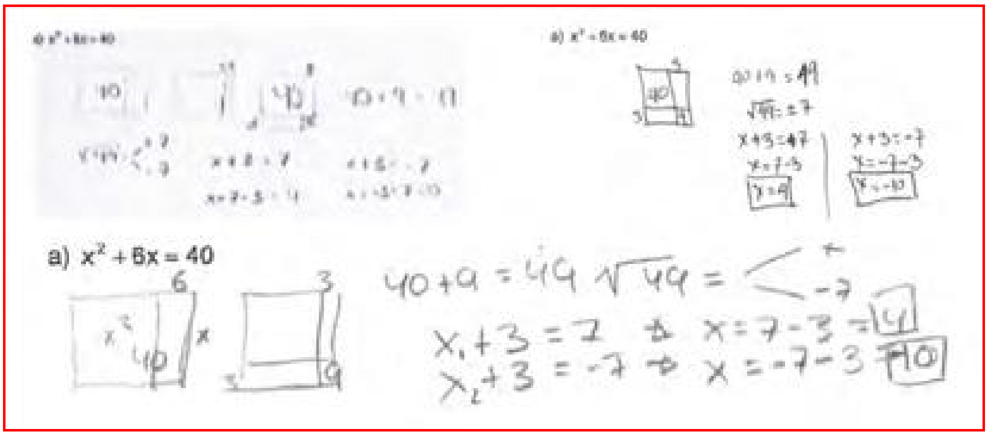


Figure 12: Some students' productions (14-15 years old) with $x^2 + 6x = 40$

By introducing the resolution of quadratic equations by completing squares throughout one school year, then waiting until the next course to introduce the resolution by using the usual formula, yielded to two relevant results about student learning. On the one hand, they discovered that the problems they were studying originated in ancient times and different cultures, while on the other they also realized that algebraic formulas could make more sense when interpreted in a geometrical manner.

ALGEBRA AND GEOMETRY IN THE MATHEMATICS CLASSROOM

This part consists of activities containing singular geometric constructions used for solving the quadratic equation in the seventeenth century. These analyses linking algebra to geometry provide students with a richer view of Mathematics and improve the teaching and learning processes. Thus, the reflection on these geometric constructions of algebraic expressions historical helps to develop the analytic and synthetic thought of students.

Indeed, the study of the origins of polynomials and their associated equations gives us a history of the geometric construction of the solution of the quadratic equation with instructive and suggestive passages for students, whether at high school or college degree level. We focus on the process of algebraization of mathematics, which took place from the late sixteenth century to the early eighteenth century (Mancosu, 1996, 84-91). This was mainly the result of the introduction of algebraic procedures for solving geometrical problems.

First geometrical justifications

In his treatise *Kitāb al-Mukhtasar fī hisāb al-jabr wa'l-muqābala* (ca. 813), Mohammed Ben Musa Al-Khwārizmī (ca. 780-850) describes different kinds of equations using rhetorical explanations, and without symbols. His geometrical justifications of the solutions of equations are given by squares and rectangles, as we have shown in the previous activity. Later, when Leonardo de Pisa (1170-1240) (known as Fibonacci) expresses these Arabic rules in his *Liber Abaci* (1202), he uses "radix" to represent the "thing" or unknown quantity (also called "res" by other authors) and the word "census" to represent the square power. This rhetorical language continued to be used in several algebraic works in the early Italian Renaissance, such as *Summa de Arithmetica, Geometria, Proportioni e Proportionalità* (1494) by Luca Pacioli (1445-1514), *Ars Magna Sive de Regulis Algebraicis* (1545) by Girolamo Cardano (1501-1576) and *Quesiti et Invenzioni Diversa* (1546) by Niccolò Tartaglia (1500-1557). All these writers used geometric squares, rectangles, and cubes to represent or justify algebraic manipulations (Stedall, 2011, 1-49).

One of the firsts to question these geometrical justifications was Pedro Nunes or Núñez (1502-1578) in his book *Libro de algebra en arithmética y geometría* (1567). After showing the classic geometrical justifications by completing squares, he claims:

> "While these demonstrations of the last three rules are very clear, by saying that in the demonstration of the first rule it is presupposed that a censo with the things of whatever number can equal any number, number being what we have defined at the beginning of this book, the adversary will be able to state that this presupposition is not true. Therefore, it will be necessary to demonstrate it." (Núñez, 1567, fo. 14r).

After this statement, Nuñez proceeds to introduce new geometrical constructions of the solutions to the quadratic equation. Although Nuñez was a pioneer in introducing new geometrical constructions, the more singular ones will occur later, as we analyse in an activity implemented in the classroom described below.

Geometrical justifications in the seventeenth century

The publication in 1591 of *In Artem Analyticen Isagoge* by François Viète (1540-1603) constituted a step forward in the development of a symbolic language. Viète used symbols to represent both known and unknown quantities and was thus able to investigate equations in a completely general form ($ax^2+bx=c$). He introduced a new analytical method for solving problems in the context of Greek analysis. This algebraic method of analysis allowed problems of any magnitude to be dealt with, and his symbolic language was the tool he used to develop this program. Viète showed the usefulness of algebraic procedures for solving equations in arithmetic, geometry and trigonometry (Bos, 2001, 145-154). He solved equations geometrically using the Euclidean idea of proportion: proportions can be converted into equations by setting the product of the medians equal to the product of the extremes. In 1593, Viète published *Effectionum Geometricarum canonica recensio,* in which he geometrically constructed the solutions of second-and fourth-degree equations. Later, in 1646, F. A. Schooten edited this book in Viète's *Opera Mathematica.* We have used this edition to design the activity for the classroom. Viète claims:

> "Proposition XII Given the mean of three proportional magnitudes and the difference between the extremes, find the extremes. [This involves] the geometrical solution of a square affected by a [plane based on a] root [$A^2 + BA = D^2$]. Let FD be the mean of three proportionals [= D] and let GF be the difference between the extremes [= B]. The extremes are to be found. Let GF and FD stand at right angles and let GF be cut in half at A. Describe a circle around the centre A at the distance AD and extend AG and AF to the circumference at the points B and C. I say that what was to be done has been done, for the extremes are found to be BF [$A + B$] and FC [$=A$], between which FD [$=D$] is the mean proportional. Moreover, BF and FC differ by FG, since AF and AG are equal by construction and AC and AB are also equal by construction. Thus, subtracting the equals AG and AF from the equals AB and AC, there remain the equals BG and FC. GF, in addition, is the difference between BF and BG or FC, as was to be demonstrated." (Viète, 1646, 234).

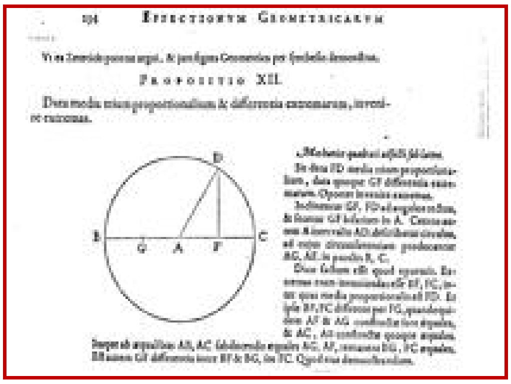


Figure 13: Viète's construction of three proportional (Viète, 1646, 234)

He sets up the equation $A^2 + BA = D^2$ by means of a proportion, which can be expressed in modern notation as $(A + B) : D = D : A$. Viète's geometric construction of the lines A, B, D satisfying this equality is set out in Figure 13. Viète draws FD = D and GF = B, making a right angle, and divides B by half AF = $B/2$. He describes a circle whose radius is equal to AC, which we can identify with the hypotenuse of the triangle formed by $B/2$ and D, AD = AC=$[(B/2)^2 + D^2]^{1/2}$. The solutions are then the segments FC = AC - AF and BF = BA + AF, which take BA = AC = radius (Massa, 2008, 295).

In the classroom, after finishing the lesson of quadratic equation, we carry out an activity taking into account these geometrical constructions in order to highlight the algebraic solution of the quadratic equation from another perspective. The procedure is as follows: after describing historical context, including Nuñez's quotation, and analysing Viète's geometrical construction, the teacher could pose the students some questions to clarify the ideas.

1) Reproduce Viète's geometrical construction and give an explanation of the procedure. 2) Could this geometrical construction be used for any quadratic equation?Give reasons. 3) What about negative solutions? 4) How are the Pythagorean and the altitude Theorem used? Explain their relationship to the solution of the equation. 5) What is the main difference between this geometrical construction and the classical construction by completing squares?

After analysing and discussing students' answers, the teacher continues by presenting a new historical text with another geometrical construction. Indeed, as Viète's work came to prominence at the beginning of the seventeenth century, mathematicians began to consider the utility of algebraic procedures for solving all kinds of problems. Thus, the other singular example is the geometrical construction in a quadratic equation found in the influential work *La Géométrie* (1637) by René Descartes (1596-1650). Descartes begins Book I by developing an algebra of segments and shows how to add, multiply, divide segments, and calculate the square root of segments with geometrical constructions (Bos, 2001, 293-305). Next, Descartes shows how a quadratic equation may be solved geometrically (see Figure 14):

"For example, if I have $z^2 = az + bb$, I construct a right triangle NLM with one side LM, equal to b, the square root of the known quantity b^2, and the other side, LN equal to ½ a, that is, to half the other known quantity which was multiplied by z, which I assumed to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line z." (Descartes, 1637, 302-303).

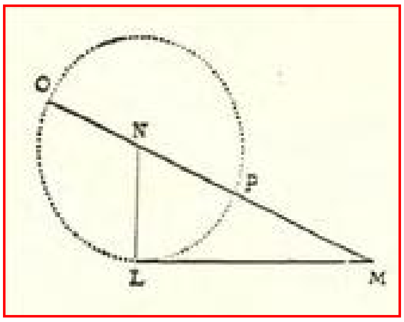


Figure 14: Geometrical construction (Descartes, 1637, 302)

In the classroom, after drawing and analysing Descartes' geometrical construction, we could hold a discussion with the students. It is important to point out that symbolic formula appears explicitly in Descartes' work. His geometrical construction corresponds to the construction of an unknown line in terms of some given lines; hence, the solution of the equation is given by the sum of a line and a square root, which has been obtained using the Pythagorean Theorem. However, Descartes ignores the second root, which is negative, and he did not quote that this geometrical construction could be justified by Euclid III, 36, where the power of a point is proved with respect to a circle.

The questions posed to students are similar to those by Viète. Moreover, students may also reflect on the meaning of both constructions. The differences from Viète are relevant because Descartes explicitly writes in the margin "how to solve" the equation, while Viète, by contrast, solves a geometric problem with a geometric figure in which a proportion is identified with an equation. Another relevant subject to consider with the students is the analytical and/or synthetic approach used in each construction.

Other possible questions: What geometrical reasoning did the author use? What is the role of the Pythagorean Theorem in solving the equation of second degree? What relation is there between this geometrical construction and the algebraic solution of the second degree equation?

All these questions enable teachers to consider the solution of quadratic equations from a geometrical point of view, as well as prompting thought about the relation between algebra and geometry through history.

SOME REMARKS

These kinds of activities are very rich in terms of competency-based learning, since they allow students to apply their knowledge in different situations rather than to reproduce exactly what they have learned. In addition, they help students to appreciate the contribution of different cultures to knowledge, which is especially important in classrooms today, where students often come from different countries and cultures.

The design of these activities also allows different levels of development and in some cases the distribution of tasks among students according to their individual skills.

The activities, based on the analysis of historical texts connected to the curriculum, contribute to improving the students' overall formation by giving them additional knowledge of the social and scientific context of the periods involved. Students achieve a vision of Mathematics not as a final product but as a science that has been developed on the basis of trying to answer the questions that mankind has been asking throughout history about the world around us.

All these activities devote an important part to geometry, which is a standard in the syllabus that students should improve, as recommended in the results of PISA assessment.

Geometry has a great visual and aesthetic value and offers a beautiful way of understanding the world. The elegance of its constructions and proofs makes it a part of Mathematics that is very suitable for developing the standard process of *reasoning and proof* of the students.

In addition, geometrical proofs have a great potential for relating geometry and algebra; that is to say, establishing connections between figures and formulas; geometric constructions and calculations.

NOTES

1. This research is included in the project: HAR2013-44643-R.

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Workshop
HISTORICAL MATHEMATICAL MODELS IN TEACHER EDUCATION - WORKSHOP ON THE DEVELOPMENT OF QUESTIONS AND CRITICAL QUESTIONING

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We present the conception and design of a seminar in the master's programme of mathematics education. The seminar starts with students' personal experiences relating to mathematical experiments, models and visualisations of mathematical objects, followed by a historical excursion around historical collections of mathematical models. On the basis of that, students undertake project work on models of drawing instruments and simple curves in historical, socio-cultural or mathematical contexts.

INTRODUCTION

The paper deals with a workshop held in the afternoon of the last day of the 7th European Summer University on History and Epistemology in Mathematics Education at Aarhus University, Campus Copenhagen. In spite of the time of the event, the workshop was well attended and met the interests of mathematics educators with various backgrounds. Our aim is to give the participants a memory of the event and to outline the concept and design of a seminar, which uses history as a tool to awaken awareness and understanding of individual development and societal change in a mathematical context. We use a comparative view on the everyday world and its past to disturb widespread routines, approaching development from an output-orientated perspective and in normative terms.

THE MAIN MOTIVATION FOR THE CONCEPT AND DESIGN OF THE SEMINAR

The aim of the presented seminar concept is to deal with normative perspectives, evaluative assessments and output-orientated categorizations of our students on education and development. Since the so-called PISA shock, the German education system has undergone subtly comprehensive restructuring, the concept of "Bildung" (usually translated as education) being replaced by the notion of “Ausbildung” (training). At least since Klafki (1994, 2000) formal education approaches in German educational sciences are believed to have been overcome. However current educational policy and real school life tells a different story. Global testing, outcome-oriented learning, competence-based redesigning of curricula and measuring and evaluating by educational standards led to a predominance of normative approaches to learning and development (Jahnke & Meyerhöfer, 2007).

Historical processes allow us to caricature such models of development. The reduction of complex relations and causal dependencies to input-output mechanisms can be taken in a cultural-historical context ad absurdum.

The gradual economisation of the educational system during the last decade also has implications on language, approaches to problems as well as on the knowledge relevant to action, prognosis and orientation of our student mathematics teachers.

Assessments of the capabilities of our students show that they master the reproduction of information and texts very well. They work hard on the perfection of presentational skills. Their strengths also include the use of modern media to access information and pattern recognition skills. Their weaknesses lie in their conceptual understanding.

Volker Ladenthin's description of contemporary student problems confirms our experience:

"Students are barely able to use abstractions. One has to speak in examples - and they will be happy to discuss on the level of examples. However, generalization and transfer of expertise hardly succeed. To transmit the statements of ancient authors (Aristotle) in contemporary parlance fails less due to fragmentary historical knowledge as to the lack of transferability. Textual analysis is done very vaguely and always very generally ("Comenius says that school is good for the people"). Syntheses is created additively and is by no means nuanced. Judgments are linear (not multi-perspective)" (Ladenthin, 2014, p. 17).

Based on this situation, the focus of the seminar is not on a historical outline of historical models and collections (as in Barbeau & Taylor, 2009) or the use of historical mathematical models and instruments in the classroom (e.g. Shell-Gellasch, 2007) but on the development of individual questions giving rise to a different contextualisation of mathematical artefacts and instruments, including the historical.

We use the notation model because it is used in connection with historical collections of mathematical artefacts, visualisations and instruments present in many European older universities. During the last decade the German research society DFG supported activities aimed at the involvement of historical scientific collections in research, teaching and the popularisation of science [1]. Perhaps that is why presently one can find in many mathematical and physics institutes expositions of historical models with the mentioned decorative purposes. We are aware that the notation model may cause irritation because of its very different use in the context of modelling. The abundant use of pictures and descriptions of activities involving the models as an object or tool should however prevent misunderstanding.

To avoid questions leading to formal reconstructions we start with our students' personal experiences with collections of mathematical models, such as visualisations, instruments and artefacts.

It seems to be useful to divide the individual school experiences of the participants with mathematical models into experiences as pupils, and experiences as mathematicians or as teachers. An excursion to a "hands-on" museum with mathematical models, for instance, raises quite different questions for the visitor, the tutor and the organiser.

FROM LEARNING MATERIALS TO HANDS-ON EXHIBITION

Contemporary student mathematics teachers can have varied experiences with mathematical models. Reform pedagogy is an important topic in educational studies. Fröbel (Klafki, 1964), Pestalozzi (Pestalozzi & Klafki, 1997), Kerschensteiner (Kerschensteiner, 2011) and Dewey (Dewey, 2007) represent sense perception and activity orientation in mathematics education and their approaches are part of the curriculum in educational science. Depending on the interests of the student group and their background, one could begin with a pedagogical-philosophical orientated introduction into reform pedagogy, starting with the 19th century or with an introduction closer to mathematics - for example, with Treutlein's collection of mathematical models (Wiener, Teubner, 1912).

For both lines of discovery learning we can use the students' personal experiences with mathematical models.

	Learning materials like the *mathematical box (Mathekoffer)* or sets of platonic solids or experimental instruments are part of the inventory of most schools and are hence discussed in courses in mathematics education.	

Fig 1: examples of working materials for mathematical lessons

A good possibility of having a common experience to build on comes with participation or even the development and organisation of a hands-on mathematical exhibition (Fig.2). Good occasions are events like the open days of mathematics departments, projects with local schools and museums and joint activities in teacher training.

	Before the introduction of the Bachelor-Master system and its rigid credit-course system it was easier to implement activities and introduce new topics and fit seminars and courses into mathematics education.	

Fig 2: Some examples of hands-on activities and materials

An even better option for the development of a common practice in preparation of our seminar on collections of historical models is to base it on another university course with practical elements; we will give an example of such a course in connection with the description of our workshop.

Nevertheless, before the introduction of the Bachelor-Master system and its rigid credit-course system it was far easier to introduce new topics and to develop fitting seminars and courses.

THE SEMINAR CONCEPTION CONCERNING CONTENT AND STRUCTURE

The development and production of mathematical models were already in use in the 19^{th} century and early 20^{th} century for the training of student mathematics teachers. Nowadays, this is an episode in the history of European science. The use of historical mathematical models and their digital images in the study, the teaching and development of mathematics allow for the relation of historical, technical, educational and information technology aspects to each other.

The workshop introduced an approach that can be used to design and produce learning materials for a seminar in university education of student mathematics teachers. There is a variety of literature with historical and pedagogical perspectives on the development of mathematical models, which constitutes the content framework for the historical research undertaken by the participants in the seminar (for instance, Bussi et al. 2010). The students actively take part in the process of material choice and organisation of the seminar. The format of the seminar supports responsible learning with initiative, learning by discovery and situational learning. However, it seems to take some time and effort to get used to it.

The participants conduct the study of historical mathematical models in the seminar by three types of contextualization. They are in particular:

1. Mathematical models as a historical artefact,
2. Historical mathematical models as a source of study and as a visualization of historical mathematical contents,
3. Mathematical models as a source of inspiration for experimentation, varying and developing new models and visualizations.

The study of the first perspective – historical mathematical models from a socio-cultural perspective – could include aspects such as:

- Teacher training before, during and after the German *Meraner Reform*,
- Intuition and perception of mathematics in the context of educational values and norms,
- Mathematical models in the context of the discussion between pure and applied mathematics,
- Historical mathematical models and patriotic education.

The second and the third perspectives focus on the mathematics behind the models and on visualisations of mathematical concepts. We introduce an additional structure through the goals of the related mathematical activities (figure 3). The concrete choice of specific models depends on the mathematical preparation of the participants in the seminar. A first attempt to organize such a seminar has been made by David Rowe, Oliver Labs and myself representing the history of mathematics, mathematics and computer science and mathematics education. The prerequisites for the students consisted of the main basic courses in pure and applied maths, courses in educational studies (pedagogy, psychology) and introductory courses in the historical and cultural roots of mathematics as well as seminars in maths education. Some of the students attended reading courses and seminars in selected topics regarding the history of mathematics.

In this first seminar, we restricted ourselves to historical models, mechanical instruments and drawing tools related to plane curves. The individual models were assigned to the themes:

- Selected static models,
- Selected kinematic models,
- Selected models closely related to school curricula.

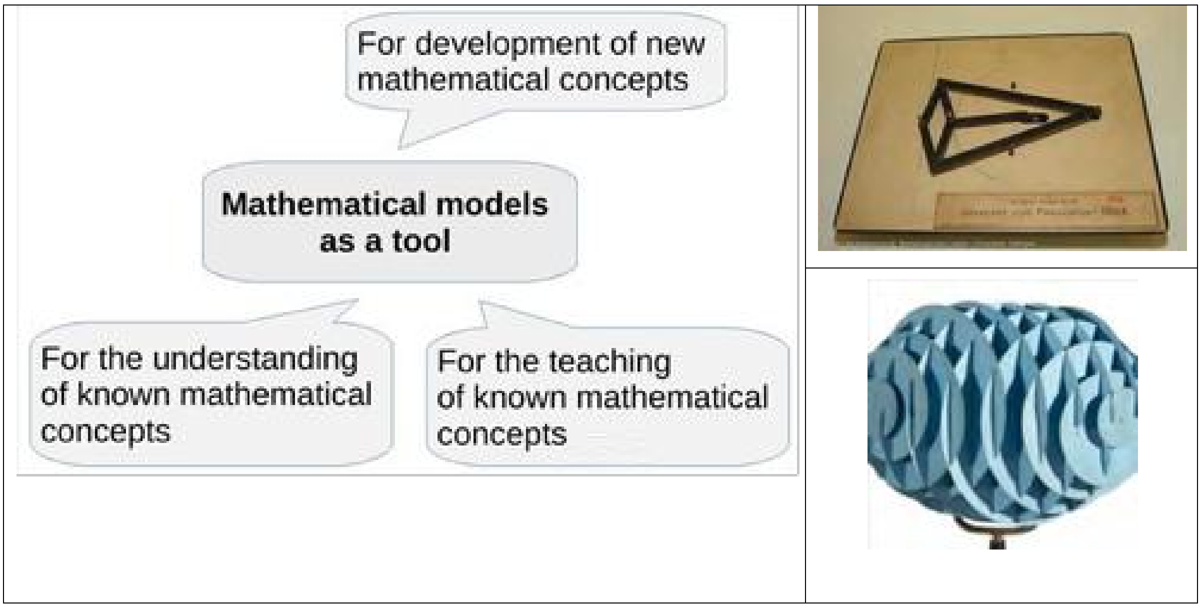


Fig 3: Perspectives on mathematical models as a tool, examples from the historical collection of mathematical models of the university Göttingen [2] und the university Halle [3].

Understanding the background of most models of the Brill and Schilling collection [4] (Polo-Blanco, 2007) is a very challenging mathematical task for students. However, the visual and tactile access when working with visualizations and real models fosters this understanding. This takes place in the realm of the third contextualization while dealing with the illustrated mathematics and the development of further illustrations. The concept of the seminar allows for the possibility of discussing different themes concerning elementary mathematics, the history of mathematics, mathematical teaching methodology and computer algebra.

From our experiences with the first seminar of this type, it seems useful to plan a prior reading course, making students familiar with the conceptual foundations of using the history of mathematics as a tool and as a goal, as well as with the corresponding concrete examples of classroom practice from different countries and time periods (Jankvist, 2009. Fauvel & Maanen, 2000).

OUTLINE OF THE WORKSHOP

The workshop started with a short presentation of the concept of the seminar. As an example of a collection of historical mathematical models we chose the *Göttinger Sammlung historischer mathematischer Modelle*, which is a historical collection of historical mathematical models closely connected with the name and activities of Felix Klein (Rowe, 2013).

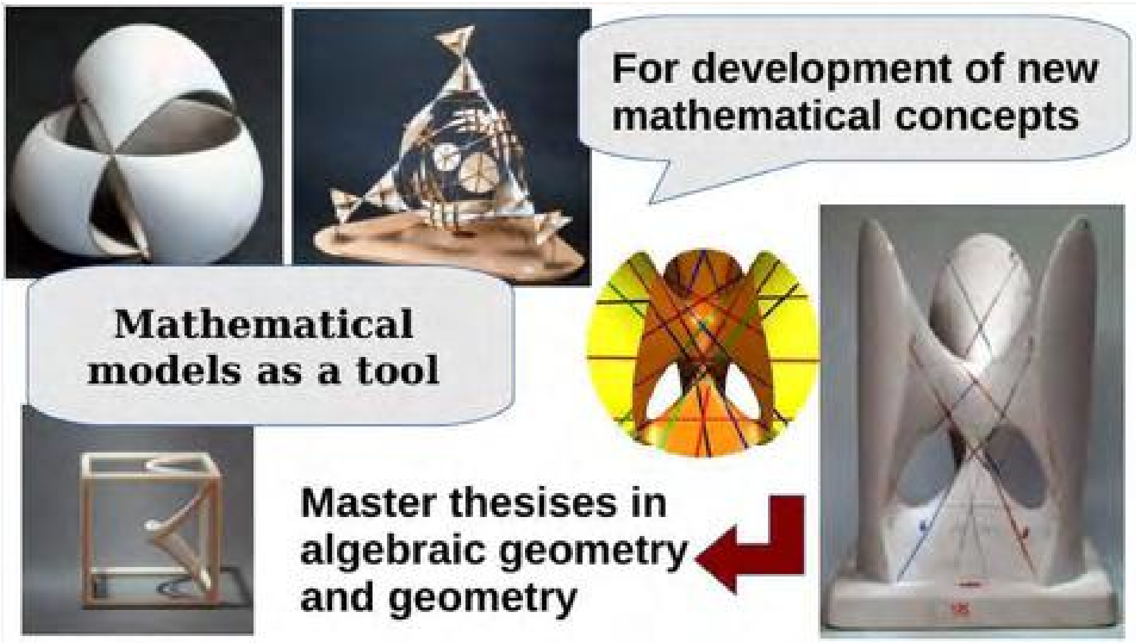


Fig 4: Some examples of historical and modern mathematical models

This collection is available digitally (http://www.uni-math.gwdg.de/modellsammlung/). Moreover, during the last year in the framework of a project at the Georg-August University Göttingen to introduce historical collections into teaching, the collection of historical mathematical models became subject of several teaching activities in the study of mathematics as well as in didactics. The topic of mathematical models is highly suited for pushing students to formulate their own research questions. After a short introduction to the background of the participants in the workshop, we started with a short presentation of the Göttinger historical collection of mathematical models. In order to illustrate the pedagogical method central to the seminar, the participants were invited to formulate research questions related to the models they would like to study. Most participants wanted to understand more of the mathematics visualised by the models.

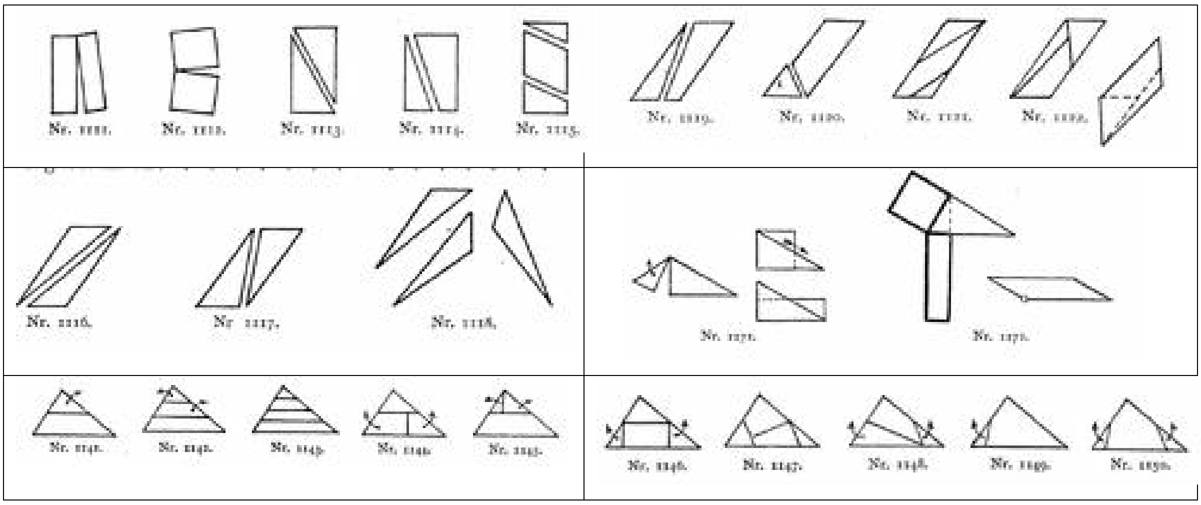
In the university seminar the situation was quite different. Most students wanted to know about the materials of the models and their durability, the form of activities using the models as tools (group work, individual work), the order and delivery times, the quantities in production and the prices of the models compared to other products. There were also approaches to studying the mathematical models as artworks and as a source of inspiration for artists.

The contextualisation of the models by the university students was much more social-cultural than the Copenhagen workshop.

One of the strengths of the university seminar was the team teaching and the interdisciplinary approach of the teaching staff. Depending on the type of questions raised by the students, they could consult either the historian of mathematics, or the specialist in algebraic curves and surfaces and their visualisations or those of the maths educator. For students interested in understanding more about the mathematical background of models of algebraic curves, there were possibilities to write a master's thesis on this topic and to study and develop digital images of the models and digital visualisations (figure 4).

For a workshop with participants of varying mathematical backgrounds, the historical models of algebraic surfaces were, however, too complex to illustrate different contextualisations.

The models that were presented were not part of the Brill and Schilling collection for university education but were models from the Treutlein Catalogue, i.e. models for use in schools.



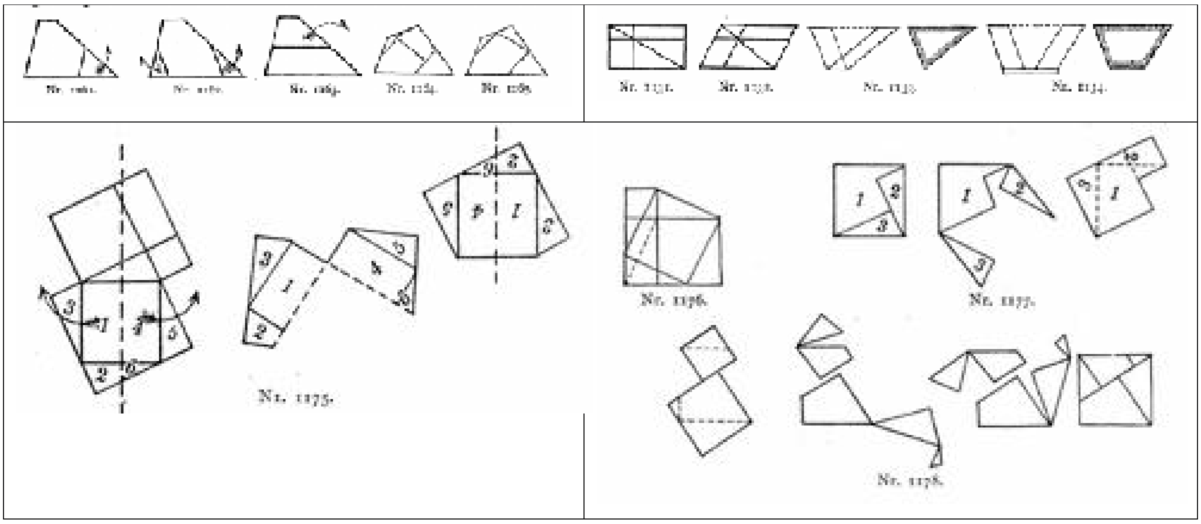


Fig 5: Samples from Treutlein's Catalogue (Wiener & Teubner, 1912)

The second exercise of the workshop sought to get the participants into the spirit of German school practices of the early 20th century. The participants were asked

1. To choose from the Treutlein's Catalogue a series of models from a chapter with geometrical flat models (fig. 5),
2. Following the pictures from the Catalogue to produce models from cardboard,
3. To give a short lesson plan on the basis of activities with the cardboard models,
4. To formulate corresponding tasks and exercises with the models.

The main implicit goal of this group-work was to develop a contextualisation of the chosen school model, which is related to individual activities and routines. The latter are the basis for formulating meaningful research questions on the subject of historical collections of (industrially produced) mathematical models for classroom activities.

Another reason to choose rather simple geometrical models from the collection is Treutlein's accompanying geometry textbook (Treutlein, 1911). The historical school textbook gives the possibility of contrasting contemporary and historical mathematical activities – modern lesson planning and problem solving with given (historical tools) and the intended activities with the same tools.

The corresponding tasks during the workshop were:

- Find in Treutlein's geometry book tasks to your crafted model.
- Compare the task with your modern versions.
- Give a formulation of a task, which could be Treutlein's.

The first part of the workshop was rounded off with a comparison of the treatment of Pythagorean Theorem in Treutlein's textbook with a modern mathematical school textbook which uses very similar puzzles for the proof.

The first part finished with a discussion of questions inspired by the making of models:

- In the historical context
- In the modern context.

Inspired by the use of models:

- From the perspective of students
- From the perspective of teachers
- From the perspective of authors.

The second part of the workshop dealt with the independent development of visualisations, models and experiments as teaching tools.

It started with a presentation of contemporary self-crafted experiments and visualisations for primary school and high school students. The presented hands-on exhibition for primary school students was a project we did in Göttingen for an annual urban science week. The project was organised by mathematics educators and students of the mathematics department of university Göttingen, it started in 2003 and continues to the present. In this hands-on exhibition students worked as tutors and accompanied experiments which they had developed on their own as part of a seminar in mathematics education (Fig.6). The activity initiated the development of similar school exhibitions in various primary schools through to a regular after school maths club [5].

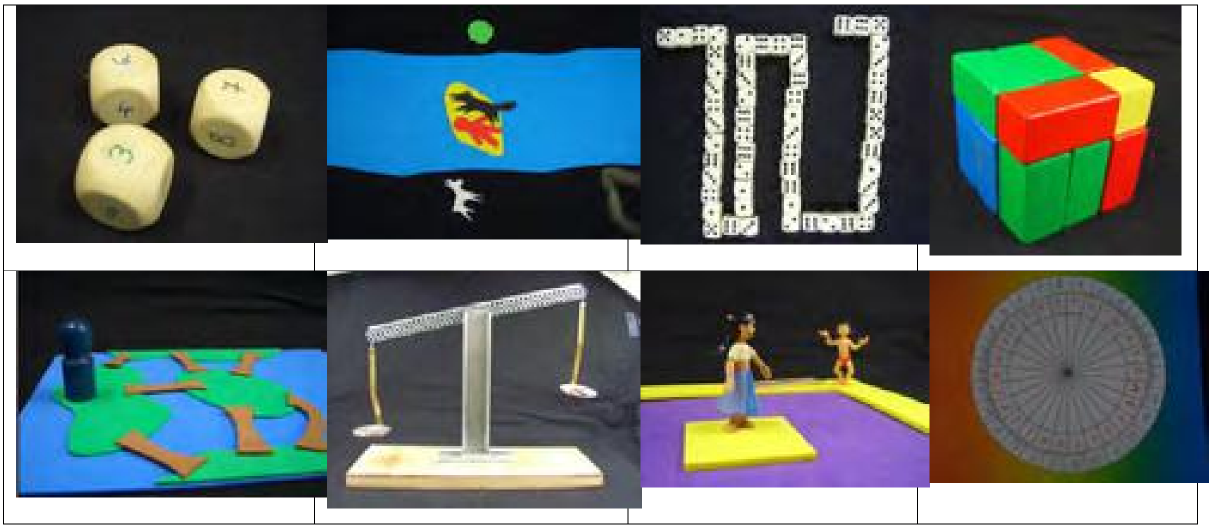


Fig 6: Experiments and models in the framework of a hands-on exhibition of mathematical experiments organized and developed for primary school students

The presented models and visualisations for high school students were results of the student work of our seminar on historical models. As we have already explained, due to the mathematical preparation of our students, we restricted ourselves to mostly simple plane curves. The mathematical themes of the seminar and related course works were:

- Models related to problems of Apollonius
- Drawing instruments
- Cycloids
- Involute and Evolute
- Curvature
- Quadrics
- Curves in space

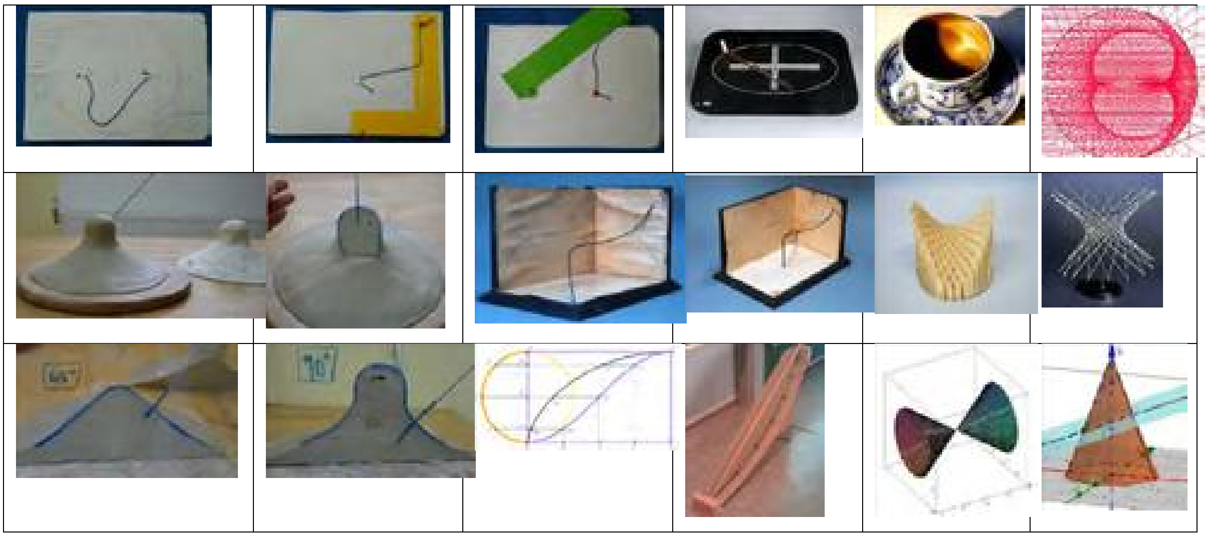


Fig 7: Models, experiments and digital visualisations produced by students in the framework of the seminar

The students were encouraged to visualise the studied mathematical objects, to create models, to develop experiments and to support digital visualisations and experiments. They had the possibility of attending an associated computer algebra course. Some of the produced visualisations and models of studied mathematics are shown in Fig.7. Similarly to the first part of the workshop, the aim of the introduction was to contextualise the making of models and visualisations by personal experiences and motives. The participants were asked to formulate historical, socio-cultural and mathematically motivated research questions based on their own interests and experiences. Because of the very different backgrounds of the participants it would have been more productive to choose only one instrument and discuss it from various perspectives. Most of the questions had to do with the mathematical background and questions of where to find relevant elementary written mathematical sources. The last task for the participants was to make their own models from paper, cardboard, coloured pencils or crayons, scissors, and binding material such as yarn or twine. For guidance, one could use the Internet, student assignments or real instruments like a pantograph.

IN RETROSPECT

After the workshop, I was asked for more material concerning the seminar, like literature, instructions for model constructions and applications for the digital experiments.

To present and illustrate the concept of an activity aimed at raising questions and not giving answers was harder than I expected. An adverse impact was the seating arrangements – one group on a very long small table. Working in small groups presenting the results of group discussions with an overhead projector (as is done in student seminars) would have given the workshop more structure in form of "answers in questions".

I am especially grateful to Jan van Maanen for his enriching comments during the discussion and his inspiring enthusiasm at the production of cardboard and twin models.

NOTES

1. http://www.dfg.de/foerderung/info_wissenschaft/2010/info_wissenschaft_10_26/index.html
2. http://modellsammlung.uni-goettingen.de/qq
3. http://did2.mathematik.uni-halle.de/modell/index.php
4. https://archive.org/details/catalogmathemati00schiuoft
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Oral Presentation

SERIES OF PROBLEMS AT THE CROSSROAD OF RESEARCH, PEDAGOGY AND TEACHER TRAINING

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Our presentation is focused on the design of a favourable environment for teachers willing to develop disciplinary or interdisciplinary reflections based on the study of a particular kind of historical sources, that we call "series of problems". In the first part we briefly explain the meaning of this notion and we outline the contents and purpose of a research project focused on their comparative study. We then describe how this project is associated with a professional teacher training session also conceived as a research seminar. In the third part we explain how we intend to reorganise the project into a collaborative edition of a sourcebook about series of problems. We finally explain the main principles of a workshop associated with this editorial enterprise that will permit the development of original reflections and pedagogical projects around the texts that will be selected for the sourcebook.

"SERIES OF PROBLEMS AT THE CROSSROAD OF CULTURES": THE FIRST STAGE OF A RESEARCH PROJECT (2011-14)

The research project entitled “Series of problems at the crossroad of cultures” has been developed within the "HASTEC labex", a cluster of several Parisian research laboratories [1]. It gathers around 15 researchers, including master and PhD students, from various disciplines: history, epistemology and anthropology of sciences and of literature, history of texts, cultural history, and educational studies. The purpose of this interdisciplinary project is to study a genre of historical texts called "series of problems". Many (though not all) of these texts can be identified as having, partially or in totality, mathematical contents.

We use the term "series of problems" to interpret historical texts having the form of a collection of questions and answers. This interpretation relies on the basic hypothesis that these texts follow, either globally of locally, some kind of principle of ordering. The term "problem" has to be understood here in a very broad sense, as referring to any kind of verbal challenge: this includes, therefore, mathematical or scientific problems in the usual sense, but also riddles (*enigmata*) or questions, in general any kind of practical, pedagogical or intellectual "task". As for the term "answers", it also refers to a wide range of possibilities, from a "solution" (in the case of mathematical problems) to quotation of authorities (in the case of questions in natural philosophy) or poems (in the case of literary riddles).

The main originality of this research project, then, consists in focusing not on individual ‘problems’ but on the principles, the characteristics and the possible roles of their collection in a certain order. The second keypoint is that this order or "principle of ordering" is not necessarily understood in the same way in each case. In other words, the confrontation and progressive clarification of the various ways, in which this ordering should be understood, is one of the basic goals in this interdisciplinarly project.

A famous example of sources entering the generic category of "collection of questions and answers" and dating back to antiquity are the (pseudo) Aristotelian *problemata*, a series of questions falsely attributed to Aristotle and proposing a wide range of intellectual and philosophical challenges [2]. Another example, which is the research subject of one of us, is a corpus related to a Hungarian tradition of mathematics education from the second half of the 20th century: mainly textbooks and teachers handbooks, partly inspired by philosophical texts and by books popularising mathematics, like Rózsa Péter’s “Playing with infinity”. In these texts, ordered series of problems play an essential role, as well as a dialogical form of presentation. In this case, divers principles of ordering can be observed, for example the variation of mathematically similar problems in different contexts, in order to guide the reader towards progressive levels of generalisation (Gosztonyi 2015).

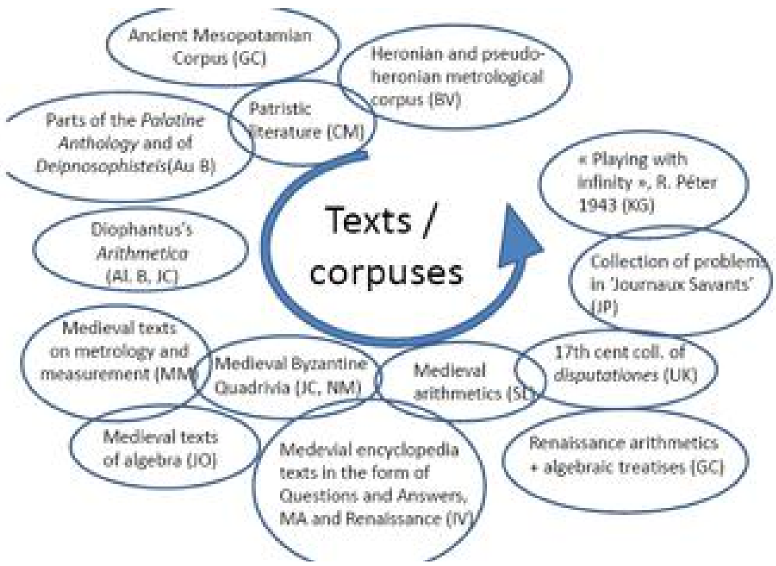

Fig.1: *the range of historical texts studied in the project (see note 3 for the abbreviations).*

Many other examples could be mentioned of course. Synoptic figure 1 gives an idea of the variety of texts and periods covered by the researchers participating in the project [3], and still many more could be added when considering other periods of cultural areas.

These texts are highly interesting research objects in several respects. Let us first insist on their interest for historical and anthropological studies. Some of these series, in their form and contents, have a long ranging history: this is, for example, the case of the pseudo-Aristotelian *problemata*, the tradition of which extends to the Middle Ages, during which they were eventually adapted, *through reordering and adaptation*, into a form of encyclopaedic knowledge (Ventura 2008). Some of them have crossed the boundaries of cultures, like Diophantus' series of arithmetical problems: originated in Greek in a coherent treatise, they have been transmitted to Byzantine and Arabic Middle Ages, before they were translated and adapted in the Renaissance periods and inspired a host of new treatises- often by way of *reordering and transforming* Diophantus's problems. Thus the study of the re-appropriation and re-ordering of such texts contributes to the understanding of their long-lasting character.

At the same time, even in the case when they are based on a long-lasting heritage, they are never organised the same way and represent an intriguing object for cultural studies: thus, Rózsa Péter's literary text can be seen as an initiation to mathematics for a wide audience, but also as a reflection of the philosophical, educational and literary concerns of a whole milieu of writers, mathematicians, educators and philosophers to whom R. Péter alludes in her book (Gosztonyi 2015). In this case, part of the book is based on problems structured in a sophisticated way, this ordering having much to do with the concerns in question. The pseudo-Aristotelian *problemata*, which cannot positively be attributed to Aristotle himself, still reflect the spirit and atmosphere of the peripatetic school. As for Diophantus' *Arithmetica*, it contains strong allusions to the background of ancient rhetoric, most notably the emphasis on the notion of *invention*. The latter is in turn related to the progressivity of his problems, which is meant to develop the reader's capacity for invention (Bernard 2011, Bernard and Christianidis, 2012). In general, these texts often pose difficult questions of interpretation: even when the intention behind their constitution is made explicit, it is not always obvious how to make it correspond to the actual structure of the text. Thus, looking into their partial or global "seriality" is one way (among others) to construct this interpretation and face this difficulty.

The third interest of such objects is the historiographical issues raised by their classification. For example, several of these series have been categorised by historians in a way that is open to dispute - in particular, while there are sometimes clues to the fact that they served didactic or pedagogical roles, in other case the positive evidence for this is lacking or, when it exists, is easily misinterpreted [4].

Finally, series of problems often represent a challenge for historical research, because in some cases (esp. in the medieval period), historical inventories of them are lacking, and many sources that fall under this category are still unedited or understudied. When an inventory is possible and expected, the criteria for building these inventories and comparing the elements of the retained corpus are also open to discussions: should the text be characterized through the contents; through the list of the statements of problems; or through the list of solutions? Finally, even in the case of the study of single series of problems, the criteria that make clear the organisation and ordering of the problems or questions have to be made clear and studied carefully, because this analytical choice has deep consequences on the interpretation of "seriality".

All in all, the primitive aim of the project has been, and still is, to improve the comparative study of these objects, not only across different ages and geographical locations, but also by taking advantage of the variety of approaches and fields represented in the research group. This variety of approaches focused on the one same object is one way to understand the notion of "crossroad of cultures": as the crossroad of intellectual approaches. Another way to understand this notion is to think about it in terms of the variety of "cultures", including intellectual and professional cultures and techniques, beyond differences of language and values, that are needed to understand "series of problems". We want to check the fundamental hypothesis that, given their complexity, taking into account several of these cultures and not only one or two of them, might bring a better historical understanding of their structure, role and relative stability in time.

Beyond their interest for historical studies, the second reason to pay attention to these objects is the fact that their study can still inspire new reflection by teachers of today working in various disciplines, or in interdisciplinarity. This is why the development of the project was very soon associated with a training session, as we shall now see.

THE ORGANIZATION AND CONCEPTION OF TRAINING SESSIONS AROUND SERIES OF PROBLEMS

Series of problems can challenge the interest of teachers for several reasons. Some of them are related to the general issues usually treated in HPM meetings: as examples of historical sources among others, they are liable to inspire pedagogical activities and reflections on mathematics in relation to the cultural context or more specifically to the interest of reading mathematics into ancient and unfamiliar texts. Also, they are potentially interesting for interdisciplinary activities: on the level of contents, the problems contained in these texts do not all concern mathematics; on the level of their interpretation, series of problems are akin to a genre of texts, that is to an interpretative tool used in literary studies.

More specifically, though, teachers may find reasons for taking interest in this literature, that are related to the pedagogical issue of teaching through problems. By studying series of problems and reflecting on questions about the order of problems, one can take into account not only the resolution of isolated problems but also structured systems of problems as well as the intellectual processes consisting in putting them in order. At an even deeper level, there is also an issue about the image and conception of knowledge which is reflected through its organisation in this serial structure.

To discuss concretely such questions, we have organized three training sessions until now (2012-15). They have been proposed to a mixed audience of teachers of mathematics, literature and history in French secondary schools, as well as to students in the human sciences through HASTEC and the associated master or doctoral structures [5]. The double purpose is (a) to offer an interesting incentive for professional development, for teachers willing to enrich their culture and knowledge, and their reflection on the teaching through problems or enigmas; and (b) to propose them simultaneously an insight into research questions about the historical sources taken as a support and point of departure of such reflections [6].

The session had each time a standard format: three days of meeting amounting to six sessions of 3 hours each. The first session is meant to expose the purpose and contents of the training, with a special emphasis on the ambivalence of the sessions, which can be seen as stages in an ongoing research seminar, and as opportunities to discuss professional issues. Then each of the other five sessions is based on the study of a particular corpus of series of problems, and consists in the presentation of their historical context on the one hand, and the collective reading of a collection of selected excerpts on the other. In principle, enough time must also be left each time for discussions with the participants on the contents of the proposed text and of its interpretation, but also on professional issues aroused by this experience of reading and understanding of ancient sources.

Let us insist here on the organisational aspects of theses sessions that make them a kind of concrete crossroad between professional and research inquiries. The first way to favour this mixture of perspectives has been already mentioned: the researchers (including students) who were called to constitute and present a collection of selected texts for discussion with the participants, were also invited to organise these excerpts according to one leading research question they had in mind. This presentation is basically meant to give an idea of the underlying research issues. For example, one of us took the opportunity of the 2013-14 session to explain the questions he had in mind about the progressivity of Diophantus's problems in his *Arithmetica*: having elaborated a first model of study of this progressivity (Bernard and Christianidis 2012) his purpose is now to improve this interpretative model through closer attention to the language used by Diophantus for the statements of his problems and the corresponding solutions (Bernard *forthcoming*).

Even more concretely, it very soon appeared that one way to explain the research questions was not only to *explain* it through a traditional kind of talk giving elements of theoretical references or historical context, but also to make it palpable through the organisation of the chosen texts. Following on the example of Diophantus's text, we were for example led to propose a translation of its problems so as to get the reading experience as close as possible to what it was in antiquity: an experience of mentalizing texts that were written in manuscripts in a "continuous" way (with no or little separation between words and sentences) through aloud reading and verbalization (*anagnôsis*). This way of preparing and presenting the text was meant to make clear and palpable the repetitiveness which is characteristic of ancient texts. It can then be explained by taking into account the concrete conditions in which reading and learning occurred in antiquity. Interestingly enough, this issue met very quickly the concerns of mathematics teachers who are developing ways to help the reading of sentences, in which algebraic symbols appear that are liable to be replaced by numerical values [7].

The other way, through which the participants were invited to participate in research inquiries and, at the same time, to question their own professional practice of concerns, is more traditional: it consists in explaining, from the outset, the origins and reasons of the session - especially the fact that it is related to a research project. At this stage the purpose is to organise a first discussion, aiming at "matching" the expectations of participants, with the purpose of the session. From this discussion generally emerge several questions and issues that participants have in mind consciously or not. Here are some examples of questions that typically emerged from such preliminary discussions: how to structure one's teaching through problems and for what purpose? How can one introduce a cultural context when discussing traditional problems? What use can be made of problems stemming from, or present in, various cultural traditions and cultures, especially in view of teaching mathematics in multicultural classes? How the same problems were formulated in various periods and languages, and what advantage can be drawn from this variety? The game, then, is to recall as systematically as possible these issues in the course of the various reading sessions. From this point of view, these sessions can be then regarded as a permanent *anamnesis* (recollection) of these key issues. This concretely calls for the presence of a moderator who should see his role as essentially maieutic, that is, as 'recollecting' the previously discussed questions. This means that to make bridges between them, reformulating them in the light of new contents, adding content, awakening new reactions and discussions, is for them the main challenge.

The limits of these procedures are, of course, time. While there is of course no limit of time for preparing a set of 'interesting' excerpts to study and read, there are obvious constraints on the time that can be devoted to open discussions in the framework of reading sessions, most of which are spent on discovering the presented material and on going beyond the usual first "shock" of meeting new and unfamiliar texts. Combining the presentation of the cultural context, open discussions and the reading of texts is a real difficulty. We thus naturally came to the idea that we should associate with these sessions a more intensive workshop. This idea also came from the recent transformation of the underlying research project, which is now turning into a reading seminar associated to an editorial project.

THE PROJECT OF A COLLABORATIVE "SOURCEBOOK" ON SERIES OF PROBLEMS

Before we come to explain how we are conceiving this associated workshop, the first edition of which began in 2014-15, we must explain the new direction the "series of problems" research project is taking, toward a reading seminar aimed at the publication of a sourcebook for the subject.

After a first 'seminal' period of three years that is now concluding with a first collective publication (to appear in 2015), the "series of problems" project is progressively taking a new turn. Since most of the participants are now willing to have a reading seminar in which excerpts of various series of problems would be examined and discussed "from close experience", the basic idea is that this seminar would be ultimately focused on the publication of a sourcebook consisting of a collection of discussions and annotations of these excerpts.

In its traditional form for history of science, a sourcebook essentially proposes historical and epistemological commentaries on the chosen texts. In our project, this would constitute the *first* layer of the expected commentaries. But, based on a previous experience of a similar editorial project associated with an experience of collective reading of the chosen materiel by teachers (Bernard *et alii*, 2010), the originality of this sourcebook would be to add a *second* layer of commentaries. This second layer would account for the lessons drawn from concrete experience of "actual encounters" between these texts, and with teachers concerns with specific professional issues. The key idea underlying these second commentaries is reflected in the beautiful narrative that Augustine proposes of this conversion, in a well-known episode of his Confessions (VIII, 29): some texts achieve an actual meaning for their readers, through the identification of its contents with the *actual* experiences and thoughts of the reader. These commentaries, then, would illustrate possible values actually given to these ancient texts by modern readers, especially teachers.

When considered on the level of research questions, the two layers of commentaries correspond to two basic kinds of issues and purposes. The first layer is oriented on historical and epistemological research on the texts themselves: the leading purpose is then to restore the adequate historical and cultural context in which the chosen excerpt might or should be understood, the reasons for choosing the excerpt and to provide elements of interpretation based on actual research, which includes bibliographical references giving an access to deeper readings. The leading questions, then, are those exposed in the first part of the present paper. The second layer is explicitly or not, related to the issues in educational research about learning and teaching through problems that have been evoked in the previous part. From this second point of view, the key issues are the meaning of "teaching through problems"; the role of seriality when building problems is considered not as an isolated activity but as building *collections* of them with a definite idea in mind, whatever it is.

Concretely, the first layer of commentaries is naturally obtained through the existence of the reading seminar called for by the participants of the project, as mentioned above. The second layer requires a slightly different kind of context and framework: for this, the workshop to which we alluded above, in relation to the training session on series of problems, seems an adequate answer.

ORGANIZING WORKSHOPS FOR PROFESSIONAL DEVELOPMENT INSPIRED BY HISTORICAL "SERIES OF PROBLEMS"

The basic aim of the workshop in question, then, is to serve as a "companion" both to the training sessions and the project of the sourcebook we discussed above [8]. As for the participants, the idea basically follows the principle of the IREM workshops [9]: researchers and teachers at various levels (primary, secondary or university) are invited to participate, provided they have some connection with the associated research project. This open framework is wholly consistent with the purpose of mixing various research and professional perspectives in one and the same framework. For teachers, it should offer the right context for professional development, that is, an opportunity to reflect about their own teaching on the basis of the historical material discussed within the project.

As compared to traditional IREM groups in France, our workshop has the particularity to welcome teachers of literature and history, who might be interested by in this material. Moreover, it is in principle open to students of the newly introduced Master's curriculum for all professions related to teaching and education (MEEF). In other words, it could become a place for meetings not only between researchers and teachers, but also between beginning and qualified teachers, and between teachers from different disciplines.

Just as we did above for the training session, we would like here to highlight how we conceived the organisation of the workshop, so as to fulfil its basic objectives. The same problems of conception mentioned above in relation to the training session, exist with this workshop in terms of organising a coherent dialogue between research perspectives and questions related to the development of professional skills. At the present moment, it is too early to draw conclusions from this nascent experience; we will limit ourselves to discussing the initial framework we considered for it, and the possible perspectives.

Our first idea is both traditional for an IREM-type workshop and an original development to our conception of the training session. Any IREM group has for its basic purpose the production of resources for mathematical teaching. Just in the same way, we thus propose that the participants develop a project that might evolve in as a possible resource for other colleagues: this might be for example an article, an academic work (a Master's thesis for example), a website, or a booklet on a definite subject, etc.

It is important to leave significant freedom in terms of the potential nature and contents of the resource they propose to build. As for the kind of project, it might evolve into personal reflections on "series of problems" as cultural, historical, literary objects, in a typically interdisciplinary perspective. This might also consist in reflections on the ways of constructing a teaching process based on problems, these reflections being inspired by the examples seen during the training session. This might be the construction and experimenting with teaching scenarios based on series of problems.

In order to leave place for the development of this diversity, and to define the projects which are the better adapted to the interests, competences and possibilities of each participants, we asked them, at the initial stage, to express some *rationale*, *emotion or desire*, that they could develop later into a more definite project. In other words, the first issue is not to define a project but to explore its roots, that is, the reasons for building a project. This notion has much to do with the Augustinian idea of "encounter with a text" underlined above.

For example, two colleagues already signalled their interest in the material that one of us proposed during the training session, in the form of translated excerpts from the Hungarian mathematics textbooks from the 1970s. What attracted their attention to this material was the fact, that part of the textbook took the form of fictive dialogues between pupils sharing their experiences and questions related to concrete problem-situations. [10] The discussion showed, that the reasons for being attracted to this idea were related to their own attempts to structure their teaching through the use of actual dialogues. More than this we cannot say at the present stage: we do not know yet, what use *they* made of dialogues and in what sense they understand this use. This might ultimately appear very different from the intentions that underlie the Hungarian texts; what counts at this stage, is that they began to identify the reasons for studying from close examination of these manuals, in relation to this particular professional experience.

The second leading idea is to leave time and freedom for the progressive development of the project. Time is again an obvious constraint: the participants, most often than not, do not have enough time and availability for developing a complete project with compelling deadlines in a short span of time. They certainly need a challenge, but not deadlines that would be incompatible with their professional activity. One of the main reasons to leave open the type of work and the support chosen is to make sure that the complexity, length and support of the project does correspond to the time constraints of the participants, and most of all that its nature and contents fit the initial desire analysed in the first place.

Ultimately, our hope and purpose is to build the concrete basis for the elaboration of the "teaching" commentaries we alluded to above: if the work led within this framework develops in the right way, it should ultimately be possible to build a synthesis making for each text the best out of various reading experiences.

CONCLUSIONS AND PERSPECTIVES

Our initial incentive for presenting this nascent work in the Copenhagen conference was to take the opportunity of an international conference in order to check whether this project could be developed on a more European level. One purely potential reason for thinking about this kind of development is that we belong to two different countries, France and Hungary: thus, while the present project is developed in France and more precisely in Paris, one could imagine in the future some 'satellite' development in Hungary, or in general in other countries.

One strong objection to such development is the language: at present, most (though not all) of the texts developed within the project, or discussed in the training sessions, were presented in French and for a French-speaking audience [11]. Thus, as usual, bringing the project at a European level would mean overcoming language barriers. It also implies difficulties in terms of finding locally enough experts available for participating in interdisciplinary discussions, not only with other specialists, but also with teachers.

In spite of this, it remains interesting to reflect about the potentials for delocalization. Given the principle of the workshop described above, there is nothing to prevent several similar groups develop in various locales, even within France for example. Indeed, what really counts is the availability of the texts studied, translated and eventually edited within (and thanks to) the project; and the possibility of inviting participants in the projects to local meetings. The development of such 'satellites' thus need time, patience and reflection.

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1 "labex" is a shortcut for "laboratoire d'excellence". The list can be found on the following webpage: http://www.hesam.eu/labexhastec/partenaires/(accessed 29 May, 2015).

2 Cf. Aristotle, *Problèmes*, éd. and transl. P. Louis, Les Belles Lettres 1991.

3 The initials refers to the various researchers involved, namely: Al. B = Alain Bernard (UPEC), Au. B = Aurélien Berra (Paris Ouest Univ.), GC = Giovanna Cifoletti (EHESS), JC = Jean Christianidis (Univ. of Athens), KG = Katalin Gosztonyi (Univ. Szeged), SL = Stéphane Lamassé (Paris 1), CM = Caroline Macé (Leuven Catholic Univ.), NM = Anathanasia Megremi (Univ. of Athens), MM = Marc Moyon (Univ. of Limoges), JO = Jeffrey Oaks (Univ. of Indianapolis), JP = Jeanne Peiffer (CNRS-CAK), Iolanda Ventura (CNRS-IRHT), BV = Bernard Vitrac (CNRS-CAK). Other contributors are not mentioned in this synoptic schema.

4 For a more detailed discussion on these issues, see the collective volume that one of use co-edited with Christine Proust (Bernard and Proust 2014), which contains several illustrations taken from various contexts.

5 Some participants belonging to the two categories. In general, though, it appeared difficult to invite teachers of history of literature to these training sessions, a difficulty which is probably due to the presence of the word "problem" in the title of the session.

6 In terms of institutional background, this means that the session was hosted by one Institute for Research on the Teaching of Mathematics (IREM, in this case the Paris IREM based on Paris-Diderot University), co-organised by HASTEC and also the new Institute for teacher training at Créteil (ESPE de Créteil) to which one of us belongs.

7 As is well known, it is difficult for students to learn how to differentiate the cases when one has to consider symbols as 'empty places' that serve to highlight the reformulation of a problem of an algorithmic procedure, from the cases where these same symbols are to be identified as "having values". This issue has strong similarities with the issue of correctly interpreting Diophantus's use of the potentialities of the Greek language for building schematic solutions of his problems.

8 The two projects are obviously coherent with each other, since traditional sourcebooks usually are the formalised and published version of the material used in academic courses.

9 "IREM" is the acronym for "Institut de Recherche sur l'Histoire des Mathématiques". See http://www.univ-irem.fr/spip.php?article6 (accessed 29 May, 2015) for a presentation (in French) and (Fauvel and van Maanen 2002, 96-97), in English.

10 The series of problems, in this case, is thus the series of questions brought about in the various stages of this dialogue, as well as the problems these pupils either discuss or (re)formulate.

11 Note, nevertheless, that following a workshop held in Athens in October 2014, an English summary of the main issues discussed within the project is now available: http://problemata.hypotheses.org/373 (accessed 29 May, 2015).

Oral Presentation

REFLECTING ON META-DISCURSIVE RULES THROUGH EPISODES FROM THE HISTORY OF MATRICES

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In this work, we present a teaching proposal about history of matrices. Our goal is to create conflictive situations in which students are encouraged to reflect upon their metadiscursive rules related to matrices, comparing them with those present in some historical writings. We have been based in the historical interpretation of Frédéric Brechenmacher and in Sfard's theory of Thinking as Communicating. The conceptual framework for using history in the teaching of mathematics was inspired by some works of Tinne Hoff Kjeldsen. We elaborated two teaching modules approaching two episodes of the history of matrices; the first has as protagonist the mathematician J. J. Sylvester and the second one has A. Cayley as protagonist. We discuss some of the results obtained in a pilot study in which the material was tested.

INTRODUCTION

Almost all Linear Algebra courses in Brazil, as well as the textbooks most often used, start with the concept of matrix as a stand-alone mathematical object. The definition is stated without reference to any problem in which the notion appears, immediately after the operations are introduced and their properties deduced in an abstract manner [1]. This sequence is thus seen as a goal in itself, no matter if it would be richer to develop further discussions about the nature or the origin of matrices and their operations.

As a consequence, when we ask students having finished their Linear Algebra courses why matrix multiplication is defined as the dot product between the rows of the first matrix and the columns of the second matrix, they generally cannot answer. The following quote shows the answer an school Mathematics teacher gave to this question (part of a questionnaire given at the beginning of our pilot study).

> Question: Imagine that a student asks you the following question during a class on matrices: "Why we have to multiply rows with columns in the matrix multiplication?" What would you answer?
>
> Answer: I would say that he should accept it as a truth. Unfortunately, this would be my answer. I wouldn't consider saying anything else.

The aim of our research is to create "conflictive situations" in which students are encouraged to reflect upon the rules that define their actions when dealing with matrices (metadiscursive rules), after comparing them with the rules that appear in some historical writings. The notion of *conflictive situation* is inspired by what Sfard calls *commognitive conflict* and the notion of *metadiscursive rule* is used here in the sense proposed by Sfard's theory of thinking as communicating (Sfard, 2008).

In the next section we explain the conceptual framework used in the research, largely inspired in the works of Tinne Hoff Kjeldsen (Kjeldsen, 2011; Kjeldsen & Blomhøj 2012; Kjeldsen & Petersen, 2014). These references made it possible to combine the historical approach we perceived as relevant to our goal with methodologies from the field of Mathematics Education.

Using historical sources about matrices, we developed teaching proposals in order to analyze the context of problems in which matrices appeared as a useful definition, so making it clear that this notion was not proposed immediately as a mathematical object. The historical work on matrices will be discussed in the third section of this paper.

Frédéric Brechenmacher (2006) is another important reference, who showed that the notion of matrix emerged and developed associated with concepts such as determinants, linear transformations and quadratic forms, to cite a few. Unlike the order in which these concepts appear in a Linear Algebra course nowadays, in history the notion of matrix was one of the last to appear. Moreover, the history of matrices shows that they come to light as a representation and their constitution as a mathematical object occurred along different mathematical practices. As Brechenmacher observed, the notion of matrix changed over time through different identities assigned to it within these mathematical practices.

The fourth section presents the pilot study carried out in the first semester of 2014 and we close the article with some initial conclusions.

CONCEPTUAL FRAMEWORK

Kjeldsen (2011) proposed a theoretical argument to integrate history in mathematics teaching based on Sfard's theory of mathematics as a discourse. According to Sfard (2008), mathematics is a well-defined form of communication or a type of discourse governed by certain rules. In this perspective, learning mathematics requires to take part in the mathematical discourse. In Sfard's words, it is necessary even for one's understanding of mathematics, since learning a mathematical discourse is "becoming able to have mathematical communication not only with others, but also with oneself" (Sfard, 2007, p. 575).

The rules that control the discourse are divided into two types: object-level rules and metadiscursive rules. The first concerns "narratives about regularities in the behavior of objects of the discourse" and the second one concerns "patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p. 201).

In the mathematical discourse, object-level rules relate to the properties of mathematical objects. Examples include: (1) in Euclidean geometry, *the interior* *angles of a triangle always add up to 180°*, and (2) in algebra, *ab = ba, where a and b are real numbers*.

The metadiscursive rules (or metarules) concern the actions of the discussants. They are usually implicit in the discourse and manifest themselves when one judges, for example, *if a particular description can be regarded as a definition or if a proof can be accepted as correct*.

Metarules govern "when to do what and how to do it" (Sfard, 2008, p. 208). So, they affect the way in which participants of a discourse interpret its content. Learning of mathematics is thus the developing of appropriate metarules. On the other hand, as these rules are contingent and tacit (Sfard, 2008, p. 203, 206), participants do not observe them in a conscious and natural way. For this reason, it is unlikely that participants can learn metarules by themselves.

The term *metarule* in Sfard's approach is quite broad, including, for example, norms, values, and goals. It can also be used to designate repetitive patterns in different activities.

> (…) it is possible to talk about the metarules regulating participation (e.g., raising hands before speaking, working in groups), or metarules characterizing participants' intentions (e.g., genuinely engaging in mathematical activity versus acting to please the teacher), or the metarules regulating the object-level rules of mathematics (e.g., using the metaphor of motion to compute limits, using graphs to realize functions). (Güçler, 2013, p. 441)

In what concerns our particular subject of research, Kjeldsen argued that history of mathematics plays a fundamental role in order to "illuminate metadiscursive rules". These kind of rules are historically established and they may thus be treated as the object-level of a historical discourse. In this way, metadiscursive rules stop being tacit and can be made explicit objects of reflection (Kjeldsen, 2011).

The idea is then to promote situations in which students are encouraged to investigate the development of mathematical practices through historical sources and to understand the vision mathematicians had about their own practices. An approach of this kind can help the students to grasp how mathematicians conceived their objects of study and how they formulated their mathematical statements. Doing so, students can have contact with discourses governed by metarules that are different from the modern ones and different from their own metarules:

> (…) the historical texts can play the role as "interlocutors", as discussants acting according to metarules that are different than the ones that govern the discourse of our days mathematics and (maybe) of the students. (Kjeldsen, 2011, p. 52)

In the present research, we developed teaching and learning situations with the aim to clarify the metarules found in mathematical texts from the past, so the participants can compare them with their own metarules. The use of historical sources can thus lead to the situation that Sfard calls *commognitive conflict*, defined as "a situation in which communication is hindered by the fact that different discursants are acting according to different metarules" (Sfard, 2007, p. 576).

Guided by such a theoretical argument, Kjeldsen and Petersen (2014) implemented, in a Danish high school, an experimental course on the history of the function concept. In addition to using Sfard's theory of thinking as communicating, the course was also designed by using a multiple perspective approach to history (Kjeldsen, 2011) and the theories related to concept image, concept definition (Tall & Vinner, 1981) and concept formation (in the sense of Sfard, as cited in Kjeldsen & Petersen, 2014, p. 32). The researchers used extracts from primary sources written by Euler (1748) and Dirichlet (1837) in order to explore two metarules:

- *General validity of analysis*. This rule assumes that results, rules, techniques, and statements of analysis are generally valid.
- *Generality of the variable*. This rule states that a variable in a function can take on all values.

These two metarules were dominant in the analysis of 18th century and Euler assumed both of them in the definition of a function he presented in 1748. His definition considered a function of a variable quantity as an analytical expression composed in any manner from that variable quantity and numbers or constant quantities (Kjeldsen & Petersen, 2014, p. 37).

Afterwards, the students get in touch with Dirichlet's definition, which departs from metarules that are different from the ones Euler assumed. In this last case, a variable quantity was used to propose a definition of a function as a relation of dependence between variables, which is not necessarily given by one same law in the whole interval; and not thought of as relations that can be expressed by mathematical operations (Kjeldsen & Petersen, 2014, p. 37).

The goal is to make the conflict to emerge between the different metarules found in the historical texts, and also between these metarules and their own. Although our research is much inspired by Kjeldsen's theoretical argument it concerns a different mathematical subject. We prepared two teaching modules focusing on episodes in the history of matrices and selected three metarules we found appropriate to provoke a conflict about the way matrices were and are conceived. In the next section, we explain the historical content and the metarules that have been selected.

HISTORICAL PRACTICES ON MATRICES AND SOME OF ITS METARULES

Two research episodes about matrices were analyzed in order to investigate the different roles that the notion of matrix acquired within two practices developed in the 1850s by the mathematicians James Joseph Sylvester and Arthur Cayley (Bernardes, 2012). The historical discussion of these works is based on the interpretation suggested by Frédéric Brechenmacher (Bechenmacher, 2006).

Sylvester introduced the word “matrix” in his research about the classification of the types of contacts between two conics. In this context, matrices were conceived as a means of representation. This role changes in Cayley’s research. In the memoir published in 1858 (Cayley, 1858), matrices offered a new language in which known problems could be treated differently and new problems could be proposed. Moreover, Cayley established the rules for operations with matrices.

In 1850 the British mathematician James Joseph Sylvester published a memoir in *The Cambridge and Dublin Mathematical Journal*, (Sylvester, 1850a) addressing one problem of a geometric nature: the classification of the types of contact between two conics. The term “contact” was used to denote an intersection point in which the two conics are tangent to each other.

The main mathematical tool used by Sylvester in order to solve the contact problem was the notion of determinant. However, he did not compute determinants of matrices, this last notion was introduced later.

In order to classify the type of contact between two conics, Sylvester analyzed the multiplicity of the roots of the equation $\det(U+\mu V)=0$, U and V being homogeneous quadratic equations in three variables that represent the conics. To let it clear:

$$U: ax^2+by^2+cz^2+2a'xy+2b'xz+2c'yz=0$$

$$V: \alpha x^2+\beta y^2+\gamma z^2+2\alpha' xy+2\beta' xz+2\gamma' yz=0,$$

and the coefficients are real numbers. The equality $\det(U+\mu V)=0$ yields a cubic polynomial equation [2].

In the articles concerning the contact problem [3], Sylvester computed determinants of (homogeneous) polynomials functions. This was a recurrent procedure in his practice and sometimes he also used auxiliary tables, in which the entries were functions of the coefficients of the conic-defining equations.

The analysis of these works motivated us to identify a metarule underlying Sylvester's practice: *determinants were tools computed from functions (homogeneous polynomials) and were useful in the investigation of properties of curves and surfaces*.

We can note immediately a huge difference between this metarule and ours, since nowadays in linear algebra determinants are defined by means of (square) matrices and seen as a property depending on these mathematical objects.

Returning to Sylvester’s practice, analyzing the multiplicity of the roots of the equation $\det(U+\mu V)=0$ was not sufficient to classify all four kinds of contact. In the case of multiplicity two or three, there are two kinds of contact, as illustrated by the examples in Figures 1 and 2.

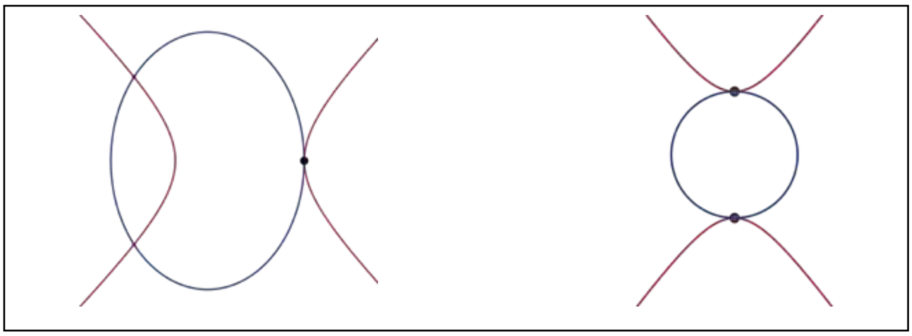

Figure 1: Simple (left) and diploidal (right) contact.

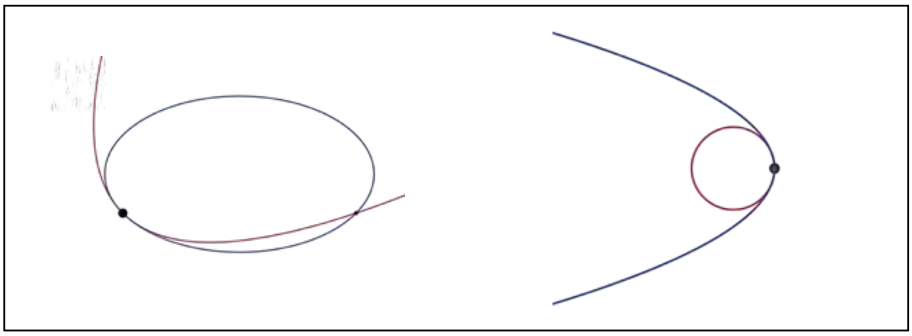

Figure 2: Proximal (left) and confluent (right) contact.

The types of contacts may be distinguished by studying the multiplicity of the intersection points in which the conics are tangent (the black dots in Figures 1 and 2). In the situation of a simple contact, there is one double intersection point (Figure 1, left); in a diploidal contact, there are two double intersection points (Figure 1, right), in a proximal contact, there is one triple intersection point (Figure 2, left); and in a confluent contact, there is one quadruple intersection point (Figure 2, right).

So, in order to solve the contact problem, Sylvester introduced the notion of minor determinants and developed a technique consisting of extracting systems of minor determinants from the complete determinant.

The term "matrix" was introduced in this context and with the goal of generalizing a property of minor determinants.

> (…) we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p and selecting at will p lines and p columns, the square corresponding to which we may be termed determinants of the pth order. (Sylvester, 1850b, p. 369)

In this quote Sylvester makes explicit his understanding of a matrix as a source of minor determinants, concisely called by Brechenmacher as "*mère de mineurs*" (2006, p. 15). This understanding was reinforced in another article:

> I have in previous papers defined a "Matrix" as a rectangular array of terms, out of which different systems of determinants may be engendered, as from the womb of a common parent (…). (Sylvester, 1851b, p. 302)

We thus propose there is a second metarule concerning matrices that underlies these works of Sylvester: *matrix is a mother of minors.* This metadiscursive rule is expressed in Sylvester's idea of a matrix as a representation from which systems of minor determinants can be generated. Before stating the next metarule that guided our work, we need to describe briefly another actor who was important in this research.

Eight years after the introduction of a matrix by Sylvester, his friend Arthur Cayley published a memoir in which he defined the matrix operations and stated their properties (Cayley, 1858). According to Cayley, matrices arise naturally from "an abbreviated notation" for linear systems. Consequently, he defined matrix operations from similar operations possible to be accomplished with linear systems.

In the first page of the article, Cayley makes an analogy of matrices with simple quantities (numbers):

> (…) It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities; they may added, multiplied or compounded together (…). (Cayley, 1858, p.17)

This analogy pushed him to consider a certain type of matrix as a simple quantity:

$$
\begin{array}{rl}
 & (m, \quad 0, \quad 0) \\
m = & |\, 0, \quad m, \quad 0 \,| , \\
 & |\, 0, \quad 0, \quad m \,|
\end{array}
$$

> The matrix on the right-hand side is said to be the single quantity m considered as *involving the matrix unity*. (Cayley, 1858, p. 20, italics in the original)

Cayley developed a practice of computation with matrices based on a dual interpretation of a matrix: either as a system of numbers and as a number (Brechenmacher, 2006, p. 20). This duality is expressed in the statement of his "remarkable theorem", announced in the first page of the memoir:

> 21. The general theorem before referred to will be best understood by a complete development of a particular case. Imagine a matrix
>
> $$
\begin{array}{rl}
M = & (a, \quad b), \\
 & |c, \quad d|
\end{array}
$$
>
> and form the determinant
>
> $$\begin{vmatrix} a - M, & b \\ c, & d - M \end{vmatrix},$$
>
> the developed expression of this determinant is
>
> $$M^2 - (a+d)M^1 + (ad - bc)M^0;$$
>
> (…) and substituting these values the determinant becomes equal to the matrix zero, (…). (Cayley, 1858, p. 23)

Cayley explains afterwards that:

$$\begin{pmatrix} a-M, & b \\ c, & d-M \end{pmatrix} = \begin{pmatrix} a & b \\ c, & d \end{pmatrix} - M\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

is the "original matrix, decreased by the same matrix considered as a simple quantity involving the matrix unity" (Cayley, 1858, p. 24).

Relying on his dual interpretation about what a matrix is in Cayley's works, we state a third metarule: *dual interpretation of a matrix*. A matrix was interpreted either as a system of numbers or as a number.

The three metarules that we have defined were explored in two teaching modules in which we presented, in an abbreviated manner, the works of Sylvester and Cayley. Original excerpts are used as much as possible, but sometimes we inserted text to make the links between parts of the text that we chose as the most relevant to our goal. We describe in the next section how these teaching modules were tested in a pilot study.

THE PILOT STUDY

We carried out an experiment in a pilot study with the goal of testing the teaching modules. We offered a mini-course for six volunteers called "Different roles of the notion of matrix in two episodes of the history of matrices". The mini-course was taught by the first author of this paper and the meetings took place on two Saturdays, lasting about five hours each.

The mini-course students were school Mathematics teachers, ranging from 6^{th} to 12^{th} grades (corresponding to students aged 11 through 17). He time of experience of the teachers varied from 3 to 12 years, and they were all taking a Linear Algebra graduate course as part of the requirements of a professional master's degree in Mathematics, offered for teachers currently teaching in the public system. In the quotes bellow, the participants will be identified by the letters M, T, Fa, Fe and J. During the meetings, they worked in groups in order to answer the historical activities proposed in the two teaching modules. Our data sources were: 1) audio recordings of the groups' discussions; 2) written answers to the activities; 3) a summary in written form explaining what they learned in each module; 4) two questionnaires, one filled before the first meeting and the other after the final meeting.

The goal of the first questionnaire (Figure 3) was to understand the profile of the participants and to get a glimpse of how they were learning matrices in their Linear Algebra course.

1. From what institution did you earn your Bachelor´s degree? When did you finish it?
2. How long have you worked as a Mathematics teacher?

3. Tell us about the Linear Algebra courses you took, at both undergraduate and graduate levels: How was the subject taught? Did you enjoy it? How hard was it?
4. Tell us about the teaching of matrices in the Linear Algebra courses mentioned above: Were matrices the first topic taught? Did it make sense to you to learn about matrices and their operations and properties?
5. Imagine that a student asks you the following question during a class on matrices: "Why do we have to multiply rows with columns in the matrix multiplication?" What would you answer?
6. Have you ever had a course in History of Mathematics?
7. Do you think it is important to learn about the history of Mathematics?
8. Do you think that mathematical notions change over time? Explain your position.

Figure 3: Questionnaire answered by the participants before the first meeting.

Based on the answers, we conclude that matrices were taught using the approach we mentioned in the introduction. Nobody answered properly the Question 5, about the definition of matrix multiplication. It seems that most of the teachers themselves did not know the reason for the rule:

> I would say that matrix multiplication is defined in that way. Each element of the matrix is determined through the inner product of a line by a column […] I would try to convince them that this theory is grounded in a higher Mathematics […] (Participant Fa, first questionnaire)

Two participants had not studied history of mathematics in the university, but this was neither a pre-requisite for the mini-course, nor did it prove to be a problem. In the last question (Question 8), two participants expressed their opinion saying that mathematical notions do not change over time but they admitted that something can change as, for instance, the way we teach the concepts, our views, etc.

> The notions did not change much over time, but they are no longer addressed in a mechanical way. Context plays an increasingly important role and the topics become closer to everyday life. (Participant J, first questionnaire)

In the questionnaire given at the end of the mini-course we asked them to write a short essay expressing their views and opinions about the study.

The teaching modules

Two teaching modules were elaborated with the following learning objectives:

i. Making participants reflect on their own metadiscursive rules when the matrix notion is at stake, by comparing them with the ones we observed in the historical writings, and
ii. Developing historical awareness about the meanings attributed to the matrix by Sylvester and Cayley.

It was not our goal to use the history of mathematics to introduce the concept of matrix or to teach linear algebra. We selected students who had already taken a first course in linear algebra and had learned about matrices.

The first teaching module was entitled "How matrices appeared in the study of conics by Sylvester". We introduced the geometric context in which the term matrix was proposed by Sylvester and explained how he solved the problem of the classification of types of contacts between two conics using determinants.

Some concepts from projective geometry were necessary, like homogeneous coordinates, projective points, projective lines, and projective conics. After introducing these notions, we presented a summary of the practice developed by Sylvester in order to solve the problem of contacts.

In the end, the students had to discuss historical questions in groups. The goal of this first block of activities was to raise a discussion among the students concerning the metarules we defined and, hence, to promote a reflection about their own metarules related to matrices. We list the activities proposed in this first module in Figure 4.

1. What is the main concept used in Sylvester's practice? Summarize how Sylvester classified the types of contacts between two conics U and V.

2. Describe the difference between how Sylvester used determinants and how we use it today.

3. Explain what a first minor determinant is according to the definition presented by Sylvester in Extract I. What is a second minor determinant? Finally, what is a minor determinant of order r?

4. Why Sylvester had to introduce the minor determinants?

5. Based on Extracts II and III, explain what a matrix was and what the role of this notion to Sylvester was.

6. Compare the definition of matrix presented by Sylvester in Extract II to the definition that is used nowadays. Write at least one similarity and at least one difference.

7. According to the text and Extract II, answer why or for what purpose Sylvester introduced the term matrix.

Figure 4: Activities proposed in the first teaching module.

The second teaching module was entitled "Cayley and the symbolic calculus with matrices". We started by giving a translation of one part of the 1858 memoir. In a second session, historical activities were proposed in order to give the opportunity for the students to reflect about the metarules.

Partial results: Discussion about metarules

The metarules selected in the historical works were explored in the teaching modules through specific questions. An explanation about the way Sylvester solved the problem of contacts, as well extracts of his articles and selected parts of Cayley's memoir were essential to support the discussion. As Kjeldsen (2011) affirms, concerning historical texts, the primary sources played the role of "interlocutors" or "discussants" acting according to certain metarules – different from our own. In the next paragraphs, we present excerpts of discussions that emerged from the metarules extracted from the works of Sylvester and Cayley.

Sylvester's conception of a matrix as the mother of minor determinants caused a bit strangeness in the participants. The transcript below is part of a dialogue that a group had when discussing the role of the matrices in Sylvester's work:

> M: From the womb of a common parent (reading Extract III) (astonishment) Jesus! (laughs) […] I think he sees it, then. In fact, the matrix is a way to organize determinants. So [...] the main thing is not the matrix, it is the determinant.
>
> M: Sylvester, he just thought in squares before. Only after he saw it was not exactly like this, right?
>
> T: I think he saw that there (matrix) should [...] solve a system.
>
> M: Yeah, after he formed the matrix. Then he did the opposite. Indeed, the matrix for him was a way to keep information. The main information: determinant. (Group discussion, first meeting)

The speech of participant M shows a conflict with the conception of a matrix as a representation, from which the minor determinants could be generated, or in other words, as a source to keep information about determinants. This idea places the determinant as the main object and emphasizes the order of development of these concepts. This contrasts with the understanding of the participant M. For him, the notion of matrices come first and then the notion of determinants (defined and computed by means of matrices).

All participants read and discussed the initial pages of Cayley's memoir (translated from English to Portuguese) together. From this activity, they became acquainted with Cayley's motivation to introduce matrix operations. In particular, they realized the origin of matrix multiplication as a composition of linear transformations.

The quote below, taken from one report, shows that some participants noticed the association of matrices with linear systems made by Cayley. This was important in the way the operations (matrix addition, matrix multiplication by a number and matrix multiplication) were defined.

> Motivated by a simpler representation of sets of linear equations, it comes to light naturally the notion of matrices. The difference [between Cayley's matrix description and the modern definition of matrices] is in Cayley's double interpretation of the matrix,

> sometimes he sees matrices like numbers. (Requested report from participants Fe, M, and J)

The issue that gave rise to the commentary above was the difference between the description of matrix presented in Cayley's memoir and the modern definition. The trio of participants F, M, and J realized that the dual interpretation of matrices determines the difference between Cayley's conception of a matrix and their own.

When requested to judge if Cayley's proof furnished for the "remarkable theorem" would be accepted as correct today, the participants F, M, and J expressed their metarule, which they saw as being in accordance with the mathematical community.

The "remarkable theorem" states that any matrix satisfies an algebraic equation of its own order. In the proof, Cayley wrote the following determinant:

$$\begin{vmatrix} a-M, & b \\ c, & d-M \end{vmatrix},$$

where M is the following matrix:

$$\begin{array}{l} M = (a, \quad b), \\ \qquad\ \ |c, \quad d| \end{array}$$

The quote below shows the response of F, M, and J to the question about the validity of the proof:

> As this proof is constructed to the particular case of matrix order 2, it would not be accepted today since that, in order to prove a theorem, you should use order "n". (Requested report from participants Fe, M, and J).

The participants argued that the proof should be made for matrices of order n. They expressed a metarule that is in accordance with the mathematical community. On the other hand, it seemed to not bother them that Cayley considered a symbolic computation involving a matrix M and numbers (the elements on the diagonal). Cayley justified his argument using the dual interpretation of the matrix either as a number or as a system of numbers, but his proof would not be accepted in the mathematical community today.

INITIAL CONCLUSIONS

Our purpose in this work was to use primary and secondary sources about the history of matrices in order to encourage participants in a pilot study to reflect upon their own metarules related to matrices, comparing these rules with those found in the historical writings. In this sense, we intended to create *conflictive situations*, in a sense similar to that of *commognitive conflicts* that Sfard proposes. Two teaching modules were developed based on two episodes in the history of matrices. They were implemented in a pilot study with six school Mathematics teachers.

The analysis of the results shows that during the discussions about the metarules appearing in the sources, the participants problematized their own metarules. The historical sources, treated through specific activities, made the participants elucidate the metarules they had in mind, thus confirming Kjeldsen's theoretical argument emphasizing the role of history as a strategy to make metarules become explicit and to convert them in objects of reflection (Kjeldsen, 2011).

The goal to develop historical awareness was reached in particular cases. One example is the observation that Sylvester used determinants before matrices were introduced, which made the participants notice the difference between the order in which matrices and determinants are presented today and the historical order in which these notions were developed. In addition, the study of Cayley's memoir of 1858 showed some motivations for defining matrix operations, in particular, the special way to define matrix multiplication.

The reflections on metarules also provided a perspective for the participants to reflect on the basic curriculum, regarding the topics of matrices, determinants and linear systems. They even discussed the ways in which matrices are treated at a basic level. One participant observed that:

> It was very interesting to know that the concept of matrix came from very different ideas of what is taught in schools today. What, moreover, allows us to take a more critical look at the math curriculum in high schools. (Participant Fa, final questionnaire)

We will continue this research by implementing additional activities and analyzing the discourses of participants while reflecting about their own metarules. The history of mathematics has proven to be an interesting way to create an environment for the participants to perceive the metarules they use and that they consider as being the right way to do mathematics.

NOTES

1. There are some different approaches. Stormowski (2008) proposed the teaching of matrices from the linear transformations in basic education. Cabral and Goldfeld (2012) presented matrices together with the topics systems of linear equations and linear transformations in their textbook for linear algebra courses.

2. For details, see Brechenmacher (2006).

3. Sylvester's research episode about the problem of the types of contacts between two conics was based on four articles (Sylvester 1850a, 1850b, 1851a, 1851b).

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Oral Presentation

ANCESTRAL CHINESE METHOD FOR SOLVING LINEAR SYSTEMS OF EQUATIONS SEEN BY A TEN YEARS OLD PORTUGUESE CHILD [1]

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Inspired by the simplicity of the fangcheng method, for solving linear systems of equations, presented on the Chinese ancient mathematics book "The Nine Chapters on the Mathematical Art", we decided to test the viability of a Portuguese ten years old child understand, reproduce and apply the method. We created a task, transposing didactically that method, to be presented to the child. By research design, the child solved the task in parts, outside the classroom and without relation with his classes. The analysis of the collected data allows us to affirm that this young child was able to use and apply this method in an a-didactical situation.

INTRODUTION

Mathematics has developed with humanity throughout the ages and in different spaces. It can be seen that within all civilizations have emerged manifestations of mathematical nature. Looking to the framework that exists today about the mathematics developed by some of the ancient civilizations is possible to infer about the mathematical knowledge of ancient civilizations of Mesopotamia, India, Egypt and China (Katz, 1998/2010, p. 4).

The ancient Chinese mathematics can be understood as an independent mathematics that was configured as a distinct area of knowledge and which was perpetuated over generations through writing (Martzloff, 1987/1997, pp. 3-13). This mathematics has remained virtually unknown to the Western world until the second half of the nineteenth century.

Swetz (1988, p.8; 1994, p.2) argues that the history of mathematics in general, and the history of ancient Chinese mathematics in particular, can provide many fruitful and challenging problems from the pedagogical point of view.

The use of history of mathematics as educational resource has been the focus of several academic studies in recent times. The available literature presents several favorable arguments to the integration of history of mathematics in mathematics education, arguing that the teaching of mathematics can be enriched through this integration (Jankvist, 2009a). Tzanakis et al. (2002) point out that these improvements can be significant in the learning of mathematics, in the development of views on the nature of mathematics, in the affinity of students with mathematics and in the social and cultural aspects of mathematics. The history of mathematics can be seen as a storehouse of issues, situations, problems and examples that may contribute to the diversification of didactic resources and consequently student engagement.

Despite favorable arguments to integrate the history of mathematics, there are few empirical studies on this true integration into mathematic teaching (Jankvist, 2009b, 2011).

Focusing on ancient Chinese mathematics, the opinion of Leng (2006) is consistent with the previous argument, when he says that the ancient Chinese mathematics has been the focus of many studies in a historical perspective, but a little has been done to investigate the role that mathematics could have on teaching and learning mathematics.

The *fangcheng,* an example of ancient Chinese mathematical method, is presented in the 8th chapter of *The Nine Chapters on the Mathematical Art* and is used to solve problems which now could be associated to a linear system of equations. It uses tables of numbers and elementary arithmetic operations between the numbers in the columns of these tables.

In his book, Martzloff (1987/1997, pp. 249-258) presented a description of this method. He starts with a problem, presented in *The Nine Chapters on the Mathematical Art,* which could be associated to the linear system of equations below

$$\begin{cases} 3x+2y+z=39 \\ 2x+3y+z=34 \\ x+2y+3z=26 \end{cases}.$$

Problem solving using the *fangcheng* method involves the distribution of the numbers that arise in the problem by columns. After identifying the first condition of the problem, the distribution of these numbers is made in the rightmost column. Then, it’s made the distribution of the numbers of second condition in the left column of the first and so on. In ancient China it was used counting rods (Martzloff, 1987/1997, p.210) in the resolution of such problems. The original representation of the problem would be as shown in fig.1:

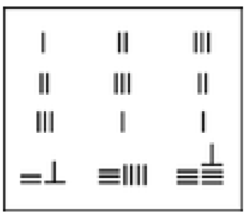

Fig. 1 – Representation of the problem using counting rods (*in* Martzloff, 1987/1997, p. 253)

The application of the *fangcheng* method produces the following sequence of tables using Arabic numeration (see Fig.2), resulting from elementary arithmetic operations on the columns, and allows us to find the solution of the problem.

1	2	3
2	3	2
3	1	1
26	34	39

1	6	3
2	9	2
3	3	1
26	102	39

1	3	3
2	7	2
3	2	1
26	63	39

3	0	3
6	5	2
9	1	1
78	24	39

0	0	3
4	5	2
8	1	1
39	24	39

0	0	3
20	5	2
40	1	1
195	24	39

0	0	3
15	5	2
39	1	1
171	24	39

0	0	3
10	5	2
38	1	1
147	24	39

0	0	3
5	5	2
37	1	1
123	24	39

0	0	3
0	5	2
36	1	1
99	24	39

Fig. 2 – Application of the *fangcheng* method to the previous system (*in* Martzloff, 1987/1997, p. 254)

In Portugal, linear systems of equations play an important role in the articulation of concepts from the domains of algebra and geometry. In Mathematics for Basic Education the content linear systems with two equations and two unknowns is taught to students with about thirteen, in 8^{th} grade (Ministério da Educação e Ciência [MEC], 2013). This topic also arises in Mathematics for Secondary Education in the 11^{th} grade, (Ministério da Educação - Departamento do Ensino Secundário [ME-DES], 2002) taught to students with about sixteen.

The method of Gaussian elimination for solving linear systems of equations is usually taught to students from eighteen, in Higher Education, in the first year of many courses.

We believe that *fangcheng* method is simplest than Gaussian elimination method because, as far as we know, that method doesn't uses algebra symbolism. Unlike this, it uses the context of the problem and arithmetic relations. Comparing the simplicity of *fangcheng* method with the Gauss elimination method we wonder if this method could be used, with advantages, by younger students. We ask ourselves if it is possible that a ten years child understands and appropriates the *fangcheng* method to solve linear systems of equations.

METHODOLOGY

To understand how a young child would react to a task on the *fangcheng* method, we designed an exploratory case study (Yin, 2009). We made the didactic transposition

(Brosseau, 1986) of this method and we created a task partitioned in three parts, with increasing level of complexity and abstraction (Stein, 1998). We emphasize that to use the history of mathematics in teaching and learning is fundamental to adjust it in order to be understood by the students. We had this fact in mind. We decided to apply it to a young child in three different moments. At the 1st and 2nd parts of the task, one of the researchers acted like mediator/teacher and interacted with the child urging dialogues where the child was able to verbalize his thoughts and the procedures used in solving the problems proposed. In the 3rd part of the task problems were provide through a website created for the purpose, and the child solved them autonomously, without any intervention of the researchers and, from the point of view of the child, the situation was not connected with the previous task.

The child, Gabriel [2], was picked occasionally. He is known by researchers, had 10 years old and he had just finished the primary school, had good marks and talent to mathematics. He didn't know yet unknowns/variables or equations and systems of equations much less.

We did the video recording of the application of the task and all productions of Gabriel were filed. After the implementation of the three parts of the task we interviewed the boy.

THE *FANGCHENG* TASK: PRESENTATION AND DISCUSSION

We start presenting the task to Gabriel on the 28th October of 2013 and we completed de implementation of the task on 10th December of 2013.

First part of the task

The first part of the task was presented and discussed in *Potentialities on the Western Education of the Ancient Chinese Method to Solve Linear Systems of Equations* (Costa, Alves & Guerra, 2014). It is synthetized in this subsection.

The first problem (see Fig.3) was designed to be the motivation for introducing the ancient Chinese method.

Gabriel didn’t know linear systems, as we said before. He managed to find a solution by using other strategies. Nevertheless he showed some insecurity in solving the problem. After this, one of us presented the *fangcheng* method to Gabriel.

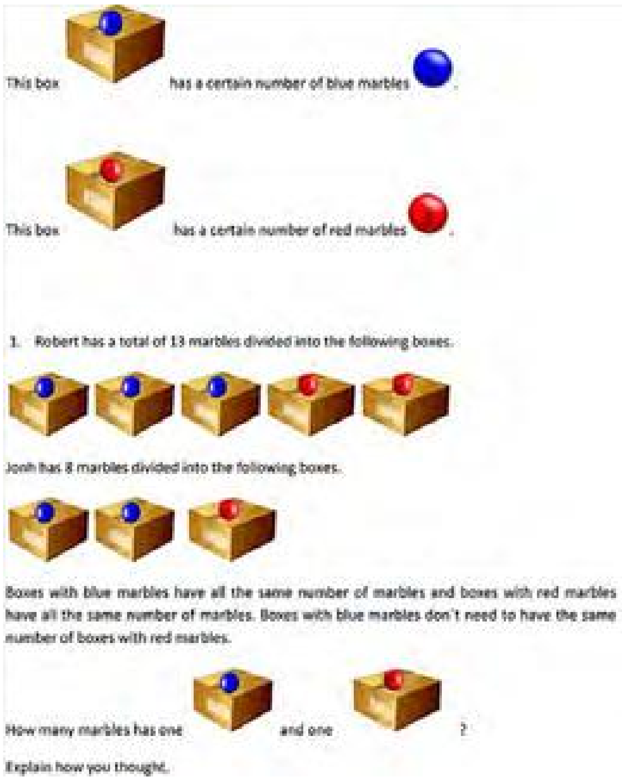

Fig. 3 - Statement of the first already solved problem presented to Gabriel that could be associated with the resolution of a linear system with two equations and two unknowns (translated to English) (Costa et al, 2014)

During this presentation of *fangcheng* method to Gabriel, we considered more appropriate didactically maintain the orientation of writing with which he was accustomed (from left to right), we used only non-negative integers and we illustrate the problem data with pictures alluding to the statement. After this initial motivation, we presented a new (but similar) problem (see Fig.4) to Gabriel and suggested him to solve it.

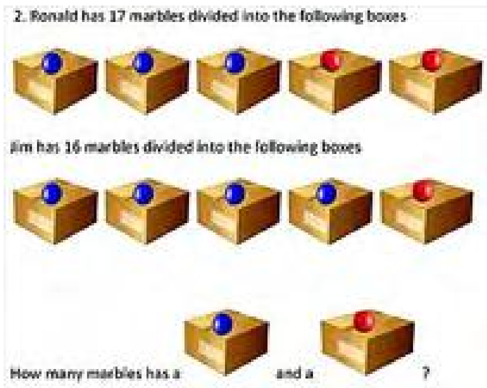

Fig. 4 - Statement of the second problem presented to Gabriel, similar to the 1st one, which could be associated with the resolution of a linear system with two equations and two unknowns (Costa et al, 2014)

Gabriel's resolution is presented in Fig. 5.

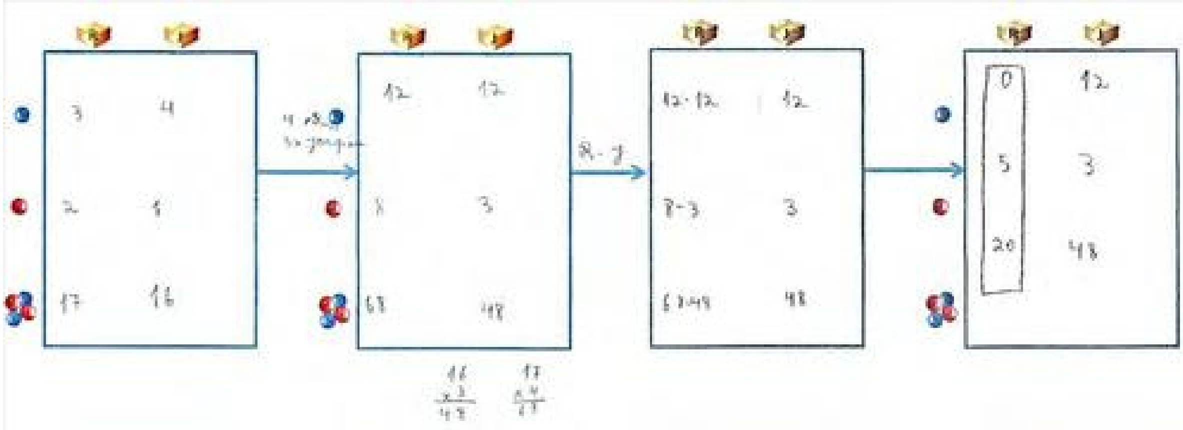

Fig. 5 – Gabriel´s resolution of the problem presented on Fig.3 (Costa et al, 2014)

After presenting these tables (see Fig.5), Gabriel wrote (in Portuguese):

5 boxes of red marbles = 20 marbles;

Red box has 20:5 = 4 marbles;

12 blue boxes + 3 red boxes = 48;

12 blue boxes + 12 red marbles = 48;

12 blue boxes = 36 marbles;

Each blue box has 3 marbles

Following we made it more difficult. This time we presented a problem with three unknowns and, following the same strategy, we present a resolution of this problem using the same method (see Fig.6).

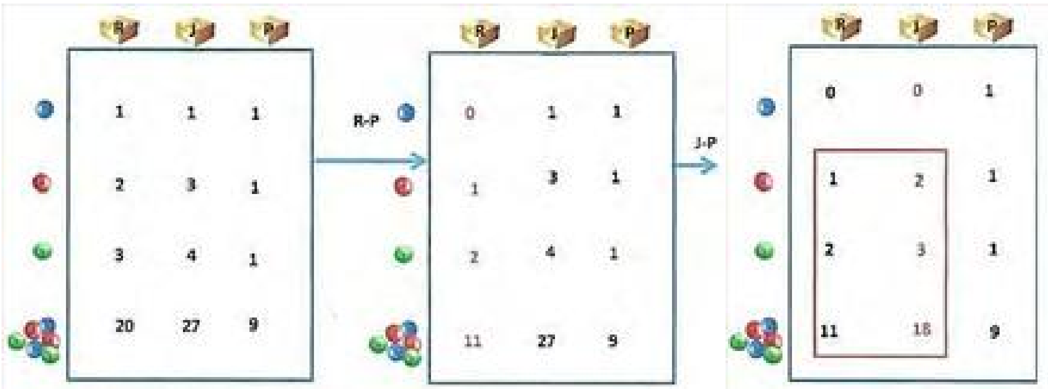

Fig. 6 - Resolution of a part of the problem that could be associated with the resolution of a linear system with three equations and three unknowns, presented to Gabriel (Costa et al, 2014)

After this, we suggest to Gabriel to solve a problem (see Fig.7), also with three unknowns.

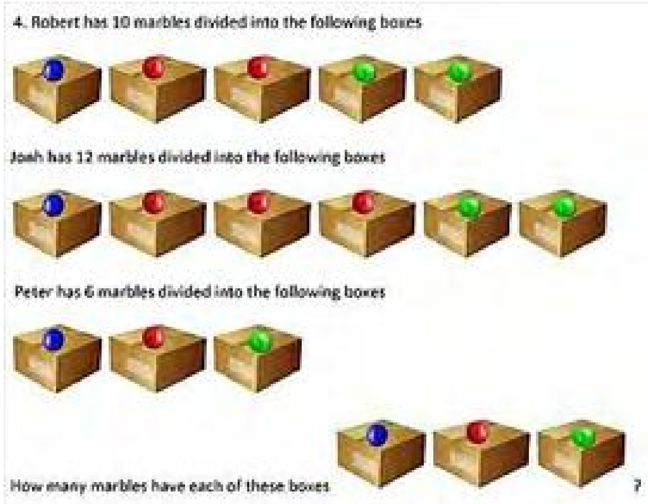

Fig. 7 - Statement of the 3rd problem presented to Gabriel, which could be associated with the resolution of a linear system with three equations and three unknowns (Costa et al, 2014)

Fig.8 shows the resolution made by Gabriel.

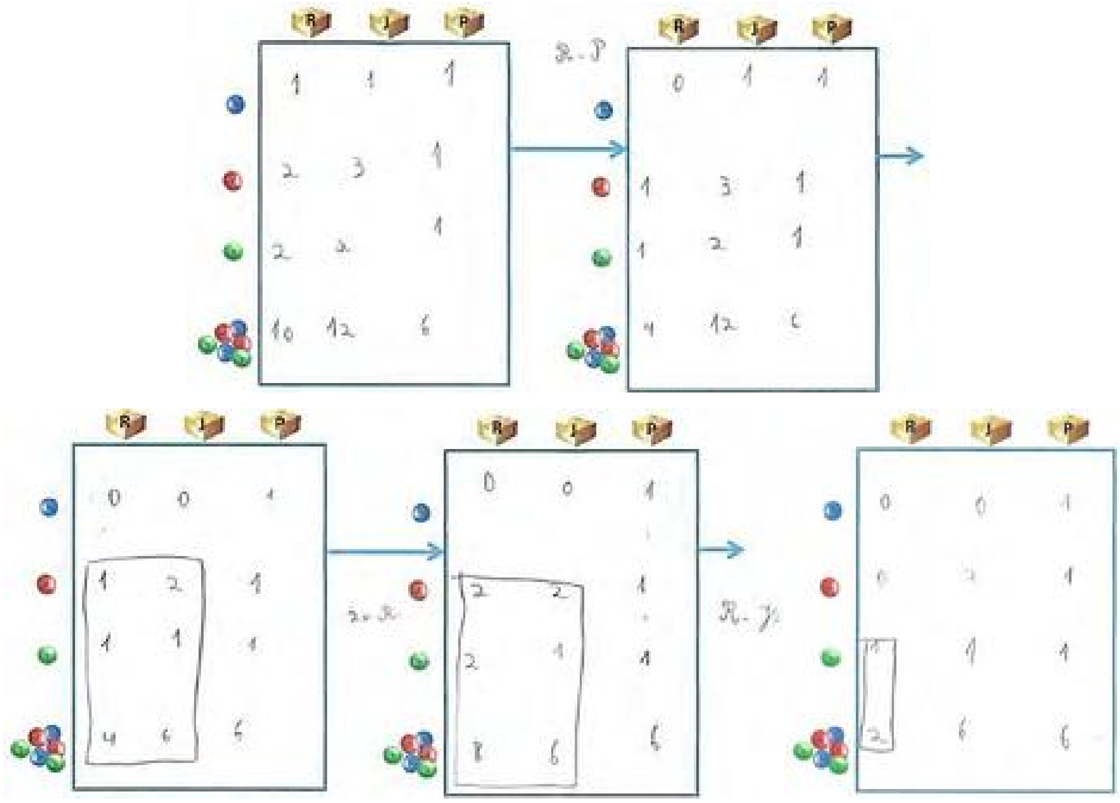

Fig. 8 – Gabriel´s resolution of the problem presented in Fig.6 (Costa et al, 2014)

After presenting these tables (see Fig.8), Gabriel wrote (in Portuguese):

1 box of green marbles = 2 green marbles;

2 red boxes + 1 green boxes = 6;

2 red boxes + 2 green marbles = 6;

2 red boxes = 4;

1 red boxes = 4:2 = 2 red marbles;

1 blue box + 1 red box + 1 green box = 6;

1 blue box= 2 blue marbles.

Second part of the task

The second part of the task was applied on 5th of December of 2013. In this second part we proposed to the child more formal statements (see Fig. 9), without pictures and therefore demanding higher degree of abstraction; however the registration tables were kept in the statement. Some images alluding to the study variables were replaced by letters (variables).

Robert wants to offer a bouquet of flowers to his mother on her birthday. He visited 3 flower shops.
The first shop had 1 bouquet of roses, 2 bouquets of tulips and 3 bouquets of daisies. Altogether, in this shop, there were 11 flowers.
The second shop had 1 bouquet of roses, 3 bouquets of tulips and 2 bouquets of daisies. There were 13 flowers.
The third shop had 1 bouquet of roses, 1 bouquet of tulips and 1 bouquet of daisies. Altogether, in this shop, there were 6 flowers.
It is known that bouquets of roses have all the same number of roses, the bouquets of tulips have all the same number of tulips and the bouquets of daisies all have the same number of daisies.
Can you find out how many flowers were in a bouquet of roses, in a bouquet of tulips and in a bouquet of daisies?

Fig. 9 - Statement of the 4th problem presented to Gabriel, which could be associated with the resolution of a linear system with three equations and three unknowns (translated to English)

Gabriel solved the problem doing the calculations presented in Fig. 10.

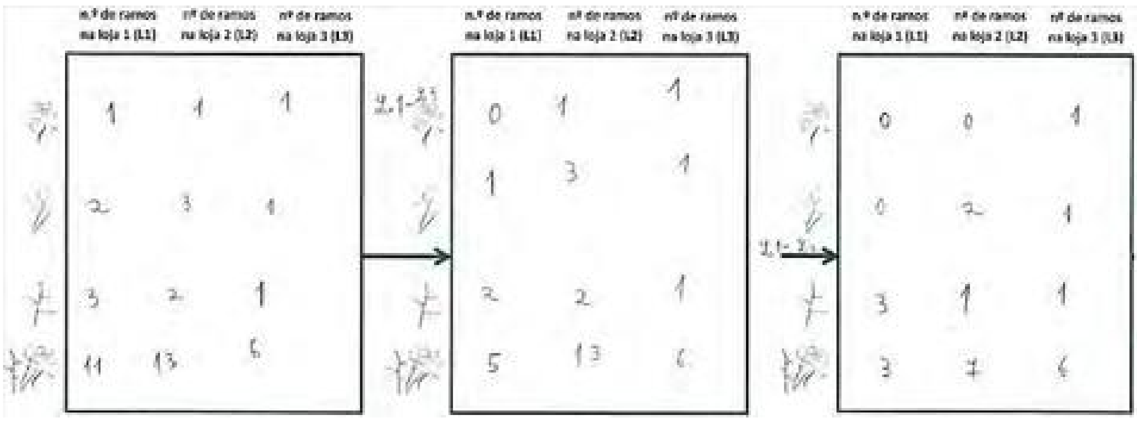

Fig. 10 – Gabriel´s resolution of the 4th problem presented in Fig.8

After presenting these tables (see Fig. 10), Gabriel wrote (in Portuguese):

1 bouquets of daisies = 1 daisy;

1 bouquets of tulips = 3 tulips;

1 bouquets of roses = 2 roses.

At that moment, it seemed that Gabriel understood and knew how to apply the algorithm associated with the *fangcheng* method. But we needed to verify if the method was learned and appropriated by him as a mental tool. For that purpose, we have carefully drafted an a-didactical situation to investigate if he mobilizes autonomously his knowledge in a different context.

Third part of the task

In creating this a-didactical situation we careful that, from the perspective of Gabriel, the situation was not connected with any of the above tasks and the problem presented could not be easily recognizable as likely to be solved through the ancient Chinese method.

The strategy we found took advantage of the Gabriel's beliefs in the existence of Santa Claus and his elves (with the consent of the child's parents).

A website (available at http://duendematematico.wix.com/concurso-natal-2013) has been created on the Internet, where a Mathematical Elf was presenting a competition for children aged between ten and twelve.

To participate, children had to solve three problems and send, by uploading on the contest website, the resolution of the problems and the video recording of task execution.

The three problems presented were similar but, only one problem (the second one) could be solved by the ancient Chinese method.

We applied the third part of the task on 10^{th} of December of 2013. Gabriel starts solving the Mathematical Elf´s task applying knowledge gained at school or at home. When he saw the statement of the problem (see Fig. 11) which could be solved by the *fangcheng* method he didn't hesitate and immediately began to draw tables and put numbers on them.

> Today I decorate 3 Christmas trees of different sizes.
> I used 1 box of red balls, 2 boxes of golden balls and 3 boxes of silver balls to decorate the biggest one. This tree was decorated with a total of 21 balls.
> I used 1 box of red balls, 2 boxes of golden balls and 2 boxes of silver balls to decorate the middle one. This tree was adorned with a total of 17 balls.
> I used 1 box of red balls, 1 box of golden ball and 1 box of silver balls to decorate the smallest one. This tree was adorned with a total of 10 balls.
> It is known that each box of each type has the same number of balls. How many balls has each box?

Fig. 11 – Statement of the second problem of the Mathematical Elf´s task. This was the only one, in the contest, that could be solved by the method under consideration.

In a few minutes Gabriel wrote his answer (see Fig. 12).

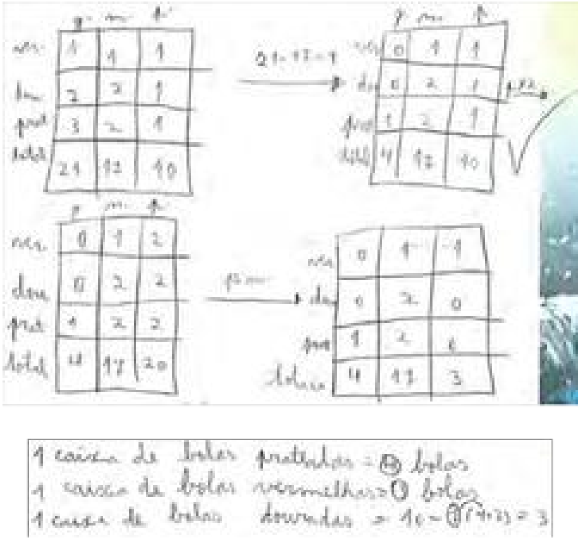

Fig. 12 – Gabriel´s resolution of the second problem of the Mathematical Elf´s contest

After presenting these tables (see Fig. 12), Gabriel wrote (in Portuguese):

> 1 box of silver balls = 4 balls;
>
> 1 box of red balls = 3 balls;
>
> 1 box of golden balls = 10-7=3 balls.

Afterwards, we interviewed the boy during 65 minutes. First we want him to explain his thinking, comparing it with his productions. In the second phase, it was required to him to solve problems with a higher degree of complexity, seeking to establish if the child is able to progress to more complex resolutions.

Focusing on the task of Mathematical Elf (a-didactic situation):

> Researcher: How did you find out that you could use the Chinese method?

Gabriel compares the statements of the problems of the various tasks and concludes:

> Gabriel: Because I thought so ... Today I decorated three Christmas trees... I think I remember something ... I thought this was familiar. I thought a bit and I remembered that the problem of the flowers began the same way.

Apparently Gabriel did not use the algorithm that has been taught and preferred to follow the resolution that seemed more appropriate and faster, since it facilitated the calculations.

> Researcher: Gabriel, you used a method that is not exactly the same method you used in the other two tasks.
>
> Gabriel: I invented a little.
>
> Researcher: Why? Why did you change the method?
>
> Gabriel: Because I think this method is easier than the Chinese method. It is more appropriate.

Researcher: Why is it more appropriate?

Gabriel: Because it's easier. I don’t need to do many calculations. It's faster.

Researcher: How did you know that the method you used is correct?

Gabriel: I didn’t know. I had my heart beating so fast!...

Researcher: And when did your heart stopped beating so fast?

Gabriel: When I started to check the problem ... When I verified that the results were correct.

Researcher: Do you think that this modified method works?

Gabriel: I think it works ... at all. I think that it always works...

Trying to analyse perception of Gabriel about the creation of *fangcheng*, the researchers were surprised. In order to answer, Gabriel associates the method to the greats of Western mathematics, like Euclid or Pythagoras, and takes them to another part of the world and a different culture, showing his vision of the universality of mathematics.

Researcher: How do you think that Chinese people discovered this method? How do you think they would thought?

Gabriel: It must have been a Pythagoras of China.

Researcher: A Pythagoras of China?

Gabriel: Yes, a Chinese Pythagoras! Or a Chinese Euclid or still a Maurits Cornelis Escher!

Researcher: In your opinion do you think kids in your age could learn this method in school?

Gabriel: Hmm... I don’t know.

Researcher: But, in your opinion, we could teach this method in maths classes for kids of your age?

Gabriel: Yes, if they are interested in learning it.

At the end of the interview the researcher suggested another problem, about “*Cakes and chocolates*”, to Gabriel. This time the statement was formal and without images.

The problem proposed could be associated with the resolution of a linear system with four equations and four unknowns however this didn’t seem to bother Gabriel. Fig. 13 shows Gabriel’s resolution.

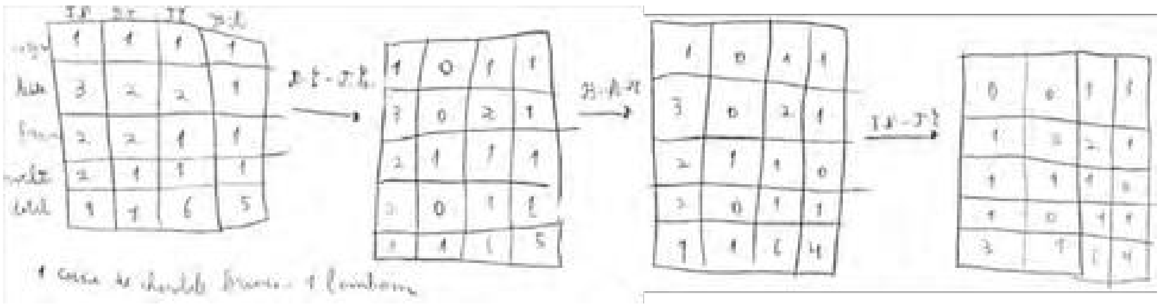

Fig. 13 – Gabriel´s resolution of the problem "*Cakes and Chocolates*" that could be associated with the resolution of a linear system with four equations and four unknowns

Gabriel concluded (in Portuguese):

1 box of bonbons of white chocolate= 1 bonbon

1 box of bonbons of milk chocolate= 1 bonbon

1 box of bonbons with hazelnuts= 1 bonbon

1 box of bonbons of dark chocolate= 2 bonbons

We hope that with the description of the *fangcheng* task and with the Gabriel's productions and comments at the interview, have reflected the way a ten years Portuguese boy sees the ancestral Chinese method for solving linear systems of equations.

CONCLUSIONS AND FINAL REMARKS

We consider that this experimental study design was adequate to the research question: Would a ten years old child understands and appropriates the *fangcheng* method to solve linear systems of equations? This exploratory case study is part of a wider investigation involving the construction and implementation of tasks based on ancestral Chinese mathematics.

From what has been presented it seems appropriate to present the initial findings of this study.

Gabriel, a ten years old boy, solves linear systems of two, three and four equations with two, three and four unknowns, respectively, using the *fangcheng* method. It should however be noted that some didactical transposition was performed in order to use only non-negative integers and to adapt the method to the occidental way of writing. We also highlight that Gabriel likes math and math challenges, which is not the standard in Portugal.

There are evidences that the *fangcheng* method had become part of the child's knowledge; meaning that Gabriel would be able to replicate it in similar situations and foremost use it in a-didactical situations.

This study shows that it is possible to learn the *fangcheng* method much earlier than in the first university year, as usually occurs in Portugal with the Gaussian elimination method.

This leads us to think that some curricular adjustments on this subject may occur, notably in the way we teach and when to teach.

We also think that this experimental case study illustrates the ideas of Swetz (1988, 1994) about the potential of the history of ancient Chinese mathematics from the pedagogical point of view. In this example it can be used to contribute to the development of pre-algebra contents, such us linear systems of equations, unknowns and matrices.

It seems to us that the case presented by us is also in the same line of thinking of the ideas of Jankvist (2011) when he says that by conducting empirical researches we can determine the true impact that the integration of history has in mathematics education. These empirical researches allow us to reassert theoretical conjectures and give new ideas for future lines of research.

From the articulation of these ideas with the feedback received during the oral presentation of this work we accept the future challenges to investigate how an ordinary class of 10 years old students would react of to this *fangcheng* task and how will Gabriel react to the introduction of algebra concepts.

NOTES

1. This work is funded by FCT/MEC through national funds (PIDDAC) and co-financed by FEDER through the COMPETE - competitiveness operational factors program under the project PEST-C/CED/UI0194/2013

2. Thanks are due to Gabriel who accepted patient and happily all our math challenges.

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Oral Presentation

A LOOK AT OTTO TOEPLITZ'S (1927) "THE PROBLEM OF UNIVERSITY INFINITESIMAL CALCULUS COURSES AND THEIR DEMARCATION FROM INFINITESIMAL CALCULUS IN HIGH SCHOOLS"[1]

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This paper discusses Otto Toeplitz's 1927 paper "The problem of university infinitesimal calculus courses and their demarcation from infinitesimal calculus in high schools." The "genetic approach" presented in Toeplitz's paper is still of interest to mathematics educators who wish to use the history of mathematics in their teaching, for it suggests a rationale for studying history that does not trivialize history of mathematics and shows how history of mathematics can supply not only content for mathematics teaching but also, as Toeplitz is at pains to emphasize, a guide for examining pedagogical problems. At the same time, as we shall discuss in our paper, an attentive reading of Toeplitz's paper brings out tensions and assumptions about mathematics, history of mathematics and historiography.

TOEPLITZ'S LIFE IN MATHEMATICS, HISTORY OF MATHEMATICS, AND MATHEMATICS TEACHING

Before starting our examination of the paper which is our focus in this paper, we ought to have some sense of who its author, Otto Toeplitz, was as an intellectual and educational figure. Toeplitz was born in Breslau, Germany (now, Wrocław, Poland) in 1881 and died in Jerusalem in 1940. His doctoral dissertation, *Über Systeme von Formen, deren Funktionaldeterminante identisch verschwindet* (*On Systems of Forms whose Functional Determinant Vanishes Identically*) was written under the direction of Jacob Rosanes and Friedrich Otto Rudolf Sturm at the University of Breslau in 1905. The following year, Toeplitz left Breslau and went to Göttingen. Heinrich Behnke (1963, p.2), describes that move as one from "a quiet provincial town to a gleaming metropolis" – an apt expression, for Göttingen at that time was blessed with the luminary presence of Klein, Hilbert, and Minkowski. Soon, Behnke tells us, Toeplitz was included among the students of Hilbert's inner circle. Toeplitz's doctoral dissertation had already touched on topics related to systems of bilinear and quadratic forms, but with Hilbert Toeplitz's interests in this direction crystalized and became the work on infinite linear, bilinear and quadratic forms and infinite matrices for which he is known. After he left Göttingen, he went to Kiel in 1913 and then to Bonn in 1928. Throughout, however, he continued the work on linear operators on infinite dimensional spaces which had been his general focus in Göttingen.

Toeplitz's mathematical accomplishments are not our main concern here though. What interests us is his involvement in the history of mathematics. Still, it is important to know that he *was* a mathematician of the first rank, for his identity as a proponent of the history of mathematics was bound together with his identity as mathematician (and, as we shall see, as teacher of mathematics). Indeed, Abraham Robinson remarks that Toeplitz "...held that only a mathematician of stature is qualified to be a historian of mathematics" [2] (Robinson, 1970, p. 428). That said, it cannot be claimed that his work in the history of mathematics as such was remarkable, at least relative to his deep mathematical work; however, he had, by all accounts, a profound interest in the history of mathematics, and he made efforts to promote its study. For example, together with Otto Neugebauer and Julius Stenzel he established *Quellen und Studien zur Geschichte der Mathematik* (*Sources and Studies in the History of Mathematics*) in 1929. It was for this journal that Toeplitz first received the papers which later became Jacob Klein's famous *Greek Mathematical Thought and Origins of Algebra*.[3] Toeplitz's enthusiasm for those papers mirrored his particular interest in the relationship between Greek thought and Greek mathematics (Robinson, 1970): "He was a classical scholar able to read Greek texts and he knew his Plato just as well as his Gauss and Weierstrass" (Born, 1940, p.617). The breadth of Toeplitz' scientific interests is also reflected in the good relations he cultivated with the philosopher and psychiatrist Karl Jaspers. Indeed, Jaspers dedicated his 1923 book "Die Idee der Universität" ("The idea of the university") to Toeplitz.[4]

Jasper's decision to dedicate his book on education to Toeplitz is pertinent to our story. For it was in his role as an educator that Toeplitz's interest in the history of mathematics was decisive. Toeplitz's father was a school teacher, and teaching was of great importance to Toeplitz. It is not by chance then that Behnke entitled his tribute to Toeplitz mentioned above, "Der Mensch und der Lehrer" ("The Man and Teacher"). In the early 1930's Toeplitz and Behnke initiated yearly "meetings for cultivating the relations between university and high school" ("Tagungen zur Pflege des Zusammenhangs von Universität und höherer Schule") (Schubring, 2008; Hartmann 2009, pp. 186-195) and they jointly founded the education oriented journal "Mathematische und physikalische Semesterberichte" ("semester reports on mathematics and physics") in 1932 (Hartmann 2009, 199-209). Toeplitz published articles on mathematics education in nearly every issue of the Semesterberichte. A study of these interesting papers is still outstanding and would be a rewarding task.[5]

THE 1927 PAPER AND THE GENETIC APPROACH

In discussing Toeplitz's intellectual life we have emphasized the three streams of mathematics, history of mathematics, and teaching. These three streams came together in what he called the "genetic method" for the teaching. It is beyond the scope of this paper to examine what records may exist of Toeplitz's actual teaching in Göttingen, Kiel or Bonn, but it is clear that he did to some degree introduce the genetic method into the classroom and that he had intended to write a textbook based on it. This Toeplitz's former student Gottfried Köthe tells us, adding that,

> He worked on [the method] for many years, pursuing intensive historical studies of the development of infinitesimal calculus. In his lectures he constantly tried out new approaches, discussing the several parts with his students and searching always for new formulations (Toeplitz, 1963, p.xii)

Köthe edited Toeplitz's lectures and in 1949, after Toeplitz's death, published the textbook that Toeplitz never finished as *Die Entwicklung der Infinitesimalrechnung: Eine Einleitung in die Infinitesimalrechnung nach der genetischen Methode, vol.1* (another volume was intended). This was later republished in English as *The Calculus: A Genetic Approach* (Toeplitz 1963) in 1963. The book, however, did not present the idea of the genetic method, its rationale and overall strategy. That was set out in the 1927 paper which is our focus here, "Das Problem der Universitätsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den höheren Schulen" (Toeplitz, 1927).

In this paper, which was the published version of an address Toeplitz delivered at a meeting of the German Mathematical Society held in Düsseldorf in 1926, Toeplitz clearly wanted to be understood as solving a specifically *educational* problem, that of designing an introductory course in calculus for beginning university students. Thus the "The problem of university infinitesimal calculus courses and their demarcation from infinitesimal calculus in high schools," mentions neither history nor the genetic approach which is based on history and which is the centerpiece of the address.

The educational problems he wishes to solve are in truth not immediately related to history. He defines these problems in terms of three basic dilemmas, or "moments." The first is contending with two existing schools of thought as to what the guiding principle should be regarding such a course. One school of thought maintains that beginning university students should have an introduction to the calculus that is exact and rigorous, the other, that the course should be intuitive and approachable. Neither "path" (*Richtung*) alone is completely satisfying, but nor is a hybrid which takes a little from each; they are, Toeplitz says, unbridgeable (*unüberbrückbar*).

The second moment is closely related to the first, in some ways, it is a version of the first. It is the tension between two aims of an introductory course in calculus, one being that students acquire necessary tools and concepts in order to ground further work in mathematics and science, and the other, that students acquire a taste for the subject. The latter is particularly important to Toeplitz. He wants students to appreciate how mathematics can be exciting and beautiful. Yet, particularly in the case of calculus, it is all too easy, he believes, to destroy any pleasure of the subject by a tendency to teach mere rules and formulas. He also stresses in this connection that he wants to reach students who are capable of studying mathematics seriously, but who are not necessarily mathematical-types.

The third moment shows, in a more immediate way, Toeplitz's concern with the intellectual character of students. The difficulty here is that there are two different groups of students who are likely to take a first course in calculus. One group comes from the science-oriented Oberrealschule and the other from the humanist Gymnasia, and each is problematic in its own way. One can easily guess the advantages of the Oberrealschule students and the disadvantages of the Gymnasium students, but Toeplitz is also astute enough to recognize that the latter have something to offer and not all is well with the former. As we remarked above, Toeplitz himself had a firm humanist training, so he could well appreciate the value of an education obtained at the Gymnasia. The problem with the Oberrealschule students is precisely what others might see as an advantage, namely, their greater exposure to the technical side of calculus. Toeplitz points out that this too often and too easily prevents them from seeing that there is more to know, that knowing some techniques of calculus is not the same as knowing calculus. These are the greater challenge for Toeplitz.

It is in this last moment that students' previous high school education comes under consideration in a concrete and pointed way. This was not by chance. Toeplitz had very much in mind a set of reforms in German high school education involving the incorporation of the infinitesimal calculus into the mathematical syllabus of the gymnasia. Bringing the infinitesimal calculus into the gymnasium was the most important project among Felix Klein's educational initiatives, and in 1925 infinitesimal analysis finally became part of the official syllabus at Prussian Gynmasia.[6] This was just one year before Toeplitz's address, which is why Toeplitz, unable to hide his objections, refers to the teaching of calculus to high school students with a discernable edge, calling it a *fait accompli*. But so it was, and, therefore, it became part of the overall difficulty in setting up a first year calculus course at the university level.

Toeplitz suggestion is that all three moments and difficulties they identify can be addressed if one gives proper attention to the history of mathematics: if one takes the development of mathematical ideas as a guide for teaching, he claims, not only will the drama of that development be revealed to students but also the logic and interconnection of mathematical ideas themselves. This is the core of Toeplitz's "genetic method." However, there are two ways the method can actually be taken up. Toeplitz calls these the "direct genetic method" and the "indirect genetic method."

The "direct method" uses the historical development of mathematical ideas to inform the presentation of ideas in the classroom – it is a way of teaching. It answers the first moment, for example, not by bridging the divide between rigor and intuition but by providing a third alternative: the student arrives at mathematical ideas by following the same slow and gentle ascent by which the ideas themselves were arrived at historically. In other words, he does want to present rigorous ideas in a softened intuitive fashion, or to use intuitive notions as a springboard for jumping to rigorous formulations, but, rather, to allow the rigorous ideas to unfold for the students according to the very same gradual process that they themselves unfolded and, by doing so, not only bring the rigorous ideas into the classroom but also to show in a *natural* way why those ideas slowly became clear and necessary.

The "indirect method" is a way to analyze problems of teaching and difficulties in learning rather than a way of teaching itself: it is, as Toeplitz puts it, "...the elucidation of didactic difficulties, ...didactic diagnosis and therapy on the basis of historical analyses, where these historical analyses serve only to turn one's attention in the right direction." The qualification at the end is crucial: the application of the "indirect method" does not necessarily mean that history itself must be brought into the classroom. In the guise of the "indirect method," history takes on a role that might be compared to the psychology of learning: teachers use it to guide their teaching strategies and decision-making, but it is not what they teach their classes. Thus, Toeplitz argues, from the historical claim that knowledge of the definite integral was possessed by the Greeks or, at any rate, predated other ideas from the calculus, the study of the calculus for modern students ought to begin with the definite integral.

Toeplitz maintained that his "genetic method" was not unprecedented.[7] Felix Klein, he recalls, adopted the biogenetic law in his teaching already in 1911. In fact, this direction in Klein's thinking was evident more than a decade and a half earlier. In 1895, Klein delivered a talk at a public session of the Göttingen Royal Association of the Sciences on the "Arithmetization of Mathematics" (Klein 1895). At the end of his talk, Klein remarked about the teaching of mathematics. He said that, in his view, a paradoxical situation exists whereby teachers at gymnasia tend to stress "Anschauung" too much while professors at universities do so too little. More to the point, university professors put "Anschauung" aside completely whenever possible. Klein argued that at least the elementary courses meant to introduce the beginner to higher mathematics should take "Anschauung" as a starting point, "*...since on a small scale the learner always passes naturally through the same development which has been passed through by science on a large scale*". ("wird doch der Lernende naturgemäß im kleinen immer denselben Entwicklungsgang durchlaufen, den die Wissenschaft im großen gegangen ist"). Although Klein did not use the term "biogenetic law" explicitly, Klein's intention was obvious to Alfred Pringsheim (1850-1941). Thus, when he spoke at the German Mathematical Society in 1897, Pringsheim asked whether it is "suitable to transfer Häckel's principle of the concurrence between phylogenesis and ontogenesis in such an unrestricted way to teaching" (Pringsheim 1897, p. 74) as Klein had done in his talk of 1895. After a lengthy discussion involving mostly non-mathematical examples, Pringsheim concluded that "the principle referred to by Herr Klein as a principle of teaching [i.e. the biogenetic law] appears to be anything but conclusive and at least needs an examination from case to case" (Pringsheim 1897, p.75). With that, Pringsheim stated his own principle, namely "Every individual passes essentially through the same development as science itself as long as he is not shown a better way" (loc. cit.). It was to continue these discussions that Klein and Pringsheim were invited to give lectures on the issue of the university courses for beginners at the meeting of the following year 1898. Toeplitz refers precisely to these lectures right at the start of 1927 paper, forming a rhetorical if not real context for his own ideas.

THE BIOGENTIC LAW AND THE SCIENTIFIC APPROACH TO TEACHING WITH HISTORY

Even though Klein did not mention the biogenetic law by name in 1895, he was clearly an adherent to the doctrine and referred to it explicitly in the first decade of the century, when Toeplitz would have known him at Göttingen. In the appendix to his *Elementary Mathematics from an Advanced Standpoint* (Klein, 1908/1939), a book meant for mathematics teachers, Klein states the law and its implications in a way that shows why Toeplitz should find it so enticing as a principle for his own pedagogical strategy:

> From the standpoint of mathematical pedagogy, we must of course protest against putting such abstract and difficult things before the pupils too early [he is referring to the theory of sets]. In order to give precise expression to my own view on this point, I should like to bring forward the biogenetic fundamental law (*das biogenetische Grundgesetz*), according to which the individual in his development goes through, in an abridged series, all the stages in the development of the species. Such thoughts have become today part and parcel of the general culture of everybody. Now, I think that instruction in mathematics, as well as in everything else, should follow this law, at least in general. Taking into account the native ability of youth, instruction should guide it slowly to higher things, and finally to abstract formulations; and in doing this it should follow the same road along which the human race has striven from its naïve original state to higher forms of knowledge. It is necessary to formulate this principle frequently, for there are always people who, after the fashion of the mediaeval scholastics, begin their instruction with the most general ideas, defending this method as the "only scientific one." And yet this justification is based on anything but truth. To instruct scientifically can only mean to induce the person to think scientifically, but by no means to confront him, from the beginning, with cold, scientifically polished systematics.
>
> An essential obstacle to the spreading of such a natural and truly scientific method of instruction is the lack of historical knowledge which so often makes itself felt. In order to combat this, I have made a point of introducing historical remarks into my presentation. (p.268).

Klein's favorable view of the biogenetic law should not be viewed as eccentricity on his part, nor for that matter on the part of Toeplitz. The biogenetic law had a strong presence in their time and long history preceding it.[8] In biology itself, it was popularized by Ernst von Haeckel (1834-1919) and was generally identified with him, as in the quotation above from Pringsheim's 1897 talk. It is fairly clear, moreover, that it is Haeckel's formulation of the rule that "ontogeny recapitulates phylogeny," that the development of the individual organism follows the development of the species, which Klein and Toeplitz refer to as the biogenetic law. Haeckel of course did not invent the law; complete and consistent statements of it can be found already in the early part of the 19th century (see Gould, 1977 and Mayr, 1994 for critical accounts). And there were variations of the law, ranging from more respectable forms in which the individual development merely paralleled species development to less respectable forms in which the latter actually caused the former. But what is centrally important is the biogenetic law was, even when rejected, viewed as a scientific matter and a serious scientific hypothesis.

Alongside the "scientific" biogenetic law was a cultural version of the same idea, namely, that the *intellectual* development an individual person follows that of civilization. This of course had educational implications and was often taken up of educational theorists and practitioners. Thus, for example, in Froebel's 1826 *The Education of Man* we read, "Every human being who is attentive to his own development may thus recognize and study in himself the history of the development of the race to the point it may have reached, or to any fixed point" (Froebel, 2005, p.40). This view was held also by the followers of the influential Friedrich Herbart (1776-1841) such as Tuiskon Ziller (1817-1883) who called it the *Kulturstufentheorie*, or the *cultural epoch theory* (see Gould, 1977, pp.149ff). More importantly for us, Florian Cajori (1859-1930), the Swiss-American educator and historian of mathematics, refers again to the law as it comes down through educational thinkers in his *History of Elementary Mathematics with Hints on Methods of Teaching* (Cajori, 1896).Cajori opens the preface of that work with a quotation from Herbert Spencer in which the genetic principle is stated:

> "The education of the child must accord both in mode and arrangement with the education of mankind as considered historically; or, in other words, the genesis of knowledge in the individual must follow the same course as the genesis of knowledge in the race" [Cajori quoting Spencer]

Cajori then uses this to justify his own use of history of mathematics for mathematics teaching:

> If this principle, held also by Pestalozzi and Froebel, be correct, then it would seem as if the knowledge of the history of a science must be an effectual aid in teaching that science. Be this doctrine true or false, certainly the experience of many instructors establishes the importance of mathematical history in teaching (p.v).

Toeplitz, curiously enough, never refers to the "cultural epoch theory" or any other of these educational versions of the "biogenetic law." One might speculate that, unlike the "biogenetic law," Toeplitz might have found these other theories were somehow unscientific. Whether or not Toeplitz accepted the "biogenetic law" in the literal way Haeckel framed it, the scientific status of the "biogenetic law" would certainly provide his own genetic approach with a firm basis as Felix Klein seemed to think regarding his own pedagogic method – for Klein, recall, referred to his approach, in the passage quoted above, "a natural and truly scientific method of instruction." This "scientific" or "natural" rationale for the genetic approach, this historical oriented method of teaching mathematics, would also imply a view of history itself as something natural, like biology. Seeing history in this way would, implicitly, allow him to approach turns in history as developments that could be rationally reconstructed on solid ground without resorting to a kind of logical axiomatic structure.

It is surprising then that Toeplitz does not embrace this view of history with conviction and, rather, denies that what he is doing has anything at all to do with history. He says (p.94) that a historian must write down everything that happens, good or bad, while he is interested in only what has successfully entered into mathematics. He tells us that his is *not* a course in history: "Nothing could be further from me than to lecture about the history of infinitesimal calculus: I myself ran away from such a course when I was a student. It is not about *history*, but about the *genesis* of problems, facts, and proofs, about the decisive turning points within that genesis" [emphasis in the original].

But can Toeplitz really separate history from this genesis of problems, fact, and proofs? The very reason why taking the genesis of problems, facts, and proofs into account should help students is that it is natural, fitting to the students' own ways of learning. Is it not for this reason that Toeplitz uses the medical language of "diagnosis and therapy" in describing how historical analysis is supposed to benefit teaching? But this is only possible if historical ideas are themselves somehow natural, just as diagnosis and therapy presuppose certain biological facts. The distinction Toeplitz makes between history of mathematics and the genesis of mathematical ideas and techniques is ultimately therefore an artificial one. Indeed, Toeplitz does not hesitate to make historical pronouncements as if they were indisputable facts – that "The Greeks discovered the definite integral" (p.96), that "The Dedekind cut is essentially in Euclid's fifth book" (p.97), that "Barrow possessed differential and integral calculus in its entirety" (p.98).

The problematic separation between history and genesis presents itself with even more force when once realizes that Toeplitz makes what are really historical claims *on the basis of his genetic approach*, even while he denies it. Thus, having argued that the *relationship* between the definite integral on the one side and the differential calculus and indefinite integral on the other is what needs to be highlighted in the calculus course, he says, "You see clearly here the difference between genesis and history. Historians place the bitter priority quarrel between Newton and Leibniz in the foreground of the historical development of the differential calculus; from the genetic perspective, completely different moments form the central focus." But even if these judgments of what should or should not be in the foreground are based on history via the "indirect genetic approach," they must, nevertheless, have some *status* as history. One cannot just dismiss the issue by saying one judgment is for history and the other for education: parallelism is a symmetric relation.

So although he purports to be taking up purely educational questions, Toeplitz finds himself unavoidably, though perhaps unwittingly, adopting historical and historiographical positions – positions that, reflexively, are also perspectives on the historical character of mathematics itself. The jury may still be out on whether this is a result of Toeplitz's own idiosyncratic way of thinking or built into the genetic method itself; however, for anyone today who wishes to use the genetic method in teaching, the question whether what we see in Toeplitz is in fact an ineluctable tendency of the genetic approach must be confronted. For even if one intends only to adopt the genetic method in the form of "history as a tool," one may be forced to adopt the method in the form of "history as a goal" (Jankvist, 2009), but not the history that one intended.

In general, the ways historiography of mathematics and teaching of mathematics, even without an immediate concern for history, may be deeply entangled should, in our view, be given much greater attention both in historical and educational research. For the latter, the issue is particularly important since the introduction of history of mathematics into mathematics teaching is taken up all too often in a purely instrumental fashion with little cognizance given to what it means to look at mathematics historically in the first place.[9] Typically, it is not asked, for example, whether the ends mathematics education aims towards are necessarily in harmony with those pursued by the history of mathematics. Of course if the genetic principle, as Toeplitz understood it, were unproblematic then such questions would lose their force; but, if not – and if one is guided by the needs of teaching modern mathematics – then one would have to confront the difficulties of anachronism and its inevitable distortions of history.

NOTES

[1] This paper has been adapted from the introduction to our translation of Toeplitz's 1927 paper soon to appear in *Science in Context* (Volume 28) under the title, "Otto Toeplitz's 'The problem of university courses on infinitesimal calculus and their demarcation from infinitesimal calculus in high schools' (1927)." Permission to use the latter was kindly given by Cambridge University Press who owns its copyright.

[2] In this regard, his relationship to the history of mathematics was similar to what one of us has called the relationship of a "privileged observer," that is, where modern mathematical knowledge is thought to provide special power in interpreting the past (see Fried, 2013)

[3] Klein (1968) mentions Toeplitz's historical work in two separate footnotes (notes 68 and 99).

[4] As for this, Uri Toeplitz, Otto Toeplitz's son, wrote in his autobiography that "In 1923, Karl Jaspers wrote a book, Die Idee der Universität, and dedicated it to my father. This demonstrates that even in Kiel father was no one-sided mathematician" (quoted in Purkert, 2012, p.111). Both Toeplitz and Behnke exchanged letters with Jaspers. These letters are analyzed in (Hartmann 2009, pp. 36-40).

[5] In 1938, on account of pressures of the Nazi government Toeplitz' name was removed from the title page of the "Semesterberichte" and in 1939 its last issue appeared. After the war, however, Behnke revived the Journal, and it still appears and is popular under the name of "Mathematische Semesterrichte".

[6] This itself has a background, for there was a long period of coexistence of algebraic analysis, based on Euler's *Introductio in analysin infinitorum*, and modern infinitesimal analysis in Prussian school mathematics, as discussed in Biermann & Jahnke (2013).

[7] Toepltiz does not have in mind here the "genetic method" to which Hilbert refers in, for example, his 1900 Über den Zahlbegriff (English translation, On the Concept of Number, in Ewald, 1996, pp.1089-1095). For Hilbert, the "genetic method" is an approach to defining numbers and other mathematical ideas in terms of more primitive concepts born in basic intuitions, for example, defining the real numbers in terms of a nexus moving through the natural numbers, integers, and rational numbers (see Corry, 1997, pp.125-130 and Ferreirós, 2007, pp. 218-222 for more about the "genetic method" in Hilbert's discussions about the foundations of arithmetic). That said, like the "genetic method" in Toeplitz's sense, the "genetic method" in Hilbert's sense stood in opposition to the axiomatic method, and, at the very end of Toeplitz's paper, Toeplitz emphasizes the difference between his "genetic method" and "Hilbert's foundational studies." If Toeplitz had in mind here works such as Hilbert's Über den Zahlbegriff, then he may have been contrasting his approach not only to Hilbert's formal ideas but also to Hilbert's notion of "genetic method." However, there is not enough evidence to make any firm claim in this connection.).

[8] A deep and thorough account of the genetic idea in mathematics education is Schubring (1978). As for the biogenetic law in biology itself, with its further ramifications, see Gould (1977).

[9] Both Jahnke and Fried have independently considered these questions in the context of mathematics education. See, for example, Jahnke (2000) and Fried (2001)

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Oral Presentation

A CONCEPTUAL AND METHODOLOGICAL FRAMEWORK ANCHORED IN SOCIOCULTURAL APPROACHES IN MATHEMATICS EDUCATION FOR THE INVESTIGATION OF *DÉPAYSEMENT ÉPISTÉMOLOGIQUE*

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Since the mid 90s, the dépaysement épistémologique *has been a recurring concept in literature regarding the use of history in mathematics education (Barbin, 1997, 2006; Jahnke et al., 2000). My research project's goal is to describe the* dépaysement épistémologique *experienced by a group of six pre-service secondary teachers who took part of seven historical texts' reading activities. Data were collected during video recordings, individual interviews and a group interview. I will focus here on how a conceptual framework anchored in sociocultural approaches in maths education (Radford, 2011, 2013), as well as a methodological framework articulated with dialogic perspective (Bakhtin, 1977, 1929/1998), helped me obtain descriptive elements of the* dépaysement épistémologique *experience.*

THE CONTEXT

For decades, many researchers have explored the contributions of history of mathematics in teacher education. In parallel, the presence of mathematics history has established itself considerably in curricula around the world. An attempt to "humanize" mathematics is increasingly present in the curricula of mathematics worldwide (Barbin, 2006; Fasanelli et al., 2000).

In the Quebec province in Canada for instance, the Ministry of Education even now prescribes the use of mathematics history in the classroom. The curriculum (both at primary and secondary level) highlights the importance for students to recognize the contribution of mathematics to science, technology and culture on societies and individuals. Cultural and historical elements form an integral part of the program implementation. This insertion of cultural references in teaching is new and characteristic of this program (Charbonneau, 2006).

These requirements concerning the presence of history in the mathematics classroom, however, raise many questions as teachers and their ability to conduct such activities and mobilize the historical aspects in their teachings. For over 20 years, the presence of mathematics history teachers in training environments has increased substantially in many countries. However, despite the various objectives associated with it, this presence, implicit or explicit, take the form of specific initiatives for each institution. Thus, the objectives and the means employed are not subject to a widely established consensus and the status of history in mathematics teachers' training does not yet seem clearly defined (see Schubring et al., 2000).

From the research side, several discourses emphasize on the positive contribution of the study of the history of mathematics, particularly in teacher education. In this context, a recurring concept is that of *dépaysement épistémologique* (Barbin, 1997, 2006; Jahnke et al, 2000). In this regard, researchers are saying that the history of mathematics “astonishes” and “troubles” our everyday customs on the discipline and highlights its cultural-historical dimension. This important experience of *dépaysement épistémologique* could bring a critical look at the fundamentally social and cultural roots aspect of mathematics.

Overall, this *dépaysement épistémologique* emphasize the historicity of mathematical objects with the astonishment of the learner facing a posture, a framework, a process or a particular argument, far from those of today. In this context, the history of mathematics is a source of encounters whose catalytic effect pushes the learner to question a naïve vision of the discipline and its objects, a vision in which they transcend eras and cultures keeping shape and immutable sense. Introducing the history of mathematics replaces the usual by the different, it makes the familiar unusual. As it occurs when someone is in a foreign context, after an initial phase of confusion, there are recovery attempts, a search and reconstruction of meaning.

These considerations about the *dépaysement épistémologique*, however, have not yet been the subject of systematic researches that truly give voice to the actors in training environments (Guillemette, 2011; Siu, 2007). Thus, I ask for my research two broad questions: “How does this dépaysment épistémologique appear and how is it manifested during training activities based on reading historical texts?” and “How does it go with the development of the ‘becoming a teacher’ of students?”.

HISTORY OF MATHEMATICS AND MATHEMATICS TEACHER’S TRAINING

My focus on these questions began with socio-cultural theories. From this perspective, new discourses have recently emerged in favour of the introduction of mathematics history in the teacher training program. From this point of view, mathematics history is a special place where it is possible to overcome the particularity of our own understanding of mathematical objects, which is limited to our own personal experiences and the sociocultural context in which we live this understanding. In other words, history of mathematics “provides tools for dialogue with other understandings [...] with those who preceded us” (Radford, Furinghetti & Katz, 2007, p. 109). It provides opportunities for meetings with ways of doing and being radically different in mathematics, ways that are historically or culturally distant from us. It is important to understand that this perspective is not carrying individuals and personal self-centred and self-sufficient discourse of empowerment and opportunities, but it is carrying the opportunity for students to explore with others new ways of being-in-mathematics and open the space of possibilities for mathematical activities that occur in classrooms.

This discourse on the potential of the history of mathematics is part of a redefinition of the teaching-learning put forward by an emerging theory in mathematics education: the theory of objectification (Radford, 2007, 2011, 2013).

ON THE THEORY OF OBJECTIFICATION

Inspired by Vygotsky's perspective, theory of objectification is a contemporary sociocultural theory of teaching and learning mathematics. It calls for a non-mentalist conception of thought. Both sensitive and historical, thinking is considered here as "a mediated reflection of the world in accordance with the mode of activity of individuals" (Radford, 2011, p. 4, my translation), that is to say, mediated by the bodies, signs, artefacts and cultural meanings.

In this context, mathematical knowledge is perceived as "movement". Knowledge is abstract and is a "set of ingrown historically and culturally process that is constituted of reflection and action" (Radford, 2013, p. 10). It is constantly changing, constantly moving. It shows itself and makes sense only through men activity, taking inevitably the trace of this cultural and historical activity.

This apparition of knowledge in the activity suggests that it is not owned nor constructed by the learner, but rather frequented. It is then expected that learners "meet" the knowledge in the classroom. Also, as we will discuss, learners can transform this knowledge, and see themselves transformed by it.

On one hand, it is a process of objectification because it is made of acts of meaning that emphasize the appearance of something that revealed itself. On the other hand, it is a process of subjectification because consciousness is also changing during learning. Thus, learning also means becoming, that is to say, the creation of a unique and particular self. They are two inextricable dimensions of learning maths.

In such context, it is now impossible to consider the class as a neutral space in which learners act according to general and invariable mechanisms of adaptation. Indeed, the classroom activity, which is centered on social interaction, does not fulfil an adaptive, facilitative or catalytic function, but is "consubstantial to learning" (Radford, 2011, p. 10, my translation). In other words, learning mathematics is not just learning how "to do mathematics", but learning ways of "being-in-mathematics". Mathematical activity, as a cultural form, is a particular way of "being-with-others".

NEW PERSPECTIVES ON *DÉPAYSEMENT ÉPISTÉMOLOGIQUE*… AND NEW QUESTIONS

From this perspective, how can we think the *dépaysement épistémologique* (Barbin, 1997, 2006; Jahnke et al., 2000) that is associated with the encounter with the history of mathematics experienced by students during their initial pre-service teacher program? The theory of objectification probably sees this encounter as an eminently social phenomenon encouraging people to take a critical look at the social aspect of mathematics to better understand the historical and cultural mechanisms of their production, to understand that mathematics are not ideologically neutral knowledge, and that all knowledge is part of ethical issues for which we need to develop our sensibility.

And what about the sphere of being, ethics and otherness here? What can mean this meeting with the history of mathematics for future math teacher? How the study of the history of mathematics and the encounter that it raises can make sense for students that are becoming teachers? And most importantly, how to give the students a voice about this experience?

Armed with a conceptual framework from the theory of objectification, while being inhabited by these epistemological questions, my research objective is to describe the experience of *dépaysement épistémologique* lived by future high school mathematics teachers in the context of training activities that involved mathematics history.

APPROACH(ES)

The need to clarify the meaning of a particular experience for the learners and the focus on the lived experience of the individual in the description of the phenomenon led me to choose a phenomenological approach. The phenomenological approach in human sciences has been developed particularly in psychology (Giorgi, 1975, 1997) and education (van Manen, 1989, 1994). It obtains, from individual testimonies, specific descriptions of the participants' experience. Descriptive and comprehensive, the phenomenological approach focuses on "the experience of the individual and his subjective experience" (Anadón, 2006, p. 19). It highlights the significant elements of the internal living world. In addition, it brings the researcher into a welcoming attitude and openness towards participants' lived experience, searching to avoid a reified, reducing or sterile description of *dépaysement épistémologique*.

That being said, the socio-cultural perspective that carries the theory of objectification on the *dépaysement épistémologique* invites me to question the kind of description that I'm searching for and the way to construct such a description. Indeed, the sociocultural perspective implies a particular view of knowledge, learning and the self. For instance, through different authors such as Bakhtin, Levinas and Heidegger, the theory of objectification emphasizes on the possibility of a divided and multiplied self. Historical, we are thrown into a world that asks us answers. As it has been discussed, learning mathematics, as a process of objectification and subjectification, is inevitably learning-with-others and implies the development of an ethical subject.

To provide a description consistent with this view, I settled up various ways "to mesh" the participants' views and to recognize the common living world that emerged from their experiences. Without rejecting the phenomenological approach that seems, at first glance, focusing on the individual and his subjective experience, I was searching for ways to articulate it to the conceptual framework. This articulation appeared to be possible through the development of a particular form for the general description of the phenomenon proposed in this study.

In other words, I was searching for a description of *dépaysement épistémologique* that tries to maintain the plurality of discourses and emphasizes on their "permeability", how they respond to each other and let them being transformed by the others. The phenomenological approach leads to a general description of the phenomenon from specific descriptions obtained by the testimony of each participant. In fact, how can we get an overview from specific descriptions? How can we avoid the simple observation of redundancy, as if, by accumulation, a general and final description could appear? This could reduce participants to simple exemplarities, culminating in statements such as; "This one, he lived it like that", "that one otherwise", "that one stands out by this", etc. Inhabited by the epistemological assumptions underlying the theory of objectification, and the prospect of "being-in-mathematics" questioned earlier, I felt the need to look for the multiplicity of students' experiences. A multiplicity that does not seek to present side by side, in rows, the experiences of each participant, but to truly provide the "common world" that emerged during the trials that took place.

LOOKING FOR COMMUNE LIVED EXPERIENCES: HELP FROM LITERARY CRITICISM

Mikhaïl Bakhtin, one of the main references in the theory of objectivation, rightly said that any movement of consciousness is itself dialogical, that is to say, penetrated by those of others, and therefore, cannot be discussed without taking into account other movements of consciousness that respond to it, and make it respond. Discourses, intimately related to consciousness here, are then perceived as "dialogic". Indeed, this dialogism "goes far beyond the relationship between the built replicas of a formal dialogue [...] it is universal and goes across all human speech, in general everything that has meaning and value" (Bakhtin, 1929/1998, p. 77). Very broadly, we can speak of dialogues both in language and in terms of ideas and social horizons.

Going further on dialogism, Bakhtin developed the concept of the "polyphonic narrative" (*ibid*.). A scientific, literary or philosophical work can be called "polyphonic" if it offers a strong plurality of discourses and understandings of the world. Bakhtin profoundly highlights an example: the novel *The Brothers Karamazov* written by Fyodor Dostoyevsky. The novel is considered emblematic of the polyphonic work. Dostoyevsky portrayed here many characters inhabited by singular personalities that take finely established roles (the bourgeois, liberal atheist, scientist, etc.). They are characters acting as "spokespersons of world views" (Sabo & Nielsen, 1984, p. 80) that are constantly in dialogue. These strong individual speeches, which escaped the author "control" through the narration, highlight the existential, ideological and socio-historical thickness of the reality. For Bakhtin, it is this polyphonic aspect of the novel that allows the readers to account for the reality of Dostoyevsky, in this case Russia after the 1860 reforms.

In a polyphonic work, "the hero and the author jointly express [...] the speech works openly, despite having two faces, like Janus" (Bakhtin, 1977, p. 198). I am myself as a writer/researcher inevitably involved in this web of meaning that bind all the "actors" of the events of the *dépaysement épistémologique*. Therefore, it is important, as Bakhtin points out, to join "the accents of the heroes (participants) and those of the author (me as a researcher) within a single linguistic construction" (*id*., 214).

For my study, polyphonic narrative appeared as a way to stage this world in common that emerged with the participants. Then, it will be possible to bring the "knowing-with-others" that emerged, the collective experience, the fabric of shared meaning on the study of the history of mathematics.

Searching for ways-of-being-in-mathematics as it claimed by the theory of objectivation, my research inscribes itself profoundly in sociocultural approaches in mathematics education. With the constitution of a polyphonic narrative as a methodological strategy to grasp the world-in-common that has arisen during these experiments, it goes deeper in research itself and looks for consistency and coherence.

"CONTEXTS" AND "DATA"

The participants' selection was conducted among those registered in the *History of Mathematics* course offered in the secondary school mathematics teachers program at the University of Quebec in Montreal. During winter 2013, I stepped in the classroom activities by providing seven reading activities (90 minutes each) of historical texts. Those texts were constituted of the writings of mathematicians associated with different eras discussed in class:

1. A'hmosè: Rhind papyrus, problem 24.
2. Euclid: Elements, proposition 14, book 2.
3. Archimedes: The quadrature of the parabola.
4. Al-Khwarizmi: The Compendious Book on Calculation by Completion and Balancing, types 4 and 5.
5. Chuquet: *Tripartys en sciences des nombres*, problem 166.
6. Roberval: *Observations sur la composition des mouvements et sur le moyen de trouver les touchantes des lignes courbes*, problem 1.
7. Fermat: *Méthode pour la recherche du minimum et du maximum*, problems 1-5.

These classical texts were read in small groups (2 or 3 students). Both synchronic and diachronic lectures (Fried, 2008) were performed. Trying, first, to understand the mathematics involved and to bring it to a modern understanding, and, second, to read the text with the worry to keep the author in his historical, social and mathematical background.

For Fried, teachers and mathematicians too often reinforce the synchronic reading of mathematical objects. In this context, the role of the teacher should precisely be to constantly switch the learner between these two visions. It is this back-and-forth work that is continuously needed and that is creating the emergence of an awareness of its own conceptions of mathematics in the learner, its individuality toward the subject and the possibility for him to confront constructively with those of others. These considerations were taken into account during the reading activities implementation, continuously trying to, not only translate the texts in modern language, but also stay with the author in his historical and mathematical background.

Six students in the group were recruited to participate in individual in-depth interviews (approximately 90 minutes) and a group interview at the end of the study session. Video recordings of classroom activities and transcripts of interviews constituted the data of my study.

Individual interviews focused on three topics: their overall experience of the course, their experience of readings historical texts and, specifically, their experiences of cultural and epistemological *dépaysement épistémologique*.

The same set of themes was taken for discussion during the group interview. This time, the goal was to encourage participants to share their experiences. Therefore, the point was not necessarily to seek consensus, but rather to refine their description of their experiences through listening to those of others. Participants were asked to respond to the comments of their colleagues in order to possibly recognize themselves or to assert their differences.

ANALYSIS (S)

Phases of analysis given here are seen as steps in writing. These analyses have allowed the collect of notes for the construction of the polyphonic narrative.

Video recordings show how activities affect learners. Students in learning situations do not know in advance how to guide their quest for knowledge. In this sense, the reading of historical texts "affects" students, and can leave them with frustration and both positive and negative emotions, because students "suffer" the objects of knowledge (Roth, 2011). Video recordings yield descriptive elements of the encounter with the history of mathematics. It could be gestures, reactions or particular expressions that emerged during the reading of historical texts. In addition, having fully participated in the readings activities as an animator, I do not exclude myself from the descriptions.

Concerning the analysis of individual interviews, they explicitly give voice to the study's participants. The goal here is to get closer to the participants, to go meet "them". Analyses of written transcripts of individual interviews were done in two steps: the extraction of meaning units and the construction of the specific descriptions. Concerning the extraction of meaning units, most phenomenologist researchers generally include four phases (Deschamps, 1993). (1) Making a general sense of the entire description of the phenomenon. (2) Identify the meaning units that emerge from the description. (3) Exploring the meaning of these units by assigning a specific category. (4) Establish the phenomenological experiences associated with meaning units. Thereafter, a summary text will be produced for each participant. This summary is called the specific description.

These phenomenological analyses recognize more accurately the experience of each participant of the study. In this particular phase of analysis, I tried to trace the process of subjectification associated with the activities of reading historical texts. As noted above, the conscience is also changing during the learning process. Learning means to frequent knowledge, but also means "becoming". This is what phenomenological analysis is pointing on.

This phenomenological approach seems appropriate here, despite the distance between the perception of the subject (including consciousness) in the phenomenological perspective and in the theory of objectification. It is not a matter of establishing facts, but to investigate the participants' experiences. I borrow to phenomenology a method, an approach, a style of analysis, but I also borrow its openness, its a-theoretical mind, and the need to leave in indecision as long as possible the establishment of the significance of the participants' experience. It is this attitude that allows to perceive the participants, not as thought by science, but as subjects received throughout the concreteness of their experiences, with all the texture, nuance and density that is implied.

TOWARDS A POLYPHONIC NARRATIVE

The transcription of the group interview forms the basis for the final description that takes the form of a polyphonic narrative. This narrative will derive its density of two previous phases of analysis. The narrative/description allows me to bring out tensions between points of view on *dépaysement épistémologique*, which overlap and influence each other, creating a sort of siphonophore, both singular and plural. Unlike the positivist position that tries to eliminate alternative discourses on the phenomenon and the subjective position of the researcher, my study rather seeks to integrate them. This narrative will be the "results" of the study. It is a way to provide the community with a rich and open description of the *dépaysement épistémologique* that occurs during the study of the history of mathematics in the context of pre-service teacher training program, a description that is consistent with the underlying epistemological theory of objectification posture.

In this perspective, my research is asking theses questions: how to stay here "on the wire" and keep a form of dialogue between individual and community, between isolated subject and multiplied subject, between singular and shared learning of participants, between inner space and group activities...?

SOME "RESULTS" FROM THE PHENOMENOLOGICAL ANALYSIS

When adopting a phenomenological stance, major themes emerge from the analysis. Two of them are the experience of *otherness* and *empathy*.

Students are saying that they are trying very hard to understand the mathematic depicted in original texts. They show great difficulties concerning language, notation, implicit argument, style, definitions, interpretations, typography, etc. Literally, they "suffer the texts". The experience of *otherness* seems brutal, from a cognitive and affective point of view, it sometimes includes shocks and violence.

From Levinas, I learned that violence is a "thematization of the Other", a reification of the Other, a way to make the Other a Mine, and that to understand something is to control it, make violence at it. I saw a few acts of violence during my experimentation, for instance, someone said: "Fermat was doing this or that".

That's why *otherness* is linked with *empathy*. Again with Levinas, and also with Bakhtin, *empathy* could be heard as an effort of a non-violent relation with the Other, in this case, a way of keeping alive the subjectivity of the authors, keeping it fragile and mysterious. The question is how to accompany the students in this ordeal, in this hardship of the experience of *otherness*? How to maintain an *empathic* relation with the authors?

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Oral Presaentation
HISTORY IN MATHEMATICS ACCORDING TO ANDRÉ WEIL

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André Weil (1906-1999) was one of the most famous mathematicians of the XXth century, working mainly in arithmetic and algebraic geometry, and he worked also on the history of mathematics, especially history of number theory. He had a very specific way to give sense to mathematical activity, in narrow relation with its history, which is useful for mathematicians at first, and perhaps also for teachers and students. Our question here is to understand, what really are history and mathematics – history linked to mathematical activity – and how each one is useful for the other, in Weil's conception ; so in this perspective we want to clarify the interest of history in teaching of mathematics.

1. SUMMER 1914: FROM CARLO BOURLET ...

In his *Apprenticeship of a Mathematician* (Weil, 1992, 22), André Weil wrote:

> At that time, the textbooks used in secondary education in France were very good ones, products of the "new programs" of 1905. We tend to forget that the reforms of that period were not less profound, and far more fruitful, than the gospel (supposedly inspired by Bourbaki) preached by the reform of our day. It all began with Hadamard's Elementary geometry and J. Tannery's Arithmetic, but these remarkable works, theoretically intended for use in «elementary mathematics» (known as *math.elem.)* course during the final year of secondary school, were suitable only for the teachers and best students: this is especially true for Hadamard's. In contrast, Emile Borel's textbooks, and later those by Carlo Bourlet, comprised a complete course of mathematics for the secondary school level. I no longer recall which one of these fell into my hands in that summer of 1914, but I still have an algebra text by Bourlet for third, second, and first form instruction, which was given to me in Menton, in the spring of 1915. Leafing through it now, I see it is not without its defects; but it must be said that this is where I derived my taste for Mathematics.
>
> From 1914 to 1916 [...] As for mathematics I had for the time no need of anyone: I was passionately addicted to it (Weil, 1992, 23).
>
> In the taupe [this is the name commonly given to courses preparing students for entrance examinations of the Ecole Polytechnique and the scientific section of the Ecole Normale], of course, the student acquires or at any rates acquired at that time – a facility with algebraic manipulation, something a serious mathematician is hard put to do without, whatever some might say to the contrary (Weil, 1992, 31).

These facts are important for us, because we think that the perspective in which Weil linked history and mathematics later, is surely related to the first influences he was subjected to, those of its professor at the lycée Saint-Louis, Auguste-Clément Grévy, and then the mathematicians Jacques Hadamard and Elie Cartan, and the books of Emile Borel and Carlo Bourlet, as we saw above.

We claim that we cannot really understand what someone says about mathematics, and eventually about its history, if we don't know who, from the beginnings, teached him what are mathematics.

2. ... TO READ ORIGINAL TEXTS …

Weil was student at the École Normale Supérieure in Paris from 1922 to 1925, where his best friends were Henri Cartan, Jean Delsarte, Claude Chevalley. In the 1920's, the first determination of "history" for André Weil (Weil, 1992) was his willing to read the masters – Bernhard Riemann, read by Weil since 1923, and then Pierre de Fermat – as the best source of inspiration, before any reading of the "auteurs à la mode du jour". In 1926, he was one of the first French mathematicians to go in Germany (Audin, 2012). He sustained the beginning of his own activity by fruitful relations (conversations) with colleagues out of France: for example, with Carl Siegel at Göttingen and with Francesco Severi (Weil, 1979, I-524-525), with Mittag-Leffler (Weil, 1992, 54). But throughout his life he travelled and met many colleagues: to know the full network of these relations, the best is to look at the index of names in *The Apprenticeship of a Mathematician* (Weil,1992, 193-197).

3. … DOING MATHEMATICS AND HISTORY OF MATHEMATICS TODAYS

A second step, after 1926 [Weil, 1979, III-400] is related to his frequenting the Institute of Mathematics of Francfort (with Max Dehn, Ernst Hellinger, Carl Siegel, Otto Toeplitz, Paul Epstein, Otto Szàsz), and the *Dehn's seminar* on History of mathematics (started in 1922).

Weil wrote: "I have met two men in my life who make me think of Socrates: Max Dehn and Brice Parain" (Weil, 1992, 52). Max Dehn wrote: "Mathematics is the only instructional material that can be presented in an entirely undogmatic way". It would be interesting to know how in the mind of Dehn, and possibly of Weil, it was related to instruction with historical sources. Perhaps that, as a kind of paradox, undogmatic attitude is possible only if we respect our source text, because effective respect of source puts in perspective any dogma.

On the link between mathematics and history, a comparison between the Weil's approach and the genetic method of Toeplitz would be useful (Toeplitz, 1964) – written in the 1930's, and started in a paper in 1927, from a talk in 1926 (Toeplitz, 1927). As quoted by Gottfried Köthe in its foreword to the German edition of 1949:

> Toeplitz does not want to have his method labelled "historical": "The historian – the mathematical historian as well – must be record all that has been, whether good or bad. I, on the contrary, want to select and utilize from mathematical history only the origins of

> those ideas which came to prove their value. Nothing, indeed, is further from me than to give a course on the history of infinitesimal calculus. I, myself, as a student, made escape from a course of that kind. It is not history for its own sake in which I am interested, but the genesis, at its cardinal points, of problems, facts, proofs" (Toeplitz, 2007, xi).

Toeplitz was convinced that "historical considerations could be useful in bridging the gap between mathematics at the Gymnasium level and at the universities" (Folkerts, 2002, 136) ; or, as said by Köthe, he was "convinced that the genetic approach is best suited to build the bridge between the level of mathematics taught in secondary schools and that of colleges courses" (Toeplitz, 2007, xi).

In 1979, Weil explained that in the Dehn's seminar, they read one classical original paper, and, if necessary, they used contemporary authors and witness:

> On ne croyait pas devoir feindre d'ignorer ce que l'auteur n'avait pas su ; au contraire, on s'en servait pour mettre en lumière les intuitions qu'il n'avait pas été en mesure d'exprimer clairement.[…]. Je ne conçois pas de plus saine méthode historique que celle-ci (Weil, 1979, III, 460).

Nonetheless he precised that "Dehn showed how this text [Cavalieri, read in 1926] should be read from the viewpoint of the author [...]" (Weil, 1992, 52).

Toeplitz assumed to modify the "objective complete history", in order to construct a tool for teaching (his genetic method). Weil also assumed to disrupt the "true historical data" – with the introduction of nowadays knowledge – but he rather did that in order to get a better reading of the history itself. Furthermore, his target was not teaching, but the development of an historical foundation or motivation for new research, for him and for any mathematician at work.

History according to Weil will be a history of mathematics for mathematicians, almost a part of mathematics. Weil was scrupulously honest with ancient texts, but he worked with the help of modern knowledge and notations, because he wanted to use past to validate mathematics of today. So he allowed this goal to historical data. In our conclusive section we will give an example of an historical analysis by Weil, where past is taken as a mirror in which we admirated our own understanding and the way in which today the things seem to be clarified. We read the past at the light of what we know today (and the history is closed because of this goal).

It is interesting, but another different way to do history of mathematics for mathematicians at work today, would be in the first instance to try to understand old texts on their own, in their own time, and in the second place, with such an understanding in hands, to try to understand some mathematical piece of today. In this case our supposition is that we do not yet understand what we are doing, and consequently history can act on our free future works.

4. THIRD STEP: HISTORICAL NOTES IN BOURBAKI'S "ÉLEMENTS DE MATHEMATIQUE"

André Weil was one of the founders of the Bourbaki group, in fact its unquestionable leader. The startìng point of Bourbaki was a conversation between André Weil and Henri Cartan, about their teaching in Strasbourg. Weil said: "Why don't we get together and settle such matters once for all, and you won't plague me with your questions any more ?" (Weil, 1992, 100).

From the beginning of Bourbaki, officially founded in 1935 (the first Bourbaki congress was held in July), he suggested to include historical notes "in order to put in a right perspective some too much dogmatical expositions". For him, this suggestion was natural, as a consequence of his experience in the Dehn's seminar:

> Ayant eu le bénéfice d"une pareille expérience, je me trouvais naturellement porté, lorsque Bourbaki commença ses travaux, à proposer d"y faire figurer des commentaires historiques pour replacer dans une juste perspective des exposés qui risquaient de tomber dans un dogmatisme excessif (Weil, 1979, III, 460).

He did the job himself for General Topology and for Infinitesimal Calculus. Then others collaborators (as Jean Dieudonné, mainly) continued the job: "The point is, for each theory, to clarify completely the directive ideas, how these ideas are developed, and how they interact each one on the others" (Bourbaki, 1984, 5).

According to Weil, the history of mathematics is a kind of "natural history", in a world of living interactive entities named "mathematical ideas": original technical gestures, formulations, methods, "theories".

> The Bourbaki's collaborators do have a conception of their collective work as attached to a tradition, represented by Poincaré and [Elie] Cartan in France, by Dedekind and Hilbert in Germany. The "Eléments de mathématiques" had been written to provide a solid foundation and an easy access to this type of research [in this tradition], and in a sufficiently general form to be applicable in many possible contexts (Dieudonné, 1977, XI).

Thus, historical notes have to be understood in relation with this tradition, in order to confirm the formal foundation by an underlying historical background, and to expose in advance a large set of possible extensions.

> In history of mathematics we very often see that a theory does start by a very specific problem, and efforts to solve it. [..] There are theories which are fruitful and still very much alive (as for examples: theory of Lie groups, algebraic topology) […] almost each idea in such theories has repercussions on other theories (Dieudonné, 1977).

A problem is important if it generates a method or a fruitful and alive general theory (as analytic theory of numbers, theory of finite Groups; but these theories are not in the Bourbaki scope).

As observed by Halmos (Halmos, 1979), in the Dieudonné's Panorama, every chapter ends with a list of *initiators*: creators of principal ideas or substantial contributors.

The introduction of history in this way, via fruitful problems and initiators, is completely in the Weil's style, including of course the perturbation of history by our today's understandings, and a somewhat naive belief in progress.

In "Two lectures on number theory, past and present" (Weil, 1979, III, 280), we read that Weil knew, or seemed to know, what are "a perfectly good and valid subject", and "perfectly good mathematics". Weil and Bourbaki developed history under this assumption that there are good mathematics, good scientists and geniuses, and today we know what these good mathematics are, and who these geniuses are. This implies a specific choice, and according to that choice, the purpose of history is to put mathematical ideas in the "right perspective".

5. STRUCTURES AND HISTORY

Through the historical notes, Bourbaki constructed a tool for teaching mathematics, although the *Éléments de mathématiques* do not constitute a manual. But this work is also of an historical nature in another aspect, as a story of structures.

Bourbaki planed to reformulate the *Éléments de mathématique*, through an axiomatic development of mathematical structures. Bourbaki took ideas on *axiomatics* from Euclid or Hilbert (but their ideas of "axiomatics" are different), and he used the logico-set theoretical system – induced by the Cantor approach's within set theory, and the research in logic at the end of XIXth century –, which is an available tool at this time (the 1930's). The axiomatics in the algebraic context is implicitely supported by the works of Emmy Nœther and Emil Artin on the natural typical types of rings in geometry. And on the side of functional spaces and analysis, it is related to the works of Maurice Fréchet or Felix Hausdorff. In Geometry or Analysis again, it is related to Elie Cartan or Jacques Hadamard works, two mathematician that Weil met (and admired) at the beginning of the 1920's.

André Weil explained that, he probably imported the notion of *structure* from the field of linguistics (Russian linguists, Roman Jakobson), that he knew by his friends or relations Claude Levy-Strauss, Brice Parain, or Emile Benveniste:

> In establishing the tasks to be undertaken by Bourbaki, significant progress was made with the adoption of the notion of structure, and of the related notion of isomorphism. Retrospectively these two concepts seem ordinary and rather short on mathematical content, unless the notions of morphism and category are added. At the time of our early work these notions cast new light upon subject which were still shrouded in confusion: even the meaning of the term of "isomorphism" varied from one theory to another. That there were simple structures of group, of topological space, etc., and then more complex structures, from rings to fields, had not to my knowledge been said by aznyone before Bourbaki, an dit was something that needed to be said. As far for the choice of the word "structure", my memory fails me: but at that time, I believe, it had already enterd the working vocabulary of linguists, a milieu with which I had maintained ties (in particular with Emile Benveniste). Perhaps there was more here than a mere coincidence (Weil, 1992, 114).

For him, in fact, the notion of structure is really a mathematical notion, or more exactly has to be mathematically determined. In any event, the notion of structure has to be mathematically defined with respect to mathematical activity, and to be recognized through the history of mathematics. When he read Fermat, he observed the invention of gestures, as “the method of descent”, or as more elementary “tricks”, and these things are in some sense “structures”, or “mathematical ideas”.

In Bourbaki, a very special *grosso modo* determination of a structure is a set equiped with constants, functional symbols, relational symbols, axioms. As mathematical objects are specified in Model theory or in Universal Algebra. But more generally, structures are also any mathematical algorithm, mathematical process, mathematical idea. Furthermore, in the Bourbaki view, as specified by Claude Chevalley (Aczel, 2009, 124), each mathematical fact has to be explained, and the result of a computation is not enough for this purpose: the true explanation is the discovery of the natural structure under a given fact.

We can say that, for André Weil, the history of mathematics is the history of emergence of structures, of mathematical delimitations of mathematical ideas. And consequently, the *Éléments de mathématique* by Bourbaki are not a treatise of mathematics, but a fictional articulation, a kind of history, made of successive treatises on chosen mathematical subjects. The story started with an “hygienic” background on sets and logics; then, some “structures-mères” were exposed, in a rational way. But the mathematical necessity of their choice and the order of these “structures-mères” is not proved, it is based on a feeling of history, and on the initial formation of the concerned mathematicians in those days.

However, in two different periods, in the Bourbaki group, two mathematicians were very attracted – in two different ways – by the question of structures in mathematics, and the research of natural structures: Charles Ehresmann and Alexander Grothendieck did not share the Weil’s perspective. For Ehresmann, it was essential to incorporate the “local structures” to the “structures-mères”, and, for Grothendieck the question was to reach the level of “categories”.

But these ideas were new, and Weil seemed very careful with new things, because he believed in the historically known facts as being the deep roots of mathematical development. The difference was between the historically constraint of mathematical meaning and the Cantor’s proclamation of a basic pure freedom in mathematics.

A history of mathematics based on problems is possible, but a history based only on our today’s view of old problems (that is to say on what we consider as convenient solutions) is too restrictive. We have to understand how in their own time problems were posed and solved. This will be a good tool for the comprehension of what we do, and what we will do.

6. LETTER TO SIMONE WEIL: MATHEMATICS AS AN ART

In the difficult context of the beginning of the war in 1940, André Weil used history to explain the meaning of his mathematical work to his sister, the philosopher Simone Weil. He did that in two letters in 1940 (Weil, 1979, I), (Weil, S., 2012). There exists an English translation by Martin H. Krieger (Krieger, 2005). It is important to notice that, there is a kind of fusional relationship between Simone and André.

In the first letter he wrote that the mathematician is an artist, similar to a sculptor, working in a very hard matter, namely the strict mathematical culture, where constraints are previous theories and problems. He suggested to examine history of mathematics from this point of view (as the history of an art).

In the second letter he adds:

> When I invented (I say invented, and not discovered) uniform spaces, I did not have the impression of working with resistant material, but rather the impression that a professional sculptor must have when he plays by making a snowman.

For Weil, it is impossible to explain the mathematical research to the layman (that means a deaf person related to mathematics), and what about history of mathematical research? Probably it is impossible too, and this explains why he only developed a history for mathematicians. He wrote:

> Quant à parler à des non-spécialistes de mes recherches ou de toute autre recherche mathématique, autant vaudrait, il me semble, expliquer une symphonie à un sourd [A. Weil, CP, I, p.255. Lettre à Simone weil] La mathématique […] n'est pas autre chose qu'un art, une espèce de sculpture dans une matière extrêmement dure et résistante […] l'œuvre qui se fait est une œuvre d'art, et par là même inexplicable […] l'histoire de l'art est peut-être possible: et l'on n'a jamais, que je sache, examiné l'histoire des Mathématiques de ce point de vue (à l'exception de Dehn, autrefois à Francfort […]). Et il est tout à fait vain de se lancer là-dedans sans une étude approfondie des textes. […] J'ai dit une fois à Cavaillès qu'il y aurait lieu d'étudier les débuts des fonctions elliptiques […] (Weil, 1979, I, p.255).

Later Weil accomplished such a study on elliptical functions (Weil, 1976).

Weil considered mathematical research as an art, and therefore as an inexplicable activity. For him, it is as a kind of sculpture, in a very hard marble or porphyry. But he thought that the history of art is possible, and after Dehn, that the history of mathematics could be done in this way. He insisted on starting with a deep study of an original text. So this type of history is reserved to mathematician, and the questions are again: Who could read it? Who can do it?

In fact, André Weil could not really discuss his mathematical works with his sister Simone, but he discussed with Simone (who was not a mathematician) on the historical and philosophical subjects of antique mathematics, as the pythagorean ideas (Weil S., 1999, 2012). For Simone, this discussion was incorporated in her teaching of philosophy of sciences. She admired her brother unconditionally.

7. LETTER TO SIMONE WEIL: DEVELOPPING ANALOGIES

In the second letter to Simone, he explained the meaning of his own work, informally but with a lot of details. Mainly he claimed to develop and to construct a triple analogy, between three mathematical domains in progress:

- the theory of numbers and *fields of numbers*,
- the *Riemanian theory of algebraic functions* on complex numbers,
- the theory of (algebraic) *functions on finite fields* (Galois fields).

These theories are described from an historical point of view. He considered that he constructed a kind of trilingual dictionary, in order to decipher a trilingual text made of desultory fragments, trying to construct mathematical analogies – see also the Weil's paper of 1960 "De la métaphysique aux mathématiques" (Weil, 1979, II). It is related to the process of "changement de cadre" (Douady, 1984). It is a case of what we call a "mathematical pulsation" (Guitart, 1999, 2008). More recently, Gérard Laumon introduced this question of analogy in his "Allocution de Réception à l'Académie des sciences" (Laumon, 2005).

8. HISTORICAL REFERENCES IN HIS MATHEMATICAL WORK

In his mathematical works, André Weil used of "historical insights" to motivate and possibly to start his mathematical gestures, and also to increase the prestige of his results (beside the main point which is that these results do solve a problem). The legitimacy of this process is clear in the area of mathematics, in the "creative phase", when we do invent – or discover – our problematics. We have to know that such a "history" is only a tool for doing mathematics.

We give one example, expressed by two quotations, from two papers of Weil.

The first quotation comes from "Sur les fonctions algébriques à corps de constantes finis" written in 1940 (Weil, 1979, I, 257):

> Les travaux de Hasse et de ses élèves; comme ils l'ont entrevu, la théorie des correspondances donne la clef de ces problèmes ; mais la théorie algébrique des correspondances, qui est due à Severi, n'y suffit point, et il faut étendre à ces fonctions la théorie transcendantale de Hurwitz.

And the second quotation comes from "On the Riemann hypothesis in function-fields" written or edited in 1941 (Weil, 1979, I, 277):

> I have now found that my proof of the two last-mentioned results is independent of this transcendental theory, and depends only upon the algebraic theory of correspondences on algebraic curves, as due to Severi.

With such observations, we understand that Weil invented a genealogy of his work for future readers, which inserts it in the great flow of the history of mathematics; and, simultanuously, objectively he gave some interesting mathematical explanations.

9. THE THREE PRINCES AND THE QUEEN, THE PROBLEMS: AN ENCHANTING STORY FOR MATHEMATICIANS

André Weil wrote in "L'avenir des mathématiques" in 1947:

> Mais si la logique est l'hygiène du mathématicien, ce n'est pas elle qui lui fournit sa nourriture ; le pain quotidien dont il vit, ce sont les grands problèmes. "Une branche de la science est pleine de vie, disait Hilbert, tant qu'elle offre des problèmes en abondance" (Weil, 1979, I, 361).

Weil thought that "logic is only the hygiene of mathematics", but the real foods for mathematics are problems. As Hilbert said: "A science is alive as long as it as abundance of problems".

So the history of mathematics does start with problems, rather than with logical foundations. The question of logical foundation is nothing else than one special problem, for "hygiene".

Hence again, we have the question of initiators, of progression of good ideas, and good problems.

> In each new special theory there are initiators, and then good contributors (Dieudonné, 1977).

> No mathematician ever attained such a position of undisputed leadership […] as Euler did […]. In 1745 his old teacher Johann Bernouilli, not a modest man as a rule, addressed him as "mathematicorum princeps" (Weil, 1984, 169).

> In 1775 he [Euler] clearly felt ready to pass the title to Lagrange "the most outstanding geometer of this century" […] In the next century the title of "mathematicorum princeps" was bestowed upon Gauss by the unanimous consent of his countrymen. It has not been in use since (Weil, 1984, 309).

"The Arithmetics is the Queen of mathematics" (Gauss). In the Weil's style, the historian knows a priori who are the geniuses or inspired initiators (e.g. Riemann, Fermat, i.e. the Princes), those creating good new theories; and then, the historian writes an informal but mathematical explanation of links between main ideas in theories.

He does or writes a kind of mathematical story telling of the living world of mathematical problems, ideas and theories; it is also a travel story of a mathematician through mathematical ideas and analogies.

The good reader for that type of history has to be himself a mathematician; in this case, such a history is useful, the reader could find a true mathematical clarification of some notions. From our point of view, the decisive point is that it is not a fairy tale (arbitrary), but an enchanting story of the real (presupposed) growing of mathematical knowledge, a story of its own mathematical clarification. We can consider that such a story is still a part of mathematics.

For Weil, it is a construction of the meaning of mathematics, and because of that, it is enchanting for a mathematician, it is the reason for which he admires mathematics. A mathematician is such an admirer.

André Weil is a specialist of Number Theory, and, furthermore he wrote five books in order to teach Number Theory.

His very basic book, written with the collaboration of Maxwell Rosenlicht, *Number Theory for Beginners* is an elementary manual, and it introduces the very basic operations of arithmetic, without historical information, only by Definitions / Theorems / Proofs / Exercises (Weil and Rosenlicht, 1979).

In *Basic number theory* (Weil, 1967), a more advanced study, Weil exposed local field, adele, class-field theory, and there he added a chronological list of initiators:

> a chronological table […] as a partial substitute for an historical survey of a chronological list of the mathematicians who seem to have made the most significant contributions to the topics treated in this volume.

Fermat (1601-1665)	Riemann (1826-1866)
Euler (1707-1783)	Dedekind (1831-1916)
Lagrange (1736-1813)	H. Weber (1842-1913)
Legendre (1752-1833)	Hensel (1861-1941)
Gauss (1777-1855)	Hilbert (1862-1943)
Dirichlet (1805-1859)	Takagi (1875-1960)
Kummer (1810-1893)	Hecke (1887-1947)
Hermite (1822-1901)	Artin (1898-1962)
Eisenstein (1823-1891)	Hasse (1898-) [1979]
Kronecker (1823-1891)	Chevalley (1909-) [1984]

We notice the name of Emile Artin, which is essential for the history of the quadratic reciprocity law in the 1920's ; this is important, because, for André Weil, the modern history of Number theory turns around this law.

In fact the subject of Adeles is also introduced by Weil in another book, *Adeles and Algebraic Groups* (Weil, 1982). In the foreword, Weil said that "it is based on lectures, which were nothing but a commentary on various aspects of Siegel's work". It is as data recorded for future historians, but also in itself it is a piece of new mathematics. We can say the same thing of the encyclopedic collection of the *Séminaires Bourbaki*. This shows us how much the different aspects of his work (teacher, researcher, author, historian) are very closely related.

The two other books are completely different, and they mix deep mathematics and deep history, they are explicitly exercises of historical reading for mathematicians.

The book on elliptic functions, *Elliptic functions according to Eisenstein and Kronecker* (Weil, 1976) is appreciated by the mathematical community: "this text undoubtedly contributes notably to the history of our science, it is also of great value to contemporary mathematical research" (P. Hilton, Chairman, Editorial board, Ergebnisse der mathematik)

The last book *Number Theory An approach through history from Hammurapi to Legendre* (Weil, 1984) leads to the reciprocity law, through a history account of technical gestures in number theory. Let us give two short explanations on this very original last book.

Starting from the works of the three princes (Euler, Lagrange, Gauss) and other initiators, the history of mathematics is at first the story of the life of the Queen (arithmetics), the mathematical comprehension of the life of its problems.

> Our main task will be to take the reader, so far as practicable, into the workshop of our authors, watch them at work, share their successes and perceive their failures (Weil, 1984. IX).
>
> In a Seminar at the Institute for Advanced Study in Princeton, Weil said that "he knew 50 proofs of the law of quadratic reciprocity, and that for each he had seen there were two others he had not » (Gerdstenhaber) "It can be said that *everything* which has been done in arithmetic from Gauss to these last years consists of variations on the law of reciprocity: one started with Gauss's law and arrived, thereby crowning all the works of Kummer, Dedekind ansd Hilbert, at Artin's reciprocity law, and *it is the same"* (Weil, "Une lettre et un extrait de lettre à Simone Weil" (Weil, 1979, I), (Lemmermeyer, 2000, v-vi, xi.).

The list of the proofs (246 proofs) and the bibliography on the quadratic reciprocity law, as given by Lemmermeyer, can be considered as an effect of the practice of Weil with history of mathematics for mathematicians.

10. HISTORY OF MATHEMATICS: WHY AND HOW (1978)

In 1978, André Weil wrote a paper "History of mathematics: why and how?" (Weil, 1979, III), explaining his conception of the practise of history of mathematics in a remarkable concise and clear way. At first he considered that we have good historians as Moritz Cantor, Gustav Eneström, Paul Tannery, and that we can discuss of their methods, with respect to Leibniz's conception:

> Leibniz wanted the historian of science to write in the first place for creative or would-be creative scientists. Its use is not just that History may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its method known through illustrious examples (Leibniz, Math. Schr., ed. C.1. Gerhardt, t.V, p. 392.).

At one moment he wrote: "A mathematician will find it appropriate to …". This sentence shows that he is doing a history *as* a mathematician, and *for* mathematicians. Also he observed that "Eisenstein fell in love with maths at only an early age by reading Euler and Lagrange": so let us read the masters (as also Weil did); so history is "some guidance to go back in mathematical readings".

For mathematics, as well as for history of mathematics, it is useful to distinguish between tactic and strategy. Tactic is the day-to-day handling of the tools of the period (with competent teachers and contemporary works). Strategy is the art to recognize the main problems, the pertinent structures, etc.

From Eneström and Tannery, history consists in following the evolution of ideas over long periods, to follow

> the filiation of idea, and the concatenation of discovery (Tannery)

to be able

> to look beyond the everyday practice of his craft (Weil).

So we get the question: "What is and what is not a mathematical idea?".

For the determination of history, according to Weil we quote the three following sentences by Weil in this paper:

> History and philosophy of maths: it is hard to me to imagine what these two subjects can have in common

> Mathematical ideas are the true objects of history of mathematics.

> Large part of the art of discovery consists in getting a firm grasp on the vague ideas which are "in the air".

Coming back to the question of anachronism or attribution of our conception to an ancient author, Weil underscored that this default is different from the use by the historian of our modern knowledge. As a comment on this opinion, we have, by Tannery:

> The greater his talent as a scientist, the better his historical work is likely to be.

11. AN EXAMPLE OF HISTORY "A LA WEIL": THE DEBATE ABOUT TANGENTS BETWEEN DESCARTES AND FERMAT

Weil concluded his paper "History of mathematics: why and how ?" (Weil, 1979, III) with a discussion on the debate about tangents between Descartes and Fermat.

The discussion lays on two ideas of Descartes and Fermat, isolated and formulated by Weil, and three observations of Weil, as below:

Idea 1 (Descartes): a variable curve [for example a circle] intersecting a given one C at a point P, becomes tangent to C at P when the equation for their intersection acquires a double root corresponding to P.

Idea 2 (Fermat): infinitesimal method, depending on adequation, or the first term of a local power-series expansion.

Weil did the following "historical observations":

Observation 1: the debate is between algebraic and mechanical curves.

Observation 2: to the "défi" of Fermat about the cycloid, Descartes do replies by the invention of the instantaneous centre of rotation.

Observation 3: At this period (XVIIth century), the distinction between differential and algebraic geometry has not been clarified. But now we can understand that Descartes'method belongs to algebraic geometry, and Fermat's method belongs to differential geometry. The first one is available with some general ground field, the second one works for more general (non-algebraic) curves.

Today, on this subject of tangents, the reader is invited to examine several analysis by Evelyne Barbin, from an historical point of view (Barbin, 2006, 2015b) and from a didactical point of view (Barbin, these Proceedings).

12. CONCLUSION: ON THE USE OF HISTORY IN THE TEACHING OF MATHEMATICS

From Weil's works and positions on history or with history, we can isolate some observations about the link between mathematics and history of mathematics, in the perspective of pursuing, transmitting and teaching of mathematics.

For various aspects of the use of a historical perspective into mathematics teaching and learning, we refer the reader to some recent published papers (Barbin, 1997, 2012, 2015c). One main idea of Evelyne Barbin (Barbin, 1997), studied again in (Guillemette, 2014), is the notion of « dépaysement épistémologique ». A decisive point is that « dépaysement » arrives if we read a mathematical text in the same way as a contemporary of the author was able to read it. Clearly André Weil contravene this attitude of mind with history, when he reads masters in order to clarify the future of theories, in order to justify the Bourbaki venture, or its own line of development in mathematics. The same observation works for the genetic approach of Toeplitz, and both, Toeplitz and Weil, admits a fictional history, eventually far from the real history, as a tool for mathematical motivation and formation. Of course any historical reconstruction is helpful for mathematical teaching, because this provides an exciting imaginary world of thinking, and in this world a personal motivation, a possibility of identification with some heroes. But a deeper insight is obtained if we work with interpretations respectful of contemporary understanding of a text; in this case we could observe the finest gestures and interpretative pulsations, and try to reconstruct the very moment of invention.

In the primary formation of André Weil, we noticed, on the one hand the influence of two great creative professionals, Hadamard and Cartan, both heirs of practize of natural care of history in science, and on the other hand the passage through preparatory classes for great schools, with Grévy. The preparatory classes (*taupe*) is an heritage from the beginnings of the École polytechnique (Barbin, 2015a), and certainly they instil the habit of taking good care of the link between mathematics and its history.

Of course for Weil, as for any mathematician, the elementary technical training inside any given closed system of calculus have to be executed as grammatical exercises, musical games, remedial gymnastics: quickly, unscrupulously and without qualms. But also the signification of such a training has to be seek out, and the natural way for that is through the reading of original mathematical texts, in the stream of an history of mathematics. And the question is the underlying conception of this history.

A point is that the core of mathematics is just its own history, and only after that point comes the questions of matters, subjects, methods which are to be considered, according to our feeling of the history. Mathematics is a culture, the history of elucidation of the necessity of mathematical ideas, rather than the history of contingent mathematical themas in which these ideas are implicated or even incarnated, or a fortiori rather than the history of its philosophical, epistemological or technical motivations. Whatever we choose to be taught from historical situations, in teaching mathematics we have to be very careful with this distinction between ideas and themes, and this is feasable especially through the reading of masters.

Another point is that mathematical works and teachings are elaborations in two directions: from problems to solutions, which are structural explanations, and conversely from structures to new problematics. So in the history appear problems, and structures, in a kind of dialectic; this dialectic is the motor of rational thinking.

From the history of mathematics we learn that the mathematician is the one which find problems where nobody could see difficulties ; and from this point derives its ability to solve problems on which everybody stumbles against. The true rigor relative to signification is there, in discovering problems; not in the process of solving, in which logical rigor is only a necessary hygiene. The point is to discover how to become subtle as far as to discover new problems. And certainly the history is the best school for that.

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Oral Presentation

TRAINING TEACHER STUDENTS TO USE HISTORY AND EPISTEMOLOGY TOOLS: THEORY AND PRACTICE ON THE BASIS OF EXPERIMENTS CONDUCTED AT MONTPELLIER UNIVERSITY

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The new context of Master's level courses for initial teacher education led in France to the development of history and epistemology courses on a larger scale, in interrelation with didactics of mathematics. We report on this phenomenon and especially on an experiment conducted at Montpellier University that aimed at training teacher students how to use history and epistemology tools. This is overall an opportunity to discuss the interrelations of epistemology and didactics, in the French culture.

I. INTRODUCTION

Due to several recent reforms of the initial teacher education system, the situation with regard to history and epistemology of mathematics for trainee teachers in France changed since the publication of the ICMI study (Fauvel & van Maanen, 2000). Indeed, the reform of 2010 offered a unique opportunity to introduce history and epistemology courses on a larger scale as Master's degrees in teacher education were created in order to raise the academic level of teachers. The national competitive examination (CAPES [1]) that regulates positions in secondary education was postponed to the second year of the Master's degree. As a consequence, a full year was gained in the training of teacher students, which gave time to teach fundamental concepts of didactics, but also history and epistemology of mathematics in more depth, in response to an increased awareness of its importance by the institution. For instance, official guidelines for Grade 10 (Terminale in France), published in 2011, make it clear that:

> elements of epistemology and history of mathematics fit in naturally in the implementation of the curriculum. To know the names of a couple of famous mathematicians, the period in which they lived and their contribution to mathematics are an integral part of the cultural baggage of all students taking scientific education. Presentation of historical documents is an aid to the understanding of the genesis and evolution of certain mathematical concepts ([2]).

The purpose of this article is to give an overview of the situation by taking Montpellier University as an example. The author will elaborate on the basis of his experience as a University lecturer in epistemology and didactics of mathematics as well as researcher in mathematics education. In particular, we will analyze teacher training in history and epistemology at Montpellier University using the whys and hows classifications available in the literature (the ICMI Study, Fauvel & van Maanen 2000; Jankvist 2009).

The core of this paper is an experiment that the author conducted with second year Master students that aimed at teaching teacher students by the practice how to use history and epistemology as a tool in the classroom. The students' project work led to the elaboration of “didactic source material” that, “compared to primary and secondary source material, seem to be the most lacking in the public domain” (ICMI Study p. 212). It is therefore also an interesting question to discuss the methodology to produce this type of resource. What kind of interactions between history, epistemology and didactics of mathematics can we expect and aim for? This is a deep question that is often debated inside the HPM community (Barbin 1997) and certainly cannot lead to a unique and definite answer. We will contribute to the debate by giving a detailed account of the epistemological and methodological domains used for the project work as well as the successes and drawbacks that we met using this approach.

II. HISTORY AND ESPISTEMOLOGY AT MONTPELLIER UNIVERSITY: FROM 2010 TO 2014

About the goals and the context of initial teacher training

The many reasons for integrating history of mathematics in mathematics education have been carefully reviewed in the ICMI Study (Fauvel & van Maanen 2000). The reader will find in Annex 1 the result of our attempt to synthesize and arrange the why arguments in a table. Considering that the education community is composed of students, teachers and didacticians (in a simplified model), we also indicated the protagonist who was mainly concerned for each of the argument. Our teacher students are both students (at University) and teachers (in the classroom for their practice work or apprenticeship). They may also adopt the posture of the didactician when they are given the task of elaborating some (simple) didactical engineering. Therefore, most of the arguments presented in Annex 1 may apply to them.

On the occasion of the 2010 reform of the teacher education system in France, local and national committees debated on the goals of initial teacher training in history and epistemology of mathematics. Arguments needed to be formalized in order to obtain from the French Ministry of Education the accreditation of the new Master's track. Several dimensions were thus combined and put forward: an epistemological, historical and cultural approach of scientific knowledge; a didactical approach of the construction of concepts in a teaching and learning environment; minimal knowledge on history of education; a practical and reflexive approach on the way a teacher may introduce a historical perspective in his lessons.

Specification and clarification of competencies with regard to history and epistemology in initial teacher education has been carried out in great detail by a local group of lecturers and researchers in history of science that piloted the new Master's track at University Paris 12 ([3]). We will list these competencies below and relate them to our typology of whys (between brackets, see also Annex 1; the sign ~ indicates an approximate matching).

- To develop one's scientific culture (~11)
- To understand how scientific knowledge is developed (6)
- To situate one's discipline in a larger context (~4)
- To acquire proficiency in the written language through the reading and studying of ancient scientific texts (5)
- To identify, in their epistemological and historical context, concepts, notions and methods met in the teaching of science at a given level (1)
- To appropriate different didactical options in the integration of a historical perspective in scientific teaching (2)
- To apprehend transverse competencies (reading, argumentation, writing, etc.) that history may work with in scientific education (5)
- To have knowledge on the history of education and the place of scientific education
- To get initiated to the questions and methods of epistemology and history of science as research fields

Initial teacher training at Montpellier University

At Montpellier University, two 50 hours-courses were devoted to history and epistemology of mathematics during the period 2010-13. The first course, a first-year Master's course, focused on tertiary level mathematics (abstract algebra, topology, probability theory, etc.) and aimed at promoting reflexive thinking on mathematical objects and methods (what is a mathematical proof, the problem of definitions and axioms, the axiomatic method, mathematical structuralism, the meta-notions of rigor, evidence, error, etc.). Students were trained to analyze and comment on a corpus of documents including primary sources or essays written by historians or philosophers. The second course was a second-year Master's course directed towards secondary level mathematics with a view to articulating history and epistemology of mathematics with didactics of mathematics. Conceptions that appeared in history were therefore connected to conceptions identified by didacticians in a learning context.

Unfortunately (or not), a second reform that took place in 2013 (after the French Presidential elections) affected the Master's degree. In an attempt to make the teaching career more attractive, the competitive examination (CAPES) was taken back to the first year of the Master, in order to facilitate a progressive entry in the career, no later than 4 years after the French Baccalauréat, with a 9 hours a week of practical training during the second year of the Master's course. As a consequence, teaching time dedicated to history and epistemology as well as fundamentals of didactics had to be diminished, which also resulted in a stronger articulation between these fields.

Nowadays in Montpellier are offered two 50 hours first-year master courses in history-epistemology and didactics of mathematics as combined subjects: a first course shares time equally between the two and corresponds more or less, as far as history and epistemology is concerned, to the previous 2010-13 course deprived of the discussion of advanced mathematics. A second course reviews the didactics of the main mathematical domains in secondary education (geometry, algebra, analysis, probability and statistics, etc.) with integrated elements of history and epistemology. Finally, a third course (24 hours) during Master 2 is focused on practical issues: learning to use history and epistemology tools in the classroom, elaboration of pedagogical scenarios and their implementation. Although 2013-14 was more of a transition year, it offered an opportunity to experiment such a course. We will report on this experimentation in the sequel.

III. THE STUDENTS' PROJECT WORK: A CASE STUDY

We will now describe in detail the tasks that were assigned to teacher trainees so that they may learn how to use history and epistemology as tools in the classroom. We will analyze a few students' productions and comment on the difficulties that they encountered in the completion of such tasks.

Description of the project work

Teaching design has been chosen as an activity for students to learn by doing how to use history and epistemology tools. This makes sense in our context since second year Master students in 2013-14 had a teaching duty of 6 hours a week, which motivated an emphasis on classroom practices. This work, carried out in groups of 3-4 teacher students, also served as a project-based assessment for the course. There are of course several possible and different ways of integrating history of mathematics in mathematics education. What are the choices made by educators at Montpellier University?

Referring to the classification of ‘hows’ presented in the ICMI study (Tzanakis & Arcavi pp. 208-213), students were asked to follow a “teaching approach inspired by history” (loc. cit., p. 209). Nevertheless, in comparison to such an approach, less emphasis was made on a “genetic approach to teaching”. To rephrase it more properly, our approach may be characterized as:

- a teaching approach supported by an epistemological analysis
- articulating history and epistemology with didactics of mathematics
- using history and epistemology as tools in the classroom
- distilling elements about the nature of mathematics or mathematics as a cultural endeavor
- producing some “didactic source material”, that is a “body of literature which is distilled from primary or secondary writings with the eye to an approach (including exposition, tutorial, exercise, etc.) inspired by history” (loc. cit. p. 212).

This approach certainly fits in the French didactical culture, in which design has always been given a central role, through the notion of “didactical engineering”

(Artigue 1992). Epistemology plays an important role, which was again acknowledged by the French school of didactics of mathematics since its foundation (Artigue 1991).

To be precise, the project work was presented to the students in the following programmatic terms:

1. Choose a theme and set up mathematical and didactical goals (parameters: the curriculum, known didactical phenomena, keeping in view the use of history and epistemology tools)
2. Epistemological analysis of mathematical notions
    - 1st goal: identify cognitive roots in history and reflect on it
    - 2nd goal: identify primary or secondary historical sources that may be used in the classroom or identify a crucial epistemological anchor point to work as a lever in the classroom

   Method: reviewing the literature in history and epistemology and also in didactics of mathematics

1. Set up epistemological goals (useful to meet didactical goals or as a combined goal: aspects related to the intrinsic/extrinsic nature of mathematical activity, the nature of mathematics/mathematics as a cultural endeavor)
2. Didactical engineering
    - History as a tool: motivation by historical questions and problems, contextualizing as a meaning-providing activity, etc.
    - Or Epistemology as a tool: through the meta lever (see epistemological domain below)

Produce a worksheet + detailed scenario + comments on didactical choices

1. Write down the *a priori* analysis
2. Classroom testing and *a posteriori* analysis (if possible)

In the context of this project work, history and epistemology are used as tools since they contribute to meet mathematical and didactical goals. The reader may also note that both tools are carefully distinguished. To our knowledge, epistemology as a tool in the classroom is given little attention in the literature. This may be due to the fact that elements related to the nature of mathematics are usually seen as history and epistemology as a goal. Moreover, epistemology may be more demanding than history to work out as a tool since it can seldom rely directly on a source document but always involves reflexive and critical thinking.

Epistemological and methodological domains

According to Radford, "the linking of psychological and historico-epistemological phenomenon requires a clear epistemological approach" (ICMI Study p. 162) as well as an adequate methodology for the design of historically or epistemologically based classroom activities. This statement is illustrated by figure 5.1 (loc. cit. p. 144) which certainly deserves to be reproduced here:

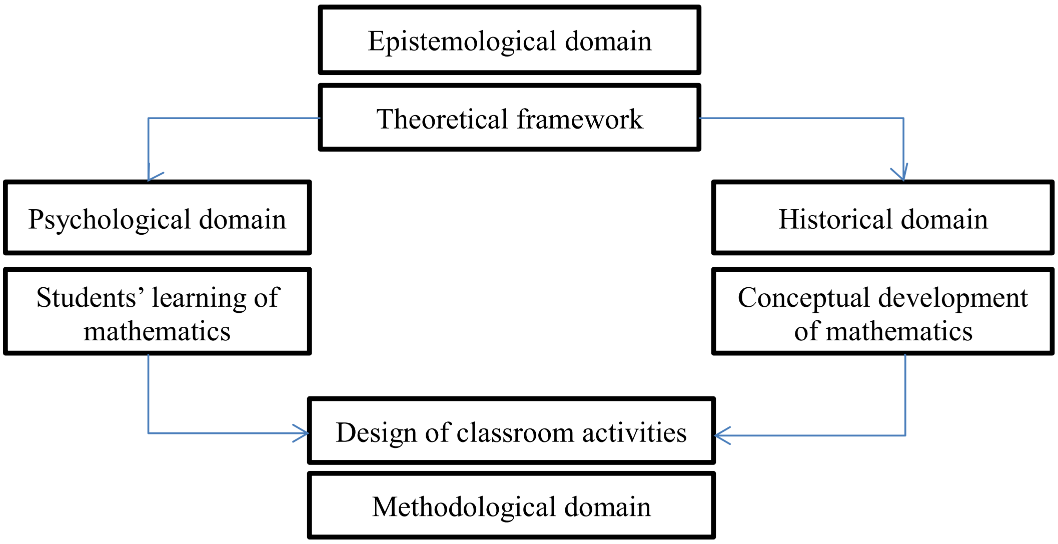

Figure 1: linking of psychological and historico-epistemological domains in the design of classroom activities, according to Radford.

Our epistemological domain may be described by the following formula:
Brousseau TDS + socio-cultural inputs + the meta lever

Brousseau's theory of didactical situations in mathematics (Brousseau 1997) is based on the idea that mathematical knowledge makes sense to the learner whenever it may be perceived as an optimal solution to a given system of constraints, in a problem-solving activity. Brousseau also incorporated in the field of didactics Bachelard's idea of epistemological obstacle. The didactical action is therefore centered on the organization of an adequate student/milieu relationship and the elaboration of "teaching situations built on carefully chosen problems that will challenge the student's previous conceptions and make it possible to overcome the epistemological obstacles, opening new avenues for richer conceptualizations" (ICMI Study, p. 163). It should be pointed out that, in this perspective, the articulation between students' learning and conceptual development of mathematics in history is not recapitulation or parallelism. History plays a role in so far as it may suggest or inspire fundamental situations as well as inform on possible misconceptions and potential steps in the conceptualization process. But any assumption should be confronted with the reality of classroom experimentation, in other words with the didactical phenomena.

In order to ponder Brousseau's paradigm, we also follow Radford and acknowledge the input of a socio-cultural perspective: mathematical knowledge is better understood "in reference to the rationality from which it arises and the way the activities of the individuals are imbricated in their social, historical, material and symbolic dimensions" (loc. cit. p. 164).

In order to make this rationality explicit, teacher students are asked to employ the meta lever (Dorier & al. 2000), that is "the use, in teaching, of information or knowledge *about* mathematics. […] This information can lead students to reflect, consciously or otherwise, both on their own learning activity in mathematics and the very nature of mathematics”. Although a large part of the meta is usually taken in charge of by the teacher, we encourage the devolution to the student of some part of this reflection, which may require a piece of didactical engineering (see examples below).

As far as the methodological domain is concerned, the project work, which leads to the production of didactical source material, uses a dedicated resource format, which may be seen as a didactical tool, since it helps students to clarify their thoughts and organize their work, to make explicit the choices that they make so that didactical action may be discussed, in particular the impact of history and epistemology and its functioning as tools. Mathematical, didactical and epistemological goals need to be carefully declared upfront and related to the curriculum, history and epistemology tools have to be described and commented: to which extent does it function as a lever? The text of the activity is complemented by a detailed scenario, and epistemological and didactical analysis are provided as annexes. Our resource format is in fact an adaptation of the SfoDEM resource documentation format (Guin & Trouche 2005), which has been designed for the purpose of collaborative elaboration, pooling and sharing of didactical material within a community of practice consisting of about 300 mathematics teachers in secondary education.

In other words, referring again to the typology of hows given in the ICMI Study and precisely the typology of examples of classroom implementation, our teacher students are building a “historical package” (Fauvel & van Maanen pp. 217-218): “focused on a small topic, with strong ties to the curriculum, suitable for two or three class periods, ready for use in the classroom”; a self-contained package “providing detailed text of activity, historical and didactical background, guidelines for classroom implementation, expected student reactions (based on previous classroom trials)”. In our case, such an extensive documentation is motivated by the development of professional skills but also by pooling and sharing since teacher students will communicate about their group-work during oral presentations in front of the assessors and their peers.

Examples

Several examples have been given to students in order to illustrate a functioning implementation of the historical or epistemological lever. We will present below a teaching sequence engineered by a team of teachers and educators at the IREM of Montpellier (Hausberger 2013, annex 3, pp. 120-158).

This sequence is devoted to a further discussion of the notion of mathematical demonstration, at the entrance of the Lycée (age 15-16). Students have already made acquaintance with standard Euclidean demonstrations during the last two years of Collège. In the sequel, history and epistemology will be called for meaning-producing activities as we tackle the following questions: what constitutes a mathematical demonstration compared to other types of argumentation? Why did mathematicians choose to set up these rules?

During a first activity, students are assigned the following tasks:

- Look up in the dictionary for definitions of the verbs “to show” ([4]) and “to demonstrate” ([5]). Give synonyms for each word. Bring into light the differences to be made between the two.
- For each of the following documents, identify the statement which is asserted and rewrite it if necessary. Is the argumentation of the statement a mathematical demonstration? If so, can you explain the different steps of the reasoning? If it isn't, can you write down a demonstration or demonstrate that the statement is false?

The documents submitted to the students include Lafontaine's poem “the wolf and the lamb” (doc. 1), the values of $n^2 - \mathrm{n}+17$ for $n \in \{0,1,2,3,4,11\}$ and the statement “$n^2 - \mathrm{n}+17$ is prime for any natural number$n$” (doc. 2), “let us show that the square of an odd number is also odd” and a proof based on the algebraic development of $(2\mathrm{n}+1)^2$(doc. 3), the graphical representation of the function $f(x) = 10\mathrm{x}^3 + 29\mathrm{x}^2 - 41\mathrm{x} + 12$ and the statement that “the equation $f(x) = 0$ has two solutions since the curve intersects the x axis in two points” (doc. 4), a puzzle inspired from Chinese mathematics that establishes that a square inscribed in a right triangle of sides$a$and$b$(apart from the hypotenuse) has side $\mathrm{c}=\mathrm{ab}/(\mathrm{a}+\mathrm{b})$ (doc. 5), and finally another puzzle by Lewis Carroll that leads to the erroneous conclusion that a square of side 8 and a rectangle of width 5 and length 13 have equal area (doc. 6).

This is an example of epistemology as a tool in the classroom: an implementation of the meta lever, involving reflexive thinking and devolution of meta-discourse to the students. No historical contextualization is given at this stage. Although a few mathematical competencies may be developed through this activity (for instance, refuting a universal statement by providing a counter-example or working out the factorization of a function), the goal is the development of competencies that our IREM team decided to set apart and explicitly describe as epistemological: being able to identify and characterize a mathematical demonstration, to distinguish induction and deduction, to distinguish the truth of a statement and the validity of an argumentation, etc. The role and status of the figure or representation is discussed with the students in the situations of “visual doubt” (doc. 4 to 6: these situations call into question the limits of our perception with the senses) as well as the validity of “cutting and pasting” procedures (doc. 5 and 6: the treatment of mathematical objects as material puzzles leads to an accurate answer in the first case but an erroneous result in the latter).

During a second activity, three historical primary sources are presented to the students: the Problem 41 of the Rhind mathematical papyrus, in which the scribe indicates how to compute the volume of a cylinder-shaped grain silo, the Yale Babylonian tablet 7289, which presents a very interesting approximation of$\sqrt{2}$ in sexagesimal numbers, and finally Euclid's demonstration of the irrationality of$\sqrt{2}$ (which requires quite a sophisticated pedagogical script in order to facilitate the reading and to make the devolution of the Greek context of magnitudes possible). Again, mathematical goals may be pursued, for instance on approximations and algorithms by asking about the obtainment of the Babylonian value by empirical measurement and the introduction of Heron's (of Alexandria) method, or on logic and mathematical reasoning (implication, contrapositive, reduction to the absurd). Once such mathematical aspects have been worked out, the pedagogical scenario puts forward the following questions for investigation and discussion in the classroom:

- On the basis of these historical documents, what distinguishes Greek mathematics from Egyptian and Babylonian mathematics?
- What might have been, according to you, the reasons that led to the development of such Greek mathematics?

This is an illustration of both history (historical contextualization) and epistemology (reflexive thinking) as a tool, again through the meta lever. Elements about the nature of mathematical objects and mathematical activity as well as mathematics as a cultural endeavor may be addressed, since historians and epistemologists identified both internal and external reasons for the appearance of the mathematical demonstration in the Ancient Greece. This socio-cultural approach makes sense both epistemologically and didactically: indeed, as stated by Balacheff, "knowledge needs to be constituted in veritable theories and be recognized as such, which means accepted by a community that renounces to take anywhere the arguments that it may use. The mathematical demonstration relies on a body of knowledge highly institutionalized, whose validity is socially shared." (Balacheff 1987, p. 160, our translation; [6]). Accordingly, the criteria for a valid argumentation in mathematics should be submitted to classroom discussion and connected to the practices of mathematicians, which is the very purpose of our teaching sequence.

Students' productions

We will now present and analyze the work of 3 groups of 4 teacher students who were involved in the project, out of a total number of 5 groups. Our main questions are the following: did they manage to implement a functioning lever? What kind of difficulties or pitfalls did they encounter? How does history-epistemology and didactics of mathematics interact in practice in the students' project works?

a) Group 1 decided to elaborate an activity dedicated to the introduction of Pythagoras's theorem at Grade 8 (4ème in France). They used history as a tool and chose to put forward the following historical problem as a motivation: how have Karnak and Luxor temples been constructed, knowing that historians consider that the masonry set square only appeared in the 15th century? The Egyptian 13-knots rope was soon introduced as an historical object, which led the classroom into an experiential mathematical activity dedicated to the construction of right-angled triangles. The pythagorean triple (3,4,5) finally emerged together with a new problem: how to characterize pythagorean triples? In order to introduce the pythagorean relation $3^2 + 4^2 = 5^2$ and interpret it in terms of square areas, the scenario used a mechanical device in plexiglass (which had been manufactured by one of the students!):

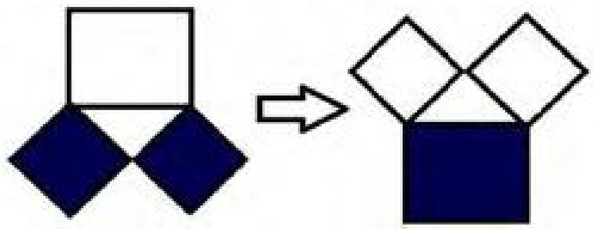

Figure 2: Students' mechanical device for the interpretation of the pythagorean relation

The blue liquid contained in the small squares flows to fill the bigger one when the device is turned upside down. Further investigations of the pythagorean relation were conducted afterward with the help of an interactive geometry software: “does other triangles fulfilling a similar relation seem to be right-angled triangles? What equality seems to exist between the sides of a triangle for it to be right-angled?”. The rest of the activity was devoted to working out a proof of Pythagoras's theorem by means of a contemporary version of the Chinese puzzle. Although it was the key to the justification of the Egyptian procedure, the reciprocal was admitted without proof.

As a conclusion, teacher students did a good job in the implementation of the history lever. Several cognitive representations or procedures that appeal to our senses and participate in the conceptualization of Pythagoras's theorem were introduced either using historical contextualization or an approach inspired by history. A weak point of the activity would be an insufficient epistemological clarification of the idea that the pythagorean relation *characterizes* right-angled triangles, which is visible in the two questions above. The current official guidelines instruct not to distinguish the theorem and its reciprocal, which troubled our teacher students. This is certainly an opportunity to work out an epistemology lever.

b) Group 2 named its project work “trigonometry and triangulation”. The genesis is the result of one student's personal encounter with the method of triangulation and the necessity to relate to the curriculum. They proposed the following situation:

About 600 years BC., Thales finds himself on a boat (point A) and wishes to know the distance to the coast. For that purpose, he sends two observers (B and C) on the (straight) coastline, separated by a known distance BC=700m, and gives them instructions to measure the angles from the coastline to the boat. The purpose is to help Thales compute the distance, the measured angles being 83.8° and 87.7°.

Teacher students described their situation in terms of an “open problem” (adidactical and non-routine task) and gave a decent *a priori* analysis. As far as history is concerned, they argued about motivation by a “historical” problem. It is of course concrete, practical... but is it historical? Contextualization is quite limited. Moreover, the mention of Thales is both historically and didactically misleading: such a question has been raised by Thales but the targeted method of resolution which involves the tangent function has nothing to do with Thales and Thales's theorem.

As a conclusion, this is diagnosed as a non-functioning history lever, both with regard to the articulation with didactics and on a social-cultural perspective. Teacher students didn't manage to mobilize the results of their research in the history of trigonometry and the triangulation principle in the elaboration of the activity and pedagogical scenario.

c) Group 3 worked on proportionality and the linear model, with a project entitled “inappropriate linear reasonings”. The starting point, once the theme had been chosen in relation to the curriculum, to lesson planning and the possibility of classroom testing, was the review of didactical literature on proportionality. Teacher students therefore got acquainted with the notion of “illusion of linearity” by reading De Bock & al. (2008). They identified the presence of an epistemological obstacle and took note of Aristotle's famous error (speed and mass are proportional) pointed out by Galileo. They decided that Aristotle's error was as a historical situation appropriate for discussing in the classroom the misuse of the “linear model”, together with a geometric situation taken from De Bock & al. (to fertilize a square field of side 200m, the farmer needs 8h. How long will it take to fertilize a field of side length 600m?) which was first presented to the students so that they (or at least some of them) may experience the illusion of linearity.

As we can see, historical-psychological parallelism is pointed out by teacher students but they do not reflect on the goals of historical contextualization in the present context. It could be an opportunity to discuss with their students the place and role of errors, to connect these with the conceptions and methods in the historical context, and state that the linear model is often advocated by application of a principle of simplicity as a heuristic rule (“nature operates in the shortest way possible”).

Among the goals declared by teacher students, the latter mention “to clarify the concept of model: proportionality models a constraint (physical, logical or social) between at least two magnitudes (which makes them dependent) and describes a functional relation between their elements”. Yet, proportionality as a linear model remains an unmet epistemological goal. The epistemological problem which relates to validating/refuting a mathematical model remains implicit in the scenario. Validation criteria for the linear model are not discussed: the teacher is the validator in the geometric situation. The experimental refutation of Aristotle's assertion is also difficult in the classroom without an appropriate protocol, which was not known to our mathematics teacher students who experimented with rulers and rubbers (although it is well known to physics teachers: for instance, take two tennis balls and fill one of the two with sand). The pedagogical scenario therefore uses a “thought experiment” and appeals to students' aptitude to argue that the result is unrealistic. Teacher students were not clear on the point that the underlying epistemology of model validation is that of experimental sciences, not mathematics! Mathematical procedures of validation would be available if tables of values were produced, which was not the case in Aristotle's situation.

As a conclusion, the teacher students' approach was quite interesting but they didn't succeed in working out the history and epistemology lever, due to lack of hindsight, particulary on an epistemological point of view (insufficient understanding of Aristotle's context and lack of expertise to discuss the notion of model, also interdisciplinarity issues in a context mixing mathematics and physics together with their specific epistemologies).

IV. GENERAL CONCLUSIONS AND PERSPECTIVES

Initial teacher education in France has been considerably impacted since 2010 by two consecutive reforms, which offered an opportunity to improve the training of pre-service teachers in history, epistemology and didactics as interrelated subjects. Courses at Montpellier University were more focused on history and epistemology as a goal during the first year of the Master degree, then as a tool in year 2 in relation to practical training. Nevertheless, the goal/tool distinction shouldn't erase dialectical aspects which were always present.

The methodology used to produce didactic source material has been carried out successfully by teacher students, although it required to search the literature in history and epistemology for connections with the curriculum, in relative autonomy, which was not straightforward. They produced simple engineerings or appropriated existing ones themselves. History as a lever has been more successfully implemented than epistemology. An analysis of the didactical material that they produced reveals quite a few epistemological issues, which suggests that epistemology as a tool should deserve further investigation within our community of practice.

Our 20 students were asked after completion of the project work the following question: “what are, according to you, the benefits of an approach that uses history /epistemology as a lever?” Their answers were interpreted and dispatched on our grid of whys (see Annex 1) as follows:

1	Re-contextualization as a meaning-producing activity	16 (students)
2	Historical genesis/ artificial genesis	7
3	Psychological motivation	6
4	Interdisciplinarity	0
5	Linguistic and transverse competencies	0
6	Nature of mathematics	5
7	Obstacles and conceptions	9
8	Illusion of transparency	1
9	Teacher's dogmatism	0
10	Humanization of mathematics and human qualities	2
11	Mathematics as a cultural endeavor	1

Figure 3: Motivations for the integration of history and epistemology of mathematics into classroom teaching, as perceived by the students after the project work

As we can see, re-contextualization as a meaning-producing activity is very well perceived. Whys connected to Brousseau's paradigm (2,7) are also reasonably acknowledged, which isn't surprising (epistemological framework of the project, French culture of didactics). Accent has not been made on the socio-cultural perspective.

To summarize, our approach is characterized by the intent that history/epistemology should explicitly meet didactical goals. For a happy and fruitful marriage, didactics of mathematics should certainly develop more specific and dedicated tools in order to integrate history and epistemology as full partners.

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NOTES

1. Certificat d'Aptitude au Professorat de l'Education Secondaire.

2. Des éléments d'épistémologie et d'histoire des mathématiques s'insèrent naturellement dans la mise en œuvre du programme. Connaître le nom de quelques mathématiciens célèbres, la période à laquelle ils ont vécu et leur contribution fait partie intégrante du bagage culturel de tout élève ayant une formation scientifique. La présentation de textes historiques aide à comprendre la genèse et l'évolution de certains concepts (Bulletin officiel spécial n° 8 du 13 octobre 2011, p.1 ; http://www.education.gouv.fr/pid25535/bulletin_officiel.html?cid_bo=57529).

3. (1) Développer leur culture scientifique (2) Comprendre comment se construisent les connaissances scientifiques (3) Situer sa discipline d'origine dans un contexte plus large (4) Travailler la maîtrise de la langue écrite au travers de la lecture et de l'étude de textes scientifiques anciens. (5) Identifier dans leur contexte épistémologique et historique les concepts, notions et méthodes rencontrés dans l'enseignement des sciences à un niveau donné. (6) S'approprier différentes pistes didactiques permettant d'inclure une perspective historique dans l'enseignement des sciences (7) Appréhender les compétences transversales que font travailler l'usage de l'histoire dans l'enseignement des sciences (lecture, argumentation, écriture…) (8) Connaître l'histoire du système scolaire et de la place qu'y occupe l'enseignement des sciences (9) S'initier aux méthodes et aux problématiques propres à l'épistémologie et l'histoire des sciences (A. Besnard, A.S. Genin, A. Mayrague, Proposition pour des formations en épistémologie et histoire des disciplines d'enseignement scientifiques et technologiques, dans le cadre des masters et parcours de masters 'enseignement' ; http://f.hypotheses.org/wp-content/blogs.dir/873/files/2014/06/08_10_15_DOCT-7-MAQUETTE-EPISTEMO-HIST-SC.pdf).

4. « montrer » in French.

5. « démontrer »: note that the two verbs differ only by a prefix in the French language, the etymology being very enlightening.

6. Elles [les connaissances mathématiques] doivent être constituées en une véritable théorie et être reconnues comme telle, c'est-à-dire acceptée par une communauté qui ne s'autorise plus à aller chercher où elle veut les arguments qu'elle utilise. La démonstration en mathématiques s'appuie sur un corps de connaissances fortement institutionnalisé, ensemble de définitions, de théorèmes, de règles de déduction, dont la validité est socialement partagée.

Annex 1: why arguments in favor of the integration of history in mathematics education (ICMI Study, Tzanakis & Arcavi, pp. 202-207)

	Why argument	Description of the argument	Student (S), Teacher (T) or Didactician (D)	Ref. to ICMI Study
1	Re-contextualization as a meaning-producing activity [didactical transposition leads to dehistorialization]	"The learning of a mathematical concept, structure or idea may gain from acquaintances with the motivations [questions and problems] and the phenomena for which it was created"	S,T	First part of a1
2	Historical genesis/ artificial genesis	" History suggests possible ways to present the subject in a natural way, by keeping to a minimum logical gaps and ad hoc introduction of concepts, methods or proofs"	T, D [didactical engineering]	Second part of a1
3	Psychological motivation	History as a resource that has the potential to "motivate, interest and engage the learner"	S	Last part of a2
4	Interdisciplinarity	History as a bridge between mathematics and other subjects, to decompartmentalize disciplines and put their interrelations into evidence.	S	a3
5	Linguistic and transverse competencies	"The more general educational values of history (reading, writing, documenting,...)"	S	a4
6	Nature of mathematics Implicit assumption: reflections about the nature of mathematics may enhance mathematical literacy	"A more accurate view of mathematics and mathematical activity" that takes into account the role of "mistakes, heuristic arguments, uncertainties, doubts", etc. i) evolutionary nature of mathematics ii) meta-concepts (proof, rigor, evidence, error, etc.) iii) also reflect on the form (notations, modes of expression and representation, etc.)	S	b)

7	Obstacles and conceptions	Identify epistemological obstacles (Bachelard) and collect conceptions. These may inform on and relate to obstacles and conceptions in the learning process.	D [epistemological obstacles, conceptions], T	c2 i)
8	Illusion of transparency	“Even when a subject may appear simple, it may have been the result of a gradual evolution”.	T, D	c2 ii)
9	Teacher's dogmatism	“Exercise sensitivity and respect towards non-conventional ways to express and solve problems through the deciphering of correct mathematics whose treatment is not modern”	T	c5
10	Humanization of mathematics and human qualities	“Mathematics is an evolving and human subject, not a system of rigid truths”. To promote (through “role models”) and develop human qualities relevant to support the learning of mathematics (perseverance, creativity, etc.)	S	d
11	Mathematics as a cultural endeavor Underlying paradigm: teaching mathematics is not only teaching mathematical knowledge but also mathematics as a culture	History may highlight the non-utilitarian driving forces of mathematical development such as “aesthetical criteria, intellectual curiosity, challenges and pleasure”, the influence of social and cultural factors, and open up to cultural diversity.	S	e

Figure 4: description of the why arguments (loc. cit.) with an emphasis on the actors (student, teacher or didactician) in the motivating area under consideration

Oral Presentation

DEGREES OF INCONSISTENCY

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Several mathematical theories, such as Newton's theory of fluxions and fluents or Peano's theory of natural numbers were originally formulated in an inconsistent form. Only after some period of time consistent formulations of these theories were found. The paper analyzes several historical cases of this "initial inconsistency". It distinguishes three kinds of inconsistency by measuring the "distance" of the inconsistent theory from its consistent form. They correspond to whether re-formulations, relativizations or recodings are needed for turning the inconsistent theory into a consistent one. The paper argues that inconsistencies of different kind have different cognitive background and so should have different role in education.

INTRODUCTION

Inconsistency is a phenomenon, which can be viewed from different perspectives. Alongside efforts to create paraconsistent logic we can study inconsistency also from a historical perspective. New mathematical theories are often born as inconsistent, but despite the inconsistencies their authors manage to derive a series of remarkable mathematical results. Sooner or later after the discovery of the inconsistency of the particular theory mathematicians usually succeed constructing a logically consistent version of the theory, in which it is possible to derive basically the same results as in the original inconsistent version. A classic example is the *differential and integral calculus* discovered by Newton and Leibniz. It took more than hundred years before Cauchy and Weierstrass presented a consistent version of this theory, nevertheless, the majority of theorems proven by Newton and Leibniz turned out to be correct also in terms of the consistent theory.

When logicians comment on this episode, they emphasize the logical inadequacy of the original theories of Newton and Leibniz. A. W. Moore, for example, in his book *The Infinite* writes: "*For all its depth and beauty, the reasoning here is, as we have seen, fundamentally flawed. It rests on a certain notion of an infinitesimal difference (as not quite nothing, but not quite something either) and this notion is ultimately incoherent.*" (Moore 1990, p. 63). This approach has the effect that as scientifically (and philosophically) relevant are considered only the consistent versions of the calculus presented by Cauchy and Weierstrass. The logical and philosophical analysis of the basic concepts of Cauchy's and Weierstrass's theory is taken seriously in the literature, while the analysis of the concepts on which Newton and Leibniz built their theories is reduced to the level of historical curiosities that do not deserve a deeper logical or epistemological analysis. By having proved their inconsistency everything relevant seems to be told. To engage in a serious philosophical analysis of an inconsistent theory seems to make no sense.

I believe that this view is mistaken. Of course, in an inconsistent theory it is possible (using the means of *our* formal logic) to derived almost anything, but we have to keep in mind that most of the results obtained by Newton and Leibniz in their "logically inconsistent" theories turned out to be correct. If we chose a random system of inconsistent axioms, we would be probably unable to ensure that we systematically derive only (or mainly) "correct" results. The problem seems to be that the derivations that Newton and Leibniz made, were not based on inferences of formal logic (and they have never claimed so), but were rather based on some contextually bound rules of inference. If this is indeed the case, then the logical inconsistency of their theory is not as a fundamental problem, as it appears to be from the point of view of contemporary formal logic. I do not want to defend here inconsistent theories in mathematics. Logical consistency is the basis of all mathematics and I do not want to question it. I'm just trying to understand how it is possible to derive within a logically inconsistent theory almost exclusively correct results. Of course, this was not always the case, but the errors were much less frequent than we might expect. Inconsistent theories, such as Newton's and Leibniz's versions of the calculus or Euler's analysis of the infinitesimals had, despite their inconsistencies, a great methodological and epistemological significance. It is therefore important to subject them to a serious philosophical analysis.

My goal is to split the broad concept of logical inconsistency into ***three narrower notions*** that can be characterized by the minimal transformations that an inconsistent theory must undergo to become consistent. All three of these narrower notions of inconsistency apply to inconsistent theories that can be converted into consistent ones by means of specific transformations. Thus I will determine the degree of inconsistency of a theory by describing how radical changes its linguistic framework must undergo in order to turn it into a consistent theory. These distinctions may offer new arguments into the debate led in the philosophy of mathematics about Euler's analysis of the infinitesimals. During the 19^{th} century the infinitely small quantities were considered illegitimate; Euler's theory was considered inconsistent and was replaced by Cauchy's and Weierstrass' theories of limits.

When Abraham Robinson discovered non-standard analysis, he considered it a vindication of Euler's theory. He was convinced that he has shown that Euler's theory *was not inconsistent* as the mathematicians of the 19^{th} century thought, because it is possible to give the concept of infinitesimal quantity a precise meaning. Some philosophers, however, reject Robinson's view. They argue that Robinson in his construction of the infinitely small quantities used mathematical means that were unavailable to Euler, and thus his construction does not show the legitimacy of Euler's methods. So Moore in the cited work writes: "*the German logician Abraham Robinson (1918-1974), who invented what is known as non-standard analysis, thereby eventually conferring sense on the notion of an infinitesimal greater than 0* *but less than any finite number. But in making this sense precise he used logical methods and techniques that went far beyond what would have been recognizable to seventeenth-century mathematicians. It would be anachronistic to see his work as a vindication of what they had been up to. It did not show that the notion of an infinitesimal, as understood by* ***them*** *had been coherent*." (Moore 1990, p. 69).

Moore makes here a remarkable argumentative move, a move that is typical. When he is confronted with the epistemological fact, that Euler's theory is *not inconsistent* – as it was considered so far – he turns from the *objective fact* of consistency or inconsistency of as theory to the *subjective fact* of how this theory was understood by mathematicians of a particular period of time. But this is, from the point of view of philosophical analysis, irrelevant. To analyze the different ways a theory was understood in a particular period is the business of the history of science, not philosophy. Thus Moore shifts the analysis of the particular theory from philosophy to history. Although he treats Euler's theory not as a *historical curiosity* (as inconsistent theories were usually treated), but he still considers the *psychological aspect of its understanding* decisive. This psychological aspect should decide about the legitimacy of an epistemological reconstruction.

In order to be able to start the analysis of inconsistent theories, we must first clarify what we are going to analyze in them. I believe that besides the "uninteresting" inconsistent theories there are three kinds of inconsistent theories that are inconsistent in an "interesting way" (i.e. they have a consistent "core" that is expressed in an inconsistent way).

1. THE NOTION OF LOCAL INCONSISTENCY

We call a theory ***locally inconsistent*** if it is logically inconsistent, but through a re-formulation it can be turned into a consistent one.[1] A *local inconsistency* can thus be regarded a mistake or an error of the author of the particular theory, because all means needed for the formulation of the consistent versions of the theory were already available. It seems that the author of a locally inconsistent theory only due to some unfortunate formulation of his assumptions, definitions, or arguments got into a contradiction. This type of inconsistency is not surprising. I introduced it just as a foil against which the two other types, the introduction of which is the main purpose of this paper, could be characterized. In many cases, and I believe that Newton's or Frege's case is included, at the time of the formulation of the theory it was not possible to create a consistent theory because the conceptual framework, in which these authors worked, did not have means for a consistent formulation of the contents, which they tried to analyze.

An example of a locally inconsistent (mini-) theory is Cauchy's proof of a theorem about the sum of a series of functions. Cauchy, one of the initiators of strict foundations of mathematical analysis proved the erroneous assertion that the sum of a convergent series of continuous functions is itself continuous. By calling this a *local* inconsistency I want to emphasize that Cauchy was the creator of the conceptual framework in which it is possible to formulate the correct version of that assertion. Thus Cauchy's proof was separated from the correct theory by a mere re-formulation. In this respect Cauchy's theory differs fundamentally from the theories of Newton or Euler, which were separated from the correct theories by a complete change of the conceptual framework of mathematical analysis.

1.1 Cauchy's theorem about the sum of a series of continuous functions

Cauchy defined continuity as: "*the function f(x) is a* ***continuous*** *function of x between the assigned limits if, for each value of x between these limits, the numerical value of the difference* $[f(x+\alpha)-f(x)]$ *decreases indefinitely with the numerical value of* α*. In other words, the function f(x) is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself.*" (Cauchy 1821, p. 26). This definition is interesting because it defines the concept of continuity of a function not for a point, but for the entire interval (for x between given borders). Before introducing Cauchy's assertion about the continuity of the sum of the series of continuous functions, I will quote three definitions given by Cauchy: the definition of a variable: "*We call a quantity* ***variable*** *if it can be considered as able to take on successively many different values. We normally denote such a quantity by a letter taken from the end of the alphabet.*" (Cauchy, 1821, p. 6), the definition of the limit: "*When the values successively attributed to a particular variable indefinitely approach a fixed value in such a way as to end up by differing from it by as little as we wish, this fixed value is called the* ***limit*** *of all the other values.*" (ibid. p. 6) and the definition of an infinitesimal quantity: "*When the successive numeral values of such a variable decrease indefinitely, in such a way as to fall below any given number, this variable becomes what we call* ***infinitesimal****, or an* ***infinitely small quantity****. A variable of this kind has zero as its limit.*" (ibid. p. 7).

Now we can state the: "***Theorem I.*** *– When the various terms of series (1)*

[*(1)* $u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$]

are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum of the series is also a continuous function of x in the neighborhood of this particular value." (ibid. p. 90).

The proposition in this form is not true. First who criticized Cauchy's proof was Abel in (Abel 1826). As a counterexample we can take the following Fourier series:

$$f(x) = +1 \qquad \text{for} \qquad 0 < x < \pi,$$
$$f(x) = 0 \qquad \text{for} \qquad x = 0,$$

$$f(x) = -1 \qquad \text{pre} \qquad -\pi < x < 0.$$

This function is obviously discontinuous at $x = 0$, but its Fourier series is[2]

$$f(x) = \frac{4}{\pi}\left\{\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \ldots\right\}.$$

This is an infinite series of continuous functions, the sum of which is discontinuous. This clearly contradicts Cauchy's assertion.

1.2 Introduction of the concept of uniform continuity

What is the problem with Cauchy's theorem is explained in many textbooks of mathematical analysis. In the case of functional series we must distinguish between the point wise convergence and the uniform convergence. Cauchy's definition captures the point wise convergence, but in order to secure the continuity of the sum of a series of continuous functions, it is not enough to have pointwise convergence, but uniform convergence is a sufficient condition. I will not discuss here the technical details of the theory of functional series; those can be found in textbooks of mathematical analysis. Similarly, I will not to burden the paper with the history of the concept of uniform convergence; this can be found for instance in (Lützen 1999). The important thing to realize is, however, that both concepts – the concept of points wise convergence as well as the concept of uniform convergence – can be introduced in the same conceptual framework. By a simple *re-formulation* of Cauchy's definition of limit it is possible to obtain a definition of uniform convergence. Thus already in the linguistic framework of Cauchy's theory ***it was possible*** to define the concept of uniform convergence and to formulate Cauchy's theorem in a consistent form. Now I do not want to raise the question, why did Cauchy not distinguish between the point wise and the uniform convergence. This is an important question that I leave open for further *historical* investigation. My point here is *epistemological*, namely the fact that Cauchy's inconsistent theory was separated from the consistent theory by a mere re-formulation.[3]

1.3 The concept of locally inconsistent theory

Shortly after Abraham Robinson discovered non-standard analysis, Imre Lakatos wrote (but did not publish) the paper *Cauchy and the continuum: the significance of non-standard analysis for the history and philosophy of mathematics* (Lakatos 1966). In this paper Lakatos expressed the view that Cauchy did not make any mistake, but he used a different concept of the continuum – the non-Archimedean continuum in which there are infinitely small and infinitely large quantities. Although this view did not become dominant, there is a stream of papers arguing in favor of this view (e.g. Laugwitz 1987, Katz and Katz 2011). In my opinion, the efforts to interpret Cauchy's theorem using non-standard analysis is inadequate. Cauchy's theory is only locally inconsistent, and therefore it is easier to interpret it on the background of the

Archimedean continuum as a theory with one small inconsistency. It is unlikely that Cauchy would use the concept of the non-Archimedean continuum and this fact would in the almost 400 pages of his book manifest itself in a single place—the proof of the above theorem.

2. THE NOTION OF CONCEPTUAL INCONSISTENCY

We call a theory ***conceptually inconsistent*** if it is logically inconsistent, it cannot be made consistent by means of a re-formulation, and the creation of a consistent version of the theory requires a new conceptual framework, i.e. a *relativization*.[4] Similarly as a locally inconsistent theory can be made consistent using a re-reformulation, in the case of a conceptually inconsistent theory this requires a relativization. An example of such a theory is Leibniz's or Newton's calculus. Newton and Leibniz used in their proofs arguments that are logically inconsistent. When I say that their theories are *conceptually inconsistent*, I want to emphasize that the inconsistency that occurs in these theories, although by an order greater than the local inconsistency, is considerably smaller than the contribution of these theories. Newton and Leibniz created a new ***instrument of symbolic representation***: they discovered the concept of a *function*, introduced the fundamental distinction between *function* and *argument*, introduced the concept of *derivative,* and discovered the *relation between differentiation and integration*. This fundamentally changed the entire mathematics.[5] That these innovations were introduced in an inconsistent conceptual framework is not so important. They were separated from the consistent version by few relativizations.

2.1 Newton's and Leibniz's versions of the differential and integral calculus

As an illustration of Newton's method of determining the derivative we can take the passage of his *The Method of Series and Fluxions*, quoted by Fauvel and Gray: „*The moments of the fluent quantities (that is, their indefinitely small parts, by addition of which they increase during each infinitely small period of time) are as their speeds of flow. Wherefore if the moment of any particular one, say x, be expressed by the product of its speed* $\dot{x}$ *and an infinitely small quantity o (that is, by* $\dot{x}o$*), then the moments of the others, v, y, z* […] *will be expressed by* $\dot{v}o$, $\dot{y}o$, $\dot{z}o$, […] *seeing that* $\dot{v}o$, $\dot{x}o$, $\dot{y}o$ *and* $\dot{z}o$ *are to one another as* $\dot{v}$, $\dot{x}$, $\dot{y}$ *and* $\dot{z}$.

Now, since the moments (say, $\dot{x}o$ *and* $\dot{y}o$*) of fluent quantities (x and y say) are the infinitely small additions by which those quantities increase during each infinitely small interval of time, it follows that those quantities x and y after any infinitely small interval of time become* $x+\dot{x}o$ *and* $y+\dot{y}o$*. Consequently, an equation which expresses a relationship of fluent quantities without variance at all times will express that relationship equally between* $x+\dot{x}o$ *and* $y+\dot{y}o$ *as between x and y; and so* $x+\dot{x}o$ *and* $y+\dot{y}o$ *may be substituted in place of the latter quantities, x and y, in the said equation.*

Let there be given, accordingly, any equation $x^3 - ax^2 + axy - y^3 = 0$ and substitute $x + \dot{x}o$ in place of x and $y + \dot{y}o$ in place of y: there will emerge

$$(x^3 + 3\dot{x}ox^2 + 3\dot{x}^2o^2x + \dot{x}^3o^3) - (ax^2 + 2a\dot{x}ox + a\dot{x}^2o^2)$$
$$+ (axy + a\dot{x}oy + a\dot{y}ox + a\dot{x}\dot{y}o^2) - (y^3 + 3\dot{y}oy^2 + 3\dot{y}^2o^2y + \dot{y}^3o^3) = 0.$$

Now by hypothesis $x^3 - ax^2 + axy - y^3 = 0$, and when these terms are erased and ***the rest divided by o*** *there will remain*

$$3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3o^2 - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3o^2 = 0.$$

But further, since o is supposed to be infinitely small so that it be able to express the moments of quantities, terms which have it as a factor will be ***equivalent to nothing in respect of the others.*** *I therefore cast them out and there remains*

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0.$$

It is accordingly to be observed that terms not multiplied by o will always vanish, as also those multiplied by o of more than one dimension; and that the remaining terms after division by o will always take on the form they should have according to the rule. This is what I wanted to show“ (Fauvel and Gray 1987, p. 385, emphasis LK). From the equation, which Newton states as the last one we can easily derive the derivative of the quantity y (i.e. of the implicit function) in the form as we are used now: $dy/dx = \dot{y}/\dot{x} = (3x^2 - 2ax + ay)/(3y^2 - ax)$.

2.2 Berkeley’s criticism

In the long quote from Newton I emphasized two steps that were relevant for the further development of the theory. In 1734, seven years after Newton’s death, George Berkeley publishes his famous *The Analyst or a Discourse Addressed to an Infidel Mathematician*, where he presented a vivid and witty criticism of Newton’s theory fluxions and fluents. Berkeley’s aim was to show that mathematical analysis that underpins the entire contemporary natural science, does not have any more solid basis than religion with its angels and miracles. Berkeley’s criticism consist in observing that Newton first operates with the quantity *o* as if it were different from zero (in order to make division by *o*), but subsequently he treats it as if it were equal to zero (when neglecting its higher powers). According to Berkeley, a particular quantity is either equal to zero, but then it is equal to zero throughout the entire calculation, and therefore it is impossible to divide by it, or it is not equal to zero, but then it is not equal to zero throughout the entire calculation, and therefore it can not be neglected. Berkeley explains the fact that mathematicians obtain correct results in spite of these errors as a *compensation of errors*: the calculations of the mathematicians are mistaken, but in mathematical analysis they use errors always in pairs, so that they can cancel each other out and the calculations give correct results. The correctness of the result is not the consequence of correct methods, but of the fortuitous circumstance that the errors cancel each other out. Thus the results of mathematical analysis are no more reliable than the miracles of religion.

2.3 The creation of the strict foundations of the differential and integral calculus

Interestingly, the way in which mathematicians obtained the consistent theory of differential and integral calculus, in a sense, followed the main idea of Berkeley's (ironically meant) interpretation of the correctness of the results of the calculus as a compensation of errors. Mathematicians stopped viewing the derivative as a ratio of two differentials (or moments) and started to see it as an indivisible whole. In other words, they combined the infinitesimal expressions into pairs so that their infinitesimal character canceled out, exactly as described by Berkeley, and for these "conglomerates" they formulated the exact rules of the calculus. So, instead of calculating with independent moments or differentials (as Newton or Leibniz would), they worked only with ratios of two moments or differentials, and thus removed from the outset the errors described by Berkeley. This combining of the infinitesimals was the idea of Lagrange, who considered d*y*/d*x* a compact expression whose value must be determined by the rules of the calculus. He rejected the notion of a limit and for expressions like d*y*/d*x* he wanted to find rules analogous to those used in algebra.

The approach of Lagrange did not succeed. The successful approach is due to Cauchy, who took the idea of Lagrange to combine the differentials into compact expressions, but instead of algebraic rules he decided to determine the values of these expressions by means of a limit transition. Cauchy's version of the differential and integral calculus is thus constructed from the same ingredients as were used by Newton or Leibniz, only these ingredients are combined differently. While Newton and Leibniz first created for each individual variable its moment or differential by means of a limit transition, and then tried to combine these individual moments or differentials, Cauchy first (making use of Lagrange's idea, the roots of which go back to Berkeley) combines these variables into certain fixed combinations and then (unlike Lagrange who rejected the limit transition) goes with these combinations to the limit. This is a fundamental change. Newton wanted to define the derivative as the ***ratio of the limit values (i.e. moments) of two variables***, but was unable to say exactly what is in that ratio. He summed up his views in the famous theory of first and last ratios, which Berkeley ridiculed calling the limit values of the variables that make up the ratio ghosts of departed quantities. Cauchy in contrast to this defines the derivative as the ***limit of the ratio of final values of the variables*** and understands it *not* as the value which the expression has for the limit values of the variables, but as the value that the expression as a whole converges towards.

I would like to interpret the differences between Newton's, Lagrange's and Cauchy's approach to mathematical analysis as relativizations, i.e. differences in the form of language.[6] Newton's approach to mathematical analysis can be interpreted as based on the ***perspectivistic form of language*** while Lagrange's approach as based on the ***compositive form*** and Cauchy's theory as based on the ***interpretative form***. It was necessary to pass several relativizations to get a conceptual framework, in which it is possible to give Newton's calculations and arguments a consistent interpretation.

2.4 The concept of the conceptually inconsistent theory

When we accept the fact that there are theories, as for instance Newton's theory of fluxions and fluents, which allow by means of heuristic schemes of reasoning discover many important theorems despite the fact that that they are logically inconsistent (i.e. an opponent like Berkeley is capable by means of the same rules arrive at paradoxical results), and at the same time in the linguistic framework in which they are formulated, they cannot be cast in a consistent form, this sheds new light on several philosophical theories of the development of science.

On the one hand, this shows the inadequacy of Popper's falsificationalism. Had Newton be a falsificationalist, mathematical analysis would probably never emerge. Newton knew that his project has many weaknesses, but fortunately he was undeterred by all the paradoxes and created a theory that after undergoing a sequence of four relativizations was finally formulated in a consistent way. Thus, the fact that scientists often ignore facts that contradict their theories is not necessarily the consequence of their complacency or irrationality. They can, like Newton, on the one hand realize that their aim (in Newton's case it was a new recoding) is of a fundamentally greater importance than the details which cause the contradictions. On the other hand, they may feel that, given the stage of development of the discipline it makes no sense to try to solve these contradictions, because in the linguistic framework that is at their disposal, it is not possible (i.e. that their theory is conceptually inconsistent, and thus any attempts to solve the inconsistencies by means of re-formulations is doomed to failure). There are several testimonies that demonstrate the awareness of this situation, as for instance the words of d'Alembert words "*Go forward, and faith will come to you*!".

On the other hand, the above fact calls into question the idea of a scientific theory as a set of propositions closed under the relation of logical consequence. This abstraction, which for the purposes of mathematical logic seems to be fully adequate, is absolutely inadequate for epistemology and philosophy of science. The phenomenon of conceptual inconsistency is important because a theory inconsistent in this particular way cannot be easily (i.e. by means of a re-formulation) made consistent. The inconsistency is part of the theory for a long period (relativizations are changes that last several decades). But despite this inconsistency theory has a core, which, thanks to the ***internal development of the inconsistent theory itself*** (i.e. without external intervention) can be consistently formulated.

3. THE NOTION OF GLOBAL INCONSISTENCY

We call a theory ***globally inconsistent*** if it is logically inconsistent, it is not locally or conceptually inconsistent, but nevertheless, by means of a recoding it can be made logically consistent. An example of such a theory is Euler's theory of infinitesimally small quantities. This theory is much "wilder" than Newton's theory of fluxions and fluents. Euler's theory could not be made consistent by means of relativizations. Cauchy's rescue of Newton's theory consisted precisely in eliminating infinitely small quantities, which for Euler's theory is not a viable path (using Berkeley's words, we could say that Euler did not care to let the errors in his theory occur in couples). Only when Robinson, using the tools of a completely different recoding (set theory), constructed a model of infinitely small quantities, he was able to give Euler's theory a logically consistent form. Thus, unlike the Cauchy, which worked still within the framework of the differential and integral calculus (i.e. within the same representational framework as Newton), Robinson left Euler's representational tool and proved the consistency of Euler's theory, so to speak, from outside, using the means of a completely different language.

3.1 Euler's theory of infinitesimals

To illustrate Euler's work with the infinitely small and infinitely large numbers, I will show his derivation of the series for the exponential function taken from (Euler 1748, par. 114-116). Let a > 1 and ω be an "*infinitely small number, or a fraction so small that it is almost equal to zero*." Then

$$a^{\omega} = 1 + \psi$$

for an infinitely small ψ. Now let us put $\psi = k\omega$. Here k depends only from a; then

$$a^{\omega} = 1 + k\omega.$$

For any real number i we have

$$a^{i\omega} = (1 + k\omega)^i .$$

Thus thanks to the binomial theorem

$$a^{i\omega} = 1 + \frac{i}{1}k\omega + \frac{i(i-1)}{1.2}k^2\omega^2 + \frac{i(i-1)(i-2)}{1.2.3}k^3z^3 + ... \quad (1)$$

For z, a finite positive number, $i = \frac{z}{\omega}$ is infinitely large. Putting $\omega = \frac{z}{i}$ into (1) we get

$$a^z = a^{i\omega} = 1 + \frac{1}{1}kz + \frac{1(i-1)}{1.2i}k^2z^2 + \frac{1(i-1)(i-2)}{1.2i.3i}k^3z^3 + ...$$

But when i is infinitely large, $\frac{i-1}{i} = 1$, $\frac{i-2}{i} = 1$, etc., and we get

$$a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1.2} + \frac{k^3z^3}{1.2.3} + \frac{k^4z^4}{1.2.3.4} + ...$$

Natural logarithms arise when a is chosen so that k = 1. Euler gives the value of a with the precision of 23 decimal places, introduces for it the symbol e which is still in use today and writes

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \dots .$$

On this and many other derivations of Euler we can see that he used infinitely small quantities not as approximations, as Newton and Leibniz did. On the contrary, he used them as numbers. Thus Berkeley's strategy of "pairing" the errors, which is the basis of Cauchy's construction of rigorous foundations of analysis, is here useless.

3.2 Elements of Robinson's non-standard analysis

Euler's formulation of mathematical analysis was never shown to be inconsistent. Nevertheless, at the end of the 18th century the mathematicians came to the conclusion that the concept of infinitely small and infinitely large quantities cannot be given a consistent interpretation. This belief, shared by some philosophers even today (see the quote from Moore in the introduction), was sufficient reason for the abandoning Euler's calculus and for its replacement by Cauchy's ε–δ analysis. It was therefore a big surprise when in the early 1960-ies Robinson managed to create non-standard analysis, and to introduce the so called hyperreal numbers, that is system containing infinitely small and infinitely large numbers. Thus, over 200 years after Euler published his *Introductio ad analysin infinitorum* (Euler 1748) it turned out, that Euler's techniques using infinitesimal numbers were fully consistent.

3.3 The concept of the globally inconsistent theory

On the above episode from the history of mathematics it is remarkable that even in mathematics it is possible to lose confidence on the basis of "defamation". No one has ever shown that Euler's calculus was inconsistent, and despite this fact the belief that the concept of infinitely small and infinitely large numbers is internally inconsistent became dominant and the theory was abandoned.[7]

In addition to the remarkable fact that in mathematics the phenomenon of social persuasion plays such an important role in the process of acceptance or rejection of a particular theory – at least in the short run, because eventually the truth came to light (although two hundred years is not that short time) – this case is interesting also in another respect. The consistency of Euler's analysis was shown by means of model theory and thanks to non-trivial contribution of set-theoretical techniques, both of which are theories having a *logical* and *expressive power* (in the strict, technical sense, as these terms are introduced in *Patterns of Change*) far outweighing the logical and expressive power of the language of mathematical analysis, in which Euler's theory was originally formulated. Thus, if we come back to the second passage from Moore's book that we quoted in the introduction, it can be argued that it is certainly not an anachronism to understand Robinson's work as a vindication of Euler's efforts. Euler's theory was rejected not because ***Euler's understanding*** of infinitely small quantity was inconsistent. The mathematicians of the 19th century, just like mathematicians today (unlike philosophers) do not care how Euler *understood* the concept of infinitely small quantities. They are interested in whether ***the concept of infinitely small quantities itself*** is or is not coherent. Mathematicians of the 19th century thought (wrongly, as we now know, thanks to Robinson) that this concept is inconsistent, and therefore they rejected Euler's theory. Robinson, in my opinion, has shown that the concept of infinitely small quantities is consistent, and thus he showed that Euler's theory was unjustly rejected. The discovery of this fact can, by all means, be considered a vindication of Euler's theory.

4. CONCLUSION

The most serious form of inconsistency is that which can not be corrected by means of recoding. This kind of inconsistency can be called ***absolute inconsistency***. A theory that contains it remains forever beyond rational discourse. The difference in evaluation of the presence of an inconsistency in a mathematical theory by mathematicians and philosophers is probably due to the fact that philosophers construe inconsistencies automatically as absolute inconsistencies. Mathematicians, on the other hand, led by the historical experience with ***(locally***, ***conceptually***, and ***globally***) inconsistent theories show more willingness to further work with such theories. The four concepts of inconsistency, which I propose to distinguish, are probably of no importance for current research. Today we have no idea which re-formulations, relativizations, and recodings will bring the future development of mathematics. Thus looking on an inconsistency we are not in the position to tell, which kind of inconsistency we are dealing with. However, the distinction of these four concepts is important in retrospect, for understanding the development of a mathematical discipline.

4.1 The rationality of the decision to ignore an inconsistency

For a historian to distinguish the different kinds of inconsistency makes it possible to give rational content to the words of d'Alembert "*Go forward, and faith will come to you*!". It can be paraphrased as: "*don't worry about local or conceptual inconsistencies when you are laying the foundations of a new recoding*." D'Alembert's encouragement can be viewed as the advice not to worry about an inconsistency of a smaller level of magnitude than the level, on the development of which the mathematician is actually working. The point is that during the 18th century the foundations of a new recoding—the differential and integral calculus – were laid. Therefore, it is rational to ignore temporally such paradoxes as the one described by Berkeley. Of course, it will be necessary to return to them, once the new recoding is sufficiently worked out.

When we see in Newton and Leibniz the creators of an ***entirely new language***, we need not emphasize the logical inconsistencies in their ***theories*** (as A.W. Moore does), and instead we can highlight their work with *functional variables* (the new type of variables that the calculus introduced into mathematics), what operations they apply to them (substitution, differentiation, and integration), etc. The fact, that they failed to find a consistent justification for their calculus, is not so important. The calculus had to be discovered first, only then could it be justified.

By this I do not want to defend inconsistencies. I want to say that the development of mathematics goes on at different levels, and inconsistencies of lower levels may be temporarily ignored in order to concentrate efforts on the construction of a higher level.[8] There are very few mathematicians who became the founders of a new recoding – besides Newton and Leibniz we can mention Frege (for predicate calculus), or Cantor (for set theory). Therefore, the fact that in the work of most of them some logical inconsistencies appeared does not reduce the value of their contribution. These inconsistencies can be repaired, and mathematicians of the next generations repaired them (in the case of Newton it was Cauchy, in the case of Frege it was Russell, and in the case Cantor it was Zermelo). I do not want to diminish the importance of the work of Cauchy, Russell, or Zermelo, yet it can hardly be compared with the importance of the work of Newton, Frege, or Cantor. The idea to calculate areas and volumes by means of inverting the operation of derivative; the idea to consider propositions as functions; or the idea to continue the number sequence into the transfinite – each of these ideas represent a breakthrough into an entirely new universe. The fact that the first reports describing these new universes contained some logical inconsistencies can be interpreted as just an annoyance.

4.2 The necessity to ignore inconsistencies

Our classification explains the legitimacy of ignoring inconsistencies also in another sense. Newton and Euler *did not have a chance to make their theories consistent*. By the linguistic tools available in their time it was simply impossible to create a consistent differential and integral calculus. They stood before the choice either to follow to the strict standards of logical consistency and to give up the development of the differential and integral calculus or to develop the differential and integral calculus in the best available form. In the first case we would have neither modern physics nor modern technology. Fortunately, it was the second alternative, which Newton and Euler, and after them hundreds of mathematicians decided to take.

The reason why by the linguistic means available at the time of Newton and Euler it was not possible to create a logically consistent differential and integral calculus can be explained using the theory of relativizations (in the case of Newton's theory) or the theory of recodings (in the case of Euler's theory). We have to realize that a consistent version of Newton's theory of fluxions and fluents was created by Cauchy using the ***interpretive form of language***. It seems, however, that the sequence of forms from *perspectivistic*, and *projective*, through coordinative and compositive to the interpretative and integrative is not accidental. Each form in this sequence is built using the resources of the previous one. The interpretative form (the first form, which is sufficiently rich to formulate a logically consistent theory of the differential and integral calculus) could not be created before the previous forms were built – the perspectivistic, the projective, the coordinative, and the compositive one. The differential and integral calculus, constructed by means of the previous forms was necessarily logically inconsistent.

NOTES

1. I introduced the notion of *re-formulation* in (Kvasz 2008) as a change that does not alter the conceptual framework in which the theory is formulated, but changes only the formulation of particular propositions. An example of a *re-formulation* was the discovery of the planet Neptune. This discovery did not changed the concept of a planet, but it changed the answer of the question how many planets has the solar system. That is why I write *re-formulation* with a hyphen. The original formulation ("The solar system has seven planets.") and the new formulation ("The solar system has eight planets.") logically exclude each other. In this respect a *re-formulation* differs from a *reformulation* (written without a hyphen) which usually expresses the same content by other words.

2. The derivation of this formula can be found in many textbooks, for example in (Courant 1927, p. 517).

3. My point here is the mere possibility of formulating the correct theory in the linguistic framework, which was used by Cauchy. I acknowledge that it is easier to define the concept of uniform convergence in Cauchy's linguistic framework, if we already have a definition of this concept. So it is possible that in order to ***discover*** this concept, it was necessary to abandon Cauchy's linguistic framework because in this framework the definition of uniform convergence is highly unnatural. I leave this ***epistemological*** question aside. From the ***logical*** point of view it is essential that in the particular linguistic framework it is possible to ***formulate*** a definition of uniform convergence.

4. The concept of relativization is discussed in the second chapter of *Patterns of Change* (Kvasz 2008). It is not possible to explain this concept in few words, but a relativization can be understood as the creation of a (cognitive) distance; as the ability to look at the particular subject matter from outside. This is when something that we are immersed in becomes an object which (often only in our imagination) we see in front of us as.

5. While relativizations are bound to the conceptual framework of a theory, recodings which are connected to the introduction of new instruments of symbolic (or iconic) representation are a more fundamental change in the language of mathematics. In *Patterns of Change recodings* are discussed in the first chapter. When Newton and Leibniz created a new representational tool, they created a new universe of objects—the universe of functions. This universe can be conceptualized in different ways and *relativizations* represent transitions between these different conceptualizations.

6. Relativizations are in *Patterns of Change* described as changes of the form of language (in the sense of the *Tractatus Logico Philosophicus* of Wittgenstein). There seems to be at least eight forms of language: perspectivist, projective, coordinative, compositive, interpretative, integrative, constitutive, and conceptual.

7. Therefore Euler's theory is probably not the best illustration of the concept of a globally inconsistent theory because, strictly speaking, it does not meet the first requirement of the definition of that concept—namely that the particular theory was indeed inconsistent. Euler's theory was only considered as such. But on the other hand, the fact that was indeed consistent did not help. It was rejected as if it were inconsistent.

8. A re-formulation is understood as a change of a lower level than a relativization, just like a relativization is understood as a change of a lower level than a recoding.

ACKNOWLEDGEMENTS

This work was supported by the Grant agency of the Czech Republic by the grant GA ČR P407/11/1740.

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Oral Presentation

A THEORETICAL FRAMEWORK OF HPM

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The relationship between the history of mathematics and mathematics education (HPM) is a special research field in mathematics education. Some of the issues in this field are:

1. Why is HPM studied?
2. What are studied on HPM?
3. How can HPM be realized?

For the first question, some scholars have made a reply from the perspective of the teachers and students respectively; for the third question, some scholars have established the theoretical framework of how to integrate the history of mathematics into mathematics teaching. In this presentation, we try to establish a theoretical framework of the relationship between history of mathematics and mathematics education on the basis of a triangle model of teaching. What's more, we discuss the first and second questions based on this framework and try to give comprehensive and holistic answers as far as possible.

Oral Presentation

KNOWLEDGE ACQUISITION AND MATHEMATICAL REASONING

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Mathematical and logical reasoning can be understood as being tautologous which makes the reasoning, informationally, empty. Mathematical and logical truths are valid, i.e., true in every possible world. That is, mathematical and logical truths do not exclude any possibilities, and contradictory statements exclude all of them. To understand how mathematics increase our knowledge, it is important to analyze concrete mathematical reasoning. In geometry, the essential element is the constructivity of the entire reasoning process. A key notion in understanding mathematical knowledge acquisition is the notion of constructivity, which is closely connected to the methodology and epistemology of mathematics. However, at the same, the constructivity allows us to understand the applicability of mathematical reasoning to experimental and empirical reasoning. The strategies of experimental and mathematical reasoning are parallel.

INTRODUCTION

The notion of reasoning, as well as the notion of mathematical reasoning, is used in everyday language. However, it is not obvious what this everyday notion is intended to mean; maybe it is, as everyday notions usually are, ambiguous. Moreover, in scientific usage, the notion of reasoning seems to be a very flexible notion. Even in the philosophy of science, there is no consensus on the meaning of the notion of scientific reasoning (Niiniluoto, 1999). In mathematics and in logic, there are different philosophical approaches that interpret the mathematical and logical reasoning in different ways (Benacerraf & Putnam, 1989).

The notion of reasoning is connected to the notion of learning: all learning is, in one sense or another, reasoning. So far, so good. However, the meaning of the notion is, once again, ambiguous; the learner learns by reasoning, but not all reasoning need be learning. Sometimes reasoning is just an explication of what we already know. There are interesting degrees of knowledge, ranging from (full) knowledge to (full) ignorance (Hintikka, 1989).

There are different kinds of reasoning, for example, Peirce characterized three kinds of reasoning, namely deductive, inductive, and abductive reasoning (Peirce, 1955). We come across deductive reasoning in logic and in mathematics, and we meet inductive reasoning in (ordinary) empirical scientific reasoning; for example, normal statistical reasoning is inductive. Abductive reasoning is more problematic, and it is met in discovery processes (Hintikka, 1998). Deductive reasoning is truth preserving, which implies that deductive reasoning does not increase our knowledge. Inductive and abductive reasoning increase our knowledge, which makes these modes of reasoning very problematic, and there are no generally accepted inference rules for inductive or abductive logic (Kelly, 1996).

Mathematical and logical, reasoning and all truth preserving reasoning that can be characterized as being tautologous (see Tractatus 6.1231). This tautologousness means that the reasoning is, informationally, empty which in terms of information theory means that mathematical and logical truths do not exclude any possibilities, that is, they are true in every possible world. On the contrary, contradictions are, informationally, full, since they exclude all the possibilities, that is, they are false in every possible world. Both logical truths and contradictions seem to be useless in any real communication; they cannot be used in conveying any factual and meaningful information.

However, the informational emptiness is not the whole story. It is true that logic and mathematics are tautologous and, hence, "“useless” in real communication, but then several questions arise: Why study mathematics? Why is mathematics so difficult to study? Can mathematics increase, in any reasonable sense, our knowledge? Why can mathematics be applied in so many fields of sciences? These questions are interesting as such, but they are closely connected to each other. As formal sciences, mathematics and logic are, informationally, empty, but this makes it possible to apply them to different fields of sciences. At the same time, as formal and abstract sciences, they are not easy to grasp.

Mathematics and logic, even if they are formal sciences, evoke emotions and passions. We have to understand that there is no pure mathematics or pure philosophy of mathematics in a sense that it would be explicit, explicitly presented and have a lack of “unintentional meanings” or “unintentional connotations”. The philosophical views are built from heteronomous sources, some ideas increasing, some decreasing. The heteronomity is a permanent condition, which has to be kept in mind while formulating a philosophy of mathematics; in particular, this heteronomity has to be recognized in mathematics and logic teaching.

It is hard to see any single fundamental opinion which could be seen as prevailing, and it is not an easy task to build a coherent picture. In a sense, the kind of practical attitude given by Beta in Lakatos (1989; 54) may seem to be the final opinion: “Whatever the case, I am fed up with all this inconclusive verbal quibble. I want to do mathematics and I am not interested in the philosophical difficulties of justifying its foundations. Even if reason fails to provide such justification, my natural instinct reassures me.” In textbooks of logic and mathematics, the emphasis has been on teaching inference rules, but not on teaching the strategic aspects of the whole reasoning process (Hintikka, 1996; 2007; Detlefsen, 1996). Teaching strategic aspects supposes that the teacher has in his or her mind a holistic picture, which he or she is intending to convey to students. However, the very nature of mathematics and mathematical reasoning is still a problem to be solved. Independently, whether we solve the problem consciously or unconsciously, we have a philosophy of mathematics. This philosophy affects the way we think about, teach, or do the mathematics. So, it is better that the philosophy of mathematics is explicit.

Mathematics and logic are understood as being formal tools that can be used in different fields of sciences. However, the notion of a tool is not as innocent as is sometimes assumed. Mathematics and logic are cultural constructs, hence mathematical and logical notions, similarly to material objects, like a hammer, carry their cultural history. Mathematics and logic are not merely tools, but part and parcel of the methodology of natural sciences; they are built into the knowledge acquisition processes (Hintikka, 2007).

In the following we are not intending to give a conclusive characterization of mathematics and logic. We are not intending to remove the multifaceted nature of mathematics and logic. The intention is to characterize one possible view which does justice to mathematical and logical reasoning. We will connect the expressed approach to some other approaches which give a richer view of the topic.

ABOUT THE PHILOSOPHY OF MATHEMATICS

The fundamental questions of the philosophy of mathematics and of logic – such as "What is mathematics?" and "What is logic?" – are open questions which do not have well characterized conclusive answers (Hintikka, 1976). Still, they are worth asking. There are several different kinds of answers in the history of philosophy, mathematics and logic. In the introduction of *The Principles of Mathematics,* Russell says that "The present work has two main objects. One of these, the proof that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles …" (Russell, 1903, p. v). The characterization is easy to accept: mathematics is a deductive science, which is based on some fundamental statements usually called axioms and on some set of rules of inference. The Russellian approach has its philosophical roots in the emergence of new mathematical logic, which "tends to identify mathematics with its formal axiomatic abstraction (...) as the formalist school" (Lakatos, 1989, p. 1). Russell and Frege can be seen as founders of the modern mathematical logic.

The late 19th century and early 20th century formed a "golden age" for modern formal logic. There is no single logic, but it has seen several different kinds of objectives. Logic has been understood, for example, as "laws of thought", a universal language or general natural science, which all have different interpretations. So, as laws of thought, logic describes how humans think (psychologism in logic) or logic tells us how to reason correctly, not how human actually or usually reason (normatism).

Mathematics and logic can be understood as just a formal study of (uninterpreted) symbols. The expressions “formal logic” or “symbolic logic” may suggest such an interpretation, which is a very problematic interpretation (Haack, 1995, p. 3). Of course, in logic, the manipulation of symbols, according to inference rules, is a central task. This manipulation is not the central content of mathematics and logic: they are rich in content and, hence, no simple idea captures their whole meaning.

The present day approach – in which logic is a field of mathematics – is compatible with the normative interpretation. Even if we do not understand logic or mathematics as part of philosophy, they are rich in content. There is no need to assume any “philosophical logic”, besides the “mathematical logic” (Hintikka, 1973, ch I). Carnap, in his early publications, emphasized the formal aspects of logic and philosophy. The notion of syntax was central for him (Carnap, 2000), and Carnap’s notion of syntax is reminiscent of Wittgenstein’s notion of grammar, which is a fundamental notion of his philosophy of language.

The fundamental idea that interconnected late 19th and early 20th century logic was formulated in logicism, which was the study of the foundations of mathematical reasoning. The basic intention was to reduce mathematics to logic. Russell was very optimistic when he said that it is possible to reduce mathematical propositions “to certain fundamental notions of logic” (Russell, 1903, p. 4). Nowadays, we may say that the fundamental idea was wrong: mathematics cannot be reduced to logic. Still, we can say that the logicist approach was very fruitful: the approach inspired research and brought together different kinds of researchers.

The more general idea behind the development of logic was the Leibnizian idea of universal language (*lingua characterica*), which was shared among the logicians of the “golden age”. The “golden age” of logic was a proper golden age; the development of logic and mathematics was something remarkable. The names like Frege, Hilbert, Russell, Carnap, Gödel, Tarski, and Genzen give an impression how rich the development in logic and in mathematics was at that time, and Frege and Russell can be seen as the founders of the modern logic.

Russell was a foundational researcher in the emerging modern logical theory. He knew exactly the ethos of modern empirical philosophy, and his logic and philosophy also had a foundational role in the emergence of this new empirical philosophy. However, at the same, Russell was anchored in the old philosophical tradition, his philosophical roots in the (criticism of) Kantian philosophy. His philosophical orientation can be seen very clearly in *The Principles of Mathematics,* which are very clear from the structure of the book[i]. Russell sees logic as a certain kind of natural science: “Logic, I should maintain, must no more admit a unicorn than zoology can; for logic is concerned with the real world, just as truly as zoology, though with its more abstract and general features” (Russell, 1929, p. 169).

Frege’s philosophical roots are in the tradition of universal language. Logic was for him the language “in the sense that, for him, something could be said if, and only if, it could be said in that very language" (Haaparanta, 1986, p. 159). His two-dimensional logical notation was pictorial and, hence, intuitively very attractive. However, the notation is very unpractical: it becomes very difficult to *see* when we consider longer sentences (see Frege, 1979). The linear notation introduced by Peano became the prevailing notion, and was used by Russell and Whithead in *Principia Mathematica*. Even if Frege never developed an explicit theory of semantics, his (semantical) analysis of language, based on his analysis on the notions of *Sinn* and *Bedeutung*, is extremely deep. The rejection of the possibility of the explicit theory of semantics is based on his opinion that it is not possible for us to look at the language outside of the language. (Haaparanta, 1986, 41). This opinion was later shared, for example, by Wittgenstein. Moreover, Russell's theory of definite descriptions is syntactic, but the intention is semantic (Hintikka & Kulas 1985, pp. 33-34). So, it is possible to agree with Wheeler (2013, p. 293), when he said: "One would be hard pressed to overestimate Frege's impact. His term logic and the invention of the predicate calculus (1879; 1893) revealed a rich, yet unified structure behind complex, quantified sentences of mathematics, and this breakthrough in logic opened the way to rigorously analyzing the meaning of mathematical statements and mathematical proof."

There was a great deal of belief in the possibilities and the power of growing logic. Gödel (1931) proved his famous and shocking incompleteness theorem for first order logic. The paper in which the theorem was proved is extremely important; it introduces several new and essential mathematical notions. For example, the method of Gödel numbering made it possible to speak about mathematics within mathematics, i.e., it made the metamathematics part of mathematics itself. The proof constructs a sentence which says that it is true but not provable. The proof clearly shows in which sense mathematical proofs can be constructive, and moreover, the theorem was something unexpected: it crushed Hilbert's original program (Nagel & Newman 1989; Hintikka, 2000).

After Gödel's result, logicians managed to formalize the notion of computability. In the 1930s, several different formalizations of the notion emerged, namely *recursivity* (Gödel, Kleene, Herbrand), *λ-definability* (Kleene, Church, Rosser), and *Turing machine computability* (Turing, Post). It was especially interesting was that all these were proved to be coextensive, which has been the basis for the *Church's thesis,* which says that an intuitive notion of computability can be identified with the notion of recursivity. Church's thesis cannot be proved, since it interconnects a nonlogical notion of intuitive computability and a logical notion of recursivity. However, the notion of computability allowed for logical proofs that prove something not-computable. In fact, the class of non-computable functions has proven to be an extremely interesting area of study (Mutanen, 2004).

The semantical or model theoretical approach has been developed extensively since the 1930s, with Carnap becoming one of the founders of the model theoretical approach. Tarski, in his papers 1933 and 1944, formulated a logico-mathematical notion of truth, which was intended to explicate the Aristotelian notion of truth. The Tarskian notion is, nowadays, known as an explication of the correspondence theory of truth (Hodges, 1986). The history of the model theoretical approach can be seen as anchored in independence and definability results in logic and in mathematics. Padoa's principle states that a predicate is not definable in a theory, if it is possible to give two different interpretations to the predicate, while all the other non-logical constants of the theory have the same interpretation. The explication of non-Euclidean geometry was a similar model theoretic proof that parallel axiom is independent of the other axioms of geometry. The modern model theoretical approach has been developed by researchers like Carnap, Tarski, but also by Löwenheim, Skolem, Henkin, and Beth. However, there is no proper disagreement between proof theoretical and model theoretical methods within first-order logic: Gödel's completeness theorem shows that a sentence is provable if, and only if, it is valid.

The difference between syntactical (proof theoretical) and semantical (model theoretical) methods is very important to keep in mind. Even if in school teaching calculating, and hence syntactical methods, are emphasized, model theoretical methods are also introduced. Maybe it could be reasonable to highlight the methodological approaches more systematically. This could enrich the conceptual understanding of mathematics and logic. The approach we are formulating in this paper gives an example of such an enrichment.

Mathematics and logic are heterogeneous disciplines in which there are several different kinds of approaches present. To get a better picture we have to consider mathematics and logic "from outside". However, this task is not so straight-forward, because it leads us to one central mathematical and logical problem: the character of metamathematics. This leads us to the lines of thought that are central for the argumentation in this paper.

LOGIC AS CALCULUS AND LOGIC AS LANGUAGE

The formal character is present in modern mathematical and logical theory, which can be seen from the works and journals of logic and mathematics. Even if logic and mathematics are expressed in different kinds of formalisms, logic and mathematics are not merely a formal game of the symbols on paper. Hilbert's famous characterization of mathematics, as a mere game played by simple rules with meaningless symbols on paper, must be understood within his more general philosophical view of mathematics. Hilbert was interested in problems of metamathematics, and his intentions were almost the converse to that of Wittgenstein.

Wittgenstein imbedded the problem of mathematics in his more general philosophy of language, when he asked the question: "Is mathematics about signs on paper?" The answer he gives is "No more than chess is about wooden pieces." (Wittgenstein, 1988, p. 290) According to Wittgenstein, mathematics is a certain kind of activity or a certain game to be played. It is not possible to take a look at the fundamentals of the game, that is, there is no metamathematics which could tell us about what mathematics really is, and it is not possible to look at the mathematical game outside of the game itself; we are bound just to play the game. That is, the only way to get to know mathematics is just to do mathematics. The meaning of the mathematical notion cannot be found from the result, but rather to understand the meaning, one must look at the proof, "the calculation actually going on in the proof" (Wittgenstein, 1988, pp. 369–370).

To get a better grasp let us consider the following distinction made by van Heijenoort (1967): [1] logic and mathematics as calculus and [2] logic and mathematics as language. The very idea is that if we understood logic and mathematics as calculus then it would appear to be interpretable and reinterpretable over and over again. The possibility of interpreting over and over again provides a great deal of practical freedom: a mathematician or a logician can decide which kind of interpretation he or she chooses, and this interpretation is developed systematically in model theory. On the contrary, mathematics and logic can be understood as language, that is, as a language with a fixed interpretation. Thus, logic and mathematics as language are languages which speak about the reality, as Russell characterized mathematics to be above. In fact, Hilbert's characterization of mathematics as a game is a game in the sense of the calculus; and for Wittgenstein, the game is in the sense of language.

The taxonomy given by van Heijenoort can be generalized as a whole language as Kusch (1989) demonstrates. The taxonomy is based on very fundamental philosophical presuppositions, which are not easily recognized. In particular, the philosophical presuppositions behind mathematics and logic are extremely difficult to recognize. Moreover, as fundamental philosophical presuppositions, they are orientating principles rather than explicit statements or norms (Hintikka, 1996).

Independently on the philosophical orientation, as Wittgenstein said, "calling arithmetic a game is no more and no less wrong than calling moving chessmen, according to chess-rules, a game" (Wittgenstein, 1989, p. 292). Wittgenstein interconnects mathematical and chess games, but at the same time, he brilliantly separates mathematics and chess from a game of billiards: "A billiards problem is a physical problem (although its solution may be an application of mathematics). (..) a chess problem is a mathematical problem" (Wittgenstein, 1989, pp. 292-293). The characterization of mathematics as a game does justice to mathematics as a dynamic computation process, which was explicated in Turing's formulation of computation (Turing, 1936).

In Turing machine computation, the starting point is a known (and usually solvable) problem, for example, what is the sum of given numbers. However, in mathematical and logical reasoning, we do not merely consider these kinds of well-defined and answerable problems; even if they seem to be over-represented in school mathematics. Mathematics is, essentially, something more than mere computation or merely following given rules. These rules allow us to formulate constructive proofs and this constructiveness is related to the demonstrativity of mathematical and logical reasoning. The strategies of mathematical and logical reasoning are the most important things to learn, in order to understand mathematical and logical reasoning. Moreover, strategic aspects are central, when mathematics and logic are applied in different fields of sciences (Hintikka & Kulas, 1985, ch III p. 17).

CONSTRUCTIVE METHODS

The philosophical background of constructive philosophy is very deep. In *Meno,* Plato demonstrates a dialectical method, which is a marvelous example of epistemic construction in which dialog proceeds via questions and answers. These questions and answers build up the knowledge of the learner (the answerer) in a factual manner. The teacher (questioner) has a strategic map of the learning situation. The dialog is extremely rich and one can find all the central aspects of constructive learning and teaching from the text. The discussion in philosophy and in pedagogy, based on *Meno,* is still going strong.

It is not obvious in what sense mathematical and logical reasoning are constructive. Carnap (1969, p. 152) says that “The basic language of the constructional system is the symbolic language of logistics. It alone gives the proper and precise expression for the constructions; the other languages serve only as more comprehensible auxiliary languages.” For Carnap, the foundation of constructions is in phenomenalism: “In this book, I was concerned with the indicated thesis, namely that it is, in principle, possible to reduce all concepts to the immediately given.” (Carnap 1969, p. vi) However, constructive philosophy does not presuppose commitment to phenomenalism or to any other ism.

Perhaps the best example of construction in mathematics can be found in geometry. Elementary geometry is known to be decidable, which means that there is a (computable) decision method for the geometry. This does not imply that it would be a trivial or a mechanical task for generating proofs in elementary geometry. One excellent example in which the geometrical constructions and their knowledge-providing character becomes evident is the slave boy example in Plato’s dialogue *Meno*.[ii] In the dialog, Socrates directs the reasoning process of a slave boy by his questioning method. The reasoning is based on the drawings made on the ground during the process. The dialog shows how these drawings increase the slave boy’s knowledge. These drawings, together with general geometrical knowledge, construct the intended result. This knowledge construction process is essential in all mathematical and logical reasoning.

The conclusion of the reasoning in *Meno* is a geometrical theorem. The proof of the theorem is a strategic search for the information needed in the proof. The strategy is realized by the Socratic questioning method. However, in the end, anyone who has followed the construction *sees* the result; that is, he or she understands the theorem and, hence, *sees* the truth of it. In fact, the Socratic method used in the dialog demonstrates a general pedagogical paradigm which can be used – and has been used – in any teaching.

The Socratic questioning method brings up the strategic level of mathematical reasoning. The questions Socrates asks are motivated by a strategy that directs the reasoning towards the intended conclusion. What knowledge is needed to lead such a process? How do such processes take into consideration the learner's level of knowledge? The process is a step-by-step process, in which each step is made obvious by giving the information needed – the questioning-answering method is designed to guarantee the success. What about the teacher's knowledge? The Socratic irony refers to idea that Socrates, in fact, knew, but he feigned being unknowing. However, there is no need to have full knowledge before the teacher leads the reasoning process; what he or she has to have is good a methodological knowledge of the problem setting. Hence, methodological knowledge is a solution to Meno's paradox (Hintikka, 2007; Kelly 1996). The explicit presentation of the reasoning process with the pictures and formulas makes the reasoning process observable. Hence, the entire audience can follow the reasoning and infer the same conclusion for himself or herself. That is, mathematically reasoned knowledge will become transmissible by such an explicit and public process (Hendricks, 2001; 2010).

The increase of geometrical knowledge in the example in *Meno* can be related to a more general problem of knowledge transmissibility. In fact, the argument shows that such geometrical knowledge is a paradigmatic example of transmissible knowledge. The reason is methodological: geometrical knowledge is constructed during the reasoning process, in a step-by-step manner. In fact, this observation can be generalized to all mathematical and even certain kinds of empirical reasoning. The pedagogical aspect of the dialog is that Socrates asks the question in a way that allows the slave boy to understand the questions and find the answers himself. So, all the steps become constructively known by the slave boy. This kind of explicit knowledge acquisition process can be followed and reproduced. Moreover, as Hendricks (2010) shows, knowledge transmissibility is closely related to public announcement that explicitly take place in the strategically led discussions like Socrates and the slave boy had in Plato's *Meno*.

The idea of the constructions is to take more and more new individuals into consideration and look at their relations to other individuals. In intuitionism, the constructive method has been an essential part of logical reasoning: "In practice, the most important requirement of the program of constructive proof is that no existential statement shall be admitted in mathematics, unless it can be demonstrated by the production of instance." (Kneale & Kneale, 1962, 675) The observation was generalized to a geometrical method of analysis and synthesis by Hintikka and Remes (1974). They characterize geometrical analysis as follows:

"Speaking first in intuitive terms referring to geometrical figures, an analysis can only succeed if, besides assuming the truth of the desired theorem, we have carried out a sufficient number of auxiliary constructions in the figure in terms of which the proof is to be carried out. (…) This indispensability of constructions in analysis is a reflection of the fact that in elementary geometry, an auxiliary construction, a *kataskeue* (…), which goes beyond the *ekthesis* (…) or the 'setting-out' of the theorem in terms of a figure, must often be assumed to have been carried out before a theorem can be proved." (Hintikka & Remes 1974, p. 2)

Geometrical constructions bring new geometrical objects into the reasoning process, and these new objects increase the information used in reasoning. This can be generalized into logical reasoning by observing that the geometrical objects behave similarly to individuals in logical reasoning. These new individuals increase the information, which can be precisely defined and even measured. The definition of the increased information is based on the number of interconnected individuals in the reasoning. The number tells us the depth of the argumentation, and it can be shown that an increase of the depth increases the logical information. Hintikka (1973) based his definition of surface tautology and depth tautology on this measure:

"Depth information is the totality of information that we can extract from a sentence by all the means that logic puts to our disposal. Surface information, on the contrary, is only that part of the total information which the sentence gives us explicitly. It may be increased by logical operations. In fact, this notion of surface information seems to give us, for the first time, a clear-cut sense in which a valid logical or mathematical argument is not tautological, but may increase the information we have. In first-order logic, valid logical inferences must be depth tautologies, but they are not all surface tautologies." (Hintikka, 1973, p. 22)

EMPIRICAL REASONING

Logical reasoning is theoretical in the sense that it can be done by paper and pencil. The results of such reasoning are statements. Such reasoning should be separated from empirical reasoning: "This means that all talk about construction, including the construction postulates, is inappropriate, for it is about *doing* things, whereas, in fact, geometry is a *theoretical* discipline that treats eternal things. Since, what Plato criticizes is just the "language" of geometers, it does not mean that all the geometer's concern with construction problems could be excluded from geometry as a science, rather, they should be reinterpreted as *theoretical* statements." (Stenius, 1989, p. 78)

If we use Hintikka's (1973) notions, we can say that this kind of theoretical reasoning is part of the indoor games. However, there is need for the logical analysis of empirical and experimental reasoning. To carry out this task, Hintikka (1973) introduces outdoor games, which are games of seeking and finding in reality. There is no essential methodological difference between indoor and outdoor games. In fact, this close interconnection is already recognized by Newton: "It is the use of the method of analysis as a model of experimental procedure of the great modern scientists, notably by Newton." (Hintikka & Remes, 1974, p. xvii)

The analysis of experimental reasoning shows that this kind of model-related logic can be a realistic reconstruction of experimental (and empirical) reasoning. Thus, we can understand why Hintikka and Remes (1974) say that:

“We do believe that in a very deep sense, Newton really practiced what he preached, and that his methodological pronouncements present an interesting general model of the experimental method at large. We have come to realize that both these claims, also the historical one, need further argument and further evidence, before we are prepared to rest our case. (…) In the case at hand, the need and unpredictability of auxiliary constructions in analysis shows once and for all that in spite of its heuristic merits, the method of analysis just cannot serve as a foolproof discovery procedure.” (Hintikka & Remes, 1974, p. xvii)

In a similar manner to how geometrical reasoning can be generalized as logical reasoning, this Newtonian reasoning can be generalized as general experimental reasoning. The theoretical foundation is the interrogative model of inquiry developed by Hintikka. The interrogative model of inquiry shows how the usual experimental reasoning can be constructive, just as logical, mathematical, and geometrical reasoning are, whilst the constructive aspects connect the interrogative model to present day discussions about causality (Woodward, 2003). The difference between the interrogative model and logical theory is in the character of the forthcoming information. With both logical and experimental reasoning, the intention is not to find singular facts or to generalize universal laws from given sets of data, but to understand the mechanisms underlying the phenomena (Hintikka & Kulas, 1985, p. x). However, the underlying logic is the same usual logic and, in particular, logical and experimental reasoning are strategically parallel (Hintikka, Halonen & Mutanen 2002).

CONCLUDING REMARKS

We have seen that there are several different interpretations of mathematics and logic which are not compatible. This is not something that should be denied or avoided. Rather, it is a symptom of the richness of the content of mathematics and logic. The heterogeneity of mathematics and logic cause polysemy into the field, which may occur in practical problem solving situations. This is challenging for teachers and researchers, but the challenging situation makes mathematics and logic extremely interesting topics to study. As we have seen, it is possible to find rich interpretations which formulate a holistic picture of the field of study and which allow for open discussion together with other interpretations.

NOTES

1. To fully comprehend this, please take a look at the table of contents of the book.

2. To see more detailed analysis of the example, see Hintikka & Bachman 1991 pp. 20-28.

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Oral Presentation

HISTORICAL EPISTEMOLOGY: PROFESSIONAL KNOWLEDGE AND PROTO-MATHEMATICS IN EARLY CIVILISATIONS

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This paper has been prompted by the work of Hoyrup (2004) and Netz (2002) that we should be cautious of the conceptual categories we use to investigate the past. Lloyd (1990) showed that a 'mentality' ascribed to a cultural group was untenable, and both Hoyrup and Netz discuss the action of using a tool as extended from physical to cognitive and theoretical objects. Regarding historical epistemology as a process of investigating the dynamics of proto-scientific activity led to considering the Vygotskian (1978) concept of tools-in-action leading to mental functions and Gibson's (1977) Theory of Affordances to consider early people as active agents embedded in a multi-layered socio-ecological environment. Some brief notes on pre-classical civilisations indicate particular points of focus.

INTRODUCTION

Pre-Classical Contemporary Historiography

Studies in pre-classical mathematics were often regarded as 'primitive' efforts toward the more sophisticated and elegant mathematics of Greece, rather than valid explorations of ideas within specific socio-cultural contexts. However, from 1959, a series of papers by Abraham Seidenberg on the ritual origins of mathematics appeared (1959, 1962a, 1962b,) which, while dismissed by some researchers as largely fanciful and unreliable, contained references to aspects of the so-called primitive world that were based largely upon cultural and religious studies, using translations of Sanskrit and other ancient texts,[i] these being the only 'non-mathematical' evidence available at the time. More recently, the critiques led by Unguru (1975) Hoyrup (1994) and Netz (1999) and followed by more consolidated accounts of Babylonian, Indian, and Egyptian mathematics (Robson, 2008; Plofker, 2007; Imhausen, 2003a) show that some of Seidenberg's opinions may have been justified, and contemporary historiography now addresses wider contexts through extended forensic interpretation of ancient texts and artefacts, using techniques from areas such as archaeology, social anthropology and linguistics. Combined with sensitive and broad knowledge of the development of past mathematics the mode of historiographic research has changed considerably in the last twenty years, offering us deeper insights into the pre-classical past.

Historical Epistemology

Epistemology makes a distinction between methods of discovery and methods of justification: that is, the way one discovers a property of a situation or a relation between objects and how a conjecture becomes a *mathematical* truth may be quite different from how it is later justified, accepted by a community, and established as a truth. We address questions like, 'what are the methods and grounds for such discoveries;' 'what is the role insight plays in these discoveries;' and 'how do interconnections between mathematical concepts lead to discoveries'? Because there are no meta-mathematical discussions in the texts made by the actors about what they are doing and why, one answer is to regard historical epistemology as investigating the *dynamics* of proto-scientific developments, insofar as they can be extracted from an analysis of texts and practices. Any analysis of the development of mathematical ideas necessarily calls for a serious approach to the social and cultural contexts and the physical environments in which the knowledge in question was generated, and should attempt to answer questions about the motivations and means (practical and theoretical) available to the agents involved. Hitherto, researchers have been looking at *mathematical*[ii] texts; namely those considered to have been written to teach or learn mathematics, but they were looking at a group of texts *already categorized in a particular way* – because they 'look like' contemporary mathematics (or parts of it) they were identified as somehow the 'same' (e.g. as equations), but neglected texts that speak *about* mathematics, or show practices being developed in some way might suggests a 'mathematical – like' activity.

Hoyrup, (1994) suggests that *sub-scientific* knowledge[iii] arises out of cultural practices, rather than taking 'already-known' knowledge being 'applied' to a problem, this knowledge can be found in craft skills like making tools, in rituals, emergent astronomy, astrology, or knowledge arising in the making of, or building significant objects by a particular group or individual.

Mathematics has not always been the 'same' because, at different periods, different kinds of mathematics were possible. The contexts, tools, and motivations, were different, and many of the problems were concerned with the immediate needs; ritual, social, and economic of the people involved. Much of this happened before there was any need for recording, since societies developed ways of handing on ideas in myths, storytelling, poetic algorithms, and practices not recognised as 'scientific' by former historians.

PRE-CLASSICAL CULTURAL STUDIES

Egypt

The work of Annette Imhausen (2007, 2003a,b) shows that original mathematical texts on papyrus, leather, or other materials are extremely rare[iv] and so it is very difficult to make an overall assessment of the Egyptian mathematical corpus. Earlier accounts of Egyptian mathematics (Gillings, 1971; Clagget, 1999) contain similar subject matter presented in contemporary mathematical terms, and consequently their commentaries have been limited by their preconceptions.[v] Apart from the more substantial Rhind and Moscow Papyrus and the Leather Roll, and some fragments of mathematical problems, there is scarce little else. However, other Egyptian texts contain a variety of everyday contexts, and their problems of calculating rations, granary volumes, the daily baking, brewing and herding activities are recorded on a variety of documents, not all obviously mathematical.[vi] Difficulties lie in the technical terms that give clues to different kinds of concepts, so that establishing the social contexts of the texts involved is problematic. (2003b: 371-373). There are administration texts that show the application of techniques, and tomb representations of everyday tasks where the actual procedure is ambiguous, but knowing the administrative, economic, and practical contexts helps to understand the problems that draw on the professional lives of scribes and use their terminology, and techniques. Imhausen shows how difficult it is to translate everyday colloquial Egyptian (2003b: 374) where not all processes describe the same steps, and although these steps may be followed, the sense of the problem is obscure because we lack information about how individual objects are actually made and we do not fully understand Egyptian scribes' own conceptions of their mathematical world. It is difficult to find out what parts of a procedure were 'sub-scientific' or 'proto-mathematical' since the true social context is not clearly known. What is needed is more information about the role of mathematics in Egyptian culture, and the satirical text *Papyrus Anastasi* (Imhausen 2007: 10-11) while not a *mathematical* text, is discussed in detail, showing how important mathematics is for Egyptian scribes[vii]. The mathematical problems found in *Anastasi* do not contain all the data needed to solve the exercises, but if they come from a well-known body of scribal mathematical tasks, the account of the type of problem was enough for an Egyptian reader to know the relevant group of problems.

There are some 14 diagrams in the ancient Egyptian *mathematical* corpus, but as yet, there are no systematic instructions for constructing them. We do not know their representational conventions, nor how those conventions relate to other aspects of Egyptian culture. There are tomb paintings representing a ritual from the Book of the Dead (Taylor, 2010) showing the use of a simple beam balance for weighing the heart of the dead person against the 'feather of truth'. If the balance reaches equilibrium, the dead person can enter Paradise. Clearly, the balance has to be accurate, and the technique employed in this ritual has to be managed by the priests, so that the outcome is beneficial (Seidenberg & Casey, 1980). On the other hand, it is apparent from a few problems (Rhind 24-29) that proportional reasoning was used in the arithmetic, and while much of the text is written as arithmetic, the conceptual background may well have been mechanical or geometrical. Having realized that other sources are relevant to a fuller understanding of the mathematics, it will take some time to collate the evidence and relevance of wider aspects of ancient Egyptian life.

Mesopotamia

This was an area where early translations regarded many of the mathematical problems to be 'the same'; or at least similar enough for researchers like Thureau-Dangin, Bruins and Neugebauer[viii] to recognise elements in the texts that enabled them to translate the problems into contemporary mathematical terms. Since then, excavations in Mesopotamia have been more extensive in chronological period and cultural areas, and new methods used to investigate not only the *mathematical* texts, but much more extensive research into the social and cultural contexts of the people living in that area. These results are now found in Nissen, Damerow and Englund (1993), Hoyrup (1994, 2010), and in Robson (2008) so that we now have extensive and detailed descriptions of different sources supporting the development and use of mathematical concepts and procedures, as well as much deeper knowledge of the languages employed.

India

The Vedic people who entered North West India were responsible for the earliest extant texts known as the Vedas, the oldest scriptures of Hinduism that became a recognised corpus of Sanskrit literature before the middle of the first millennium BCE. These texts contain hymns, formulas, and spells for rituals, and part of these were the *Śulba Sūtras* that provide the 'cord-rules' for the construction of sacrificial fire altars (Seidenberg 1962a). They give the instructions for building brick altars used in ritual sacrifice. Most mathematical problems considered in the *Śulba Sūtras* spring from a single ritual requirement; namely that of constructing altars that have different shapes but occupy the same area. (Plofker 2009: 13-28). How this ritual geometry became integrated into the process of sacrificial offerings is unknown, the rules may have emerged through trying to represent cosmic entities physically and spatially, or perhaps existing geometric knowledge was incorporated into ritual to symbolise some universal truth about spatial relationships. Whatever their deep motivations, the basic tools were simply 'peg and cord' to make arcs of a circle. The origins of this geometry can be seen in the use of the shadow-stick gnomon to set the East-West equinoxes and record the daily passage of the sun (Keller, O. 2006: 28-41). Many of the instructions contain transformation rules for preserving areas, such as changing a rectangle into a square of the same size, and one of the elementary perceptions involved are the proportional properties of the right angled triangle and the division of a rectangle by a diagonal. (Keller, 2006: 125-166) (Plofker, 2009:13-42) From the instructions in the texts, the following diagrams (Figs. 1a and 1b) show that removing the small red square from the large blue square produces the equal areas of the blue rectangle and the green square.

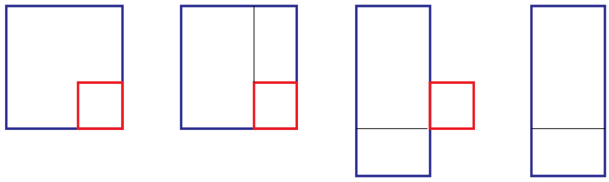

Fig 1a

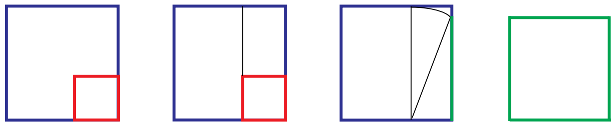

Fig 1b

The ritual is preserved in a film made by Frits Staal, now available on the internet[ix].

China

Divine origins of arithmetic and geometry are repeatedly stressed in the earliest records we have of Chinese history. The earliest Canonical[x] account of the past is the creation of a socio-political order by a human ruler in the *Shu Jing*, *The Book of Documents* (c. 500 BCE) traditionally compiled by Confucius from earlier sources where the legendary Emperor Yao commissioned two star-clerks Xi and He, to "accord reverently with august Heaven and its successive phenomena, with the sun, and the moon, and the stellar markers, and thus respectfully to bestow the seasons upon the people." (Cullen 1996: 3) The times of the summer and winter solstices and the spring and autumn equinoxes were recorded, eventually establishing a solar calendar of about 366 ¼ days. Chinese mathematical astronomy appeared as a functioning system in the Han Dynasty (c. 207BCE - 220CE) where the use of a carpenter's square, a compass, and other astronomical tools were already well practiced. Chinese astronomers had to deal with a luni-solar calendar, having to work with months that followed lunations quite closely, as well as keeping a civil year of a whole number of months in step with the seasons, and months in step with a cycle of a 'week' of 10 days (Cullen 1996: 7-26). The calendar had an important *Ritual function* because the Chinese emperor was responsible for looking after the people, as well as the world order, and disorder in nature was a sign of a malfunction of the human order thereby causing criticism of the emperor's rule. The astronomer's task was to reduce as many phenomena as possible to rules and thus to predictability, so that state rituals should be carried out at the proper times, and if mistimed or not performed correctly, harm could come to the population. It was therefore expected that the motions of the visible planets should be tabulated in detail, and that lunar eclipses, and other phenomena should be predicted. Irregular events could be ominous; comets, meteor showers, and novae could predict disaster. Almanacs were published to detail these events, and virtually all activities had to be considered in terms of the calendar. As yet, there is very little in translation about mathematical techniques in the centuries before the Han, but the use of the *Gnomon* and plumb line as essential tools must have appeared well before the written data, and the 'out-in' principle in geometry appeared as a very powerful method of using the idea of equivalence of areas.[xi] The diagram (Fig. 2) where the upper and lower areas are equal is visually 'obvious' was well established in the *Zhou bi suan ji*ng in the Zhou dynasty (1046 - 256 BCE) (Cullen, 1996).

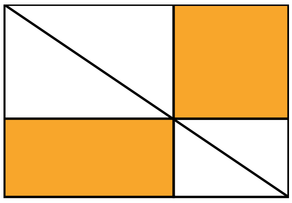

Fig. 2.

Earlier practical examples of these ideas have so far not been found, yet in the earliest known bamboo text *Suan shu shu – A Book on numbers and computation* put together in the Qin dynasty*,* (221-186 BCE) which is a collection of problems showing examples of the use of fractions, distribution of goods using proportion, and problems on areas of fields, shapes and volumes. (Cullen, 2004). Applications and developments of these algorithms are found in more detail later in the *Nine Chapters* (Chemla & Shuchun 2004) where, in particular, we find the dissection of a cube into three square pyramids (388-406, Diagram 5.15). Fig 3

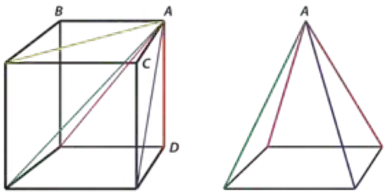

Fig 3

We have learnt to be much more cautious in our interpretations of ancient evidence, but also realise that many of the ideas found in the well-documented activities in Mesopotamia, India, Egypt and China indicate that at the most basic level; namely ideas of ratio and proportion, dissection and rearrangement, and area conservation, that appeared as professional knowledge of the scribe, priest, or shaman, be it 'sub-scientific' or 'proto-mathematical', in some of the earliest activities in human development.[xii]

INVESTIGATING MATHEMATICS PAST AND PRESENT

Jens Hoyrup: Canons and Taboos (2004)

Hoyrup (2004) addresses a perennial problem: were historical concepts really different, and the historical actors unable to think or express themselves in our terms, or is everything just a question of terminology and notation? Citing the example of

Unguru's (1975) paper on rewriting Greek mathematics showed that historians of mathematics, being mathematicians, tacitly assumed that mathematical entities were Platonic, sharing the same ideal forms, having no connection with the thought of the individual or the historico-cultural context into which the historical writer's ideas were being broadcast. Hoyrup points out that this kind of debate is unduly simplistic, and that more careful reading of early sources indicates that early mathematical writers might have other reasons than inadequate conceptual capacity or unhelpful terminology to be able to express themselves in ways different from our own.

By deconstructing the idea of a "mode of thought" as intangible[xiii] that "does not in itself assist us in understanding whether, why or in which respect this mathematics differed from ours". Hoyrup suggests that talking about the *mathematical concepts* of a culture is less elusive, but we should not identify a concept with the words that are used to describe it. He describes a mathematical concept as a *mental tool* that is being used for specific operations, together with the connected network of concepts and their properties, characterises a particular 'mode of thought' (Hoyrup 2004: 131)

Hoyrup's metaphor is very similar to the ideas of Vygotski who proposed that just as physical tools extend our physical abilities, mental tools extend our mental abilities, enabling us more able to solve problems. Before we learn to use mental tools, learning is largely controlled by the environment; being a matter of reaction to various stimuli. Once we learn to master mental tools, we become learners, who by attending and remembering in an intentional and purposeful way, can transform cognitive behaviors, and also use other mental tools to transform our physical, social, and emotional behaviors. These mental tools can also transform our minds, leading to the emergence of *higher mental functions*. This idea first appeared in Vygotski's *Thought and Language* (1962) and was further developed in his *Mind in Society* (1978). Mental functions are cognitive processes acquired through learning and teaching within a system of practices common to a specific culture. Hoyrup points out that structures of mathematical operations emerge from operations with physical tools or particular cultural practices: (for example, bamboo sticks, or tokens on a counting board, using a dust abacus, or practicing routines for accounting, or solving equations). These practices are never identical with the abstracted mathematical structure because *the mathematical structure is essentially abstract.*

> "it cannot be excluded that mathematical conceptual structures that are fairly congruent with something *we* know grow out of manipulations of tools that are quite different from those from which we are now accustomed to see them evolve. Identifying underlying tools that differ from ours does not prove that the corresponding concepts were also fundamentally different." (Hoyrup 2004: 133)

After offering a variety of examples from Egyptian and Old Babylonian culture, he concludes,

> … that much of what the texts do not say or do not do must be explained, not from what their authors *could not think* but instead in terms either of *what they did not*

> *find it professionally fitting to say, or what they found it incoherent to say.* (Hoyrup 2004: 142)

Practices can be built up by a powerful economic or political clique and defended to exclude others, or to preserve a practice in the face of modification or opposition. Thus the role and social status of the scribe in ancient Egypt, as evidenced by the Egyptian *Papyrus Anastasi,* or the concentration of the Chinese astronomers on prediction of events to defend their status, or the priests maintaining the rituals of the *Sulbautras*, all provide examples where certain practices may be built up and maintained for reasons other than immediately useful or 'scientific', so that

> … the absence of such conceptualizations from ancient sources as a modern mathematical reader might expect to find there does not prove that the ancient authors *were not able* to think more or less in our patterns – it may also be due to an explicit rejection of this way of thinking, either because of the existence of some canon or because they deemed it conceptually incoherent. Only close analysis of the sources at large will, in the best of cases, allow us to distinguish between cognitive divergence and cognitive proscription." (Hoyrup 2004:144-45)

Reviel Netz: It's not that they couldn't.

In this paper, Reviel Netz (2002) takes up the discussion by demonstrating that transforming an old piece of mathematics into its *contemporary equivalent* is misleading, because it conceals the idiosyncratic features of the old mathematics that prevented it from becoming contemporary mathematics. He insists that it is not just a matter of notation.

It is difficult to see what is meant by 'similar' or 'equivalent' here.

> "The standard example - the equivalence of Euclid's *Elements* II with algebraic equations - seems to suggest a meaning of 'equivalence' along the following lines: historians of mathematics often take two theorems to be equivalent when, from the perspective of the modern mathematician, the proof of any of the theorems serves to show, simultaneously, the truth of the other." (Netz 2002: 264 footnote)

In my opinion, this means at the very least, being able to make the mental transformation from one medium to another, and 'seeing the algebra in the geometry', which comes from ignoring the true socio-cultural context of the work.

Netz (2002: 265) also cites Unguru's (1979) response to the opposition led by van der Waerden (1976) involved wide-ranging historiographical and philosophical comments that conclusively settled the argument. At the heart of Unguru's reply lies his critique of Jacob Klein's (1968) *Greek Mathematical Thought and the Origins of Algebra,* where Unguru claimed that Greek mathematics could not be interpreted to be the same as modern mathematics, because the Greeks did not possess the right kind of concepts: for algebra, *one needs 'second-order' concepts that refer to other concepts, but the Greeks had only first-order concepts, referring directly to reality.* (Italics, mine)[xiv] Thus, modern mathematics, in the Greek context, was *conceptually impossible*. In Klein's work, concepts constitute the 'mode of thought' of a group of individuals, without any historical account of why their mental possibilities should be limited in a particular way. So the historiography of conceptual structures is no more than a version of the history of mentalities. The idea that Greek science could be explained as expressing an abstract mentality was attacked by Lloyd (1990), by showing the contradictions and inconsistencies when particular mentalities are assigned to an individual, a group, and even to a whole nation.

Netz draws on several studies of Greek mathematics where exceptions to the rules appear, claiming differences in mathematical *practice*[xv]. While examples may be rare, they exist, and taken together, they show the inadequacy of the argument from conceptual impossibility. Furthermore, Netz points out that there are extra-mathematical concerns influencing the definitions of unit and number (Euclid, VII. 1-2). Definitions interact with the intellectual world where they serve philosophical goals; so Netz claims that readers of *Elements I* would feel that a discussion of what'points' were, was philosophically necessary, so Euclid put the definition, 'a point is that which has no part', at the beginning of his work. Furthermore when, in the *Sand Reckoner* Archimedes introduced a new numerical system for a non-mathematical audience, it was necessary to name, in a natural language sense, extremely large numbers, Netz (2002: 277)[xvi].

Netz refers to Hoyrup's use of concepts as tools and maintains (2002: 282) that it is clear that as people produce artifacts, and have recourse to several tools that are culturally available, so the *action of using the tool* can be extended from material objects, to cognitive and indeed theoretical objects.[xvii] The possibilities opened up by a tool, whether physical or cognitive are considerable, and unpredictable. The unsuspected possibilities of applying a tool to a task set up an interaction between the tool and the task itself, with often unpredictable results. Many values influence any particular activity, and depending on the different value, different practices might arise. For the sake of efficient calculation, typographic representations of numbers are preferable; for the sake of proximity to natural language, verbal representations are preferable. Practices are determined not by the *totality* of values brought to bear, but by the *most important* of such values: the value of efficient calculation was important to Archimedes, but in the context of a literary treatise, it has an even more important value for a non-mathematical audience, that of proximity to natural language. So Netz puts forward the following explanation of the non-arithmetical nature of Greek mathematics:

> Greek literary production is marked by a hierarchy of values always related to a certain 'literary' or 'verbal' preference: literature is ranked above science, inside science philosophy is ranked above mathematics; persuasion (to the Greeks, the central verbal art) is ranked above precision, and natural language above other symbolic domains. Hence it is easy to understand Euclid's deference to philosophy in his definition of number. More significant, inside Greek mathematical writings, the qualitative statements of geometrical demonstration become the norm against

> which arithmetical representations of the same object come to be seen as marked. (Netz 2002: 287)

Thus reinforcing Hoyrup's point about other external influences from practices built up by tradition or a powerful economic or political clique that are defended to exclude others, or to preserve a practice in the face of modification or opposition.

ASPECTS OF HUMAN COGNITION

Gibson's Theory of Affordances.

Gibson's principal idea is that cognition is not isolated from all the other attributes that may influence a learning agent at a particular time, place, or context. It is a way of looking at cognition that considers *an active agent embedded in a multi-layered socio-ecological environment.* Gibson first applied his theory to psychology as 'A way to understand how learning takes place through perception of, and interaction with, an environment' (Gibson, 1977), where he conceived the individual actor (animal or human) in a general ecological environment, and considered the options available in terms of possible awarenesses, perceptions and their consequent actions.

The *Affordances* of an environment are what it *offers* the agent, what it *provides* or *furnishes;* the consequences of which could be good or bad. As an affordance of support for a species of animal, they have to be measured *relative to the animal*. They are unique for that animal or agent, and not just abstract physical properties. Affordances have a unique unity relative to the nature, physical attributes, and behavior of the agent being considered. Affordances are "action possibilities" latent in the environment, objectively measurable, and independent of the individual's ability to recognize them, but always *in relation to the actor* and therefore dependent on their capabilities. *Constraints* may be actual, physical, environmental, or perceptual, depending on the context and the abilities of the agent. Knowledge emerges through the *primary agent's bodily engagement with the environment*, rather than being simply determined by and dependent upon either pre-existent situations or personal construals. Greeno (1994) took up Gibson's agent-situation interactions in ecological psychology in his 'situated cognition' research because its holistic approach rejected the assumptions of individual 'factors' in current psychology. This perspective focused on 'perception-action' instead of memory and retrieval. A perceiving-acting agent is coupled with a developing-adapting environment and what matters is how the two interact. Greeno also suggested that affordances are "preconditions for activity," and that while they do not determine behavior, they increase the likelihood that a certain action or behavior will occur. These ideas continue to be developed in an active school of ecological psychology, where Harry Heft (2003) writes,

> At a basic, pre-reflective level of awareness, prior to the abstractions (e.g. categorization, analysis) all humans so readily perform on immediate experience, we perceive our everyday environment as a place of functionally meaningful objects

> and events. In their immediacy, the "things" of our everyday environment have perceivable psychological value for us in terms of the possibilities they offer for our actions and, more broadly, for our intentions. This aboriginal mode of awareness runs through the flow of our ongoing perceiving and acting, constituting its experiential bedrock. ……. Perceiving the affordances of our environment is, if you will, a first-order experience that is manifested in the flow of our ongoing perceiving and acting. By *first-order experience* I mean experience that is direct and unmediated; it is the experiencing of *x,* in contrast to experiencing *x* through the intercession of *y* or *z*." (Heft, 2003: 151)

In this sense it is the *intuitive and unmediated experience* that gave rise to the Vygotskian perceptions of the emergence of activities and use of physical tools leading to the development of mental functions. Gibson is clear that the environment offers affordances and constraints (physical, social, or mental) that may act upon or be acted upon by the agent, thus illustrating the variety of possible outcomes. From what we now know of ancient cultures, we can recognise individuals in a *milieu* of affordances and constraints available in their environments, ecological, social, and cognitive, acting upon individuals and groups that resulted in the artefacts, products, and writings that have been (and continue to be) discovered, analysed and debated.

Visualisation and Diagrammatic Reasoning

The brief survey by Hanna and Sidoli (2007) shows how recent interest in visualisation has grown in both mathematics, philosophy of mathematics, and mathematics education, and while they refer to Mancosu's chapter on visualisation in (2005: 13-30) which is valuable itself as an investigation into relatively recent mathematics, Mancosu there considers the *re-emergence* in modern terms of what I regard as an *ancient cultural-historical human ability*. After the denigration of visual evidence in mathematical proof in the nineteenth century, he sees the recent interest in visualisation as a change in mathematical style (2005: 17). On the other hand Giaquinto (2007: 35-49) addresses what he sees as the acquisition of "… basic geometrical knowledge …not acquired by inference from something already known or some external authority …" and furthermore, in (262-263) he describes different aspects of visualized motion that contribute to our ability to act upon and transform our mental images.

In the contexts explored above we see that archaeologically recovered materials from Egypt and Mesopotamia, India and China provide some of the earliest written sources of astronomy and mathematics known to us today. By the middle of the first millennium BCE the cultures discussed here had reached a high level of socio-economic organization and technical expertise, developed from the use of simple tools for plotting objects in the sky or measuring the ground for both practical and ritual purposes. The use of simple practical tools inspiring many developments were motivated by a variety of purposes that depended on the affordances that the agents perceived in their environments. The manipulation of constructed objects like manufactured bricks or wooden frameworks was transferred to local media (sand table, clay, papyrus, painted surfaces) that may be used for demonstrating a particular practice or the transmission of technical knowledge, as a visual record to be passed on to others.

In all of these representations it is inevitable that some tacitness remains and unarticulated aspects are 'taken-for-granted' by the actors. The observation of and contemplation upon the dynamic effects of manipulation of the 'object-image' affords the possibility of using these properties in a different context as a new tool to solve what may be a totally unrelated problem. Access and conviction grew from 'hands-on' practical activities, manipulation of actual objects and their transformations by developing the use of representations of these objects, mentally, and in terms of locally available media as iconic likenesses so that operations on these representations led to the 'dissection and re-arrangement' of indexical (Peircean) fluidity of the icon. But while it seems clear that at some stage the contextual affordances gave rise to the comparison and reinforcement of intuitive properties of, for example, right-angled triangles and the emergence of proportional relations, some important questions remain.

If, as I have stated at the beginning of this paper we regard historical epistemology as investigating the *dynamics* of professional knowledge and proto-scientific developments, insofar as they can be extracted from an analysis of texts and practices, to what extent can I justify the transfer of practical knowledge to theoretical knowledge and higher mental functions as evidenced by the texts we already have?
Both Hoyrup and Netz allow the development of a *functional dynamic* of mental operations with uncertainties about the outcome: Hoyrup considers the "metaphor that a mathematical concept is *a tool*: a mental tool, but a tool only by being a tool for operations. The shared properties and conditions of the whole network of connected mathematical concepts with participating operations then characterize the corresponding mode of thought." (Hoyrup 2004:131)

And Netz agrees that outcomes cannot be pre-determined: "There is always a grey area of what a tool can do, depending on which task you put it to: grey area which is not fully determined by the tool itself-so that a dialectic of tools and tasks ensues." (Netz 2002: 282-283)

Mental tools contain knowledge representation structures that allow for drawing inferences from prior experiences about complex objects and processes even when only incomplete information on them is available, and so the epistemic function of visualisation in mathematics can go beyond the merely heuristic one and become a means of discovery of new ideas - and even become belief-forming dispositions. (Giaquinto, 2007: 35-49). Allied to the affordances, we can recruit a conception of an emergent *community of learning* which emphasizes various processes of socialization, involving communities and their values with not only the acquisition of skills and participation in activities, but a third stage where individual and collective learning goes beyond mere information given, and advances knowledge and understanding by a collaborative, systematic development of common objects of activity into shared *knowledge-creation*. (Sami and Hakkarainen 2005)[xviii]

Why should we restrict the creativity of our ancestors with an attitude restricting the possibilities of what was available to them? Is it not better to allow that "we come up with an account where mathematics is not always the same, while people are: which forms, I believe, the historian's intuition." (Netz 2002: 288) After all, I could remark that the field is open, and "absence of evidence is not evidence of absence." (Sagan 1995: 221).

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NOTES

i "text" here applies (as in linguistics) to inscriptions of any kind, on any kind of object.

ii The word *mathematical* here may be pre-emptive, determining the 'mathematical' context in advance

iii See Hoyrup 1994: 25. "… specialists' knowledge that (at least as a corpus) is acquired and transmitted *in view of its applicability*. Even sub-scientific knowledge is thus knowledge beyond the level of common understanding…"

iv It appears that no new mathematical sources have been discovered in the last 70 years.

v Clagett (1999) for example, sees arithmetic and geometric progressions, and geometric problems, while Gillings (1972) sees equations of the first and second degree.

vi An overview of the individual problems and their classification can be found in the introduction of Imhausen 2003.

vii The well-known *Satire of the Trades* compares the position of the scribe to other professions where a scribe has pleasant work and a higher place in society, referring to mathematics and tax collection as part of the scribal profession (Lichtheim 1976). See UCL website
http://www.ucl.ac.uk/museums-static/digitalegypt/literature/satiretransl.html

viii Much of this work was not easily accessible. For details see Robson 2008 Chapter 1. 1-26.

ix Seidenburg & Casey (1980: 336 footnote 53) states that Frits Staal made the film of the *Atiratra Agnicayana* ritual, now available at https://www.youtube.com/watch?v=UnbqnMhbB44 x By *Canonical*, is meant the officially accepted 'traditional' account.

xi The 'in-out' principle is also applied in the Nine Chapters (Chemla & Shuchun 2004: 661-693).

xii Of course, whether or not there was any transfer of ideas between these cultures remains an open question.

xiii There are deeply embedded influences from de Tocqueville, Levy-Bruhl and others that attribute styles of thinking as a characteristic of different social or national groups.

xiv There is a problem with the labeling of and attributing 'second order concepts' to writers in the past. How do we know what 'first order concepts' they had? These categories are all of our own making.

xv See Euclid I,4 for 'superposition' as part of the proof that triangles are congruent which is used again in I, 8 and III, 24 but rarely found elsewhere. 'Superposition' may be intuitively obvious, implies a physical action on an ideal object.

xvi In Vedic literature there are names for *each* of the powers of 10 up to 10^{62}. In the Buddhist tradition, the *Lalitavistara Sutra* recounts a competition between the mathematician Arjuna and the Buddha for naming very large numbers. Today, the words *lakh* and *crore* referring to 100,000 and 10,000,000, respectively, are in common use in newspapers and among English-speaking Indians.

xvii The role played by culture and language in human development is an essential aspect of the Vygotskian framework which examines the relation between learning and mental development through (a) social sources of individual development, (b) semiotic mediation, and (c) genetic (developmental) analysis.

xviii Scandinavian Cultural-Historical Activity Theory developed from Vygotski's (1978) Cultural-Historical Psychology and Leontev's (1977) Activity Theory. See Engestrom, Yrjo (1999).

Oral Presentation

ON THE UNDERSTANDING OF THE CONCEPT OF NUMBERS IN EULER'S "ELEMENTS OF ALGEBRA"

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Many problems encountered by students during the time of transition from arithmetic to algebra are based on different understandings and usages of the concept of number and variable. The analysis of the historical development of algebra and within this field the nature of the discussed objects can be helpful to understand the problems for students nowadays. In the following article Leonhard Euler's understanding of algebra in his textbook "Elements of Algebra" will be discussed. Unlike modern mathematics Euler considers numbers as objects grounded in an empirical subject area. Numbers are defined as the ratio of measurable quantities. In conclusion Euler's understanding of the concept of number will be discussed with the help of the idea of empirical theories.

"ELEMENTS OF ALGEBRA"

The analysis of the understanding of algebra in course of the historical development is based on Euler's textbook "Elements of Algebra". The choice of this textbook is firstly justified by Euler's position in the development of mathematics in general and his contribution to teaching of mathematics in particular and secondly by the significance of the textbook itself. The importance of his textbook has to be seen in the chronological context in which it was written. Most likely Euler started to write the "Elements of Algebra" in Berlin. It was published 1768 in St. Petersburg at first in a Russian translation before it was released 1770 in the original German version. The textbook was translated and reprinted several times, especially with the additions of Lagrange. 1774 the "Elements of Algebra" appeared in a French translation of Johann III Bernoulli. This French edition became a source of the English version. The following analysis applies to this English translation by John Hewlett from 1828. Even though Euler's Textbook had only a small positive impact on science after the appearance (Schubring, 2005, p. 258), it was widely read. In the German edition of Reclam, the textbook was printed from 1883 till 1942 in 108.000 copies. Therefore, Euler's Algebra was really a bestseller (Fellmann, 2007, p.120f). Because of this great demand and the many translations in other languages the "Elements of Algebra" played a major role for the learning of algebra.

The circumstances of the appearance of the textbook are affected by Euler' blindness. Euler needed the help of his servant to write the book. In accordance to an anecdote Euler's non-skilled servant understood the mathematics Euler dictated to him and was in the end able to do algebra by himself (Euler, 1828, Advertisement).

The textbook is addressed to a mathematical interested audience. According to the Advertisement, Euler's intention was to

> "compose an Elementary Treatise, by which a beginner, without any other assistance, might make himself a complete master of Algebra." (Euler, 1828, Advertisement)

Judging from today's point of view the standard set in this textbook and also the treated subjects are beyond the capability of an untrained learner. Nevertheless, the textbook "Elements of Algebra" is a progressive introduction from the natural numbers to Diophantine equations. As set out by Fellmann, the textbook is still

> "-in the judgment of today's foremost mathematicians – the best introduction into the realm of algebra for a "mathematical infant." (Fellmann, 2007, p.121)

The "Elements of Algebra" are a systematic introduction into the arithmetic and elementary algebra. The book is subdivided in two parts. The first part contains the initiation of different kinds of numbers, the basic arithmetic operations, the calculation with variables and the calculation of interests. The second part deals mainly with solving equations of different degrees.

The following analysis is a systematic and text-based approach in order to obtain an understanding of the concept of numbers within the "Elements of Algebra". The achieved insights will be used to discuss the broader understanding of algebra as presented in this textbook. These results will indicate what kind of understanding the reader of Euler's textbook will possibly develop.

THE CONCEPT OF NUMBER

Euler starts his presentation with an ontological explanation of mathematics and the processed objects. He writes at the beginning of the first chapter:

> "[…] And this is the origin of the different branches of the Mathematics, each being employed on a particular kind of magnitude. Mathematics, in general, is the *science of quantity*; or, the science which investigates the means of measuring quantity." (Euler, 1828, part 1 § 2)

This definition of mathematics can be seen as a programmatic fundament for the following contents in the textbook. Contrary to today's understanding of mathematics as an abstract formal science, Euler considers mathematics as a science of concrete measurable quantities. A quantity is defined as follows.

> "Whatever is capable of increase or diminution, is called *magnitude*, or *quantity.*" (Euler, 1828, part 1 § 1) [1]

Euler introduces quantities not as an element of a formal axiomatic structure, but as quantity founded empirically. As examples for quantities Euler names weight, length and the sum of money. The given examples indicate that Euler refers quantities to real subject area. Euler's definition of quantity can be traced back to Euclid. Thiele describes the introduction and use of the concept of quantities in Euclid's Elements in this way:

> "There is no definition of the concept of magnitude (Greek *μεγεθος*, megathos) because there is no superior concept for this fundamental concept. Nevertheless, Euclid is dealing with magnitudes throughout the Elements; […] Magnitudes are generally characterized by the property of being able to increase and decrease." (Thiele, 2003, p. 6)

Like Euclid Euler defines quantities in reference to their capability of increase and diminution. Therefore, Euler assumes a definite order of the quantities, which he does not discuss explicitly. The same applies for the properties of an axiomatic domain of quantities, as transitive and irreflexive. The domain of quantities should be considered as an algebraic structure with an operation addition and an order relation less-than. The quantities in Euler's "Elements of Algebra" are given by empirical examples and it seems like the properties of the quantities are also given based on the empirical foundation and require no formal definition.

To compare and calculate with quantities it is necessary to be able to measure or determine a quantity. Euler remarks to this:

> "Now, we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and point out their mutual relation." (Euler, 1828, part 1 § 3)

The determination of a quantity requires a unit, a quantity of the same kind, which can be put in a ratio to the proposed quantity. The following given examples are once again real quantities as weight, length and the sum of money.

Natural Number, Whole Numbers and Rational Numbers

Based on the concept of quantity Euler defines numbers as the ratio of one quantity to another:

> "So that a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit."(Euler, 1828, part 1§ 4)

The definition of numbers by Euler is based on the fundamental idea of partition and measurement. Nowadays in mathematics school courses numbers are defined as cardinal numbers, ordinal numbers or measure values. However, Euler introduces natural numbers as the ratio of quantities of the same kind. Therefore, natural numbers are characterized according to the empirical origin of the underlying quantities.

After the introduction to numbers Euler initiates the basic arithmetic calculation for the new objects. He starts with an explanation of the symbols + and – and the use of these symbols related to the natural numbers. Within this approach Euler mixes the symbols as operation signs and as algebraic signs of a number. He states:

> "Hence it is absolutely necessary to consider what sign is prefixed to each number: for in Algebra, simple quantities are numbers considered with regard to the signs which precede or affect them. Farther we call those *positive quantities*, before which the sign + is found; and those are called *negative quantities*, which are affected by the sign –." (Euler, 1828, part 1 § 16) [2]

It can be seen that a change of the ontological state of the signs happens here. Before the sign stood for an operation, which connects two numbers with each other. In the context of positive and negative numbers the sign is part of the name of the quantity itself. Euler pays no particular attention to this fact.

Also nowadays the whole numbers are defined as difference a – b of to natural numbers a and b. The subtraction of the field ℤ is introduced with the inverse element regarding to the addition.

In Euler's approach this is just a step further towards the extension of the number system to whole numbers. Based on the characterising of negative quantities Euler introduces negative numbers with regard to the empirical quantities:

> "The manner in which we generally calculate a person's property, is an apt illustration of what has just been said. For we denote what a man really possesses by positive numbers, using, or understanding the sign +; whereas his debts are represented by negative numbers, or by using the sign –." (Euler, 1828, part 1 § 17)

It becomes clear at this point, that Euler does not strictly distinguish between a quantity and a number, which is defined as ratio of quantities. Euler considers negative number as quantity itself. In this sense Euler also proceeds with numbers as if they were quantities of a material world. According to the relation of whole numbers to the domain of quantities of a sum of money, numbers can be ordered linearly on the number line. Euler argues that:

> "Since negative numbers may be considered as debts, because positive numbers represent real possessions, we may say that negative numbers are less than nothing." (Euler, 1828, part 1 § 18)

In this explanation zero stands for the case when someone has no property of his own or in other words it represents nothing. Euler himself does not name zero directly as number in this chapter, but includes it in the series of natural numbers and also in the series of negative numbers. Nevertheless, in the summarization of whole numbers Euler does not name zero as a possible value of numbers. The given context and also the handling in the subsequent chapters show that zero as a possible value has to be included. He describes whole numbers as follows:

> "All these numbers, whether positive or negative, have the known appellation of whole numbers, or integers, which consequently are either greater or less than nothing." (Euler, 1828, part 1 § 20)

This characterisation corresponds to the law of trichotomy for ℝ or more generally for ordered sets.

> "If $x \in S$ and $y \in S$ then one and only one of the statements $x < y$, $x = y$, $y < x$ is true." (Rudin, 1964, p.3)

A formal introduction of negative numbers does not occur. Euler's justification of negative numbers and the existence of the order of whole numbers is based on the presented quantities.

In the same manner Euler initiates rational numbers. Here again Euler refers to a concrete domain of quantities to justify the new numbers. Special attention should be given to the fact, that Euler defines rational numbers not directly as ratio of two natural numbers, but rather introduces rational numbers by the help of lengths. The example leads Euler to an idea of the concept of rational numbers and justifies the ontological existence of the number at the same time. Euler states:

> "When a number, as 7, for instance, is said not to be divisible by another number, let us suppose by 3, this only means, that the quotient cannot be expressed by an integer number; but it must not by any means be thought that it is impossible to form an idea of that quotient. Only imagine a line of 7 feet in length; nobody can doubt the possibility of dividing this line into 3 equal parts, and of forming a notion of the length of one of those parts." (Euler, 1828, part 1 § 68)

The "number" we gain by dividing a quantity by a number is a quantity with a given unit, which is contrasted by Euler's definition of numbers in general. The given problem is based on the fundamental idea of distribution and not of the proposed fundamental idea of partition and measurement, as indicated by the definition of number. Euler just introduces numbers by empirical examples and does not define the ordered field ($\mathbb{Q}$, <,+). But like for the whole numbers Euler presupposes a natural order of the rational numbers.

Euler's formulation "nobody can doubt" emphasises the self-evident character of his explanation. Euler uses the knowledge and laws of the everyday life to introduce and also justify new contents. Vollrath points out that also for today's students it is obvious that a division of a distance leads to another distance. The recurrent problem is only to determine the length of the parts (Vollrath, Weigand, 2007, p. 40).

In summary, the numbers underlying concrete quantities are the basic concepts, which require no definition. They are defined by the capability of increase and diminution and are clarified by examples. Euler's understanding of mathematics is highly related to science.

Properties of Numbers

The properties of numbers are gained by the interpretation of the numbers as quantities of an empirical subject area. The justification of the properties relies on empirical examples. Euler refers to obvious characteristics of the quantities, which are transferred to the numbers, equally to his approach by the extension of the number systems. This is clearly evidenced in Euler's explanation of density:

> "For instance, 50 being greater by an entire unit than 49, it is easy to comprehend that there may be, between 49 and 50, an infinity or intermediate number, all greater than 49, and yet all less than 50. We need only to imagine two lines, one 50 feet, the other 49 feet long, and it is evident that an infinity number of lines may be drawn, all longer than 49 feet, and yet shorter than 50." (Euler, 1828, part 1§ 20)

Euler does not only introduce new concepts and properties referring to empirical quantities, but also justifies operational rules and laws by reference to concrete quantities. For the justification of the operational rule (+)(–) = – Euler observes:

> "Let us begin by multiplying –a by 3 or +3. Now since –a may be considered as debt, it is evident that if we take the debt three times, it must thus becomes three time greater, and consequently the required product is –3a." (Euler, 1828, part 1 § 32)

The phrase "evident" suggests for the student an implicitness of the obtained rule. Euler apparently considers this rule as an evident statement, which is an extract from an empirical observation and for this reason does not require a formal proof.

The validation of the commutative law is illustrated by empirical examples. Contrary to the previous rule Euler explains the commutative law not by referring directly to quantities but under the specification of concrete number values. He argues:

> "It may be farther remarked here, that the order in which the letters are joined together is indifferent; thus ab is the same thing as ba; for b multiplied by a is the same as a multiplied by b. To understand this, we have only to substitute, for a and b, known numbers, as 3 and 4; and the truth will be self-evident; for 3 times 4 is the same as 4 times 3." (Euler, 1828, part 1 § 27)

Euler's example is so simple and common in the everyday life, that he also states this fact as self-evident. Similar to the other presented introductions and explanations Euler abdicates a formal derivation or proof.

Euler's approach resembles the methods and access in nowadays school mathematics. As Padberg points out, in the primary school the properties are obviously not formulated in an abstract way. The students will rather experience them as computational advantageous. The justification of the properties can be obtained by example-attached strategies of proof (Padberg, 2009, p.125). The given explanations refer mainly to dot patterns or arrangements of objects.

For Euler numbers simply have their properties because of the fact, that the underlying relevant quantities have these properties and this is so obvious and common knowledge, that there is no need for any kind of proof. It seems as if the numbers inherit the characteristics from the basic empirical entities.

Imaginary Numbers

Of great importance for Euler in the "Elements of Algebra" is the concept of the imaginary number. It should be pointed out here that it should be distinguished between the complex number as an element of the field $\mathbb{C}$, like it is understood today, and the square root of a negative number as imaginary number of Euler's days. In Euler's days there was no theory of complex numbers and therefore there had not been an axiomatic approach, on which it could fall back on. The first documents bringing up complex numbers date back to the Renaissance. In 1645 Cardano published his book "Ars Magna", where the process to solve cubic equations had been generalised by the help of the square root of negative numbers (Remmert, 1991). By means of some equation the presented process provides imaginary numbers as solutions. Cardano suggested that square roots of negative numbers have a "sophisticated nature" since they are neither near the "nature of a number" nor near the "nature of a quantity". Cardano concluded that the results of equations, which include square roots of negative numbers, are useless (Cardano, 1545, p. 288). Even years later, when imaginary numbers were used in calculations a systematic analysis of the imaginary numbers is missing. Although Euler handled imaginary numbers in calculations and actually invented i as notation for $\sqrt{-1}$, the ontological state of imaginary number was undetermined. Only 1831, Gauss was able to interpret imaginary numbers as points in a plane and founded them in geometry. Several years later Hamilton described imaginary numbers as an ordered pairs (x,y) of real numbers and defined for them arithmetic calculations as addition and multiplication (Remmert, 1991).

As written above, Euler was well aware of imaginary numbers, but nevertheless

> "Euler had great difficulty in explaining and defining just what the imaginary numbers, which he had been handling so masterfully during the past forty years and more, really were." (Remmert, 1991, p. 59)

For Euler the imaginary number represents an expression without any relation to the real subject area. Nevertheless, this expression has to exist, due to the fact, that he gains them by applying allowed calculation rules on negative numbers. Thus the term of the square root of a negative number appears in this sense in a natural way. But the new term is not compatible with Euler's understanding of a number, since the definition of number, as ratio of quantities of the same kind, does not apply to the square root of negative numbers. Especially the properties of numbers that result from their definition do not refer to the new terms. Euler notes that:

> "All such expressions, as $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}$, ... are consequently impossible, or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, or greater then nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible." (Euler 1828, part 1 § 143)

As pointed out above, a main characteristic of numbers is that they can be ordered linearly on a number line. The law of trichotomy must be fulfilled. Thus every kind of number has to be less than zero, equal to zero or greater than zero as condition to be a possible number. The square root of negative numbers, however, does not follow any characteristic of a linear order. Moreover, the unknown expressions cannot even be approximated. The value of the square root of a negative number can not be qualified. Euler points this out as follows:

> ""[…] whereas no approximation can take place with regard to imaginary expressions, such as $\sqrt{-5}$; for 100 is as far from being the value of the root as 1, or any other number." (Euler, 1828, part 1 § 702)

The square root of a negative number is a result of solving an equation, but is not even considered as possible number. Imaginary numbers do not refer to empirical objects and therefore, they are not part of our material world. There is no empirical quantity, which is expressed by the imaginary number, as it is the case for the negative number and the debts.

Nevertheless, Euler considers imaginary numbers to be important for addressing algebra. Euler justifies his considerations regarding the imaginary numbers against the widespread opinion that they are useless expressions and do not need to be discussed. Euler identifies the benefit of imaginary numbers as indicator whether an equation is solvable or not. He states:

> "For the calculation of imaginary quantities is of the greatest importance, as questions frequently arise, of which we cannot immediately say whether they include any thing real and possible, or not; but when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible." (Euler, 1828, part 1§ 151)

This opinion on the imaginary numbers as an indicator for solvability of problems is not new. Before Euler, Newton only had understood the imaginary expression as symbol for the impossibility to solve the equation (Remmert, 1991, p. 58) and Descartes had actually understood imaginary numbers as geometric impossibility:

> "To see how Descartes understood the association of imaginary numbers with geometrical impossibility, consider his demonstration on how to solve quadric equation with geometric constructions. He began with the equation $z^2 = az - b^2$, where a and b^2 both non-negative, […]." (Nahin, 1998, p. 34)

Another point in which imaginary numbers show themselves to be of use for Euler is as provisional result, since after operating with them they can lead to possible numbers. Thus Euler uses imaginary numbers later on to find the factorisation of the equation $ax^2 + bxy + cy^2$. The integration of imaginary number in the calculation is one further step to a theory of complex numbers.

Euler's solution in handling the imaginary expressions is to transfer the well-known operations and calculation rules from the real numbers to the new expressions. This is done without a formal definition of the potential operations regarding imaginary numbers. For Euler it is natural that the normal calculation rules also apply to imaginary numbers due to the fact that we can have an idea of them:

> "But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number which, multiplied by itself, produces -4; for the reason also, nothing prevents us from making use of these imaginary numbers, and implying them in calculation." (Euler, 1828, part 1§ 145)

Euler's proceeding resembles his examination of real numbers. Real numbers also appear by extracting the square root of numbers, which are no square themselves. In the same way as for the imaginary numbers Euler gains an idea of the real numbers. He writes as follows:

> "These irrational quantities, though they cannot be expressed by fractions, are nevertheless magnitudes of which we may form an accurate idea; since, however

> concealed the square root of 12, for example, may appear, we are not ignorant that it must be a number, which, when multiplied by itself, would exactly produce 12; and this property is sufficient to give us an idea of the number, because it is in our power to approximate towards its value continually." (Euler, 1828, part 1§ 129)

Despite the parallels in these two remarks the differences are obvious. Although Euler is not able to obtain a concrete perception of the square root of 12, he may approximate the value of the real number by rational numbers and especially he can order the real numbers linearly. Both qualities do not apply for the imaginary numbers, as pointed out above.

The application of the empirically founded calculation methods to imaginary numbers without a proper definition is problematic. The missing definition of the basic arithmetic operation for the square root of negative numbers leads to an ambiguity of the multiplication. On the one hand Euler writes:

> "In general, that by multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or by taking the square of $\sqrt{-a}$ we obtain –a." (Euler, 1828, part 1 § 146)

On the other hand Euler states two paragraphs later:

> "Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$;" (Euler, 1828, part 1 § 148)

Like Neumann points out, the attentive reader will have to ask himself how $\sqrt{-a}\sqrt{-a}$ has to be determined (Neumann, 2008, p. 118). Firstly it can be calculated $\sqrt{-a}\sqrt{-a}$ = $(\sqrt{-a})^2$ = –a, and secondly like this: $\sqrt{-a}\sqrt{-a}$ = $\sqrt{(-a)^2}$ = $\sqrt{a^2}$ = –a. Euler does not clarify this issue. [3] Remmert remarks to this problem, that "Euler occasionally makes some mistakes" (Remmert, 1991, p. 59). This is, however, not tenable, because it implies the existence of the definition of the multiplication of imaginary numbers. But an algebraic definition of the multiplication did not exist until Hamilton.

The question of the ontological status of imaginary numbers was not sufficiently answered. Scholz points out correctly, that the question has to be whether the knowledge about calculation methods is reason enough to award imaginary numbers with their own ontological status (Scholz, 1990, p. 294). It seems in the "Elements of Algebra" that any kind of algebraic expression based on empirically founded arithmetic operation, are ontologically justified due to the fact that these exist simply because of this operation. Besides the investigation of imaginary numbers Euler discusses algebraic expression such like $\frac{1}{0}$ in his Algebra. His statement has to been seen critically:

> "For $\frac{1}{0}$ signifying a number infinitely great and $\frac{2}{0}$ being incontestably the double of $\frac{1}{0}$, it is evident that a number, though infinitely great, may still become twice, thrice, or any number of times greater." (Euler, 1828, part 1 § 84)

In this regard, Jahnke draws attention to the fact that an abstract quantity simply can be determined by its occurrence as a variable in a formula. And due to this fact also objects, which cannot be interpreted empirically, can be referred under this concept (Jahnke, 2003, p. 106).

In contrary Kvasz does not believe that for imaginary numbers a complete detachment from the empirical subject area is possible. He writes:

> "Thus for Euler too these quantities exist only in our imagination. But this subjective interpretation of the complex numbers cannot explain how it is possible for computations involving these non-existent quantities to lead to valid results about the real world. […] If the complex numbers make it possible to disclose new knowledge about the world, they must be related to the real world in some way. A purely subjective interpretation is therefore unsatisfactory." (Kvasz, 2008, p. 182)

Euler's discussion of the imaginary numbers clearly shows that the ontological status must not be fully determined and that an axiomatic access to operate with expression as objects is not necessarily required. The known and established operations, which were initiated on the basis of empirical quantities, can be transferred to new, undefined expressions. The imaginary numbers do not belong to any known and empirically justified number system. Nevertheless, they exist, since they result by taking the square root of a negative number.

As it has been made clear in this chapter, Euler deals with symbolic expressions without referring directly to a real subject area. It is indeed wrong to assume that Euler justifies each step in his Algebra by referring to empirical objects. Furthermore, Euler introduces new concepts with regard to his basic objects, the empirical quantities, but subsequently handles them without reference to the domain of quantities. Euler handles and uses the concepts algebraically. At the beginning of his textbook he points out:

> "In Algebra, then we consider only numbers, which represent quantities, without regarding the different kinds of quantity." (Euler 1828, part 1 § 6)

The foundation of his approach remains the localisation of Algebra in the context of empirical quantities.

THE CONCEPT OF VARIABLE

The word "variable" is a term from the present day and is not used by Euler in the "Elements of Algebra". Euler describes the variables in terms of a sought number, unknown quantity or known numbers. Since for Euler a number is the ratio of two quantities, it could be expected that the unknown itself is no abstract entity to him.

Euler introduces the variables at the very beginning of the Algebra during the initiation of the basic arithmetic operations. Euler discusses arithmetical laws in this manner generally. He characterises the variable as follows:

> "All this is evident; and we have only to mention, that in Algebra, in order to generalise numbers, we represent them by letters, as a, b, c, d etc." (Euler, 1828, part 1 § 10)

In Euler's Algebra variables represent numbers. The use of these variables is to generalise a proposition and to be able to examine equations. Therefore, Euler needs a general symbolism and syntactic rules to operate with the letters. After demonstrating every arithmetic operation for examples they are applied to letter as variables.

Euler does not specify the conditions for arithmetical operations and laws. Therefore, he does not introduce a set to which the operation or law applies. He neither discusses the closure under the operation. Even for the generalisation of a ratio Euler does not limit the domain for the variable. In this context, Heuser mentioned in his introduction to real analysis that the calculation with letters can be handled as used from school, since there does not exist something newly learned regarding to the basic arithmetical operations (Heuser, H. (2009), p. 40). It can be said that in the same way, in Euler's Algebra, the transfer of the operations to the variables is familiar, because the variables just represent the numbers or quantities sought.

In German secondary schools nowadays the variable as concept is usually introduced as representation of a number or quantity. Also, known operations from the presented number system are transferred to the variables without further formal explanation, but with a visualisation of the validation regarding concrete quantities.

Euler uses letters as variables not only for the number sought, but also for given unknown numbers. During the discussion of solving quantities Euler states:

> "And, in general, if we have found $x + a = b$, where *a* and *b* express any known number, […]." (Euler 1828, part 1 § 574)

In order to solve the equation, Euler demonstrates the calculation methods for exemplified problems. During the problem solving Euler handles the variables as if they were concrete numbers. This can be clearly seen by this example:

> "In order to resolve this question, let us suppose that the number of men is = x; and, considering this number as known, we shall proceed in the same manner as we wished to try whether it corresponded with the conditions of the equation." (Euler, 1828, part 1 § 567)

Euler does not justify every transformation step during a calculation with regard to empirical quantities. As already described above, Euler discusses mental representations of empirical objects and uses empirically founded operations. Thus, a justification is given implicitly all the time by the nature of the processed objects.

EULER'S UNDERSTANDING OF ALGEBRA

The manner in which Euler introduced the concepts in the textbook as well as the introduction of properties provides justified conclusion about Euler's understanding of algebra. The previously gained insights into the understanding of the concept of numbers and variables shall be discussed with the help of the idea of empirical theories. [4]

Contrary to the modern understanding of algebra, which is focussed on the structure, Euler's ambition in the "Elements of Algebra" is to describe and explain empirical phenomena. He wants to develop a theory of algebra, which can help to solve problems of the natural environment. Since the "Elements of Algebra" is constituted as a textbook with the intention that an unskilled student can learn the algebra without further help, Euler starts his description with the basic objects of his theory. The basic objects are concrete, measurable quantities, because Euler defines the quantities through their empirical characteristic of the capability of increase and diminution. A natural number is defined through the ratio of quantities of the same kind. Based on this concept of numbers Euler extends the number system to the whole numbers and also the rational numbers with reference to a domain of quantities. Thus Euler introduces the numbers as representation of empirical objects. They are Elements of a real subject area. In the introduction of other concepts and laws Euler refers to the underlying empirical quantities. His justifications as shown above are intuitive. He calls on the common knowledge of the reader of empirical quantities and numerical examples. Thus it can be said that Euler fulfils in his Algebra the characteristics of an empirical algebraic theory regarding to a subject area.

Euler does not define his theory of algebra like modern mathematicians. The objects in this textbook are not composed abstract elements of a set, but rather representation of empirical objects. The properties of the numbers are not deduced from stated axioms but are derivated from the properties of the quantities. Similar the calculation laws are constituted related to a subject area. Thus, a formal proof or logical derivation from axioms is not required. Euler's characterisation of the imaginary numbers as indication of insolvability of a problems shows the necessity of verifying the statements empirically. This contrast modern understanding of mathematics, in which verification of a statement can only be attained by a formal proof.

He experiments with symbols like a scientist. Fraser's opinion about analysts can be transferred to Euler:

> For the 18th century analyst, functions are things that are given ‚out there', in the same way that the natural scientist studies plants, insects or minerals, given in nature." (Fraser (2005), p. 246)

In the "Elements of Algebra" natural numbers are defined by the ratio of two quantities. In this sense the numbers are for Euler given objects of his Algebra with which he can experiment.

Imaginary numbers have a unique status in Euler's Algebra. They are not possible numbers, since they are not less than, equal to or more than zero. Imaginary numbers do not represent empirical objects and therefore they are imaginary. Nevertheless, they are created by the application of allowed operations on negative numbers. Thus, imaginary numbers exist and Euler applies the current operation on the undefined expressions. Imaginary numbers have no independent ontological status and thus cannot be discussed isolated from the operation, which creates them. Imaginary numbers only have a meaning in the context of the theory of algebra, since they exist through the defined operation within the Euler's theory of algebra. In this manner it can be said that Euler's approach in Algebra is empirical. [5]

NOTES

1. In this article the terms magnitude and quantity are used as synonyms.

2. It should be pointed out here, that in the original German version the formulation of the sentence leads to a stricter interpretation of Euler's understanding of numbers. The understanding of numbers as quantities themselves is more clearly expressed. Euler writes: "Hence, they used to consider in the algebra numbers with the preceding sign as a single quantity." (Ibid. Euler, 1770)

3. Indeed Neumann notes correctly that an ambiguity of the multiplication exists here, but he himself makes a mistake by formulating the two different ways to handle the equation. Instead of taking the square of $\sqrt{-a}$, he calculates $\sqrt{-a}\sqrt{-a} = \sqrt{(-a)^2} = -a$ for the first possibility.

4. Compare the approach of empirical theories by Balzer & Moulines & Sneed (1987) and Burscheid & Struve (2010).

5. For further discussion compare Reimann & Witzke (2013).

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Oral Presentation

THE PROBLEM OF THE PARALLELS AT THE 18TH CENTURY: KÄSTNER, KLÜGEL AND OTHER PEOPLE

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During the 18th century a lot of work on the problem of the parallels – that was in the traditional understanding, proving Euclid's parallel postulate on the base of his other axioms and postulates – was done. There was G. Saccheri with his remarkable work "Euclidis ab omni naevo vindicatus" (1737) – which remained more or less completely unnoticed - on one hand and A.M. Legendre with his widespread demonstrations ("Eléments de géométrie" (1794)) on the other hand. But in between there was also a remarkable dissertation written by Georg Simon Klügel ("Conatuum praecipuorum theoriam parallelarum demonstrandi recensio" (1763)) under the guidance of Abraham Gotthelf Kästner at Göttingen; it was the latter who draw some skeptical conclusions of the work done by Klügel in his remarks in his "Anfangsgründe" (first published 1758-1764; there are different later editions).

In my conference I want to retrace in a short way the history of the problem of the parallels in particular the proposals made by Saccheri and Legendre. Its main purpose is to describe Klügel's critical work and the conclusions derived from it by Kästner. We will see that the convictions of the mathematicians were surprisingly enough very important in this history – the main difference between J. Bolyai and N. Lobachevsky on one hand and their precursors laying exactly in this respect, Klügel and Kästner being remarkable forerunners of them.

The dissertation by Klügel is now available in a German translation by Dr. M. Hellmann (Weilheim) in Volkert, K. "Das Unmögliche denken. Die Rezeption der nichteuklidischen Geometrie im deutschsprachigen Raum 1860 – 1900" (Heidelberg: Springer, 2013).

Oral Presentation

HPM IN MAINLAND CHINA: AN OVERVIEW

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Primarily five themes on HPM are discussed in Mainland China: 1) Discussion on “Whys” & “Hows”. The “whys” are categorized corresponding to three mathematics teaching objectives, and four approaches to using the history of mathematics in teaching are identified based upon teaching practice. 2) Education-oriented researches on the history of mathematics. The history of specific topics on school mathematic is studied, for example, the history of the concept of ellipse, the history of using the letters to represent the numbers, etc. 3) Empirical studies on the “historical parallelism”. For example, students’ understanding of the concept of the tangent line is surveyed, and the historical parallelism id examined. 4) Integrating the history of mathematics into mathematics teaching: classroom practice or experiments. Many teaching experiments are carried out and many teaching materials have been built so far. 5) HPM & mathematics teachers’ professional development. In this presentation, we mainly focus on the four approaches to using history in teaching based on some lessons, such as the linear equation with one unknown, the application of similar triangles & congruent triangles, the concept of complex numbers, etc.

Oral Presentation

DIFFERENT UNDERSTANDINGS OF MATHEMATICS:

AN EPISTEMOLOGICAL APPROACH TO BRIDGE THE GAP BETWEEN SCHOOL AND UNIVERSITY MATHEMATICS

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A survey in Germany amongst students who have chosen to obtain a teaching degree shows that the transition from school to university mathematics is experienced in the context of a major revolution regarding their views about the nature of mathematics. Motivated by the survey, a team of researchers is currently working on a historically-motivated concept for an undergraduate course to help bridge the gap.

THE PROBLEM OF TRANSITION: STILL OF IMPORTANCE

A classical problem of mathematics education certainly is the problem of transition from school mathematics to university mathematics and back again. It is a problem all high school teachers around the world encounter during their training. Even Felix Klein (1849-1925), prominent mathematician and mathematics educator, in this context complained about the phenomena he coined as "double discontinuity":

> The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (Klein, 1908/1932, p. 1, author's translation)

In the following we focus on the "first discontinuity", postulating an epistemological gap between school and university mathematics. As the problem is at least more than 100 years old, definitive solutions do not seem to appear on the horizon (cf. Gueudet, 2008). Contrarily, dropout rates (especially in western countries) remain on a constantly high level – in Germany about 50% of the students studying mathematics or mathematics-related fields stop their efforts before having finished a bachelor's degree (Heublein et al., 2012). This again leads to an at least perceived intensification of research in this field. In 2011 the most important professional associations regarding mathematics (education) in Germany (DMV-Mathematics, GDM – Mathematics Education & MNU – STEM Education) formed a task force regarding the problem of transition (cf. http://www.mathematik-schule-hochschule.de). In February 2013 a scientific conference with the topic "Mathematik im Übergang Schule/Hochschule und im ersten Studienjahr" ("Mathematics at the Crossover School/University in the First Academic Year") in Paderborn (Germany) attracted almost 300 participants giving over 80 talks regarding the problematic transition-process from school to university mathematics. The proceedings of this conference (Hoppenbrock et al., 2013) and its predecessor on special transition-courses (Bausch et al., 2014) give an impressive overview on the necessity and variety of approaches regarding this matter. Interestingly a vast majority of the studies and best practice examples for "transition-courses" locate the problem in the context of deficits (going back as far as junior high school) regarding the content knowledge of freshmen at universities.

> In the "precourse and transition course community" it seems to be consensus by now that existing deficits in central fields of lower-secondary school's mathematics make it difficult for Freshmen to acquire concepts of advanced elementary mathematics and to apply these. Fractional arithmetic, manipulation of terms or concepts of variables have an important role e.g. regarding differential and integral calculus or non-trivial application contexts and constitute insuperable obstacles if not proficiently available. (Bieler et al., 2014, p. 2, author's translation)

The question of how to provide first semester university students with the obviously lacking content knowledge is certainly an important facet of the transition problem. But as the results of an empirical study suggest, there are other, deeper problem dimensions which aid in further understanding the issue.

MOTIVATION: A SURVEY

To investigate new perspectives on the transition problem, approximately 250 pre-service secondary school teachers from the University of Siegen and the University of Cologne in 2013 were asked for retrospective views on their way from school to university mathematics. When the questionnaire was disseminated the students had been at the universities for about one year. Surprisingly, the systematic qualitative content analysis of the data (Mayring 2002; Huberman & Miles 1994) showed that from the students' point of view it was not the deficits in (the level and amount of) content knowledge that dominated their description of their own way from school to university mathematics. To a substantial extent, students reported problems with a feeling of "differentness" of school and university mathematics than with the abrupt rise in content-specific requirements. Three exemplar answers to the question,

> What is the biggest difference or similarity between school and university mathematics? What prevails? Explain your answer.

illustrate this point quite clearly.

> Student (male, 20 years): "The biggest difference is, that university mathematics is a closed logical system, constituted by proofs. School mathematics in contrast is limited to applications. Regarding the topics there are more similarities, regarding the process of reasoning more differences." (author's translation)

> Student (male, 19 years): “The fundamental difference develops as mathematics in school is taught ostensibly (“*anschaulich*”), whereas at university it is a rigid modern-axiomatic structure characterizing mathematics. In general there are more differences than similarities, caused by differing aims.” (author’s translation)

At this point we can only speculate on the term “aims” but in reference to other formulations in his survey it seems possible that he distinguishes between general education (*Allgemeinbildung*) as an aim for school and specialized scientific teacher-training at universities.

The third example is impressive in the same sense:

> Student (female, 20 years):

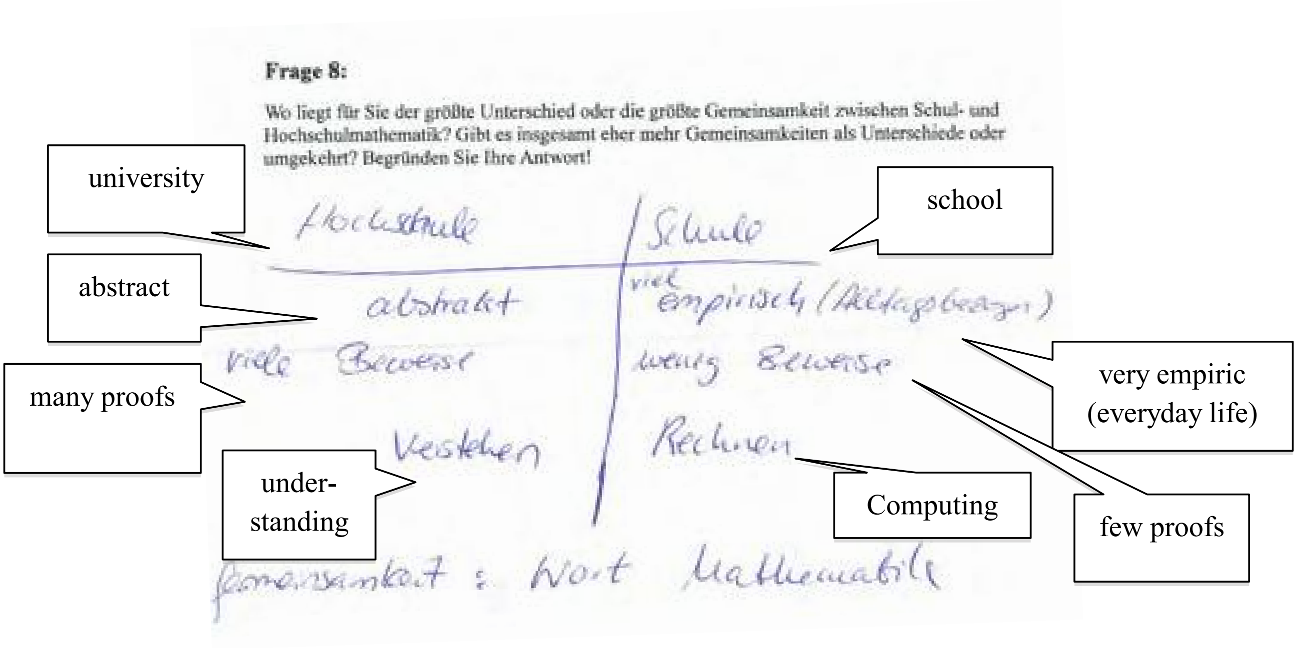

Figure 1: A student’s appreciation of difference or similarity between school and university mathematics.

In all three cases the students clearly distinguish between school and university mathematics, which is most prominent in the last example (see Fig. 1): for this student school mathematics and university mathematics are so different, that the only remaining similarity is the word ‘mathematics’. This “differentness” encountered by the students is specified in further parts of the questionnaire with terms as *vividness, references to everyday life, applicability to the real world, ways of argumentation, mathematical rigor, axiomatic design, etc.*

Using additional results of studies with a similar interest (e.g. Gruenwald et al., 2004; Hoyles et al., 2001) the author comes to the preliminary conclusion that pre-service teachers clearly distinguish between school and university mathematics regarding the *nature of mathematics*. In the terms of Hefendehl-Hebeker et al., the students encounter an “Abstraction shock.” (Hefendehl-Hebeker et al., 2010)

This sets the framework for further research concerning the problem of transition: following the idea of constructivism in mathematics education, students construct their own picture of mathematics with the material, problems and stimulations teachers provide in the classroom or lecture hall (Anderson et al., 2000; Bauersfeld, 1992). Thus it is helpful to reconstruct the *nature of mathematics* communicated explicitly and implicitly in high school and university textbooks, lecture notes, standards, etc., with a special focus on differences.

REFLECTIONS ABOUT THE NATURE OF MATHEMATICS & MATHEMATICAL BELIEF SYSTEMS IN SCHOOL AND UNIVERSITY

Beliefs

The terms *nature of mathematics* and *belief system* regarding mathematics are closely linked to each other if we understand learning in a constructive way. Schoenfeld (1985) successfully showed that personal belief systems matter when learning and teaching mathematics:

> One's beliefs about mathematics [...] determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. The belief system establishes the context within which we operate […] (Schoenfeld, 1985, p. 45)

From an educational point of view beliefs about mathematics are decisive for our mathematical behavior. For example, there are four prominent categories of beliefs concerning mathematics as a discipline distinguished by Grigutsch, Raatz, and Törner (1998): *the toolbox aspect, the system aspect, the process aspect and the utility aspect.* Liljedahl et al. (2007) specified this wide range of possible aspects of a mathematical worldview as follows:

> In the "toolbox aspect", mathematics is seen as a set of rules, formulae, skills and procedures, while mathematical activity means calculating as well as using rules, procedures and formulae. In the "system aspect", mathematics is characterized by logic, rigorous proofs, exact definitions and a precise mathematical language, and doing mathematics consists of accurate proofs as well as of the use of a precise and rigorous language. In the "process aspect", mathematics is considered as a constructive process where relations between different notions and sentences play an important role. Here the mathematical activity involves creative steps, such as generating rules and formulae, thereby inventing or re-inventing the mathematics. Besides these standard perspectives on mathematical beliefs, a further important component is the usefulness, or utility, of mathematics. (p. 279)

Very often these beliefs are located within certain fields of tension (*Spannungsfelder*): there is, for example, the *process aspect* which is always implicitly connected to its opposite pole the *product aspect*. Another pair of concepts in this sense is certainly an *intuitive aspect* on the one hand and a *formal aspect* on the other, having even a historical dimension: "There is a problem that goes through the history of calculus: the tension between the intuitive and the formal." (Moreno-Armella, 2014, p. 621) These fields of tension may help to describe the problems the students encounter on their way to university mathematics. Especially helpful when looking at the survey results, representing one important facet, seems to be the tension between what Schoenfeld calls an *empirical belief system* and a *formal(istic) belief system* – a convincing analytical distinction following the works of Burscheid and Struve (2010). The *empirical belief system* on the one hand describes a set of beliefs in which mathematics is understood as an experimental natural science, which of course includes deductive reasoning, about empirical objects. Good examples for such a belief system can be found in the history of mathematics. The famous mathematician Moritz Pasch (1843-1930) who completed Euclid's axiomatic system, explicitly understood geometry in this way,

> The geometrical concepts constitute a subgroup within those concepts describing the real world […] whereas we see geometry as nothing more than a part of the natural sciences. (Pasch, 1882, p. 3)

Mathematics in this sense is understood as an empirical, natural science. This implies the importance of inductive elements as well as a notion of truth bonded to the correct explanation of physical reality. In Pasch's example Euclidean geometry is understood as a science describing our physical space by starting with evident axioms. Geometry then follows a deductive buildup – but it is legitimized by the power to describe the physical space around us correctly. This understanding of mathematics as an empirical science (on an epistemological level) can be found throughout the history of mathematics – prominent examples for this understanding are found in many scientists of the 17th and 18th centuries. For example, Leibniz conducted analysis on an empirical level; the objects of his *calculus differentialis* and *calculus integralis* were curves given by construction on a piece of paper – not as today's abstract functions (cf. Witzke, 2009).

Now, how does all of this come together with students and the transition problem? If we take a closer look at the survey results, and combine this with a look at current textbooks we see that students at school are likely to acquire an empirical belief system – which on epistemological grounds shows parallels to the historical understanding of mathematics. These epistemological parallels were fundamental for the design of our 'transition seminar' for students. The main idea is that the recognition and appreciation of different conceptions of mathematics in history (i.e., those held by expert mathematicians) can help students to become aware of the own belief system and may guide them to make necessary changes.

SCHOOL & UNIVERSITY

If we look at the most recent National Council of Teachers of Mathematics (NCTM) standards and prominent school books we see that for good reasons, mathematics is taught in the context of concrete (physical) objects at school: The process standards of the NCTM and in particular "connections" and "representations" (which are comparable to similar mathematics standards in Germany) focus on empirical aspects. At school it is important that students "recognize and apply mathematics in contexts outside of mathematics" or "use representations to model and interpret physical, social, and mathematical things" (NCTM, 2000, p 67).

The prominent place of illustrative material and visual representations in the mathematics classroom has important consequences for the students' views about the *nature of mathematics*. Schoenfeld proposed that students acquire an *empiricist belief system* of mathematics at school (Schoenfeld, 1985; 2011). This is caused by the fact that mathematics in modern classrooms does not describe abstract entities but a universe of discourse ontologically bounded to "real objects": *Probability Theory* is bounded to random experiments from everyday life, *Fractional Arithmetic* to "pizza models", *Geometry* to straightedge and compass constructions, *Analytical Geometry* to vectors as arrows, *Calculus* to functions as curves (graphs) etc.).

At university things can look totally different. Authors of prominent textbooks (in Germany as well as in the U.S.) for beginners at university level depict mathematics in quite a formal rigorous way. For example in the preface of Abbott's popular book for undergraduate students, *Understanding Analysis*, it becomes very clear where mathematicians see a major difference between school and university mathematics: "Having seen mainly graphical, numerical, or intuitive arguments, students need to learn what constitutes a rigorous proof and how to write one" (Abbott, 2000, p. vi). This view is also transported by Heuser's popular analysis textbook for first semester students (Heuser, 2009, p. 12, author's translation).

> The beginner at first feels […] uncomfortable […] with what constitutes mathematics:
>
> - The brightness and rigidity in concept formation
> - The pedantic accurateness when working with definitions
> - The rigor of proofs which are to be conducted […] only with the means of logic not with *Anschauung*.
> - Finally the abstract nature of mathematical objects, which we cannot see, hear, taste or smell. […]

This does not mean that there are no pictures or physical applications in the book; it is common sense that modern mathematicians work with pictures, figural mental representations and models – but in contrast, to many students it is clear to them that these are illustrations or visualizations only, displaying certain logical aspects of mathematical objects (and their relations to others) but by no means representing the mathematical objects in total. This distinction gets a little more explicit if we look at a textbook example. In school books the reference objects for functions are curves. Functions are virtually identified with empirically given curves. Consequently, schoolbook authors work with the concept of graphical derivatives in the context of analysis (see Fig. 2). At university, curves are by no means the reference objects anymore; they are only one possibility to interpret the abstract notion of function. The graph of a function in formal university mathematics is only a set of (ordered) pairs.

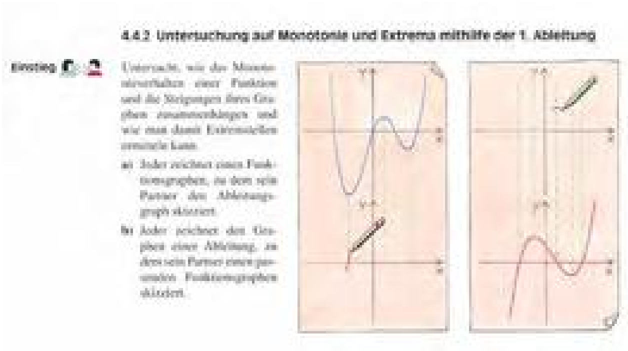

Figure 2: Graphical derivatives in a schoolbook.

If we, in a theoretical simplification, contrast the *empirical belief system* many students acquire in classroom with the *formal(ist) belief system* students are supposed to learn at university we have one model that explains the problem of transition. For example, in this model the notion of proof differs substantially in school and university mathematics. Whereas at universities (especially in pure mathematics) only formal deductive reasoning is acceptable, non-rigorous proofs relying on "graphical, numerical and intuitive arguments" are an essential part of proofs in school mathematics where we explain phenomena of the "real world". In the terminology of Sierpinska (1992), students at this point have to overcome a variety of "epistemological obstacles", requiring a big change in their understanding of what mathematics is about.

HELPING TO BRIDGE THE GAP: SEMINAR CONCEPTION

The findings of the survey and the theoretical discussion are essential for the author's design of a course for pre-service teachers which will be taught, evaluated and analyzed for the first time in spring 2015 together with the University of Cologne (Horst Struve) and the Florida State University (Kathleen Clark). [1]

The overall aim of the course is to make students aware and to lead them to understand of crucial changes regarding the nature of mathematics from school to university. The different conceptions of mathematics in school and university can be reconstructed as well for the history of mathematics, as we previously stated. Thus, an understanding of how and why changes regarding the nature of mathematics (for example from empirical-physical to formal-abstract) took place may be achieved by an historical-philosophical analysis (cf. Davies 2010). This is the key idea of the course. Thereby we hope that the students then can link their own learning biographies to the historical development of mathematics. This conceptual design of the course draws upon positive experience with explicit approaches regarding changes in the *belief system* of students from science education (esp. "Nature of Science", cf. Abd-El-Khalick & Lederman, 2001).

The undergraduate course designed to cope with the transition problem is organized in four parts:

1) Raising attention for the importance of beliefs/philosophies of mathematics.
2) Historical case study: geometry from Euclid to Hilbert. Which questions lead to the modern understanding of mathematics?

3) What characterizes modern formal mathematics? (Exploration of Hilbert's approach.)
4) Summarizing discussion and reflection

1) *Raising attention for the importance of beliefs/philosophies of mathematics.*

In the first part of the seminar we want to make the students aware of the idea of different belief systems/natures of mathematics. Here we start with individual reflections and work with authentic material such as transcripts from Schoenfeld's research that clearly show the meaning and relevance of the concept of an empirical belief system. In a next activity we will compare textbooks: University textbooks, school textbooks, and historical textbooks.

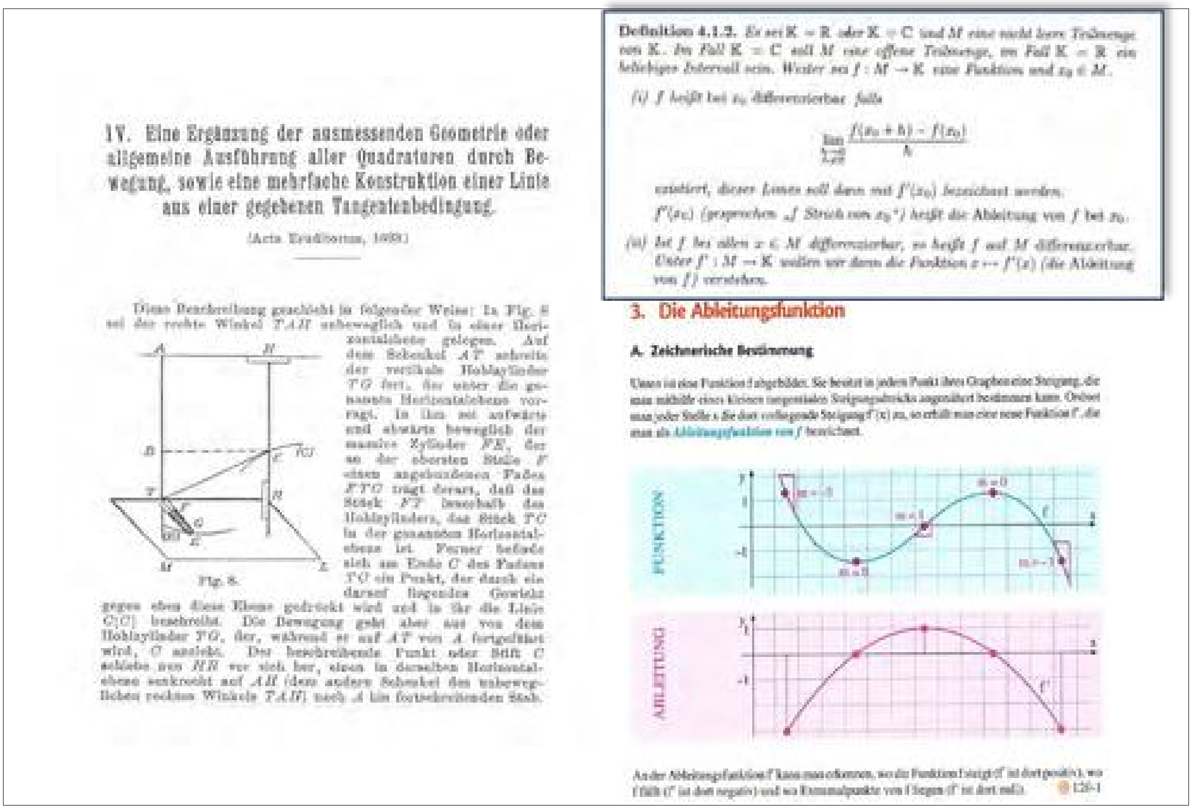

Figure 3: Three excerpts of different textbooks for comparison.

The three excerpts (Fig. 3) illustrate how we will work in this comparative setting. In the upper right-hand corner of Fig. 3 is a formal university textbook definition of differentiation. It is characterized by a high degree of abstraction: the objects of interest are functions defined on real numbers and even complex numbers. We see a highly symbolic definition where the theoretical concept of limit is necessary. Just below we see in contrast, is an excerpt from a popular German school textbook. Here the derivative function is defined on a purely empirical level: the upper curve is virtually identified with the term 'function'. Characteristic points are determined by an act of measuring and the slopes of the triangles are then plotted underneath and constitute the red curve. Interestingly for students, should be that the theoretical abstract notion of function – as it is presupposed in the university textbook – did not always characterize analysis.

If we look back to Leibniz (one of the fathers of analysis), with his *calculus differentialis* and *intergalis*, he conducted mathematics in a rather empirical way as well (cf. Witzke, 2009): his objects were curves given by construction on a piece of paper – properties like differentiability or continuity were read out of the curve…and not only there seem to be parallels on an epistemological level between school analysis and historical analysis. For example, Leibniz presented (published in 1693) the invention of the so-called integrator (left-hand side of Fig. 3), a machine that was designed to draw an anti-derivative curve by retracing a given curve. So here, as in the schoolbook, it is on an epistemological level that the empirical objects form the theory. Combined with selected quotes from schoolbooks emphasizing its experimental and empirical access to mathematics, quotes like Pasch's regarding Euclidean geometry as an empirical science on the one hand and Hilbert's statement,

> If I subsume under my points arbitrary systems of things, e.g. the system: love, law, chimney sweep ..., and then just assume all my axioms as relationships among these things, then my theorems, e.g. also the Pythagorean theorem, are true of these things, too. (Hilbert to Frege, 1980, p.13, author's translation)

on the other, it becomes clear that something revolutionary had changed regarding the nature of mathematics at the end of 19th century mathematics. This change is a revolution, which on an epistemological level has parallels to what students encounter when being faced with abstract university mathematics.

2) *Historical case study: geometry from Euclid to Hilbert. Which questions lead to the modern understanding of mathematics?*

An adequate description of the development of the conception of mathematics in the course of history requires more than one book. We refer to the following ones: Bonola (1955) for a detailed historical presentation; Grabe (2001), Greenberg (2004) and Trudeau (1995) for a lengthy historical and philosophical discussion; Ewald (1971), Hartshorne (2000), and Struve & Struve (2004) for a modern mathematical presentation. Additionally, Davis & Hersh (1981 & 1995) or Ostermann & Wanner (2012) presented aspects of the historical and philosophical discussion in a concise manner, relatively easily accessible to students.

The overall aim of the historical case study is to make students aware of how the nature of mathematics changed over history. Regarding our theoretical framework, we aim to make explicit how geometry – which for hundreds of years seemed to be the prototype of empirical mathematics, describing physical space – did develop into the prototype of a formalistic mathematics as formulated in Hilbert's foundations of Geometry in 1899 (cf. Fig.4). And thus, we can help students on their way to develop an understanding for different mathematical conceptions, in particular, modern ones taught at the university level.

In the course we start with Euclid's *Elements*: they show what a deductively built piece of mathematics looks like in a prototype manner. Here we will induce the students, e.g., to display in a graphical manner how Pythagoras' theorem can be traced down to the five postulates – as the 2013 survey results showed that most students were not familiar with a deductive structure after one year at university.

It is quite important for the overall goal of the seminar that the *Elements* give reason to discuss status, meaning and heritage of axiomatic systems. Thereby we will focus on the self-evident character of the axioms (or, postulates) describing *physical space* in a true manner – as undoubtedly provides insights on the surrounding real space which were accepted without proof (cf. Garbe, 2001, p. 77).

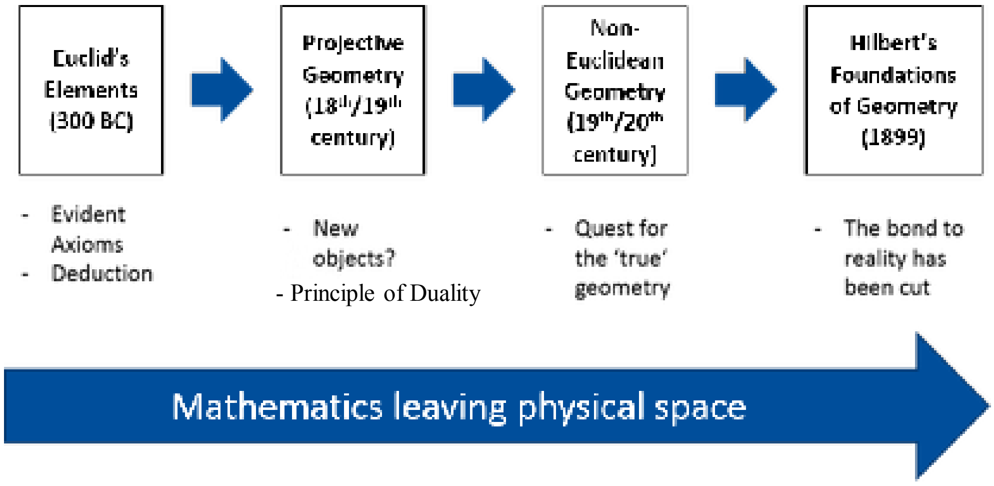

Figure 4: The historical and philosophical development of mathematics along the development of geometry

Projective geometry is the next stop on our way to a modern understanding of geometry (cf. Ostermann & Wanner, 2012, pp. 319-344). Starting with the question of whether other geometries, besides the Euclidean one, are conceivable, projective geometry seems to be an ideal case. Related to the overall aim of the course, the notion that there is more than one geometry can foster the idea that there is more than 'one' mathematics, leading away from the quest for one unique mathematics describing physical space (cf. Davis & Hersh, 1985, pp. 322-330).

Well, on the one hand, projective geometry seems to be so intuitive and evident if we look at its origin in the arts in the vanishing point perspective. On the other hand it adds new abstract objects to the Euclidean geometry (the infinitely distant points on the horizon) and familiarizes us with the idea that all parallels may meet eventually. With projective geometry the students encounter a further axiomatizable geometry – which has irritating properties that finally influenced Hilbert (cf. Blumenthal, 1935, p.

402) to ultimately design a geometry *free of any physical references*. Julius Plücker saw in the 19th century for the first time, that theorems in projective geometry hold if the terms "straight line" and "point" are interchanged: the so-called principle of duality – giving a clear hint why it became reasonable in mathematics to focus on mere structures of theories.

A decisive revolutionary step towards a formalistic abstract formulation of geometry can then be seen in the development of the so-called non-Euclidean geometries. This development is connected in particular to the names Janos Bolyai (1802-1860), Nikolai Ivanovitch Lobatchevski, Carl Friedrich Gauß (1777-1855) oder Bernhard Riemann (1826-1866) (cf. Garbe, 2001, Greenberg, 2004, Trudeau, 1995 on their historical role regarding non-Euclidean Geometries).

In fact, the non-Euclidean geometries developed from the "theoretical question" around Euclid's fifth postulate, the so-called parallel postulate:

> Let the following be postulated: [...]
>
> That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles. (Heath, 1908)

Compared to the other postulates like the first, "to draw a straight line from any point to any point," the fifth postulate sounds more complicated and less evident. This postulate cannot be "verified" by drawings on a sheet of paper as parallelity is a property presupposing infinitely long lines. In the words of Davis & Hersh (1995), "it seems to transcend the direct physical experience" (p. 242). In history this was seen as a blemish in Euclid's theory and various attempts have been undertaken to overcome this flaw. On the one hand, different individuals tried to find equivalent formulations, which are more evident (e.g. Proclus (412-485), John Playfair 1748-1819)[1]. On the other hand, several mathematicians tried to deduce the fifth postulate from the other postulates so that the disputable statement becomes a theorem (which does not need to be evident) and not a postulate (e.g. Girolamo Saccheri (1667-1733), Johann Heinrich Lambert (1728-1777)). (cf. Davis & Hersh, 1985, pp. 217-223; Garbe, 2001, pp. 51-74; Greenberg, 2004, pp. 209-238)

In contrast in the 18th and 19th century, Bolyai, Lobatchevski, Gauß, and Riemann experimented with negations and replacements of the fifth postulate guided by the question of whether the parallel postulate was logically dependent of the others (cf. Greenberg, 2004, pp. 239-248). If this would have been true – Euclidean geometry

[1] To Proclus, who was amongst the first commentators of Euclid' Elements in ancient Greece, already formulated doubts on the parallel postulate and formulated around 450 an equivalent formulation (cf. Wußing & Arnold 1978, p. 30). Playfair's formulation (1795), "in a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point", is quite popular today (cf. Prenowitz & Jordan 1989, p. 25; Gray 1989, p. 34).

should actually work without it – what it does, in a sense that no inconsistencies occur. But this logical act leads to conclusions that differ from those in Euclidean geometry.

For example:

- In the so-called hyperbolic geometry the sum of interior angles in a triangle adds up to less than 180°, in elliptic geometry to more than 180° (see Fig. 6).
- The ratio of circumference and diameter of a circle in hyperbolic geometry is bigger than π, in elliptic smaller than π.
- In hyperbolic as in elliptic geometry triangles which are just similar but not congruent do not exist.
- In hyperbolic geometry there is more than one parallel line through a point P to a given line g and in elliptic geometry there are no parallel lines at all (see Fig. 5).

(cf. Davis & Hersh, 1985, p. 222; Garbe, 2001, p. 59)

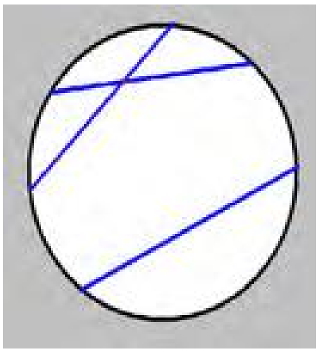

Figure 5: Klein's Model for hyperbolic geometry: More than one parallel line to a straight line through a given point.

Working with texts and sources regarding the process of discovery of the non-Euclidean geometries may have an important impact on students' beliefs system, as it tackles the so-called "Euclidean Myth" (Davis & Hersh, 1985) which was widespread within the 2013 survey results: to many first-year students mathematics is a monolithic block of eternal truth; a theorem, once proven, necessarily holds in every context.

With the discovery of the non-Euclidean geometries, it became apparent in history that there is no such truth in a total sense. In contrast, there seems to be more of such truths, depending on the context you are working in. A discussion of Gauss's qualms to publish results on non-Euclidean geometry implicitly emphasizing this aspect, afraid of being accused of doing something suspect, or the (probably legendary) story that he tried to measure on a large scale whether the world is Euclidean (cf. Garbe, 2001, pp. 81-85), can make the students amenable to the revolutionary character of this discovery for changing natures of mathematics. Following Freudenthal's (1991) idea of guided reinvention, recapitulating the history of humankind seems to bear quite fruitful perspectives for the development of individual belief systems in this context.

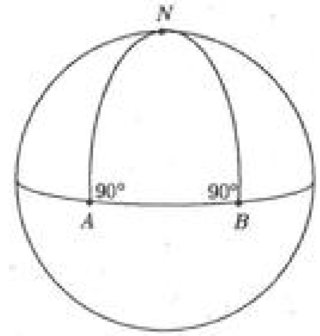

Figure 6: Angle sum in an elliptic triangle.

Finally, from the discussion of the non-Euclidean geometries students will approach the questions which lead to Hilbert's formal(istic) turn. *If there was more than one consistent geometry, which one is the true one?* This question is closely linked to the question*, what is mathematics?*

3) What characterizes modern formal mathematics? (Exploration of Hilbert's approach.)

Hilbert actually gave an answer to this problem – not only in a philosophical and programmatic way but also by formulating a geometry "exempla trahunt" (Freudenthal, 1961 p. 24), a discipline that was seen for ages as the natural description of physical space, in a formalistic sense and characterized by an axiomatic structure. The established axioms are fully detached and independent from the empirical world, which leads to an absolute notion of truth: mathematical certainty in the sense of consistency. With Hilbert the bond of geometry to reality is cut. This becomes very vivid when reading Hilbert's *Foundations of Geometry* (1902; see Fig. 7) in detail, as we plan to do with the students in the seminar.

> § 1. THE ELEMENTS OF GEOMETRY AND THE FIVE GROUPS OF AXIOMS.
>
> Let us consider three distinct systems of things. The things composing the first system, we will call *points* and designate them by the letters A, B, C,...; those of the second, we will call *straight lines* and designate them by the letters a, b, c,...; and those of the third system, we will call *planes* and designate them by the Greek letters α, β, γ,... The points are called the *elements of linear geometry*; the points and straight lines, the *elements of plane geometry*; and the points, lines, and planes, the *elements of the geometry of space* or the *elements of space*.

Figure 7: The famous first paragraph of Hilbert's *Foundations of Geometry.*

Hilbert does not give his concepts an explicit semantic meaning; he speaks independently from any empirical meaning of distinct systems of things. Consequently, intuitive relations like *in between* or *congruent* do not have an empirical meaning but are relations fulfilling certain formal properties only. (cf. for example, Greenberg, 2004, pp. 103-129)

As we all know, the development of mathematics did not come to an end with Hilbert; the seminar is intended to finish with discussions of texts taken from *What is Mathematics, Really?* (Hersh, 1997). Hersh understands "mathematics as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context" (p. xi), which creates a balanced view.

However, nobody will deny that formalism in Hilbert's open-minded version had a lasting effect on the development of mathematics. As a consequence, today's university mathematics has the freedom to be developed without being 'true' in an absolute sense anymore (cf. Freudenthal, 1961), but nevertheless including the possibility to interpret it physically again.

> In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. And it seems to me that the numerous and surprising analogies and that apparently prearranged harmony which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science, have their origin in this ever-recurring interplay between thought and experience. (Hilbert, 1900)

It is the openness and freedom of questions of absolute truth, which Hilbert replaced by the concept of logical consistency that made mathematics so successful in the 20th century (cf. Freudenthal, 1961, p. 24; Garbe, 2001, pp. 100-109, Tapp, 2013 p. 142). In Einstein's words:

> Geometry thus completed is evidently a natural science; we may in fact regard it as the most ancient branch of physics. Its affirmations rest essentially on induction from experience, but not on logical inferences only. We will call this completed geometry "practical geometry," and shall distinguish it in what follows from "purely axiomatic geometry."[…]As far as the propositions of [modern axiomatic] mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.[…] The progress achieved by axiomatics consists in its having neatly separated the logical-formal from its objective or intuitive content […] These axioms are free creations of the human mind. The axioms define the objects of which geometry treats. […] I attach special importance to the view of geometry, which I have just set forth, because without it I should have been unable to formulate the theory of relativity. (Einstein, 1921, as cited in Freudenthal, 1961, p. 16; for a readable article on exactly this point compare with Hempel (1945))

This makes again quite clear that modern mathematics after Hilbert is on epistemological grounds completely different than (historical) empirical mathematics and of course mathematics taught in school. Whether the first is grounded on set axioms and the notion of mathematical certainty (inconsistency), the second and third are grounded in evident axioms – thus describing physical space including a notion of (empirical) truth, resting essentially on induction from experience.

4) Summarizing discussion and reflection

In the last part of the course we want to initiate discussions connecting the insights gained from the historical perspectives with the individual biographies. We plan to remind the students about the preliminary discussions regarding different personal belief systems that occurred in the first part of the course. The intention is that the transparency on the historical problems that led to a modern abstract understanding of mathematics leads to an understanding of what happens if students live on epistemological grounds through this revolution as individuals, thus opening differentiated views on the transition problem. For school purposes – from a well-informed mathematics educator's point of view – nothing speaks against doing mathematics in an empirical way (when including deductive reasoning, of course, otherwise it would just be phenomenology). History has shown that empirical mathematics was a decent way to develop mathematical knowledge and the experimental natural sciences generate knowledge comparably. Yet approaches to bring formal(istic) mathematics into school classrooms have failed miserably. Moreover, we cannot step away from teaching mathematics in a theoretical way at universities. In contrast, the course described here intends to make tangible, understandable, and explicit (that) to first-year students the transition from school mathematics to university mathematics is an epistemological obstacle. Hefendehl-Hebeker (2013, p. 80) sees quite comparably

> […] a principle difference between school and university is at university with the axiomatic method a new level of theory formation has to be reached, and thus it follows that the discontinuity cannot be avoided.

So if the discontinuity cannot be avoided, what may teachers and students at university take from a course like the one described here?

1) The historical excursions do not only focus on the beliefs aspect but also demonstrate crucial mathematical activities – especially regarding deductive reasoning within the frameworks of consistent mathematical theories.

2) Teachers and students should be sensible about the dimension of the problem: it is not as easy as repeating some lower secondary school mathematics, as many approaches seem to suggest. Instead a revolutionary act of conceptual change is required that does not occur overnight and needs guidance. The historical questions that lead to the modern understanding of mathematics are too sophisticated and waiting for students to develop these for themselves is a particular burden on top of all the other factors of beginning at university. The approach of initiating these questions explicitly within the described framework may support a more adequate and prompt change of belief system.

3) The course should sensitize for crucial communication problems. Teachers and students should acknowledge that when talking about mathematics, using the same terms might not imply talking about the same things. For example, students may come from school to university having learned calculus in an empirical context such that functions might be equivalent to curves. This might imply that properties like continuity or differentiability are empirical and can be read from the sketched graph of the function (comparable to 17th century mathematicians). The lecturer at university on the other hand probably has a general abstract notion of function implying a completely different notion of mathematical reasoning and truth. In particular, lecturers should repeatedly check if the knowledge of their students is still bonded to

(single) objects of reference. The same holds for the students eventually leaving university and starting as secondary school mathematics teachers: they should be aware that what they consider from an abstract point of view their students may instead possess visualizations of abstract notions as the reference objects.

CONCLUDING REMARKS/PERSPECTIVES

An in-depth study based on data collected from surveys containing both standardized and open-ended items and student interviews accompanying the seminar course describes here will follow in 2015. A follow-up course will be conducted at Florida State University in spring 2016. The data, along with the personal evaluations of the involved researchers, will clarify whether explicitly discussing historical epistemological obstacles regarding changes on mathematical belief systems supports students on their way through university mathematics. Much will depend on if we succeed in initiating thinking-processes which bring the historical and personal perspectives together. Only then will it be possible to determine if the historical-philosophical elements of the course have a lasting effect.

NOTES

1. Many elements of the course discussed here have been tested in isolated settings in Cologne and Siegen but not in a coherent course to face the problem of transition.

2. Also, there is another dimension to the axioms as fundamentals of a platonic construct of ideas, called "geometry".

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Oral Presentation

PARALLELS BETWEEN PHYLOGENY AND ONTOGENY OF LOGIC

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The principle of a parallel between the ontogenetic and the phylogenetic development of knowledge is a known principle in mathematics education (see Schubring 1978). If teaching omits some of the stages, that were important in the historical development of the particular discipline, it can become an obstacle for students understanding. Logic is in this respect a very special discipline. Its origins are linked to ancient philosophy; during the Middle Ages it became an instrument for theological disputations. It was not until the late 19th century that a specific kind of logic, which we today call mathematical, started to develop. In our research we try to find out whether there are any parallels between the historical development of logic and the spontaneous growth of logical thinking in children.

THE GENETIC PARALLEL

The idea of a parallel between the ontogenetic and the phylogenetic development is established internationally (Schubring 1978) as well as in the Czech didactics of mathematic (Hejný 1984, Hejný & Jirotková 2012). Also P. Erdnijev's words are often cited:

> "The growing of the tree of mathematical knowledge in mind of each person will be successful only if repeats (to a certain extent) history of growing of this science." (Erdnijev 1978, p. 197)

Freudenthal expressed the same idea more precisely:

> "Children should repeat the learning process of mankind, not as it factually took place but rather as if would have done in people in the past had known a bit more of what we know now." (Freudenthal 1991, p. 48)

If the teaching process does not respect some important developmental stages of a particular discipline or topic, students can have problems to understand it. As one example for all, we can take the calculus. The history of this mathematical discipline is linked with Newton's and Leibniz's theory of infinitesimals. But nowadays teaching of derivatives typically starts with the so-called ε-δ calculus, which is mathematically more correct, but much less intuitive (Toeplitz 1949). Many students then have problems to grasp the essence of this topic; they learn to repeat theorems and proofs only in a formal way.

Logic is a very special discipline in this respect. As we indicated in abstract, the historical development of this discipline has crossed the spheres of philosophy, theology and other humanities. And only a short period of its modern development is connected with mathematics. But – contrasting to that – the first (and often the only) logic students meet is the mathematical one; Boole's truth-functions and Frege's quantifiers. These logical lessons take place typically during the first year of higher secondary education.

RESEARCH QUESTIONS AND AIMS

The idea of the genetic parallel seems to be ignored in teaching/learning logic. Should we see this as a problem? What is the level of logical abilities of students leaving lower secondary education? Is there a significant progress in logical abilities during the lower secondary education? And finally: Are there any parallels between the historical development (phylogeny) and the ontogeny of logic? For example, is it possible to say, that a child, that cannot solve a syllogism, will be unable to understand the idea of implication?

These questions and all of them are rather broad and so it is not possible to give precise and clear answers. This is a beginning of our research and we hope to formulate some more specific questions (and answers) in our future research and experimental work.

THE THEORY OF ONTOGENETIC DEVELOPMENT

Why do we think, that we should be able to recognize a development of logical abilities of children during the lower secondary education? One of the reasons is Piaget's theory of stages of cognitive development. According to him, the last stage of cognitive development, which is called "stage of formal operations", starts about the 12th year of age.

> "Subject starts to be able to draw logical consequences from *possible truths*. (...) He/she should be able to use new propositional operations, such as implication (*if then*) disjunction (*either or*)..." (Piaget 1971, p. 98)

The important part is the "possible truths". It means that children in this stage should be able to determine the truth value of statements only on the basis of the logical form of the sentences.

RESEARCH TASK EXAMPLES

We will discuss two concrete tasks and examples of their evaluation from our research. The first task is concerned on the very idea of implication. From the ontogenetic point of view, it is generally known, that children (but not only them) do not distinguish the relation of logical implication from that of equivalence. There are many researches on this topic, e. g., Shapiro & O'Brian (1970), Hoyle & Küchemann (2002). This tendency even received its own label; it is known as "child logic".

Focusing on the phylogenetic approach, implication has been a big issue since ancient times. There were some alternatives to the classical material form of implication, e. g.

Diodoros or Chrissipos attempts (Kneale & Kneale 1962), but all of them used modal logical functors. A few centuries later, C.I. Lewis deals with this topic in the realm of modern logic; his conception of conditionals is widely called “a strict implication” (Lewis & Langford 1932). This concept, just like those of Diodoros and Chrissipos, is situated in the field of intensional logic.

From the practical point of view, a very important issue is the influence of context. In the wider sense – we have no other possibility than to use our natural language to describe the task situation. And there problems could appear caused by the ambiguity of some terms of natural language. And in a more strict sense context is closely connected with the issue of motivation. More information about the role of context in logical tasks can be found in O’Brian, Shapiro & Reali (1971).

The concrete form of our implication-task was inspired by the Wason selection task. It is a very famous task from the area of psychology, when researchers tried to disprove Piaget. Wason selection task was submitted to different groups of adults, but almost in all cases the number of correct answers was only about 10 %. A very interesting study about using this task in the rather specific population of mathematics teachers and students is in Inglis & Simpson (2006).

Let us formulate the first task:

> A brave Prince entered a mysterious castle and after a while he found himself in a special room. There were no windows and the light of a few torches fell upon a large book that lay on a pedestal in the middle of the room. The book was opened on the first page and the prince read:
>
> Brave visitor, the door can hide great danger!
>
> Choose well: **If there is a tiger behind the door, there is the letter T on this door.**
>
> In addition to the door by which the prince entered the room, there were three doors:
>
> (1) door with the letter D,
>
> (2) door with the letter T and
>
> (3) door without any letter.
>
> Make a decision for every door whether:
>
> a) *there is a tiger,*
>
> b) *there is not a tiger,*
>
> c) *we cannot decide, whether there is a tiger.*

To find the correct answer two mental steps are needed. At first – in the first and in the third doors cannot be a tiger, because in that case there was one, there would be a “T” on these doors. From the logical point of view, it is the rule of *modus tollens*.

The second operation is much more difficult. About the second door we cannot decide, because in either case, if there is or is not a tiger behind the door, the rule (tiger $\rightarrow$ T) is not broken.

The correct answer is 1b-2c-3b. That means: behind the door with the letter D there is not a tiger, about the door with the latter T we cannot decide whether there is a tiger and behind the door without any letter there is not a tiger.

Table of respondents of our research follows:

Grade	5^{th}	6^{th}	7^{th}	8^{th}	9^{th}
Number of respondents	38	47	51	55	37

Table 1: Numbers of respondents

In our research sample only 2 % of respondents chose the correct combination (1b-2c-3b). The semi-correct answer (1b-2b-3b or 1b-2a-3b) chose 20 %. The most common wrong answer was, as you can guess, 1b-2a-3c; chosen by 35 % of respondents.

But this is a very general classification of the answers. We tried to introduce a finer and deeper classification ascribing a score value for every part of all possible answers. This score value expresses the difficulty of the particular answer. In the following table you can see the scores and also the concrete percentage of given parts of the answers. In the table below there are the score values we assigned to these answers.

	Door (1)			Door (2)			Door (3)		
Answer	a)	b)	c)	a)	b)	c)	a)	b)	c)
Percentage	7 %	62 %	29 %	84 %	10 %	5 %	7 %	40 %	50 %
Score value	0	3	1	1	2	20	0	4	1

Table 2: Percentages and scoring values for task no. 1

Let us explain how we ascribed the score values. The simplest is the explanation of the score values 1. For the second and the third door we used this value for answers, which are the most intuitive. In the first door we used this value for the answer 1c, which is logically equivalent with the choice of 3c.

We decided to assign a slightly higher value to the answer 2b. It is still a wrong answer, but the respondent seems to suspects that the implication form cannot be reversed without loss of a generality.

Next higher values – 3 and 4 – we used for the correct answers in the first, respectively in the third door, which are logically equivalent. But the values are not the same, because the answer 1b is much more intuitive than the choice of 3b.

Finally we have to justify the high score value for the answer 2c. As we already mentioned, a very difficult mental operation is needed to do this choice. To determine the concrete value we compute how many “points” we gave in total in the first and in the third door. In both of these doors the total score was about 200 (205 and 210 respectively). To have a similar total score in the second door, we decided to use the value 20.

The zero values were used for answers, which are wrong and contra-intuitive; we are not able to identify the concrete mental operations leading to these answers.

After ascribing score values to each answer we can calculate the table of average scores in each grade. As can be seen from the table below, no significant progress can be seen there.

Grade	5th	6th	7th	8th	9th
Average score	6,03	5,11	7,53	5,85	6,94

Table 3: Average scores for task no. 1

The second task we used was based on the logical form of a syllogism. From the ontogenetic point of view, a syllogism is a very common logical form, but usually we use it implicitly, some of the premises are unspoken. To justify the choice of a syllogism from the phylogenetic point of view, we can, of course, mention Aristotle and his very impressive logical system. But we will cite the research of the Russian psychologist Alexander R. Luria (1976). In his study on the historical development of cognitive processes we read:

> "The emergence of verbal-logical codes allows to abstract the essential symptoms of objects ... leads to the formulation of complex logical apparatus. These logical devices allow getting the conclusions from the premises without immediate clearly achievable (known) reality. They allow acquiring new knowledge discursive, verbal-logical manner." (Luria 1976, p. 116)

To map the historical development of cognitive processes Luria did a research during the 1930-ties in the least developed areas of Soviet Union, on the territory of present-day Kyrgyzstan and Uzbekistan. People living here were usually illiterate, in most cases they have never left their native valley. And Luria gave them tasks, in which different levels of abstraction were needed. Some of these tasks also deal with syllogistic form. For example:

> "Cotton can grow only where it is hot and dry. In England the weather is cold and damp. There can grow cotton?" (Luria 1976, p. 122)

For most of Luria's respondents it was typical that their thinking very strongly bond to their common everyday concrete reality. Using modern terminology, they were unable to make a hypothetical judgment. We can quote one typical answer:

> "I do not know, I was just in Kaschgaria ... if there was a man who was everywhere, well he could have answered to that question." (Luria 1976, p. 122)

We expected similar phenomena when we gave tasks with syllogism to children. The context we were using was familiar to the children, so this kind of argumentation (from experience) was possible. But, of course, we hoped to see an increasing portion of use of logical argumentation. Let us formulate the task:

> In a class, for all boys the following two rules apply:
>
> 1st: Anyone who plays football can run well.
>
> 2nd: Somebody of those who plays hockey plays football too.
>
> Can we surely say that there is a hockey player, who can also run well in this class? Write why.

And there are two answers representing a kind of experience-argumentation. Both of them were written by 5th graders.

> "Yes, hockey player knows how to run, because if he could not, he cannot skate fast."
>
> "No, he is a hockey player, so he may not be able to run well, but he must skate well to be able to play the game."

Even if these two answers are, from the logical point of view opposite – both of them are on the same cognitive level. Let us describe the evaluation process of this task. After repeated reading of all the answers we sorted them to five, respectively six categories. These categories we ordered to according to cognitive performance needed to give this kind of answer and we assigned them corresponding values.

In the 0th category we included the respondents, who didn't give any answer.

The 1st category we used for students, who answered yes or no, but without argumentation.

Next category included the "experience-argumentation".

Category no. 3 we called "stepping out of experience". It is not really a judgment, but some hints can be seen in that direction.

Next – the 4th category – is a logical judgment.

And the last category is an if-judgment. Because the really right answer should include a condition of non-empty set of boys in that class and similarly a condition of non-empty set of football players.

Category	No answer	No argumentation	Experience argumentation	Stepping out of experience	Judgment	If-judgment	Average score inc. 0th cat.	Average score exc. 0th cat.
Value	0	1	2	3	4	5		
5th grade	14 %	8 %	35 %	3 %	43 %	0 %	2,91	2,53
6th grade	38 %	24 %	20 %	2 %	20 %	0 %	2,27	1,45
7th grade	22 %	10 %	33 %	6 %	33 %	0 %	2,75	2,16
8th grade	19 %	11 %	30 %	4 %	39 %	0 %	2,84	2,33
9th grade	19 %	6 %	8 %	8 %	56 %	6 %	2,57	2,89

Table 4: Percentages and average scores for task no. 2

We used two averages in this task. If we want to know which of them fits better the situation, we need to know why some of our respondents didn't give us any answer. This can be because they didn't understand it or because they really didn't know. There many possibilities.

Seeking any trend in this table, we can compare the column titled "Judgment" and the last column, "Average score excluding 0th category". In both of these columns we can see an increasing trend starting in the 6th grade. But our data do not allow us to say whether these trends are statistically significant.

CONCLUSIONS AND FUTURE ORIENTATION OF RESEARCH

Our research, a small part of which was described here, consisted from seven tasks. We tried to map logical abilities of children in the several areas such as classification, negation, syllogism and implication. At first we were a really surprised by the absence of any remarkable trend.

Nowadays we believe that the idea of genetic parallel in logic needs to be grasped in another way. We have at least two two possible explanations:

a) We can find no continous ontogenetic development, because the phylogenetic development wasn't fluent too. After a very fruit-full era of ancient logic there were several centuries, in which the development of logic was incomparably slower. And from the end of 19th century, this development again rapidly accelerated by the onset of modern logic.

b) We cannot rely on the spontaneous ontogenetic development, because the phylogenetic development wasn't spontaneous too. Aristotle's *Organon* emerged from his confrontation with the social problem of sophistic philosophy. The origin of modern logic was stimulated and conditioned by requirements of the development of mathematics.

This indicates that our approach to this topic has to be wider. Now we are studying possible connections between logic and cognitive sciences. We can see some concepts which can be very useful for our effort to describe development of logical thinking in children; e.g. conceptual metaphors and theory of embodied mind, that were introduced by Lakoff & Núñez (2000) in the book "Where the mathematics comes from" or the very impressive theory about cognitive tools, which was worked out for the field of logic by Novaes (2012) in her book "Formal languages in logic – A Philosophical and Cognitive Analysis."

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THEME 2:

CLASSROOM EXPERIMENTS AND TEACHING MATERIALS, CONSIDERED FROM EITHER THE COGNITIVE OR/AND AFFECTIVE POINTS OF VIEW, SURVEYS OF CURRICULA AND TEXTBOOKS

Plenary Lecture

HISTORY IN THE CLASSROOM: EDUCATIONAL OPPORTUNITIES AND OPEN QUESTIONS

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Using the history of mathematics in everyday classroom activities is difficult because of various reasons, but it is an intriguing aim. This paper will report examples of activities, mainly inherent to interpretation of original texts, developed in my classes. Opportunities and problems, achievements and failures will be analysed.

With the aim of carrying out a critical analysis, theoretical considerations will be taken into account. The purpose is to introduce an ongoing discussion with regard to the complexity of everyday classroom activities. Ultimate answers are not the main aims of my analysis.

INTRODUCTION

It is widely recognized the importance of introducing history in mathematics teaching at all school levels, see (Furinghetti, 2012; Barbin & Tzanakis, 2014). In this paper I report on the activities I have recently carried out in my classroom. My motivations for the use of history rely on the conviction that history is a carrier of culture in student's view of mathematics (Jankvist, 2015) and that important educational goals of mathematics teaching may be achieved through history (Kjeldsen & Jankvist, 2011). In planning my activities I followed Janhke et al. (2000) who support the use of original sources in the classroom as a demanding task that may be carried out according to different perspectives. I considered group work among students and role of the teacher as tools for transformation of knowledge (Radford, 2011).

My paper will illustrate the theoretical background orienting my choices, the school context in which I realized my project, the main steps of the implementation into the classroom and the analysis of the outputs with some preliminary conclusions. My experiment will be presented almost as a narration to allow the reader hearing the voice of a teacher who tries to combine his educational goals in teaching mathematics with his passion for history. This narration is going to highlight the facts that, in my opinion, are really significant for discussing the issues related to the use of history and make my experiment transferable to other situations.

The problem of the transferability of experiments to different situations with different teachers is really crucial. In particular, when dealing with the introduction of history of mathematics in mathematics teaching, there is the problem of the teachers who do not believe that this introduction is possible or really suitable to reach their teaching goals. Many teachers are not familiar with history of mathematics and, even more, with original sources. All these difficult cases were discussed in the workshop I carried out during the conference ESU7 (see Demattè, to appear) on the ground of the paper (Siu, 2006). To meet the need and the perplexities of these teachers I devote a section of this paper to the presentation of materials and teaching sequences that may be developed in a mathematics laboratory.

In recent years, some authors, Jankvist (2009) for one, have raised the question of promoting empirical research to better understand the potentialities of this use. Through the description of my experiment and the analysis of the doubts raised by the results of my experiment I hope to offer materials for facing the following research questions:

- What kinds of activities are most suitable to involve teachers in using history of mathematics in their classroom?
- What educational goals regarding the use of originals could teachers consider relevant goals for their mathematics classes?

THEORETICAL BACKGROUND

Due to the fact that in my experiment I try to combine the need of achieving my educational goals and my confidence on the efficacy of history in my teaching, the theoretical underpinning of my work is inspired both by the research in mathematics education in general and by the particular research which concerns the relation between history and pedagogy of mathematics.

Classroom culture and mathematical discourse

Let me start from the educational side by quoting my personal experience as a young teacher. I remember an author whose works I got to know during a training course at the very beginning of my career. That is Carl Rogers (1951), the American psychologist who is considered the founder of the client-centred approach in psychology. Nowadays, I am able to quote only little of his thought, the following sentence for example: "A person cannot teach another person directly; a person can only facilitate another's learning". Some key words of this statement, or suggested by it, synthesize my ideal approach to teaching: facilitate, students' autonomy, learning with meaning and consciousness.

To explain my approach to teaching I start from the drawing of figure 1 where an Italian pupil of grade 3 answers the task: "Draw your mathematics class, that is the teacher and your classmates in a mathematics lesson. Use bubbles for speech and thought to describe conversation and thinking. Mark yourself (Me) in your drawing", see (Laine, Näveri, Ahtee, Hannula, & Pehkonen, 2012).

The student's perception of the atmosphere in the classroom expressed by this drawing is in line with the description made by Lampert (1990, p. 31) of the school experience in which: "*doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical *truth is determined* when the answer is ratified by the teacher." As a consequence of this school experience we have a cultural assumption which associates mathematics with the idea that knowing mathematics means to be able to get the right answer, quickly and following the rules given by an authority (the textbook, teachers). I try to challenge this common assumption by changing the roles and responsibilities of teacher and students in classroom discourse so that, as advocated by Lampert (1990), the practice of knowing mathematics in school becomes closer to what it means to know mathematics within the discipline.

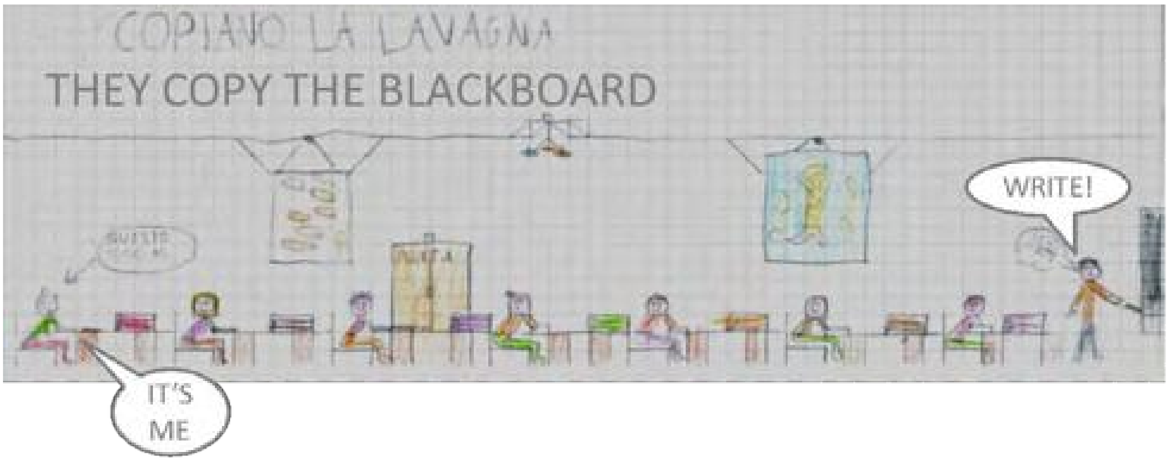


Fig. 1. Drawing by a pupil of grade 3

As a teacher I try to create a classroom culture in which student activity can occur through participation in the doing and learning mathematics so that students learn not only contents of their curriculum but also "what counts as knowledge and what kind of activities constitute legitimate academic tasks" (Lampert, 1990, p. 34). In the classroom discourse I implement teacher-student interaction and content, such as: learning as (re)discovery, group work, discussion among peers and between teachers and students, exploring mathematical phenomena, generating conjectures, verifying and, in case, refuting and refining them. Students become authors of their ideas and responsible of their intervention in the mathematical discourse. As discussed in (Demattè & Furinghetti, to appear), the laboratory is a good place where to realize my project since it allows actual participation to mathematics activities.

Using original historical sources

Taking into account the large amount of literature and my previous experiments, I decided that the history of mathematics is a good tool for realizing my ideas. In particular, as pointed out in the Introduction , original sources have shown their pedagogical efficacy in mathematics teaching, see (Furinghetti, Jahnke, & van Maanen, 2006; Pengelley, 2011; Jankvist, 2014).

Barbin (2006) has pointed out that there are different ways of reading original sources. My way, in line with previous experiments in the classroom, see (Bagni, 2008; Glaubitz, 2011a), is based on the hermeneutic approach. This approach changes the strategy of teaching/learning. Students use original mathematical documents and are asked to use the mathematics they have learnt, in a new way. Jahnke (2014, pp.83-84) outlines the basic guidelines of the hermeneutic procedure as follows:

> "(1) Students study a historical source *after* they have acquired a good understanding of the respective mathematical topic in a *modern* form and a *modern perspective*.
>
> The source is studied in a phase of teaching when the new subject-matter is applied and technical competencies are trained. Reading a source in this context is another manner of applying new concepts, quite different from usual exercises.
>
> (2) Students gather and study information about *context* and *biography* of the author.
>
> (3) The historical *peculiarity* of the source is kept as far as possible.
>
> (4) Students are encouraged to produce *free associations*.
>
> (5) The teacher insists on *reasoned arguments*, but not on accepting an interpretation which has to be shared by everybody.
>
> (6) The historical understanding of a concept is contrasted with the modern view, that is the source should encourage processes of reflection".

The points of the guidelines may be grouped according to different types of action. (2), (3) and (6) concern history of mathematics in its "strong role" (Demattè, 2006a). In contrast with the "weak role" that confines the use of history to mathematics, the "strong role" is based on didactical activities that are directly inherent to history and aims not only at learning the mathematical subject-matter. Then it requires an additional amount of time. Definitely, (2) refers to giving historical knowledge, while (3) may be integrated with mathematics teaching/learning since "the historical peculiarity" may regard the use of unusual procedures and concepts that reinforce previous students' skills by provoking the *dépaysement*, that is the alienation and reorientation (see Janhke et al, 2000). The point (6) furnishes a synthesis of deep reasons for reading originals instead of doing ordinary exercises.

Developing points (4) and (5) means that through the originals it may happen that students' capabilities and though of ancient mathematicians meet. Point (4) suggests that students should get some ways to become protagonists, autonomously with respect to the leading teacher's role. What (5) suggests is far from the current situation in Italian schools. In order to be implemented, it requires a radical change in students' assessment criteria. Judgments would be referred to individual advancement instead of to a set of abilities required by institution. Most teachers agree that students have to be involved in "reasoned arguments", but they aim at contents and conclusions which have to be shared by all students. These teachers would not accept the proposal to use originals according to the point (5). Points (4) and (5) introduce a way to personalize teaching, suggesting students to build their learning directly on personal previous knowledge (this reminds us the ideal teaching we spoke at the beginning). However (4) and (5) do not suggest how to establish students' levels of proficiency. On the contrary, class tests, national and international achievement inquiries require all students to achieve common competencies (PISA, for example). (6) could suggest that the teacher should establish common levels of proficiency during “the processes of reflection” on “the modern view” of mathematical subjects.

When the hermeneutic approach in Jahnke (2014) makes it difficult to cope with the requirements of the school situation, that is to reach a homogeneous level for all students, the teacher may resort to socio-cultural resources of interaction between peers and teacher, in line with Vygotsky (1978).

By combining the previous issues concerning theoretical background and practical requirements I pinpointed the following goals of my experiment:

- reinforcing abilities regarding solution of quadratic equations;
- using previous knowledge and abilities for discovering the familiar into the unfamiliar (Jahnke et al., 2000);
- using a text to gather mathematical information;
- knowing peculiarities of an original text (types, rhetoric aspects, choose of terms, lack of symbolism);
- viewing the document in broad sense including students' personal connections and remarks.

Since I assume, as Leatham (2006) does, that “teachers are seen as complex, sensible people who have reasons for the many decisions they make” (p. 100), in the present paper we will mainly concentrate on the reasons behind my decisions. In doing that, I feel I am in the situation so well illustrated by Donald Schön (1987) in his *The reflexive practitioner* where he highlights how professionals do not always feel at ease reflecting on their action because of the turbulence of environments like schools. In fact, reflection in action increases complexity of teachers’ task and, as an extreme consequence, situations might even slip out of control. The other side of the coin is that teachers have to look for proper circumstances suitable to promote reflection on what they are doing, otherwise their work risks to become at least an unfruitful ritual.

A CLASSROOM EXPERIMENT USING ORIGINALS

My teaching context

I teach in a “Liceo delle Scienze Umane” [Human Science Lyceum]. Students can choose between two branches: a) ‘human sciences’, *strictu sensu*, b) ‘social-economic’ in which the study of human sciences is less widely treated, in favor of economics and mathematics. The school lasts 5 years. The students' score in mathematics is quite low. An ‘entry test’ is prepared by the school department of mathematics teachers and is handed out at the beginning of the first year. The average score is around 60% of correct answers. At the end of the second year a test provided by the National assessment organization INVALSI-Istituto Nazionale per la VALutazione del Sistema educativo di Istruzione e di formazione (National Institute for evaluating the educational system of instruction and education) is administered. It gives information to schools and teachers but, at the moment, do not officially certify students' levels. At the end of the fifth year, students take their final state national examination. Together with three or four other school subjects, mathematics is part of a one out of three written tests and with five other subjects is part of an oral examination.

The classroom activity with original sources

The activity took place in a third class of the socio-economic lyceum (students aged 17-18). The average level of proficiency in National achievement examination of these students was under the average score. Students were weak in algebraic manipulation and were having difficulties in problem solving so I was looking for new kinds of problems and I considered interpretation of texts a good option. In previous lessons they met history of mathematics in various circumstances: references to Archimedes works about levers, use of Arab combinatorial reasoning, arithmetic triangle in Chinese, Arab, Tartaglia's versions, solution of quadratic equations by al- Khwarizmi, a quadratic function from a medieval treatise etc. For some topics I used the materials in (Katz & Michalowicz, 2004).

An example of a document I proposed is in figure 2. It concerns an exercise which requires calculations less complicated then most of those required by students' textbook, but not trivial.

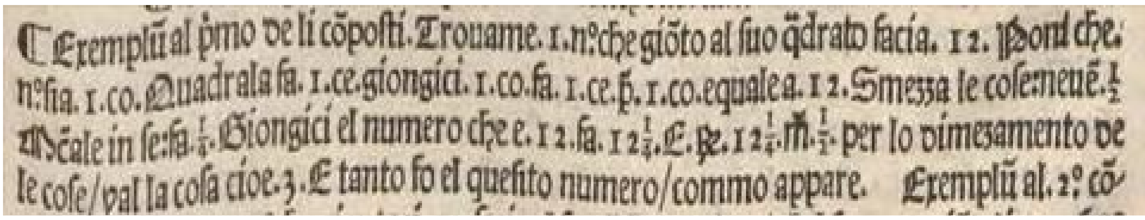

Exemplũ al p̃mo de li cõposti. Trouame .1. n°. che giõto al suo q̃drato facia .12. Poni che n°. sia .1. co. Quadrala fa .1. ce. giongici .1. co. fa .1. ce. p̃ .1. co. equale a .12. Smezza le cose: neuẽ ½ Mcale in se: fa ¼. Giongici el numero che e .12. fa 12¼. E R. 12¼ m. ½ per lo dimezamento de le cose/ val la cosa cioe .3. E tanto fo el quesito numero/ commo appare. Exemplũ al .2°. cõ

Fig. 2: From Pacioli's *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*, Venice 1523, edition of 1523 (first edition 1494), folio 145.
"[…]

Find for me a number that, if joined to its square, makes 12. Imagine that the number be a thing. Square it. It makes 1 census. Join 1 thing. It makes 1 census plus 1 thing equals 12. Halve the things. It becomes ½. Multiply by themselves. It makes ¼. Joint the number which is 12. It makes 12¼ . And the square root of 12¼ minus ½, because of the halving of the things, equals the thing that is 3. And the required number makes this amount, as it appears. […]."

Giving to the students the short document from Pacioli's *Summa*, I asked them to read it and interpret it with respect to mathematical content. I underlined that the main goals that students were requested to achieve were to use personal resources and that the interpretation of the document aimed at: using previous knowledge and reinforcing abilities regarding quadratic equations, using a text to gather mathematical information, knowing peculiarities of that ancient text, formulating personal remarks, connecting with Italian literature whose origins students were treating in class.

I considered that history allows introducing humanistic aspects in mathematics teaching (mainly those regarding communication) according to the peculiarities of school (a human sciences lyceum). I reflected on the opportunity to use a different version of the original, changing, in case, some aspect but I decided to maintain all the following characteristics:

- types, because they show that it is an ancient document, at a glance;
- use of abbreviations, because it poses questions regarding printing in 15th and 16th Century;
- archaic Italian words, because they show example of evolution of language;
- absence of separation between problem and solution, because it recalls a feature present in mathematical treatises since early Middle Ages;
- redundancies, because also the manner to communicate mathematics has changed;
- unusual manner to write rational numbers, because it shows an example of “the familiar found inside the unfamiliar” in elementary mathematics.

The students worked in pairs. I asked them to translate the document in modern Italian language, to conjecture about the meaning of the full document and its specific parts and to compare personal explanations of the mathematical passages in order to write a shared version of the interpretation. After a few minutes, I listened to their questions and I answered through hints or other questions that could help them to reflect upon the document and the (partial) explanations they had found at that moment. In order to give further opportunities for better understanding and for reviewing, I requested them to compare their explanations in this manner: standing in front of mates, most of them read their interpretations; they could also briefly criticize interpretations of other students.

Actually, the activity based on Pacioli's document showed controversial outcomes. Supplementary explanations by the teacher followed the group activities and in the written test students got an average score similar to previous tests, some of them even better. Students considered it as a meaningful experience. On the contrary, the part of the activity in which students were requested to work autonomously showed unsatisfactory results. The use of the written text did not fully succeed, with respect to the goal regarding solution of quadratic equations and use of previous knowledge and abilities for discovering familiar concepts into the unfamiliar document. The document highlighted students’ incapability to use their mathematical knowledge to interpret the text or, in the same sense, to link their previous abilities to the content of the document. After the experiment, I met other class situations that had similar outcomes (even if regarding different kinds of documents such as an Euler’s excerpt, or a graph). I have argued it could depend on the operational nature of my students’ mathematical conceptions (Sfard, 1991) so that their knowledge was not actually at their disposal for interpreting the text. Many students required supplementary explanations about algebraic skills, which they were able to apply in routine exercises but they did not think to use for interpretation.

Here it is a list of difficulties that several students had.

1. Typefaces. For example: "What does T mean, inside the word Trouame?" The current Italian word *Trovami* suggests the right answer: "**T**". At the same time, u appears as a different manner to write ***v***.
2. Contractions. In the word q̄drato, "quadrato", two letters, that is ***u a***, are omitted and the specific mark ¯ highlights this fact.
3. Exposition of the statement. "Where is the question? Where does the solution begin? No modern symbols!". The sequential exposition in the document conflicts with the modern formula which shows all operations together.
4. From words to symbols. "Find a number that, if joined to its square, makes 12: what equation can I obtain?" [By a really weak student].
5. Unknown mathematical procedure. "Why do I have to halve the 'things' (coefficient of the linear term)? It is not even!"
6. Search for information to guess meanings (general meta-cognitive competency). Some students looked lost in front of difficulties 1) and 2): they did not think to read the text again, in order to find in the document the specific meaning of letters and words.
7. Meta-cognitive competencies regarding mathematical tools. Many students did not think to use the modern formula in order to interpret the document (to understand the meaning of specific elements like the letter ℞ for "Root", as well as passages in the reasoning).

Comments

I shortly describe my students' difficulties as lack of willingness to guess, to produce conjectures, or to check them autonomously searching reasons inside the document. It seems that they have not internalized the *hermeneutic circle* which concerns the idea that the interpreter's understanding of the text as a whole is established by reference to the specific parts and his/her understanding of each individual part by reference to the whole. I considered that students had experienced the hermeneutic circle using different kinds of texts, in various disciplines.

About the lack of disposition to make conjectures, we can identify a diagnostic role of hermeneutic approach. Conjecturing shows one's competences; students who produce conjectures reveal their being. It is really different from repeating a piece of the teachers' lesson! These remarks are based on the works of an Italian author, Bertagna (2000). He notes that from Parmenides and Aristotle until Heidegger, Fromm and Marcel an anthropological dilemma regards the distinction between *to be* and *to have*. The somebody's *being* is her/his *essence* or *substance*. The Italian term "capacità" (*capacity*), in psycho-pedagogical context, recalls individual potentialities. The term *competence* has been used by Chomsky in contrast to *performance*: when teachers want to see whether students know a procedure, they create a task that requires a performance. A valid performance sometimes hides a lack of underlying competence. Both capacity and competence are inherent to the being of a student. On the contrary, *knowledge* and *skill* belong to the *having*. It is remarkable that from the Latin verb *habeo* (to have) derived: *habitus* (to behave), *habilis* (to do something well), *habitare* (to live in a place). About *knowledge*, we know *that*, *where/when/why*, *how*. The meaning of *skill* is strictly connected with that of the Greek *techné* (craft or art).

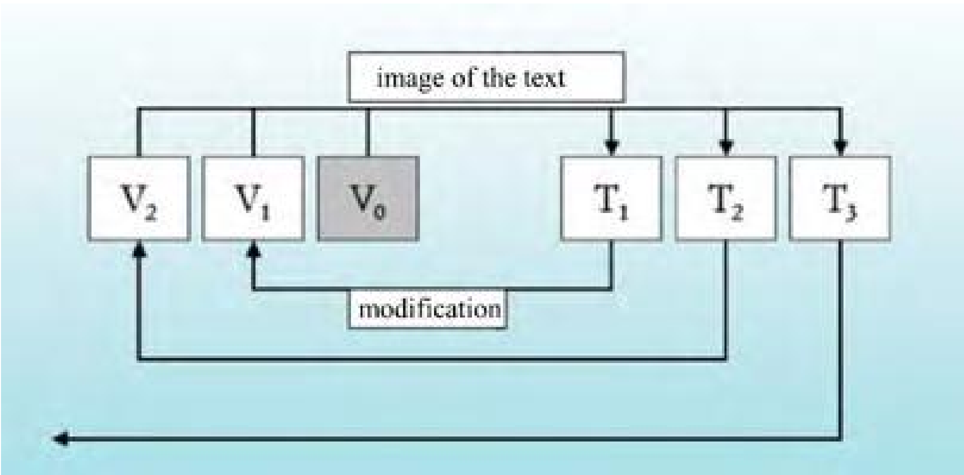


Fig. 3: Interpreting a text; from (Glaubitz, 2011b).

In interpreting a text, "you start with a certain image of the text reflecting your expectations about what it might be about. Then you read the text and realize that some aspects of your image do not agree with what is said in the source. Thus, you have to modify your image, read again, modify and so on until you are satisfied with the result or simply do not like to continue […] the hermeneutic circle can be considered as a process in which a *hypothesis* is put up, tested against the source, modified, tested again and so on until the reader arrives at a satisfying result" (Jahnke, 2014, pp.84-85), with reference to figure 3. Notice that the term "hypothesis" is used. I agree with Lampert (1990) who, quoting Lakatos, indentifies a conjecture with a "conscious guess". I believe that making hypothesis could be an unconscious guessing, also in the case in which students interpret originals.

When students are requested to interpret the text without any suggestion by the teacher, as in the first part of the experiment, they have to take a risk and guess or make conjectures, because they do not have the opportunity to choose the right specific knowledge or the skill they have got by the teacher. They have to abandon the reference to what has come to them from outside, and instead use their inner sources (the fact that they know to be requested to interpret the text with respect to its mathematical content does not significantly influence the situation). We call these inner sources *competences*.

I presume that only self confident students, i.e. those who believe to have a chance to give a “rewarding” answer to questions asked by teacher (a right one or, in any case, one well accepted by the members of class), have the willingness to conjecture. The students involved in the experiment got quite low test scores. Therefore I was not completely surprised when they did not conjecture trying to interpret Pacioli's original. As a teacher, this fact has posed me the problem on how to help students. Beyond individual cognitive difficulties, I think that they are influenced by other factors, mainly by the *didactical contract* discussed by Brousseau (1984), so that students consider more rewarding to give a performance quickly, rather than to “waste” time in personal efforts. It mostly happens in written tests. In addition, students could consider that leaving interpretation to the teacher has positive consequences: the quality of performance will be better, so they will get a better mark.

These reflections cannot be general because of the way in which the experience had been realized. However, some characteristics of my students are the same of other Italian students: they attend a state school which has similar curricula with respect to other kinds of secondary schools, the performance standards in mathematics and literature are similar to a significant percentage of Italian students, almost all of them have personal interests especially in new technologies etc. This leads to guess that some of their difficulties could be common to many other students.

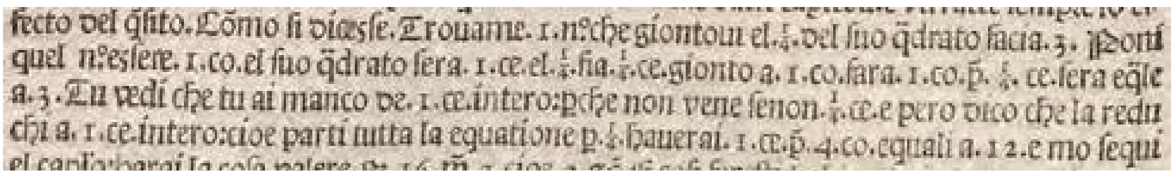
HISTORY IN THE MATHEMATICS LABORATORY: A PROPOSAL TO THE TEACHERS WHO DO NOT USE HISTORY; A WAY TO ACT ON THE *ZONE OF PROXIMAL DEVELOPMENT*

Reflection on the questions reported in Introduction about teachers’ reluctance to use history in their mathematics teaching, led me to design proposals of activities that are now collected in the textbook (Bergami & Barozzi, 2014). Almost every chapter of this textbook contains a section called *historical laboratory* divided in three parts: the first one is printed in the paper book as input; the second regards “supplementary exercises” and is on-line, like the third which is devoted to cultural context (where and how mathematicians operate, mathematics and other subjects or applications, a research on the internet). As an example I describe the activity entitled “Equations with Friar Luca Pacioli”.

EQUATIONS WITH FRIAR LUCA PACIOLI

[In translating into English I kept the reduction of old words used in the original passage: “co” literally means “thing”; “ce” means “census”.]

[The students are requested to read the following historical document.]



fecto del qsito. Como si dicesse. Trouame .1.n°. che gionto ui el.¼. del suo qdrato facia .3. Poni quel n°. essere .1.co. el suo qdrato sera .1.ce. el.¼. fia .¼.ce. gionto a .1.co. fara .1.co.p.¼. ce. sera eqle a .3. Tu vedi che tu ai manco de .1.ce. intero: pche non vene se non .¼.ce. e pero dico che la redu chi a .1.ce. intero: cioe parti tutta la equatione p.¼. hauerai .1.ce.p.4.co. equali a .12. e mo sequi

"[…] Find for me a number such that, if ¼ of its square is added, makes 3. Let that number be 1 co; its square will be 1 ce. Its ¼ be ¼ ce which added to 1 co will make 1 co p[lus] ¼ ce, it will be equal to 3. You see that you have less than 1 whole ce because it results ¼ ce, but I say that you [can] reduce it to 1 whole ce, that is divide all equation by ¼; you will have 1 ce p 4 co equals 12 [...]"

Luca Pacioli, *Summa*, p. 146.

Let's interpret the document together.

[In the original worksheet, the problem "Find for me a number such that, if ¼ of its square is added, makes 3" is written in contemporary Italian here.]

a) Search the problem inside the original. How is written the word "quadrato" [square]? How is shown the omission of letters?

b) Write the associated equation.

c) Delete denominators [using equations' properties].

d) Use the answers a), b) to interpret the other lines of the document; then, answer the three following questions:

I. What do "co" and "ce" mean?

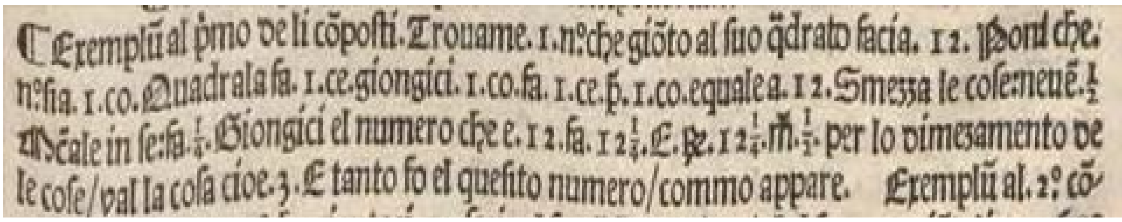
II. Divide the equation you wrote (previous point b) by ¼: what do you get? Check that you are right looking at the original text.

How does Pacioli write the addition sign?

Extra exercises

1. Use the reduced formula to solve the equation you have got from p.146 of the Luca Pacioli's *Summa*.

2. What we name "reduced formula" describes a procedure friar Luca used even when the linear coefficient is not an even number. Analyze the following original [students are requested to use the same formula to interpret the document].



Exemplũ al p̄mo de li cõposti. Trouame. 1. n°che giõto al suo q̄drato facia. 12. P̄ori che. n°sia. 1. co. Quadrala fa. 1. ce. giongici. 1. co. fa. 1. ce. p̃. 1. co. equale a. 12. Smezza le cose: neuē. ½ Mēzale in se: fa ¼. Giongici el numero che e. 12. fa. 12¼. E. R. 12¼. m̄. ½. per lo dimezamento de le cose / val la cosa cioe. 3. E tanto fo el quesito numero / commo appare. Exemplũ al. 2° cõ

[See the English translation in figure 2].

Luca Pacioli, *Summa*, p. 145.

a) Focus on the problem.

b) Write the corresponding equation

c) Solve it using the "reduced formula" (the linear coefficient is equal to 1 and can be written as 2 ½).

d) Compare the reckonings you have made to the part of the document which starts from "Smezza le cose" [halve the things]: first of all, find the manner Pacioli uses to write the square root and the minus sign, then calculate the value of 12 ¼.

e) Take note that Pacioli obtains only one solution: which one doesn't he consider?

[The correct answer to the e) question is that Pacioli does not consider the value -4 as a solution of the equation. After that answer I propose to students the following historical remark, in order to explain the reason of this fact.]

In millenarian tradition, the solution of equations was based on geometric figures.

Research activity

IS MATHEMATICS THE SAME EVERYWHERE? [This is the main question of the task that is explained in the following two points.]

1) Babylonians, Greeks (Euclid), Indians, Arabs, Europeans (for us, Luca Pacioli): this is a short list of peoples who have made the history of algebra, particularly of quadratic equations. Search further details in the web or in books. Consider that these peoples are from different places so, in this case, mathematics looks the same everywhere.
2) On the contrary, point out the fact that different peoples had their own specific mathematics, with their own connotations.

Keywords for web research

Luca Pacioli, quadratic equations in the history of mathematics [in English also in the original for search on English sites], Babylonian-Arab-Indian mathematics, al-Khwarizmi, completing the square.

Suggested readings

Demattè Adriano, *Vedere la matematica – Noi con la storia,* UNI Service Trento, 2010 (http://www.uni-service.it; some pages in http://books.google.it)

Joseph G. Gheverghese, *C'era una volta un numero*, Il Saggiatore Milano, 2003. [Joseph G. Gheverghese, *The Crest of the Peacock: Non-European Roots of Mathematics*].

Suggested sites

www.e-rara.ch/zut/content/titleinfo/2683230

http://www.matematicamente.it/tesi-didattica/Lungo-Equazioni.pdf

http://www-history.mcs.st-and.ac.uk/HistTopics/Quadratic_etc_equations.html

[In the Publisher's website, students can find the answers to the previous questions]

Fig. 4: From the textbook (Bergamini & Barozzi, 2014)

The role of questions posed by the teacher

In the *historical laboratory* (figure 4), some questions for students accompany the document to specify with respect to what issues interpretation is required. They are aimed at facilitating the task and they:

- take into account the hypothesized students' prerequisites
- construct prerequisites (like in the case of the original we analyzed at the beginning of my presentation with respect to the other);
- address the main points of the document;
- suggest how to create a bridge between modern solution and ancient solution;
- propose situations so that students could infer by themselves the meaning of specific elements in the document (for example, question d) I.);
- could be used for tests.

Asking questions to help pupils to acquire knowledge reminds us of the Socratic maieutics as a pedagogical method based on the idea that truth is latent in the mind of every human being. I consider it a fundamental part of the deep, ancestral teachers' role. It is a way to act on the *zone of proximal development* of Vygotsky (1978).

An example of asking questions can be found also in (Pinto, 2010) and regards the description of a workshop based on Pedro Nunes' shadow instrument (figure 5). The author asked four questions to participants so that they could "understand and validate the functioning of this instrument":

"1. Show that the triangles [*S'TS*] and [*S'TO*] are congruent and that <*SS'T* = <*OS'T*.

2. Show that <*OS'T* = <*AOX* .

3. Show that the plan *SS'T* is perpendicular to the horizontal plan.

4. Show that the angle that the sun rays make with the horizontal plan is equal to <*AOX*, i.e. equal to the angle marked in the circle by the shadow of the hypotenuse of the triangle."

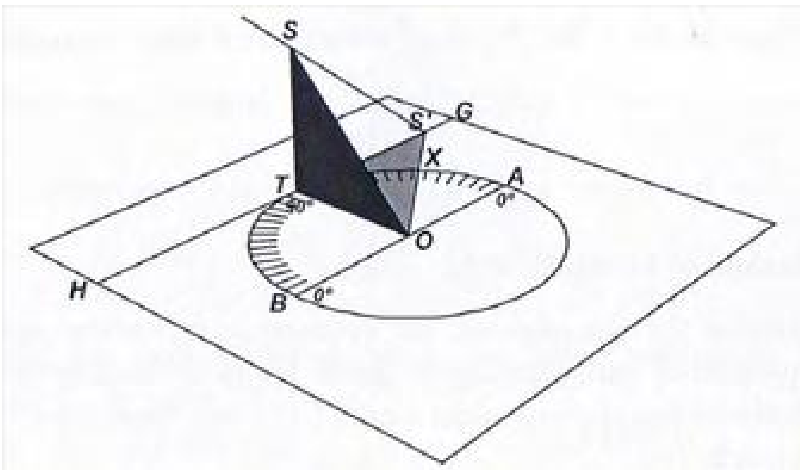


Fig. 5: The shadow instrument

Pinto's workshop, such as my laboratories for students, engages participants in a double level of text analysis: the original and the questions. In my opinion, questions can facilitate the understanding but not remove all obstacles (in case students are not good at reading, for example). Students have the obligation to follow a supplementary reasoning, just the one sketched by questions. It, paradoxically, requires no interpretation. More precisely, the students' attempt to refine their image of the text constituted by questions would introduce an "impossible" task, because they would have to understand the reasoning made by experts in mathematics like the teacher or the writer who prepared the written material.

Jahnke (2014, previous quotation) suggests that there could exist many kinds of reasoning in interpreting a document. In fact, expressions like "certain image of the text", "expectations about what it might be about", "satisfying result" implicitly suggest that, for example, a mathematics historian and a student do not have the same image and expectation about the Pacioli's document they never saw before. In addition, he states that "different readers with their different backgrounds arrive at different interpretations". As a teacher I know that students with low motivation consider satisfying the result that is unsatisfying for other mates; students like those in (Demattè, 2006b) got the same result but paying attention to different steps of reasoning. I have to precise that, in this paper, the term "reasoning" refers to individual processes aimed to acquire mathematical notions, nothing saying about mathematical objects.

In different classes of mine, using different documents accompanied by written questions, it happened that some students required supplementary explanations just regarding those questions. In this case, we can not say that every question facilitated their understanding. Another problem derives from the fact that questions usually regard specific parts of the text but students have to understand the whole meaning of the document, according to the concept of *hermeneutical circle*. In (Demattè, 2006b), I reported a case study of a student (grade 12) who focuses on specific parts of an original regarding al-Khwarizmi's graphical resolution of quadratic equations. This way, he did not get the whole view of the document. I argued that he was worried by reckoning and by dealing with algebraic passages, so that he did not acquire the capability to reason and operate with reference to an aim, that is to establish connections among data into the final geometric figure, in which also the solution is represented. Differently, a female student of the same class operated trying to interpret the figure: she was able to explain it with reference to the meaning of each part and to remedy her reckoning mistakes. In general, I consider that the reference to an aim is a manner to reconstruct a mathematical reasoning in an "almost narrative way", an "elementary" but necessary way to understand, because abstract reasoning is based on it. Aims inside mathematical reasoning suggest a track for identifying the whole meaning of a document (see Solomon & O'Neill, 1998; Thomas, 2002; Zazkis & Liljedahl, 2009).

FINAL REMARKS

In this paper, examples of class experiences as well as proposals of activities are described. I have considered this plenary an opportunity to let know a teacher's point of view. I am aware that not many teachers use history in their classes, but I believe that every teacher could agree about the relevance of educational problems like, for example, the way to involve students or to help them in learning, which have been analysed with reference to the history in mathematics class. With respect to educational research, I hope that the classroom episodes I have described in these pages could be useful for discussing the role of history in mathematics teaching, specifically for discussing what kind of mathematical or interdisciplinary abilities history can contribute to develop. I consider that this is one of the main contributions that teachers who participate in ESU-HPM Group activities can give, according to the goals we can find in the history of the Group written by Fasanelli and Fauvel (2006):

- “To produce materials which can be used by teachers of mathematics to provide perspectives and to further the critical discussion of the teaching of mathematics.
- To facilitate access to materials in the history of mathematics and related areas.
- To promote awareness of the relevance of the history of mathematics for mathematics teaching in mathematicians and teachers”.

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Workshop

MATHEMATICS, ALGORITHMICS AND HISTORY: AN INTEGRATED APPROACH IN TWO CLASSROOM EXPERIMENTS

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Université Paris Diderot

In 2009, algorithmics was explicitly introduced in the new mathematics curriculum for the first year of secondary education in France. This introduction was extended to the new curricula for the second and third year published in 2010 and 2011. In the latter, the intentions of the curricula developers regarding algorithms are clearly stated: algorithms should be part of a problem-solving approach integrated in the other topics of the mathematical curriculum (analysis, geometry, statistics and probability, logic) and they could also be connected to other disciplines.

Algorithmics is therefore not meant as an independent sub-part but as a spiral work throughout the high school mathematics curriculum. Having this in mind, we had the idea to integrate history of mathematics in this approach to algorithmics. The following article presents two classroom activitites based on the reading of original sources and experimented with first and third year students in two different classroom contexts. The first activity is a computer-assisted exercise session meant as an introduction to the chapter on quadratic functions for first-year students and based on a problem by Al Khwarizmi. The second activity is a guided research session based on Heron's method for the approximation of the square root of a number. It was intended for third year students enrolled in the scientific section (Terminale S) as a conclusion to the chapter on sequences and limits and was carried out in small groups. After describing the pedagogical intentions and conceptual process, we review the activities and summarize the pupils' work. We end up with an assessment of these two classroom activities from both pupils' and teacher's standpoints. In particular we try and assess the relevance, in these two cases, of the use of historical material and of the introduction of a historical perspective in teaching mathematics.

WORKSHOP

'TELLING MATHEMATICS' REVISITED

Jan van Maanen

Utrecht University & Groningen University

The history of mathematics has a strong oral tradition. People tell each other problems and methods, and not so much in classrooms but rather in coffeehouses and during walks and parties. I studied this phenomenon before and reported about it at the European Summer University at Louvain, calling it "Telling mathematics". Interest in these problems continued, as can be seen from the recent book Mathematical Expeditions - Word problems across the ages by Frank Swetz (2012).

I will shed some new light on this culture, of passing mathematical problems and knowledge by sharing it with others. In Louvain my focus was on the role that such problems and especially the act of telling it to fellow students, could have in the classroom. In this presentation I will take a more historical and anthropological point of view. An experiment with two groups of about 40 mathematics teachers each will provide information about the repertoire of professional mathematicians, as far as 'telling mathematics' is concerned. And some of these problems I will trace through history. Many of them originated in Asia, and entered Europe in the Middle Ages and Renaissance. And they continue to be told.

An interesting didactical question arises, which is why this spontaneity of sharing problems with each other is observed rather outside school. What can we, teachers, learn from that?

Workshop

WORKSHOP ON THE USE AND THE MATHEMATICS OF THE ASTROLABE

Wilfred de Graaf[a] & Michel Roelens[b]

[a]Utrecht University, [b]University Colleges Leuven-Limburg

For more than one thousand years the astrolabe was one of the most used astronomical instruments in both the Islamic World and Europe. It was used to locate and predict the positions of the sun and stars, for instance to schedule prayer times, and to determine the local time. In the first part of the workshop, the participants learn how to use (a cardboard model of) the astrolabe. In the second part they study the mathematical and astronomical principles on which the astrolabe is based. We explain the mathematical properties of stereographic projection and we show how the lines and circles on the astrolabe can be computed.

INTRODUCTION

Background of the workshop

The astrolabe workshop is based on an idea by Prof. Dr. Jan P. Hogendijk, University of Utrecht. The workshop has been held in recent years on several occasions in Iran, Tajikistan, The Netherlands, Turkey, Syria and United Kingdom by Wilfred de Graaf. In Belgium the workshop has been held by Michel Roelens for high school students and (future) teachers. Recently a detailed instruction on the use of the astrolabe and on the mathematical method of stereographic projection has been published by Michel and Wilfred in *Uitwiskeling*, a Belgian journal for high school mathematics teachers.

Classroom use

The workshop as a whole is suitable as an interdisciplinary project for high school students aged between roughly 15 and 18 years that have a keen interest in the school subjects of physics and mathematics. The first part of the workshop, on the actual use of the astrolabe, can also be given to a wider audience with interest in history, geography and culture, and younger pupils (13-14 years old). For mathematics education the astrolabe is of particular interest since the instrument is based on the mathematical concept of stereographic projection. In a classroom situation the pupils could for example be asked to derive certain formulas related to this projection, thereby using such things as Thales' theorem and the inscribed angle theorem of plane geometry. About the use of the astrolabe in the classroom, see also de Graaf and Roelens (2013) and Merle (2009).

The original astrolabe

Based on mathematical principles which date back to Greek antiquity, the astrolabe flourished in the Islamic World from the year 800 CE onwards. In this workshop the participants learn how to use the astrolabe of the renowned mathematician and astronomer Abū Maḥmūd Khujandī.

Figure 1: The astrolabe of Abū Maḥmūd Khujandī

He constructed this astrolabe in the year 985 CE at the observatory in Baghdad. It is one of the oldest and most beautiful decorated astrolabes extant today. It is currently displayed at the Museum of Islamic Art in Doha, Qatar. The distributed astrolabe model has been recalculated for the latitude of Antwerp, i.e., 51° N.

Principles

The astrolabe is based on the mathematical principles of the *celestial sphere* and *stereographic projection*. The celestial sphere is an imaginary sphere concentric with the earth on which the stars and the apparent one year path of the sun are projected from the centre of the earth. Stereographic projection is a method to map a sphere onto a plane. In this case the celestial sphere is mapped from the celestial south pole onto the plane of the celestial equator.

FIRST PART: THE USE OF THE ASTROLABE

The astrolabe model consists of two parts.

On the overhead slide: the spider

The spider contains the stereographic projections of the *ecliptic*, which is the apparent one year trajectory of the sun along the sky, and of 33 stars. These stars are the same as on the astrolabe of Khujandī. The positions of the stars are recomputed for the year 2000 CE, showing the effect of precession of the equinoxes if the model is compared to the original astrolabe. The precession is about 15 degrees in a 1000 year interval. In the model, the position of a star is indicated by a dot in the middle of a small circle.

On the sheet of paper: the plate

The plate has been combined with the rim, which is a circular scale divided into 360 degrees. The plate displays (parts of the) the stereographic projections of the following points and circles.

- The centre of the plate is the *celestial north pole*, which is the centre of three concentric circles: the Tropic of Cancer, the celestial equator and the Tropic of Capricorn.
- The *horizon*, whose projection is visible on the plate in Eastern, Northern and Western directions. The *twilight line* is 18° below the horizon.
- The *almuqanṭarāt* (altitude circles) are the nearly concentric circles 3°, 6°, 9°… above the horizon.
- The *zenith* is the point directly above the head of the observer, i.e. 90° above the horizon.
- The *azimuthal circles* or circles of equal direction. Its projections are drawn for 5° intervals and are numbered at their intersections with the horizon. The *first vertical* is the azimuthal circle through the East and the West point. It is the reference circle for the other azimuthal circles. Note that all azimuthal circles pass through the zenith.

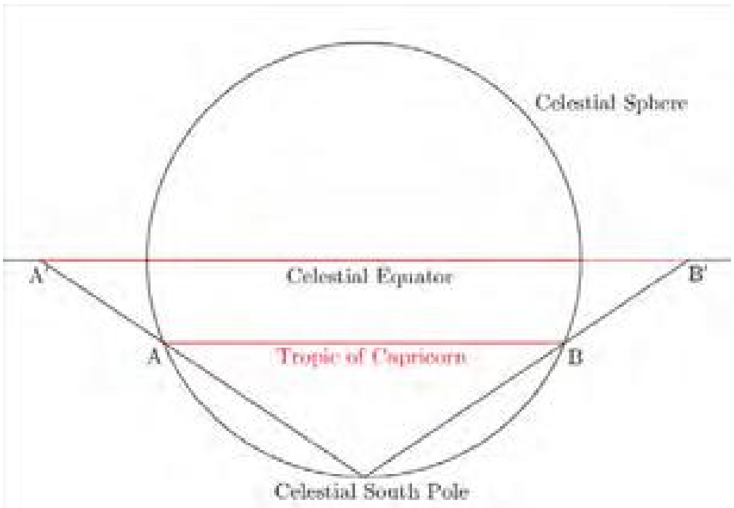

Figure 2: The stereographic projection of the Tropic of Capricorn from the celestial sphere onto the plane of the celestial equator

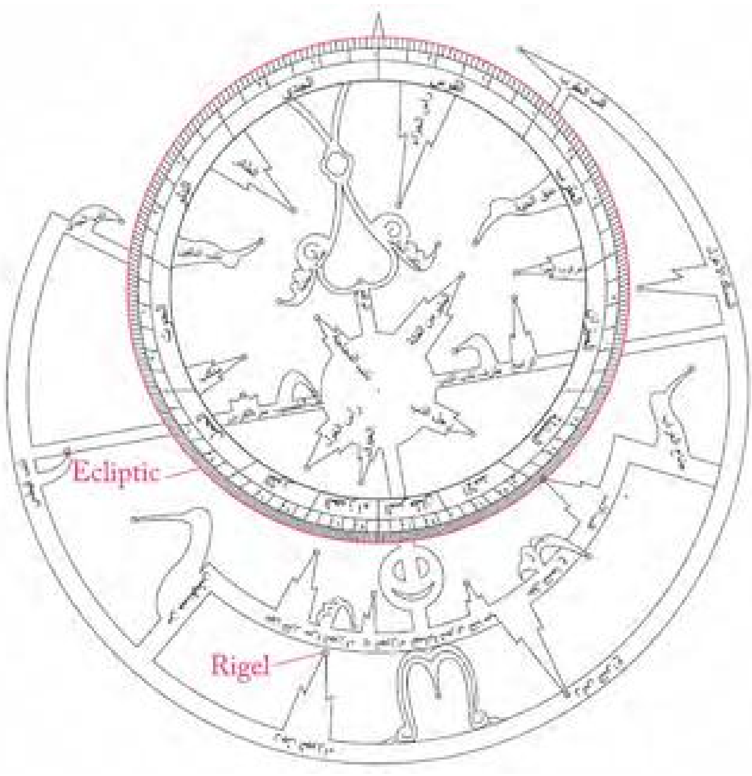

Figure 3: The spider with the ecliptic and the star Rigel highlighted

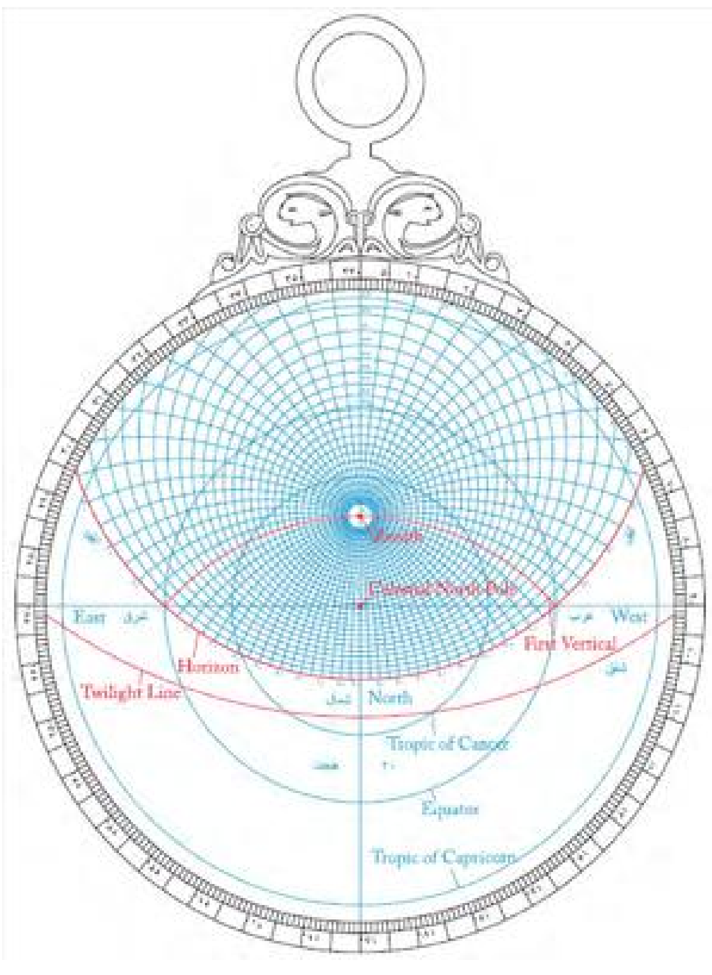

Figure 4: The plate for 40° N (i.e. the latitude of Khujand, Tajikistan)

The back side of the astrolabe is not shown in this model. It contains the alidade: a metal strip with two sights and a pointer. An alidade can be used to measure the altitude of the sun or a star in degrees, if the astrolabe is suspended vertically. The altitude can be read off on a circular scale.

The use of the astrolabe

If one knows the position of the sun in the ecliptic on a given day, the astrolabe can be used to tell the local time. It can then also be used as a compass. The position of the sun can be estimated using the fact that the sun moves through the twelve zodiacal signs, into which the ecliptic is divided, in the course of one year. Every sign is divided into 30 degrees. The sun moves with a velocity of approximately one degree per day.

For any day, the position of the sun in the ecliptic can be marked on the spider by a non-permanent marker. The altitude of the sun can be measured using the alidade on the back side of the astrolabe. The spider can now be set to represent the actual position of the celestial constellations with respect to the horizon. By means of the azimuthal circles one can read off the direction of the sun, for example 10° S or E. To determine the local time, note that the pointer of the spider indicates a number on the rim. A full rotation of the spider corresponds to 24 hours, so 1 degree of rotation corresponds to 4 minutes of time. By rotating the spider, one can determine the interval of time between the moment of observation and, for example, sunset, noon, and sunrise.

Aries	March 21	- April 19
Taurus	April 20	- May 20
Gemini	May 21	- June 20
Cancer	June 21	- July 22
Leo	July 23	- August 22
Virgo	August 23	- September 22
Libra	September 23	- October 22
Scorpio	October 23	- November 21
Sagittarius	November 22	- December 21
Capricornus	December 22	- January 19
Aquarius	January 20	- February 18
Pisces	February 19	- March 20

Table 1: The signs of the zodiac and their corresponding dates

The assignments of the workshop are divided into three levels: the calculation of the length of daylight on a given day of the year (level 1), the use of the astrolabe as a clock and as a compass (level 2) and the determination of the direction of Mecca (bonus level).

Workshop on the use of the astrolabe, level 1

1. The date of your anniversary is: day .... month ....
2. Then the sun is in the sign of the zodiac: ....
3. And in the degree: .... In case the degree is 31, write 30.

Now mark the position of the sun on the ecliptic on the spider. Be sure to mark it on the outer rim of the ecliptic!

4. At sunrise on your anniversary, the position of the pointer is: ....

Recall that the sun rises on the Eastern horizon.

5. At sunset on your anniversary, the position of the pointer is: ....
6. The difference between the position of the pointer at sunset and the position of the pointer at sunrise is: .... degrees. When encountering a negative difference, add 360 degrees to the position of the pointer at sunset.
7. The length of daylight on your anniversary is: ....

Recall that 15 degrees corresponds to 1 hour.

Workshop on the use of the astrolabe, level 2

Suppose you have measured with the alidade on the back side of the astrolabe that the sun is 9 degrees above the horizon. You have done the measurement in the afternoon of your anniversary date.

8. The position of the pointer at that moment is: ....
9. The position of the pointer at noon (12.00 true local solar time) is: ....
10. The difference between the position of the pointer at the moment that the sun is 9 degrees above the horizon and the position of the pointer at noon is: .... degrees.
11. The true local solar time at the moment that the sun is 9 degrees above the horizon is: ....
12. The direction of the sun at that moment is: ....

Workshop on the use of the astrolabe, bonus level

Suppose you are at a place with the same latitude as Antwerp. The geographical longitude of this place is 15 degrees East of Mecca. You know that the sun passes through the zenith of Mecca on the days when it is in 7 Gemini and in 23 Cancer.

13. Use the astrolabe to find the direction of prayer, *qibla*, at your place.

SECOND PART: THE LINES ON THE ASTROLABE

Stereographic projection

Each line on the astrolabe is the stereographic projection onto the equatorial plane of a circle on the celestial sphere. We project from the south pole of the celestial sphere. This means that the circle that we want to project is connected by straight lines with the celestial south pole so that an oblique[1] circular cone is created. The stereographic projection is the intersection of this cone with the plane of the equator.

A major advantage of the stereographic projection is that the circles are projected as circles (we will prove this later). The lines of the astrolabe can thus be drawn with a compass!

Figure 5 shows the stereographic projection of a set of circles on the celestial sphere that lie in a set of planes that are parallel to each other, but not parallel to the plane of the equator. This is for example the case with circles at a fixed altitude above the horizon (e.g. all the points 20°,40°… above the horizon).

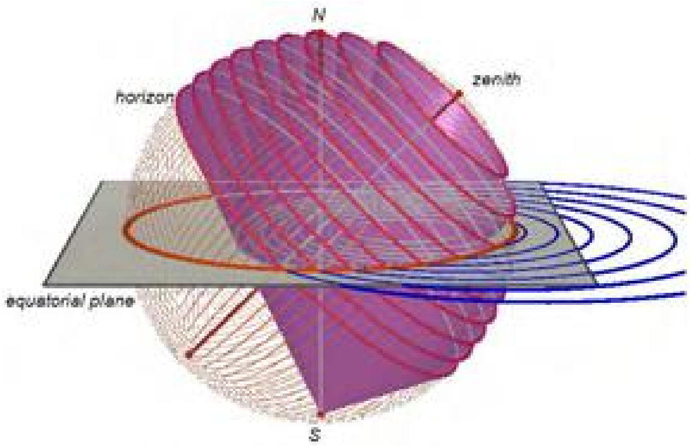

Figure 5: Stereographic projection of circles parallel to the horizon

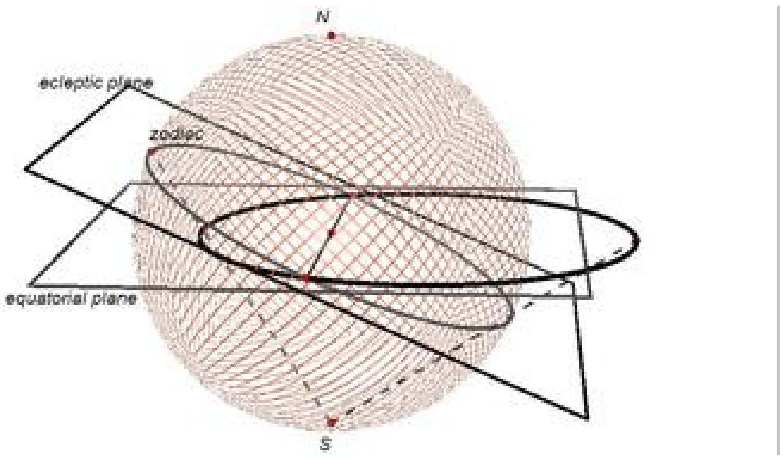

Figure 6: Stereographic projection of the ecliptic

Figure 6 shows the stereographic projection of the zodiac. The zodiac is the apparent path of the sun around the earth in one year. It is the intersection of the celestial sphere with the ecliptic plane. Since the rotation axis of the earth is not perpendicular to the ecliptic plane, the projection of the zodiac is decentred with respect to the centre of the astrolabe.

Circles on the celestial sphere remain circles on the astrolabe

We want to prove that the stereographic projection on the equatorial plane of a circle on the celestial sphere is again a circle.

Take any circle c on the celestial sphere. The stereographic projection of c is the intersection c' of the cone of base c and apex S (the south pole of the celestial sphere) with the plane of the equator. We want to prove that c' is also a circle.

Apollonius of Perga (3rd century BCE) proves in *Conica* the following two propositions.

(Conica I.4) The intersection of an oblique circular cone with a plane parallel to the basis is a circle.

(Conica I.5) The intersection of an oblique circular cone with a 'subcontrary' plane is also a circle.

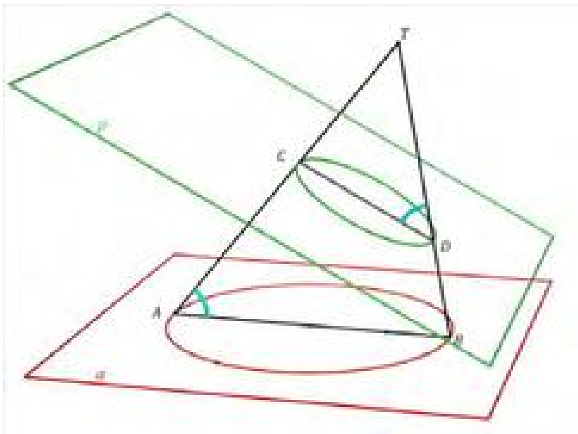

Figure 7: Intersecting a cone with a 'subcontrary' plane

Apollonius explains what he means by 'subcontrary'. For this purpose, he uses the intersection ABT of the cone with the plane perpendicular to the basis and containing T and the centre of the basis (see figure 7). In this plane, AB is the diameter of the basis and CD is the diameter of the intersection, with C on AT and D on BT. If we cut the cone parallel to the basis, then the angle $\hat{C}$ is equal to the angle $\hat{A}$. Now, cutting with a subcontrary plane means cutting in such a way that the angle $\widehat{D}$ is equal to the angle $\hat{A}$.

Proof based on Apollonius' theorem

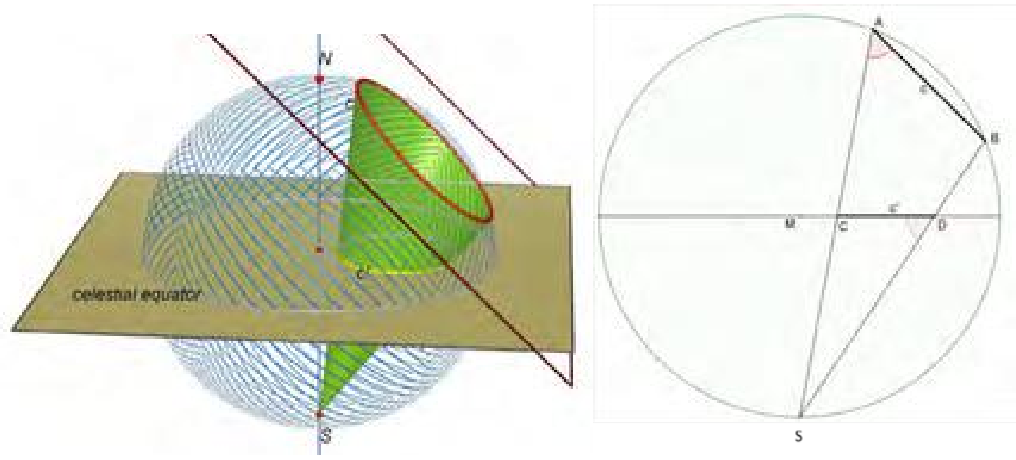

Figure 8

We operate in the plane passing through the centre M of the sphere, the celestial south pole S and the centre of the circle c. This plane is then automatically perpendicular to the equatorial plane. We have to prove that the angles $\hat{A}$ and $\widehat{D}$ are equal (figure 8). Indeed, if this is the case, it follows from the theorem of Apollonius that the intersection *c'* of the cone with the equator plane is also a circle.

Using figure 9, we can prove the equality of the angles $\hat{A}$ and $\widehat{D}$. We have: $\hat{A}$ is equal to $\widehat{N}$ because they are inscribed in the same circle. Now, $\widehat{N}$ is the complement of $\hat{S}$ because $\hat{B}$ is inscribed in a semicircle. Finally, $\hat{S}$ is the complement of $\widehat{D}$ in the right angled triangle *DMS*. This proves that $\hat{A} = \widehat{D}$.

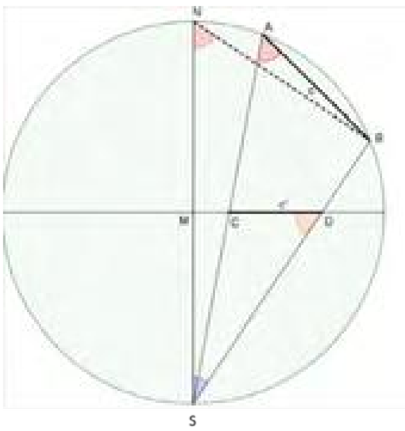

Figure 9

With theorem I.5 of the Conica we have proved that the stereographic projection of a circle on the celestial sphere is a circle on the equatorial plane (and thus on the astrolabe). We now give a proof of Apollonius' theorem I.5.

Proof of Apollonius' theorem

Given is an oblique cone, with circular base in the plane α and apex T. This cone is cut by a plane that is perpendicular to the plane ABT, in such a way that the angle $\widehat{D}$ is equal to the angle $\hat{A}$, as in figure 7. We have to prove that the intersection with β is a circle too.

Apollonius takes an arbitrary point P on this intersection and he proves that the angle $C\hat{P}D$ is right. He considers the intersection of the cone with a plane α' parallel to α through P. In a previous theorem (Conica I.4), Apollonius proved that this intersection is a circle. Since the planes α' and β are both perpendicular to the plane ABT, their intersection line is also perpendicular to this plane. Denote by M the intersection point of this line with the plane ABT (figure 10). The triangles CEM and DFM are similar. So we have:

$$CM \cdot MD = EM \cdot MF$$

In the right angled triangle EPF, we have

$$EM \cdot MF = PM^2$$

Hence

$$CM \cdot MD = PM^2$$

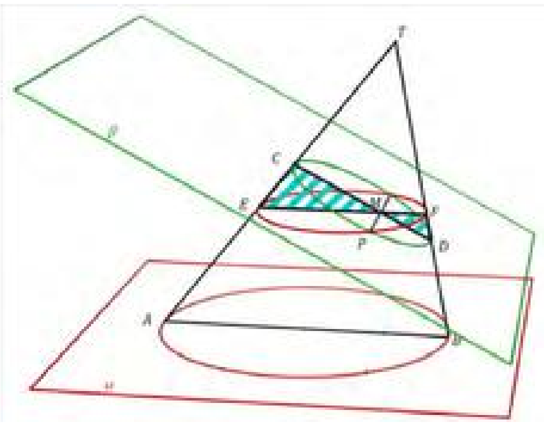

Figure 10

From this it follows that the triangle *CPD* is rectangular in *P*. Note the use in the two directions of the property "the triangle *CPD* is rectangular in *P if and only if* the height on *CD* is equal to the product of the length of the segments *CM* and *MD* in which it divides *CD* ".

This proves Apollonius' theorem.

Drawing the Horizon

Using some trigonometry, students can draw some circles on the astrolabe themselves. In the workshop below, we will do this for the special case of the horizon.

Workshop: Drawing the Horizon

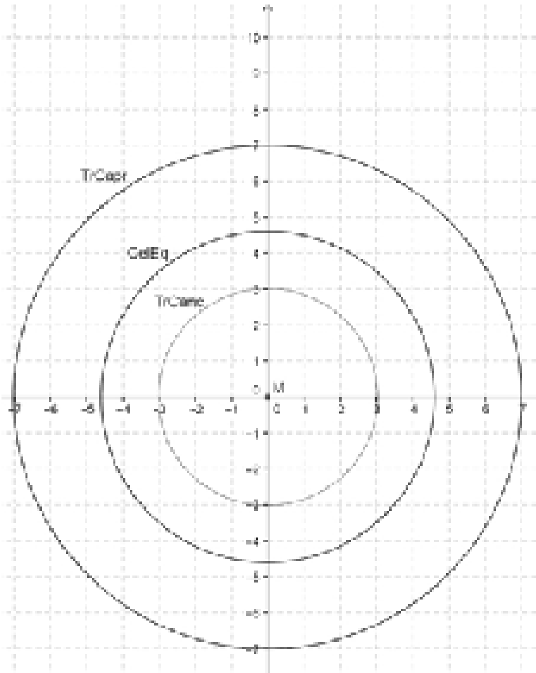

Figure 11: Blank astrolabe

We want to draw the horizon on the plate of the 'blank' astrolabe of figure 11. The projections of the celestial equator and the two tropics are already drawn. Just like on the model, we have taken r = 4.6 cm as the radius of the celestial equator. Using that the latitude of the tropics is at 23°26'16" N and S, we can calculate that the radius of the Tropic of Cancer on this model is 3.0 cm and that of the Tropic of Capricorn is 7.0 cm.

The horizon of an observer at a certain latitude on earth is projected on the plate. We assume that the observer is at the latitude of Antwerp, 51° N. All circles on the plate are stereographic projections of circles on the celestial sphere. In order to draw the horizon, we first identify what circle on the celestial sphere represents the horizon; then we determine its stereographic projection. Because we know that the stereographic projection is a circle again, it suffices to determine its centre and radius.

On an earth globe, we locate Antwerp at 51° N. The plane tangent to the earth at this point is the plane of the horizon for an observer in Antwerp (figure 12).

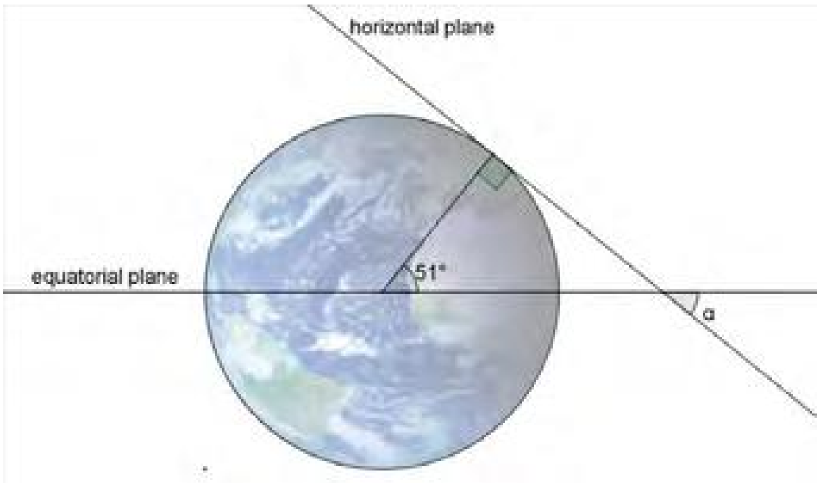

Figure 12: Earth globe with plane of the horizon

Exercise 1 What is the angle α between the plan of the horizon and the plane of the equator?

Because the earth is negligibly small compared to the celestial sphere, we can regard the plane of the horizon going through the centre *M* and having an angle α with the plane of the equator. The horizon is the intersection of this plane with the celestial sphere. The earth is represented as the point *M* (figure 13).

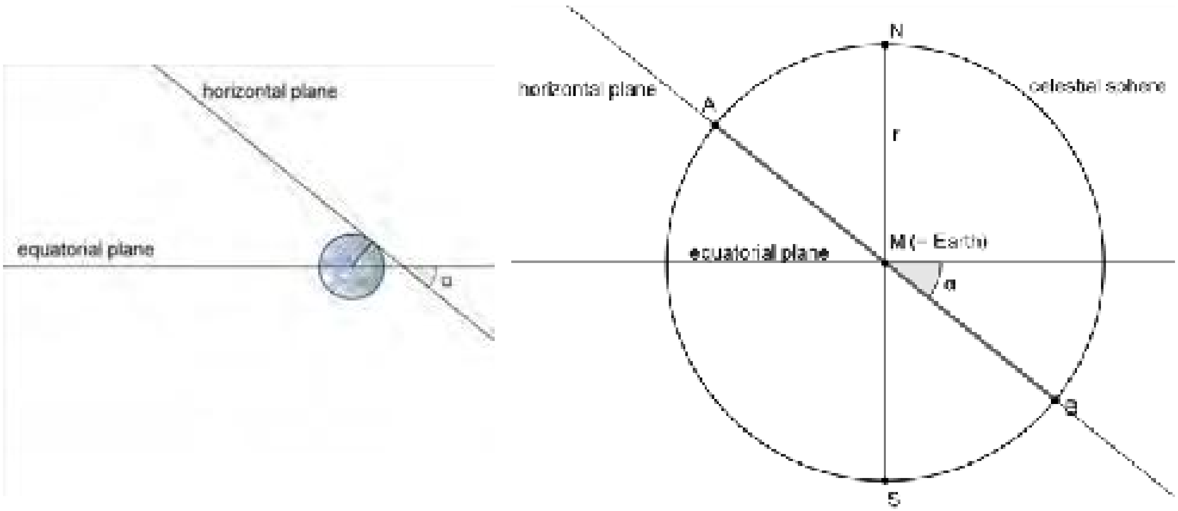

Figure 13: Horizon in the celestial sphere

We have to determine the stereographic projection of the horizontal circle. In figure 14, the horizontal circle is represented by the line segment *AB* and its stereographic projection by the line segment *A'B'*.

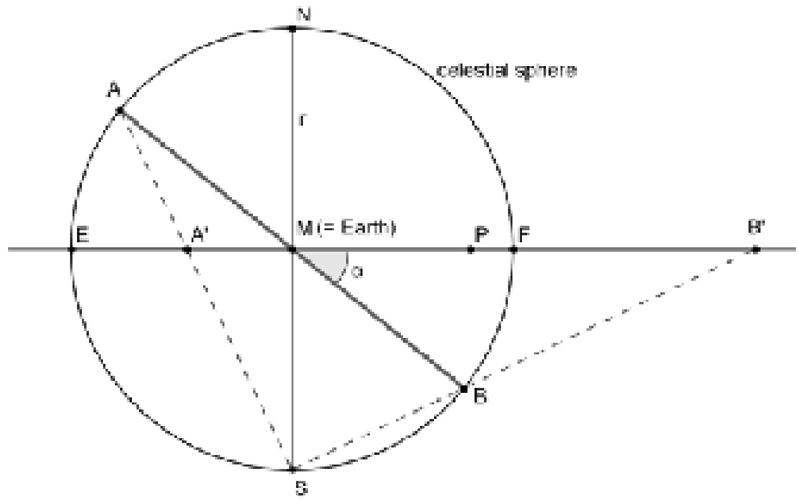

Figure 14: Constructing the projection of the horizon

Let us now write r for the radius of the celestial sphere in general. For the astrolabe drawing we will take $r = 4.6$ cm at the end of the computation.

Exercise 2 Express the distance *A'M* and the distance *B'M* in terms of the radius r. Make use of the right angled triangles *A'MS* and *B'MS*. What is the radius r_h of the projection of the horizon on the astrolabe? How far from the centre M of the astrolabe should the centre P of the projection of the horizon be drawn?

Did you find

$$r_h = \frac{r}{2}(\tan 64.5° + \tan 25.5°) \approx 5.9 \text{ cm};$$

$$|PM| = \frac{r}{2}(\tan 25.5° - \tan 64.5°) \approx 3.7 \text{ cm?}$$

Exercise 3 Draw the projection of the horizon on the blank astrolabe. Note that P should be 'above' the midpoint M (that is southward) on the astrolabe.

In an analogous manner (other) altitude circles can be drawn. You can do this at home for example for the altitude circle 30° above the horizon (figure 15). It is more complicated to draw the azimuthal circles. (It involves another feature of stereographic projection, namely that it preserves angles.)

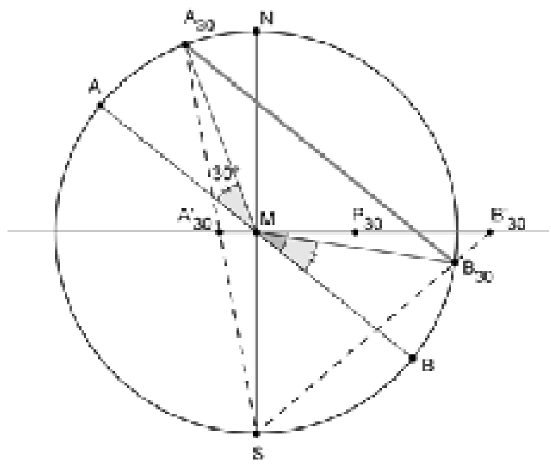

Figure 15: Construction for the projection of the altitude circle 30° above the horizon

FINAL REMARK

We believe the astrolabe is a very powerful didactic instrument to learn on the one hand about the movements of the earth, the sun and the stars, and on the other hand about the mathematics that is behind the method of stereographic projection. Also, we believe, the astrolabe is a wonderful historical tool to enthuse young students for the study of mathematics and natural sciences.

NOTES

1. An oblique (circular) cone is a cone of which the apex is not situated directly above the centre of the (circular) base. It may also occur that the cone is not oblique but right. This is the case when the circle on the celestial sphere happens to be in a plane parallel to the equatorial plane.

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http://www.ibttm.org (Istanbul Museum for History of Science and Technology in Islam)

http://www.jphogendijk.nl/ (website of Jan Hogendijk with many relevant links)

Workshop
PLAYING WITH EULER

Caterina Vicentini
Liceo Artistico "Max Fabiani" & Università degli Studi di Udine

This workshop has been based on the work my students and myself have made during the school year 2011–2012 and we have presented at the "Mathematical Games' Corner" of the Third Edition of the Scientific Communication Festival "Scienza under 18" that took place in May 2012 in Monfalcone (Italy).

We played with other students and the general public a new game which was born during the work on original sources examined during some extra-curricular workshops on History of Maths. The participants were all volunteer students (aged from 15 to 18 years) frequenting the ISIS "D'Annunzio-Fabiani" in Gorizia, coming from the linguistic, scientific and artistic sections of our Institute.

During the workshop in Copenhagen we have played the difficult version of the game and we have examined the pages of Euler's Algebra we had taken into account. Then we have discussed about the transposition of original sources in a game as well as the value of social games in increasing of students' motivation in the learning of mathematics.

INTRODUCTION

In Italian Schools, Mathematics is generally taught and learned in a way that comes with an almost always implicit Platonic Epistemology (1995). Even though nowadays some problem solving activities are faced in textbooks and classroom practice, they are often coming from the INVALSI [1] and OCSE-PISA tests and going in the direction of merely guarantee acceptable learners' performances in these assessment occasions. Some pills of History of Maths appear here and there too, usually exiled in the textbooks' chapters' last pages, often neglected by teachers and learners. History of Mathematics, even when present is mostly not integrated with the core of the subject, as one can see consulting the syllabi by Bergamini, Trifone, Barozzi (from 1997 to 2015) or Sasso (from 2006 to 2015) that, all together seem to cover the 90% of Mathematics textbooks national sales. There are of course some virtuous exceptions and they are well known in literature, but they unfortunately are neither systemic nor close to the average classroom habit.

Going against the flow, I have always felt the need to conceive tasks based on original sources to let students become aware that Mathematics and its teaching have a historical development and are therefore a human cultural product. As a teacher, I usually only had the opportunity to do so in my own classrooms, but since 2010 the Science Communication Festival "Scienza under 18" has given to my students and myself the chance to reach all the education community of the Gorizia's province as well as the general public at least during three days a year.

Being quite difficult to find some financial support to realize extracurricular initiatives during this economic crisis period , a good way to obtain some funds from the Friuli Venezia Giulia Region has been to connect the introduction of an original source with the popularization of cultural heritage that enriches the local institutions and the well-known European CLIL [2] Project.

THE WORKSHOPS AT SCHOOL

As already said, the first phase consisted in 6 sessions of extracurricular two hours workshops on an original source. The book chosen was "The Elements of Algebra" by Leonhard Euler, whose original version, "*Vollstandige Anleitung zur Algebra*", dates back to 1770. The main reason for this choice is that a copy of its second French edition dating back to 1807 is kept by the Biblioteca Statale Isontina in Gorizia and its third English edition (1828) is available for free in the site of the Harvard College Library. So we had the possibility to visit the ancient books section of the library and could touch and smell the ancient book. This experience had an important emotional impact on learners. What is more, during the Festival we popularized the existence of the generally unknown important ancient collection of mathematical books stored in town, becoming cultural promoters.

The students were all volunteers, coming from the linguistic, artistic and scientific sections of the Istituto Statale d'Istruzione Superiore "D'Annunzio–Fabiani" in Gorizia: quite heterogeneous group. For this reason the choice of the topic needed to be quite accessible, generally unknown, but not trivial. This fact also in consideration of our final goal: the construction of a social mathematical game that would have been played by pupils, students and the general public. So, we took into account the third section of the first book, in particular the chapters three and four concerning arithmetical progressions and their summation (1828). This topic in fact, even though present in the program guidelines and textbooks, is very often neglected by teachers due to the structural lack of time in Italian Mathematics schedule of all types of high schools.

The workshop sessions were organised in this way: at the beginning the students individually read a small fragment followed by some questions proposed by the teacher, then shared their ideas with a partner and in the end two pairs were asked to come together in a group of four to work out the task.

The first quote considered was:

CHAP. III.

Of Arithmetical Progressions.

402. We have already remarked, that a series of numbers composed of any number of terms, which always increase, or decrease, by the same quantity, is called an *arithmetical progression.*

Thus, the natural numbers written in their order, as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. form an arithmetical progression, because they constantly increase by unity; and the series 25, 22, 19, 16, 13, 10, 7, 4, 1, &c. is also such a progression, since the numbers constantly decrease by 3.

403. The number, or quantity, by which the terms of an arithmetical progression become greater or less, is called the *difference*; so that when the first term and the difference are given, we may continue the arithmetical progression to any length.

For example, let the first term be 2, and the difference 3, and we shall have the following increasing progression: 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, &c. in which each term is found by adding the difference to the preceding one.

404. It is usual to write the natural numbers, 1, 2, 3, 4, 5, &c. above the terms of such an arithmetical progression, in order that we may immediately perceive the rank which any term holds in the progression, which numbers, when written above the terms, are called *indices*; thus, the above example will be written as follows:

Indices.	1	2	3	4	5	6	7	8	9	10	
Arith. Prog.	2,	5,	8,	11,	14,	17,	20,	23,	26,	29,	&c.

where we see that 29 is the tenth term.

accompanied by the following questions:

1. Let a be the first term, and d the difference, how would you write the 10^{th} term if the progression is increasing?
2. and if the progression is decreasing?
3. How would you write the n^{th} term in both cases?

The second fragment:

405. Let a be the first term, and d the difference, the arithmetical progression will go on in the following order:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a, & a \pm d, & a \pm 2d, & a \pm 3d, & a \pm 4d, & a \pm 5d, & a \pm 6d, \text{ \&c.} \end{array}$$

according as the series is increasing, or decreasing, whence it appears that any term of the progression might be easily found, without the necessity of finding all the preceding ones, by means only of the first term a and the difference d; thus, for example, the tenth term will be $a \pm 9d$, the hundredth term $a \pm 99d$, and, generally, the nth term will be $a \pm (n - 1)d$.

406. When we stop at any point of the progression, it is of importance to attend to the first and the last term, since the index of the last term will represent the number of terms. If, therefore, the first term be a, the difference d, and the number of terms n, we shall have for the last term $a \pm (n - 1)d$, according as the series is increasing or decreasing, which is consequently found by multiplying the difference by the number of terms *minus* one, and adding, or subtracting, that product from the first term. Suppose, for example, in an ascending arithmetical progression of a hundred terms, the first term is 4, and the difference 3; then the last term will be $99 \times 3 + 4 = 301$.

was followed by the questions:

4. Suppose to have an increasing progression of 7 terms, whose first is 2 and last 26, find the difference.
5. If in a finite increasing progression we know that the terms are n and the first term a and the last term z are given, how to calculate the difference?

Then students discussed on the third quote:

407. When we know the first term a, and the last z, with the number of terms n, we can find the difference d; for, since the last term $z = a \pm (n-1)d$, if we subtract a from both sides, we obtain $z - a = (n-1)d$. So that by taking the difference between the first and last term, we have the product of the difference multiplied by the number of terms *minus* 1; we have therefore only to divide $z - a$ by $n - 1$ in order to obtain the required value of the difference d, which will be $\frac{z-a}{n-1}$. This result furnishes the following rule: Subtract the first term from the last, divide the remainder by the number of terms *minus* 1, and the quotient will be the common difference: by means of which we may write the whole progression.

408. Suppose, for example, that we have an increasing arithmetical progression of nine terms, whose first is 2, and last 26, and that it is required to find the difference. We must subtract the first term 2 from the last 26, and divide the remainder, which is 24, by 9 − 1, that is, by 8; the quotient 3 will be equal to the difference required, and the whole progression will be:

1	2	3	4	5	6	7	8	9
2,	5,	8,	11,	14,	17,	20,	23,	26.

that went with:

6. How to find the number n of terms if the first and the last terms are given together with the difference?
7. Would question 6 always have an answer?
8. If yes explain why, if not, give a counterexample.

Afterwards the scrap examined was:

409. If now the first term a, the last term z, and the difference d, are given, we may from them find the number of terms n; for since $z - a = (n - 1)d$, by dividing both sides by d, we have $\frac{z-a}{d} = n - 1$; also n being greater by 1 than $n - 1$, we have $n = \frac{z-a}{d} + 1$; consequently the number of terms is found by dividing the difference between the first and the last term, or $z - a$, by the difference of the progression, and adding unity to the quotient.

For example, let the first term be 4, the last 100, and the difference 12, the number of terms will be $\frac{100-4}{12} + 1 = 9$;

410. It must be observed, however, that as the number of terms is necessarily an integer, if we had not obtained such a number for n, in the examples of the preceding article, the questions would have been absurd.

Whenever we do not obtain an integer number for the value of $\frac{z-a}{d}$, it will be impossible to resolve the question; and consequently, in order that questions of this kind may be possible, $z - a$ must be divisible by d.

Moreover, after having examined the hereafter paragraphs 412, 413, the students were asked for a proof that the sum of any two terms equally distant, the one from the first, the other from the last term, is always equal to the sum of the first and the last. After that the following paragraph 414 was given for a check.

CHAP. IV.

Of the Summation *of* Arithmetical Progressions.

412. It is often necessary also to find the sum of an arithmetical progression. This might be done by adding all the terms together; but as the addition would be very tedious, when the progression consisted of a great number of terms, a rule has been devised, by which the sum may be more readily obtained.

413. We shall first consider a particular given progression, such that the first term is 2, the difference 3, the last term 29, and the number of terms 10;

1	2	3	4	5	6	7	8	9	10
2,	5,	8,	11,	14,	17,	20,	23,	26,	29.

In this progression we see that the sum of the first and last term is 31; the sum of the second and the last but one 31; the sum of the third and the last but two 31, and so on: hence we conclude, that the sum of any two terms equally distant, the one from the first, and the other from the last term, is always equal to the sum of the first and the last term.

414. The reason of this may be easily traced; for if we suppose the first to be a, the last z, and the difference d, the sum of the first and the last term is $a + z$; and the second term being $a + d$, and the last but one $z - d$, the sum of these two terms is also $a + z$. Farther, the third time being $a + 2d$, and the last but two $z - 2d$, it is evident that these two terms also, when added together, make $a + z$; and the demonstration may be easily extended to any other two terms equally distant from the first and last.

In a successive moment the proposed aim was finding the general formula for the sum in two cases:

1. knowing the first, the last and the number of terms;
2. knowing the first term, the difference and the number of terms.

As a result, students read the Euler solution in paragraphs from 415 to 420:

415. To determine, therefore, the sum of the progression proposed, let us write the same progression term by term, inverted, and add the corresponding terms together, as follows:

$$\begin{array}{l} 2+\ 5+\ 8+11+14+17+20+23+26+29 \\ 29+26+23+20+17+14+11+\ 8+\ 5+\ 2 \\ \hline 31+31+31+31+31+31+31+31+31+31 \end{array}$$

This series of equal terms is evidently equal to twice the sum of the given progression: now, the number of those equal terms is 10, as in the progression, and their sum consequently is equal to $10 \times 31 = 310$. Hence, as this sum is twice the sum of the arithmetical progression, the sum required must be 155.

416. If we proceed in the same manner with respect to any arithmetical progression, the first term of which is a, the last z, and the number of terms n; writing under the given progression the same progression inverted, and adding term to term, we shall have a series of n terms, each of which will be expressed by $a + z$; therefore the sum of this series will be $n(a + z)$, which is twice the sum of the proposed arithmetical progression; the latter, therefore, will be represented by $\frac{n(a+z)}{2}$.

Afterward they were asked to particularize the formula in the cases of progressions starting with 1, having n terms and whose difference varies from 1 to 10 and then they checked the various particular formulae in paragraphs from 421 to 424.

421. If it be required to add together all the natural numbers from 1 to n, we have, for finding this sum, the first term 1, the last term n, and the number of terms n; therefore the sum required is $\frac{n^2+n}{2} = \frac{n(n+1)}{2}$. If we make $n = 1766$, the sum of all the numbers, from 1 to 1766, will be 883, or half the number of terms, multiplied by $1767 = 1560261$.

422. Let the progression of uneven numbers be proposed, 1, 3, 5, 7, &c. continued to n terms, and let the sum of it be required. Here the first term is 1, the difference 2, the number of terms n; the last term will therefore be $1 + (n-1)2 = 2n - 1$, and consequently the sum required $= n^2$.

The whole therefore consists in multiplying the number of terms by itself; so that whatever number of terms of this progression we add together, the sum will be always a square, namely, the square of the number of terms; which we shall exemplify as follows:

Indices,	1	2	3	4	5	6	7	8	9	10,	&c.
Progress.	1,	3,	5,	7,	9,	11,	13,	15,	17,	19,	&c.
Sum.	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	&c.

423. Let the first term be 1, the difference 3, and the number of terms n; we shall have the progression 1, 4, 7, 10, &c. the last term of which will be $1+(n-1)3=3n-2$; wherefore the sum of the first and the last term is $3n - 1$, and consequently the sum of this progression is equal to $\frac{n(3n-1)}{2} = \frac{3n^2-n}{2}$; and if we suppose $n = 20$, the sum will be $10 \times 59 = 590$.

424. Again, let the first term be 1, the difference d, and the number of terms n; then the last term will be $1 + (n-1)d$; to which adding the first, we have $2+(n-1)d$, and multiplying by the number of terms, we have $2n + n(n-1)d$; whence we deduce the sum of the progression $n + \frac{n(n-1)d}{2}$.

And by making d successively equal to 1, 2, 3, 4, &c., we obtain the following particular values, as shewn in the subjoined Table.

If $d = 1$, the sum is $n + \frac{n(n-1)}{2} = \frac{n^2+n}{2}$

$d = 2$, - - - $n + \frac{2n(n-1)}{2} = n^2$

$d = 3$, - - - $n + \frac{3n(n-1)}{2} = \frac{3n^2-n}{2}$

$d = 4$, - - - $n + \frac{4n(n-1)}{2} = 2n^2 - n$

$d = 5$, - - - $n + \frac{5n(n-1)}{2} = \frac{5n^2-3n}{2}$

$d = 6$, - - - $n + \frac{6n(n-1)}{2} = 3n^2 - 2n$

$d = 7$, - - - $n + \frac{7n(n-1)}{2} = \frac{7n^2-5n}{2}$

$d = 8$, - - - $n + \frac{8n(n-1)}{2} = 4n^2 - 3n$

$d = 9$, - - - $n + \frac{9n(n-1)}{2} = \frac{9n^2-7n}{2}$

$d = 10$, - - - $n + \frac{10n(n-1)}{2} = 5n^2 - 4n$

THE SCIENTIFIC COMMUNICATION FESTIVAL

The three last workshops were dedicated to imagine how to share with others the school work at the Third Edition of the Scientific Communication Festival “Scienza Under 18” which was announced for the first week of May 2012.

The Festival “SU 18” is a three days meeting directed to all the schools of the Gorizia’s province, in which classes or other groups of students explain to mates and the general public a peculiar scientific work they have done during the year. The main feature of the festival is its being composed by hands-on exhibits, i.e. you may touch and interact with the stand.

During the previous editions, we had verified that to let people play a game is an appreciated way to set up a mathematical exhibit (2012, 2014) while other type of fittings had led to a lower rate of presence and enthusiasm.

Therefore we concentrated in designing the game.

The main goals were:

- to let a large amount of people have fun while doing Mathematics in order to encourage everybody to try to understand it,
- to enhance mathematical self-esteem of all the participants to convince people that it is worth to do the effort needed to succeed in Mathematics,
- to endorse mathematical reasoning and mental calculating,
- to convey an important topic usually neglected, namely the arithmetical progressions and their sum,
- to introduce the figure of Euler to the general public and to present the book “Elements of Algebra”,
- to popularize the presence of a generally unknown important ancient collection of mathematical books stored in the Biblioteca Statale Isontina,
- to be attractive and quite understandable at a certain distance,
- to be usable also by pupils and students coming from Nova Gorica, the Slovenian part of Gorizia town.

The game should subsequently have certain peculiar features.

First of all, it needed to be social, i.e. to be played in teams in order to give the possibility of sharing the displeasure of a failure as well as the joy of a victory.

Secondly, each game session had to last no more than half an hour, otherwise not all the visiting group classes would have the opportunity to participate.

On the other hand, the chance component should be present to let open the prospect to win for the average or even the worse students, but not be prevalent, because one of the aims was to promote mathematical thinking.

What is more, people of a very large range of ages should be given the opportunity to participate.

Eventually it should be played on a colourful billboard and guided by the students who had participated to its design both in the Italian and English version.

To try to fulfil all the characteristics, we conceived the following structure:

- a vivid billboard and one die are needed to play, as well as a person who asks the questions and controls the answers giving the right ones if they are wrong,
- the board has 28 cubbyholes,
- with the first throw one simply enters the play, then, after a throw the participants have to answer a question about progressions in 2 minutes: if they answer correctly, they will advance the points obtained, otherwise they will advance three points less,
- the winner is the player that first overcomes the 28^{th} slot,
- there are three versions of questions: easy (for 8 to 11 years old pupils), medium (for girls and boys from 11 to 15 years old), difficult (from 15 years old to adults),
- an English version of the questions and answers is available on request.

In addition on the Festival's Games Corner walls there were posters telling about Euler's life and work, and showing pictures of the Euler's books conserved in the Biblioteca Statale Isontina with particular attention to "Elements of Algebra" both in English and Italian. Students were charged in turn to guide the participants to the poster section visit and to explain a bit more about the work done during the workshops or to drive the game in English or Italian.

Hereafter you can read the questions and the answers of the difficult version of the game:

1) 5/8 5/4 15/8 … Which is the next term?

2) 7/2 –1 –11/2 … Which is the next term?

3) 3/7 2 25/7 … Which is the next term?

4) 0 –3/5 -6/5 … Which is the next term?

5) –1/5 –3/5 –1 … Which is the next term?

6) 3 8 13 18 … Which is the 10^{th} term?

7) 5 0 –5 –10 … Which is the 8^{th} term?

8)	5	8	11	14	…	Which is the 12^{th} term?
9)	1	3/2	2		…	Which is the 1000^{th} term?
10)	3	13/3	17/3	7	…	Which is the 11^{th} term?

11) In an arithmetical progression of 12 terms, the first is 7 and the last is 51. Find the third.

12) In an arithmetical progression of 16 terms, the first is 11 and the last is 356. Find the 12^{th}.

13) In an arithmetical progression of 7 terms, the first is 208 and the last –2. Find the 5^{th}.

14) In an arithmetical progression of 9 terms, the first is 8 and the last is –4. Find the 5^{th}.

15) In an arithmetical progression of 7 terms, the first is 4 and the last 23/2. Find the 3^{rd}.

16) In an arithmetical progression of 6 terms, the first is 8/7 and the last –97/7. Find the 4^{th}.

17)	4	6	8	…	Which is the sum of the first 10 terms?
18)	15	12	9	…	Which is the sum of the first 6 terms?
19)	24	32	40	…	Which is the sum of the first 7 terms?
20)	18	22	26	…	Which is the sum of the first 15 terms?
21)	–1/8	1/8	3/8	…	Which is the sum of the first 12 terms?
22)	2	3/2	1	…	Which is the sum of the first 8 terms?

23) How many and which are the arithmetical progressions that start by 3, end by 25, and have as common difference a whole number?

24) How many and which are the arithmetical progressions that start by –20, end by 17, and have as common difference a whole number?

25) How many and which are the arithmetical progressions that start by 11, end by 2, and have as common difference a whole number?

26) How many and which are the arithmetical progressions that start by –11, end by 5, and have as common difference a whole number?

In the easy version the questions were essentially about the tables of multiplication going up, down, and beyond the traditional tenth term; while in the medium one, beside the multiplication tables with natural numbers as in the easy version, the questions were extended to negative numbers and to fractions and sometimes we asked for an intermediate term having given the first and the last.

FINAL REMARKS

To close let me make only a few remarks.

- The over 400 people who visited our Games' Corner appreciated the activity and were happy to play with us and some of them went to visit the stand twice or three times bringing with them their parents.
- Our exhibit had the attention of the local newspaper "Il Piccolo" that wrote about our work.
- The Old Books section of the Biblioteca Statale Isontina had an increase of visitors after the Festival.
- The teachers who brought their classes to our stand continued to visit us during the following years.
- All the students who have participated to the workshops improved their school mathematics performances as well as their personal motivation in studying Mathematics and together with their parents asked for repeating the experience in the Consiglio di Classe's [3] meetings during the following years.
- The subsequent autumn the podium of the most known Italian Mathematics Competition, the Giochi di Archimede, at the Istituto d'Arte "D'Annunzio-Fabiani" was entirely covered by students that had participated to the workshops (3 over 12 participants in a school of 800 students, one of them was voluntarily taking part for the first time, the two others had already competed and never had been in the first three positions before).
- The group was invited by the Mathematics teachers of various schools to let other classes play the game.

To sum up I would like to conclude with a personal opinion: I think that, beside the good or excellent scientific work researchers do in their specific field, the widespread dissemination of a more humanistic idea of what Mathematics is and how to think mathematically should be felt by everybody in the Mathematics Education domain as a moral duty going into the direction of accelerating the overall improvement of the learning of Mathematics.

NOTES

1. Istituto nazionale per la valutazione del sistema educativo di istruzione e di formazione.
2. Content and Language Integrated Learning.
3. The periodical meetings among students, parents and teachers in which the advancement of the class activities is discussed.

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Workshop

TWO INTRODUCTORY UNIVERSITY COURSES ON THE HISTORY OF MATHEMATICS

Robin Wilson

Pembroke College, Oxford University& The Open University

Over the past five years I have been involved with the preparation and presentation of two courses at a basic level on the history of mathematics. The first was for interested adults, while the second was for liberal-arts college freshmen in North American Universities. Both courses were based around Marcus du Sautoy's award-winning BBC-Open University television series 'The Story of Maths' and on a booklet that I wrote to accompany this series.

The first course was an Open University 10-point Level 1 Course, designed to 'teach the maths behind the programmes' to adult students studying at home. This course used a 200-page OU booklet that I wrote, and has now been successfully presented a dozen times to a total of about 2000 students. I shall describe the course content and the motivation for producing this course, and also describe the results.

The second had two forms, and was presented as 'total-immersion' courses to first-year liberal-arts students in Western Canada (over 18 days) and in Colorado, USA (over 36 days). Like the Open University course, it was based on Marcus du Sautoy's television programmes and my course notes, but also (in order to emphasize the differences between 'the history of mathematics' and 'the history of mathematicians') on R. Flood and my recent illustrated book 'The Great Mathematicians'. Again, I shall describe and analyse the results of teaching these courses.

Oral Presentation

THE DIFFERENCE AS AN ANALYSIS TOOL OF THE CHANGE OF GEOMETRIC MAGNITUDES: THE CASE OF THE CIRCLE

Mario Sánchez Aguilar & Juan Gabriel Molina Zavaleta

CICATA Legaria, National Polytechnic Institute

In this paper we present a didactical proposal focused on the study of some of the existing relationships between the radius, area and circumference of a circle. The proposal is inspired by historic elements of the genesis of calculus and makes use of the software GeoGebra. Although the proposal could add dynamism to the teaching of geometry and even have some motivational value for students, it would be necessary to do some field research to illustrate its scope and limitations.

INTRODUCTION

The use of history in the teaching and learning of mathematics is a well-established area of research within the international community of mathematics educators; this can be verified through the many publications, conferences, and study groups that have specialized in this area in recent years. The sixth volume of ICMI studies (Fauvel & Van Maanen, 2002), and the groups History and Pedagogy of Mathematics (HPM) and History in Mathematics Education at the European conference CERME are just a few examples that illustrate the interest of our community in the use of history of mathematics as an element of mathematics instruction. However, not all the uses made of the history of mathematics in teaching are of the same nature. In his categorization of the "whys" and "hows" of using history in mathematics education, Jankvist (2009) proposes three categories to organize the different uses of history in mathematics education: (1) the illumination approaches, (2) the modules approaches, and (3) the history-based approaches.

In this manuscript we present a didactical proposal framed in the category of history-based approaches; it is a proposal inspired by the historical development of mathematics. In particular, we consider the idea of the *difference* between two quantities x_1 and x_2, which was a tool for analysing the variation of quantities that was used during the genesis of calculus. Our proposal is inspired by the interpolation method called *methodus differentialis* which was first used by Isaac Newton as a tool for predicting the behaviour of some celestial bodies (Newton, 1686, pp. 287-288).

One aim of our proposal is to help students to discover some existing relationships between the length of the radius of a circle, and the area and perimeter of that circle. As we will see, the proposal includes the use of the software GeoGebra, which is perceived as an instrument of knowledge mediation.

In the next section we explain the context that gave rise to our proposal, referring to a recent reform of secondary education in Mexico. Next, we briefly introduce the *methodus differentialis*. In the next two sections we detail the purpose and operation of the proposal, but we also clarify the links between the proposal and Newton's *methodus differentialis*. We conclude the manuscript with some reflections on the didactical proposal.

THE CONTEXT OF THE PROPOSAL

The didactical proposal originates in the context of a recent reform of secondary education in Mexico. The implementation guidelines of the reform recommend the use of dynamic geometry software to support the study of geometric bodies (see Secretaría de Educación Pública, 2006).

In the guidelines, the following instruction caught our attention:

> As with the study of the other figures, the aim is not only is to calculate the area and perimeter but also, given the perimeter and the area, to calculate the length of the radius or diameter, as well as to find areas of shaded regions (annulus); **the relationship between the length of the radius and the area of the circle must also be analysed, and compared with the relationship between the length of the diameter and the length of the circumference**. (Secretaría de Educación Pública, 2006, p. 54, our translation, our emphasis)

These guidelines tell the teacher what to teach, but do not clarify how it should be taught. For this particular lesson, the guidelines recommend consulting the supplementary material "Geometría dinámica" (Dynamic Geometry) (Secretaría de Educación Pública, 2000). One would expect this material to include specific instructions on how the teacher can implement the required exploration using a dynamic geometry software; however, as stated on pages 68–70 of the supplementary material (which deals with the analysis of the magnitudes of a circle), only a pen-and-paper activity is proposed, but not one using a dynamic geometry package. Our didactical proposal was designed with the intention of filling this gap.

ILLUSTRATION OF THE OPERATION OF THE METHODUS DIFFERENTIALIS

The *methodus differentialis* is a mathematical method originally used by Isaac Newton as a tool for predicting the behaviour of some celestial bodies; it is essentially an interpolation arithmetic method in which a finite set of points in a plane, A, B, C, D, E, F, etc., is considered. From these points the line segments AH, BI, CK, DL, EM and FN are drawn. These segments are perpendicular to another line segment HN (see Figure 1).

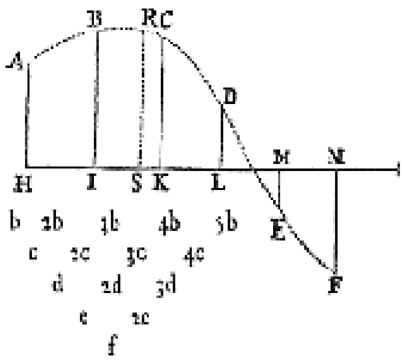


Figure 1: Graph illustrating the operation of the *methodus differentialis* (Newton, 1686, p. 288).

The main purpose of this method is to find the length or height corresponding to an unknown point that is located in any intermediate position between the points A, B, C, D, E, and F. In Figure 1 this unknown height is represented by the line segment RS. Clearly the represented lengths could be interpreted today as the ordinates corresponding to elements H, I, K, L, M and N in the domain of a function.

The method makes use of differences and quotients of these differences. Such quotients are represented in Figure 1 by the expressions including the lowercase letters a, b, c, d, e and f. The quotients are defined as follows:

$$b = \frac{AH - BI}{HI}, 2b = \frac{BI - CK}{IK}, 3b = \frac{CK - DL}{KL}, 4b = \frac{DL - ME}{LM}, etc.$$

$$c = \frac{b - 2b}{HK}, 2c = \frac{2b - 3b}{IL}, 3c = \frac{3b - 4b}{KM}, etc.$$

$$d = \frac{c - 2c}{HL}, 2d = \frac{2c - 3c}{IM}, etc.$$

$$e = \frac{d - 2d}{HM}$$

It is important to clarify that the terms $2b$, $3b$, …, $2c$, $3c$, …, $2d$, etc., do not carry the meaning of multiplication by 2, 3, etc., but rather they carry the meaning of the modern day notation of subscripts like b_2, b_3, etc.

Our main interest in this manuscript is not to detail the operation of the interpolation method, as it is not the underlying principle upon our didactical proposal is based; instead we want to emphasize an idea behind our proposal, namely that the nth order differences of a polynomial of degree n are constant. Later we will illustrate this principle in our proposal (see for example figures 3 and 4).

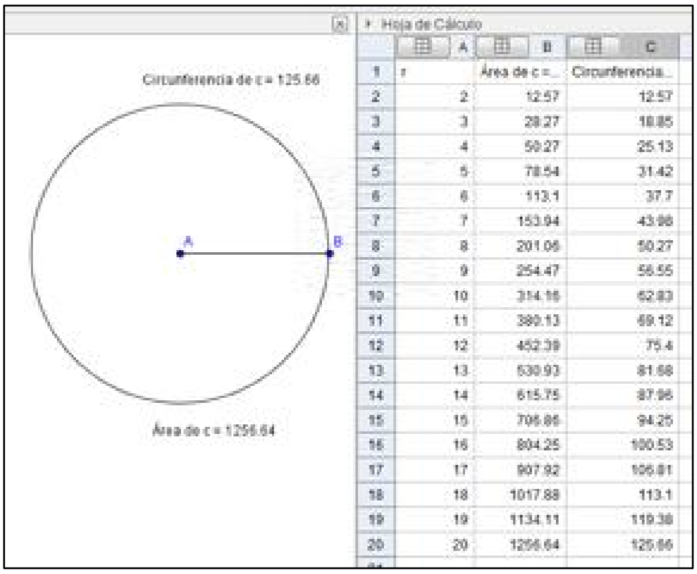
THE DIDACTICAL PROPOSAL TO ANALYZE RELATIONSHIPS BETWEEN SOME MAGNITUDES OF THE CIRCLE

The GeoGebra software allows dynamic linking of the geometric and numeric contexts of representation, which is ideal for analysing the relationships between the magnitudes of the circle specified in the reform. The aim of the didactical proposal is twofold: firstly, it aims to help students to discover that the area and length of the circumference increase when the length of the radius increases, but not in the same way or with the same speed; on the other hand, it is also intended to help mathematics teachers to explore these kind of mathematical relationships along with their students in a dynamic way.

We begin with the following situation:

> *Consider a circle whose initial radius is 1 unit long. As the length of the radius increases in steps of 1, what happens to the area of the circle? What about the length of the circumference?*

This situation is represented by an animation made with GeoGebra in which the length of the radius of a circle varies discretely, while the magnitudes of the area and the circumference when the radius changes are recorded. Column A in Figure 2 shows the values of the radius r, while column B shows the respective areas and column C the values of the circumference.



	A	B	C
1	r	Área de c =...	Circunferencia...
2	2	12.57	12.57
3	3	28.27	18.85
4	4	50.27	25.13
5	5	78.54	31.42
6	6	113.1	37.7
7	7	153.94	43.98
8	8	201.06	50.27
9	9	254.47	56.55
10	10	314.16	62.83
11	11	380.13	69.12
12	12	452.39	75.4
13	13	530.93	81.68
14	14	615.75	87.96
15	15	706.86	94.25
16	16	804.25	100.53
17	17	907.92	106.81
18	18	1017.88	113.1
19	19	1134.11	119.38
20	20	1256.64	125.66

Figure 2: Variation of the radius of a circle and the magnitudes of the area and the circumference using the graphical and spreadsheet capabilities of GeoGebra.

Column D in Figure 3 shows that the calculation of the differences applied to the values of the circumference during the animation generates the constant value of 6.28 ≈ 2π. In the animation, it can be seen that as the radius grows, so do the area and circumference, but how do the area and circumference grow? To address this question we use the spreadsheet application in GeoGebra to calculate the differences between consecutive numbers in a similar way as performed in the *methodus differentialis*. We do this for both values: the values of the circumference and the values of the area of the circle.

Hoja de Cálculo

	A	B	C	D
1		Área de c ...	Circunferencia...	Diferencias Circunferencia
2	2	12.57	12.57	6.28
3	3	28.27	18.85	6.28
4	4	50.27	25.13	6.28
5	5	78.54	31.42	6.28
6	6	113.1	37.7	6.28
7	7	153.94	43.98	6.28
8	8	201.06	50.27	6.28
9	9	254.47	56.55	6.28
10	10	314.16	62.83	6.28
11	11	380.13	69.12	6.28
12	12	452.39	75.4	6.28
13	13	530.93	81.68	6.28
14	14	615.75	87.96	6.28
15	15	706.86	94.25	6.28
16	16	804.25	100.53	6.28
17	17	907.92	106.81	6.28
18	18	1017.88	113.1	6.28
19	19	1134.11	119.38	6.28
20	20	1256.64	125.66	

$$
\begin{aligned}
C_3 - C_2 &= 18.85 - 12.57 = 6.28\\
C_4 - C_3 &= 25.13 - 18.85 = 6.28\\
C_5 - C_4 &= 31.42 - 25.13 = 6.28\\
C_6 - C_5 &= 37.7 - 31.42 = 6.28\\
C_7 - C_6 &= 43.98 - 37.7 = 6.28\\
C_8 - C_7 &= 50.27 - 43.98 = 6.28\\
C_9 - C_8 &= 56.55 - 50.27 = 6.28\\
C_{10} - C_9 &= 62.83 - 56.55 = 6.28\\
C_{11} - C_{10} &= 69.12 - 62.83 = 6.28\\
C_{12} - C_{11} &= 75.4 - 69.12 = 6.28\\
C_{13} - C_{12} &= 81.68 - 75.4 = 6.28\\
C_{14} - C_{13} &= 87.96 - 81.68 = 6.28\\
C_{15} - C_{14} &= 94.25 - 87.96 = 6.28\\
C_{16} - C_{15} &= 100.53 - 94.25 = 6.28\\
C_{17} - C_{16} &= 106.81 - 100.53 = 6.28\\
C_{18} - C_{17} &= 113.1 - 106.81 = 6.28\\
C_{19} - C_{18} &= 119.38 - 113.1 = 6.28\\
C_{20} - C_{19} &= 125.66 - 119.38 = 6.28
\end{aligned}
$$

Figure 3: Calculating the differences between the values of the circumference using the spreadsheet application in GeoGebra.

The constant value of the differences (6.28) indicates that the circumference of the circle changes linearly when the length of the radius increases; if we plot the radius-circumference ratio, we obtain a linear function graph. Since we calculated the differences of the data once, we can say that we obtained a variation of first order and in this case the result is a constant value.

A purpose of our proposal is to illustrate how the arithmetic difference may be used as a tool to analyse the variation of geometrical magnitudes, and we believe such mathematical technique is accessible to lower secondary school pupils; however, some aspects of the relationship between the length of the radius of a circle and its circumference and area can serve as an introduction to more advanced concepts belonging to differential calculus. For example, if we proceed similarly for the case of the radius-area ratio, when calculating the second differences (i.e. the differences of the differences contained in column B in Figure 3), we will obtain again the constant value 6.28 ≈ 2π. In this case we have a second order variation (Figure 4).

E	F
Diferencias de área	Seg. Diferencias de área
15.71	6.28
21.99	6.28
28.27	6.28
34.56	6.28
40.84	6.28
47.12	6.28
53.41	6.28
59.69	6.28
65.97	6.28
72.26	6.28
78.54	6.28
84.82	6.28
91.11	6.28
97.39	6.28
103.67	6.28
109.96	6.28
116.24	

Figure 4: Calculating the first differences (column E) and second differences (column F) for the values of the area of the circle using the spreadsheet application in GeoGebra.

The fact that we obtain a constant value (approximately equal to 2π) after calculating the first differences of the length of the circumference and the second differences of the values of the area is related to the mathematical concept of the derivative, since the idea of difference is the foundation of the structure of this mathematical concept: it is known that the derivative of a function $C = C(r)$ with respect to r is $C'(r) = \lim_{h \to 0} \frac{C(r+h)-C(r)}{h}$, provided that this limit exists; if we deliberately omit applying the limit as h approaches 0, then an approximation of the derivative is obtained.

In the case of the values of the circumference analysed in this work, $h = 1$ and the expression $\frac{C(r+1)-C(r)}{1}$ is equivalent to the differences $C_{r+1} - C_r$ calculated in the spreadsheet. In the case of the area we would have $B_{r+1} - B_r$ for r = 2, 3, ... 19; these are the calculations that we performed in the spreadsheet and that are shown in Figure 3 and Figure 5. A similar rationale is used for the second differences: the result of these calculations is $6.28 \approx 2\pi$. Furthermore, it is known that the area of a circle is $A(r) = \pi r^2$ and its derivative is $A'(r) = 2\pi r$; the result of the second derivative is $A''(r) = 2\pi \approx 6.28$. If both functions are graphed, the results are a parabola and a straight line.

LINK BETWEEN THE METHODUS DIFFERENTIALIS AND THE DIDACTICAL PROPOSAL

As previously mentioned, our proposal to explore how the changes in the magnitudes of a circle is based on the *methodus differentialis*, but we consider only the first part of the method in which quotients of difference are calculated. Next we illustrate the application of this part of the method to the values of the areas that we have worked in GeoGebra (these values are shown in Figure 3).

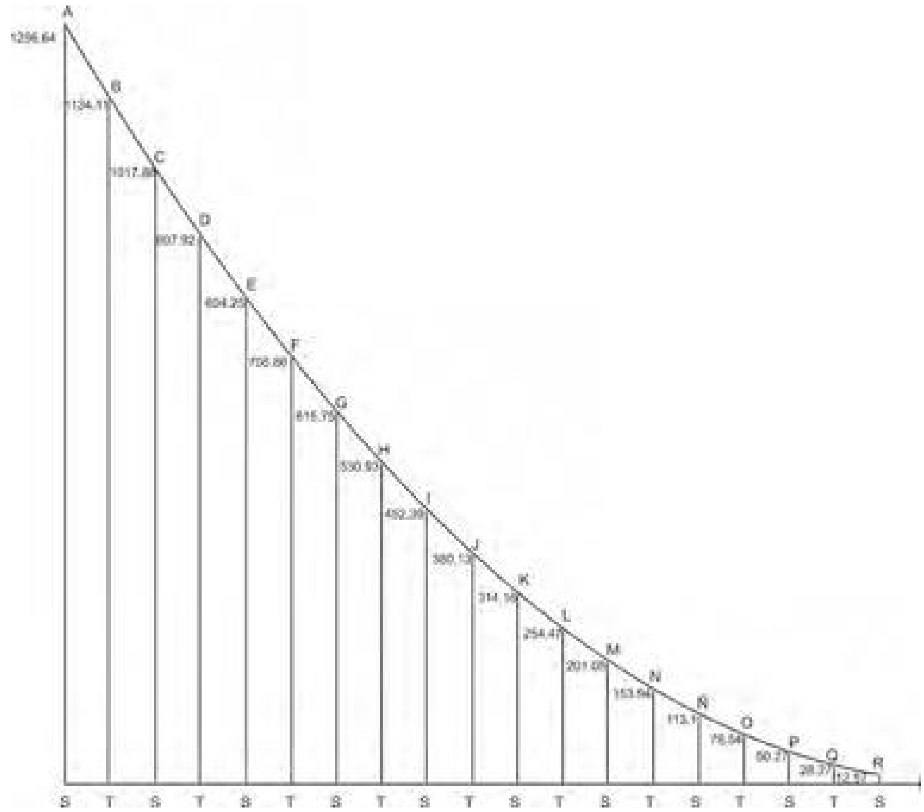


Figure 5: Representation of the variations of the area similar to the graphical representation used in the *methodus differentialis* (see Figure 1).

In this case we would have:

$$b = \frac{AS - BT}{ST} = \frac{1256.64 - 1134.11}{1} = 122.53$$

$$2b = \frac{BT - CS}{TS} = \frac{1134.11 - 1017.88}{1} = 116.23$$

$$3b = \frac{CS - DT}{ST} = \frac{1017.88 - 907.92}{1} = 109.96$$

$$4b = \frac{DT - ES}{TS} = \frac{907.92 - 804.25}{1} = 103.67, etc.$$

FINAL CONSIDERATIONS

This combination of historical elements of mathematics with the use of technological tools is not new; there are proposals like the one by Kidron (2004), in which software with graphical and algebraic capabilities is used to teach the topics of approximation and interpolation according to their historical development. We believe that these types of proposals that combine history and the use of technology should be further developed since, on the one hand, the use of history in the teaching of geometry can have a motivational value to students (Gulikers & Blom, 2001), and, on the other hand, the use of technology can add meaning to the concepts studied in the mathematics classroom by making evident the relationship between the different contexts of representation (such as the numerical, algebraic, and geometric). It is necessary that proposals such as the one presented in this paper be tested in real mathematics classrooms to learn more about the scope and limitations of this type of teaching approach.

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Oral Presentation

MENO BY PLATO: AN ANCIENT INEXHAUSTIBLE MINE OF KNOWLEDGE

Panagiotis Delikanlis

Greek Ministry of Education and Religious Affairs

Meno by Plato is a middle period dialogue. It was written about 385-386 B.C. The dialogue begins with Meno asking Socrates whether virtue can be taught. Socrates states that he does not know the definition of virtue. Meno is in aporia (puzzlement) and responds with a paradox. Socrates introduces the recollection as a theory of knowledge (since, it would seem, research and learning are wholly recollection, Plat. Meno 81d). Afterwards, he illustrates his theory by posing a geometrical problem: "A square of side two feet has area four square feet. Doubling the area, we draw another square of eight square feet. How long is the side of the new square?"

In this oral presentation I would like to discuss:

- Solutions using different types of square paper
- Socratic method solutions
- Pick's formula solution

Square paper is a mediating tool. Tools are mediators of human thought and behaviour. The use of square paper supported ingenious geometry solutions.

Oral Presentation

TEACHING THE AREA OF A CIRCLE FROM THE PERSPECTIVE OF HPM

HONG Yanjun[a] & WANG Xiaoqin[b]

[a]East China Normal University, [b]Shihezi University

In China, importance is attached to three objectives in mathematics teaching: knowledge & skills, process & methods, affect & attitude, corresponding to which we have the following "whys" of integrating the history of mathematics into mathematical teaching: (1) The history of mathematics is helpful for deepening students' understanding of mathematics; (2) The history of mathematics provides a lot of problem-solving methods and can broaden students' thinking; (3) The history of mathematics increase students' interest and create their motivation. The above values had been really achieved in the mathematics classroom when the history of mathematics was integrated into the teaching of the area of a circle at the sixth grade in a junior high school, and the benefits were identified from the students' viewpoints.

INTRODUCTION

In the field of education in China, great importance has been attached in recent years to the Three-Dimension Instructional Objectives, i.e., knowledge & skills, process & methods, affect & beliefs. From the perspective of HPM, we have designed teaching projects to accomplish those objectives by integrating the history of mathematics into mathematical teaching. The reasons we use this approach are listed as follows:

(1) The history of mathematics is helpful for deepening students' understanding of mathematics;

(2) The history of mathematics provides a lot of problem-solving methods and can broaden students' thinking;

(3) The history of mathematics increases students' interest and creates their learning motivation.

Can those goals really be achieved? Next, we will introduce an experiment of using history of mathematics in the teaching of mathematics of one middle school in shanghai, and share our experience as well.

THEORIES RELATED TO HPM TEACHING

Principles of HPM teaching design

Italian scholar Fulvia Furinghetti (2000) introduced a general process to integrate history of mathematics into mathematics teaching. Based on her idea, we made some adaptations and set the following teaching process: choosing a teaching subject → investigating related history → selecting suitable materials → analyzing classroom requirements → developing classroom activities → implementing teaching design → evaluating the course.

The key to success in HPM teaching design is selecting proper materials of history of mathematics. In our opinion, the historical materials selected must be interesting, scientific, effective, learnable and innovative. **Interesting**: the historical materials should raise students' interests in study, that's why we need to select stories closely related to the teaching design. **Scientific**: we mean the materials must comply with facts or historical backgrounds. As the HPM teaching is not only using history of mathematics for the sake of history of mathematics. **Effective**: we mean that the materials should serve for the objectives of the teaching. **Learnable**: we mean that the materials should be provided in accordance with students' cognition level and could be readily accepted by them. **Innovative**: we mean that the materials should be new to students, the teaching design has distinguishing features, and can promote teachers' professional development.

Approaches of integrating history of mathematics into classroom instruction

One of the important questions in HPM is the study on approaches to integrate history of mathematics into mathematics education. Under the frame of mathematics education, researchers have constructed many integrating approaches, considering the relations between history of mathematics and teaching factors. Fauvel (1991) has generalized 10 ways. Tzanakis and Arcavi (2000) have 3 different approaches, including (1) "Learning history, by the provision of direct historical information", (2) "Learning mathematical topics, by following a teaching and learning approach inspired by history", and (3) "Developing deeper awareness, both of mathematics itself and of the social and cultural contexts in which mathematics has been done". Jankvist (2009) outlined another three ways, illumination, the modules and the history-based approaches. Professor Wang Xiaoqin (2012), by integrating and adapting the above-mentioned two grouping methods, labels them as complementation, replication, accommodation, and reconstruction, (Table 1).

Table 1: Approaches of using history of mathematics in teaching

Approaches	Description	Tzanakis&Arcavi	Jankvist
complementation	Display mathematicians' pictures, give an account of related stories,etc.	Direct historical information	Illumination approach
Replication	Directly using mathematical problems, methods, etc.	Direct historical information	Illumination approach; modules.
Accommodation	Problems adapted from historical ones or based upon historical materials	–	–
Reconstruction	Genesis of knowledge based on or inspired by the history of mathematics	Teaching approach inspired by history (Genetic approach)	History-based approaches

Direct using of the historical materials is the first level of using the history of mathematics, while mathematics teaching and learning in the perspective of HPM is the second level, which means to learn from, replay, and reconstruct the history of mathematics.

The next we'd like to present the teaching of the area of a circle as an example to better elaborate the above mentioned problems.

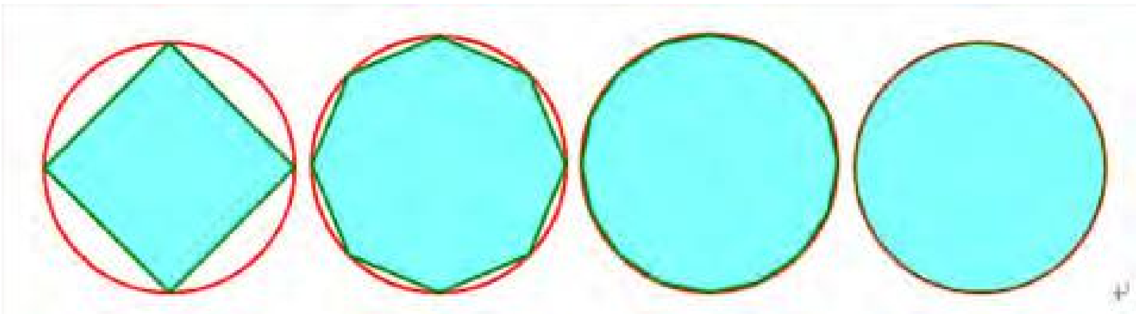
TEACHING THE AREA OF A CIRCLE INTEGRATED WITH HISTORY OF MATHEMATICS

The area of a circle is a knowledge point in the 6th graders' mathematics textbook in Shanghai. Previously the students have had a rough idea of the circle and the circumference, after having studied the areas of the linear graphics such as rectangles, squares, triangles, parallelograms and trapezoids. In this teaching design, Kepler's method of calculating the area of a circle is programmed and detailed in classroom teaching. This is the approach of accommodation in HPM teaching.

Investigating the history of the area of a circle

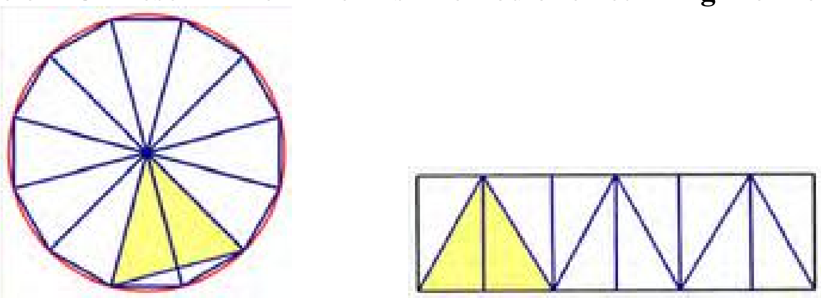
The ancient Babylonians and Egyptians encountered the problem of the area of a circle when measuring land, but they did not produce a calculation formula (Liang, 1995, p164-165). As shown on the Babylonian tablets YBC 7302, the area of a circle would be 1/12 times the square of its circumference. The Egyptian Rhind papyrus of 1800 BC gives the area of a circle as (64/81) d^2, where d is the diameter of the circle. In ancient Greece, Antiphon (c.480-411BC) originated the idea of squaring a circle with an inscribed regular polygon (Liang, 1995, p255). As in Figure 1, we inscribe a square in a circle, and then double its number of sides repeatedly. When the sides are infinite, this regular polygon eventually 'becomes' a circle. Then we have the area of a circle. Learning from Antiphon's idea, Archimedes (287-212BC) used inscribed and circumscribed regular polygons, applying the method of exhaustion to prove that the area of a circle is half its circumference times its radius (Heath, 1949).

Figure 1: Antiphon's method of squaring a circle



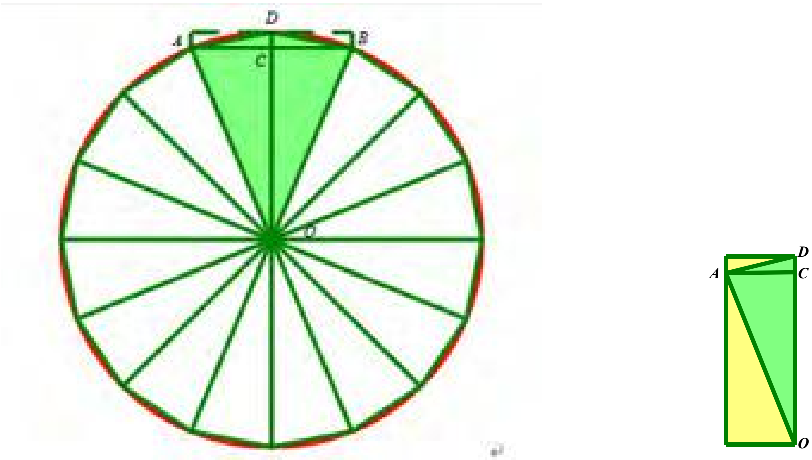
In China, a book named *The Nine Chapters on the Mathematical Art* written before the 2nd century BC also tells that the area of a circle is half its circumference times its radius. However, ancient Chinese mathematicians used a quite rough method to compute the area of a circle. They took the circumference of an inscribed regular 6-gon as a circle's circumference, and the area of an inscribed regular 12-gon as that of a circle, applying *the Out-In Complementary Principle* to patch the regular 12-gon into a rectangular which has half of the regular 6-gon's circumference as its length and radius of the circle as its width (Figure 2). Hence the area of a circle is 3 r , where r is the radius of the circle. Here π is 3, very roughly in terms of its real value (Guo , 2007).

Figure 2: Ancient Chinese Mathematicians' method of calculating the area of a circle



To find a formula for the area of a circle, Liu Hui (c.225-295AD) used the cyclotomic method (Wang, 2013), (Figure 3). The area of inscribed regular 2n-gon is added up by *n* times of the deltoid OADB. As every deltoid is a rectangular consisting of 4 parts, its area is $\frac{1}{2}a_n R$. We then have the formula for the area of the regular 2n-gon $S_{2n} = \frac{1}{2}na_n R$. Liu Hui commented that when the sides of the inscribed regular n-gon increase, its circumference is more approximate to the circumference of the circle and its area is more approximate to the area of the circle. In today's mathematical language, we have $S = \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} \frac{1}{2}na_n R = \frac{1}{2}CR$.

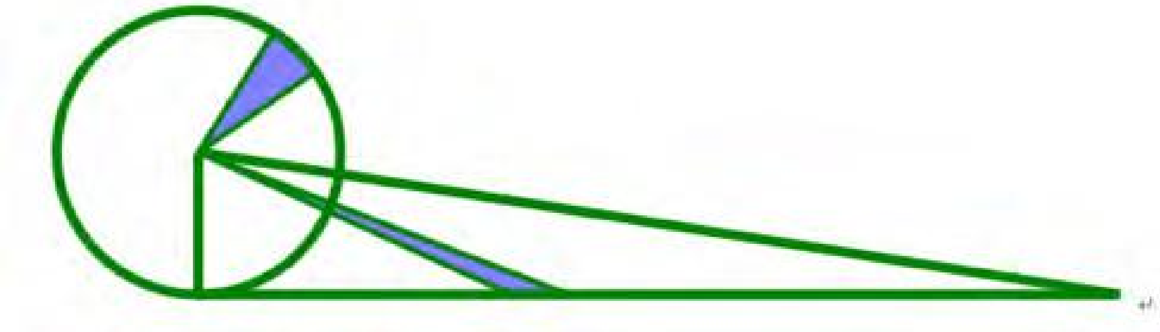
Figure 3: Inscribed regular 2n-gon is made up of n deltoids



It is worthwhile to note that the ancient Greeks had no notions of limit. In their opinion, the inscribed polygon approximates a circle, as close as one's mind can reach. However, there must be tiny parts missing between the area of a circle and that of a polygon. They proved their thinking with the technique of double reduction to absurdity, not the method of limit (Boyer, 1977). Liu Hui's cyclotomic method, in his opinion, will eventually lose no parts of the circle. This idea is quite close to modern concept of limit (Guo, 1983).

The German mathematician Johannes Kepler (1571-1630) came out an idea to compute the area of a circle on his second wedding while calculating the volume of a wine barrel. As in Figure 4, he divided a circle into countless small triangles with vertexes at the circle centre and radius as their heights. In fact, these triangles are small sectors. As the number of the circle divided is getting increasingly greater, the sector is coming closer to a triangle (Struik, 1948). If we change these small triangles into triangles with same base and height, they would form a right triangle. Hence we have $S=\sum\frac{1}{2}c_i r=\frac{1}{2}Cr=\pi r^2$.

Figure 4: Kepler's method of calculating the area of a circle



Selecting proper teaching materials

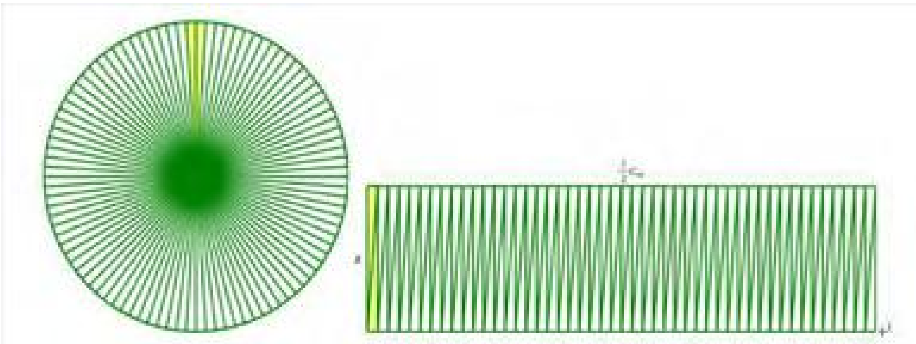
For the sixth graders, the area of a circle, upgraded from linear graph to curve graph, is a qualitative leap in both learning contents and methods, especially the inference of the area of a circle. Antiphon's method of squaring a circle is just an idea. Archimedes' calculation uses the double reduction to absurdity and the method of exhaustion, which is quite a complicated process. Liu Hui's cyclotomic method connecting 4 parts of a deltoid into a rectangular is also a little bit complex compared to Kepler's forming of triangles. These methods are not suitable for teaching sixth graders.

In Kepler's method, the area of a circle is the sum of all countless small triangles with vertexes at the center of the circle and base at its circumference. Hence the area of a circle is equal to the half circumference multiplied by radius. This method cannot be called precise if we have no concept of limit. Besides, it's not easy for a sixth grader to understand. For that reason, we use the method of accommodation， make some adaptations to Kepler's method and program it as follows: Substitute the inscribed n-gon with right triangles with the same area, and continually increase n, thus the area of the right triangle will get closer to that of a circle.

Kepler's method meets the five principles of our teaching design. The story itself is fun. The method Kepler used is in line with that of Archimedes and Liu Hui, and in compliance with sixth graders' cognition level, which means scientific. Kepler's method, without complicated calculation, makes it easier for the students to understand the formula of the area of a circle, which means our teaching process and objective can be effectively realized. As the students need only to know that triangles with same base and height have the same area, it is easy for them to learn Kepler's area equation process. And last but not least, Kepler's method itself is quite innovative.

Besides, the textbook introduces a way to calculate the area of a circle, by connecting little sectors into a rectangle in Shanghai (Shanghai Education Publishing House, 2011). As in Figure 5, the circle is equally divided into small sectors, and connected to create a graph that is approximately a rectangle. The more sectors the circle is divided into, the closer the area of the rectangle is to that of the circle. Then we can have the formula of the area of a circle.

Figure 5: Method in textbook (the circle is divided into 96 equal sectors)



Though the method of joining rectangles in the textbook is simple, straightforward, and easy to understand, it might still mislead students into believing that the formula is a rough and ready one,. If we do not make it clear, it would be puzzling for some students to comprehend this method.

In view of this, based on the above-mentioned five principles and an easy-to-difficult logical approach, we design our teaching by combining the methods mentioned in the mathematics textbook and used by Kepler. We start by using the method in the textbook, employing intuitive means to activate students' thinking. Then we bring in Kepler's method, preparing them with the idea that the number of a circle divided is increasingly greater, the sector is coming closer to a triangle. Then we teach them the method of connecting regular right triangles. In this way the students would be much more ready to accept that the formula of the area of a circle is an exact value.

The combination of these two methods enables the students to know not only the area of a circle, but also how that is realized. Apart from revealing knowledge's internal relations, it can also help students to understand the concepts of infinite approximation, to develop their spatial imagination, to motivate them to pick up more scientific methods, to acquire new knowledge with what they have learnt, to deepen their understanding of surroundings, and improves their learning ability.

Developing and implementing teaching design

This teaching project, the area of a circle, was designed by HPM team from Shanghai East China Normal University and teachers from a local middle school, under the guidance of Project for HPM and Professional Development of Mathematic Teachers in Junior High Schools by Shanghai Hutai Road Education Development. It was carried out in a common grade 6 class of 47 students by a young teacher under 5-year teaching experience. The teacher gave a simulated lecture to the project team, and then the design was discussed by all participants. At last, the teacher carried out the teaching project in the class. The teaching process is as follows.

(1) Introduction

Play an animation about a goat tying to a woodpile eating grass; introduce the topic of the area of a circle.

Design objective

If a teacher creates life situations in classroom to merge intangible emotions with tangible situations, the students would feel that what they study are from the life and for the life. This would boost their interest for learning, and lead them into the first step of studying a new topic, i.e., the concept of the area of a circle.

(2) Exploratory I

Introduce the concept of cut-supplement method; students assemble the jigsaw puzzle prepared beforehand; deduce the formula of the area of a circle. This is the method used in the textbook.

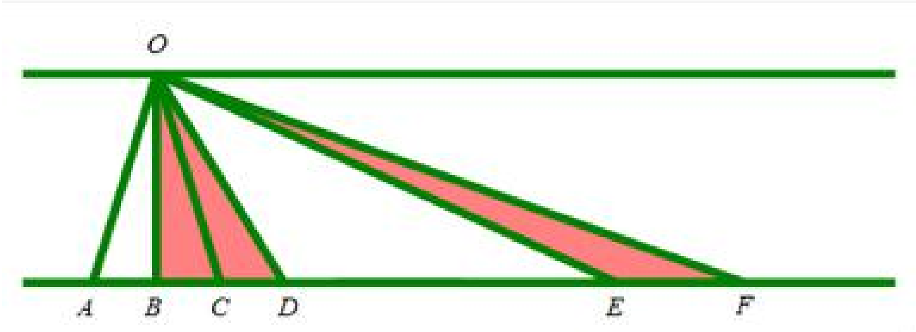
Design objective

Students participate, research and discover by themselves. they go through the development of the mathematical knowledge, and experience mathematical ideas and methods.

(3) Exploratory II

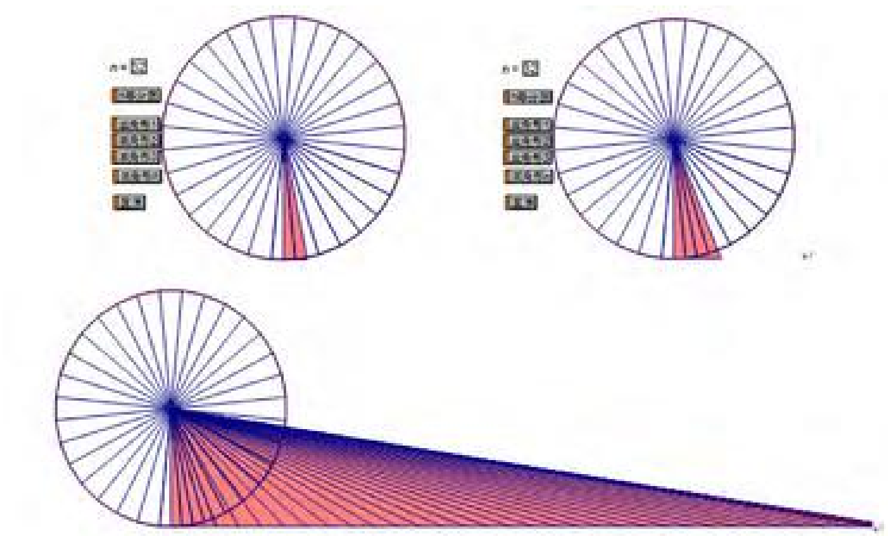
Narrate the story of Kepler calculating the area of a circle. Create graphs to show that triangles with same base and same height have same areas. As in Figure 6, triangles OAC, OBD and OEF have same base and same height, so their areas are also the same.

Figure 6: triangles with same base and same height have same areas



After that, we use authalic triangles to replace small triangles of the inscribed regular n-gon, and connect them one by one. In the end we will get a right triangle which has the same area as the inscribed regular n-gon. And we come up with the formula of the area of a circle: (Figure 7).

Figure 7: The process of Kepler's calculating of the area of a circle, n= 32



Design objective

These two methods have different starting points. The former guides the students to form an approximate rectangle, while the latter a regular right triangle. It enables the students to understand the nature of mathematical activities from different perspectives, and also to develop the fluency, flexibility and uniqueness of their thinking.

(4) Examples and exercises

(Omitted)

(5) Summary

(Omitted)

Teaching feedbacks

After the teaching, all 47 students took part in a survey, while some of them were interviewed.

Question: Do you have any trouble in understanding Kepler's method of calculating the area of a circle taught by your teacher?

45 students (96%) said they could understand fairly well or completely. Only a tiny minority of the teachers present at the class doubt that some students could understand Kepler's method, which was refuted by the survey.

Question: In comparison with previous teaching styles, do you like this one integrated with the history of mathematics?

46 students (98%) said they liked this style or liked it very much.

Question: Is there any difference in your attitude toward mathematical study after this teaching project on the origin of the formula for the area of a circle?

39 students (83%) said they were more interested or much more interested in mathematics.

As to subjective question: What impressed you most in this teaching project? Why?

Over one third of the students mentioned that the story of Kepler and his method impressed them most. Here are some of the students' responses.

Student A: I like mainly that part with mathematical history, for it is interesting and informative. (Figure 8)

Student B: What impressed me most is the method Kepler used to find the formula of the area of a circle. The story makes me more engaged to this class. (Figure 9)

Student C: I was most impressed by Kepler's method. It's simple and easy to understand.

Student D: How to calculate the area of a circle impresses me most. It is interesting, comprehensible and unique.

Figure 8: Feedback from Student A on this teaching design

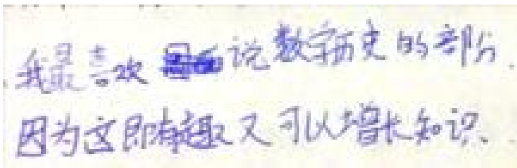
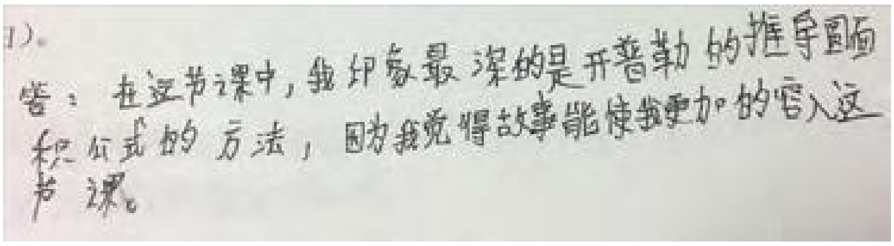

我最喜欢说数学史的部分
因为这既有趣又可以增长知识.

Figure 9: Feedback from Student B on this teaching design


答：在这节课中，我印象最深的是开普勒的推导圆面积公式的方法，因为我觉得故事能使我更加的容入这节课。

Generally speaking, the following four benefits of integrating the history of mathematics into classroom teaching are identified from the students' feedbacks:

(1) The history of mathematics is very interesting and can capture students' attention;

(2) The methods integrated with history of mathematics are easy to understand for the students;

(3) The history of mathematics can extend students' knowledge of mathematics;

(4) The history of mathematics tells the students that there are a variety of methods to solve a problem.

CONCLUSIONS

This teaching design integrated with the history of mathematics is quite a success. We rearranged the historical materials related to Kepler's method. And we also used the computer to demonstrate its process, which makes it more comprehensible and acceptable. Besides, Kepler's optimistic attitude toward his hard life also had positive effect on the students.

The survey and interview reveal that most students like the teaching with the history of mathematics. They are interested in Kepler's story and are impressed deeply by his method, and understand well the thinking of cyclotomic method.

This project proves that teaching from the perspective of the HPM can well serve China's Three-Dimension Instructional Objectives. It also brings some inspirations on mathematical teaching and textbook compilation. For example, we can create lively teaching situations with historical materials to make the learning more interesting, enable students to know about the development of essential mathematics, and strengthen students' understanding and improve their problem-solving ability by means of appropriate computer aided teaching. With the help of a textbook integrated with proper historical materials, right arrangement of teaching resources, and an intuitive-to-abstract, easy-to-difficult process, it will be of great use for students to reflect what they have learned and acquire new cognitive experience by analogy.

In a word, it's worthwhile for us to further study, explore, and apply HPM teaching into our classroom instructions.

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Oral Presentation

STATISTICS AND FREE WILL

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Abstract *In this paper we present our teaching work based on Quetelet's texts on "Moral Statistics" and Free Will aiming to motivate and stimulate relevant discussion with students. The work done allowed them to obtain significant insights on the Free Will debate, statistics and their relation. We provide evidence supporting the position that with adequate teaching design and implementation, it is possible to explore fruitfully existing links among statistics, probability and important philosophical issues, even with novice students in statistics.*

INTRODUCTION

All along the historical development of mathematics and philosophy, there have been deep links between them, developed and operating fruitfully in both directions. In particular probability and statistics are connected to concepts such as *uncertainty* and *chance* that also convey an important philosophical meaning (for the historical relations among probability, statistics and philosophy see, e.g., Hacking 1975, Porter 1986, Hald 2003, Chandler & Harrison 2012).

Though historically probability, statistics and philosophy have been strongly linked, their rich interrelations have been very little explored in the conventional teaching of these disciplines, and even less (or not at all) at an introductory level.

We argue that: (a) With adequately designed and implemented teaching activities, it is possible to explore links among probability, statistics and philosophy even with novice students in statistics and probability. (b) Exploring these links can contribute significantly to discuss with students deep philosophical issues, which are often related to important aspects of everyday life and in most cases are nontrivial for the students. (c) Appropriate teaching activities exploring such links for discussing philosophical issues may have an important motivational and emotional impact on the students, raising their strong interest and involvement. (d) Such teaching activities enrich students' concept image of what statistics is about, how it works and why it is interesting and meaningful. Moreover, combining (d) and (c) may also improve students' affective predisposition towards statistics.

To support (a)-(d) above, we present an example of teaching work realized during an introductory seminar on probability and statistics with prospective elementary school teachers.

HISTORICAL TEXTS USED

A key element in our teaching was the use of original historical texts that present statistical work and link it to a fundamental philosophical issue; namely, that of *Free Will* (FW). The texts we used with the students were chosen among Quetelet's works concerning Moral Statistics (MS)[1] and FW (we used mainly Quetelet (1847); but also Quetelet (1833), Quetelet (1842) Book 3, ch3 and Quetelet (1848)).

Quetelet's writings on MS and FW, as well as, the rest of his statistical works on the study of social phenomena are among the pioneering works that used statistics in social sciences. Educated as a mathematician and astronomer, Quetelet was familiar with probability theory of his time[2], as well as, with observational methods in astronomy, geodesy and meteorology and the associated error theory.

In analogy to these methods and the underlying theory, he thought that quantitative data of social phenomena could be understood as consisting of average values related to constant causes and variation around these averages related to accidental causes. Also, he thought that if one observes sufficiently large populations, then, because of the Law of Large Numbers (LLN) and the de Moivre-Laplace Central Limit Theorem, the cumulative influence of accidental causes on statistical figures is practically neutralised and thus it would be easier to identify regularities and relations between average values and the underlying constant causes. Quetelet believed that this approach involving statistics in social sciences has a great potential for uncovering regularities and relations concerning social phenomena (which he often called "*social laws*") and thus it would permit their much deeper understanding. He was able to find such important regularities on social phenomena, like marriages and crimes. However, he was often criticized of being over-optimistic concerning his opinion on the generality and validity in time of "*social laws*" that could be uncovered by the use of statistics, and, even more, for his vision that a coherent system of such laws – that he called "*Social Physics*" - could be established. Nevertheless, the promising results in his work, as well as, Quetelet's enthusiasm and energy, significantly inspired scientists to systematize the use of statistics in social sciences (Porter 1986 chs.4-6, Stigler 1986 ch.5, Stigler 1999).

In his works on MS, from 1829 till 1869, Quetelet pointed out that events like crimes, suicides and marriages present a remarkable statistical stability from year to year, provided that social conditions in a given country or state remained approximately stable. This stability allowed a quite accurate anticipation of statistical results for the years to come, on the condition of social stability as well. On the other hand, in such events, human FW plays an important role, and according to conceptions at the time, events in which FW is involved, should escape any possibility of prediction. Quetelet considered that the observed stability of statistical results and the resulting possibility of prediction, point to the existence of restrictions on the large-scale ("macroscopic") influence of human FW and call for revising existing ideas on FW. These statistical results and Quetelet's interpretation stimulated at the time, the debate on human FW and its limitations and Quetelet was criticised that his work promoted ideas close to fatalism and materialism. He worked hard to refute this criticism by elaborating on his argumentation further and providing new statistical results to support it (Lottin 1911, Seneta 2003, Porter 1986 ch.6).

In addition to their interest of linking statistics to philosophy, Quetelet's texts we chose, have significant educational advantages,: (i) the mathematical treatment of statistical data is simple enough and thus adequate to be discussed with novice students in statistics; (ii) the proposed interpretation of statistical results is explained in detail, often accompanied by illustrative examples; and (iii) the texts themselves reflect the enthusiasm, passion and excitement that usually accompany new important and promising discoveries.

OUTLINE OF COURSE WORK

Our teaching work was realized during an introductory seminar on probability and statistics (with classroom meetings 3 hours per week; however, see note 5 below) for 29 3rd and 4th-year students (26 girls and 3 boys) of our Department of Education. Their sole previous education on probability and (descriptive) statistics was some rudiments they have been taught in high school, so the first three weeks were devoted to revise and complete this knowledge.

From the 4th to the 8th week, Quetelet's paper on the statistics of marriages (Quetelet 1847) was discussed. There he presented his point on the link between the observed stabilities of statistical figures and the limitations of men's FW. This was the first part of classroom discussion. For the second part, the teacher asked students to look for different philosophers' positions and ideas about FW and to present in the classroom elements of their personal study, thus enriching the classroom discussion in connection with results of the first part. This second part of the discussion lasted from the 8th week until the end of the course (12th week) [3].

Moreover, the teacher asked each student to prepare a written essay, of at least 6000 words, that should be delivered one month after the end of the classroom meetings and in which they should present and comment both on elements of the classroom discourse and of their personal study concerning philosophers' positions on FW.

After the end of the classroom meetings, the teacher interviewed each student individually, focusing on what they found (or did not find) interesting and attractive in the course, as well as, their motivations and feelings about the work they had done.

Since the first part involves more work on statistics, and at the same time, is essential for the reader to understand our approach, in this paper we present in more detail elements of the first part of the classroom discussion, while elements of the second part are presented more briefly due to space limitations.

BACKGROUND INFORMATION

As already mentioned, our students' previous education on probability and statistics consisted of only some rudiments of descriptive statistics and probability that they had been taught in high school[4]. This knowledge was revised and completed during the first three weeks[5]. We talked about data organization and their (graphical and numerically tabulated) representation, measures of central tendency (mode, median, mean) and variation (range, interquartile range, mean absolute deviation and standard deviation), the shape of a distribution and skewness. We also talked about the probability multiplication and addition laws, the binomial distribution and examples of its applications (e.g. chance games, newborns' sex, simple insurance models) and the LLN and the normal distribution accompanied by adequate examples.

THE FIRST PART OF THE CLASSROOM DISCUSSION

Introducing the problem

During the 4th week, the teacher gave information on the important scientific developments in the 19th century and the corresponding intellectual atmosphere and enthusiasm. In this context he explained the great interest of the scientific community on probability and statistics, whose main successful applications at the beginning of the 19th century were in astronomy and geodesy, while later on, their use was extended to all natural and social sciences. Then he presented elements on Quetelet's education and work, paying due attention to its pioneering character in social sciences, and Quetelet's point of view and ideas about the virtue and possibilities offered by the use of statistics in the study of social phenomena. Furthermore, he discussed with students the concept of accidental and non-accidental causes of variation, together with adequate examples[6]; a key concept in the development of 19th century statistics, as well as, in Quetelet's ideas for the use of statistics in social sciences (Stigler 1986).

Then, the teacher presented Quetelet's introduction in his 1847 paper (Quetelet 1847[7]), where he remarks that moral statistics is criticized for attempting to measure man's passions and inclinations, which is not only impossible, but also absurd; moreover, that this is an effort "…to chain up (men's) future in an inflexible mathematical formula.." (Quetelet 1847, p.135). For those studying only individual cases - he wrote - FW acts in a way so capricious, disordered and unpredictable that it seems absurd to suppose regularities and laws in the series of facts realized under its influence. However, he remarks, when observing large populations, the influence of peculiarities of individuals' FW vanishes and the series of general facts because of which society exists and lasts become dominant. When a large population is observed the effects of peculiarities of individuals' FW on statistical results are mutually neutralized and fall under the category of effects due to purely accidental causes[8]. This fundamental property of human FW allows establishing moral statistics and obtaining useful results. Moreover, Quetelet emphasizes that this property is also remarkable from a philosophical point of view, since it informs us that the influence of individual peculiarities of man's action is limited in a sphere such that the underlying laws of nature escape from it forever (Quetelet 1847, p.136); moreover, it points out that conservation laws may exist in the moral world, as they exist in the physical world. Quetelet remarks that a main question is to prove this fundamental property of human FW, and that in previous works he had pointed out that the neutralization of effects of peculiarities of individuals' FW is indeed observed when the examined data concern a sufficiently short period of time so that social conditions remain essentially unchanged. Then, he mentions his works on criminal acts, in which a remarkable stability and regularity of statistical figures is observed from year to year and notes that in the present work the same question is examined for marriages, based on data from Belgium.

Discussing variation and effects of peculiarities of individuals' Free Will

Subsequently, Quetelet provides examples of data in support of the existence of a remarkable stability of statistical figures of marriages for 1841-1845, a period of social stability in Belgium. The teacher asked students to examine these data and formulate their own considerations and opinions about their stability.

Quetelet presents the annual number of marriages among widowers and widows, which were for the towns 231, 221, 224, 244, 226 and for the rural communities 498, 474, 492, 482, 514. Students calculated the average, the differences maximum - average, average - minimum, the range and the Mean Absolute Deviation (MAD)[9], first for towns, and then for rural communities. After that, they calculated the variation measures as ratios (percentages) of the corresponding average values. So they found that the maximum and minimum values for towns differ from the average by 6.5% and 3.6% of the average, with the corresponding percentages for rural communities being 4.5% and 3.7%, and that MAD is 2.9% of the average for towns and 2.3% for rural communities. Given these results, students agreed that Quetelet was right to consider that there is a small variation, hence stability in the annual number of this category of marriages.

Then students worked on the second example given by Quetelet; the number of men and women 25 to 30 year-old, married in towns. For the period 1841-45, they are 2681, 2655, 2516, 2698, 2698 for men, and 2119, 2012, 1981, 2120, 2133 for women. They found that for men, the differences maximum-average, average-minimum and the MAD are 1.8%, 5% and 2% of the average value, and for women the corresponding value are 2.9%, 4.4% and 3%. Once again students expressed the opinion that these results indicate a small variation and thus considerable stability. Subsequently, they continued to work on other data from Quetelet's paper (the table on p.143), where he gives the distribution of marriages per year according to the grooms' and brides' age category. In the four larger categories (with average values between 2495 and 12752) the ratios of variation measures divided by the corresponding averages were not far from those observed in the previous examples[10], and for the total annual number of marriages these ratios were about the same size[11]. In the smaller categories, however, there were cases for which these ratios were larger than the aforementioned ones, especially in categories with average value less than 150. Quetelet remarks on this, that for small categories it is more likely that accidental causes destroy stability and thus show a larger (relative) variation.

In the follow-up discussion, students expressed the opinion that the data examined so far, were compatible with Quetelet's interpretation that in a period of social stability, the variation of men's FW generates a small variation in the large categories of population concerning the annual number of marriages and thus considerable stability is observed in these statistical figures.

Then the teacher posed the question: Even in a period of social stability there are many economic, emotional or social reasons, because of which individuals may change their will and disposition from year to year about getting married. What do you think it may happen so that, despite all these possible reasons, the annual number of marriages in the country did not change substantially? Students assumed that compensation processes were at work and proposed relevant examples such as:

- In a given country in the course of a year, some people loose their job, which may affect their will to marry that year; however, if stable social conditions are prevailing, about an equal number of people will find a job, affecting their will to marry the other way round. So, although in both groups there are changes concerning people’s will to marry, the annual number of marriages may very well remain practically unaffected.
- In a big country under stable social conditions, it may be assumed that each year about the same number of unmarried people are in grief because of their parents' death, but they are not the same individuals each year, since some enter a state of grief and approximately the same number leaves it. Concerning their will to marry, the first are affected negatively, while, the others positively. However, it is likely that the annual number of marriages remains unaffected.

The teacher remarked that, under stable social conditions, these compensation processes produce variations of the kind Quetelet called accidental variations. He also discussed with students about accidental variation and the fact that compensation processes, like those mentioned above, work better concerning statistical figures of large categories or groups of population (e.g. a big city), than for small ones (e.g. a village). This happens because there are accidental causes of variation which in small groups can easily produce important variations compared to the corresponding average values, while, this is not likely to happen in large groups. Once again, students proposed a variety of adequate examples such as:

- In a village in which 10 marriages per year happen on the average, divorces of engaged couples or job loss can easily yield a 10% or 20% decrease of the number of marriages from year to year. But, in a big city, this cannot happen. In fact, in big cities, especially under stable social conditions, the number of divorces of engaged couples or of men that loose or find a job does not change much from year to year …

Comment: This discussion allowed students to form a better qualitative understanding of accidental variation and how it works in social phenomena; in particular, concerning its influence on the stability of statistical figures of large and small groups. Also, the teacher noted common aspects between accidental variation and variation of random samples that obey the LLN. Moreover, in this way students came to better understand Quetelet's position that in conditions of social stability the variations in time of individuals' FW do not produce important variation of statistical figures for large populations.

After that, the teacher noted that Quetelet's remark that the period 1841-1845 was socially stable in Belgium does not mean that this should be understood too literally, but only as an approximation. This means that it would be possible for some important social factors to exhibit small, though significant changes, without however disturbing the overall image of social stability. Such changes may lead to small, but non-accidental variation of the statistical figures of marriages in that period. Hence, part of the observed variation could be non-accidental. Furthermore, with the aid of adequate examples, the teacher remarked that the LLN does not apply to non-accidental variations, which often do not depend on the size of the observed population categories.

Then, he invited students to search if there are indications of such non-accidental variations in the data. Students, despite their limited formal background in probability, were able to make some sensible relevant observations. Notably: (i) They observed that three out of the four maximum values of the four larger categories previously examined concern 1841. They considered that this was an unusual result if the year of maximum for each category was determined randomly in the five-year period. (ii) The total population constitutes a category far larger than the four large categories examined previously, so students thought that if all the observed variation was accidental, then the measures of relative variation of the total population should be substantially smaller than those of the four larger categories. However students considered that the ratio of the difference average-minimum over the average did not decrease as expected (see endnotes 10 & 11) and that this could be also an element indicating the existence of non-accidental variation[12].

People's tendency to follow social habits and requests

Quetelet writes that people have the strong tendency to follow customs, habits and requests of the society to which they belong and that this is a main element influencing their will in general and on marriage in particular, and determining the relevant statistics[13]. This is the second main element he proposes for interpreting the observed statistics in connection to men's will. Moreover he provides empirical evidence to support and illustrate his position, such as: (i) The modal age of brides differs as much as two years from one province to another and the difference is observed each year. This difference, he remarks, is due to the difference of customs of different provinces and not to individuals' FW. (ii) The number of marriages between young and aged people is small, but quite stable from year to year. Quetelet comments on this arguing that a man less than 30 married with a woman more than 60 did so not because of fate or blind passion; he was in a position to think about it and to fully use his FW; however, he finally decided to pay his debt to the needs of the existing social organization. This kind of debt, he remarks, is paid each year more regularly than the taxes paid to the State.

The teacher presented Quetelet's position and examples regarding the influence of social factors on men's will to marry and then asked students to express their thoughts and opinions on this issue. They identified a large variety of such factors. Based on their knowledge and experience, they presented a considerable number of examples to illustrate the influence of these factors, which can be classified as factors concerning: familial environment, economic situation, social environment and in particular other people's opinion, and education. Students remarked that through these factors, social habits and ethics, as well as, moral and religious beliefs are often strongly expressed, and underlined the importance of education in cultivating men's ability to critically evaluate and consider the influence of these factors. Many students described the influence of social factors in terms of pressure to which men's will yields, or is subordinated. Other students reacted to this, noting that for many people, their FW is in agreement with ethics, social habits and common moral; hence, in this case, there is no question of such yielding or subordination. Others remarked that people from their earliest days are subjected to strong influences from their family, education and social environment, creating stereotypes and beliefs that determine their future will on issues like marriage. So, even if they are willingly in agreement with ethics and social habits, it is questionable if this will is free will. These remarks led to the following important question: To what extent a man creates and controls his own will?

This question was raised during the 6th week, but was discussed mainly in the 7th. Many students thought that a large part of ideas and beliefs determining men and women's will are determined by social factors, but there is also a significant part which is their own. Others remarked that even referring to their own crucial decisions[14], they could not identify any important ideas or beliefs underlying these decisions that were completely theirs. They said they found ideas and thoughts that they initially considered being their own, but upon deeper examination they found that these were strongly influenced by preexisting ideas and beliefs which in turn, were formed under the strong influence of their family, education and social environment. They agreed that this is a difficult issue to clarify, but that it is important to keep on trying, because any clarification may be important for revising possible illusions on men being masters of their own will.

SECOND PART OF THE CLASSROOM DISCUSSION

In the 6th week the teacher said that it would be interesting to read about other scholars and philosophers' ideas on FW, proposed some reading sources on the past and current debate on FW, and mentioned some key personalities, who have significantly contributed, like St Augustine, St. Thomas Aquinas, Newton, Hume and Kant. He further suggested to start with an overview of the subject, but that the students should feel free to continue focusing on one or more philosophers or lines of thought that they would find interesting and attractive in relation to their own ideas and thoughts. The students actively worked on this task as they found the subject very attractive. So, from the 8^{th} week on, till the end of the course (12^{th} week), they orally presented in the classroom, elements of their study and their own comments that substantially enriched the discussion there.

Ideas of St Augustine and St. Thomas Aquinas were often presented and commented. An important element introduced in this way was the discussion on the interrelations among FW, personal responsibility of one's own actions, and the aim and role played by punishment and reward. This covered a considerable fraction of the second part of the seminar. Students brought in this debate, ideas of many other philosophers; either classical, like Hume, Kant, Schopenhauer, or modern – hence less known to the wider public - like Frankfurt, Strawson, Kane and others (cf. the next section as well). This is a strong indication that although students had no specialized knowledge on philosophy, they were strongly stimulated by the issues raised in Quetelet's text and their elaboration in the classroom discourse, and they intensively worked on them, searching into the existing literature by themselves. Below we describe some characteristic aspects of the classroom discourse:

St Augustine and St Th. Aquinas remark that FW is not the only condition for attributing moral responsibility; it is also necessary that one is aware of the consequences of his/her choices. In particular, they stressed that children and fools cannot be held responsible for their actions because of lack of this awareness. This was vividly discussed among the students, who remarked that very often someone could not have any satisfactory knowledge of the long-term consequences of its choices because of existing objective and/or subjective uncertainties. Some students said that moral responsibility should be attributed to a person according to its knowledge of the consequences of its choices. Other students remarked that this is not the only thing to be taken into account; social conditions that have played a determinant role on the formation of a person's will and character must also be taken into account. Moreover, some students referred to and commented on elements of Quetelet's work on crimes (excerpts from Quetelet 1833, 1842, 1848). Quetelet observed a remarkable stability in time of the statistical data of the different kinds of crimes, as well as of suicides, though he found important differences among different provinces and countries. One aspect of Quetelet's interpretation that students underlined, is that the different kinds of crimes and their frequency are determined by social conditions and organization, while criminals are just the tools for realizing these crimes[15]. These elements fed the discussion on extenuating circumstances that should be considered and some students argued that in fact it is very difficult to fairly attribute moral responsibility to someone for its choices and actions. Later on, many students considered as satisfactory moderate answers to this question contained in the so-called compatibilist ideas of Hume and Kant.

Another interesting issue is that students presented ideas about Newton and Laplace's hard determinism physically based on Newtonian mechanics, and indeterministic ideas stemming from Quantum Theory. According to Newton and Laplace, the future is fully determined once the initial conditions are given; hence, there are no alternative possibilities and therefore, both uncertainty and FW are only illusions. In this context, theories and ideas on FW, as well as probability theory, are only conceptual models for managing parts and aspects of our ignorance. The teacher remarked that although uncertainty objectively exists according to Quantum Theory, our ignorance is also a reality, and part of the use we make of probability theory is due to our ignorance and not to any objectively existing uncertainty. In this sense Laplace's conception of probability is partly valid. Some students thought that a similar idea holds also for FW; although FW may very well exist, part of the potential we attribute to it, is due to ignorance about restrictions on its influence, as well as, to lack of awareness of the influence of social factors determining our will[16]. Students were not convinced by Newton and Laplace's deterministic ideas, but the fact that it was these great men, who supported these ideas, strengthened their quest on the limitations of human FW and, for some, their quest about the very existence of FW.

ON THE STUDENTS' ESSAYS

The teacher asked each student to provide a written essay of at least 6000 words, within a month after the end of the classroom meetings, presenting and commenting on aspects both of the classroom debate and of their own study of other philosophers' positions on FW. He also encouraged them to feel free to develop their own thoughts and ideas on them.

In all essays, students discussed limitations of the influence of men's FW, as well as factors that influence the formation of men's will. Some of them considered the discussion on these limitations and factors in the context of their own quest about the central question of the existence of FW. There were also students who discussed the importance of critical awareness of these limitations and factors and the role played by education, family and society in the development of a person's critical thinking. Six students focused on the relation between FW and personal responsibility, and three on the relation among uncertainty, chance and FW.

Below we list the philosophers/scientists, whose ideas were more frequently discussed in students' essays (the number of essays referring to a philosopher's ideas follows his name): Quetelet, 26; St Th. Aquinas, 11; St Augustine, 10; Kant, 9; Hume, 7; Hobbes, 6; Aristotle, 3; Newton, 3; Frankfurt, 3; Laplace, 2; Fichte, 2; Schopenhauer, 2; Everett, 2; Steiner, 2; Strawson G., 2; Kane, 2. Another 23 philosophers' ideas were mentioned, though each one of them appears in one essay only.

It is also worth mentioning students' positions in the essays on the question of the existence of men's FW:

(a) 10 students were not convinced for its existence; (a_1) 2 of them expressed the opinion that it is an illusion; (a_2) the other 8 were skeptic about its existence;

(b) 19 were convinced that it exists, but that there are also important restrictions about it. (b_1) 7 of them emphasized that there are people without FW on essential points (like people who are subjected to systematic totalitarian or manipulative education from childhood). (b_2) The other 12 did not emphasize what is mentioned above in b_1.[17]

FINAL COMMENTS

On the discussion with students about Free Will

The philosophical discussion on FW has been lasting more than 2000 years and has formed part of the central philosophical debate on the basic characteristics of man as an individual and as a social being. Hence, the seminar could not aim at formulating and discussing any definitive answers, but rather, at raising questions and bringing to light issues that till then, students had considered little, or not at all. Quetelet's statistical data and his interpretation was an important asset for posing such questions and stimulating a debate that motivated students' further study and thinking.

In the first part of the classroom debate, students identified and discussed at an initial level, both limitations on the influence of men's FW and on factors that form and determine men's will. Students gradually realized that (i) this is a complex and deep issue; (ii) they had little knowledge and had thought little about it; (iii) gaining knowledge and insights of it, is not only interesting for philosophical and social questions, but also important for personal fulfillment. These three elements together, generated a strong motivation for students to work on and look for this issue further. In the second part of the debate, ideas of all philosophers' points of view discussed, underlined the importance of critical awareness on this issue that further enhanced students' interest and motivation to search for it. It is because of this motivation and interest that, in many cases, students' work by far exceeded the course's typical requests (for a sample of students' opinions in the final interviews see Appendix, excerpts 1-3).

On Statistics and Free Will

On the one hand, the work done allowed students to improve their understanding on specific issues in statistics; in particular, the distinction between accidental and non-accidental variation and on how variation works in social phenomena in connection with the LLN. On the other hand, it allowed students to enrich their concept image on what statistics is about and how it works. More specifically:

(i) Students realized that statistics is not just the technical treatment of data, but it may concern issues like FW, which is not only a fundamental philosophical issue, but also has important implications on everyday life and personal behavior and attitude.

(ii) They had the opportunity to realize that on issues such as FW, statistics can provide macroscopic information of critical importance, which cannot be accessed if one is limited to examining the subject only at the individual (microscopic) level.

(iii) They realized that an important part of statistics is the interpretation of statistical results and how it works. During their work on interpreting data, students linked statistical results with ideas and beliefs they disposed, as well as with elements of their experiential background. This linking often led to the evolution of ideas and beliefs, the emergence of new ideas and the rise of new questions. In fact, interpretation work was for students the most interesting part of statistical work that made statistical results meaningful.[18] [19]

Furthermore, because of the conjoint presence of the three aforementioned elements many students' poor affective disposition and opinion on statistics was improved.

APPENDIX: Excerpts from students' final interviews by the teacher[20]

(1) Maria: I never believed that a course on statistics could refer to such interesting issues; I mean not just academically interesting, but interesting for each one of us personally... There are all these decisions and choices that I thought to be my own, and then, after discussing and thinking about them, I realized that there are so many influences that determine our will! I did a lot of work wondering which of my choices and decisions are really mine, and which responsibility is really mine. This is a difficult question, but it is also important to find at least some answers; I mean, it is important not just philosophically, but personally… Because of these questions, I did a lot of work voluntarily and not because of the course's requests.

(2) Katherina: I found the discussion on the restrictions and potential of our FW very important… For example, Quetelet is very right saying that we tend to follow and do what our environment and others say. This point, that is, "do what the others like", is an issue to which I devoted a lot of thought, not only in general, but also examining myself, my own behaviour and attitude… Also, Quetelet's statistics points out that despite our FW, society - like a "well-oiled" machine - produces the same results each year. His statistics confirm that our FW has a "limited sphere of influence", as Quetelet says. Based on the whole discussion and study on this issue, I came to believe that this "sphere of influence" is small. But how small? This is an important question that still remains unanswered to me… The discussion during the seminar posed questions and burning issues that are not going to be extinguished any time soon. In fact, as far as I understand, we have just started struggling with these questions.

(3) Anna: …. Another important issue was the one on FW and responsibility. On the basis of the discussion, I realized how complex and difficult is to judge people for their choices and actions justly; still, we keep doing it easily and superficially every day. But, by doing so, it is very probable that we become unfair without even being aware of it. This is not just an academic discussion. If one succeeds to understand better this issue, it is very likely that he will change his attitude while judging others…

(4) Photini: …I had never thought that statistics could be so interesting. I mean it is so interesting because it is linked to the issue of FW, which - as we have seen - is important philosophically and socially and personally. Additionally, it was the way that we worked on the statistical results. At school, most of the time devoted to statistics, we were finding averages and graphs only, so I believed that statistics is a very boring subject. Here we discussed a lot on the statistical results, trying to explain them. We discussed examples and individual cases in connection with the statistical results. There were Quetelet's statistical results but there were also Quetelet's ideas for explaining them and then we brought in our own ideas and we could discuss our examples and even our related personal experiences. And then, with all this, we started grasping the problem of FW; I mean we did not find any definitive answers, but we raised deeper and wider questions and saw aspects of the subject that we did not even suspect they exist. … This was a really exciting course; if this is doing statistics, then statistics is far more interesting than I had thought.

(5) Eva … With all these causes and factors that may influence and change a person's will about getting married, I could not guess the existence of such an annual stability in the number of marriages; not just in general, but for each category and in each region. And the stability of the number of crimes; this is even more impressive. What Quetelet says in his text is important; that without statistics, people could believe that peculiarities of the individuals' FW can produce important changes in the number of marriages, or crimes from year to year. But this is a wrong idea, which overestimates the power and potential of an individual's FW; moreover it is an idea that describes society more disordered than really is. This is what I found important with statistical results; they allow to clarify things and to avoid certain important wrong ideas…

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[1] Quetelet considers *Moral Statistics* to be the domain of statistics that concern phenomena, like crimes, suicides, marriages, which are phenomena that may be subjected to moral characterization (Hankins 1908 ch 4., Lottin 1911).

[2] Two key elements being the Law of Large Number (LLN) and the DeMoivre-Laplace Central Limit Theorem (CLT), both permeating explicitly, or implicitly his work. In view of our students' elementary knowledge of Statistics however, his papers selected for the seminar's purpose are mathematically more elementary, hence not directly referring to the latter.

[3] In the second part students also presented elements of other works of Quetelet; in particular, of his statistical works on crime and suicides.

[4] We note that high school teaching gave students the impression that statistics is mainly the technical treatment of data (computations, creation of graphs etc). As a result, many of them considered it as an unattractive subject.

[5] Classroom work lasted for 6 (teaching) hours in the first week, 3 hours in the next, and so on (so on the average it lasted 4.5 hours per week).

[6] E.g. the analysis of error measurements that are due to accidental and systematic errors; or the analysis of reparations paid by insurance companies, as due to constant causes determining the average values of the reparations and to accidental causes responsible for deviations from these averages.

[7] Quetelet's text is available online (http://www.edc.uoc.gr/~tzanakis/Quetelet1847Marriages.pdf)

[8] When Quetelet refers to men's FW, he actually means the peculiarities of individuals' FW, not the common aspects of men's will. Obviously, the effects of these common aspects are not mutually neutralized and do not vanish for a large population. As Lottin puts it, Quetelet considers FW as a reaction force that creates peculiarities and individual specificities (Lottin 1911). On the other hand, students used the term FW in an ordinary sense that encompasses both individual peculiarities and

common aspects of human will. Therefore the teacher clarified this point to avoid misunderstanding Quetelet's text.

[9] Students chose to use MAD as a global measure of deviation instead of the Standard Deviation (SD), because they felt it is simpler and better understood than SD. Though the teacher accepted this choice, on several occasions later, he asked them to compute also the SD so that they gradually became acquainted with it, and benefited from the comparative consideration of MAD and SD.

[10] The four ratios concerning MAD were between 2.2% and 4.3%, those concerning the differences maximum - average were between 3.2% and 5.5%, and those concerning the differences average - minimum were between 3% and 5%.

[11] The ratio for MAD was 1.4%, that concerning the difference maximum-average was 2.6%, and the one for the difference average-minimum was 3.1% (the average of the total annual number of marriages was 29131).

[12] Then the teacher remarked that it would be interesting if they could find a way to estimate the size of variation measures that it is probable to result by accidental variation and then to compare it with the variation measures calculated from Quetelet's data. Six students worked on this issue with teacher's assistance, in activities independent of the rest of the course. They did interesting work using - among other things - large numbers of random samples as informal tools to answer questions that were raised. Their work is not presented here, because of space limitations.

[13] Furthermore, Quetelet relates this strong tendency to people's inherent sociability, which leads humans to voluntarily cede part of their individuality in order to become members of the society.

[14] E.g. selecting a subject for their tertiary studies and a profession.

[15] Furthermore, students provided recent statistical data pointing out a very important increase (about 30%) of suicides in Greece during the current economic crisis. They considered this to be also in line with Quetelet's interpretation.

[16] Moreover, other students made an interesting analogy: Though Quetelet considered macroscopic stability of statistical figures, he thought that, at the individual level, there are capricious and unpredictable peculiarities of human FW, which, however, are not powerful enough to destroy this stability. Students considered that there is an analogy between this and the deterministic regularity of macroscopic phenomena as described by the laws of Newtonian mechanics and the uncertainty of physical phenomena at the microscopic level inherent in Quantum Theory. It is worth noting that Herschel's presentation of Quetelet's research was inspiring for Maxwell, who thus conceived an analogy close to the aforementioned that stimulated him to introduce a statistical approach to microscopic phenomena; in particular, his introduction of the normal distribution to derive the molecular velocity distribution of gases, a key step for the systematic development of Statistical Physics and Kinetic Theory since then (Porter 1986, pp.115-116, 118, 121, 123).

[17] However, we should note that many students remarked that their position in the essay was a first one, susceptible to change after further study.

[18] See some of students' own opinions; appendix, excerpts 4, 5.

[19] This is a point that deserves to be examined as a more general characteristic of learning (other topics of) mathematics: work on understanding and interpreting selected data (e.g. measurements of

physical or geometrical magnitudes) acts as a strong motivation for learning a particular subject and modifying positively the learner's affective disposition to it.

[20] Names have been changed.

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Oral Presentation

FORMAL PROOF AND EXPLORATORY EXPERIMENTATION: A LAKATOSIAN VIEW ON THE INTERPLAY BETWEEN EXAMPLES AND DEDUCTIVE PROOF PRACTICES IN UPPER-SECONDARY SCHOOL

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This paper investigates conceptions of mathematical investigation and proof in upper-secondary students. The focus of the paper is an intervention that scaffolds the interaction between open explorative activities and the development of proof sketches through explorations of lattice polygons, aiming at proving Pick's theorem. In the process we investigate whether and how the conceptions of proofs and explanations in mathematics change. We work with the hypothesis that the problem of supporting the transition to deductive proofs in upper-secondary school students can at least partly be explained as a problem of bringing their empirical investigations into the deductive proof process in relevant and productive ways. Through our analyses of the portfolios and deliberations of the students, we are able to assess their performance of proofs and the conceptions of mathematical methodology before and after the intervention.

EXPERIMENTS AND PROOFS IN MATHEMATICS AND MATHEMATICS EDUCATION

Empirical and deductive proof schemes

The tendency among upper secondary students to "prove" mathematical statements by examples rather than by universal deductive reasoning has been established as a robust research result in mathematics education research (Arzarello et al., 2011). This educational problem is described as students possessing "empirical proof schemes" opposed to "deductive proof schemes". Phrased in these terms, a large amount of empirical studies have shown that students have difficulties performing and internalizing the movement towards deductive proof schemes, and that empirical proof schemes, and more broadly work with examples gives rise to difficulties (such as misunderstanding, difficulties and confusions) with the acquisition and performance of deductive proofs.

Such conflict results from "*the concept of formal proof is completely outside mainstream thinking*" (Arzarello et al., 2011, p. 51) suggesting an irreducible gap between everyday empirical thinking and formal mathematical thinking. The existence of such a gap is well supported by results from cognitive science (Kahneman, 2011), but little is known about specific approaches to overcome this gap and especially: "*the evidence about the transition from empirical to general proof schemes is based on* *limited evidence collected in suitable environments"* (Arzarello et al., 2011, p. 53).However a few distinctions can already be made: (1) empirical proof-schemes can be seen either as a necessity or as a problem in the transition to deductive proofs, and (2) the transition to deductive proofs can be seen either as a radical change in the mode of reasoning or as a natural continuation and refinement of empirical proof schemes.

In this project we suggest a continuous approach, activating rather than suppressing example work and the empirical proof schemes inherent in the students. These choices are informed by new "maverick" trends in the philosophy of mathematics, suggesting that investigations and heuristics are closely connected to more formal justificatory practices in mathematics.

Exploratory experimentation as a maverick approach to mathematical justification – Lakatos on the mathematical proof

Much traditional philosophy of mathematics has focused on providing accounts of the certainty of mathematical results. However, over the past decades, a new 'maverick' trend has been focusing on a broader and practice-informed philosophy of mathematics (Davis & Hersh, 1981; Lakatos, 1976; Mancosu, 2008). Among the insights thus produced is that the sharp context-distinction between a *context of discovery* and a *context of justification* does not square well with actual practice. In particular, Imre Lakatos' (1922-1974) book *Proofs and Refutations* put great focus on the informal aspects of mathematical knowledge production and on the epistemic roles played by examples and counter examples (Lakatos, 1976). Lakatos argued by a rational reconstruction of the history of Euler's polyhedral formula that counter examples and proof analysis play crucial roles in shaping mathematical concepts and developing increasingly refined proofs.

On Lakatos' account, the dialectic process of proofs and refutations (counter examples) can be used to develop mathematical knowledge about initially naively defined or partially understood concepts. Thus, if the classic context-distinction was to be imposed, Lakatos' dialectic belongs partly to the realm of heuristics in gaining insights about those concepts and partly to the realm of justification in providing and shaping the proofs of the theorem as they develop.

Thus, Lakatos implored us, mathematical statements are not static and do not epistemologically predate their warrants; and conversely a mathematical proof is not an analytical afterthought warranting a previously existing mathematical insight. Rather, proof-practices are active in creating the mathematical landscape of theorems and claims.

Recently, new practice-oriented trends in the philosophy of mathematics have investigated how the present availability of desktop computers with flexible mathematical software systems increases the interplay between proving, investigating examples, and suggesting new theoretical concepts. Using computers not only to verify proofs or generate data for heuristic conjecture formation, it is possible to undertake what has been described as "exploratory experimentation" in mathematics in which concepts are formed through experimentation and in which experiments critically inform (if not warrant) proof (Sørensen, 2010 see also e.g. Borwein 2012).

In this paper we bring these two conceptions from the recent philosophy of mathematics – the continuous overlap between empirical and deductive proof schemes involved in exploratory approaches to mathematical research and the specific role of computer-assisted experimentation to bear on the didactical situation where the two proof schemes are often (misguidedly, we claim) separated. We do so by first detailing the discussion of different proof schemes and their potential overlap, before we discuss the role of computers in exploratory experimentation. We then describe the context and content of our intervention and the data produced, which is subsequently analyzed, bringing to the fore both some of the successes in integrating exploratory experimentation in mathematics education and some problems which students experienced in completing the transition to deductive proof schemes.

DISCOVERY AND JUSTIFICATION: TWO DISTINCT CONTEXTS OR BLENDED DOMAINS?

The transition from experimentation with specific examples to formal proof can be studied as a change from a heuristic context of discovery to a justificatory context of proof or as a matter of drawing upon both empirical and deductive proof schemes (Arzarello et al., 2011). Lakatos suggests that even though we can talk about a transition *to proving,* there is no such thing as a *transition away from working with examples.*

What Lakatos thus points out is that, in mathematics, the contexts of discovery and justification are not to be too sharply distinguished, neither temporally nor methodologically. Initially, Lakatos' analyses were aimed at research-level mathematics and the production of new mathematical knowledge, but they also have important implications for mathematics education, such as have been pursued by contemporary mathematics educators (see e.g. Ernest, 1991).

Therefore we suggest using the notion of contemporaneous empirical and deductive proof schemes to conceptualize not different contexts that students move in and out of in a binary fashion, but as relating to different domains influencing their experiences of working with mathematics. In our analysis we will discuss whether or not students *frame their activities towards the domains of examples or towards the domain of formal proofs.* This distinction is an analytical one inspired by (Hanghøj, Misfeldt, Bundsgaard, Dohn, & Fougt, n.d.), and we expect to see that students express references to both domains in their work with constructing proofs..

The domain of tasks and mathematical examples

When students in Danish upper-secondary mathematics classes work with word problems and similar tasks, they work almost exclusively with examples. Hence, such examples represent tasks and situations where the student is expected to apply mathematical theory. Formulas are *tools for working with examples* and proofs are hardly relevant when considering examples. The type of reasoning applied when working with word-problems is deductive, but specific: students need to use rules, theorems and formulas to calculate the solution to a certain problem.

Moreover, in their textbooks examples are often used to show how a certain type of task is performed or how a mathematical result is activated. And finally, examples can be used in a theory-generating fashion – typically as motivational devices preceding a theoretical construct. Hence in the domain of examples *the objects* are *specific rather than general, formulas* are *tools,* and *proofs* are of *little relevance. Such a view* promotes a process going from *problem situation* to *solution* by *using mathematical theory,* as well as a tendency to describe the involved objects in *specific* rather than general *terms.*

The generative uses of examples described above come close to the way examples will be used in our material: Focusing on the transition from examples to proofs, we will use examples (of lattice polygons) both in order to motivate, as specific stand-ins for general objects, and as objects unto which the general theory is to be applied Yet, our intervention is designed so as to facilitate a *continuous* transition in which knowledge acquired in the empirical investigation of examples is to feed constructively into the shaping of deductive proofs.

The domain of formal proof

The domain of formal proof differs from the domain of examples in a number of ways. On the object-level *proofs* and *theorems* are at the *center of the activity*, and correspondingly, on the meta-level, the involved objects are described *as generally as possible* and the argumentative schema goes *from theorem* (stating a result) *to proof* (warranting the result). When students in Danish upper-secondary schools work with formal proofs they are usually expected to read and understand these proofs and in some cases also memorize and perform them. In this context, formal proofs usually have to them the flavor of "divinely informed calculations" with little explanatory motivation given. Understanding the proof largely consists in remembering a few main ideas, typically developed over generations of mathematicians to a very elegant and condensed form. It is much rarer for these students to develop their own mathematical proofs. Hence in the domain of formal proof, the *official mathematical text is at the center;* whereas in the domain of examples the *student's own voice is acknowledged*.

EXPLORATORY EXPERIMENTATION IN A LAKATOSIAN FRAMEWORK

As mentioned, Lakatos' original description of the proof process saw it as a perpetual dialectic between what we call domains of examples and domains of formal proof. Building on this, we suggest to add a process of *presenting codified proofs* such as they typically come to appear in accepted mathematical communication, including textbooks. Obviously, as is one of Lakatos' main points, such a codified proof could still be subjected to further dialectic treatment, but it seems to us an important part of the process of teaching mathematics to reach a recognizable, relatively stable notion of a (written, codified) proof. Furthermore, still building on the Lakatosian approach, we wish to emphasize the wide applications for exploratory experimentation, some of which (examples and counter-examples) are to be found also in works by Polya (Pólya, 1945) and Lakatos. Such experimentation is readily available through the use of software, the use of which is, itself, a goal of Danish upper-secondary mathematics education and, of course, a topic of educational research (see also Conner et al, Guven et al., Guin, Ruthven, & Trouche, 2005) However, as we aim to show, such experimentation can also be important (beyond the roles of mere motivations or illustrations) for shaping sub-arguments of larger mathematical proofs, thereby also giving rise to proof-generated concepts as emphasized by Lakatos. The result of these considerations is an envisioned process of moving from idea generation to experiment to proof. This process is shown in figure 1 and represents our envisioned learning trajectory (Cobb & Gravemeijer, 2008) for the movement from example to proof in the students of our intervention.

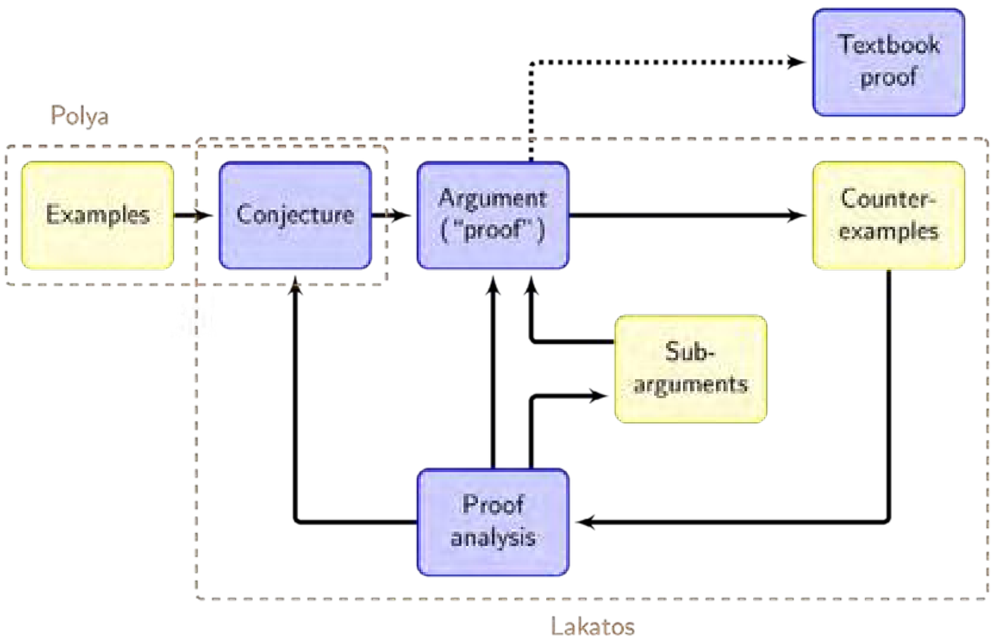

Figure 1: The envisioned process of experiments and proofs (and our intended learning trajectory) based on inspiration from Polya and Lakatos. This scheme shows

the Lakatosian dialectic of conjectures, refutations, and proof analysis leading to refined conjectures and proofs. It also suggests how examples (yellow boxes) have multiple functions, both in forming conjectures (Polya), as refutations that prompt proof analysis (Lakatos) and as means to calibrate proof analysis and lemmas (Lakatos). Added to the Lakatosian framework is our suggestion of a process of presenting codified (textbook-like) proofs that transcend the dialectic of proofs and refutations.

Hence we view the classical context-distinction not as transition, nor as just two complementary views on the subject, but as a process with repeated feedback loops where activities are framed towards different domains at various stages (see figure 1).

THE EDUCATIONAL SCENARIO

Our intervention is centered on a beautiful, yet somewhat atypical and slightly complicated theorem about areas of polygons in a lattice. In this section we describe this result – Picks theorem – and suggest why it is an interesting case to support the development of formal proof strategies. In the next section we then proceed to describe the teaching material and the classroom intervention we have conducted. For further documentation, we refer to (Danielsen, Misfeldt and Sørensen, 2014).

Pick's Theorem

The theorem at hand is known as Pick's Theorem named after the Austrian mathematician Georg Alexander Pick (1859-1942), who first described the result in 1899. Published under the title "Geometrisches zur Zahlenlehre", Pick's theorem is located on the intersection of geometry and arithmetic that was cultivated around 1900, in particular by the German mathematician Hermann Minkowski (1864-1909). Educated in Vienna, Pick spent his entire career in Prague, where he also published his result in a relatively obscure journal of the German-language scientific and medical association (Pick, 1899). During the 1930s, Pick became a victim of Nazi persecution, and he perished in Theresienstadt in 1942. Over the years, the theorem has been proved repeatedly and in various ways; it has also been used to train mathematics teachers at various levels, but it is (we believe) relatively rarely taught to students. As a theorem, it is remarkable for a number of reasons that include the following:

a) It can be inductively approached using either physical lattices (in Danish: "sømbræt") or computer-based experimentation (see figure 2).
b) It links two domains of mathematics by showing that in some cases, you can actually *count* an area which is normally something to be *measured*.
c) It involves a number of basic geometrical ideas such as triangulation and knowledge about basic geometrical concepts such as polygons, areas of triangles etc.
d) Its proof is slightly more complicated and intricate than proofs by traditional derivations; yet, it is at a level of complexity where it can be taught to students.

The theorem provides a way of computing the area of a lattice polygon, i.e. a polygon whose vertices are located in the grid (lattice) $Z \times Z$. If P designates such a lattice polygon, i(P) counts the number of interior lattice points in P, and b(P) counts the number of lattice points located on the boundary of P, then Pick's theorem states the relation $A(P) = i(P) + b(P)/2 - 1$, where A(P) is the area of the polygon P. The proof traditionally operates by three important steps:

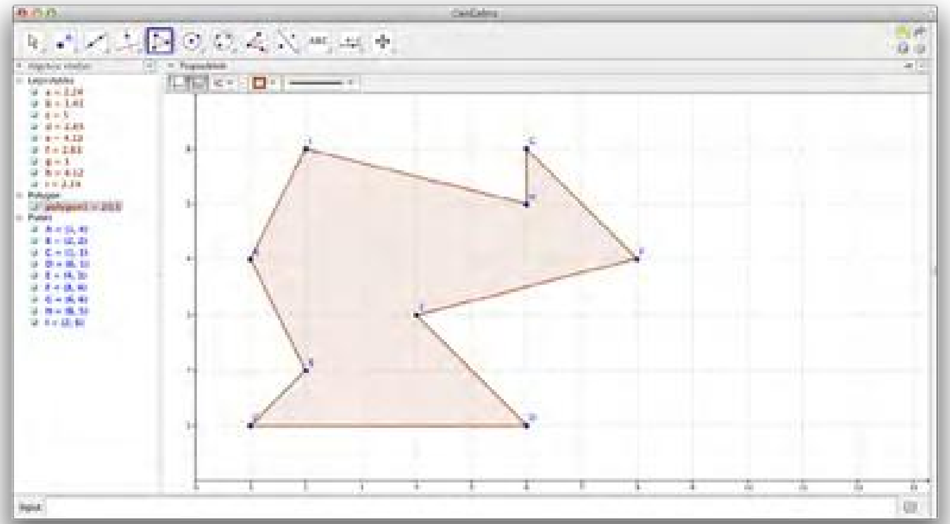

Figure 2: Example of a lattice polygon drawn in GeoGebra.

1. Proof that the Pick function defined by $\Pi(P) = i(P) + b(P)/2 - 1$ is additive when two adjacent lattice polygons are merged into one.
2. Proof that any lattice polygon can be triangulated into lattice triangles.
3. Proof that for any lattice triangle T, $\Pi(T) = A(T)$.

This three-step proof scheme might appear complicated or foreign to students, since it does not reduce to either a calculation or a traditional Euclidean proof scheme. It purports to show a complicated identity by showing that the identity holds for atomic configurations and that it is preserved when complex configurations are built up from such atomic building blocks. Although such proofs are relatively rare in teaching on the upper-secondary level, similar proofs are actually abundant in mathematics, and students will also encounter them, for instance when it is shown that any (sufficiently simple) function is differentiable.

Teaching material and educational intervention

The educational intervention was situated in one upper-secondary class (senior year, 3.g STX MAT-A) taught by the second author of this paper. It consisted of 10 one-hour lessons and was planned to consist of 5 modules. As an answer to the students' needs, the teacher added two more modules. Six of the seven modules were built on the same template (described below) and the last module was a blackboard-based proof of Picks theorem serving an institutionalization purpose showing the students how the knowledge that they had explored and the propositions that they had justified fit into a larger landscape of official, codified mathematical knowledge (Brousseau, 1997). Each module contained the following elements:

- Introductory activity: a simple activity introducing one of the ideas in the module in a simple way
- Closed task introducing an important tool or concept
- Investigation prompted by an open task/invitation
- Buffer activity to make sure that everyone had something relevant to do
- A collaborative reflection activity

The rationale behind this template was to scaffold (1) individual or small group investigations of a specific aspect of mathematics, and (2) collective reflection and formulation of results of that activity. For instance, in one module (module 5) on triangulation of lattice polygons, the work sheet involved the following activities:

- Activity 5.1: What happens to points and areas when a polygon is divided into two (or more) polygons?
- Activity 5.2: What happens to points and areas when two (or more) polygons are put together?
- Activity 5.3: Formulate some rules for the number of points in a lattice polygon when you divide a polygon or put polygons together. You should introduce some suitable names and notation for the elements you use. Make the rules as simple as possible and save all your suggestions for later (also the ones that turn out to be wrong).
- Activity 5.4 (buffer activity, intended to make sure that students are at same pace, when starting activity 5.5); Try your best rules on a lot of different cases, using the computer to produce the cases. Correct the rules if necessary. Save all your suggestions (also the one that turn out to be wrong).
- Activity 5.5; Do your rules hold in all cases? Do you think you have made a theorem? Do you think the theorem is proved?

The teacher was mainly acting as guide and supervisor with respect to the mathematical aspects and as a process facilitator with respect to the progress of the modules.

The topics dealt with in the individual modules were:

1. Module 1: Areas of polygons
2. Module 2: Lattice polygons

3. Module 3: Areas of simple lattice polygons
4. Module 4: Generals aspects of the area of lattice polygons
5. Extra Module: Formulas
6. Module 5: Triangulation of lattice polygons
7. Extra Module: proof conducted on the blackboard as a combination of lecture and plenary discussion in the class.

Upon completion of the work originally intended to prove Picks theorem (module 5 and the extra module), the teacher decided to change the work method towards whole-class discussion and lecture. The rational for doing that was that the students were increasingly working without direction and with the need of so much teacher guidance that the idea of working individually and in small groups collapsed and a whole-class discussion seemed like the healthy pedagogical choice. This “collapse” happened in the transition from experiments generating and verifying formulas for the area of lattice polygons to establishing a mathematical theorem with a proof and as such is very interesting for our design; therefore this “collapse” is further described and discussed in the data and results sections that follow.

DATA AND RESULTS

As data from the invention, we can draw on the teacher’s impressions and experiences teaching the students combined with various products of students and their answers to a questionnaire with six qualitative questions:

1. How did the activities in the modules work? How can they be improved?
2. Did you learn anything about mathematical objects (triangles, polygons, functions, etc.)?
3. Did you learn anything about formulas?
4. Did you learn anything about proofs?
5. Did anything surprise you?
6. Do you have other comments?

The students generally enjoyed working with the material and considered it a nice variation away from the typical classroom work. They found it nice to work in a different way with a subject, and they liked the structure with more independent work enjoying the opportunity to take active part in developing mathematical theory; Representative of their evaluations, they expressed that:

"It was a good and different way of working with polygons" and similarly *"It was good with more independent work, which was followed up afterwards."*

Although the students liked to work independently, some of them also lacked the overview gained from a lecture structured by a teacher. In several cases, the students were able to express rather elaborately what they had learned. On the topic of creating mathematical results as consequences of considering, combining and dividing into simple examples, several different students expressed view such as:

"I have learned how to come up with results in different ways. For example by looking at a triangle, it is possible to say something about a square. I have also learned that there is a relationship between the dots in a lattice polygon and its area ... "

"We have been working on how a complex object can be simplified by dividing it into simpler shapes."

"We have learned something about how to calculate the area of polygons by dividing them into triangles. We also learned something about the connection between the formulas you can use to calculate the area of lattice polygons, and how to derive such formulas."

Several students highlighted the fact that they were able to develop their own formula to describe a rather surprising and strong mathematical result. Making your own formula was considered fun:

"It was fun to try to come up with your own formula to solve a particular problem. It is probably this activity, which I liked best."

It was also considered less complicated than the students would have thought, and directly connected to inductive reasoning:

"I've learned that by sitting and trying various possibilities, you can relatively easily come up with your own formula, and it need not be very complicated."

Furthermore the students expressed that they had leaned something about mathematical reasoning and proving. Some noticed that proofs can consist of many independent parts: "*Yes, [we have learned] that a proof can easily contain smaller elements of proofs that together form the basis for proving the same theorem."* And similarly "*The proof we did was different from the ones we normally do, because the proof was divided into several parts (various geometric shapes) to eventually cover all polygons."* The open nature of the proof-process was also explicitly noticed by some students as being different: *"[We have learned] that there are many ways to prove a claim; and this is different from the classical proofs we have done in the past."*

ANALYSIS

Two positive results in terms of the students' mathematics learning are suggested by the data:

1. The students are able to come up with a formula based on experimenting with various cases. The students expressed that it is new to them that formulas in that sense can be created.
2. The students participate in warranting their formula by considering how the formula is true for various examples and classes of examples.

The aim of this project was to create a situation where the inductive reasoning would suggest structure and propositions strongly enough that some students would eventually start proving without strong teacher support. This did not happen to the degree we had envisioned. The students experienced a lot of difficulties with the transition from considering (classes of) examples to a formal proof. However finding the mathematical result and starting the line of argumentation warranting this result in simple examples was possible for the students.

There are several aspects of the two results mentioned above that suggests that our intervention has activated and blended the two domains of proofs and examples. The students are surprised that they are able to come up with an official mathematical result themselves; they expected this to be "*much harder*". This can be described by observing that official mathematical theory resides in the domain of formal mathematics where students, by the prototypical conception, are not able to contribute. However examining a large number of examples is relatively easy for the students using suitable software, and when they do that using our material, it is easy to propose the general formula.

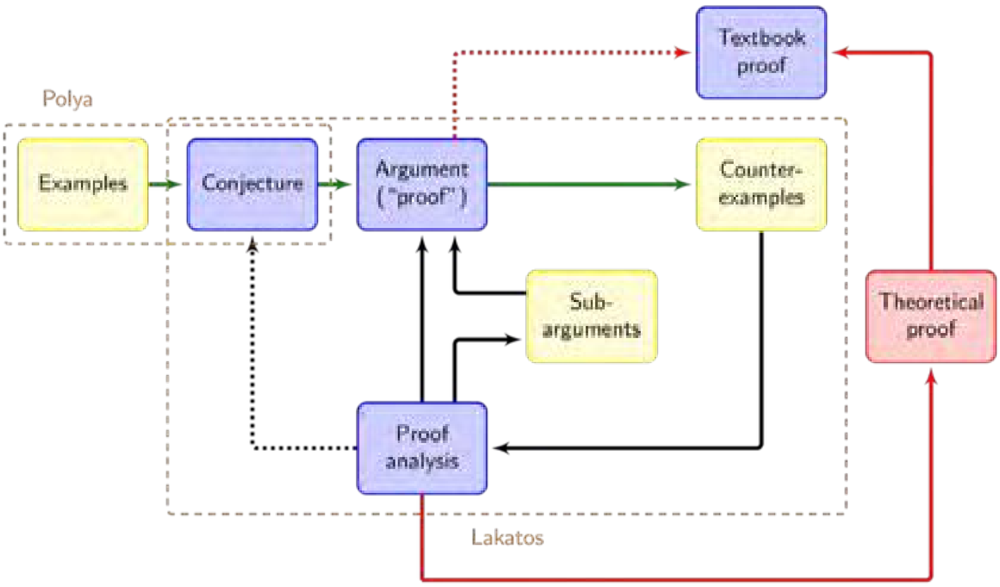

Figure 3: Refined schema of the learning trajectory, (figure 1) under the influence of our intervention. The arrows in green represent paths that the students were able to follow and perform with only minimal guidance. The red arrows point to the conflict that emerged when students were asked whether the proof was complete. That conflict

points to a difficulty in bringing isolated exploratory experiences in the domain of examples in the form of a codified mathematical text in the domain of deductive proofs.

Verifying the formula in simple situations can be viewed as residing in the domain of examples. Checking that a formula holds for a specific geometrical shape is a typical task for students at this level. This part of the activity was performed successfully by most students. Considering the augmentation of one example to a whole class of examples is also a relatively doable activity for the students. However, understanding the reason for doing this, and especially seeing how one can be assured that all cases are covered by securing a number of simple cases and procedures of augmentation, requires the student to move into the domain of formal mathematics. In that sense it is not surprising that the students' independent work collapsed at this stage. This suggests that the difficulties that students faced when bringing their empirical findings into the realm of deductive proofs was not just a matter of reasoning mode or of moving from a small number of examples to a general statement and proof. Instead, what they found difficult was the structure of the theorem to be proposed: They did not internalize how the different experiments fit together to prove the generality of the theorem.

Thus, when the students were expected to change domain to formal mathematics and construct their own proof (rather than reading, understanding and performing official proofs) based on insights gained in the domain of examples and tasks, a number of potential difficulties became visible:

Can I use examples in a formal argument? In the domain of tasks and examples students are interested in the specificity of the example, not in the general class of objects dealt with. This change in view is rather radical and is likely to confuse students. When reporting on their learning, the students point to the fact that a proof can consist of smaller parts when describing what they learn; this seems to suggest that this particular use of multiple examples and cases in a complicated proof is new to these students. Thus, it may not be the examples as much as the structures of a proof consisting of elaborate sub-arguments that actually were in contrast with the expectations of the students.

Can I construct a proof? As simple as this might seem, students do express a big surprise that they can contribute in the formal domain, both with proposing a formula (which they actually found easy) and with a proof. This specific proof is not so easy and hence perhaps not the best starting point for creating formal mathematics.

Can a proof be non-algebraic? Apart from consisting of various cases and examples, the proof that the students were asked to contribute to is also very multimodal. We do not have strong empirical evidence here, but the explicit use of non-algebraic (but logical) reasoning makes the proof of Pick's theorem quite different from the textbook proofs these students have seen, perhaps also feeding into their conflict when asked to present a codified proof.

CONCLUSION

In this paper we have investigated a case where students were brought to use empirical strategies or example-driven reasoning in proposing, constructing and proving mathematical results. We suggest analyzing the difficulties that the students have as difficulties with combining a domain of mathematical tasks and examples and the domain of formal mathematics. Using this lens we came to see how the mutual activation of proofs and examples gave rise to certain conflicts that reside not with the empirical mode of investigation, neither with the deductive reasoning mode as such when applied to sub-arguments in the proof, but with the ability to gain a comprehension of the structure of more complex deductive proofs such as the proof of Pick's Theorem.

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Oral Presentation

INTEGRATING THE HISTORY INTO THE TEACHING OF THE CONCEPT OF LOGARITHMS

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There are lots of theoretical discussion on the role of history in mathematics education and the relation between history and mathematics education. It seems that we lack the empirical studies which could link the history with mathematics education. In this lecture, we would like to report a case study in which the history of mathematics is integrated into a normal class in China.

The mathematical concept of logarithms plays a crucial role in many aspects of human existence. In the traditional curriculum, logarithms are introduced as exponents, just as Euler's definition of logarithms. However, historically, the invention of logarithms was completely independent of exponents. That's a miracle in the history of mathematics and the fact has been ignored in our teaching before. As many mathematics teachers in senior high schools said, students are always confused with why they have to learn the concept of logarithms and what the meaning of the sign of "log" is, and they are inclined to forget the logarithmic properties.

There are four approaches to integrating the history into mathematics teaching. They are complementation approach, replication approach, accommodation approach and reconstruction approach. In this lecture, we would like to present how we, a team of researchers, cooperated with a team of experienced senior high school teachers, to develop the design of the lesson, how to cite the story about Napier, and how the four approaches mentioned above are used in the first lesson about the concept of logarithms to help students experience the necessity of the invention, deepen their understanding of the concept and remind them to remember and appreciate the efforts and contributions of past mathematicians and learn from them.

What's more, we'd like to show the results of questionnaires and interviews on students, we could easily see that the goals of knowledge and affection are both attained. Though there are some aspects needed to improve, the results are generally satisfactory.

Besides students, the teacher also harvests a lot during the process. It's quite helpful for the teacher's professional development. Some of the changes with knowledge and attitude are presented in this lecture. In fact, there are lots of excellent teachers in

China, we would like to share some ideas about the use of history in Mathematics education from some experienced Chinese mathematics teachers.

THEME 3:

ORIGINAL SOURCES IN THE CLASSROOM AND THEIR EDUCATIONAL EFFECTS

Workshop

SELECTING AND PREPARING ORIGINAL SOURCES FOR PRE-SERVICE MATHEMATICS TEACHER EDUCATION IN TURKEY: THE PRELIMINARY OF A DISSERTATION

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[a]Ahi Evran University, [b]Middle East Technical University

In this paper, we aim to introduce two historical teaching modules based on the use of original sources of mathematics, namely Euclid's Elements and al-Khwarizmi's al-Jabr Wa'l-Muqabala, in order to improve junior and senior pre-service middle school mathematics teachers' mathematical knowledge for teaching, and their beliefs about mathematics and mathematics teaching. It is a preliminary study of the first author's dissertation including the introduction of the essential criteria for selecting the original sources and preparing the accompanying tasks for these sources. The study was discussed in a workshop where the participants worked on the historical modules and provided valuable inputs for their improvement and practice.

INTRODUCTION

The use of history of mathematics has been included in Turkish middle school (grades 5 to 8) mathematics curriculum with 2005 reform (Ministry of National Education [MoNE], 2005) and the following revisions (MoNE, 2009, 2013). The current curriculum (MoNE, 2013) states that history of mathematics is able to change middle grades students' attitudes towards mathematics and their learning of mathematics in a positive way. Moreover, it asserts that the history can show the discipline of mathematics as a cultural heritage. Despite the ideas on the value of history of mathematics, their integration into the relevant formal mathematics textbooks is limited to the historical information provided by secondary sources. Biographies, dates and names are some examples (MoNE, 2009, 2010). This type of integration cannot go beyond *the illumination approaches* of using history of mathematics in mathematics education (Jankvist, 2009a).

Bringing history of mathematics in mathematics classroom is a responsibility mainly on teachers' part. Mathematics teachers can be educated for this purpose in their pre-service training process or through specifically designed in-service training programs. In this study, we focus on the pre-service training because it is a relatively long formal education process before starting the teaching career. Among the undergraduate courses suggested for Turkish pre-service middle school mathematics teachers, History of Mathematics appears to be the most relevant for our focus. Alpaslan, Işıksal and Haser (2014) found that this course had a potential to increase the pre-service teachers' knowledge of history of mathematics, yet the quality of this knowledge did not seem to be promising for their future teaching. Alpaslan and Haser (2012) investigated the content and pedagogy of such a History of Mathematics course in their case study, and they concluded that the course was ineffective for developing pre-service middle school mathematics teachers' beliefs about the use of history of mathematics. Therefore, there appears to be problems due to the nature of the existing History of Mathematics courses in pre-service middle school mathematics teacher education programs.

One of the most effective ways of using history of mathematics in teacher education is the study of original sources of mathematics (Jahnke, 2000). The original sources are defined as primary mathematical products of the studies made by great mathematicians for the purposes of (i) doing mathematics for its own sake and/or (ii) teaching mathematics (Knoebel, Laubenbacher, Lodder, & Pengelley, 2007). They show reflections of the milestones on the evolution of mathematics, which has taken place in different cultural contexts in history. Jahnke (2000) addresses that original sources can provide three distinctive benefits for both learners and teachers: (i) *replacement* of the typical perceptions about mathematics; (ii) *reorientation* of ideas about mathematical concepts through making the familiar unfamiliar; and (iii) *cultural understanding* that deals with the evolutional process of mathematics linked with technological and scientific developments in the context of various societies. The study of 'unpolished' definitions, algorithms, representations, and problems in the authentic context of the original sources can be quite valuable for pre-service mathematics teachers because it provides different mathematical and pedagogical perspectives to the learning and teaching of modern mathematics concepts (Barnett et al., 2008). In such a study, the reader is guided to understand, interpret, and discuss the mathematical reasoning of the 'masters' (van Maanen, 1997). He/she attempts to comprehend the meaning of the old mathematics through stating hypotheses. Focusing on a particular mathematics topic can yield hints about how the ideas about that topic have evolved. Original sources can also challenge the accuracy of the contemporary 'polished' forms of mathematical concepts found in the secondary sources (e.g., textbooks) which are mostly utilized in modern mathematics education today (Jahnke, 2000). Additionally, possible benefits and restrictions of contemporary mathematical forms can be noticed by students and teachers through the study of the original sources (Tzanakis & Arcavi, 2000). Herein, it should be noted that these sources require careful interpretation considering historical, mathematical, and cultural context of their own time and language. Otherwise, we may confront with the problem of Whig interpretation of history, that is, considering the past from the modern perspective only (Fried, 2001).

The Study of Original Sources in relation to Pre-Service Mathematics Teachers' Mathematical Knowledge for Teaching

In the educational context of the US, Hill, Rowan and Ball (2005) studied mathematics teachers' knowledge for teaching through a practice-based research study, and they basically defined it as “the mathematical knowledge used to carry out the work of teaching mathematics” (p. 373). Ball, Thames and Phelps (2008) later categorized this knowledge mainly under two: *subject matter knowledge* and *pedagogical content knowledge*. *Subject matter knowledge* was formed of three components: (i) *common content knowledge* that stands for the general mathematics knowledge of the settings in and out of the classroom; (ii) *specialized content knowledge* that meets the necessary knowledge and skills particular for the mathematics classroom; and (iii) *horizon content knowledge* that involves mastering the further mathematics behind classroom mathematics. *Pedagogical content knowledge*, which collated the content of a subject and the distinguishing pedagogy for its teaching, it included three components as well: (i) *knowledge of content and students* that contained understandings about students’ learning, their preconceptions, difficulties and affective dispositions; (ii) *knowledge of content and teaching* that consisted of how methods and strategies of teaching mathematics can work in the classroom to result in a meaningful understanding; and (iii) *knowledge of content and curriculum* that referred the subject within teaching programs, the related textbooks, manipulative models and the like.

Even if Ball et al.’s mathematical knowledge for teaching framework has US origin, we find it useful for our context because all of its components are regarded as necessary for Turkish mathematics teachers as well. MoNE (2008) states that a middle school mathematics teacher have to master knowledge of numbers and operations, algebra, geometry and measurement, data processing, and probability as five main learning areas (i.e., *subject matter knowledge*) in the curriculum, and to have ideas about how to use this knowledge in the teaching process (i.e., *pedagogical content knowledge*). In the relevant documents, an exemplary performance indication is that “[A mathematics teacher] knows […], representations of mathematical concepts and procedures.” (MoNE, 2008, p. 13).

The study of the original sources is able to enhance pre-service mathematics teachers’ *common content knowledge* on the grounds that it may broaden the meaning of the prevalent mathematics structures, concepts and algorithms through “unpolished” definitions, narratives, presentations, and approaches (Mosvold, Jacobsen, & Jankvist, 2014). Such supplementary knowledge is likely to dispute with the pre-service teachers’ existing conceptions and understandings about mathematical concepts (Ball, 1990; Furinghetti, 2000). Original sources can go beyond serving *common content knowledge* by supplying symbols, proofs, representations, and illustrations which do not have widespread use today (Jankvist, 2009). This kind of knowledge addresses *specialized content knowledge* that is likely to enable pre-service teachers to connect the important ideas for their related understandings at first. Gaining such alternatives in the teaching repertoire is also acknowledged in the modern view of teaching mathematics through developmentally appropriate practices (van de Walle, Karp, & Bay-Williams, 2010). In this sense, improving *specialized content knowledge* through history of mathematics can assist pre-service teachers in choosing more appropriate perspectives in a classroom of students with different background knowledge and various learning styles. If the pre-service teachers intend to develop *horizon content knowledge*, which is advanced knowledge of the particular topics and procedures related to teaching, they may consult the origins of those topics and procedures because the historical sources are full of further definitions, explanations, reasons, proofs, methods, and examples by the 'masters' (Knoebel et al., 2007). This process might make their mathematical work more meaningful (Ball et al., 2008). Nevertheless, such an advanced work may not be liked by the pre-service teachers because it may seem useless for teaching purposes (Mosvold et al., 2014). At this point, it is crucial to find ways to make pre-service teachers engage in the activity and opportunely using original historical sources is shown to provide such an engagement (Jahnke, 2000).

Original sources have a potential to support pre-service teachers' professional repertoire regarding their future students' ways of thinking about mathematics topics. This is related to pre-service teachers' *knowledge of content and students*. Sfard (1995) states that conceptual problems in constructing mathematical knowledge may echo in the related epistemological difficulties faced in the historical development of that knowledge. Being aware of such epistemological difficulties seems to be important for pre-service teachers because they enable pre-service teachers to better 'guard' against the common student difficulties and misconceptions, and to utilize these as opportunities to construct new knowledge (Brousseau, 1997). Additionally, the study of original sources might provide possible reasons and discussions behind such obstacles through the primary definitions and representations of concepts (Mosvold et al., 2014). Concerning *knowledge of content and teaching*, Mosvold and colleagues (2014) assert that mathematics textbooks often cover the concepts in the order of general to specific (e.g., formulas to drill). However, this is not in accordance with the nature of the development of mathematics topics in history. Abstraction gradually increased for those doing mathematics in the former times. The original sources can provide varying degrees of abstraction that can also be employed in the modern mathematics classroom. In order to serve pre-service teachers' *knowledge of content and curriculum*, original sources can be utilized to provide ways for achieving curriculum goals and objectives through enriching the available teaching materials (e.g., manipulative models, textbooks) such as developing materials considering the primary texts. Another opportunity may be studying the historical order of the subjects in the curriculum through the original sources (Jankvist, Mosvold, Fauskanger, & Jacobsen, 2012). Interdisciplinary characteristics of the concepts may also be studied with reference to these first-hand materials. In this way, pre-service teachers can give meaning to the organization of the subjects and the related objectives in the curriculum. Also, they may make connections to other subject areas such as physics. Finally, historical study of various mathematics curricula and the relevant materials can also be helpful in understanding the different conceptions of the identical topics in these sources.

In addition to the theoretical arguments in favor of the potential benefits of using original sources for enhancing pre-service mathematics teachers' mathematical knowledge for teaching, the history and pedagogy of mathematics (HPM) literature also have some experimental studies that support these arguments. In Clark's (2012) study from the context of US, pre-service mathematics teachers stated that the study of al-Khwarizmi's *al-Jabr Wa'l-Muqabala* enhanced their understandings about quadratic equations. In particular, the geometric interpretation of the method of completing the square assisted them to comprehend the rationale behind solving quadratic equations. In Italy, Furinghetti (2007) provided pre-service teachers with cognitive roots of algebra concepts, methods and procedures by means of following their historical development through certain original sources written by the masters of algebra (e.g., Alcuin, al-Khwarizmi, Descartes, Viète). This experimental kind of studies, which have positive experiences and findings about improving pre-service teachers' mathematical knowledge for teaching through the study of original sources, has motivated the dissertation discussed in this paper.

The Study of Original Sources in relation to Pre-Service Mathematics Teachers' Beliefs about the Discipline of Mathematics and Mathematics Teaching

Pre-service mathematics teachers' beliefs can be identified as their judgments regarding the issues about mathematics and mathematics teaching (Hart, 1989), which are formed through their existing knowledge and experiences as a product of schooling and professional education (Calderhead & Robson, 1991). In this study, one of our foci is pre-service teachers' beliefs about the discipline of mathematics and mathematics teaching. Considering Jankvist (2012) and Niss and Højgaard (2011), pre-service mathematics teachers' beliefs about the discipline of mathematics can be categorized into three: (i) beliefs about application and sociologically oriented aspects of mathematics (e.g., the influence of mathematics on society), (ii) beliefs about historical and developmental aspects of mathematics (e.g., the driving forces of the historical development of mathematics), and (iii) beliefs about philosophical issues about mathematics as a discipline (e.g., whether mathematics is product of discovery or invention). The study of original sources has a potential to develop this kind of beliefs by revealing the *metaperspective issues of mathematics* within their authentic contexts (Jankvist, 2014). The other focus of the study is pre-service teachers' beliefs about mathematics teaching, particularly using history of mathematics in mathematics education as an alternative teaching strategy. These beliefs are judgements such as whether or not history of mathematics is a useful cognitive tool for teaching middle grades mathematics topics.

METHODS

Criteria for Selecting the Original Sources

In order to explain the rationale behind selecting specific original sources for the historical teaching modules intended for pre-service middle school mathematics teachers, we have set certain criteria and essential characteristics by considering the relevant workshops conducted in the previous HPM Study Group meetings, the related HPM literature, and the particular requirements of the Turkey context.

- The sources should fit into the curricula addressing the participants of the study. Herein, middle school mathematics curriculum (MoNE, 2013) and higher education policy documents regarding the pre-service teachers ([Council of Higher Education] CHE, 2007a, 2007b) constitute these curricula of interest. The use of original sources should correspond to:
    - Middle school mathematics curriculum (MoNE, 2013) by
        - serving the general aims of mathematics education (e.g., interconnecting mathematical concepts, including problems of real-world situations),
        - addressing the adopted teaching and learning approaches (e.g., teaching through problem solving, using concrete experiences to make abstraction),
        - being within the scope of the five learning areas that are Numbers and Operations, Algebra, Geometry and Measurement, Data Processing, and Probability,
        - considering the level of mathematics taught in middle grades (students aged 11 to 14), and
        - having a potential to make the curriculum objectives (e.g., to write a verbal statement in an algebraic expression) achievable.
    - CHE (2007a, 2007b) documents in relation to History of Mathematics course for pre-service middle school mathematics teachers by
        - developing the pre-service teachers' general knowledge about mathematics (e.g., mathematics as a heritage of various cultures in history), and
        - dealing with the prescribed content (e.g., analytic and modern geometry) and pedagogy (e.g., stress on mathematical aspects) of the course.
- The sources should be appropriate for the pre-service teachers' background knowledge (Barnett et al., 2008; Jankvist, 2009b). In other words, they can "show sth., that is accessible to the ordinary student and at the same time strange and different from what he has known hitherto" (Jahnke, 1991, p. 11).
- The sources should address *inner issues of mathematics* such as mathematical ideas, concepts, algorithms, methods, theories, proofs and argumentation (Jankvist, 2009a).

- The sources should able to display *metaperspective issues of mathematics*, for instance, human, social and cultural effects on the historical development of mathematics (Jankvist, 2009a).
- The sources should be milestones (i.e., initiating a field in the discipline of mathematics, being among the greatest works by 'the masters', having a great influence, exhibiting the emergence of a key concept, involving crucial ideas and fundamental problems in the development of a concept) in the historical development of mathematics (Barnett et al., 2008).
- The English translations of the sources must be reliable (Siu & Chan, 2012).

The Historical Modules

Two historical teaching modules are discussed in this preliminary study: Euclid Module and al-Khwarizmi Module. Euclid Module starts with an introduction regarding the *Elements* (3rd century BC) in which the pre-service teachers can have general ideas about where and when the book was written, what mathematical subjects it covered, and which purposes it had. The introduction is followed by open-ended questions asking the story behind the writing of the book, the topics that it contained, and the originality of the work that Euclid did. The questions here serve raising *metaperspective issues* of mathematics. After that, four selected texts from Potts' (1871) school edition of Book-I of the *Elements* are provided along with some probing questions revealing *inner* and *metaperspective issues* of mathematics and mathematical pedagogy. In particular, the four texts are Proposition-9: "Bisecting a given rectilinear angle", Proposition-18: "The greater side of every triangle is opposite to the greater angle.", Proposition-20: "Any two sides of a triangle are together greater than the third side." and Proposition-32: "In a triangle, the exterior angle is equal to the two interior and opposite angles.". The questions about *inner issues* of mathematics ask how the propositions are justified and which geometrical definitions, concepts and arguments are utilized in this justification. The pre-service teachers are also equipped with the whole Book I to be able to respond these questions. The questions regarding *metaperspective issues* of mathematics ask the nature of the axiomatic approach used in the propositions. As for the questions about mathematical pedagogy, they invite the pre-service teachers to look the propositions from a teacher's perspective (e.g., how a middle school mathematics teacher can benefit from the texts to enrich his/her teaching repertoire).

Al-Khwarizmi Module initially focuses on reading the preface written by Rosen (1831) to his translation of Mohammed Ben Musa al-Khwarizmi's *al-Kitab al-Mukhtasar fi Hisab al-Jabr Wa'l Muqabala (The Compendious Book on Calculation by Completion and Reduction)* (9th century). In this preface, it is aimed to give the related historical insights for the pre-service teachers. Accompanying open-ended questions ask the story of the writing of the book, the mathematical content of the book, driving forces for the historical development of mathematics in general, al-Khwarizmi's particular role in the evolution of algebra, the other sciences related to mathematics and the originality of al-Khwarizmi's work when it is compared with those of Greeks and Hindus. After the translator's preface, the preface originally written by al-Khwarizmi is presented together with open-ended questions posing *metaperspective issues* of mathematics such as the appreciation of the work done by the fellow scientists, the relationship between science, religion and authority. The core of al-Khwarizmi Module follows this through excerpts about three cases of equations for completion and reduction and the demonstrations of these cases. The related questions about *inner issues* of mathematics are for understanding the algorithms that al-Khwarizmi followed in the solution of the cases, geometrical representations that he made to further clarify the cases, and the conceptual links between these algorithms and demonstrations. The questions about *metaperspective issues* of mathematics intend to enable the pre-service teachers to consider various representations of mathematics in history (e.g., verbal statements of quadratic equations) and how a mathematical concept can change and develop (for instance, quadratic equations by showing their old forms of different cases and their solution sets including only positive numbers). Lastly, pedagogy-related questions ask the pre-service teachers to evaluate the style of teaching mathematics that al-Khwarizmi adopted such as supporting algorithms with relevant demonstrations.

The Workshop

The workshop of two-hours was initiated with a brief presentation about the criteria for selecting the original sources. After that, the aim was announced as getting feedback from the workshop participants in relation to the potential of the historical modules for accomplishing their target purpose. The participants were mathematics teachers, graduate students in mathematics education master's or doctoral programs, and HPM advisory board members. The participants formed groups of two in order to work on the designed historical teaching modules. At the beginning of each module, they read the related historical contexts (e.g., social conditions of the time, mathematics of the time, why the source may be written, stories about the source) through the relevant introductions and prefaces. Then the participants read the mathematical content interested in the original sources. For each of the historical excerpts, they answered some probing questions about *inner* and *metaperspective issues* of mathematics as well as mathematical pedagogy. This was intended to be in line with the pedagogy of discovery and inquiry which could enable one to explore, understand, explain, interpret, and justify the mathematical and pedagogical content in the original sources (Barnett et al., 2008). At the end of the work, each group shared their experiences in their own work, and they pointed out various advantages and disadvantages of using the historical modules in the existing format. Considering that the target group was actually pre-service middle school mathematics teachers, the participants made valuable suggestions for the improvement of the design and implementation of the historical teaching activities.

DISCUSSION AND SUGGESTIONS

In this section, we present the discussion and suggestions made at the end of the workshop, and develop criteria for preparing the accompanying tasks for the original sources. One of the main topics of discussion was the *originality* of the original sources. The experts on Euclid's *Elements* noticed that the excerpts from Potts (1871) were slightly changed by himself for teaching purposes. Hence, they suggested using another translation which was not edited in this manner. They also added that comparing such a school edition of *Elements* with the original one may reveal the pedagogical aspects behind writing that school edition. Consequently, this may have inputs for the pre-service middle school mathematics teachers' mathematical knowledge for teaching and their beliefs about mathematics teaching. Another discussion topic was the need for using (i) brief and appealing introductions for the original sources rather than long pages of historical information (e.g., the preface by the translator), (ii) contextualizing the original sources into the relevant culture, (iii) more detailed instructions throughout the study of the original sources, and (iv) more open-ended questions following the extracts in order to clarify the *inner* and *metaperspective issues of mathematics* hidden in the original sources (e.g., "Describe al-Khwarizmi's geometric method for solving quadratic equations in your own words."). All of these needs uttered the necessity of employing a particular approach (such as *the genetic approach* or, *the hermeneutic approach*) for using original sources in our context of pre-service mathematics teacher education. Lastly, there was a debate about whether to use or not to use the marginal notes by the translators when presenting a historical excerpt. In our opinion, this decision depends on the approach adopted for the use of original sources.

Criteria for Preparing the Accompanying Tasks for the Original Sources

In the light of the above discussions and suggestions, and the practice of the workshop in general, we set the following essential criteria for preparing the accompanying tasks for the original sources.

- A genuine historical account containing the relevant mathematics and culture of the time under interest should contextualize the original sources. This account can be provided by pictures, biographies, anecdotes, and so on. It seems important that such an account has to be concise in order not to exhaust the pre-service teachers.
- It is crucial that the selected original sources are in agreement with the actual purposes of the historical modules. In other words, the purpose of the modules should be reconsidered in giving the final decision about whether the selected

sources are really helpful or not (as our experience on the need for choosing the appropriate translation of the *Elements* indicates).

- Instructions should be clear and understandable for the pre-service teachers' studies on the tasks accompanying the texts selected from the original sources. For example, putting a prescription about what the pre-service teachers are expected to do in a task appears to be helpful.
- The questions asked in the tasks should be able to make the pre-service teachers critically think about the potential *inner* and *metaperspective issues* of mathematics embedded in the texts. Moreover, the questions should have a potential to put pedagogical issues on display.
- Instructor plays an important role in guiding the pre-service teachers' readings of the original sources as well as their studies in the accompanying tasks.

The above mentioned criteria for preparing the tasks for the original sources and the discussions and in the workshop have led us to think about adopting *guided reading* (see, for example, Barnett et al., 2008; Knoebel et al., 2007) as a methodological approach for the dissertation. In particular, *guided reading* seems to be coherent with our intention of using original sources from the aspects of (i) situating the sources in their authentic historical context, (ii) supporting the pre-service teachers' access to the source content by giving the right amount of support (i.e., guiding), and (iii) creating a discussion environment among the pre-service teachers. The workshop experience has contributed to the dissertation also in selecting proper original sources for the modules and revising the questions asked in the tasks.

NOTES

1. The authors wish to thank Evelyne Barbin, Tinne Hoff Kjeldsen and Uffe Thomas Jankvist for motivating this study to become a workshop.

2. The authors are also thankful to Constantinos Tzanakis, Gert Schubring, Jan van Maanen, Kathleen Michelle Clark, Leo Rogers, Luis Puig, Man-Keung Siu, Michael Fried and the other workshop participants for providing valuable comments for the improvement of the content and pedagogy of the historical modules.

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Workshop

ABSTRACT AWAKENINGS IN ALGEBRA: A GUIDED READING APPROACH TO TEACHING MODERN ALGEBRA VIA ORIGINAL SOURCES

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This workshop will explore a particular approach to the use of original historical sources in the undergraduate university mathematics classroom. The cornerstone of this approach is the extensive use of excerpts from original source material close to or representing the discovery of key mathematical concepts and theory as a means to develop the material in question. Through guided reading of the excerpts and completion of exercises based upon them, students are prompted to explore these ideas and develop their own understanding of them. Each excerpt is introduced by brief historical comments and biographical information about its author to set it in context and offer students a view of the humanistic aspect of mathematics. By placing a problem in its historical context, the student is also able to follow the thought process that led to the discovery of complicated and subtle mathematical concepts while simultaneously being guided to construct their own understanding of the present day ideas evolving via the project exercises.

Examples of this approach are found in a compendium of projects developed and tested since 2008 by an interdisciplinary team of mathematicians and computer scientists for the teaching of topics in discrete mathematics with support from the US National Science Foundation. All our projects are available via our two web resources:

(http://www.cs.nmsu.edu/historical-projects/,
http://www.math.nmsu.edu/hist_projects/).

More specifically, the workshop will focus on student projects designed for use in a first course in Abstract Algebra in the standard US undergraduate curriculum, including portions of the project Abstract awakenings in algebra: Early group theory in the works of Lagrange, Cauchy, and Cayley (project # 11 at http://www.cs.nmsu.edu/historical-projects/). Successfully tested as a textbook replacement for a significant portion of an abstract algebra course at three US institutions to date, this project exemplifies the benefits of using select original source excerpts to draw attention to mathematical subtleties which modern texts may take for granted. In keeping with the historical record, this often leads to a more concrete approach than is typical of current texts. For example, to provide context for the abstract group concept first defined in an 1854 paper by Arthur Cayley, this project begins by studying specific mathematical systems (e.g., roots of unity in Lagrange, permutations groups in Cauchy) which were well-developed prior to Cayley's explicit recognition of their common structure. Cayley's own references to these and other specific nineteenth century appearances of the group concept render his paper a powerful lens on the process of mathematical abstraction that more standard textbooks do not provide.

Additionally, workshop participants will explore portions of a student project in ring theory, based on excerpts from works by Dedekind, Fraenkel, E. Noether and Krull, which is currently under development for use in a first course on abstract algebra. The workshop agenda will also include a discussion of implementation and evaluation issues, an overview of the general pedagogical goals guiding our work, and analysis of specific features of these particular abstract algebra projects relative to those goals and other theoretical frameworks.

Workshop

ALGORITHMS: AN APPROACH BASED ON HISTORICAL TEXTS IN THE CLASSROOM

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The new curriculum in French High schools, which is currently being implemented, highlights the importance of algorithms in mathematics, promoting a kind of algorithmic way of thinking. It explicitly requires that elementary but varied works on algorithms be carried out in the classroom.

In this paper, we attempt to show that this work could be carried out on the basis of historical sources, and not only with a computer.

USING HISTORICAL TEXTS: WHY AND HOW?

We have been working on introducing a historical perspective into the mathematics classroom for some thirty years. The main reasons for that are that we think that:

- The history of mathematics allows us to motivate the introduction of a concept and to see the use of it.
- It is an inexhaustible source of problems.
- It shows mathematics in the process of being done, in relation with its time and culture, and not as dogmatic objects.
- It can raise interest in interdisciplinary projects. These enable students to become aware that mathematics contributes to the culture of an age.

Our way of implementing these ideas follows these principles:

- The history of mathematics is integrated into the mathematics curriculum. It is not treated chronologically, but in step with the concepts as they are taught.
- We have students read the original texts because these allow them to become aware of the evolution of the notion of rigour, the multiple attempts which lead to the notations that they use, and the long gestation of concepts. And also because a text written by a mathematician, since it is not written *a priori* for students, demands understanding in depth (in contrast to the automatic reactions when facing ritual and stereotyped exercises).
- The reading of the texts is often supplemented by a set of commentaries presenting the document (context, methods and vocabulary) and exercises restating the problems of the text in modern terms.

A NEW CURRICULUM IN FRANCE

For this workshop, we have chosen the theme "algorithms" because a new curriculum (beginning in 2009) in French High schools highlights the importance of algorithms in mathematics, promoting a kind of algorithmic" way of thinking. It explicitly requires that elementary work on algorithms be carried out in the classroom more or less difficult, in a variety of situations.

> "Algorithmics have a natural place in all fields of mathematics, and the associated problems have connections to other parts of the curriculum (functions, geometry, statistics and probability, logic), as well as to other disciplines or to everyday life."
>
> "The algorithmic process has been, since the beginning of time, an essential part of mathematical activity. In the first years of secondary education, pupils encounter algorithms (Algorithms of Elementary Arithmetic Operations, Euclid's Algorithm, Algorithms in Geometrical Constructions). What is proposed in the curriculum is **formalization** in natural language". (10^{th} grade)

The first use of the word "algorithm" that we know comes from Carmen *de algorismo,* by Alexandre de Villedieu (circa 1220): "This new art is called the algorismus, in which we derive such benefit out of these twice five figures of the Indians: 0 9 8 7 6 5 4 3 2 1." Here "algorismus" refers to the "art" of Algus (or Argus, or Aldus), the latinized name of Al-Kwhārizmī, whose *"The Book of Addition and Subtraction According to the Hindu Calculation"* survived only in its Latin translation. The first words of an untitled manuscript are *Dixit algorizmi* (so said al-Khwārizmī). Beginning in the XIIth century, this positional system of numeration spread into the medieval Europe, in competition with the then-common use of "abacus (counting tables) and tokens." Thus, in the beginning, the word "algorithm" only referred to arithmetical operations. During the course of time, its meaning was extended from routine arithmetic procedures "to mean, in general, the method and notation of all types of calculation. In this sense we say the *algorithm of integral calculus*, the *algorithm of exponential calculus*, the *algorithm of sinus*, etc." as D'Alembert wrote in the article "Algorithme" of the *Encyclopédie* .

Now, here is the definition in our curriculum[1]:

> "An algorithm is defined as an operational method allowing one to solve, with a number of clearly specified steps, all the instances of a given problem. This method can be carried out by a machine or a person".

This definition involves that the number of steps is finite and that the result is the right answer!

[1] 7 th year students majoring in « Informatique et Sciences du Numérique »

AN EXAMPLE OF DIDACTICAL SITUATION: FERMAT ABOUT THE FACTORIZATION OF LARGE NUMBERS.

By 1631, Fermat was councillor at the parliament in Toulouse, and greatly interested in mathematics. He met the mathematician Carcavi, who was also a councilor; it was Carcavi who put Fermat into contact with Mersenne and his group. In the early seventeenth century, there was no scientific journal. Exchanges between scientists were by letters. Mersenne helped to coordinate correspondence between all Europeans scientists; he had nearly 140 correspondents, including astronomers and philosophers as well as mathematicians. Most of Fermat's work in number theory is known by his correspondence with Mersenne.

We investigate with the students a method of factorization of large numbers developed by Fermat. (Martine Bühler worked about this text with) These students are engaged in a scientific curriculum (17-18 years old). In France, such students have six weekly hours in mathematics, and some of them have two additional weekly hours (called "specialty mathematics"). For this "specialty mathematics", the curriculum has two parts: number theory and matrices.

In 1643, Mersenne challenged Fermat to find "in less than a day" whether the number 100,895,598,169 is prime or not; and, if not, to find the factorization of this number, and to give a general method of factorization. Fermat gave a reply stating "this number is the product of 898,423 by 112,303, which are both prime". He later explained, in another letter, his method of factorization. It's this second letter that we studied in class (in January 2014). The letter is given in Appendix 1. We give below in modern language the method explained in the text.

We want to factorize a non-square odd natural number N. If N is even, it is easy to factorize, and if N is known as a square, N is already factorized. We use ordinary algebraic identities to explain the method.

If we can write N as a difference of two squares, then $N = a^2 – b^2 = (a+b)(a-b)$. Then we have factorized N: $N = p*q$ with $p = a+b$ and $q = a – b$. Conversely, if $N = p*q$, then p and q are odd numbers (because, if p or q is even, the product is even). But N is odd. So, we can write $N = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2$: with $p+q$ and $p – q$ even, so that $\frac{p+q}{2}$ and $\frac{p-q}{2}$ are natural numbers. The problem of writing N as a difference of two squares is therefore equivalent to the problem of factorizing N.

We thus have a method for factorizing N. We want now to find natural numbers a and b so that $N = a^2 – b^2$, or to obtain an integer a so that there is a b such as $a^2 – N= b^2$; that is to say, an integer a so that $a^2 – N$ is a square. We must have $a^2 – N \geq 0$, so that $a \geq \sqrt{(N)}$. There are well-known algorithms for the calculation of the square root of an integer N "by hand". We can then calculate the greatest integer less than $\sqrt{(N)}$, namely $\left[\!\left[\sqrt{(N)}\right]\!\right]$. If N is a square, we stop there. If not, we start our searching with

$a = \left[\!\left[\sqrt{(N)}\right]\!\right] + 1$. If $a^2 - N$ is a square, we stop there. If not, we try $a+1$: is $(a+1)^2 - N$ a square? And so on with $a+2$, $a+3$, etc. until we reach a square.

This is clearly an algorithmic method and the algorithm returns a value of a such that $a^2 - N = b^2$, with $b = \sqrt{a^2 - N}$. We can formalize the algorithm as follows:

Input: N, non-square odd integer
Procedure: a is set equal to $\left[\!\left[\sqrt{(N)}\right]\!\right] + 1$
While $\sqrt{a^2 - N}$ is not integer, do
a changes to $a + 1$
WhileEnd
Output: a and $\sqrt{a^2 - N}$.

We now describe the project that was assigned to the students: "***Large numbers factorization***"

Part I: Difference between 2 squares and factorization: the Fermat's method
In 1643, Fermat responded to Mersenne who challenged him – Fermat – to find "in less than a day" a factorization of 100,895,598,169. Fermat found this factorization (898,423 by 112,303), and explained in a later letter a general method for factorizing large numbers. We'll read this letter together.

Let N be an odd non- square natural number.

1) We suppose that $N = a^2 - b^2$ with a and b natural numbers. Determine, depending on a and b, two natural numbers p and q such as $N = pq$.
2) We suppose that $N = pq$ with p and q natural numbers such as $p > q$.
   a) Which is the parity of p et q?
   b) Determine, depending on a and b, two natural numbers p and q such as $N = pq$.
   c) Give all the factorizations of 45 as products of natural numbers, **and** as differences of two squares of natural numbers.
   d) Formulate into one logical equivalence everything that was demonstrated in 1) and 2) b.
3) Reading Fermat's text.

Notice that Fermat uses the following definitions: the *parts* of a number are its *divisors.* And a number is *composed (*product) of its *parts*. For instance, if 45 = 9 * 5, 9 and 5 are the *parts* of 45 and 45 is composed of 9 and 5.

Fragments of a letter by Fermat, <1643> Translation by Christian Aebi & John Steinig *Every non-square odd number is […] the difference of two squares as many times as it is composed of two numbers, […]* *It is quite easy to find the adequate squares, when we are given the number and its parts, and to have its parts when we are given the squares.*

Explain the link between these sentences and the questions 1) and 2).

Comments on Student Responses to Part I:

The second question posed difficulties for the students. Finding a and b in relation to p and q is difficult: some students did not see that we have a system to solve (with unknown a and b). Others saw the system, but were not able to solve it.

Another difficulty: we are looking for two adequate integers (i.e. integers p and q which are solutions for the problem) but not **all** adequate integers. Thus, it is sufficient to take $a=\frac{p+q}{2}$ and $b=\frac{p-q}{2}$, but it is not necessary. This provides an opportunity to work on logic, which is also part of the curriculum.

None of the students (even the best) managed to write the logical equivalence in 2d. We discussed this in class and, finally, all the students actually saw that this is what Fermat asserts in his letter.

After correction of Part I, we worked on the questions in Part II. We answered question 1 together without any difficulty, and students then worked in groups for the other questions.

Part II. The factorization algorithm:

In this Part II, N is a odd non- square natural number.

1) Explain why determining a factorization of N reduces to determining an integer a such that $N-a^2$ is the square of a nonzero natural number b.
2) Write an algorithm that determines an integer a such that $N-a^2$ is the square of an integer b, with N being an input from the user and a and b given as outputs.
3) Does the algorithm end for every odd non-square natural number N ?
4) What is the output of the algorithm if N is a prime number?
5) In his letter Fermat uses his method in order to factorize $N = 2{,}027{,}651{,}281$.
   a) Do the same by running the algorithm "by hand". You can use a table or your calculator if you wish.
   b) How many steps are required?

c) The algorithm requires the computation of a square at each step of the conditional loop. By using the expansion of $(a + 1)^2$, modify the algorithm in order to that the test requires only a first grade computation.

6) When running the algorithm, you have to test whether some numbers are the squares of primes. Fermat asserts in his letter: "the *remainder is 49619, which is not a square, because no square ends with 19*"

   a) Is it possible that a square ends with 7? Justify your answer.

   b) How would you justify the assertion of Fermat?

Comments on Student Responses to Part II:

1) Writing the algorithm:

One group immediately wrote an appropriate algorithm, initializing the variable a appropriately. Three groups started pretty quickly, after the teacher (Martine Bühler) told them to think how they would deal with 45 "by hand". Two groups found it difficult to start.

Of the latter five groups: one group tried a conditional testing through an IF/THEN/ELSE structure to be sure that a^2-N is greater than 0, but the group did not succeed in this way. It is easier to initialize a correctly. One group initialized a at 1, and did not see any problem. One group initialized a at 1, but became aware of the problem when they tried question 5a and then made the correction. One group had great difficulties to write the WHILE loop. One group used another variable b, equal to a^2-N, initialized b, but did not change the value of b in the WHILE loop.

2) Other observations

- Question 3: some groups immediately saw that the algorithm always stops, as $N = 1 \times N$, hence N is always the difference of two squares with $a=\frac{N+1}{2}$ and $b=\frac{N-1}{2}$. So, even if N is a prime number, the algorithm stops and then the output is $a=\frac{N+1}{2}$ and $b=\frac{N-1}{2}$. The converse was studied at the following class session, but I gave the result. As we ran out of time, I asked the students to answer questions 5a and 5b at home and to go directly to questions 5c and 6.
- Question 5c: We can optimize the initial algorithm; to do so, we must calculate a square at each step. But, having calculated a^2-N, it is easier to calculate $(a+1)^2-N = a^2-N + 2a + 1$.Thus, it is sufficient to add $2a + 1$ to the previous number and it is not necessary to calculate $(a+1)^2$. We need an additional variable, but the calculation is simpler. Some groups had no problem in revising the algorithm after I told them that they need another variable, but others could not see where to put the loop and we went over this together.

- Question 6: Students had no difficulty with this. It's a traditional question using congruences and "disjunctive cases". Students are used to make congruences tables. To justify Fermat's assertion, you may work modulo 100.

Optimized algorithm
Input: N odd integer, not a square
Procedure: a changes to $\lfloor \sqrt{N} \rfloor + 1$
c changes to $a^2 - N$
While $\sqrt{c}$ is not an integer, do:
c changes to $c + 2a + 1$
a changes to $a + 1$
WhileEnd
Return: a and $\sqrt{c}$

We remarked that, at each step, we add a term of an arithmetic sequence with 2 as common difference.

Thus, with this problem of factorization, we can deal with the problem of time complexity. The efficiency of an algorithm can be measured by time complexity and by space complexity. Run time analyses is a classification that estimates the increase in running time of an algorithm as its input increases. This is a topic of great interest in computer theory.

At the end of the session, I gave students Fermat's text (Appendix 1: we read it at the following session in class) and the second part of the problem (see below), as homework. The aim of the problem is the factorization of 250,507, using a sieving method, and the study of Carissan's device.

Part III. Factorization of large numbers and Carissan's device.

The aim of this part is the factorization of N =250,507. *This is done by determining an integer x such that $x^2 - N$ is the square of an integer.*

1°) Working modulo 7.

a) Complete the following table with the remainder of X^2 modulo 7 depending on the values taken by X modulo 7.

X mod7	0	1	2	3	4	5	6
X^2 mod7							

Is it possible for the number 7 x 113 + 3 to be a square? Why? (It is imperative to use the previous table, but **not** the calculator; the numbers 0, 1, 2, 4 are called quadratic residues modulo 7).

b) Determine the remainder of the Euclidean division of 250,507 by 7.

c) We are looking for an integer x such as $x^2 - N$ is the square of an integer. Thus, if the integer x is suitable, then the number $x^2 - 250{,}507$ must be a square. Using the precedent table, determine the possible values of $x^2 - 250{,}507$ modulo 7. Deduce the possible values for x^2 modulo 7.

d) But x^2 must be a square; thus the same table allows us to restrict again the possible values of x^2 modulo 7. Do this, then give the possible values for x modulo 7. Could the number 778 be a solution of the given problem?

2°) Do the same work modulo 9.

3°) Do the same work modulo 15.

4°) Determination of an integer x such that $x^2 - 250{,}507$ is a square.

a) Justify the assertion: if x is a solution of the problem, then, $x^2 \geq 250{,}507$. What is the smallest possible value for x ?

b) Let $x_0 = 501$. Calculate the remainders of x_0 modulo 7, modulo 9 and modulo 15. Is the number x_0 a solution of the problem?

c) Complete the following table until you find a value of x which fits the conditions found in the questions 1°), 2°), 3°).

x	501	502	503	504	...
Mod 7					
Mod 9					
Mod 15					

Can we be sure that the value found with this method is a solution of the given problem?

Verify that this value is actually a solution and deduce a factorization of 250,507.

Comments on Student Responses to Part III: We started with the end of Fermat's text.

$N = 2{,}027{,}651{,}281$.

$$\left[\left[\sqrt{N}\right]\right] = A = 45029$$

$N = A^2 + R$ with $R = 40{,}440$.

We have to calculate $(A + 1)^2 - N$ to see whether it is a square or not.

$(A + 1)^2 - N = 2A + 1 - R$.

That is: We subtract the remainder R = 40,440 from the double plus 1 of the square root of N (the translation in the appendix 1 is not quite correct there).

$(A + 1)^2 - N = 49{,}619$ (this number is not a square).

We add 90,061= 90,059 + 2. Students understood this because they knew that, at each step, we add a term of an arithmetical sequence of common difference 2 (as had been seen in the part II of the exercise).

Another question that naturally arises about the algorithm concerns the number of steps (namely the question of time complexity).

Set $N = pq = a^2 - b^2$, with $a = \frac{p+q}{2}$ and $b = \frac{p-q}{2}$. One starts with a = $\lfloor \sqrt{N} \rfloor + 1$

The number of steps is

$$\frac{p+q}{2} - \left(\left[\left[\sqrt{N}\right]\right] + 1\right) \approx \frac{p+q-2\sqrt{pq}}{2} \approx \frac{\left(\sqrt{p}-\sqrt{q}\right)^2}{2}$$

This number is even smaller than the difference of the divisors p and q. So the method gives the divisors which are the closest to the square root of N. We can thus deduce that, if we obtain, $a = \frac{N+1}{2}$(that is, $a = \frac{p+q}{2}$ with $p = N$ and q = 1), then we are sure that N is a prime number, because, if not, we would have obtained a divisor p of N closer to the square root of N.

This method can be used as a primality test, but it is a bad primality test. If N is prime, the number of steps is about $N/2$, far more than with the elementary method of trying of odd integers between 3 and N.

But, in the case studied by Fermat, the example is well selected, because we need only 11 steps. It is easy to create such examples, by choosing two prime numbers close to one another and calculating their product.

Principle of Carissan's Device

The method is a sieving method.

X mod7	0	1	2	3	4	5	6
X^2 mod7	0	1	4	2	2	4	1

$N \equiv 5 (mod7)$

If $x^2 - N$ is a square, then $x^2 - N$ is equivalent to 0 or 1 or 2 or 4 modulo 7. namely, $x^2 \equiv 5$ or 6 or 0 or 2 mod 7

These are the possible values modulo 7. We can do the same work modulo 9 and 15, and we will obtain the possible values mod 9 and mod 15.

It remains to complete the last table, and to stop when the **three** value are possible values for x.

This is useful because we can then mechanize the algorithm.

The choice of the moduli is a pedagogical choice. I wanted three moduli, because it is necessary to understand the method to have at least 3 moduli (2 are not enough). I also did not want the preparatory work to be too long. I thus choose numbers that are not too large.

In the last classroom session, I showed the beginning of a film explaining the historical context of the Carissan's device. The film is an amateur's one, lasting about 15 minutes. You can see the film on the IREM site:
http://www.irem.univ-paris-diderot.fr/videos/la_machine_des_freres_carissan/
We quickly corrected the problem, rather successfully completed by the students. I showed them a Carissan's device with 3 disks using an overhead projector[2]. We then watched the last 10 minutes of the film, showing the Carissan's device at work.

SOME OTHER EXAMPLES OF HISTORICAL ALGORITHMS WHICH CAN BE CARRIED OUT IN THE CLASSROOMS

Algorithms for geometrical constructions based on Euclid Elements

You will find some very basic geometrical constructions, in Euclid *Elements* which are clearly algorithms. Each of them is a step by step procedure, with a set of rules. The problem is enounced in natural language, then the data are named, and the procedure you apply. You have only to be sure that you know (or the machine knows) the elementary tools. If not, you have to insert a subprogramme. We have chosen some propositions from Euclid Elements as illustrations.

Proposition 1, Book 1

To construct an equilateral triangle on a given finite straight line.

Let *AB* be the given finite straight line. It is required to construct an equilateral triangle on the straight line *AB*.

Construction:

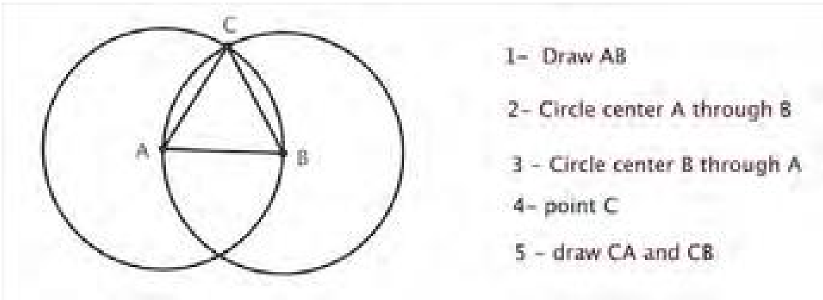
Describe the circle *BCD* with centre *A* and radius *AB*. (I post 3)

Again describe the circle *ACE* with centre *B* and radius *BA (*I post 3*)*.

Join the straight lines *CA* and *CB* from the point *C* at which the circles cut one another to the points *A* and *B*. *(*I post 1)

Conclusion: Therefore the triangle *ABC* (I, Def 20) is equilateral, and it has been constructed on the given finite straight line *AB*. Q.E.F.
Then, you will find, of course a demonstration. But this is apart from the algorithmic construction. The tools you need are pointed as, for instance (I post 3) or (I, Def 20), that means: Book 1 postulate 3 or definition 20.

[2] See the website of the irem Paris (groupe MATH).



Proposition 11, Book 1

To draw a straight line at right angles to a given straight line from a given point on it.

Let *AB* be the given straight line, and *C* the given point on it.

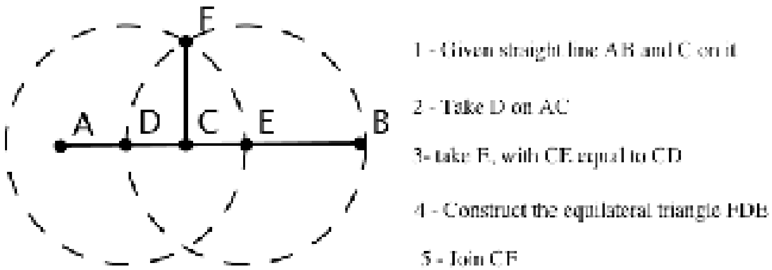
It is required to draw a straight line at right angles to the straight line *AB* from the point *C*.

Construction:

Take an arbitrary point *D* on *AC.* Make *CE* equal to *CD.* (I 3). Construct the equilateral triangle *FDE* on *DE,* and join *CF* (I 1). (I post 1)

I say that the straight line *CF* has been drawn at right angles to the given straight line *AB* from *C* the given point on it.

Conclusion: Therefore the straight line *CF* has been drawn at right angles to the given straight line *AB* from the given point *C* on it. Q. E. F.



Proposition 9, Book 1

To bisect a given rectilinear angle.

Let the angle *BAC* be the given rectilinear angle.

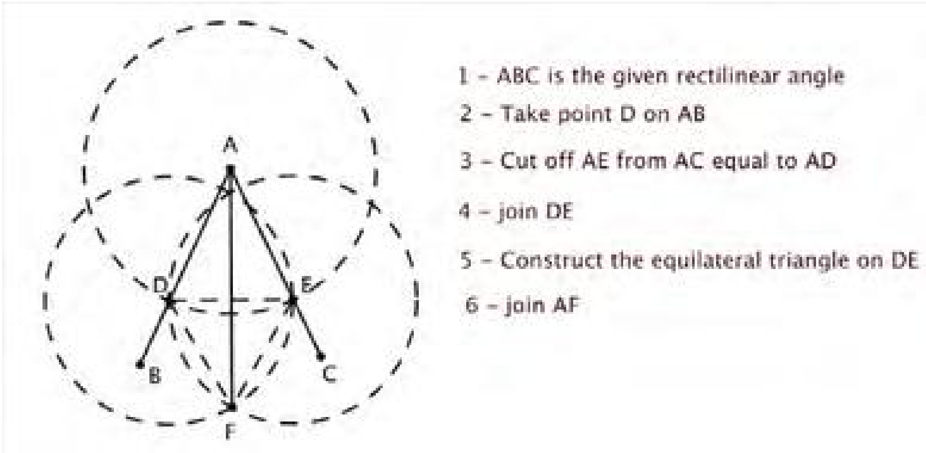
It is required to bisect it.

Construction

Take an arbitrary point *D* on *AB.* Cut off *AE* from *AC* equal to *AD,*(I 3) and join *DE.*(I post 1*)* Construct the equilateral triangle *DEF* on *DE* (I 1*),* and join *AF.*

I say that the angle *BAC* is bisected by the straight line *AF.*

Conclusion: Therefore the given rectilinear angle *BAC* is bisected by the straight line *AF.*



You can find here the useful definitions and postulates for these three algorithmic constructions:

> I def 20: Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.
> I post 1: To draw a straight line from any point to any point.
> I post 3: To describe a circle with any centre and radius.
> I prop 3: To cut off from the greater of two given unequal straight lines a straight line equal to the less.
> I prop 8: If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

HERON OF ALEXANDRIA: FORMULA VERSUS ALGORITHM

Heron of Alexandria was an important geometer and worker in mechanics. A major difficulty regarding Heron was to establish the date at which he lived. From Heron's writings it is reasonable to deduce that he taught at the Museum in Alexandria. His works look like lecture notes from courses he must have given there on mathematics, physics, pneumatics, and mechanics. Some are clearly textbooks while others are perhaps drafts of lecture notes not yet worked into final form for a student textbook.

We propose here some excerpts from *Metrica* (ca 50 A. D.)

About area of triangles:

Book I of his treatise *Metrica* deals with areas of triangles, quadrilaterals, regular polygons of between 3 and 12 sides, surfaces of cones, cylinders, prisms, pyramids, spheres etc. Usually you will find Heron's formula about the area A of a triangle whose sides of length are a, b, c, and the half-perimeter $p=\frac{(a+b+c)}{2}$ is given as follows:

$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

Actually in Heron's *Metrica* you find the result formulated as an algorithm on a generic example:

Heron's *Metrica*	Modern algebra
Let the sides of the triangle be of 7, 8, 9 units.	$a = 7$, $b = 8$, $c = 9$
Compose [add] the 7 and the 8 and the 9: the result is 24;	$a+b+c = 24$
from this take the half: the result is 12;	$(a+b+c)/2 = 12$ $p = 12$
subtract the 7: 5 remaining.	$(a+b+c)/2 - a = 5$ $p - a = 5$
Again from the 12, substract the 8: 4 remaining;	$(a+b+c)/2 - b = 4$ $p - b = 4$
and again the 9: 3 remaining.	$(a+b+c)/2 - c = 3$ $p - c = 3$
Make the 12 by the 5: the result is 60;	$p(p - a) = 60$
these by the 4: the result is 240;	$p(p - a)(p - b) = 240$
these by the 3: the result is 720;	$p(p - a)(p - b)(p - c) = 720$
from these take a side and it will be the area of the triangle.	$\sqrt{p(p-a)(p-b)(p-c)} = A$

Then, Heron gives a method in natural language, on a generic example, to approximate the square root

Heron gives this in the following form:

> Since 720 has not its side rational, we can obtain its side within a very small difference as follows. Since the next succeeding square number is 729, which has 27 for its side, divide 720 by 27. This gives $26\ ^2/_3$. Add 27 to this, making $53\ ^2/_3$, and take half this or $26\ ^5/_6$. The side of 720 will therefore be very nearly $26\ ^5/_6$. In fact, if we multiply $26\ ^5/_6$ by itself, the product is $720\ ^1/_{36}$, so the difference in the square is $^1/_{36}$. If we desire to make the difference smaller still than $^1/_{36}$, we shall take $720\ ^1/_{36}$ instead of 729 (or rather we should take $26\ ^5/_6$ instead of 27), and by proceeding in the same way we shall find the resulting difference much less than $^1/_{36}$.

So you can notice:

The explanation on a generic example
The algorithm is iterative
The explanation is given in natural language
The quantities are expressed with fractions (epistemic context)
The basic idea for the process is the notion of arithmetical mean
The algorithm gives the same results as Newton's method[3]

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[3] See an example on the website of the APMEP (French Association of Teachers of Mathematics), by Martine Buhler

ABOUT HORNER'S ALGORITHM

In mathematics, the algorithm known as Horner's method is described in many textbooks. Horner's method is an economical way of evaluating a polynomial for a given value of the argument. It is efficient, too, for polynomial division, polynomial root finding, and very fast for derivatives evaluation.

William George Horner (1786?-1837) was a school master who ran his own school at Bath, from 1809 until his death . In 1819, he published a paper on the numerical solution of equations: *A new method of solving numerical equations of all orders, by continuous approximation*.

In this paper (philosophical transactions, 1819), he gave a tabular scheme for computing a real root of a polynomial equation, but it was substantially different from the basic algorithm so called now Horner's method. The paper was also written in a very obscure style.

Anyway, Julian Coolidge (1949) wrote:

> A great mathematician he certainly was not. He offers a fine example of what an amateur can accomplish by dogged industry, and his method is surely the best we have for solving numerical equations.

In fact, the basic algorithm conventionally called Horner's method was first given in England by Theophilus Holdred (a londonian clock maker) in 1820, and was not described by Horner until 1830 (published in 1845, post mortem). The method had been anticipated by Paolo Ruffini(1765-1822) in Italy(1804), and François-Désiré Budan de Boislaurent (1761-1840) in France (1807). And long before, related techniques were known to chinese and arabic mathematicians.

Asking why this method is primarily known as Horner's method, we note that the popularization of Horner's process was due to Augustus de Morgan (1806-1871), a prominent figure in 19^{th} century English mathematics.

Description of the Algorithm

The so called basic method, of this algorithm, as explained by Thomas Stephen Davies, in *The mathematician*, 1845 is**:**

Given the polynomial:

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + etc. + A_2 x^2 + A_1 x + A_0$$

We wish to evaluate it at a specific numeral value of x, say x_0.

To accomplish this, we define a new sequence of constants as follows:

$$B_n = A_n$$

$$B_{n-1} = A_{n-1} + B_n x_0$$

…

$$B_0 = A_0 + B_1 x_0$$

And B_0 is the value searched.

To see why this works, note that the polynomial can be written as follows:

$A_0 + x(A_1 + x(A_2 + etc. + x(A_{n-1} + A_n x)))$

Thus, by iteratively substituting, you obtain:

$$A_0 + x_0(A_1 + x_0(A_2 + etc. + x_0(A_{n-1} + A_n x_0)))$$
$$= A_0 + x_0(A_1 + x_0(A_2 + etc. + x_0 B_{n-1}))$$
$$etc.$$
$$= A_0 + x_0 B_1$$
$$= B_0$$

Examples

First: give the value of $x^4 + 2x^3 - 22x^2 + 7x + 42$, for x = 3.

We use the so called ***synthetic division***, as follows

	1	**2**	**- 22**	**7**	**42**
3		**3**	**15**	**- 21**	**- 42**
	1	**5**	**- 7**	**- 14**	**0**
	B_4	B_3	B_2	B_1	B_0

The entries in the third row are the sum of those in the first two. Each entry in the second row is the product of the *x*-value (3 in this example) with the third-row entry immediately to the left. The entries in the first row are the coefficients of the polynomial to be evaluated.
The value for 3 is 0.
And the remainder in the division by (x – 3) is 0.
3 is a root.
And you can write: $x^4 + 2x^3 - 22x^2 + 7x + 42 = (x - 3)(x^3 + 5x^2 - 7x - 14)$
Second example:
Divide $x^3 - 6x^2 + 11x - 7$ by (x – 2)

	1	**- 6**	**11**	**- 7**
2		**2**	**-8**	**6**
	1	**- 4**	**3**	**- 1**
	B_3	B_2	B_1	B_0

The quotient is $x^2 - 4x + 3$, and the remainder is -1, so that:
$x^3 - 6x^2 + 11x - 7 = (x^2 - 4x + 3)(x - 2) - 1$
Horner's method is optimal, in the sense that any algorithm to evaluate an arbitrary polynomial must use at least as many operations. Alexander Ostrowski_proved in 1954 that the number of additions required is minimal. Victor Pan proved in 1966 that the number of multiplications is minimal. In fact this method is very efficient, so that it is used for computers and calculators.

Use for Derivatives

In their first publications, Ruffini and Horner used the differential calculus in expounding their methods. Later, both authors gave simplified explanations, using ordinary algebra, as it is shown above.
But it can be used to evaluate derivatives.
Given a polynomial $P(x)$, and a real number x_0.
Using the preceding method, you write $P(x)$ as: $P(x) = (x-x_0)Q(x) + P(x_0)$**,** where $Q(x)$ is a new polynomial.
Now, you can repeat with $Q(x)$, and obtain: $Q(x) = (x-x_0)Q_1(x) + Q(x_0)$**.**
That gives: $P(x) = P(x_0) + (x-x_0)Q(x_0) + (x-x_0)^2\, Q_1(x)$**.**
Then you'll obtain:
$P(\mathrm{x}) = P(x) = P(x_0) + (x-x_0)Q(x_0) + (x-x_0)^2\, Q_1(x_0) + (x - x_0)^3 Q_2(x)$.
And so on.
Actually the Taylor's theorem, gives
$P'(x_0) = Q(x_0)$; $P''(x_0) = 2!\, Q_1(x_0)$; … ; $P^{(\mathrm{k})} = \mathrm{k}!\, Q_{\mathrm{k}-1}(x_0)$

Example:
Given: $P(x) = x^5 - 6x^4 + 8x^3 + 8x^2 + 4x - 40$
We wish to evaluate $P(3)$, then $P'(3)$, $P''(3)$, $P'''(3)$.

	1	-6	8	8	4	-40	
3		3	-9	-3	15	57	
	1	-3	-1	5	19	17	P(3) = 17
3		3	0	-3	6		
	1	0	-1	2	25	← Q(3)	P'(3) = 25
3		3	9	24			
	1	3	8	26	← $Q_1(3)$	P"(3) = 2! 26	P"(3) = 52
3		3	18	78			
	1	6	26	← $Q_2(3)$	$P^{(3)}(3)$ = 3! 26	$P^{(3)}(3)$ = 156	

Appendix 1

Translation of *Fragment d'une lettre de Fermat, Œuvres*, éd. Tannery et Henry, tome II, 1894, pp.256-258. There are a few annotations between square brackets to help the reader. Translation by Christian Aebi and John Steinig.

Site: http://gradelle.educanet2.ch/christian.aebi/.ws_gen/9/Fermat_english.pdf

Fragments of a letter by Fermat <1643>

Every non-square odd number is […] the difference of two squares as many times as it is composed [*written as a product*] of two numbers, and if the squares are prime to one another then the same may be said of the two composition numbers [*factors*]. But if the squares have a common divisor, the number in question will also be divisible by the same common divisor, and the composition numbers will be divisible by the side [*square root*] of the common divisor.

For example: 45 is composed of 5 and of 9, of 3 and 15, of 1 and 45. So, it will be thrice the difference of two squares: according of 4 and 49, who are prime to one another, as are the corresponding composers 5 and 9; plus of 36 and 81, which have 9 as common divisor, and the corresponding composers, 3 and 15, have the side of 9, meaning 3, as common divisor; finally 45 is the difference of 484 and of 529, which have 1 and 45 as corresponding composers.

It is quite easy to find the adequate squares, when we are given the number and its parts[*divisors*], and to have its parts when we are given the squares.

[…]

That settled, let a number be given to me, for example 2 027 651 281; we are asked if it is prime or composed, and in that case, of which composers.

I extract the square root to find the smallest of the preceding numbers, and find 45 029 with 40 440 as remainder, from which I substract the double plus 1 from the preceding root, meaning 90 059: the remainder is 49 619, which is not a square, because no square ends with 19, from there I add 90 061 to it, meaning 2 more than 90 059 which is the double plus 1 of the root 45 029. And since the sum 139 680 is still not a square, as can be seen by the ending [*final digits*], I add to it once again the same number increased by 2, meaning 90 063, and I continue to add in that manner until the sum is a square, as can be seen here. This happens only at 1 040 400, which is the square of 1 020, and thus the given number is composed; because it is easy, by the examination of the preceding sums, to see that there is no other that is square except the last, because squares cannot bear the ending they have, except for 499 944 which nevertheless is not a square.

Now, to find out the numbers that compose 2 027 651 281, I remove the number that I had first added, meaning 90 061, from the last added 90 081. There remains 20, from which half plus 2, meaning from 12, I add the root previously found, 45 029. The sum is 45 041, to which number by adding and removing 1020(the root of the last sum 1 040 400), we have 46 061 and 44 021, which are the two nearest [*side-by-side*] numbers that compose 2 027 651 281. These are also the only ones [*factors*], for they are, one as well the other, prime.

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Workshop

MAKING (MORE) SENSE OF THE DERIVATIVE BY COMBINING HISTORICAL SOURCES AND ICT

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To complement a teaching module on the introduction of the derivative, we designed three worksheets based on historical sources. The three worksheets illustrate different aspects of the derivative; select and use historical sources in different ways; and make different (but systematic) use of ICT (information & Communication Technology): dynamic geometry, programming languages, computer algebra system, and spreadsheet.

CONTEXT AND CONTENTS OF THE PAPER

We would like to present three sets of historical texts, and three classroom activities based on them. Our goal is to help high school students make sense of the new and thorny concept of the derivative. Needless to say, the literature on this topic is huge: it shows quite clearly where the main difficulties lie, and offers many fruitful leads [1]. We retained an approach emphasizing the importance of task solving, and designed a learning path which gradually expands the concept by unveiling new and *efficient* aspects: local straightness, limit-position of secants, affine approximation, and iterated affine approximation. The use of original sources *complements* these key stages.

More specifically, our target population is that of students in the French "Première" class who choose to major in maths and the sciences (the so-called scientific stream). As far as the teaching of mathematical analysis is concerned, the curricular context is the following:

Seconde Age 15-16	Basic notions on functions
Première Age 16-17 Scientific stream	The derivative • As an object: definition, geometrical interpretation, standard formulae • As a tool to study the qualitative behaviour of a function (variations, extrema...) Transversal methodological goals • Emphasis on algorithms (in the natural language or in a programming language) • Emphasis on proof and reasoning
Terminale Age 17-18	Integral calculus Transcendental functions: ln, exp, sin, cos

With high-school teacher and former teacher-trainer Sylvie Alory we designed a teaching module for the "derivative as an object" chapter. Among the various well-known ways to introduce the derivative we chose the "local-straightness" approach, which we feel provides the necessary feedback when students are to engage in rather open-ended tasks. To make a long story short, the two-week module starts and builds upon a new mathematical "experimental fact": if you use Geogebra to zoom in hard enough onto a point on a functional graph, you quickly get something that you cannot visually distinguish from a line-segment.

Two weeks later, we aim at reaching the following:

- Some functions have a derivative, some do not (at least not everywhere, for various reasons which can be illustrated graphically)
- A definition for the derivative $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, based on a loose and intuitive notion of what a limit is.
- The generic equation of a tangent to the *f*-graph is $y = f'(a)(x-a) + f(a)$
- Some basic formulae, such as $[x^n]' = n\,x^{n-1}$, $[uv]' = u'v + uv'$ …

This teaching module is not based on the history of mathematics [2]: it does not explicitly use historical sources; its design was not based on any form of rediscovery/genetic approach; when Sylvie and I discussed what we would consider to be a satisfactory way to teach this thorny chapter, the history of mathematical analysis was never mentioned. This is the reason why the present paper is not about this teaching module *per se*, but about how we chose to *complement* it.

Our background knowledge of the history of mathematics suggested that several topics could be investigated at various points of the "Première" year, in order to *flesh out* the derivative concept. In the main teaching module, we felt that two much context would stand in the way from a cognitive viewpoint; in contrast to that, we know that to make real sense of a new, tricky and rich concept such as that of the derivative, it is useful to study its role in several different contexts, to see it from a variety of angles and in various semiotic environment. We do not claim to cover all – not even all the important – contexts in which the derivative plays a part: we did not include kinematic aspects, or the reflection of light-rays, or the notion of visual horizon for an observer sitting on a curve etc... It so happens that for the three topics which we retained, we felt we could benefit from an ICT-rich environment (ICT = Information & Communication Tehcnology); in fact, the final worksheets cannot be implemented without a dynamic geometry software, a programming language, a computer algebra system and a spreadsheet. The rich environment provided by the original sources suggested that we could also address methodological teaching goals pertaining to proof and reasoning, and algorithmic thinking.

Let us give the outline of the three topics:

- Lines tangent to a circle / lines tangent to other curves
  Historical sources: Euclid, Clairaut (1713-1765).
  ICT: Dynamic geometry
  Maths contents: Tangents as limit positions of secants (chords). Proof and reasoning (reading, analysing, and assessing proofs).
- A Babylonian method to approximate square roots
  Historical source (indirect use): Cuneiform tablets BM 96957 and VAT 6598
  ICT: Programming. Dynamic geometry.
  Maths contents: Algebra and inequalities. Derivative as best local affine approximation. Proof and reasoning.
- An iterative method to approximate the roots of a polynomial
  Historical source: Euler's Elements of Algebra
  ICT: Computer algebra system. Spreadsheet.
  Maths contents: Linear approximation. Iterative algorithms and recursively defined sequences.

Needless to say, our selection of original sources depends heavily on the teaching goals. Here, historical documents are used as means to teach the derivative from a variety of angles, and tackle general methodological goals as well. In fact, we did not use any of the sources which represent milestones in the history of the calculus (tangents in Descartes, Fermat, Roberval; the calculus according to Newton and Leibniz); in contrast, these landmark texts are those we cover in our history of maths courses for pre-service teachers.

For all three topics, we designed a thematic worksheet, with a student version and a teacher version; only the student versions are presented here, with a few introductory remarks (italicized). The main teaching module has been implemented several times in the classroom, and feedback is being analyzed; the thematic-worksheets, however, have not yet been tried out. The package will be made available to teachers and teacher-trainers in France in the fall of 2014.

THEMATIC WORKSHEET #1: LINES TANGENT TO THE CIRCLE.

Students learn about the tangents to the circle in middle-school, but this first encounter with the notion of tangent is not usually very helpful when it comes to introducing a new (or, rather, more general) notion of tangent in high school: students generally remember one fact only, namely that the tangent is the perpendicular to the radius drawn from a point on the circle; a property which cannot be generalized. In this worksheet, we provide material on the basis of which the notion of tangent can be studied from a variety of angles; in particular, in Clairaut's text, a deep theorem is proved in a context where the tangent is seen as the limit position of a one-parameter family of chords with one fixed endpoint.

Dynamic geometry is used to help students visualize invariants and re-enact dynamic arguments such as Clairaut's. In this worksheet, a strong emphasis lies on methodological goals, in particular proof and reasoning. In the first part, excerpts from Euclid's Elements are analyzed, in order to exemplify the notions of "existence theorem", "uniqueness theorem" and "proof by contradiction". In the second part, students (and teachers) are confronted with a text which most of us would not consider a bona fide proof. It is interesting, and to some extent convincing, but several of its features are clearly non conventional: it relies on some form of dynamic geometrical intuition, on an implicit continuity principle; its style is very rhetorical.

Session 1: Definition and characteristic property of tangents to a circle, in Euclid's Elements (Heath, 1908).

1. In middle-school you studied the notion of a tangent to a circle. Can you recall its definition?

At the beginning of Book III of the Elements (written circa 300 BC), Euclid gave the following definition: *A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle*.

2. Can you reformulate this definition with your own words or with diagrams; in particular, explain the difference between "touch" and "cut".

In Book III, proposition 16 reads: *The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed*.

Here is the proof of the first part of the proposition:

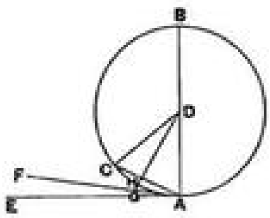
Let *ABC* be a circle about *D* as centre and *AB* as diameter;

I say that the straight line drawn from *A* at right angles to *AB* from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as *CA*, and let *DC* be joined.

Since *DA* is equal to *DC*,

the angle *DAC* is also equal to the angle *ACD*. [I. 5]



But the angle *DAC* is right;

therefore the angle *ACD* is also right:

thus, in the triangle *ACD*, the two angles *DAC*, *ACD* are equal to two right angles; which is impossible. [I. 17]

Therefore the straight line drawn from the point *A* at right angles to *BA* will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference;

therefore it will fall outside.

3. a. Draw a diagram showing only those elements which are relevant for this part of the proof.

3. b. The proof refers to propositions 5 and 17 from book I. Can you make a conjecture as to what these propositions state?

3. c. What makes this proof a proof by contradiction (also called *reductio ad absurdum*)?

Here is the proof of the second part of the proposition:

> Let it fall as *AE*;
> I say next that into the space between the straight line *AE* and the circumference *CHA* another straight line cannot be interposed.
> For, if possible, let another straight line be so interposed, as *FA*, and let *DG* be drawn from the point *D* perpendicular to *FA*.
> Then, since the angle *AGD* is right,
> and the angle *DAG* is less than a right angle,
> *AD* is greater than *DG*. [I. 19]
> But *DA* is equal to *DH*;
> therefore *DH* is greater than *DG*, the less than the greater: which is impossible.
> Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

4. a. In the diagram, the position of line AF doesn't seem to be quite right. Is it a mistake, ascribable either to the author or to the publisher?

4. b. A key argument in the proof comes from proposition 19 of Book I, the content of which may not be familiar to you. This proposition states an intuitive relationship between the longest of the three sides and the greatest of the three angles, in any triangle. Can you suggest a statement of this proposition?

4. c. In what respect is proposition 16 of Book III an existence theorem? In what respect is it a uniqueness theorem?

Session 2: The alternate segment theorem, according to Clairaut (1713-1765).

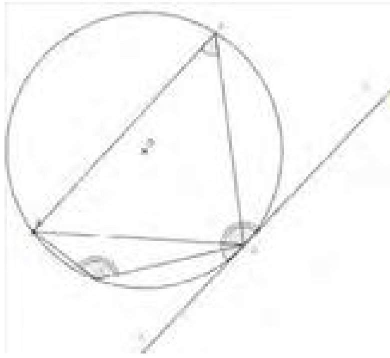
Let us consider a circle **C** with centre O; let AB be a chord (but not a diameter), and E another point on the circle. Draw triangle ABE.

1. a. You studied in middle-school a theorem about two angles subtended by the same arc AB: an angle at the centre of the circle, and an angle at the perimeter (or circumference) of the circle. Can you state this theorem?

[In Euclid's Elements, this theorem is proposition 20 of Book I]

1. b. Create a *Geogebra* file in order to illustrate this property.

1. c. In the particular case when AB is a diameter, which well-known property does this general theorem boil down to?



After proving this theorem, Euclid stated a new result on tangents, which we now call the alternate-segment theorem. Proposition 32 of Book III reads: *If a straight line touches a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle*.

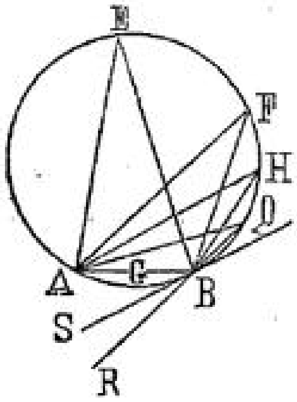
In this diagram, line SB touches the circle at B, and AB is the line "cutting the circle".

In Euclid's Elements, the proof of this proposition did not rely on proposition III.20 (studied above). However, in his Eléments de Géométrie, Alexis Clairaut (1713-1765) derived the alternate-segment theorem from proposition 20.

Here are Clairaut's proposition and his justification (Clairaut, 1853) [3]:

The tangent to a circle is the line which touches it at only one point. The angle to the segment is that between the chord and the tangent. Its measure is half that of the arc of the segment.

Fig. 87.



Since we saw that the angles on the perimeter AEB, AFB, AHB (fig. 87) are all equal, one wonders what becomes of angle AQB as its vertex Q coincides with point B, the extremity of its base. Would this angle then vanish? One cannot see after which point this angle would cease to exist; how, then, could we measure this angle? The only way out of this conundrum is to resort to the geometry of the infinite; a geometry of which all men have some (maybe imperfect) grasp, and which we aim at improving.

Let us first observe that, as point E approaches point B, thus becoming F, H, Q etc., line EB gradually decreases, as the angle EBA which it makes with line AB increases ever more. But, however short line QB may become, the angle QBA will not cease to be an angle, since, to make it perceptible, we only need to extend line QB to point R. Will the same hold for line QB once it has decreased to the point of vanishing? What has then become of its position? What about its extension QR?

It is obvious that it becomes no other than the line BS which touches the circle only at B, without meeting it at any other points; for this reason, this line is called the tangent.

Moreover, it is clear that as line EB continuously decreases and eventually vanishes, the line AE, which successively becomes AF, AH and AQ etc., comes ever closer to AB, and eventually coincides with it: hence the angle AEB subtended at the *perimeter, after becoming AFB, AHB and AQB, eventually becomes the angle ABS between chord AB and tangent BS; and this angle, which is called the alternate-segment angle, must retain the property of being half of the measure of arc AGB.*

In spite of the fact that this proof may be a little abstract for the beginner, I thought fit to include it, since it will be very useful for those who will further their study into the geometry of the infinite to become accustomed to these considerations fairly early on.

2. a. Illustrate Clairaut's reasoning on your *Geogebra* file, using E as a moving point.

2. b. Would you call Clairaut's reasoning a *proof*?

2. c. In his reasoning, Clairaut never mentioned the fact that a tangent to the circle is perpendicular to a radius. What are the two features of the tangent to a circle that he mentioned or used?

2. d. Compare Clairaut's notion of tangent to the notion used in your lesson on the derivative of a function.

2. e. Clairaut wrote that his reasoning could help accustom beginners to the *geometry of the infinite*. In your opinion, did he mean "infinitely small" or "infinitely large"?

THEMATIC WORKSHEET #2: A BABYLONIAN PROCEDURE TO APPROXIMATE SQUARE ROOTS

In this worksheet, an approximation method is first studied from a mathematical viewpoint independently from the derivative context (only basic algebra and the algebra of inequalities are required); the method is also implemented in a programming language (ALGOBOX being the one most commonly used in French schools). The connection with the derivative is made in the third part, where the intuitive notion of "best linear approximation" is brought into the picture.

Here, the use of historical sources is quite unusual. We felt the Babylonian tablet was too difficult to study, which is why, in the second part of the worksheet, we decided a secondary source could be studied instead. We chose an excerpt from Fowler and Robson's paper on tablet YBC 7289 to discuss a possible geometric argument accounting for the approximation method. Of course, we would not object to anyone using the original source in the classroom! This is why it is included in an Appendix, along with reading tips.

Part 1

Some Babylonian clay tablets from the 2nd millennium BC display a procedure to approximate the square root of a number. This procedure can be summed up as follows:

To find the square root of N, look for the largest integer B whose square is less than (or equal to) N. We have $N = B^2 + A$. An estimate for $\sqrt{N}$ is given by $\sqrt{N} = \sqrt{B^2 + A} \approx B + \frac{A}{2B}$.

1. What output values does this procedure yield for the following $\sqrt{N}$:

$$\sqrt{104} \qquad \sqrt{4{,}5} \qquad \sqrt{10} \qquad \sqrt{81} \qquad ?$$

2. As we know, an approximate value can be either above (an overestimate) or below (an underestimate) the target exact value.

2. a. In the examples from question 1, does this procedure provide underestimates or overestimates? Can you answer this question without using the "square root" button of your calculator?

2. b. Work out the square of $B + \frac{A}{2B}$. How can you generalize your answer to question 2.a?

3. We would like to write an algorithm implementing this procedure for any input integer chosen by the user.

3. a. In the first part of the algorithm, when looking for the value of B, will we need a FOR-loop or a WHILE-loop?

3. b. Write the complete algorithm. Check it with the values studied in question 1, and for perfect squares.

4. The Babylonian method is deeply connected to the following approximation formula:

T formula: if a is close to zero, then $\sqrt{1+a} \approx 1 + \frac{a}{2}$

4. a. Interpreting "close to zero" as "lying between 0 and 1", check that formula T is a special case of the what the Babylonian procedure yields.

4. b. With the same interpretation, show that, in the Babylonian method, $\frac{A}{B^2}$ is close to zero when B is greater than 2.

4. c. Under this condition, factorize B^2 in $\sqrt{N} = \sqrt{B^2 + A}$, and show that the Babylonian formula is a special case of formula T.

Part 2

5. Historians of mathematics David Fowler and Eleanor Robson (Fowler & Robson, 1998, pp.370-372) reconstructed a geometrical argument which they think could have led Babylonian mathematicians to their procedure.

5. a. Up to now, we focused on the following numerical problem: "to find the square root of a given number". Can you think of a geometrical problem which would – to a large extent – be equivalent to this numerical problem?

Here is an excerpt from the Fowler and Robson paper:

(...) to help the reader, we shall use lower-case names such as "approx", "new approx" for lengths, and capitalized names such as "Number" and "Bit" for areas.

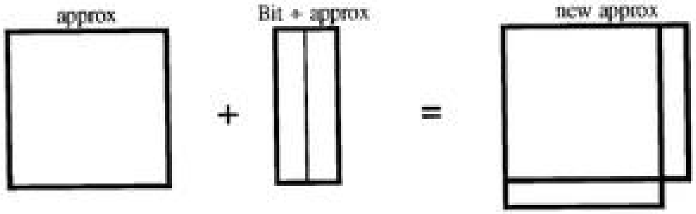


Figure 2

So suppose we want to evaluate the "side of a Number" (our square root). We start from some approximation, and let us first examine the case where this is an underestimate, so

Number = Square of approx + Bit

which, geometrically, can be represented by the sum of a square with sides approx and the leftover Bit. Now express this Bit as a rectangle with sides approx, and therefore, Bit ÷ approx, or, Old Babylonian style, Bit × IGI approx [IGI means *reciprocal*]*; cut this in two lengthwise, and put the halves on two adjacent sides of the square root of approx, as shown in Fig. 2. Hence*

new approx = approx + half of Bit x IGI approx,

and it will clearly be an overestimate because of the bite out of the corner.

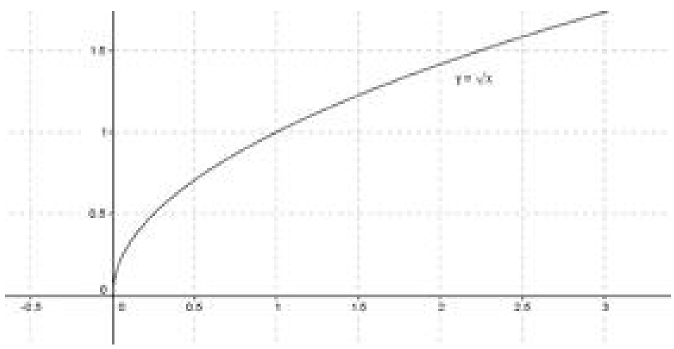
5. b. Draw the diagrams which illustrate Fowler and Robson's argument for the case $N = 104 = 10^2 + 4$

5. c. Use the letters N, B and A in the diagrams to recover what we called the Babylonian formula.

5. d. Explain "it will clearly be an overestimate because of the bite out of the corner", and compare with question 2.b.

6. We saw that the Babylonian procedure was, to a large extent, equivalent to formula T. Leaving aside Babylonian-style arguments, we will see that the study of the square root function (and its graph) can lead to formula T.

Let us define f by the formula $f(x) = \sqrt{x}$, where x denotes a non-negative real number. Let C denote the graph of f.



6. a. Work out $f(1)$ and $f'(1)$.

6. b. Show that the tangent line to curve C at point (1,1) has equation

$$y = \frac{1}{2}(x-1)+1$$

6. c. For which values of x do you think the following formula would be relevant, and why?

$$\sqrt{x} \approx \frac{1}{2}(x-1)+1$$

6. d. Substitute $1 + a$ for x in the formula. What formula do you get, and for which values of a would it be relevant?

6. e. Both formula T and the Babylonian procedure yield overestimates. Can you make sense of it geometrically?

6. f. One could come up with a wealth of linear approximation formulae similar to formula T, such as

if a is close to zero, then $\sqrt{1+a} \approx 1+\frac{a}{3}$

if a is close to zero, then $\sqrt{1+a} \approx 1+2a$

if a is close to zero, then $\sqrt{1+a} \approx 1-\frac{a}{2}$

Could a quick look at the graphs suggest that they yield poorer estimates than formula T?

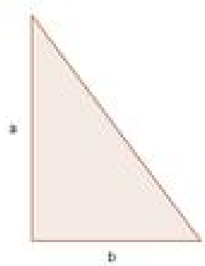
Appendix: A Babylonian worked exercise

Clay tablets BM 96957 and VAT 6598 display a series of worked exercises. Here is a translation of one of them:

A gate, of height ½ <rod> 2 cubits, and breadth 2 cubits. What is its diagonal? You: square 0;10, the breadth. You will see 0;01 40, the base. Take the reciprocal of 0;40 (cubits), the height ; multiply by 0;01 40, the base. You will see 0;02 30. Break in half 0;02 30. You will see 0 ;01 15. Add 0 ;01 15 to 0;40, the height. You will see 0;41 15. The diagonal is 0;41 15. The method.

Reading help:

To work out the length of the diagonal of a rectangle with sides a and b units of length, assuming $a > b$, the procedure corresponds to the following formula:



$$diagonal = \frac{(b^2 \times reciprocal(a))}{2} + a$$

Which can be interpreted as a combination of Pythagoras' rule, and the approximation method studied above:

$$diagonal = \sqrt{a^2 + b^2} \approx a + \frac{b^2}{2a}$$

Additional help:

- Units of length : 1 rod = 12 cubits (≈ 6m) (see Fowler & Robson op. cit., p.369 footnote 8)
- The numerical system used in this tablet has base 60 (sexagesimal system). In the transcription chosen by Fowler and Robson, the semicolon separates the whole part from the fractional part, and the 59 “digits” are separated by blank spaces.

  Example: 0 ; 01 40 stands for 0+ 01/60+40/3600
- Breadth = 2 cubits = 2/12 rod = 10/60 rod = 0 ; 10 rod.
- $(10/60)^2$ =100/3600=(60+40)/3600=60/3600+40/3600=1/60+40/3600 = 0 ; 01 40
- Height = ½ rod 2 cubits = 6 cubits + 2 cubits = 8 cubits = 8/12 rod = 40/60 rod = 0 ; 40
- The reciprocal of 40/60 is 60/40=(40+20)/40=1+20/40=1+1/2=1+30/60 = 1 ; 30

To know more about Babylonian mathematics, and learn how to use on-line sexagesimal calculators, you can Google *mesomath.*

THEMATIC WORKSHEET #3: AN ITERATIVE METHOD TO APPROXIMATE THE ROOTS OF A POLYNOMIAL

The French and English editions of Euler’s Elements of Algebra include a very clear exposition of Lagrange’s version of the Newton-Raphson approximation method (also known as the method of tangents). In addition to introducing a standard and powerful approximation method, it enables us to focus on two different mathematical topics.

The first one is that of iterative methods: these can be studied from an algorithmic viewpoint, or, if formulated in terms of formulae, through recursive sequences. Here, the algorithmic aspect is not studied with a programming language but with a computer algebra system, which is used step-by-step in an iterative way. The “sequence” point of view is studied both with a spreadsheet, and on pen and paper, to yield formulae such as $x_{n+1} = \frac{2(x_n)^3+2}{3(x_n)}$.

The other topic is, of course, that of the derivative. Here the notion of linear approximation is studied from the numerical angle, in a polynomial context. Euler does not mention the more general context in which the notion of derivative becomes necessary. We leave this to the teacher if he/she pleases, since the method of tangents is studied in most textbooks, usually from a graphical viewpoint.

Session 1: Discovering a new method

We shall use – and try to account for – a method for solving polynomial equations by approximation. This method can be found in many texts; we will use Leonard's Euler (1707-1783) *Elements of Algebra* (Euler, 1822, p.289).

756. We shall illustrate this method first by an easy example, requiring by approximation the root of the equation $x^2 = 20$.

Here we perceive, that x is greater than 4 and less than 5; making, therefore, $x = 4 + p$, we shall have $x^2 = 16 + 8p + p^2 = 20$; but as p^2 must be very small, we shall neglect it, in order that we may have only the equation $16 + 8p = 20$, or $8p = 4$. This gives $p = \frac{1}{2}$, and $x = 4\frac{1}{2}$, which already approaches nearer the true root. If, therefore, we now suppose $x = 4\frac{1}{2} + p'$; we are sure that p' expresses a fraction much smaller than before, and that we may neglect p'^2 with greater propriety. We have, therefore, $x^2 = 20\frac{1}{4} + 9p' = 20$, or $9p' = -\frac{1}{4}$; and consequently, $p' = -\frac{1}{36}$; therefore $x = 4\frac{1}{2} - \frac{1}{36} = 4\frac{17}{36}$.

And if we wished to approximate still nearer to the true value, we must make $x = 4\frac{17}{36} + p''$, and should thus have $x^2 = 20\frac{1}{1296} + 8\frac{34}{36}p'' = 20$; so that $8\frac{34}{36}p'' = -\frac{1}{1296}$, $322p'' = -\frac{36}{1296} = -\frac{1}{36}$, and

$$p'' = -\frac{1}{36 \times 322} = -\frac{1}{11592}:$$

therefore $x = 4\frac{17}{36} - \frac{1}{11592} = 4\frac{4473}{11592}$, a value which is so near the truth, that we may consider the error as of no importance.

1. Euler stated without justification that $\sqrt{20}$ "is greater than 4 and less than 5". Can you justify it?

2. Using pen and paper only, carry out Euler's calculations up to the $4\frac{17}{36}$ value. You should use different symbols (such as = and ≈) to distinguish between "equal" and "approximately equal".

3. Is $4\frac{17}{36}$ a better estimate of $\sqrt{20}$ than $4\frac{1}{2}$? You may use your calculator.

4. Euler repeatedly claimed that p^2 is "very small", hence can be "neglected". Does this sound reasonable to you?

5. To carry out Euler's procedure, all we need to do is to expand squares of sums, and solve linear equations; a computer algebra system can do this for us.

5. a. Carry out Euler's computations using *Geogebra's Algebra View*.

5. b. Euler stops at x=$4\frac{4473}{11592}$. Is this what you find with *Geogebra*? Is it the exact value of $\sqrt{20}$?

5. c. Carry out Euler's procedure one more time, to get an even better estimate of $\sqrt{20}$.

5. d. Work out $\sqrt{20}$ with your calculator. How many times do you need to carry out Euler's procedure to get the same number of decimal places?

Session 2: Applying the method to a variety of equations

After this detailed exposition of the case of equation $x^2 = 20$, Euler quickly explained how to adapt the same method to other equations.

6. For instance, for $x^3 = 2$ he wrote (Euler, 1822, p.291):

> For example, let $x^3 = 2$; and let it be required to determine $\sqrt[3]{2}$. Here, if n is nearly the value of the number sought, the formula $\frac{2n^3+2}{3n^2}$ will express that number still more nearly; let us therefore make
>
> 1. $n = 1$, and we shall have $x = \frac{4}{3}$,
> 2. $n = \frac{4}{3}$, and we shall have $x = \frac{91}{72}$,
> 3. $n = \frac{91}{72}$, and we shall have $x = \frac{162130896}{128634294}$.

6. a. How can you be sure that there is a number between 1 and 2 whose cube is equal to 2? Is there only one?

6. b. Use the computer algebra system to check Euler's calculations.

6. c. Use algebra to justify Euler's claim: "if n is nearly the value of the number sought, the formula $\frac{2n^3+2}{3n^2}$ will express that number still more nearly". You may use either pen and paper, or *Geogebra*.

6. d Use the $\frac{2n^3+2}{3n^2}$ formula in a spreadsheet software to display a sequence of ever more accurate approximations of $\sqrt[3]{2}$.

7. Same questions as in (5), for the following excerpt (Euler, 1822, p.291):

> 790. In order to apply this operation to an example, let $x^3 + 2x^2 + 3x - 50 = 0$, in which $a = 2$, $b = 3$, and $c = -50$. If n is supposed to be nearly the value of one of the roots, $x = \frac{2n^3+2n^2+50}{3n^2+4n+3}$, will be a value still nearer the truth.
>
> Now, the assumed value of $x = 3$ not being far from the true one, we shall suppose $n = 3$, which gives us $x = \frac{61}{21}$; and if we were to substitute this new value instead of n, we should find another still more exact.

NOTES

1. The list of references could go on forever. For a classical list of references, see (Artigue, 1991).

2. For a nuanced approach to the teaching of analysis, enriched on the basis of the historical knowledge of task designers, see (Hauchard & Schneider, 1996).

3. Free translation by R. Chorlay.

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Workshop

USING AUTHENTIC SOURCES IN TEACHING LOGISTIC GROWTH: A NARRATIVE DESIGN PERSPECTIVE

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In this paper, we present and discuss newly designed materials intended to teach logistic growth in Danish upper-secondary mathematics classes using an authentic source. The material was developed in conjunction with teaching a university-level course for future mathematics teachers and published in the form of a small booklet that introduces, contextualises, explains and discusses an original source by the Belgian mathematician Pierre François Verhulst (published 1838). The material offers multiple strategies for classroom implementation, ranging from a basic introduction to the topic to a full-scale teaching unit, as well as detailed suggestions for its cross-disciplinary integration with other subjects such as history, languages, art and social science. Our approach has shown potential for engaging students in enquiry-driven learning of both mathematics and its history.

HISTORY OF MATHEMATICS IN DANISH MATHEMATICS EDUCATION

Since the 1980s, history of mathematics has been required as an integrated part of the mathematics curriculum in the Danish upper-secondary schools. The arguments for teaching history of mathematics include its potential for showing mathematics as a human and creative activity, embedded in broader culture and responding to relevant cultural questions, rather than a fossilised or sterile discipline of formal reasoning and rote learning (Jankvist, 2009; Kjeldsen & Carter, 2014). Since a reform in 2004, cross-disciplinary collaboration between subjects have also been mandatory, and history of mathematics is a vehicle for integrating mathematics in its cultural and historical contexts.

With such aims in mind, the university courses in history (and philosophy) of mathematics offer opportunities for both becoming acquainted with the context and contents of important mathematical concepts and episodes, for initiating methodological reflections, and for stimulating identity-formation in students, whether they are future teachers or researchers.

For the history of mathematics courses at Aarhus University aimed at future teachers, we have identified four focal points (in addition to the learning objectives) that will aid the teacher in using history relevantly in mathematics teaching: tools for critical reading of i) primary and ii) secondary sources; and iii) historiographical reflections, in particular concerning iv) biographies of mathematicians.

In order to implement these goals, two challenges therefore arise for us: How do we (best) educate the teachers and how do we devise suitable material for classroom usage? Historians and educators of mathematics have addressed these two questions for decades (see e.g. Fried, 2001; Jahnke et al., 2002; Pengelley, 2011; for a discussion of the use of primary sources in mathematics education, see also Jankvist 2014). We have combined these mathematics-bound discussions with inspiration drawn from teaching Nature-of-Science (NOS) through a narrative-based approach (see, for example, Alchin 2012) as this approach proposes to lead students to independent enquiry of complex questions while providing relevance and identification through the narrative scaffolding. In this paper, we present our perspective on the design of teaching material using authentic sources and based on a situating narrative and a historical contextualization. Although the material has been tried and is being used, this paper does not purport to systematically analyse results from its actual class-room usage.

In the Danish upper-secondary mathematics curriculum, differential equations and the logistic model are core topics. We think that, ideally, successful historical material should be chosen among the core topics or in close relation to them. Such choices make it possible to simultaneously teach mathematics and history of mathematics. And although many topics are not as prone to historical contextualization, a fair number of good historical sources exist that can be used in teaching elementary geometry, trigonometry, function theory, proofs, etc.

LOGISTIC GROWTH THROUGH VERHULST' AUTHENTIC SOURCE

Based on our teaching of a university course in the history of mathematics aimed at future teachers, we have developed teaching material about logistic growth. Central to the material is a short authentic mathematical source, namely Pierre-François Verhulst's article from 1838 in which he introduced what is now called the "logistic curve" (Verhulst, 1838; translated into English in Vogel et al., 1975). Although quite short, this source invites discussions of four important aspects of the use of history in upper-secondary mathematics education: teaching a central mathematical topic, illustrating and discussing the mathematical modelling process, addressing issues of philosophy of mathematics, and informing the students about the historical complexity of mathematics.

Our material (Danielsen & Sørensen, 2014) includes a translation of the main source into Danish as well as a structured narrative providing the students with a window into the historical context and an understanding of Verhulst's thought process. The approach of working directly with the primary, authentic source forced us to situate it in a historical narrative that briefly introduces the main protagonists, their social and political contexts, and the mathematics of the early nineteenth century to the teachers and, eventually, to their students. Such a narrative approach is employed also to aid student identification with the problem at hand and make accessible the methods of solving it (see also Allchin, 2012).

After the translated source is reproduced *in extenso*, our material also contains a section-by-section elaboration of it intended to aid the teacher in acquiring the relevant information and perspectives. Along the way, numerous exercises interspersed with the sections of the source point to tasks to guide students in their engagement with the source. Finally, a number of broader perspectives are presented which lend themselves to either detailed mathematical treatment or to cross-disciplinary collaborations with other subjects in the upper-secondary school. While parts of the material are directly useful for students, the booklet as a whole is intended for teachers who should appropriate its content to their specific usage.

Verhulst's short text can be structured into four sections with different content and different perspectives on the source.

Part 1: Background of the problem (§1-4)

In the first paragraphs of the article, Verhulst introduced the background of his model by referring to Malthus' idea that populations grow exponentially. While this could be maintained if no restrictions were put on the growth, this was not realistic in comparison to most European societies where cultivation of the land had already been cultivated to near its capacity. Thus, a refined model would be required.

Concerning these passages, we can pose enquiry-enabling questions in the form of an exercise based on reading the source and using the internet to find relevant information:

1. The name of the author is Pierre-François Verhulst (1804–1849). What can you find out about him on the internet?
2. How is Thomas Robert Malthus (1766–1834) described in the source? What can you find out about him online?
3. What can you find out about the context in which the source was produced?
4. How is the influence of agriculture and cultivation on populations described?

Part 2: Outlining the model (§5-9)

In light of the insufficiency of exponential growth to model realistic populations, Verhulst in the following paragraphs set out to formalise a more refined model. He wanted to conceptualise the "hindrances" to the exponential growth, and he drew on an analogy with Quetelet's "social physics" which likened the hindrances to the resistance on the motion of a body falling through a liquid. Thus, Verhulst obtained assumptions that would ultimately underpin his mathematical model such as his realisation that populations have a capacity:

> All the formulas by which one may attempt to represent the law of population must therefore satisfy the condition of allowing for a maximum which will only be reached in an extremely remote era. This maximum will be the total of the population which tends to become stationary. (Translated from Verhulst 1838, 114-115)

He then went on to describe how he had not been completely successful in this endeavour since his assumptions and the available data were not sufficient to completely determine the model:

> I have tried for a long time to determine by analysis a probable law of population; but I have abandoned this type of research because the available data is too limited to allow the verification of my formula, so as to have no question about its accuracy. However, the course which I followed seemed to lead me to the understanding of the actual law, which when sufficient data becomes available, will support my speculations. (Translated from Verhulst 1838, 115).

As seen from these quotes, Verhulst's paper provides a rare snapshot of the modelling process in mid-action. In order to assist the students in engaging with this piece of text, we pose questions such as:

5. How is exponential growth described?
6. Population growth is compared to an example from physics. How?
7. Why do models of population growth require the inclusion of an upper limit?
8. Which problems have Verhulst had to overcome to build his model?

Part 3: The mathematical model (§10-11)

Verhulst proceeded to formulate a mathematical description of population growth which is equally interesting in its details. He set out from the case of exponential growth, $dp/dt=mp$, where m is a constant, and sought to "slow" it by an unknown function, thus yielding $dp/dt=mp-\varphi(p)$. He then proceeded:

> The simplest hypothesis that can be made on the form of the function φ is to suppose that $\varphi(p) = np^2$. (Translated from Verhulst 1838, 115)

And for this case, he could solve the equation by separation of variables and eventually deduce a formula for the population p as a function of time, t, from which also followed an expression for the upper limit of the population.

This is the most technical of the sections in Verhulst's paper, and depending on the mathematical level of the students, various approaches can be taken. Among the questions to be posed are:

9. Which equation is eventually used to describe population growth?
10. Can you rework the equation into a form that is recognizable and understandable from a modern, school perspective?
11. How is the solution formulated in the source?

12. Use a CAS tool to find a solution. What do you find?
13. Can you find a relevant formula in your textbook? What is the textbook's solution?

    Compare the various solutions obtained.
14. Which limit is contained in the formulas?

Part 4: Verhulst's discussion of the model (§12-16)

Finally, Verhulst provided some discussion of his own model. For instance, he noticed that he had tried various alternatives for the function φ but that these were all indistinguishable from the simplest one given the limited amount of data at his disposal. He appended to his articles four tables of populations in Belgium, France, Essex and Russia, and these all show demographic statistics was still in its infancy and data was scarce.

Concerning this part of the article, we pose questions such as:

15. Is the description of population growth unique?
16. In Verhulst' paper, he introduced four sets of data for populations in France, Belgium, Essex and Russia. These data can be downloaded from www.matematikhistorie.dk/logistisk-vaekst. How would you treat the data that Verhulst presented?

Working with authentic data is another requirement for upper-secondary mathematics education and this can be achieved either by using e.g. Verhulst's historic data or by trying to apply logistic growth to e.g. the population census of Denmark during the twentieth century.

META-PERSPECTIVES GAINED FROM THE AUTHENTIC SOURCE

The material has been tested in different classes in different modules. In the design, it was important to allow for flexibility in the scope: the historical perspective could be used to briefly introduce logistic growth, or used throughout as a means of teaching logistic growth, or as the nexus for various collaborations with other subjects. These experiences have all been positive, and we are still receiving feedback, which we look forward to analysing, in particular concerning the full-scale student project in history and mathematics.

Discussions about mathematical models

Mathematical models and the ability to critically assess them are key topics in mathematics classrooms. By using historical sources such as Verhulst's paper, the authentic modelling process with its incompleteness and contingent choices become visible to the students. In particular, Verhulst's aesthetic and pragmatic choices in building the model are otherwise difficult to illustrate. Thus, students get to "look into the workshop" of the mathematical modelling process, and what they see may nuance the picture often presented in textbooks of a finished, fossilised model. The fact that historical sources can help teachers identify what Anna Sfard denotes *commognitive conflicts* (Sfard 2010) has also been remarked in (Kjeldsen & Petersen, 2014) where it is shown how exposure to an authentic historical source brought to light some key problems about mathematical symbols that were otherwise invisible to the teacher. Thus, when students are presented with Verhulst's original notation and wording, they will have to come to accept, for instance, that his p' is *not* the derivative of p but rather the population at $t=0$. In performing such translations – which are not often present in stream-lined textbooks – their understanding of the conceptual level might well be revealed. And when they confront the contingencies present in the authentic source with the often sterilized product found in textbooks, they can be led to a deeper understanding of the processes and practices involved in mathematical modelling and, more generally, in mathematical research (for more on the role of history in teaching mathematical modelling, see also Kjeldsen & Blomhøj, 2013).

In the source, Verhulst provided remarks on the process of building his model and on the factors that influenced it. In an exercise, we put these into perspective and discuss the modelling process and the status of the model in relation to data:

1. First outline the model of mathematical modelling as presented in figure 1. Be sure to explain how the mathematical model is related to reality by focusing on how the model is built and what kinds of statements about reality that it warrants.
2. Now analyse Verhulst's source to find those places in it where he explains or comments upon his model. What does he, for instance, say about the relation between the model and the actual population sizes? Again, you should focus on the construction of the model and the types of warrant that it provides.
3. Discuss the perspective on the process of building mathematical models which you have achieved in the first two tasks by discussing the types of knowledge that mathematical models can provide of the real world.

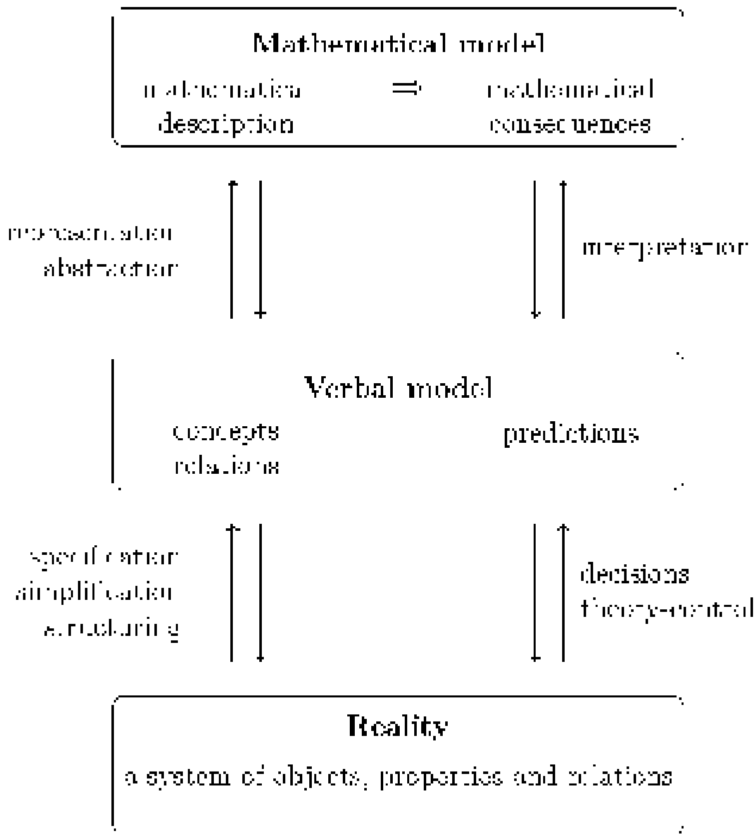

Figure 1: A simple model of the mathematical modelling process. The process starts by delineating a section of “reality” to be modelled. The arrows then indicate dialectic processes, and the entire modelling process is also to be considered iterative (see also Johansen & Sørensen 2014, chap. 10).

Illustrating historical complexity and context

The material is designed to include sufficient historical context to aid the teacher in situating the source. And the source is given in translation quite close to the original in order to allow the teacher to address historical complexities such as differences in notation or conceptions about mathematical notions and objects. These are, we believe, very important aspects that history can bring to the mathematics classroom. Students will thus be challenged – and eventually benefit – from having to appropriate knowledge from the source into their own mathematical frameworks; Elaborating on the commognitive conflicts alluded to above, we hope that in reappropriating the source into their modern framework, students are led to both an appreciation of the historical development and a richer understanding of the modern theory. Obviously, since logistic growth is a core component of the curriculum and can be included in the written exams, students must also be comfortable with such translations to the extent that they can solve relevant exam questions about the topic. If the historical source is indeed used as the focal point of teaching logistic growth, these translations will have to be explicitly stressed in order to achieve the abilities at problem-solving required for the exam.

In order to address the contextual aspects of the authentic mathematical source, we also pose exercises that can either be discussed directly in the mathematics class or can lead to integration with history, for instance in the form of individual reports which the students are required to write in the final, third year.

This is a good opportunity to explain that any mathematical creation is formed in a specific context, and it is often very important to know about this context in order to understand and explain the mathematics created, its purpose and its relevance. In the case of Verhulst's modelling, some of the relevant factors can be probed based on the following guiding tasks and questions:

1. Start by comparing two maps of Europe from, say, 1800 and 1900 and describe the changes in nations and their borders that you see.
2. Search the internet for information on the so-called "Congress of Vienna" of 1814-1815 and describe in more detail the changes to the nations of Denmark and The Netherlands.
3. Several European nations introduced new constitutions in the decades between 1830 and 1850: Belgium in 1831 and Denmark in 1848. The Danish constitution was partly modelled on the Belgium one. Analyse the constitutional rights of individual citizens granted by the Danish constitution of 1848 (chapter IV). Who were granted such rights? What shift of democratic power did this entail?
4. The new states were to a larger extent formed by (emerging) criteria of national belonging. Describe differences between the absolutist state and the nation state by focusing on the demarcation of the state.

The many new European states had a need for forming their own identity. This was partly a conscious process and partly the background for the emergence of new states in the first place.

5. Analyse some Danish national romantic works (poems and paintings) from the period 1830-1850, focusing on their role in forming a national identity. You should include knowledge from other subjects (Danish and history) in analysing the context and interpretation of these works.

An important part of the formation of new states consisted in the institutional development of a bureaucratic system for supporting parliamentarian and democratic decision processes.

6. The Danish Statistical Bureau was formed in 1849 as the precursor of the modern *Danmarks Statistik*. What can you find out about this bureau and the background for its establishment?

7. The geographic survey of Denmark was begun under the auspices of the Royal Danish Academy of Sciences and Letters towards the end of the eighteenth century. What can you find out about this project and the mathematicians who were involved in it?

Integration with other subjects

In the Danish upper-secondary school system, integration of different subjects is very important, both in the day-to-day teaching and in special projects. Thus, it becomes a special challenge to provide material suitable for the itegration of mathematics with , for instance, history. The above exercise on the context of Verhulst' model provides one example of how this can be accomplished, but other possibilities also exist based on different views of how to integrate the two topics.

Thus, whereas the content of the above exercise was essentially to situate the mathematical source in its relevant context, it is also possible to use the mathematical source – and the ideas that it entail – to situate discussions about demographics, statistics, and the governing of states and people through quantification. Although this is perhaps further from the core mathematics curriculum, it integrates well with a variety of other subjects in the upper-secondary school, including history and Danish, of course, but also spanning classical culture, social science, art, and languages. Among the topics suggested, we mention:

A central element of Verhulst's modelling is the assumption that social and human relations are regulated by laws and can be studied in the same way as (natural) science studies Nature.

1. The conception of a "social physics" as the study of social relations subjected to laws analogous to the laws of nature goes back to the scientists and philosophers Adolphe Quetelet (1796-1874) and Auguste Comte (1798-1857). What does Quetetelet mean when he in *Sur l'homme et le développement de ses facultes, ou essai de physique sociale* (1835) talks about "the average man"? What else can you find out about the social physics of Quetelet and Comte?

Human beings and their properties have been quantified in many different ways over time. One of the purposes of associating numbers to individuals has been to define what it means to be 'normal' – and sometimes 'ideal'.

2. In the arts (paintings and in particular sculptures), human proportions have been associated with numbers in order to describe 'the ideal body'. Analyse selected works by e.g. Vitruvius (ca.75BC-ca.25BC), Leonardo Da Vinci (1452–1519) or Le Corbusier (1887-1965), focusing on their description and use of ideal human proportions.
3. Attempts have also been made to quantify non-physical aspects of human beings like personality, intelligence or crime. This became most explicit in the

pseudo-science phrenology which was taken quite seriously in the nineteenth century. What can you find out about phrenology?

4. In order to illustrate how quantification can be used to define normality, do some research on the notion of 'body mass index' which was first suggested by Quetelet.

Thus far, students have been asked to explore some applications of quantification in social and human domains. These are not value-neutral as the following questions explore:

5. Present Malthus' political and philosophical position which is the foundation for his analysis of population growth. Which views of revolution and democracy are behind his position?
6. The quantification of individuals is of course an important part of democratic government in which all votes are equal (with some provision for representative democracy). But it is not (and has not been) unproblematic to decide which votes and voices are to be counted. Analyse selected democracy-critical thinkers with special attention given to their presentation of the quantification of individuals.
7. The quantification of individuals can also lead to alienation and the loss of individuality and identity. Analyse selected works (fiction and movies) for their presentation of this form of alienation through quantification. Examples could include Dickens' novel *Hard Times* (1854) or episode of the TV-series "The Prisoner" (1967). Sometimes, such alienations are related to certain political doctrines, so discussions should also include the previous task.

CONCLUSIONS AND PERSPECTIVES

In this paper, we have outlined the contents and design of our material for teaching logistic growth in Danish upper-secondary mathematics classes based on the authentic source of Verhulst's 1838-paper. Although short, the paper allows for a variety of perspectives and avenues for students to follow in their engagement with the source: Sometimes, they will have to search for factual information about the context of the source; sometimes, they will have to work with the mathematical description to bridge the gap between the historical presentation and contemporary school mathematics; sometimes, they will be hard-pressed to understand Verhulst's ideas about the modelling process and the relation between his "laws" governing population growth and the actual statistical data; and sometimes, they will be asked to consider much broader questions about the interactions and relations between mathematics and the social and cultural context in which it is produced and used. All these perspectives (and more) belong to what we consider relevant mathematics education in the upper-secondary level: they all contribute to understanding the processes that led to technical advances in mathematics as well as its applications to relevant problems, and if historically contextualised, they also contribute to showing how mathematics (and quantification) came to be such a dominant approach to understanding modern society. All this can be approached through unwrapping of a single historical source, as we show with our material.

We conclude with several remarks concerning the design of our materials. First, because of the open-endedness of the student-driven, enquiry-based approach and the vast variety of issues that can be taken up, the material had to be designed in an open way that invites and helps teachers to appropriate it for their specific needs. This was achieved through the use of narrative contextualisation and explanatory sections that provide pieces of information, pointers, and perspectives for the teacher to use. Second, the historiographical approach has been to contextualise and show the complexity of a *single* historical source, rather than teaching history through episodic sources connected by some theme. This is deliberate, as we both found that a well-chosen single source can allow for a richer contextualisation and that longitudinal connections are often harder to teach without resorting to simplified connections.

Finally, we have reasons to believe that this approach can be very successful in stimulating mathematical as well as historical competences in the students. When set to work with authentic mathematical texts, students are required to use both linguistic and mathematical translations that activate their representational competences. And, although the case of Verhulst's paper is ideal in many respects, because it is short, addresses a core topic in the curriculum, and opens for such vast philosophical and historical perspectives that are also on the curriculum, we believe it is not unique. We are currently working with other projects to produce other materials by the same method; and we have come to believe that although it is challenging to find the relevant sources and prepare them for teachers to use, it is also worthwhile and certainly in demand in the Danish system, where teaching history of mathematics is mandatory but where teachers complain about the lack of suited materials and about their own insecurity in teaching what is still often considered a different and difficult subject.

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Workshop

YES, I DO USE HISTORY OF MATHEMATICS IN MY CLASS

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In this article I report on the two parts of the workshop I carried out during ESU7. The first one was aimed at gathering and discussing the opinions of the participants to the conference about the use of history in mathematics classroom. During the workshop, Man Keung Siu's article "No, I don't use history of mathematics in my class. Why?" has been illustrated. People who did not participate in the workshop could post their answers into a specific BOX standing in the hall of the conference site, during all the five ESU7 days. The main outcome was that history is believed to be a resource for students to improve their way to work in mathematics.

In the second part of the workshop, some documents from the work of Italian mathematicians Francesco Ghaligai and Rafael Bombelli were provided to participants who were asked to analyse the originals and to plan some activities, for example laboratories or short exercises, for students aged 14-16. Discussion regarded both historical and pedagogical remarks.

PREFACE

At the beginning of the conference until the end, a BOX and a poster were put in the main hall of the conference. The BOX was aimed at collecting the ESU7 participants' answers to the questions proposed by the following poster, which was hung up aside:

TEN YEARS AGO:

"No, I don't use history of mathematics in my class. Why?" - Observation and thought of school teachers collected by Man Keung Siu (2006)

[The content of column (*) of the table that follows was also reported here]

NOW:

WHY DO WE USE, OR SHOULD WE USE,

THE HISTORY OF MATHEMATICS IN CLASS?

Please, post your answers here

[Part of these answers is available in the Table 1, column #**]**

Fig. 1: The poster standing in the main hall

INTRODUCTION

Before analyzing some excerpts from the participants' answers that tackle the "observation and thought of school teachers", it is meaningful to read what Siu says about the reasons for using the history of mathematics in the class:

"First of all, the basic tenet I hold is that mathematics is part of culture, not just a tool, no matter how useful this tool might prove to be. As such, the history of its development and its many relationships to other human endeavors from ancient to modern times should be part of the subject. Secondly, through my own experience in teaching and learning I have found that knowledge of the history of mathematics has helped me to gain a deeper understanding and to improve my teaching. Now, integrating the history of mathematics with teaching is only one of many ways to do this. Anything which makes students understand mathematics better and makes students get interested in mathematics may be a good way. The history of mathematics may not be the most effective choice, but I believe that, wielded appropriately, it can be an effective means." (Siu, 2014, pp. 27-48). Some articles quoted in references in the same work discuss reasons to use history of mathematics in class (for example: Jankvist 2009, Pengelley 2011; see also: Haverhals and Roscoe 2012).

Inspired by (Siu 2006), Tzanakis (2008; see also: Tzanakis & Thomaidis, 2012) proceeded to classify/structure the various objections by grouping together arguments as follows (see numbers in column (*) of table 1):

A. Objections of an epistemological – philosophical character

A.1: Related to the nature of mathematics: No 2, 9, 13;

A.2: Related to difficulties inherent in the attempt to integrate history in mathematics education: 14, 15, and "Students may have an erratic historical sense of the past, which makes historical contextualization of mathematics impossible without their having a broader education in general history" (Fauvel, 1991).

B. Objections of a practical and didactical character

B.1: Related to teachers' background and attitude: 1, 10, 11, 12;

B.2: Related to difficulties in assessing the impact of a historical dimension in mathematics education: 3, 4 plus 16 (which could merge into one argument);

B.3: Related to students' background and attitude: 5, 6, 7, 8.

Furinghetti (2012) has grouped the objections under these main points:

integration (1, 2, 3, 4);

cultural understanding (5, 6, 7, 8);

looking for meaning (9, 13, 14, 15);

teacher training (10, 11, 12).

Integration refers to the fact that history has not to be considered an additional subject, but it has to be embedded in the teaching. *Cultural understanding* refers to a way of looking at mathematics as a vivid matter embedded in the socio-cultural process. *Looking for meaning* has to be one of the aims of mathematics teaching and learning. *Teacher education* gains from the point of view of cultural understanding and of fostering pedagogical reflection.

PARTICIPANTS' ANSWERS

The following table shows the "sixteen unfavorable factors" collected by Siu and some excerpts from the answers posted during ESU7 (in total, four snippets whose length varies from three to 30 lines).

(*) A list of sixteen unfavorable factors	**(#)** To tackle the "sixteen unfavourable factors" From the participants' answers
1. I have no time for it in class! 2. This is not mathematics! 3. How can you set question on it in a test? 4. It can't improve the student's grade! 5. Students don't like it! 6. Students regard it as history and they hate history class. 7. Students regard it just as boring as the subject mathematics itself! 8. Students do not have enough general knowledge on culture to appreciate it! 9. Progress in mathematics is to make difficult problems routine, so why bother to look back? 10. There is a lack of resource material on it! 11. There is a lack of teacher training in it! 12. I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition? 13. What really happened can be rather tortuous. Telling it as it was can confuse rather than to enlighten! 14. Does it really help to read original texts, which is a very difficult task? 15. Is it liable to breed cultural chauvinism and parochial nationalism? 16. Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?	I. I start teaching probability with the problem of dividing the bet if the game is interrupted. Success is sure. I can find in old textbook a lot of interesting tasks. II. Studying history of maths can open up a student's mind to new ways of thinking about and solving problems. III. Using historical problems may bring even fun to the lectures. Humour and poetry can be brought through the stories from the life of scientists (Archimedes run naked through Syracuse, Newton was hit by a falling apple...). IV. I use history of mathematics in my class because it allows to work in interdisciplinarity. V. I agree that such an integration is helpful to the attainment of the so-called "3-D teaching objectives", namely, knowledge and skill, process and method, affective attitude and value judgement. VI. In my opinion, history of specific topics is a necessary part of the whole subject (not only in mathematics): teacher who does not know history is not a good teacher. VII. History makes more sense of curriculum contents, as well. Many scientists were international several centuries ago. Was Euler Swiss or Russian? Kepler was German but he stated his theorems in Prague... If we use the historical approach, we can explain to the students why was the formula (or method etc.) developed. Example: using historical tasks to solve quadratic equations students understand that the formula does not fall from the sky.

Table 1.

Many of the "sixteen unfavorable factors" constitute real objections participants could, at least partially or indirectly, contribute to refute. Here I propose a correspondence of them and the seven excerpts from participants' answers, accompanied by my synthetic interpretations.

1. I. (Problem solving is part of school mathematics) II. (History of mathematics can facilitate students' reasoning)
2. II. (By means of history, students do mathematics) V. (Attitudes and values have to be taken into account for a better understanding of mathematics)
3. VII. (See the last part of the excerpt as an example)
4. II. and V. (History creates premise to improve the grade) VI. (Teacher's competence helps students' achievement)
5. III. ("Humor and poetry" with history) I. (Students like to be successful in problem solving!)
6. III. (History of mathematics can be a lovable part of general history)
7. IV. (Students can not find only "the subject mathematics itself" in history) I. (History of mathematics offers various opportunities to involve different students) III. (Mathematics in history gives more resources then pure mathematics)
8. VII. (Students appreciate that mathematics can acquire more sense) IV. (History of mathematics is culture)
9. II. (Students who believe that mathematics is "routine" do not have an open mind)
10. I. (Ancient texts, such as their modern revisions, are rich of ideas for mathematical activities)
11. VI. and I. (Teachers can autonomously find material in ancient texts)
12. I. (Originals are the best source, not only for researchers but also for teachers)
13. V. (History clarifies the aspects of mathematics which is an intrinsically complex science) II. (By means of history, students get instruments for mathematical thought)
14. I. (Finding problems in the originals is not an impossible task)
15. III. and VII. (Mathematics is international)
16. I. (Teachers who use history in class can say 'yes')

STUDENTS' OPINIONS

Table 1 reports some of teachers' points of view. During the workshop we also briefly discussed students' opinions about the history of mathematics in class. Comparing some hints from Hong-Kong and Italy we have agreed that students who usually get high assessment levels in mathematics tend to criticize the use of history. Here I report the opinions of a female student, Anna. Her level of competence in mathematics was good but not always satisfactory in the other school subjects. Anna and her classmates were requested to write a report (at least 10 lines) imagining to talk about the use of history in math classes to a friend of another school. In brackets, you can see a suggestion of correspondence to some points of the "list of sixteen unfavorable factors".

"HISTORICAL COURSE IN MATHEMATICS

About the program implemented this year in mathematics, I believe that all the mathematical-historical course we learned is not useful [No 4] because I do not see it as mathematics [2] and, considering that it is quite a difficult subject [9], if I have something to suggest to the teacher, in the years to come he should try to engage in real mathematics, that is exercises with expressions, equations etc., everything in a mathematical way and without inserting history because if we have to make reckonings, which I believe it is a fundamental thing for everyday life, neither history nor those problems with strange words inside will remain [8, 14].

Happily enough, past years my teacher was much engaged in his work and he taught us the real mathematics [2], and I believe that this must be done at school; then, if you are interested in something regarding ancient mathematics, everybody can do it by oneself using a computer or something else.

Numbers learned this year are:

EGYPTIANS, BABYLONIANS, MAYANS [capitals in the original].

May be, it is because I do not get on well with history [5, 6], in contrast to another person who likes the history and could find it funny".

Another female student, Anny, hardly got a positive level of competence in mathematics. She answered the same request.

"This year, at school we learnt mathematics, applying history to it. In my opinion, it is a thing which is useful in order to understand the bases and is interesting, too, but also a little difficult because I was not always able to apply those rules to problems of today. We have learnt methods that have been discovered by famous mathematicians like Bombelli and Fibonacci, ancient problems and equations and also the number systems of Egyptians, Babylonians and Maya. As I said before, it is a very useful but also a bit difficult and I was not able to understand why they studied the things of the past but not ours".

I consider the last line as the most obscure in Anny's snippet. Anyway, I like to interpret her remark as a consequence of her personal work with historical documents. She looks like having found more difficult working with - I believe - Mayas numbers than with ours. I consider her naive historical question a consequence of *replacement* caused by using original documents (Jahnke et al., 2000).

MATHEMATICS WITH GHALIGAI AND BOMBELLI

I think that some teachers' beliefs on the use of history could be modified if the problem of identifying materials suitable to work in the classroom is weakened. For this reason I complemented the workshop on the theoretical discussion with the presentation of materials that can be used in the classroom. Then, the attendants the field what the use of history implies.

Ghaligai's *Pratica d'Arithmetica* was first published in 1521. The author does not preface his treatise with an autobiographical introduction, in fact after the frontispiece we only know that he was a Florentine.

Fig. 1: Ghalighai, *Pratica d'Arithmetica*, frontispiece

Like other medieval or renaissance manuals, its audience was merchants. The treatise is divided into thirteen books. The last four are devoted to algebra, which includes explanations of methods for the extraction of roots. It also includes operations with binomials regarding radicals, as found in other 15^{th} and 16^{th} century's works. Ghaligai quotes from other authors' writings, for instance Euclid, Fibonacci and Pacioli. In regards to the symbols he introduces, he says they belong to "Giovanni del Sodo who uses them for his algebra".

Rafael Bombelli (Bologna 1526 – Rome 1572) in his ***L'Algebra***, gives an account of the algebra known at the time. The treatise is divided into three books and includes his contributions to complex number theory.

Bombelli's *L'Algebra* was intended to be in five books. The first three were published in 1572. Unfortunately Bombelli died shortly after the publication of the first three volumes. In 1923, however, Bombelli's manuscript was discovered in a library in Bologna by the Italian mathematician and historian Ettore Bortolotti. As well as a manuscript version of the three published books, there was the unfinished manuscript of the other two books. Bortolotti published the incomplete geometrical part of Bombelli's work in 1929 (http://www-history.mcs.st-andrews.ac.uk).

In the preface "to the readers" Bombelli makes mention of al-Kwarizmi, Diophantus and Pacioli. Leibniz praised Bombelli as an "outstanding master of the analytical art".

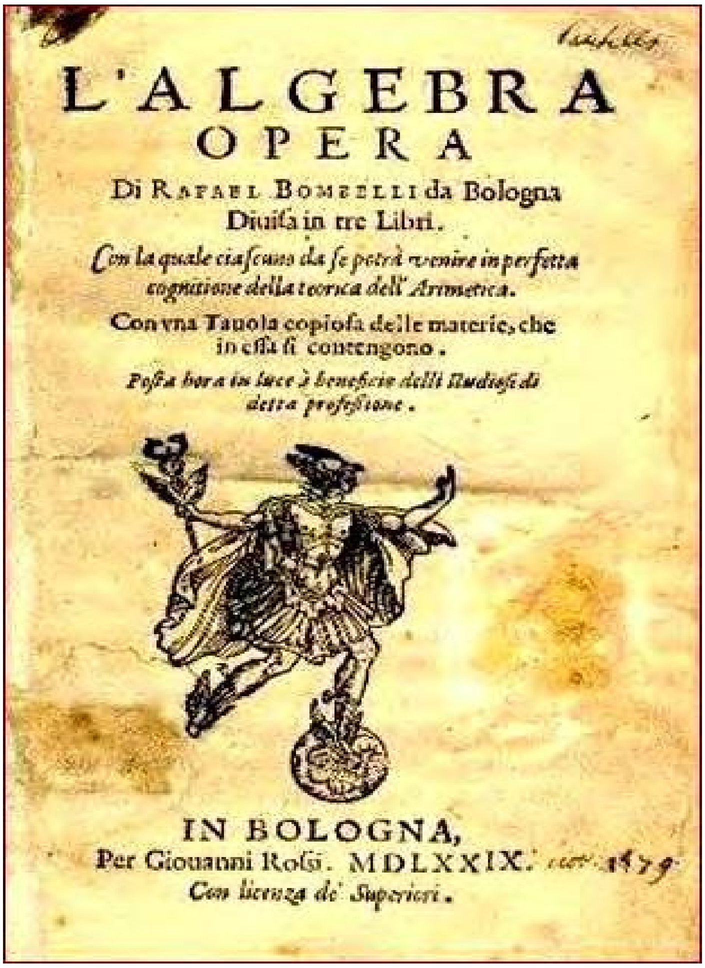
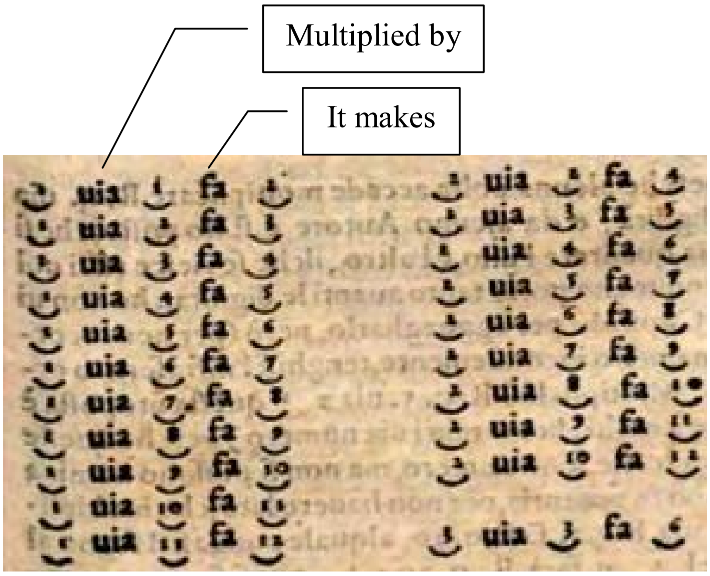


Fig. 2: Bombelli's *L'Algebra*, frontispiece

WORKSHOP MATERIALS

From Bombelli's *L'Algebra.* Bombelli's documents that were proposed for the workshop aimed to give examples of addition, subtraction and multiplication of polynomials. Opening examples regarding powers have to be considered prerequisites (figure 3).



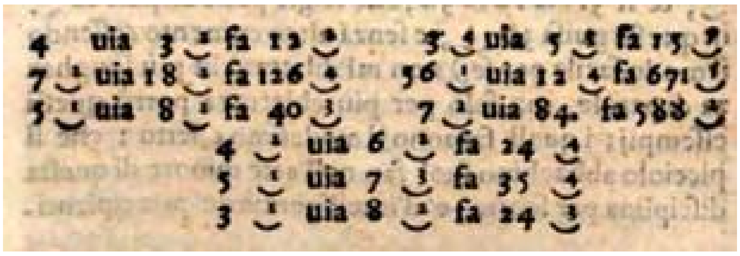


Fig. 3: *L'Algebra,* Book second, 205, examples regarding powers
Fig. 4: *L'Algebra,* Book second, 206, examples regarding monomials

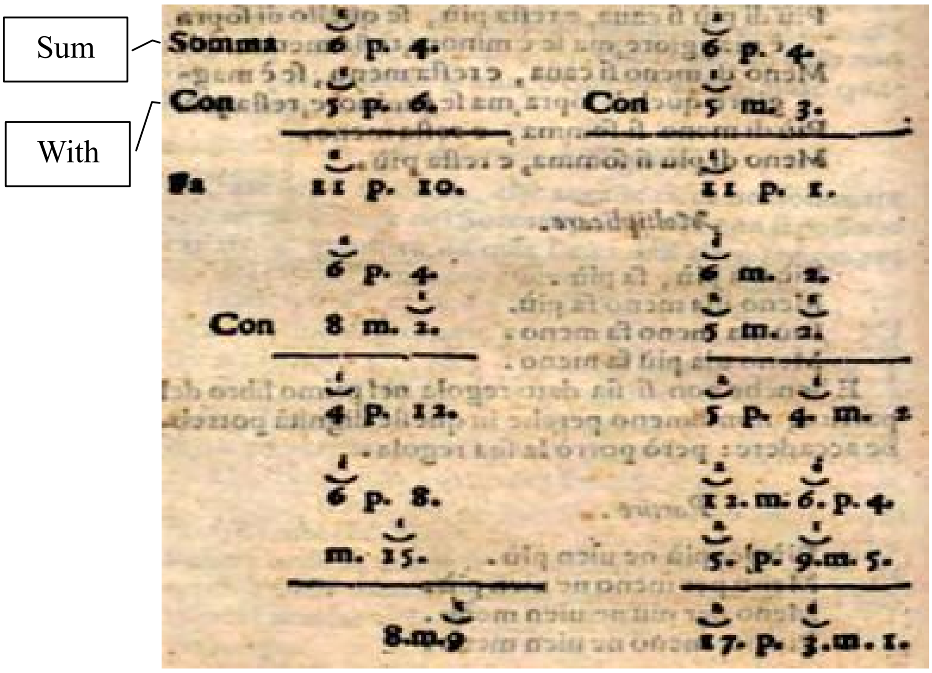


Fig. 5: *L'Algebra,* Book second, 212, examples regarding sums

Fig. 6: ***L'Algebra,*** **Book second, 213, examples regarding subtractions**

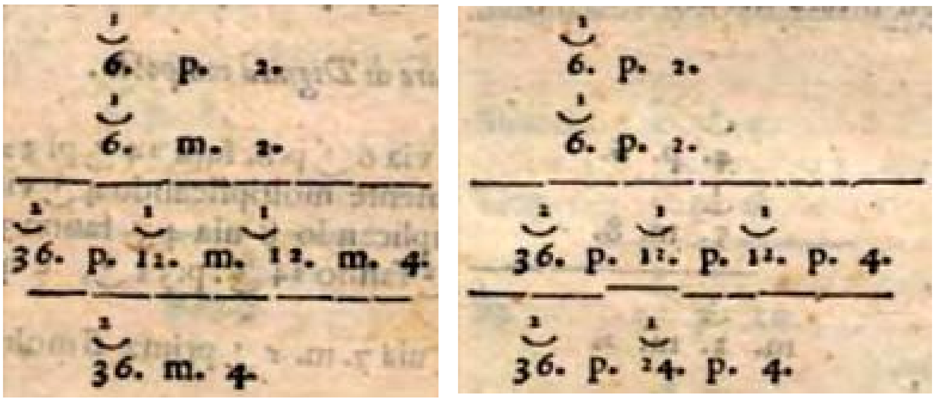


Fig. 7: ***L'Algebra*****, Book second, 214, examples regarding multiplications**

This material from *L'Algebra* was previously used in class, in an Italian Liceo delle Scienze umane (Human Sciences Lyceum, 9th grade, with 14-15 aged students). In this experiment we observed an outcome similar to that regarding the interpretation of a Pacioli document, illustrated in (Demattè, to appear). In this case students often get stuck when they were not able to read some symbols, for example the exponents written in small dimension, seen in the column on the left, in the first document from *L'Algebra,* presented in Fig. 3. Note that they are widespread in the same document so students could get an opportunity to infer the right value reading all the original text (*hermeneutic circle*). In that situation they often asked the teacher. This shortcut suggests some reflections regarding which previous educational circumstances made this choice more preferable for students, instead of insisting on finding the correct answer by themselves. This issue is particularly relevant considering that, among the possible answers, one regards aspects closely inherent to mathematical reasoning. In fact, when students conjectured the right answer, they were in a situation of uncertainty: they could foresee it or not. The choice between these two alternatives was left to them. Asking the teacher allowed students to avoid any risks with respect to the teacher's approval. The conclusion is quite disconsolate: necessity to have an authority who masters education (the teacher) reinforces performances that are different to those which qualify mathematical reasoning, for example: pleasure in personal endeavour, conjecturing and checking conjectures.

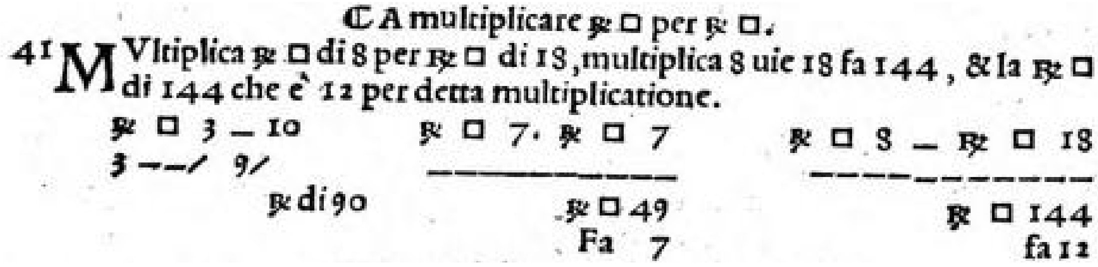
The students' task analysed here has to be considered rather easy because it does not require a wide amount of prior mathematical knowledge. This suggests that among different reasons for getting stuck, their lack of self confidence in mathematics should not be considered. I prefer to address the problem of how a teacher has to act in order to improve students' high level competencies without losing their volition to learn.

I would like to compare the situation regarding the document in Fig. 3 with the situation in which students were requested to discover the operation Bombelli used in the calculation shown in materials “For participants’ working groups-2.a” (see next paragraph). Also in this circumstance, students had to examine some possible solutions, specifically different operations: addition, subtraction and multiplication, without others, considering what they had previously learnt. Why did not they conjecture which one would work? Beyond the answer regarding the teacher's role, I suppose that this lack of disposability to conjecture could have a social origin, considering that in everyday life a false statement could produce damage to personal prestige or standing (in fact I believe that most of us would usually not knowingly risk to say something which could be wrong).

From Ghaligai’s *Pratica d’Arithmetica.* Ghaligai's documents was introduced in the workshop as proposals to analyse in order to plan activities for the class. The whole *Pratica d'Arithmetica* was at disposal in a .pdf file (in Italian).

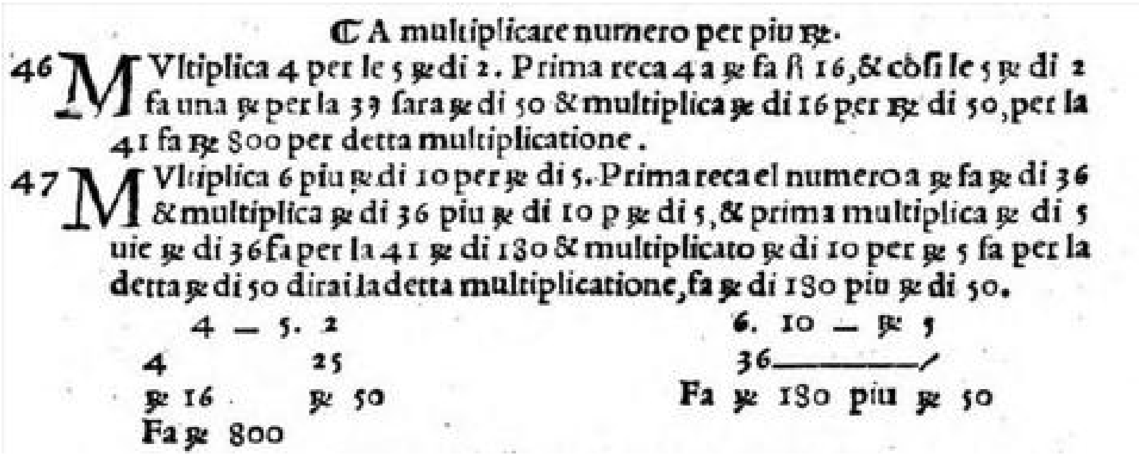
The first part “LE FIGVRE” [the figures] was considered self explanatory, so no translation was given, apart from “di” means “of”. The main idea about possible use of this part in class was that Ghaligai's symbols show an alternative way to represent powers and to work with their properties. I believe that also the fact that there are specific symbols for the powers having a prime number as exponent (“censo, cubo, relato, pronico, tromico, dromico”) could be profitable for class. The other short Chapters introduce a brief itinerary about operation with radicals (fig. 9 and fig. 10).

Fig. 8: Francesco Ghaligai’s *Pratica d’Arithmetica,* Book tenth, folio 72.


℄ A multiplicare ℞ □ per ℞ □.

41 MVltiplica ℞ □ di 8 per ℞ □ di 18, multiplica 8 uie 18 fa 144, & la ℞ □ di 144 che è 12 per detta multiplicatione.

Fig. 9: ***Pratica d'Arithmetica,*** **Book tenth, folio 75. "**C About multiplying square roots by square roots. Multiply the square root of 8 by the square root of 18; multiply 8 by 18: it is 144, and the square root of 144 is 12 for that multiplication".


℄ A multiplicare numero per piu ℞.

46 MVltiplica 4 per le 5 ℞ di 2. Prima reca 4 a ℞ fa ℞ 16, & cosi le 5 ℞ di 2 fa una ℞ per la 39 sara ℞ di 50 & multiplica ℞ di 16 per ℞ di 50, per la 41 fa ℞ 800 per detta multiplicatione.

47 MVltiplica 6 piu ℞ di 10 per ℞ di 5. Prima reca el numero a ℞ fa ℞ di 36 & multiplica ℞ di 36 piu ℞ di 10 p ℞ di 5, & prima multiplica ℞ di 5 uie ℞ di 36 fa per la 41 ℞ di 180 & multiplicato ℞ di 10 per ℞ 5 fa per la detta ℞ di 50 dirai la detta multiplicatione, fa ℞ di 180 piu ℞ di 50.

Fig. 10: ***Pratica d'Arithmetica,*** **Book tenth, folio 75.** "C About multiplying a number by several square roots. Multiply 4 by the 5 square roots of 2. First of all, bring 4 to square root: it is square root of 16, such as the 5 square roots of 2 is one square root for the [proposition] 39 it will be square root of 50, and multiply square root of 16 by square root of 50, for the [proposition] 41 it is square root of 800 […]".

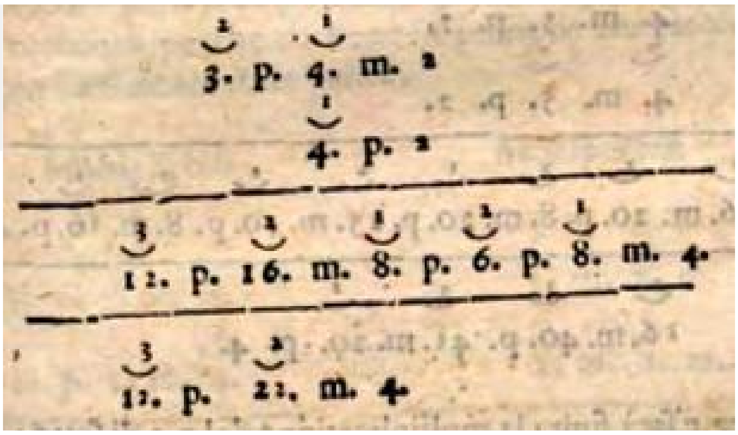
Worksheet for participants' working groups

Among the following questions/activities, feel free to choose and discuss those you prefer.

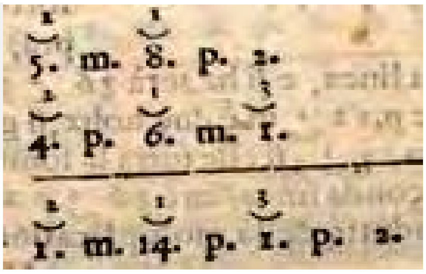
1. Compare Ghaligai's and Bombelli's notation. Why would students have to analyze it, that is: why could it be useful for students? What prerequisites do students need to interpret Ghaligai's and Bombelli's originals?

2. Two exercises from Bombelli's original (hermeneutic approach):

a) Write the following multiplication using modern symbols.



b) Which operation does Bombelli use in this calculation? Explain your answer.



Write other exercises.

3. I guess that Ghaligai's symbols could help students to tackle some difficulties. For example: using symbols with reference to formal properties and being able to control the meaning of symbols. Agreement/disagreement and why.

4. Working with Ghaligai's or Bombelli's documents, what kind of students' capabilities could the hermeneutic approach improve?

5. Write proposals of laboratories using Ghaligai's, Bombelli's or other authors' algebraic works (main points, keywords, activities, resources etc.).

Transcription of the previous documents in modern symbols

From Bombelli's *L'Algebra* (fig. 3, fig. 4, fig. 6, fig. 5, fig. 7):

205 $x \cdot x = x^2$ $\quad x^2 \cdot x^2 = x^4$

$x \cdot x^2 = x^3$ $\quad x^2 \cdot x^3 = x^5$

$x \cdot x^3 = x^4$ $\quad x^2 \cdot x^4 = x^6$

$x \cdot x^4 = x^5$ $\quad x^2 \cdot x^5 = x^7$

[...] [...]

206 $4 \cdot 3x^2 = 12x^2$ $\quad 3x^4 \cdot 5x^5 = 15x^9$

$7x^2 \cdot 18x^2 = 126x^4$ $\quad 56x \cdot 12x^4 = 671x^5$

$5x \cdot 8x^2 = 40x^3$ $\quad 7x^2 \cdot 84 = 588x^2$

$4x^2 \cdot 6x^2 = 24x^4$

$5x \cdot 7x^3 = 35x^4$

$3x \cdot 8x^2 = 24x^3$

212 $(6x + 4) + (5x + 6) = 11x + 10$ $(6x + 4) + (5x - 3) = 11x + 1$

$(6x + 4) + (8 - 2x) = 4x + 12$ $(6x - 2) + (5x^2 - 2x) = 5x^2 + 4x - 2$

$(6x + 8) + (-15x) = 8 - 9x$ $(12x^2-6x+4)+(5x^2+9x-5) = 17x^2+3x-1$

213 $(4x + 6) - (2x + 5) = 2x + 1$ $(4x + 6) - (5x - 8) = -x - 2$

214 $(6x + 2) \cdot (6x - 2) = 36x^2 + 12x - 12x - 4 = 36x^2 - 4$

$(6x + 2) \cdot (6x + 2) = 36x^2 + 12x + 12x + 4 = 36x^2 + 24x + 4$

From the material "For participants' working groups, 2 a) and 2 b)":

215 $(3x^2 + 4x - 2) \cdot (4x+2) = 12x^3 + 16x^2 - 8x + 6x^2 + 8x - 4 = 12x^3 + 22x^2 - 4$

213 Subtraction: $(5x^2 - 8x + 2) - (4x^2 + 6x - x^3) = x^2 - 14x + x^3 + 2$

From Ghaligai's *Pratica d'Arithmetica* (fig. 9, fig. 10):

f.75 **41** $3\sqrt{10} = \sqrt{9}\,\sqrt{10} = \sqrt{90}$ $\sqrt{7}\,\sqrt{7} = \sqrt{49} = 7$ $\sqrt{8}\,\sqrt{18} = \sqrt{144} = 12$

46 $4 \cdot 5\,\sqrt{2} = \sqrt{16}\,\sqrt{25}\,\sqrt{2} = \sqrt{16}\,\sqrt{50} = \sqrt{800}$

47 $(6 + \sqrt{10})\sqrt{5} = (\sqrt{36} + \sqrt{10})\sqrt{5} = \sqrt{180} + \sqrt{50}$

DISCUSSION DURING THE WORKSHOP

Participants worked above all on the interpretation of the documents. I collected their opinions which were homogeneous about the fact that Ghaligai's documents would not be easy for students. Some participants were oriented to sketch a proposal of activities for the class after ESU7 Conference. They also found a good opportunity for students to reflect on a calculation in a historical document and to discover the operation used. This task was considered quite meaningful because 15th and 16th century originals do not yet use our modern symbols for arithmetical operations. On the contrary, somebody else believed that it is more profitable to use historical problems instead of introducing different symbolisms in class.

During the presentation, a participant posed the following question: Why did Bombelli write in Italian, unlike Cardano, for instance, who used Latin, the language of science at that time? We shared the hypothesis that the answer has to be looked for in their biographies: the former lived in Italy (was born and dead in the Papal States), the latter travelled to Germany, France, Scotland and England. Despite this, we observed similarities between Bombelli's and Stevin's symbolism about the manner to write powers.

Fig. 11: From Calandri's *Arithmetic*

HISTORICAL AND PEDAGOGICAL REMARKS

I would like to focus on the schemes that appear at the end of each text in Ghaligai's documents, comparing them with that in the document (Calandri, 1491/2, figure 11): "A tower is 40 braccia high and at its base runs a river which is 30 braccia wide. I want to know how long a rope which runs from the top of the tower to the other side of the river will be?" (Katz 2000, p.65). Ghaligai uses the schemes as a summary of the text. In Calandri's document the scheme encompasses the whole solution which is not accompanied by any words, apart from "la radice di" i.e. "the root of". In Ghaligai, the symbol of square root appears in the text and in the scheme as well. On the contrary, Calandri does not use a specific symbol. At the beginning in paragraph 41 (fig. 9), Ghaligai makes explicit that he is using the square root. Note that a preceding part of his *Pratica d'Arithmetica* was devoted to cubic root.

There is a similar use of lines connecting numbers which have to be combined in calculation. Note that in Ghaligai they are sometimes replaced by dots. In Calandri as well as in Ghaligai, numbers that are used in the same calculation are not always connected. Horizontal lines are also used for another goal: in Calandri, just before the result of an addition in which the two addends are written one above the other; in Ghaligai, before the product of roots. Some questions arise and regard the role of publishers and printers. How could they take into account the fact that the author was moving toward symbolism (if this would be the case)? How could they coherently interpret the meaning of each sign contained in the author's manuscript? What kind of typographic solutions could their possible misunderstandings produce? With respect to our analysis, the previous explanation regarding the role of a horizontal line might eventually be changed: like in the last scheme of paragraph 41, is the dashed line to be interpreted as implication, considering the application of a general rule, in this case regarding multiplication of roots with the same index? We cannot hide that these questions implicitly depends on the assumption that in the European Renaissance conditions for the new, efficient symbolism emerged (Radford, 2006).

CONCLUSION

The most common feature of the participants' answers is that the history of mathematics can be a way to achieve high level educational aims. More precisely, it is believed to be a resource for students to improve their way to work in mathematics. Narratives, problems, suggestions for laboratory activities can be only some of the many possible resources.

The image of mathematics which appears in the answers is that of an 'open science' either from the point of view of method or content. This shows the innovative role which the history of mathematics can play. In my opinion, it can address effectively some unsatisfactory aspects in our classrooms: those regarding, in short, the difficulties students have in managing mathematics when it is presented only in a formalized manner.

The "unfavorable factors" can be also part of students' opinions. This fact shows how the use of history in a profitable manner requires considering the different school actors. The perspective to achieve high-level aims is motivating but involves a wide investment of resources.

Considering the lack of a coherent symbolism, suggestions in favour of students who are requested to interpret schemes like those in paragraph 41 could be appropriate. A suggestion is to highlight the numbers, to search how to combine them and what operation to choose to get the result (see, for example, the first and the second one in paragraph 41 – Fig. 9 that are not accompanied by explanations). In my opinion, this way, students get a basic key to get a grasp of the hermeneutic approach. In fact, interpretation is achieved starting from an incomplete set of data. It is incomplete because the interpreters have to reconstruct other parts of reasoning in order to obtain a frame that has to be satisfactory with respect to their personal previous knowledge. Supplementary historical endeavour is necessary to interpret the document with respect to the author's viewpoint and knowledge about the topic to which document refers.

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Workshop

USING HISTORICAL TEXTS IN AN INTERDISCIPLINARY PERSPECTIVE: TWO EXAMPLES OF THE INTERRELATION BETWEEN MATHEMATICS AND NATURAL SCIENCES

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In this workshop two sets of historical texts will be studied with the aim of providing the participant with examples of effective interrelations between mathematics and natural sciences (physics and astronomy) according to the way these two scientific areas are defined today. These two sets of documents will refer to two different times: the Greek and Chinese Ancient World and the Italian Classical Time; they will include the following texts:

First set (Ancient Greek and Chinese cosmology)

- Ho Peng Yoke. (1966). The Astronomical chapters of the Chin shu. Paris: The Hague, Mouton & Co.
- Weir, J. (1931). The method of Eratosthenes. The Journal of the Royal Astronomical Society of Canada, 25, 294-297.

Second set (Galileo's theory of free fall)

- Galilei, G. (1638). Mathematical discourses and demonstrations, relating to Two New Sciences, trans. H. Crew and A. de Salvio (1914). Macmillan. (3rd day, th. II, prop. II).

In the historical texts forming the first set of documents two different cosmological models leaning on the same empirical observation are described and used. The participants will be introduced to the specificities of the experimental procedures (and the associated measuring instruments) and their relationships with the mathematical tools involved. A pedagogical use of the two texts will be proposed with the aim of providing students with elements of understanding what a "model" is (from an epistemological point of view).

The second set of documents concerns Galileo's discovery of the law of free fall and its implication to inclined planes. They will be used in order to explore the manner in which purely mathematical considerations entered into Galileo's working on movement. Starting from Aristotelian's philosophy of movement, the participants will access Galileo's intellectual procedure where the conceptualization process is intrinsically connected to a mathematical processing.

The general objective of the workshop is to illustrate the intrinsic links connecting mathematics and natural science for the modelling process of the natural phenomena that forms an essential step for a rational comprehension of the world. Using historical grounds in the classroom could be a fruitful way of making students (and teachers) aware of aspects of the science enterprise (in terms of measuring, modelling, conceptualizing, etc.). In this regard, the math/science complex as revealed by some discovery processes has a promising part to play.

A CABINET OF MATHEMATICAL WONDERS: IMAGES AND THE HISTORY OF MATHEMATICS*

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The Pennsylvania State University

Most mathematics teachers agree that the incorporation of mathematics history into their instructional process would enrich the lessons. The one most common objection to this innovation is the expenditure of teaching time. Compatible but nonintrusive ways must be found to overcome this objection. One such method, which has proved successful, is the use of historical images in the classroom: both during the instructional process and as passive reminders of the human involvement with mathematics. This presentation introduces the concept, supplies illustrative examples and renders advice on the securing of appropriate images. In particular, the historical archive, "Mathematical Treasures" maintained by the Mathematical Association of America in its e-journal <u>Convergence</u> *is recommended*

INTRODUCTION

A few years ago while visiting the British Museum, I was attracted to a display case containing cuneiform tablets from the Old Babylonian Period (1800-1600 BCE). One tablet in particular attracted my attention; it was a palm size oval with several columns of characters. Consulting the information supplied about this tablet, I learned that it was a sexagesimal multiplication table. Here, four thousand years ago, a young student, probably a scribe in training was learning his multiplication facts. This realization impressed upon me the continuity of mathematics and its learning tasks over a period of 4000 years. But what would impress be even more deeply, was the imprint of a human finger that accompanied the numerals and was preserved in the hard baked clay surface. It jarred me both emotionally and conceptually. This mark served as evidence of the human involvement with mathematics, it reinforced the fact that a person, an individual, did this mathematics. Despite further years of study and research on the history of mathematics, the impact of this image affirming the need to acknowledge and attempt to understand the persistent human involvement with mathematics has remained with me. An old adage says that "One picture [or image] is worth a thousand words," I certainly believe this. The statement is not merely a cliché as neuropsychological investigations have shown that the majority of reality-based information processed by the brain is obtained visually. Further, much of this processing is affective-it influences attitudes. Educational practice has always utilized this facility. In particular, visual imagery and its interpretation is a vital part of mathematical discourse (O'Halloran, 2005).

One of my major involvements over my career has been to convince mathematics teachers, at all levels, to incorporate the history of mathematics, the record of human involvement, into their teaching. I feel like such an endeavor helps to humanize the subject, that is, remove its aura of mystery and better reveal mathematics as a natural, human, activity. Usually, teachers appreciate the implications of this association but then the practical task of just how to incorporate history into mathematics teaching arises. Teaching time is valuable and examination standards must be met. If historical insights are to be provided they must be effective and minimally intrusive. Working with teachers, we have explored several appropriate strategies including: use of occasional anecdotes, brief stories from the history of mathematics; employing historical related mathematical learning tasks in small group settings (Swetz, 1994) and the classroom assignment of actual historical word problems (Swetz, 2012). Generally, such attempts, if diligently undertaken, have produced favorable results. Still, the expenditure of "valuable" classroom time remains a deep concern for teachers. One effective solution to this issue is the use of visual centered displays: images and posters that pertain to relevant historical personages and achievements. Images can be used as part of the classroom instruction, illustrating, enriching and reinforcing the specific concept being discussed or as a passive learning aid, displayed to attract student attention, arouse curiosity and perhaps prompt further investigation. Extra credit reports can be assigned on an image-"Here is a copy of a page from a 1520, British geometry book. What do you recognize? What unusual things do you see?"

TO KNOW HOW TO SEE

Let me give two examples of instances where the use of historical images resulted in fruitful discussions and prompted further learning interaction. During the ESU4 session in Sweden, Leo Rogers gave a talk, "Robert Recorde, John Dee, Thomas Diggs, and the "Mathematical Artes" in Renaissance England" (Rogers, 2004) in which he employed several illustrations. One simple image, from Robert Recorde's *Pathway to Knowledge* (1551) excited the audience: "Where did you get that? How can I get a copy?" See Figure 1. The situation depicted is elementary: it shows a man ascending a ladder to reach the top of a tower. The geometry reveals a right triangle,

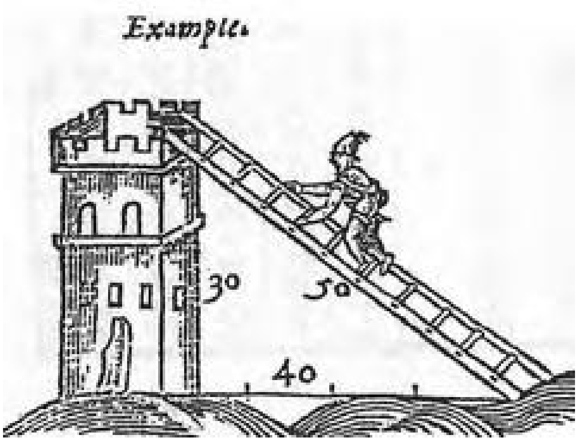

Fig 1: *Pathway to Knowledge*, Tower Problem

actually a 3-4-5 right triangle, and the viewer is asked to determine the length of the ladder when the height of the tower is given as 30 feet and the foot of the ladder is 40 feet away from the tower. What is the pedagogical appeal here?

- The ladder against a wall problem is familiar to almost all young algebra and geometry students.
- The illustration is from the 16th century-years ago, “they were doing the same problems as us five hundred years ago.“
- The Pythagorean theorem is being demonstrated: the lengths of two legs of the triangle are given; the viewer must find the length of the hypotenuse.
- A 3-4-5 right triangle is involved.
- The illustration appeals to the imagination of a young viewer-a castle is being stormed.
- The more perceptive student would note that for better climbing safety, the bottom of the ladder should be placed closer to the base of the tower. This is a pseudo-realistic situation, contrived for the mathematical convenience.
- A follow up exercise for this scenario would be “Determine the shortest ladder that could be used to scale this tower.”

At a talk I gave in the United States at a national conference of mathematics teachers, I briefly discussed and illustrated the galley method of division using the image from a 1604 manuscript, presented to King James I of England as a gift [1]. See Figure 2. After this talk, the student aids, all secondary school students, solicited me to show them this diagram again

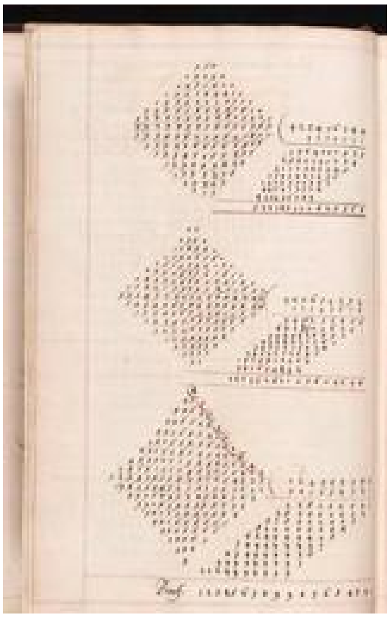

Fig 2: Galley Division in King James I’s Manuscript

and to explain it in detail-‘Just how was the operation of division being carried out?’ Firstly, they were visually attracted to the number configuration: a large triangle of numbers. Then, they became conceptually involved: if this is a division exercise, ‘How is the process being undertaken?’ ‘Where is the divisor, the dividend?” I explained the algorithm, line- by- line, pointing out the significance of the crossed out or “scratched out” terms and noted that due to its appearance, the technique was sometimes referred to a “scratch division”. Also in this discussion, I interjected the fact, that at this time in history, there were several division algorithms in use and this one, more popularly known as “Galley division” was the most frequently performed. I left my audience with the provocative comment that I believe this algorithm to be more mathematically efficient than the one we have actually adapted for school teaching, the “downward” or ‘Italian division” method. Hopefully some of them would investigate and attempt to substantiate my claim. If the reader has never attempted a “galley” division exercise, I suggest he or she do so. It is a very enlightening experience.

In selecting images for instructional purposes, two requirements should be fulfilled:

1. The material should appeal to the viewer, that is, be visually or conceptually attractive or intriguing. It should seduce the viewer “to want to find out more”.
2. It should have purpose and power by pertaining to the mathematical situation, topic or concept, you wish to promote and clarify.

View the object in Figure 3. Can you see any mathematics? Ask a companion to

perform the same task and compare what each of you see. Hopefully, each of you has perceived some similar images: a circle, trapezoids, hexagons, a star, etc. and would probably eventually query: “What is this object? Where did it come from?” If so, I have set you (the viewers) up, prepared you, to pursue the topic of ethnomathematics.

The object in question is a container for sticky rice as served during a meal in northern Thailand. The tribal people who made the basket never had formal schooling

Fig 3:Topview of Thai Basket for Serving Rice

or studied geometry and yet they exhibit knowledge of the geometric shapes you have discovered and are certainly concerned with the concept of volume, in this case, the volume of a circular cylinder of rice.

SELECTING IMAGES

Appropriate computer searches reveal an abundance of images: of instruments for measuring and computing; of people who contributed to mathematics and actual mathematical works such as books and notes, that can be used to enrich mathematics teaching and learning. If known by name, they can be sought out, for example, the Ishango Tally Bone, the oldest existing human mathematics artifact. See Figure 4. Its age, 20, 000- 25,000 years, testifies to the long human involvement with mathematics and the need to record mathematical data and its function prepares the audience for a discussion on numeration in general. In observing this image two questions immediately arise: "How do we know these are human markings?' and 'What might these notches be enumerating?' The latter question has actually resulted in quite a controversy. Of course, the situation opens the topic of tally sticks, European

Fig 4: Ishango Tally Bone [2]

experiences and the rather interesting/amusing story of the British Exchequer's Office 1826 fire. Due to their disposal of centuries of accumulated tally sticks by burning, the House of Lords was set afire and burned down. A very effective and information laden image which has appeared in several books on mathematics and the history of mathematics is YBC 7289, i.e. Yale Babylonian Cuneiform 7289, a cuneiform tablet from the Old Babylonian period. [3] See Figure 5.First, one notes the physical size, it fits into a man's hand, and indeed it was a palm tablet, held conveniently in the palm of one hand while the scribe, using his other hand,

Fig 5: YBC 7289 [4]

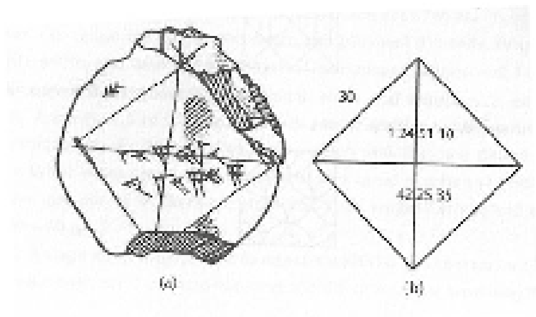

Fig 6: YBC 7289 Deciphered

wrote on it. Modern students visualize cuneiform clay tables as being quite large-perhaps they are influenced by the stylized biblical illustrations of Moses holding the tablets of the Judaic-Christian Ten Commandments. But, the most striking feature, to the novice viewer, is that the diagram clearly shows a square with its inscribed diagonal. Yes, the ancient Babylonians drew accurate geometrical diagrams in clay as part of their mathematical discussions. Once, when I offered this illustration in an article for publication, I was personally challenged by the journal's editor that I had drawn the diagram myself and was falsely attributing its origins to the Babylonians. Even very sophisticated people do not appreciate the fact that ancient mathematicians drew geometric diagrams to assist in problem solving. Next, a deciphering of the cuneiform inscription reveals two numbers in sexagesimal notation: what we know as $\sqrt{2}$ and $\sqrt{2}$ X 30, the specified length of the side of the square. See Figure 6b. At this point in the inspection, two implications emerge: at this early period before the rise of Greece and the Pythagoreans, the Babylonians possessed what we generally know as "the Pythagorean Theorem" and that these ancient people could extract the square root of a number to several decimal points accuracy. Another obvious question then arises: "How did they do it?" Further, if one looked closely at the image, the viewer would see the "tell-tale" human fingerprint!

The finding of some appropriate images for instruction may require a more formal search and research on the part of the instructor. In an instance where I desire to talk on the subject of linear measure and would like to use actual examples of societal devised units of measure, the image shown in Fig.7, taken from the German humanist, Peter Apian's, *Cosmographia* (1524), Folio 15r, would provide a learning inducement. It illustrates and demonstrates the use of fingers and feet to designate distance measures of the 16th century, ones not usually known or recognized by the modern viewer: one foot or *pes* equals four *palmus* or palms of the hand; a *leuca, or league,* the distance a person can walk in one hour, equals 1500 *passus* or paces

Fig 7:Distance Measures Given in *Cosmographia* (1524) [5]

and an Italian military mile is 1000 paces whereas a German military mile comprises 4000 paces. If appropriate, students could perform these measuring postures, record and compare results. Through such activity, the personal, human origins, of measurement are recognized. Another image that I have employed with great success in working with secondary school students is page 60 from the Swiss mathematics teacher, Johann Alexander's algebra book, *Synopsis Algebraica* (1693). See Figure 8. In the history of mathematics texts, this book is rather unique; it is a text/workbook, where instruction is provided on one page and the opposite page is left for notes and computations. It is the first such book of this kind that I know of. On page 60,

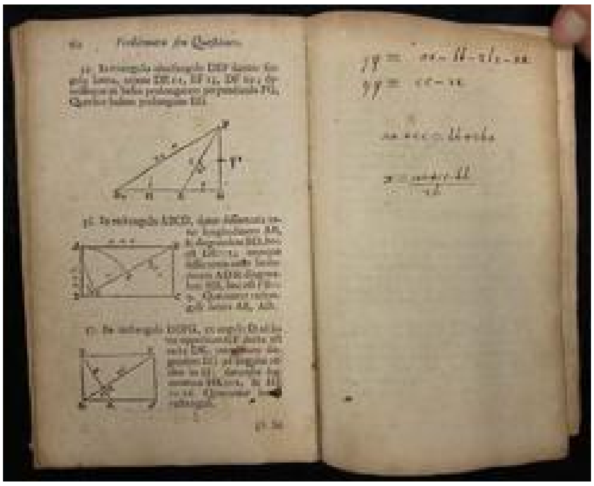

Fig 8; Student Exercise, 1693

three geometric situations are given requiring algebraic solutions. Although the book, including its exercises, is written in Latin, an astute student of today can understand the problems. A student from the 17^{th} century offers a solution for problem 37. Is he correct? My audience checks the work and finds out, there is a mistake! 'Ah, this privileged student of the 16^{th} century made a mistake. What is it?' The students realize that the mistake is similar to one many of them would make themselves; even the mistakes of problem solving hundreds of years ago are familiar to a modern audience. I have them correct the problem. . Then I ask them to solve the remaining problems. They are thrilled to be interpreting and solving "400 year old problems".

Throughout history, millions of such diagrams and illustrations have been devised to promote mathematical learning and understanding, they can be resurrected to serve immediate teaching needs and, further, they bear the added feature of an historical dimension which increases their attraction for the modern viewing audience. One last example of an image inspired encounter, a calculus student in a colleague's class was directed to an image of Isaac Newton's *The Method of Fluxions and Infinite Series with its Application to the Geometry of Curved Lines* (1736). She was enthralled to see the actual book that introduced calculus, albeit with fluxions, to her historical peers and instigated the mathematical field of analysis. Interpreting the frontispiece reveals an interesting story [6]. See Figure 9. But reading the title page

Fig 9:Gentleman Bird Hunters

THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, Kt.
Late President of the Royal Society.

Translated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is subjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Consisting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatise
A compleat Institution for the use of LEARNERS.

By JOHN COLSON, M.A. and F.R.S.

LONDON:
Printed by HENRY WOODFALL;
MDCCXXXVI.

Fig 10: Title Page

information, she was prompted to investigate the circumstances resulting in the appearance of this text. See Figure 10. It was attributed to Newton as the author but was actually composed by John Colton after Newton's death. 'Who was Mr. Colton and what was his association with Isaac Newton?' 'Why didn't Newton, himself, publish the book while he was alive?' Such questions and the resulting revelations their answers supply add much to the understanding of how mathematics is developed, transmitted and refined. Just seeing this image of Isaac Newton's work makes him and his accomplishments less remote- "Yes, Newton lived and he wrote this book!" Such a visual encounter reinforces the perception of mathematics as a dynamic, evolving discipline.

In each of the visually focused examples considered above, a different picture of mathematics was presented. Each image allowed us, the viewers, to briefly travel back in time, to realize that mathematics has a past. It came from somewhere and was devised for a purpose. Mathematics students introduced to similar visual excursions, when confronted with a new mathematical concept or idea, are less apt to inquire: 'What is this good for?' or 'When are we ever going to use this?' They have seen the past and know the answers. Of course, before introducing an historical image to a class, an instructor, himself or herself, must be historically knowledgeable of the significance of the item in question.

OBTAINING IMAGES

As the previous discussion has revealed, there are many available resources from which to seek historical mathematical images for instructional purposes. We all have our personal collection of mathematics texts and reference books that can supply us with images. Library searches, especially among large university holdings, also prove fruitful; however, computer searches of existing digital collections similarly provide many visual treasures [7]. In such search efforts, one must be mindful to comply with all legal and/or cost requirements attached to items. However, the Mathematical Association of America, MAA, has complied an archive of historical images specifically intended for instructional purposes that is available to all mathematics teachers and researchers free of cost. It is found online in the Association's e-journal *Convergence* as the feature "Mathematical Treasures". All the images discussed here, together with over a thousand others, are available for use from 'Mathematical Treasures". I sincerely hope that many readers of this article will avail themselves of this wonderful resource.

* The initial workshop paper was entitled "Pantas' Cabinet..."; *Pantas* is an old Indo-Malayan word meaning "teacher". It was the author's nickname when he taught in Southeast Asia and was used to personalize the title: however, it caused confusion and has been deleted from the revision.

NOTES

1. In 1604, George Waymouth, English explorer and navigator presented his King, James I, an illustrated manuscript: Jewell of Artes. Written and illustrated by Waymouth, the manuscript showed all the novelty, beauty and utility of the mathematical arts of this period. In 1605,Waymouth did receive royal funding for an expedition to the East coast of North America.
2. Unearthed in 1950 in the Belgian Colony of the Congo, this tally bone is the oldest known, existing, mathematical artifact. Discovered by the Belgian anthropologist jean de Heinzelin de Braucourt (1920-1998), it is named after the region in which it was dug up. The Ishango Bone is now housed at the Museum of Natural Sciences in Brussels, Belgium.
3. For example, David burton *The History of Mathematics: An Introduction*, New York: McGraw-Hill, 1997, p.77; Hodgkin *A History of Mathematics: From Mesopotamia to Modernity*, Oxford: Oxford University Press, 2005, p.25.
4. Yale Babylonian Cuneiform 7289 (ca.1800-1600BCE) is contained in the Yale Babylonian Collection. It was donated to the University in 1912 by the American industrialist J.P. Morgan. An excellent presentation on this tablet can be found @ www.math.ubc.ca/~cass/Euclid/ybc/ybc.html .
5. Peter Apian's (1495-1552) *Cosmographia* was intended to supply student instruction in astronomy, geography, cartography, navigation and instrument making. This image of Folio 15 recto was obtained from the History of Science Collection, University of Oklahoma Libraries.
6. The scene shows two country gentlemen bird hunting. They judge the flight of the birds and allow a lead in their aim, that is, they judge the velocity correctly. At the lower corner of this scene is a group of classical philosophers debating the action. The literal translation for the Latin inscription pertaining to the hunters is:

"Velocities perceived by the senses are measurable by the senses." While the lower classical Greek reads as:" What was common then is the same now [Things don't change]. Colton was mindful of the controversies surrounding the calculus, particularly George Berkley's charges in the *Analyst* (1734).

7. Some rich sources for images are:
    a. The US Library of Congress: lhttp://www.loc.gov/library/libarch-digital.html
    b. The British Library: http://www.bl.uk/
    c. Schoenberg Coll.,The University of Pennsylvania: http://www.library.upenn.edu/collections/rbm/bks/
    d. History of Science Collection, UO: http://ouhos.org/2010/06/19/digitized-books/
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Workshop

WHEN HPM MEETS MKT – EXPLORING THE PLACE OF HISTORY OF MATHEMATICS IN THE MATHEMATICAL KNOWLEDGE FOR TEACHING

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In mathematics education, one way to approach the question of "What should a mathematics teacher know?" is through the framework of Mathematical Knowledge for Teaching (MKT), based on the work of Deborah Ball and others. In most articles on MKT, history of mathematics is barely mentioned or not mentioned at all. However, there are exceptions pointing out that history of mathematics has a role in several – or indeed all – the subdomains of the MKT model. A further exploration of this can give important insights into the discussion on the role of history of mathematics in teacher education and in mathematics teaching, as well as enhance the MKT theory with insights from the work on history of mathematics.

In this article, I discuss didactical examples from the literature related to The International Study Group on the relations between the History and Pedagogy of Mathematics (HPM), to illustrate and discuss the role of history of mathematics in the framework of MKT.

INTRODUCTION

A fundamental question for mathematics teacher educators is "What should a mathematics teacher be able to do?" An important sub-question is "What should a mathematics teacher know?" Of course, the mathematics teacher needs to know everything that the students are supposed to learn, and more than that. What is this "more than that"? Is it more advanced mathematics, is it mathematical games, is it the history of mathematics, or is it simply everything the teacher educator finds interesting?

One way to approach the question of "What should a mathematics teacher know?" is through the framework of Mathematical Knowledge for Teaching (MKT), based on the work of Deborah Ball and others (Ball, Thames, & Phelps, 2008). This framework (often referred to as "the oval") is presently very popular in mathematics education research, and the article I just cited is referred to in hundreds of scholarly articles per year. The purpose of the present article (and of the workshop on which it was based) is to discuss whether MKT is useful for the discussions of what history of mathematics may contribute to in mathematics teaching; but also, how can history of mathematics contribute to the development of the MKT framework? The discussions are based on investigation of particular examples of the use of history in mathematics teaching. In the workshop, the group discussions were briefly summarized to everyone, and these summaries have informed the discussions in this article. First, I will give a short introduction to the MKT framework.

INTRODUCTION TO MKT

In developing their framework, Deborah Ball and others have taken Shulman (1986) as their starting point, with his concepts of "subject matter knowledge" and "pedagogical content knowledge". In the MKT framework, these two domains have been subdivided further, based on research on practice in the US.

I will introduce each of these domains shortly, but I would like to stress at the beginning that I see the mathematical knowledge for teaching as dependent on the context. What fits into which domain depends on the teacher, on the students, on the curriculum etc. In my examples, I will mostly think of a Norwegian teacher who is teaching grade 6 or 7 (students aged 12-14) or something similar. A teacher teaching other students, in another grade level, at another point in time or in another country, may need different knowledge. When discussing MKT, it is important to be explicit about which teacher and context we are thinking of. It can also be argued that the contents of the domains move around at times of curricular change. (Smestad, Jankvist & Clark, 2014)

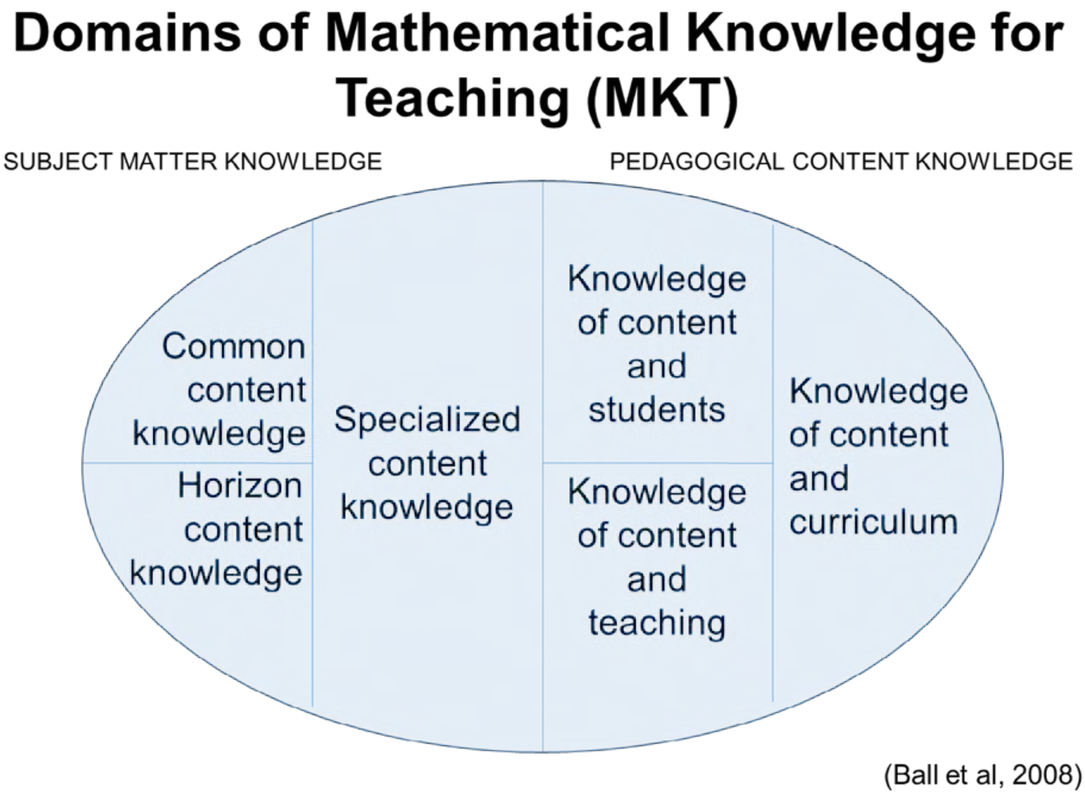

Figure 1: Domains of Mathematical Knowledge for Teaching, from Ball et al. (2008).

On the left-hand side of the diagram (Figure 1), we have subject matter knowledge:

- *Common content knowledge* is knowledge in mathematics that is not special for teachers. This includes the mathematics the students are supposed to learn, such as being able to add two integer numbers.
- *Specialized content knowledge* is knowledge in mathematics that is primarily necessary for teachers. Ball et al. (2008, p. 404) observed: "it is hard to think of others who use this knowledge in their day-to-day work". Ball et al. mentioned as an example the ability to see a new algorithm and decide whether it is sound, which is something a teacher needs when evaluating students' attempted solutions. The importance of this domain is to point out that there are things teachers need to know about mathematics that other professionals using mathematics do not need.
- *Horizon content knowledge* can be described as "a sense for how the content being taught is situated in and connected to the broader disciplinary territory" (Jakobsen, Thames, Ribeiro, & Delaney, 2012, p. 4642) This includes how the content taught is connected to other mathematical topics the students will meet later, but also for instance how the mathematical content has developed.

On the right-hand side, we have pedagogical content knowledge:

- *Knowledge of content and students* is "focused on how *students* think about, know and learn mathematics" (Mosvold, Jakobsen, & Jankvist, 2014, p. 50, my emphasis), for instance, identifying student misconceptions.
- *Knowledge of content and teaching* concerns "design of instruction" (Mosvold, et al., 2014, p. 50) in mathematics, for instance how to design a lesson with the use of examples, tasks and discussions.
- *Knowledge of content and curriculum* is "a particular grasp of the materials and programs that serve as 'tools of the trade' for teachers". (Shulman, 1987, p. 8, cited in Mosvold et al., 2014, p. 50). This includes both knowledge of the curriculum documents – which mathematical topics have the students met before and which will they meet next year – and which resources are available.

I also stress that all mathematical knowledge for teaching does not fit nicely into these categories. Often, there will be knowledge that fits into more than one. The point of the model, the way I see it, is not to make everything nice and orderly, but to give us additional tools to use while discussing.

Next, I give some examples of mathematical knowledge for teaching, and discuss what would be the "right" domains to place them in. (In the workshop, this was discussed in groups.)

Example 1: Knowing how to calculate 325:25

Of course, no teacher should attempt to teach division knowing only this. Thus, it is tempting to try to list (or map) other bits of knowledge we would like mathematics teachers to have for teaching division, including on its history, for them to be able to teach it in a meaningful way. Indeed, the MKT domains may serve as starting points for such a list, and this could form a meaningful activity in teacher education. However, the present task was to look at just this isolated piece of knowledge. In that case, this would clearly be common content knowledge, as everybody in our society is supposed to learn how to divide (disregarding arguments that this is a task better done by computers or calculators).

Example 2: Knowing how to simplify $(4x^3+2x^2-x) : x$

This is an example of subject matter knowledge. As everyone is supposed to learn this at some point, it cannot be said to be special for teachers, so it is common content knowledge. However, as it is not something the teacher's students are supposed to learn at this age (12–14) in Norway, but is connected to what they are supposed to learn at a later stage, it could be argued that it should rather be placed in the domain of horizon content knowledge for this teacher. The knowledge that the students will have to learn this later, however, would be knowledge of content and curriculum for the teacher.

Example 3: Knowing a little about the historical origin of the Hindu-Arabic numeral system.

I argue that this is common content knowledge. Knowing a little about our numeral system should be part of the mathematics curriculum for everyone. In the 1997 curriculum of Norway, it was explicitly so, and even in the new curriculum, there are enough general remarks on the importance of history that it should be included.

This is a good opportunity to discuss what "mathematics" is in the MKT context. In the original articles by Ball and her colleagues, I cannot find "mathematics" or "mathematical" defined. Examples of common content knowledge are, for instance, "a simple subtraction computation" (Ball, et al., 2008, p. 396), and common content knowledge is defined in terms of computations, "simply calculating an answer or, more generally, correctly solving mathematics problems." (Ball, et al., 2008, p. 399). However, when discussing what a mathematics teacher should know, it would not make sense to define "mathematics" narrowly as only mathematical algorithms or mathematical concepts. Arguably, at least in some parts of the world the history of mathematics is an intrinsic part of the subject mathematics (Fauvel & van Maanen, 1997), and in so far as history of mathematics is part of what students are supposed to learn in mathematics class, I would regard this as subject matter knowledge, more precisely common content knowledge.

Here, we can also touch upon Jankvist's concepts of history of mathematics as tool vs. goal (Jankvist, 2009). If we see history of mathematics as a goal, meaning that students should learn something from history of mathematics that they cannot learn otherwise, it is reasonable to think of the history of mathematics in question as part of common content knowledge. If it is just a "tool" for learning mathematics (in the sense of algorithms and concepts), it becomes purely a pedagogical device, perhaps better placed in knowledge of content and teaching or knowledge of content and curriculum.

Thus, we see that decisions on where to place parts of teacher knowledge in the various MKT domains depend on our context, including our goals for teaching mathematics, which in its turn are connected to our view of what "mathematics" is.

Example 4: Knowing how to find two fractions with different (and small) denominators, that adds up to a number less than 1.

This knowledge is certainly useful while teaching fractions. I argue that not many others need exactly this knowledge. Therefore, it is specialized content knowledge.

Example 5: Knowing that the Egyptians mostly used unit fractions (that is, fractions with numerator 1).

Keeping to my point of view that history of mathematics is part of mathematics, I would regard this as knowledge of mathematics that is useful for the teacher, as it may influence him when introducing fractions to the students. Thus, it is either horizon content knowledge or specialized content knowledge – and it could be considered both.

Example 6: Finding the mistake in a calculation and considering whether it could be a sign of a usual misconception.

Being able to find a mistake in a calculation (as opposed to just finding that the answer is wrong) is mathematical knowledge that you rarely need outside of the teaching profession. Thus, it should be considered as specialized content knowledge. Knowing about typical misconceptions, on the other hand, is knowledge of content and students. In this way, threads of knowledge that are closely knit together and used simultaneously when planning or performing teaching, may well belong to different domains of the MKT model.

Example 7: Finding a useful counterexample to the sentence "Division of a number by another number makes the original number smaller."

Finding a counterexample is specialized content knowledge as other professions rarely need counterexamples, while they are necessary for teachers. However, the word "useful" is important. A teacher cannot use just any counterexample; he needs the counterexample that is exactly right for his students. To find that, he also needs to know his students well, so knowledge of content and students is needed. As in example 6, we see how the knowledge from different domains work together even in simple tasks.

Example 8: Being able to figure out whether the method of multiplication in Figure 2 works in general. (From Smestad, 2002)

26·41	
13·82	82
12·82	
6·164	
3·328	328
2·328	
1·656	656
	1066

Figure 2: Method of multiplication

Being able to figure out why or whether methods work, is not normally part of the mathematics that everybody learns and needs, but it is central knowledge for teachers, who need this to be able to give relevant feedback to students using different methods. Thus, this is specialized content knowledge. In some countries, this method (which is inspired by the Egyptian method for multiplication given in the Ahmes papyrus) could be part of the curriculum, and would then be common content knowledge.

Example 9: Knowing the origin of the words "algebra" and "algorithm".

Again, I consider history of mathematics as an intrinsic part of mathematics. In most countries and curricula, the etymology of words will not be considered part of what everybody needs to learn, thus it is probably not common content knowledge. However, to be able to answer reasonable questions from students, a teacher needs to know this part of mathematics. Therefore, it should be considered specialized content knowledge. Since it is part of the background of the mathematics teachers teach, it could also be argued that it is horizon content knowledge. However, if you do not regard history of mathematics as part of mathematics, this knowledge would be relegated to the pedagogical content knowledge domains, and would perhaps belong in knowledge of content and teaching.

The first nine examples were decontextualized sentences about knowledge. Some participants in the workshop found it very frustrating to work on such examples, stressing that the context is so important. The remaining examples are (parts of) teaching materials concerning history of mathematics that are designed for teacher education. Here, I discuss what knowledge could result from working on the examples and in which domain(s) such knowledge belongs. In the text, I will refer to the pre-service teachers as PSTs, while their future students in school will be referred to simply as "students".

Example 10: Biographical introductions

Below is an example from Haanæs & Dahle (1997, p. 97), a textbook for 6th grade. (It was mentioned in Smestad (2002, p. 36). The translation is mine.):

> We consider Florence Nightingale the founder of modern nursing, but she was also one of the first female statisticians in the world. She considered statistics a way of changing society, and she contributed to making statistics a subject on its own at the university of Oxford in England. She tried to help people who were ill and suffering in the world by showing how many they were. She made statistics herself that lead to a new way of treating patients all over the world.

Imagine that we work with PSTs on preparing such introductions: what knowledge could result, and which domain would the knowledge belong to?

Writing such biographical texts should result in some knowledge of the biography of a mathematician. Is this just "spice" that the PSTs can later use to engage their students? In that case, it is purely a pedagogical tool, probably belonging to knowledge of content and curriculum. However, one of the PSTs' goals may be to develop in the students a sense that mathematics is a human activity and to develop students' epistemological points of view. In that case, the PST needs biographical information "on the horizon" of the mathematics to do that, thus the biographical details are horizon content knowledge. Some PSTs may even think that knowledge of a particular mathematician's life is part of what everyone should know (for instance Abel in Norway or Newton in England) – just like some teachers think everyone should know about Ibsen (in Norway) or Shakespeare (in England). In that case, they will consider it common content knowledge.

Again, we see that knowledge cannot be placed in MKT domains without regarding the personal epistemology and the goals of the teacher.

Example 11: al-Khwarizmi

The following is an excerpt from Clark (2012, pp. 72-73), where she described part of a Using History for Teaching course. In this task prospective mathematics teachers (PMTs) are

> …solving quadratic equations using the methods of al-Khwarizmi (early ninth century CE). PMTs in the UsingHistory course were provided an excerpt from the History of Mathematics: A Reader (Fauvel & Gray, 1987, pp. 228-231) in which an English translation of al-Khwarizmi's rhetorical and geometric explanation for how to solve quadratic equations was given. A similar excerpt is:
>
> > … a square and 10 roots are equal to 39 units. The question therefore in this type of equation is about as follows: what is the square which combined with ten of its roots

> will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this, which is 8, subtract from it half the roots, 5 leaving 3. The number three therefore represents one root of this square, which itself, of course is 9. Nine therefore gives the square. (O'Connor & Robertson, 1999, para. 12)

> […] PMTs were not given the accompanying figure at the outset of the exploration. Instead, they were to use the rhetorical solution to develop the geometric argument. After exploring each of al-Khwarizmi's explanations and reporting out the whole class, groups continued with their investigation of solving quadratic equations from a historical perspective […].

What knowledge could such work lead to, and which domain(s) would it fit in?

In Clark (2012), an important point was that the PSTs learned how to solve equations geometrically. That knowledge is specialized content knowledge, if not common content knowledge. However, we hope that PSTs will not only remember the method of solving, but also realize something about the long history of mathematics and the point that different cultures have contributed to its development. The cultural component is horizon content knowledge.

Example 12: Fibonacci

Here is the start of the introduction of Fibonacci's *Liber abaci* (1202):

> After my father's appointment by his homeland as state official in the customs house of Bugia for the Pisan merchants who thronged to it, he took charge; and in view of its future usefulness and convenience, had me in my boyhood come to him and there wanted me to devote myself to and be instructed in the study of calculation for some days.
>
> There, following my introduction, as a consequence of marvelous instruction in the art, to the nine digits of the Hindus, the knowledge of the art very much appealed to me before all others, and for it I realized that all its aspects were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter, while on business.
>
> I pursued my study in depth and learned the give-and-take of disputation. But all this even, and the algorism, as well as the art of Pythagoras, I considered as almost a mistake in respect to the method of the Hindus. (Modus Indorum). Therefore, embracing more stringently that method of the Hindus, and taking stricter pains in its study, while adding certain things from my own understanding and inserting also certain things from the niceties of Euclid's geometric art, I have striven to compose this book in its entirety as understandably as I could, dividing it into fifteen chapters.
>
> Almost everything which I have introduced I have displayed with exact proof, in order that those further seeking this knowledge, with its pre-eminent method, might be instructed, and further, in order that the Latin people might not be discovered to be

> without it, as they have been up to now. If I have perchance omitted anything more or less proper or necessary, I beg indulgence, since there is no one who is blameless and utterly provident in all things.
>
> The nine Indian figures are:
>
> 9 8 7 6 5 4 3 2 1
>
> With these nine figures, and with the sign 0 ... any number may be written.

What knowledge could work on the introduction of *Liber abaci* contribute to, and into which domain(s) does this knowledge belong?

I have used this example with my PSTs, and I had two main goals: The first was that PSTs should have some idea of where the Hindu-Arabic numeral system comes from. I would regard this as horizon content knowledge. (At least as long as the curriculum does not mandate that everyone should learn about this, in which case it would be common content knowledge). The second was more general: that the PSTs should realize that the mathematics we use is not predetermined but evolves, depending on human choices. This would also be horizon content knowledge.

Example 13: The Pascal-Fermat correspondence

The Pascal-Fermat correspondence, which many regard as the beginning of probability theory, provides interesting questions. I have developed a few exercises based on the history, which can be found in Smestad (2012). They lead PSTs through some of the problems found in the correspondence. In the workshop at the Seventh European Summer University, we looked at what is known as the problem of points, and the PSTs' work on different mathematicians' attempts at solutions.

Again, I have used these exercises in my own teaching. My main goal in using them was that the PSTs should learn probability theory while doing them. This would be common content knowledge. But I also included these exercises because they showed how mathematics has developed, and in particular how mathematicians interact, using trial and error and providing counterexamples to develop theories. This I would regard as horizon content knowledge. But again, the exercises could also be seen as mainly motivational, with a little "human interest" added to spice up the mathematics. In that case, I would regard it as knowledge of content and curriculum as "tools of the trade" to teach probability.

Concluding discussion

The examples show that what fits in which domains is context-dependent. This would be a bad thing if the point was to sort knowledge neatly into domains (or to test PSTs). For use in teacher education to foster discussion, it may not be. In discussing examples in connection to MKT, PSTs need to be explicit about the way they intend the history to be used and in which context they intend to use it. It becomes clear that the same knowledge can be used for different goals, depending on the teachers' personal epistemology and his goals for teaching mathematics.

In the workshop, we also discussed what could come out of connecting HPM and MKT. Already, several articles have been written about how looking at history of mathematics can enrich the MKT framework. (Mosvold, et al., 2014; Smestad, et al., 2014) In this article, I argued that researchers should be explicit about what they regard as "mathematics" or "mathematical" when looking at "mathematical knowledge for teaching" to include history of mathematics. Also, from a more strategic point of view, the HPM community should engage with the theories that are considered important in the general mathematics education community, as seen, for instance, in the PME conferences. If we do not, we risk being seen as irrelevant, which will make it more difficult to get our ideas across and to attract newcomers. In addition, in teacher education it is useful to connect to theories that PSTs already know when discussing history of mathematics.

The advantages of using MKT in HPM research are less clear. It is important that we are explicit about how history of mathematics may contribute to mathematics teaching, but there are other frameworks for this that may be as useful as the MKT framework, for instance Jankvist's idea of history as a tool vs. history as a goal.

We should not just look at what can be gained by using the MKT framework, but also at what can be lost. In his 2014 book *The Beautiful Risk of Education*, Gert Biesta argued forcefully that too much ink is spent on the qualification of teachers, and not enough on the socialization and – most importantly in his view – the "subjectification" of teachers. "Subjectification" is connected to his concept "becoming educationally wise", which he argued is very different from obtaining knowledge. (Biesta, 2014) Instead of looking at what teachers should know (which the MKT framework helps us do), we should perhaps look at how teachers can use what they know in the art of teaching. As with any frameworks of interest to mathematics teacher education, the discussions on what it does not encompass are as important as the framework itself.

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Workshop

ORIGINAL SOURCES IN TEACHERS TRAINING POSSIBLE EFFECTS AND EXPERIENCES

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The advantages of using original sources in school mathematics have been discussed widely. Within academic studies for prospective teachers, history of mathematics is also attributed an important role, with a great variety of possible implementations. At Siegen University the integration of history of mathematics is one of the key features: In addition to special courses on history of mathematics, we integrate historical aspects and work with original sources in didactics courses as well as in lectures on mathematics and elementary mathematics. From a normative point of view, working with original sources can contribute to widely accepted aims of teachers' education, such as understanding the process character of mathematics and fostering authentic experiences in mathematical research on an elementary level. Furthermore, the alienation by historical sources can provoke cognitive conflicts that can enhance the perspective on current mathematical processes of conceptualizations. After discussing advantages and problems of an approach to original sources on an abstract and theoretical level, we will present concrete examples and experiences from courses on elementary mathematics as well as didactic seminars, which show the broad use of original sources.

At Siegen University, there is a long tradition of courses on history of mathematics and implementing an historical perspective and historical sources in the context of teachers' education. In our paper we want to present and discuss these experiences. We believe that the general context of the course – university mathematics, elementary mathematics, mathematical didactics or courses on history of mathematics – is of crucial relevance for the impact made. In our paper, we therefore discuss advantages and problems of an approach to original sources first on an abstract and theoretical level. Naturally, our personal experiences influence this account, however they are in accord with empirical results from studies focusing on various levels of mathematics education at school and at university. With that said, we will present concrete examples and experiences from courses on elementary mathematics as well as didactics seminars. In each case, we will focus on the connection between context and aims. By this, we want to thoroughly differentiate between different impact on students, which original sources can have, and propose a possible classification.

USING HISTORICAL SOURCES – GOALS AND OBSTACLES

> "It is unbelievable how ignorant the young students come to the university. If I calculate or do geometrical constructions for only 10 minutes a quarter of the audience will fall asleep."

We started this section with an original source[1], a quote from 18th century – written by Georg Christoph Lichtenberg, mathematician, physicist, philosopher and man of letters. Its content, however, fits well into current debates. These debates articulate a widely spread dissatisfaction with the level of General Mathematical Education (Allgemeinbildung). In particular, this refers to

- a robust and flexible elementary knowledge,
- the propaedeutics: first year students of mathematics and other disciplines heavily relying on mathematics do not seem to be well prepared to their studies,
- but also - which is often forgotten, but important to add - with respect to reflection, to the ability of judging about mathematics and its role for cultural history and for present societies.

For the universities and their mathematics (or physics or engineering) departments, the second type deficits might be the easiest to compensate by just implementing specific first year courses for necessary training. Concerning the other two, it is only the schools (elementary as well as high schools) where this knowledge could be imparted. Certainly, it is the duty of schools to teach the elementary knowledge. But also the ability of reflection *about* mathematics stems mostly from school experiences. All other efforts of popularization are based on these first experiences[2].

An all too easy implication of the above observation could state that our school education is insufficient, or pupils as well as mathematics teachers are simply to stupid for mathematics. However, a slightly more sophisticated analysis might ask for the quality of mathematics teachers' studies – thus the destination of the critique changes to the universities and their professors: The quality of teachers' studies is in fact one key element of the problem.

Here an integration of history as well as philosophy of mathematics3 - within the special focus of our paper based on original sources - can help in various ways to improve these studies. In the following we will substantiate this claim. In the first part of this theoretical introduction we will focus on historical sources as **a mere tool f**or better teaching mathematics. The aim is thus fostering a deeper understanding of the mathematical content. Moreover, we will also shed some light on certain limits and risks of this usage. In the second part, however, we will illustrate why and how knowledge in the reflection discipline is also **a goal in itself**.[4]

History as a tool

We start with describing different ways how history of mathematics could be used as a teaching tool. The most frequent usage of the history of mathematics is **telling anecdotes**. Sometimes also a professor feels the necessity to pep up his boring stuff.

Anecdotes are living from the contrast to the "normal presentation", the emphasis lies, e.g., on the biographic and personal, the human touch. **Original sources** in this function usually come along as **longer or shorter quotes** (see, e.g., our initial quotation). A specific form is the **comforting usage**. Here one might tell the students about great mathematicians struggling for years and years before they found a solution or the fruitful concepts. So, the beginner should not be too much frustrated if it takes weeks or even months for him to understand the given solutions.

A problematic variant of the anecdotal type presents a purely **heroic picture**. Mathematical development is thus created by inimitable geniuses. Living in the shadow of these giants there is no way for own mathematical activity. The contrary picture is equally unsatisfactory. We will call it the **jovial picture**. Here the historical progress is overstressed, we look back from our state of the art and describe in a more or less patronizing way what former times "already knew". We thus completely forget the historical context.

The next form is much more ambitious. Going back at least to Otto Toeplitz (1881-1940) it might be called the **genetic use**. Here the historical-genetic approach is only one facet of a more general concept being in contrast to a 'deductive' or 'formalistic' presentation of mathematics. Within the historical-genetic approach one can distinguish an explicit and an implicit form. An **implicit historical-genetic** approach relies on some historical knowledge on the side of the teacher but does not explicitly present the history of the mathematical subject. The main function of historical awareness is becoming sensitive to barriers of understanding which were historically effective and therefore probably also for the individual learner. In contrast, the **explicit form** will present the mathematical subject by a presentation of its historical development. Using **original sources** can thus almost automatically be classified as explicit. If, however, the historical process is presented all too exact this form can become rather problematic if not counterproductive. The complexity of the history of mathematics – containing errors, dead end streets, deserted roads (from our point of view!), intended and now forgotten applications etc. - might become quite *confusing*, especially for the beginner. The explicit-genetic tool may then be switched into an autonomous goal. Though it does not make the *learning* of mathematics easier, we could obtain a more substantial picture of *mathematics* by knowing the historical way. We will come back to this later.

A specific function for students' future teaching duties is the **de-familiarizing effect** of historical sources. Especially with respect to the elementary school a student normally does not have a vivid memory of his own learning processes. Here the historical context can change the appearance of elementary mathematics in a way that students are forced to repeat the learning of the basic concepts. Moreover, knowing the historically realized alternatives we see that, e.g., our way of denoting numbers and calculating is by far not the only or even the canonical way of doing it (cf. the aspect of "reorientation" (Jahnke et.al., 2000)). And, finally, we learn to appreciate the value of a historically grown clever notation.

Finally, we would like to mention the **paradigmatic use** of history. In most cases teachers' students will hardly come into contact with authentic, actual mathematics –

or if so, they will probably experience only total ignorance. Intensively discussing a historical example - based on original sources - can bring them into successful contact with authentic, though not contemporary, mathematical research. In general the mathematical difficulties will shrink with the time distance but the historical difficulties will grow. Therefore one will make some concessions with respect to the historical precession, e.g., will use **translations** and **secondary sources**. Within this paradigmatic use another most important experience is possible namely learning about interdisciplinary connections of mathematics within authentic settings (e.g. Habdank-Eichelsbacher B. & Jahnke H.N. (1999)).

History as a discipline for reflection

Besides the above discussed variety of supporting function there is an independent role of history of mathematics for any teacher's studies. First of all, it is a mere triviality that any faithful picture of mathematics encompasses the genesis of its problems, its concepts and results thus the history of the discipline. Really *knowing* mathematics in a proper sense implies knowledge of its history and this is more than a mere collection of historical facts – a systematic orientation about the history is needed. Besides a training of mathematical skills any school education should provide also this type of knowledge[5]. In that spirit a **historico-critical perspective** on mathematics will present a historically grown discipline and therefore the history of mathematics as an autonomous subject. Among others one goal will be showing that there is usually a large contrast between the canonical, systematic and often formalized version of mathematics taught in our lecture halls and the much more involved, in part also quite chaotic history of its genesis. Only with respect to that background we can appreciate the huge accomplishment of that systematization, but also the losses[6]. Moreover, it is our conviction that questions for motivation and heuristics within mathematics itself can hardly be discussed without reference to history.[7] The adequate presentation and discussion of any subject within the history of mathematics will require much more than telling a nice anecdote, it takes time and a special sort of diligence - and it will almost necessarily refer to original sources. On the side of the professor at least some professionalism is needed and on the side of the students a solid and flexible, though not yet a historical knowledge of the mathematical subject.

Widening the horizon we observe that mathematics is not just important for the history of sciences – but it is an essential power influencing the whole **history of ideas**, the political, cultural and social history (cf. the aspect of "cultural understanding" (Jahnke et.al, 2000) in a broad sense). This aspect is often neglected or dramatically underestimated. Certainly this is due to the fact that historians in general do not look at mathematics and mathematicians and also historians of mathematics do not transgress the boundaries of the mathematical discipline. A central question for teachers-to-be, namely a justification for the choice of the mathematical subjects and the form of presentation in the classroom, however, can only be answered with reference to this cultural history. Of course, neither the school curriculum nor the university studies could encompass a complete picture of the role of mathematics for cultural history. Therefore we should teach and learn by examples or case studies. For this „macro-perspective“ probably the use of original sources might be a to narrow look and **secondary texts** might be more suitable.

USING HISTORICAL SOURCES – EXAMPLES AND EXPERIENCES

In Germany, courses for prospective mathematics teachers do not only differ in content (geometry, algebra, calculus and so on) or method of teaching (lecture, seminar, working group). They are also very different concerning the perspective on the mathematical content and its impact for the teaching and learning of mathematics. In Siegen, alongside to the regular courses on advanced mathematics, we present courses on elementary mathematics – e. g. school-mathematics from an advanced point of view[8] –, courses on mathematics education as well as courses on history and philosophy of mathematics.[9] In each of these parts of teachers' education a fruitful integration of work on original sources is possible – of course with different goals, impact and obstacles like described above.

In courses on history and philosophy of mathematics original sources are used quite naturally, e.g., to train text apprehension. At the university of Siegen, a seminar in history of calculus will require the students to present the important steps in the development of analysis by reference to sources from Euclid, Archimedes, Torricelli, Leibniz, Riemann and others …

Students will work intensively with original sources, when writing a master thesis on history or philosophy of mathematics. Among others, Siegen's students wrote theses on Bernard Bolzano's *Paradoxes of Infinity*, George Berkeley's *Analyst*, Blaise Pascal's *Wager*, Lewis Carroll's (1972)[10] *Symbolic Logic* and *The game of logic* or infinite series in the middle ages using the example of Nicole Oresme.[11]

In the following we will describe our experiences by using original sources. We will present an example for each of the different parts of teachers education mentioned above, to show the wide range of possible applications and will discuss the similarities and differences concerning the sources impact on teachers' education. In order to avoid problems with different degrees of difficulty or different periods of development, we focus on the same mathematical discipline throughout all examples. Thus all original sources used are well known historical texts concerning real analysis. Therefore the sources will not be analyzed and described in detail. Our focus will lie on their function in teachers' education and their possible impact according to the different types of courses.

Euler's Introductio in a lecture on real analysis

Infinite sequences and series are an established content in lectures on real analysis. For students, however, it is the first time they are confronted with the mathematical concepts of infinity and convergence in a rigorous way. Due to its prominent position in curriculum, raising the students' awareness for possible obstacles and ways of handling them (mathematically) has to be the main goal. One possibility to achieve this aim is to take a glance at the concept's history. Original historical sources give authentic impressions of history (of mathematics) and an illuminating contrast to the current and rather technical ways of handling the concepts. Moreover they emphasize the importance of a more or less intuitive way of grasping a concept, especially if the sources mark the state of the art at their time.

Getting to know the genesis of key concepts is not only important for prospective teachers, like the discussion about parallels of historical and individual epistemic "roots" may show (cf. Furinghetti, 2007, p. 133) , but for all mathematics students. We are convinced that - on the one hand - there is a positive effect on the process of understanding. On the other hand as discussed above, knowing at least a little about mathematics' history has to be part of *Allgemeinbildung*. So, historic facts as a self-contained content should be compulsory within every lecture on advanced mathematics. One way of initializing resp. supporting a deeper understanding as well as a mathematic-historical *Bildung* is using original sources (cf. Jahnke et. al., 2000). For example, within a lecture of real analysis, an extract of a textbook by Leonard Euler (and additional information on the historical background) can be used as a "tool" to enlarge the comprehension of the modern notion of infinity as well as a "goal" for mathematical *Bildung* at university (cf. Jankvist, 2009). The following extract of a homework on infinite series and the logarithm may concretize both:

Read chapter 7 of Leonard Euler (1748): Introductio in analysis infinitorum[12].

a) Investigate the historic background of Euler's book.
b) Read chapter 7 carefully with 'paper and pencil': Which assumptions does Euler make? Explicate the goal of the chapter.
c) Which series expansion does he use for the logarithm? What does the text say about the connection between exp and ln?
d) Mark at least two passages in which Euler's handling of infinitely small resp. big quantities could be described as intuitively (e.g. in which Euler's handling doesn't work correctly with respect to our contemporary rigor).

In part a) the students are asked to do, what a historian would do quite naturally when analyzing an original source. However, within a lecture on advanced mathematics, this modus operandi is quite unusual and therefore has to be initialized carefully. By embedding Euler and his oeuvre in mathematics history in general and the given text in the history of real analysis in this particular case the students get valuable insight in the genesis of their discipline. The autonomous research of the historical background within an exercise focusing on the mathematical content as well, will naturally link the historic facts with the mathematical notions of the lecture. Furthermore it will raise awareness for the differences to the modern presentation and therefore will help to avoid treating the text anachronistically or even in a completely ahistorical way.

While part a) is a contribution to the aspect of mathematical *Bildung* as described above, parts b) and c) focus on the underlying mathematical content. Due to the historic and therefore unfamiliar notation it will be difficult for the students to identify the concepts known from the lecture or current textbooks. They are forced to go through the text step by step – using the modus of a historian once again. And it won't suffice to grasp the modern notions only in an algorithmic way. The effect of alienation by the original source is used to initiate a process of repetition and deeper comprehension of the subject matter. Reading carefully through the text, the students will face obstacles, since Euler's approach to the infinitely small resp. large quantities contains some confusing passages. It can be described as quite informal, however successful – i.e. his use of infinitesimals and his notion of convergence doesn't correspond to today's rigor but the results do. Part d) should draw attention to these differences. To identify the crucial passages the students need a flexible comprehension of the current concept of infinite series and their convergence. Conversely, Euler's way of handling provides informally accessible examples for the rigorous definitions but underlines the benefit of today's rigor as well – the students will meet examples where Euler's reasoning won't work. Altogether, working with the original source is supposed to initiate a deeper comprehension of one of the key concepts of real analysis.

Lecture on school calculus from an advanced point of view

The curriculum of the university of Siegen contains lectures on school mathematics from an advanced point of view. Similar to Felix Klein's (1932) concept of his *Elementary mathematics from an advanced standpoint*, the aim of these lectures is to embed school mathematical contents into its mathematical background on a higher, more rigorous level. But contrast to Klein, the intention is not to dig into advanced mathematics deeply and to show possible contiguous research fields, but stick very closely to the actual school mathematical content and to focus on the multiplicity of various aspects and points of view.

A main part of the lecture on school calculus deals with extreme problems and the variety of methods to solve them – with the differential calculus as well as by geometric and algebraic methods. In order to compare the slope and the opportunities of the different methods, the isoperimetric problem for rectangles came in handy as a productive and versatile example. After presenting and discussing this well known antique problem, the students were given an extract of a translation of Pierre de Fermat's *Oeuvres I*, accompanied by the following assignment:

1. Explain, how the example, that Fermat uses to illustrate his method can be identified as the isoperimetric problem for rectangles.
2. Read Fermat's solution carefully and describe his method in your own words, using the modern notation.
3. Fermat is led by the intuition that function values hardly differ in the neighbourhood of a maximum (or minimum).
    a) Is Fermat's intuition viable – that is, does it characterize extrema sufficiently?
    b) In some way, Fermat's method and the modern method of differential calculus are led by the same intuition – explain!

As a main goal, in part 1 and 2, students will become familiar with a historical argumentation that varies from the rigorous explanation, they are used to. Similar to the usage of original sources in lectures on advanced mathematics, the effect of alienation by the original source is used to initiate a learning process on another level. It wasn't easy for the students to identify Fermat's example with the already known isoperimetric problem, simply because the notation used is quite unusual.

By making the students describe the method used by Fermat, we accepted the hazard of an anachronistic interpretation. Indeed some students interpreted Fermat's procedure in the modern way. However, the assignment aims to focus on the benefit of intuition and core ideas that have been developed over the years (especially in part 3)[13]. The students were able to experience mathematics as a human enterprise and a developmental process on the one hand, but learned to value the intuition itself. This led to a fruitful discussion on the core ideas and the final (rigour) elaboration.

Seminar on subject matter didactics of calculus

Seminars on the teaching and learning of mathematics can discuss general topics like mathematical problem solving or mathematical learning processes as well as a special mathematical discipline. In either case, original sources can be used fruitfully: First of all students get to know examples of original sources that can be used in mathematics classroom as well.[14] Within seminars on general topics in addition, they can be used to show the procedural character of mathematics or to give authentic examples for mathematical modeling for example. Within subject related didactic courses, original sources are an appropriate tool to initiate discussions about the nature of the subject and the consequences for the teaching and learning. Original sources force to link mathematics to the situation at school and therefore persuade the students to concentrate on the subject matter *didactically*. The following assignment was given in a seminar on the teaching and learning of school calculus:

Students got a translation of *exercise one* from Johann Bernoulli's *Lectiones de calculo differentialium (1691/92)*[15] (How to find the tangent on a parabola) and as additional information his *postulates* and the *rules about sums and products of differentials.* Some biographical information about the author and his oeuvre[16] and the following tasks completed the worksheet, on which they had to work during the seminar lesson in groups of three or four:

1. Read the excerpt from Johann Bernoulli's „Vorlesung über das Rechnen mit Differentialen" (1691/92) carefully (with 'paper and pencil'):
    a) Draw your own sketch while reading. Use the one of Bernoulli for aid.
    b) How can the quantities dx and dy be interpreted within this construction? What is the so called 'subtangent'? Which geometrical considerations are used?
2. Compare Bernoulli's text and current textbooks in school:
    a) What is the starting question resp. the starting problem in each text? How is the problem formulated and motivated?
    b) Which mathematical 'tools' (geometrical or algebraic considerations, a coordinate system, sketches, computer, …) are used?

Questions one a) and b) should help to analyze the source and to understand the underlying mathematical content. The questions help to focus on the geometrical character of Bernoulli's formulation of the problem and the solution. Even if all of the students know how to find the slop of a tangent in a given point of a quadratic function, it seemed to be this geometrical character of argumentation which make it difficult to grasp the arguments given by Bernoulli: Most students started to draw a coordinate system and reformulated the arguments by their notions of functions – which in fact is more complicated. Other difficulties arose when dealing with the infinitely small triangles.

In the previous lesson the students discussed different established approaches to the introduction of the derivative at school[17] and identified them within popular German textbooks. Question two leads the students to recapitulate these and to compare them with Bernoulli's approach. The aim of Bernoulli (finding a second point on the sketchpad to draw the tangent vs. determine the slope of the tangent in any point to get the derivative function) as well as the discussed objects (curves on a sketchpad vs. functions or graphs of functions) and the mathematical tools (Euclidian theorems about congruence of (infinite small) triangles vs. algebraic rules and the limit value of the ratio of the differences) are very different from the modern differential calculus. But on the second glance there are also parallels: In both cases, the parabola resp. quadratic function is used as an introductory example. Maybe less obvious, a deep comprehension of the concept is complicated by dealing with infinitely small quantities – actually this is often disregarded by prospective teachers because of being too familiar with the concepts in a way and especially because of the rather schematic and technical use of rules for calculating derivatives. The effect of alienation by the original source can help them making these obstacles relive and becoming aware of possible difficulties, their prospective pupils might have. Another parallel can open a discussion about different domain-specific beliefs of mathematics: Even if current textbooks often start interpreting the derivative as instantaneous rate of change, the argumentation is prominently supported by pictures of graphs of functions with their secants and tangents. This may lead to an "empirical belief system" (cf. Schoenfeld, 1985, p. 161), in which the function is identified with the sketch of the graph in the way Bernoulli did by arguing by means of the curve on the sketchpad. (Cf. Spies & Witzke, forthcoming) Of course a profound reflection about such topics needs additional information and quite some time of plenary discussion after working with the original source.

Main differences of the usage in the different types of courses

Some impact can be equally obtained in all kinds of mathematical courses. Whenever original sources are used in university courses, they contribute to mathematical *Allgemeinbildung*, as described above, by giving an insight into the historical development of the studied subject. Students will also always train the text comprehension and experience the alienation, when comparing the different representations, verbalizations and sometimes different underlying intuition.

However, we believe, that in every course the usage of original sources may have a different focus and therefore different impact:

In courses on *advanced mathematics* original sources will help to contrast the often deductively presented content by underlining the procedural character of mathematics. This might have a positive effect on the process of understanding the instructed notions.

Courses on *elementary mathematics* not only aim to embed the school-mathematical content into the mathematics taught at universities and to present it in an abstract and mathematically correct language, but to call the students attention to the multitude of different approaches to an object, definition, theorem or prove. History of mathematics provides us with a number of different approaches. The often intuitive presentations, help to emerge the core ideas, that may be concealed in the modern, deductively structured and abstract notation.

Within courses on *didactics of mathematics*, original sources can initiate discussions about the nature of the subject. In these courses the focus will lie on the discussion of consequences for the teaching and learning. On the one hand the occupation with original sources might help to allude to some epistemological obstacles (cf. Brousseau, 1997) and possible consequences for the mathematics education. On the other hand original sources force to link mathematics to the situation at school and therefore persuade the students to concentrate on the subject matter didactically.

HISTORY OF MATHEMATICS – A DEMAND WITH TRADITION AND FRUSTRATION

The positive impact on teachers' studies is well known and intensively used at certain places. However, these places are still rare and the history of its use is as long as the history of the resigned comments about the ignorance of the "rest of the world". It suffices to quote just a few examples: Vollrath (1968) pleads for a direct genetic method (see our examples for thesis subjects) and mentions a lack of time and knowledge on the side of the professors[18], Artman et al. (1987) complain that mathematics fails to use the excellent historic examples[19], while Jahnke et al. (1996, p. VIII) state that "[h]istory [...] has an indispensable role [...] However, in spite of these well-established reasons in favour of introducing history into the mathematics classroom, both in schools and universities, the idea has not been very successful, as yet. [...] Somehow, history is considered alien to everyday classroom work".

Finally, Mosvold et al. (2014) look back at Arcavi (1982) and state: "Furthermore, the role of history of mathematics in general mathematics education research appears equally limited today as it was back then." So it seems important to realize and address the question: Why is the role of history of mathematics in general so much underestimated for teachers' studies? Why is it so plausible for most mathematicians *and* mathematics educationalists to abstain from an historical approach? Of course, the spectrum of reasonable answers will be out of proportion to the scope of our paper. One important aspect, however, might be that it is indispensable to reflect about the various specific aims one could pursue by presenting historical sources to teachers' students. Here, the general context of the course – university mathematics, elementary mathematics, mathematical didactics, or history – is of crucial relevance as we tried to show by our examples.

NOTES

1 Quoted from Georg Christoph Lichtenberg (1983). *Aphoristisches zwischen Physik und Dichtung.* Vieweg, Braunschweig, p. 135 (author's translation).

2 For further reading about the role of reflection on mathematics and its integration into school curricula see for example Ole Skovsmose (1998) or Roland Fischer (2006).

3 In this paper we focus only on history of mathematics. In our workshop at ESU 7, however, we discussed also the impact of studying philosophical sources. For further reading cf. Nickel (2015).

4 With respect to this differentiation cf. Jankvist (2009).

5 This conviction is in accordance with the widely shared concept of Allgemeinbildung. With respect to mathematics see, e.g. Winter (1995).

6 It may be necessary to emphasize here that approaches to transform the bulk of mathematical knowledge into a systematically ordered form are almost as old as the history of mathematics itself (Euclid's elements being the most prominent example).

7 In accordance to this general perspective working with historical examples and reflecting on contrasts is also supposed to be helpful for getting aware of ones own strategies of doing mathematics - from a theoretical point of view (cf. Fried, 2007) as well as by empirical evidence (cf. Kjeldsen and Petersen, 2014).

8 This special course, established in Siegen, is based on the curse of Felix Klein (1908): Elementary mathematics from an advanced standpoint. Among a detailed analysis of Klein's oeuvre, differences of these two concepts are discussed in Allmendinger (2014).

9 The curriculum is not the same all over Germany. Elements of the described perspectives are established in nearly every German university, but with very different weights of the different parts and organizational forms. So for example there are universities where history and philosophy of mathematics are only optional parts of courses in advanced mathematics. And sometimes elementary mathematics and mathematics education are toughed together in the same course. Never the less the authors are convinced of the necessity of an equilibrated role within teachers education of all these perspectives (cf. Beutelspacher et. al., 2011).

10 Originally the *Symbolic Logic* was published 1862 and the G*ame of Logic was* published 1887.

11 Some more examples can be found in Beutelspacher et al. (2011).

12 We use the German translation of Maser (Berlin, 1885), made available as copy for the students.

13 This part of the assignment was inspired by the interpretation of Fermat in Danckwerts & Vogel (2005, pp. 94-97). Going deeper into the details of Fermat's approach one can argue that there are alternative - and even by today's standards rigorous - heuristics which could account for his procedure (we owe an anonymous referee this remark).

14 For examples of successful usage of original sources in mathematics classroom and the discussion of didactical strategies and possible impact see Jahnke et. al. (2000, pp. 307-316) or Glaubitz (2010).

15 Schafheitlin, Paul (ed.) (1924): Die Differentialrechnung von Johann Bernoullie aus dem Jahre 1691/92. Oswalds Klassiker der exakten Wissenschaft. Leipzig: Akademische Verlagsgesellschaft.

16 Due to lack of time within the seminar lesson the students are not asked to find this information on their own – even if this would be more helpful by means of the aim of Allgemeinbildung coupled with the use of original sources (see above).

17 To introduce the different interpretations again original sources can be helpful. Comparing the definition by August Louis Cauchy (Cours d'Analyse (1815)) with the one of Karl Weierstrass can open up the horizon on a well known subject and help to subsume the schools' notion of derivation into the historical development (cf. Danckwerts/Vogel 2006, pp. 45ff.). Furthermore the students get to know two important protagonists in the history of rigorous analysis.

18 "Die direkte genetische Methode erfordert natürlich viel Zeit im Unterricht. Sie setzt auch einen Lehrenden voraus, der selbst in der Entwicklungsgeschichte der Mathematik zu Hause ist. Darin sieht H. Behnke (1954) die Ursache dafür, da diese Methode an der Hochschule kaum praktiziert wird. (...) Auch die Schule praktiziert kaum die genetische Methode. Sie vergeudet damit eine Quelle für Motivationen von der Sache her."

19 "Weder der Musiklehrer noch der Deutschlehrer lassen sich von den Schwierigkeiten abschrecken: Die Schüler lernen an erstklassigen Beispielen unserer kulturellen Tradition, ihren Geschmack zu bilden. Hier hat die Mathematik ein Defizit, in der Schule wie auch in weiten Bereichen des Hochschulunterrichts."

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Workshop

GAMES OF PIET HEIN & TRIBUTE TO MARTIN GARDNER

Bjarne Toft

The University of Southern Denmark

The workshop will exhibit some of the mathematical games of the Danish poet and designer Piet Hein, in particular SOMA and HEX, and material related to them. The games are remarkable because of their simplicity of rules, in contrast to their difficulty of play. They can be enjoyed at many different levels and by all ages.

The games were made famous worldwide by Piet Hein's friend Martin Gardner, who wrote about them in his Scientific American column. Martin Gardner occupies a special place in twentieth-century mathematics, inspiring young and old, amateurs and professionals, to think about mathematical problems and have fun. His writings are culturally broad, clear and contagious. He made recreational mathematics a respectable discipline and argued for its use in education. His close friend, Persi Diaconis, wrote about him: Warning: Martin Gardner has turned dozens of innocent youngsters into math professors and thousands of math professors into innocent youngsters. The workshop will pay tribute to Martin Gardner in the centennial year of his birth.

Oral Presentation

SOLVING SOME OF LAMÉ'S PROBLEMS USING GEOMETRIC SOFTWARE

Maria Zoraide Martins Costa Soares, Otília Terezinha Wiermann Paques, Rosa Maria Machado & Douglas Daniel

Instituto de Matemática, Estatística e Computação Científica

Laboratório de Ensino de Matemática

The mathematics teaching laboratory (LEM/UNICAMP), Brazil, currently has as aim to research the use of history of mathematics in teaching, using geometry software. This paper concerns two problems of the French mathematician Gabriel Lamé (1795-1870) with solutions using geometry software. We can find these problems in the book Examen des diferentes methodes employées pour résoudre les problèmes de géometrie (Paris-1818).

The view of Lamé for the problems was only algebraic. With the geometry software, we have dynamic solutions that complete the Lamé's solution and brings us a global analysis by means of the movement. These new view of the problem take us to add real conditions for Lamé's solutions.

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THEME 4:

HISTORY AND EPISTEMOLOGY AS TOOLS FOR AN INTERDISCIPLINARY APPROACH IN THE TEACHING AND LEARNING OF MATHEMATICS AND THE SCIENCES

Plenary Lecture

PROMOTING AN INTERDISCIPLINARY TEACHING THROUGH THE USE OF ELEMENTS OF GREEK AND CHINESE EARLY COSMOLOGIES

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Abstract: Most of the curricula, at an international level, encourage an interdisciplinary approach for the teaching of both mathematics and sciences. In this context, interdisciplinarity is often promoted as a fruitful way of making students aware of the links existing between mathematics and the sciences. History of science can be considered as an inspiring ground for the elaboration of teaching sequences where mathematical and scientific knowledge and skills are integrated. In this paper, examples of such integration are presented through the use of two distinct historical episodes dealing with Greek and Chinese early cosmologies. From these cosmologies teaching sequences (involving historical elements mixed with non-historical ones) have been elaborated in order to provide students with elementary astronomical knowledge dealing with scientific and mathematical knowledge and skills.

INTRODUCTION

Most of the curricula, at an international level, encourage an interdisciplinary approach for the teaching of both mathematics and sciences (see for example AAAS 1989, Rocard 2007). In this context, interdisciplinarity is often promoted as a fruitful way of making students aware of the links existing between mathematics and the sciences. As an example, the third pillar of the French *common base of the knowledge and skills* for primary and lower secondary school claims for “concrete and practical approaches to mathematics and sciences” that should allow students to acquire the “scientific culture needed to develop a coherent representation of the world and an understanding of their daily environment” and help them grasp that “complexity can be expressed in fundamental laws” (French MEN 2006 – my trans.). Here, mathematics and experimental sciences are considered altogether in a global enhancement project of scientific literacy.

Nevertheless, nothing is easy about effectively integrating mathematics and science in the classroom since the disciplinary isolation of the two disciplines in the traditional teaching organizations has to be overcome (Czerniak et al. 1999). Indeed, in most cases, the separation between science and mathematics is rigorously maintained, even in primary school where both mathematics and science are taught by a unique teacher. Moreover, few teaching materials involving both mathematics and science have been developed (Davison & al. 1995). Nevertheless, research addressing interdisciplinarity issues show that even young students are able to acquire skills in the domains of mathematics, science, and scientific processes such as measuring, modeling, etc. (Munier & Merle 2009). The lack of teaching resources of that kind may be puzzling if one considers the interrelations between science and mathematics in their historical developments. In this regard, history of science can be considered as an inspiring ground for the elaboration of teaching sequences where mathematical and scientific knowledge and skills are integrated.

In this paper an example of such integration is presented through the use of three distinct historical episodes dealing with early Greek and Chinese cosmologies. From these cosmologies teaching sequences (involving historical elements mixed with non-historical ones) are elaborated in order to provide students with elementary cosmological knowledge dealing with scientific and mathematical knowledge and skills (quasi-parallelism of Sunrays, shape and size of the Earth, Sun-Earth distance, measuring and computing, etc.).

MATHEMATICS AND PHYSICS AS INTERRELATED AREAS OF KNOWLEDGE

Claims for bridging mathematics and physics in science teaching often refer to Galileo Galilei who wrote:

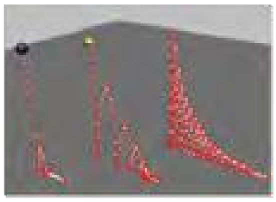
> Philosophy is written in that great book which ever is before our eyes - I mean the universe - but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is **written in mathematical language, and the symbols are triangles, circles and other geometrical figures**, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth (Galileo, *Il Staggiere*, 1623).

Actually, physics embraces much more mathematics than Euclidian geometry and the intricate connection between mathematics and physics have been valued by lots of scientists before and after Galileo (see Siu, 2009). In a few words, physics can be defined as a domain of knowledge which explores “inanimate nature” (Wigner 1960, p. 3) from infinitely large to infinitely small. This exploration can concern structure, organization, movement of matter; it can involve elementary objects or interactions between objects, etc. The generic process of the discovery of laws of nature is to translate natural phenomena - observable or not, combining measurable quantities in order to establish laws expressed mathematically; these quantities refer to formal concepts (ie: concepts that don’t have empirical correspondences and which are defined by their attributes – such as Force, Energy, Field, etc.). All the laws of nature are conditional statements which permit a prediction of some future events on the basis of the knowledge of the present. As a consequence, the validation of knowledge in physics is absolutely based on both “reproducibility” and “predictability”:

Historically, physics progressively passed from a construction activity interrelated to (Euclidian) geometry to an activity that describes the variation of matter and radiation in space and time. Consequently, most of phenomena the physicist is interested in are described by the second derivatives of positional or temporal coordinates.

To sum up, the language of the physics applies to systems extracted from the real-world. It is structured by figures, graphs, mathematical symbols, or proposals formed by words. It allows predictions and relies on causal relationships established through measurements. In this context, problems under physics are diverse (explanation, creation of phenomena, of objects, predictions of behaviour, etc.) but globally, their solutions take the form of laws which are assumed to govern the reason for the in-animate nature (why nature - matter, radiation - is as it is?), the how of its past (how did it get there?) and its future (what would happen if…).

Let's consider an example: the bouncing of balls. Here one can focus on "why" balls bounce or on "how" they bounce. If one focuses on "how" balls bounce the physics enterprise consists in looking for a relationship between different quantities on the basis of conservation laws. We find that a constant quantity exists which connects the bounce height and is the drop height. And what is very interesting is that this quantity - the coefficient of restitution (*k* in fig. 1), allows predicting the total distance *D* travelled by the ball according to the number n of bounces and the associate length of time *T*.



$$D = d_0 + 2d_1 \frac{1 - k^{2n}}{1 - k^2}$$

$$T = t_0 + \sum_{t=1}^{n} 2t_n = t_0 + 2t_1 \frac{1 - k^n}{1 - k}$$

$$k = \frac{v_2}{v_1} = \frac{t_2}{t_1} = \sqrt{\frac{h_2}{h_1}}$$

Fig. 1: Mathematics modelling of the bouncing ball. *D* represents the total distance covered by the ball and *T* the length of time of the movement of the ball.

Looking at the formula (fig. 1) one can mathematically admit an infinite number of bounces and find a finite distance *D*, but this infinite limit does not make sense from a physics viewpoint, since the ball stops its movement after a while. This aspect can be puzzling for students.

MATHEMATICS IN PHYSICS LEARNING

Several researches in physics education have been carried out where interplays between mathematics and physics have been questioned (Artigue & al. 1990, Gill, 1999, Albe & al. 2001, Melzer, 2002, Hestenes, 2003). Some researchers have focused on the difficulties encountered by students when using/facing mathematics in physics (manipulation of vector quantities); others have shown that the process of conceptualizing in physics strongly takes advantage of a good mastery of mathematics: the meaning of the constant of integration allows to understand the importance of the initial conditions of velocity for example in mechanics, or, a single point in a space-time diagram allows to better grasp the deep signification of what an event is in the framework of Special Relativity:

> There is a significant correlation between learning gain in physics and students' pre-instruction mathematics skill (Meltzer, 2002)

As a conclusion, considering mathematics as a part of the teaching of physics is an epistemological reality and a cognitive opportunity since the mathematical abstraction favours the conceptualization process in physics (and *vice versa* ?). Thus, it appears as a real didactic necessity[i]. In the following the potential of history of science in the promotion of teaching sequences based on effective mathematics-physics interplays is highlighted.

HISTORY OF PHYSICS AND PHYSICS LEARNING

Today researches involving science education and history of science follow two different orientations – which are not mutually exclusive. The first one aims to provide students with element associated with Nature of Science (science activity, elements of epistemology, etc. see Abd el Khalick & Lederman, 2000); the second one searches for elements that may favour a better appropriation of concepts and laws. In the way I work, the spontaneous reasoning of students plays a determining part. Actually, the common sense is very powerful in providing operational and coherent explanation while facing empirical phenomena. Today, the whole community of researchers in physics education agrees that learning physics is based on a negotiation process between the rationality of the common sense[ii] and the rationality of physics. In this regard, using history of science for physics learning in a conceptual perspective should take into account the type of reasoning a student can hold concerning a phenomena to be studied. And a way of managing this is to search within history of science ideas that could, to some extent, echo with common students' ideas or conceptions in order to create a problem directly inspired by an historical episode that could meet student's interest and thus, be accepted by them.

First example: Earth is spherical

My first example leans on a teaching sequence for 10 year-old children elaborated by the French science education researcher Hélène Merle (2002).

The context is Greek astronomy, also qualified by historians of science as "mathematical astronomy" (Neugebauer, 1957) or "geometrical astronomy" (Coveing, 1982). In Ancient Greece, the prevalent assumption is that the movements of celestial bodies are circular and uniform:

> Pythagorism turned geometry into the instrument for astronomy as a contemplative science of the natural being. (Coveing, 1982, p. 146).

The historical text that inspired the teaching sequence is an excerpt of Aristotle's *Treaty of the Sky* in chapter 14. In this text, Aristotle argues for a spherical Earth on the basis that:

> According to the way celestial bodies show themselves to us, it is proved that not only the Earth is round but what is more it is not very big; because we just have to make a

> small travel either at the South or at the North, so that the circle of the horizon becomes obviously quite different. So the celestial bodies which are over our heads undergo a considerable change, and they do not seem to us any more the same, as we go to the South or to the North. **There are certain celestial bodies that we see in Egypt and in Cyprus, and that we do not see any more in the northern parts of the country**. On the contrary, some celestial bodies that we see constantly in the northern countries lie down when we consider them in the parts of the country which I have appointed. This proves not only that the shape of the Earth is spherical, but still that its sphere is not big (Aristotle, *Treaty of the sky*, chap. 14).

The sequence addresses children's ideas about "horizon" and takes into account the idea that vertical and horizontal notions are only considered by 10 year-old children locally. As an example, children represent the level of a given quantity of water contained in a bottle, as a straight line, and its direction is perpendicular to the boundaries of the bottle itself (Ackermann, 1991) – whatever the orientation of the bottle.

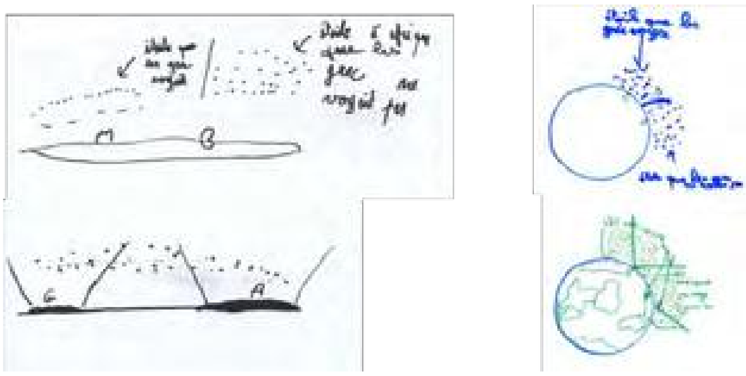


Fig. 2: Drawings provided by children involved in Merle's research (2002) who are asked to explain the reason why some stars disappear when travelling to the South. ***Translation of children comments: "stars seen by Greeks" / "stars Greeks cannot see".***

First, the problem is transformed and adapted so that children are asked to explain the reason why some stars disappear when travelling to the South. They provide some relevant drawings, relevant in the sense that they fit with Aristotle observation (fig. 2). But some drawings involve a flat Earth, while others involve a spherical Earth because the way children represent the "field of vision" (vertical or oblique lines) fit with both shapes (fig. 3). In other words, Associated with the way children represent the "field of vision", Aristotle observation is not sufficient to discriminate a spherical Earth from a flat one.

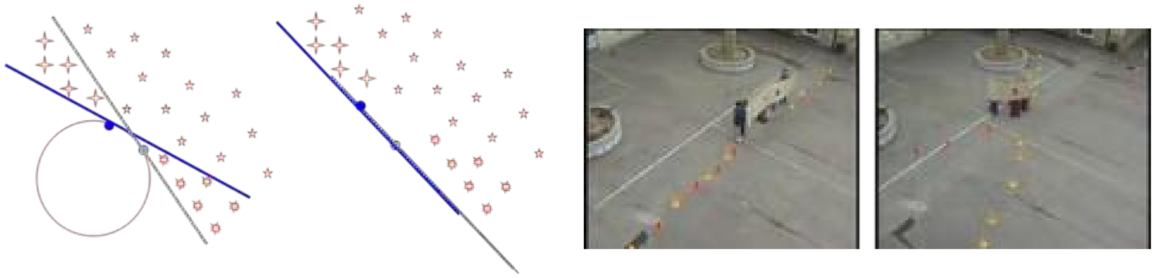


Fig. 3a: Horizon seen as a tangent line to the terrestrial sphere allows Aristotle to explain why some stars disappear when travelling to the South on a curve ground. In figure 3b, two children are placed behind a cardboard where two windows have been opened so that they can see other children transporting coloured cones (red and yellow). Each child is responsible for a given coloured cone and is piloted by one of the children placed behind the cardboard. Children transporting a cone are asked to drop it when it cannot be seen through the window. In the pictures two geometries of Earth's ground are modelled: a flat one and a curve one; to each geometrical model corresponds two lines of cones (ie: two horizon lines) differently arranged.

In other words, for Aristotle, the differences, in visible stars from two different places is a hint of the spherical shape of the Earth, but not for children.

The interesting thing here is that history of science provides a fruitful problem-to-be-solved: the problem of the stars can be solved by children and make their conception of the field of vision (as a delimitation for the visible space) be expressed. Because children know that Earth is round, they can use this knowledge in order to initiate a conceptual change and pass from an inappropriate modelling of the visible space to a correct one. Note that there is no parallelism between a (supposed) historical path and children's conceptual development, since horizon is a geometric tool that allows Aristotle to argue in favor of a spherical Earth, while children use a spherical Earth hypothesis to build the concept of horizon. In this regard, the way the sequence is conducted in order to make children conceptualize the horizon line is totally a-historical (see fig. 3b).

Second example: cosmological distances are measurable (see de Hosson & Décamp, 2014).

The second example leans on two different cosmologies: the Chinese one and the Greek one seen in the light of various historical sources: *the Zhou Bi* in the Cullen (1996) and Kalinowski (1990) translations, Ho Peng Yoke's *Astronomical chapters of the Chin shu* (1966), and Clemoledes' *De Motu circulari corporum caelestium* in the Weir translation (1931).

A current astronomical activity in primary school in France (carried out in both mathematics and science courses (see for example, di Folco and Jasmin 2003; Kuntz 2006) consists in exploiting the procedure supposedly used by Eratosthenes in the 3rd century BC in order to measure the perimeter of the Earth. This procedure leans on two observations: (1) A gnomon located in Alexandria (northern Egypt) at noon the day of the summer solstice casts a shadow of a certain length, (2) At the same time a gnomon located in Syene (middle Egypt) casts no shadow since the Sun appears at the zenith. From the pedagogical use of Eratosthenes procedure, some researchers questioned children's difficulties while modeling the Sunrays (Feigenberg et al. 2002, see fig. 4). Children are asked first to explain why during summer solstice at noon, a gnomon located in Alexandria (northern Egypt) casts a shadow while another (identical to the previous one) located in Syene (middle Egypt) casts no shadow.

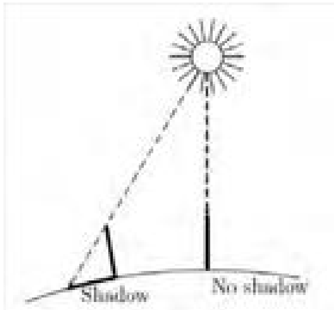


Fig. 4: Drawing provided by students who are asked to model the reason why gnomon placed at Alexandria casts a shadow, whereas at the time the day of summer solstice (at noon) a gnomon placed at Syene casts no shadow.

As an explanation (of what we will call the 'shadows observations'), some children draw non-parallel rays coming from a sketched Sun down onto a curve (or a plane) surface of the Earth. This drawing is also typical of those proposed by most of the primary teachers (target of the following sequence) explaining the same observation (Merle 2000).

The Chinese text presented hereafter (Doc. 1) is taken from the Chin Shu, a book written around 635 A.D.

> According to the Chu Li (Rites of Zhou), the shadow of the Sun at midday during the summer solstice was 1 chi 5 tsun. The place where this particular observation was made was known as the 'Earth centre'. Cheng Chung said that the length of the gnomon shadow template was 1chi 5 tsun and that the place where a vertical pole 8chi in length at midday of the summer solstice cast a shadow the same as that of the shadow template, was called the 'Earth centre'. The place corresponds to the present location of Yangchen, in Yingchuan. Cheng Huan said that the shadow cast by the Sun on the Earth surface changed by a length of 1 tsun for every change of 1000 li in the horizontal distance (north or south). Since the length of the shadow is 1chi 5 tsun, the Sun is 15000 Li away and to the south of the observer. From this it can be deduced that the vertical distance of the Sun is 80000 Li from the Earth's surface"

Doc. 1: *The Chin Shu*, Ho Peng Yoke (1966), p.65 - Units of length: 1chi = 10 tsun = 35.8 cm; 1 tsun = 3.58 cm, 8chi = 2.86 m and 1li = 560 m.

The astronomical part of this book has been written by Li Shun-fêng. The proposed excerpt refers to the astronomical knowledge under the Zhou dynasty that began about a thousand year B.C. Another historical text, *the Zhou bi* (namely, the gnomon of the Zhou) gives similar elements to those found in this Chin Shu. The proposed excerpt presupposes children and teachers main type of explanation of the ‘shadows observations’ and based on it computes some measurements: the shadow of a vertical eight chi long gnomon (2.86 m) located in Yangchen is 15 tsun (53.7 cm) long, at noon, on the day of the summer solstice. The same day at the same time, an identical gnomon located 1000 li (560 km) south of Yangchen will cast a 14 tsun (50.1 cm) long shadow, and if it is located 1000 li (560 km) north of Yangchen, this gnomon cast a 16 tsun (57.3 cm) long shadow. The text presented in doc. 1 deduces from these measurements that the Earth-Sun distance is 80,000 li (44,800 km). Figure 5 helps us to understand this result.

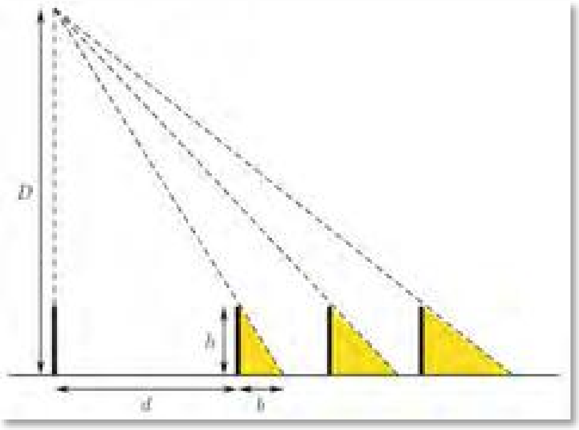


Fig. 5: Distance *d* is computed knowing that each 1000 li, a *h* height gnomon casts a shadow whose length decreases of 1 tsun. Since *b*=15 tsun, *d*=15 000 li. Today, one can compute Distance *D* by using similar triangles property: (*D*-*b*)/*d*=*h*/*b*; since *h*=8chi, *D*≈80 000 li.

A similar observation supports a spherical geometry, which seems to be at the root of Eratosthenes measurement of the Earth’s perimeter. Unfortunately, the original writings of Eratosthenes (2 books) were lost. We have access to his work only through authors of antiquity such as Cleomedes, Pliny, Strabo. The most detailed among these writings is a short review by Cleomedes. We are told by Cleomedes (see the translation of *On the circular motion of the celestial bodies*, book 1, Chap. 7, by Weir 1931) that Eratosthenes made measurements with a gnomon that cast a shadow onto the graduated inner surface of a hemispherical sundial named *scaphe*. Eratosthenes knew that on a certain day (summer solstice) at noon in Syene, the gnomon of a *scaphe* cast no shadow, whereas the same day at the same time in Alexandria (located at 5000 stadia -800 km- at the north of Syene) the shadow cast by the gnomon of an identical scaphe reaches an arc equal to 1/50th of a circle from the base of the gnomon (Fig. 6). Assuming the parallelism of the sunrays that reach Syene and Alexandria and the fact that both cities are on the same meridian, it is easy to deduce that the distance between Syene and Alexandria is also equal to 1/50th of Earth’s circumference and then to compute this measurement. The method Eratosthenes used to compute the distance between Syene and Alexandria has been subject to debate. It seems to be based on maps of Egypt or on accurate distance estimations made by bematists. These men were trained to make regular paces when marching from one place to another and to record their numbers (Dutka, 1993).

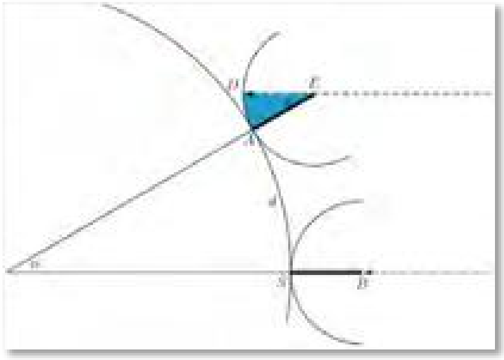


Fig. 6: Illustration of Eratosthenes' procedure as described by Cleomedes in *On the circular motion of the celestial bodies* (Weir 1931). The shadow *AD* cast by the gnomon in the hemispherical sundial reaches an arc equal to 1/50th of a circle of radius *AE*. The ratio of 1/50th of the circumference of the Earth corresponds to the distance AS between Alexandria and Syene (de Hosson & Décamp, 2014).

There are many similarities between the Chinese and Greek chosen measures. In both cases, the astronomers have chosen noon of the summer solstice to make their measurements. This is probably not by accident: midday (solar time) of the summer solstice corresponds (in the Northern hemisphere) actually to the moment at which the shadow of the gnomon is the shortest during the year. In both cases, they also computed a terrestrial surface measurement and derived from it a vertical measurement using a strategy based on proportionality. Both used a sundial but an interesting difference is the fact that Chinese and Greek instruments are not exactly the same. The Greek *scaphe* and its gnomon are an hemispherical sundial. It gives direct access to the searched portion of the circle (we would say to the angle in modern terms) and this is the useful measurement in a spherical Earth cosmology. The Chinese *bi* is on the contrary a flat sundial which gives access to the angle tangent, a more adapted measure for a flat Earth cosmology. This is an interesting illustration of Bachelard's thought:

> A measuring instrument always ends up as a theory: the microscope has to be understood as extending the mind rather than the eye. (Bachelard 2002, p. 240).

The Chinese astronomical model promotes a hypothesis very close to prospective primary teachers' ideas about the propagation of the Sunrays. This model (flat Earth/close Sun sending divergent Sunrays) was chosen as an anchoring situation that would echo students' prior knowledge. Prospective teachers were then engaged in operating this model through an experimental activity. By confronting Chinese data (e.g. the Sun-Earth distance) with the current one, they were prepared to elaborate an alternative way of modeling the shadows situations, based on parallel Sunrays. Nevertheless, the majority of these future teachers did not understand how a single point could send parallel rays as illustrated in the following piece of transcription between a prospective teacher T and the researcher R:

> [T] I still don't understand why the light sent by a single luminous point can be modeled using parallel rays
>
> [R] Actually, the Sunrays are not exactly parallel. But if I stretch out two long strings from the same mooring, their extremities can be considered as parallel lines under certain conditions? Which one?
>
> [T] Hum… if it is nearly parallel it is very different!
>
> [R] Why?
>
> [T] Because in the case of nearly parallel lines from a single point it is obvious that they can have the same origin, whereas if they are really parallel they will never cross each other; they cannot come from the same point
>
> [R] Ok. But what could allow considering these two lines having the same mooring as parallel?
>
> [T] If the extremities are very close
>
> [R] Do you think Syene and Alexandria are close enough to consider the Sunrays reaching them as parallel lines?
>
> [T] Well… oh… ok… Yes, since the Sun-Earth distance is much larger than the distance between the two towns! The distance between the extremities of the Sunrays should be very very small

Actually, the Sun is not a point source of light but an extended one and its angular diameter is about 0.5°.

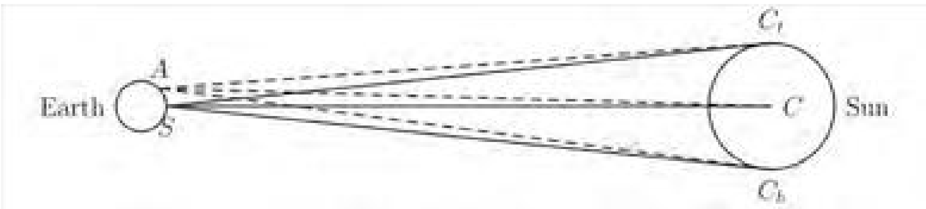


Fig. 7: Illustration of two cones of light coming from the Sun and reaching Syene S and Alexandria A. Figure not drawn to scale: $\widehat{C_tAC_b} = \widehat{C_tSC_b} \approx 0,52°$whereas $\widehat{ACS} \approx 1,1''$, (Décamp & de Hosson, 2012).

Half of the Earth's surface is struck by an infinite number of cones, each containing an infinite number of rays sent out by each point of the Sun. In Fig. 4 we have illustrated two of these cones reaching the towns of Alexandria and Syene. One must consider that the angle $\widehat{C_tAC_b} = \widehat{C_tSC_b} \approx 0{,}52$. This angle is small because the diameter of the Sun [C_tC_b] is small with respect to the distance between Sun and Earth. This is the reason why the Sun can roughly be considered as a point source. In the same picture we see that the angle $\widehat{C_tAC_b}$ is even smaller. Therefore the sun's rays can be considered parallel. This clarification illustrates the complexity of the process underlying the hypothesis of parallelism of sunrays, which at the scale of a portion of 1/50th (or 1/48th) of the Earth's circumference remains an approximation. Similarly the assimilation of the Sun to a single point stems from an identical complexity. Yet, the fact that the Sun is an extended source explains that the shadows cast by gnomons are surrounded with a partial shadow area.

Only a rigorous geometrization of the astronomical construction legitimates the approximation usually presented to students. In that perspective mathematics gives sense to physics, not only for the understanding of concepts but also for the grasping of the deep meaning of what physics is: a construction of theory and models validated through the predictions they allow within certain domain of validity and taking into account measurement uncertainties.

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[i] In France, something quite surprising happens concerning physics teaching. In order to make physics more attractive, in 2011, the curriculum designers have make mathematics almost disappear and we can read in the K-12 curriculum that "using mathematical tools is not the first target of physics learning, even though it is sometimes necessary to carry a study out". It seems that physics teaching in France is a paradigm where mathematics reduces physics attractiveness…

[ii] Common sense can be considered as a set of representations of the world shared by most of people and capable to generate operative and relevant explanation concerning natural phenoma but inappropriate according to scientific rationality (see Viennot, 2001, 7-11).

Workshop

WHAT CAN WE LEARN FROM JACQUES ROHAULT'S LESSONS IN MATHEMATICS AND PHYSICS?

Frédéric Metin & Patrick Guyot

University of Nantes, University of Dijon

Jacques Rohault (1618-1672) was often referred to as 'the famous Monsieur Rohault' in contemporary English books about natural philosophy. He was regarded as the most important Cartesian philosopher, and his posthumous works were published several times in London, in Latin as well as in English, even after his own homeland had forgotten him.

Rohault was also a famous teacher: his public lessons in Paris were attended by a variety of persons, as Malebranche witnesses. The lessons on physics were especially appreciated, for they were spectacular and based on experimentation; but we must not forget that Rohault was also a teacher of mathematics for the Dauphin of France, and as such he taught Euclid's Elements (the first six books), trigonometry, practical geometry, fortifications etc. The course for the Dauphin has been taught in other private lessons, the manuscripts of which still exist in public libraries.

The reading of Rohault's writings, pointing out the differences between printed and manuscript versions, offers a view on a great mind of his century, inventor of pedagogical choices based on the personal implication of the one who teaches, on debate and controversy.

Suggested texts (to be studied during the workshop; the text in French will be "made English"):

Œuvres posthumes de Jacques Rohault. Paris, 1680.

Course on trigonometry and fortification (manuscripts), ca 1670.

Rohault's System of Natural Philosphy, illustrated with Dr Samuel Clarke's notes, taken mostly out of Sir Isaac Newton's Philosophy. London, 1723.

Workshop

SOME STEPS OF LOCATION IN OPEN SEA

HISTORY AND MATHEMATICAL ASPECTS FROM ASTROLABE TO GPS

Xavier Lefort

IREM des Pays de Loire, Université de Nantes

Quelques étapes du repérage en haute mer: atelier pour ESU7
Se situer en haute mer, loin des côtes, a toujours été le principal problème de la navigation hauturière. Les seuls repères possibles se trouvent dans le ciel, le soleil, la lune et les étoiles. L'utilisation d'un système géocentrique a perduré jusqu'à aujourd'hui, conduisant à d'intéressants calculs en géométrie sphérique.
L'atelier proposait un historique des méthodes de repérage, en s'arrêtant à quatre étapes: Pedro de la Medina (L'Art de naviguer 1569), Simon Stevin (De l'histiodromie 1634), Etienne Bézout (cours de navigation 1781), Thomas Sumner (Finding a ship's position at sea 1851). Lire et refaire les calculs issus des ouvrages de ces auteurs pour retrouver leurs résultats et les commenter est peut-être un bon moyen d'ouvrir les mathématiques au grand large!

Some steps of the location in open sea: workshop for ESU7
On the open sea, far from land, has always been the main problem of navigation on the high seas. The only possible marks are in the sky -- the sun, the moon and the stars. The use of a geocentric system has endured to the present, leading to interesting calculations in spherical geometry. The workshop presented four episodes in the history of the methods of location, drawing on the works of Pedro de la Medina, (L'Art de naviguer, 1569), Simon Stevin (De l'histiodromie, 1634), Etienne Bézout (Cours de navigation, 1781), Thomas Sumner (Finding a ship's position at sea, 1851). Reading and carrying out the original calculations from the works of these authors to find their results and comment on them is maybe a good way to open up mathematics to the wide horizons.

THE ASTROLABE, THE COMPASS, LATITUDE, LONGITUDE

When did offshore navigation begin to be practised? Not before the fifteenth century as far as we know, although there is possible evidence of earlier open sea navigation from North America and Northern Europe and elsewhere.

To find a position on the Earth, we need a model: in the second century, Ptolemy conceived a geocentric model of the universe, which remained the undisputed standard until Copernicus and Galileo. This geocentric model is still the one used today for the location of any point on the surface of the Earth, including and especially for navigation. We also need fixed points, such as the stars, and some instruments: the astrolabe, the compass, etc.

While the astrolabe has been in use for navigation only since the fifteenth century, and then only in a very simplified version, the compass appeared much earlier, around 1100. It consists of a magnetized needle, originally used in China and in the Mediterranean and then, during the following century in England, before its adoption throughout Europe and Arabia. Initially, it consisted of a needle resting on a pivot above a compass rose (see figure 1) but was later set on universal joints inside a wooden box. The compass rose is graduated in directions or 'rhumbs', that is in bearings. The problem is that this graduation is not in degrees, but obtained by successive divisions between the four cardinal points, so that between North (N) and East (E) we have NE and then NNE, ENE and so on.

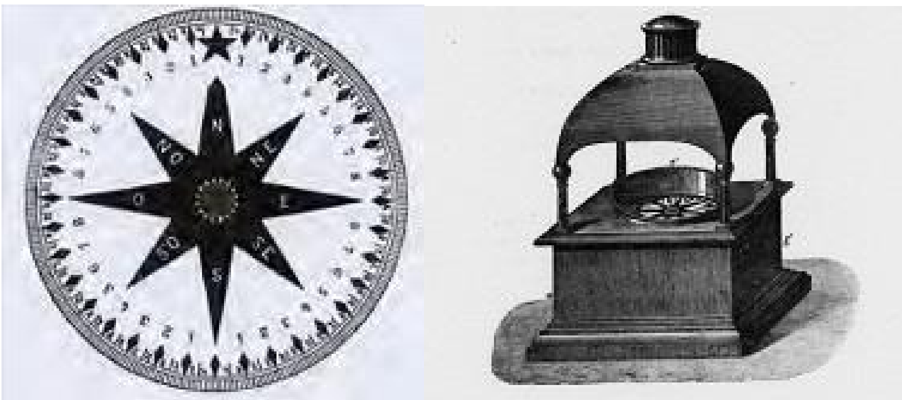


Figure 1: Wind rose and compass, Dubois, E. (1869). *Cours de navigation et d'hydrographie*, Paris: Arthus-Bertrand.

Apart from the lack of refinement, a compass bearing is not the best solution to go from A to B except for short distances. Indeed to travel on a constant course is an excellent way not to arrive there. The trajectory determined in this way is a loxodrome or rhumb line, which will wind around the pole, as shown in figure 2. The most direct trajectory is an orthodrome, a section of a great circle, but this requires regular adjustments to the course.

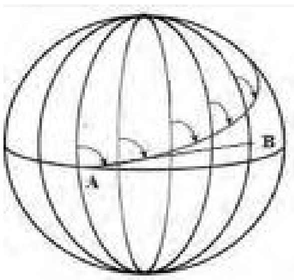


Figure 2: Rhumb line on terrestrial sphere

Any attempt at navigation must begin by identifying a position on the globe. A spherical coordinate system uses latitude (the distance from the Equator), and longitude (the distance from a meridian, or fixed great circle through the poles). In fact these ‘distances’ are measurements of angles, given by α and β in the figure. Lines of latitude are Northern or Southern as required and longitudes are East or West of an agreed prime meridian (universally adopted as the Greenwich meridian in 1884).

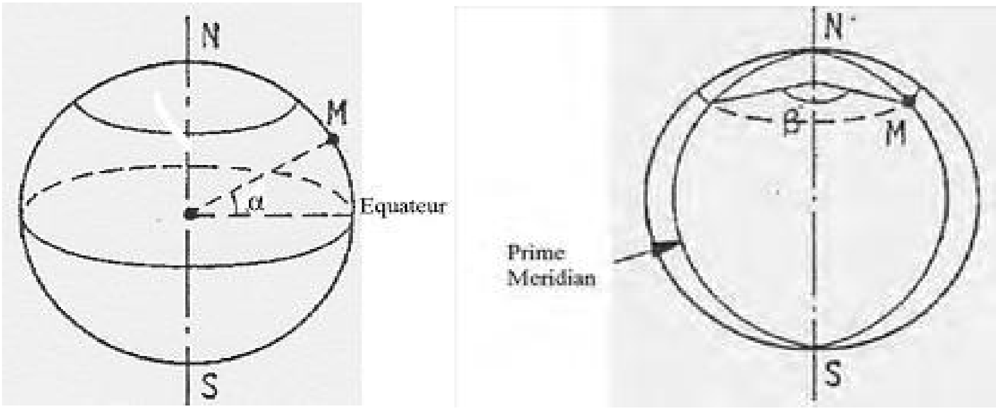


Figure 3: latitude and longitude of a point M

We can determine latitude from observing a fixed point in the sky, the polar star or the sun at noon, say. The polar star was a popular choice because when it is at its maximum the angle the North Star makes with the horizon is the same as the latitude of the observer. For other fixed points we need to take account of the apparent tilting of the polar axis with respect to the plane of the ecliptic, called the declination, which changes throughout the year (see figure 4). For the sun at noon at any point P we have: Latitude = 90° - altitude + declination

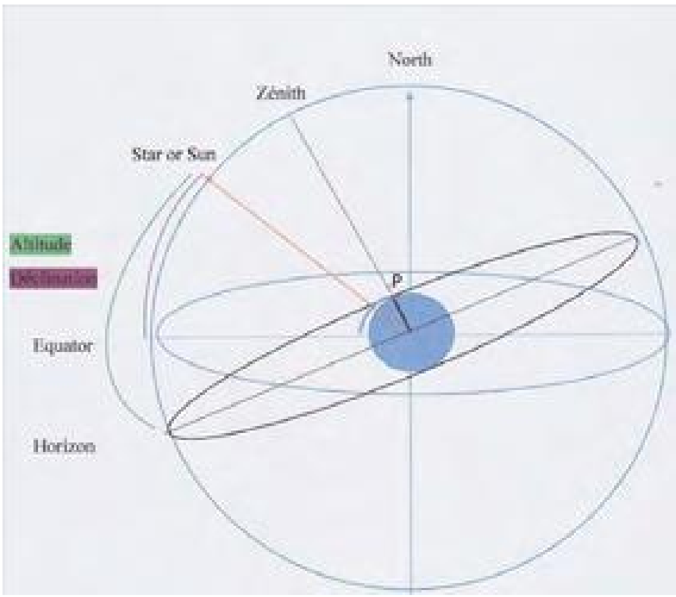


Figure 4: Declination on terrestrial sphere

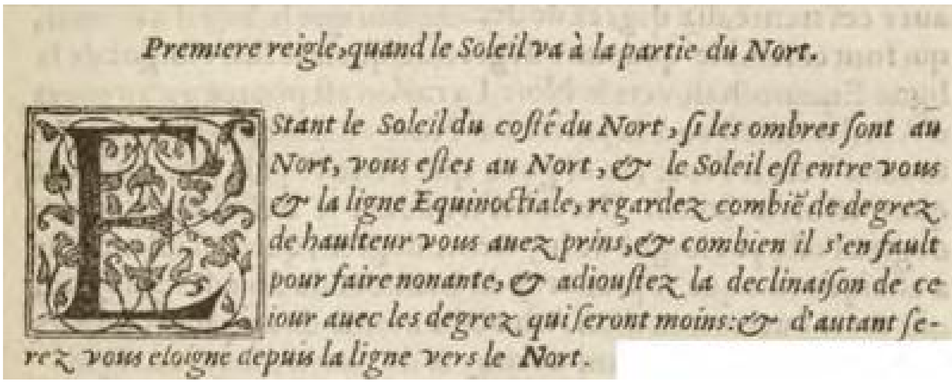
The declination being taken as negative when the sun is north of P. When using this method, the values of declination for each day would be have to be found from previously calculated tables.

Thus calculating latitude is a fairly simple procedure but determining longitude is much more difficult. It is proportional to the time difference between a point P and a reference point, originally the port of departure. But to do that requires carrying an accurate timepiece set to the reference point. Before that time some astronomical measurements were in use. The general idea was to know the time when an astronomical phenomenon seen from a reference place occurs, and to note the local time when one observes the same phenomenon. There were at the end of the fifteenth century tables giving this information, along with the declination of the sun. The time difference in hours multiplied by 15 will give the difference of longitude.

THE AGE OF DISCOVERY

Determining position at sea was always a problem. In 1487, the Portuguese sailor Bartolomeu Diaz sailed south along the African coast. He went too far south to avoid storms and then sailed east. He went back north and found land, having passed the Cape of Good Hope unawares. In 1492, the first voyage of Christopher Columbus was along a fixed latitude from the Canary Islands. Columbus described many difficulties in determining his longitude position, and his calculations are very erroneous. Amerigo Vespucci, later, explained his methods for determining latitude and longitude, but the results he gave were obviously wrong and his explanations not very convincing. (He made four trips between 1497 and 1504.)

A contemporary of Christopher Columbus, Pedro de Medina (1493--1567) is the author of one of the first books about navigation in the open sea. His *Ars de Navigar* (1569), was very quickly translated into Portuguese, French, English and Italian.


Premiere reigle, quand le Soleil va à la partie du Nort.

EStant le Soleil du coſté du Nort, ſi les ombres ſont au Nort, vous eſtes au Nort, & le Soleil eſt entre vous & la ligne Equinoctiale, regardez combiẽ de degrez de haulteur vous auez prins, & combien il s'en fault pour faire nonante, & adiouſtez la declinaiſon de ce iour auec les degrez qui ſeront moins: & d'autant ſerez vous eloigne depuis la ligne vers le Nort.

Figure 5: First rule when the sun is going to the North.Pedro di Medina. (1569) *L'Art de naviguer*, trans. Nicolai, Lyon: Roville (ed), pp 97-98.

The sun being on the North side, if the shadows are at the North, you are in the Northern hemisphere, and the sun is between you and the equator; watch how much degrees in altitude you've measured, how much it needs to make ninety and add the

> declination of this day to the degrees which will be less: you will be as far away from the line northwards. (pp. 97-98)

This is pretty self-explanatory. The sun being on the north side meaning north of the equator and the declination of the day would have to be found from tables.

> Example
>
> The sixth of April, taking the altitude of the sun, the shadow comes to fall to the North, and finds the sun in sixty degrees on the astrolabe, ten degrees of declination which the sun has this day. (pp. 97-98)
>
> What is the latitude? (Solutions to all problems are at the end.)

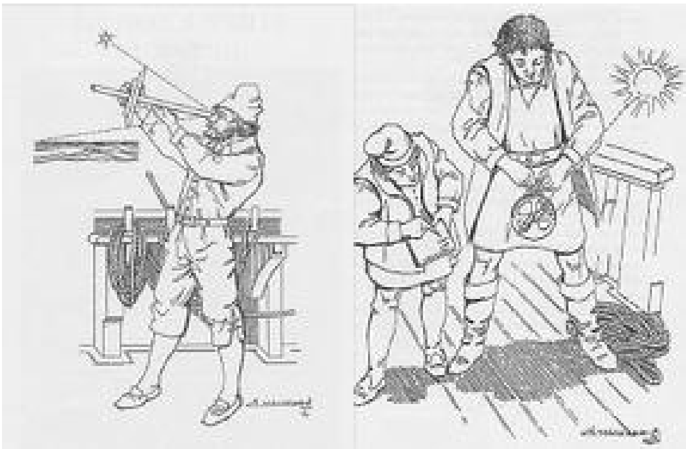


Figure 6: Use of the cross staff and astrolabe (private collection)

To measure the altitude of a celestial body, different instruments were used (see figure 6), some taking a sighting of the sun and so dangerous for the eyes; observers were frequently affected by blindness.

Figure 7: (By courtesy from the association *La Méridienne*, Nantes)

It is also important to know how far the ship has voyaged. Until the nineteenth century, speed at sea was measured using a 'chip log' (see figure 7) which was a wooden board cast over the side. The board was attached with a rope and knots were made in the rope. To measure the speed of the ship, one sailor counted the knots as the rope passed between his hands while another marked the time using a sand glass. The number of knots in 30 seconds was the speed of the vessel.

As long as the navigation remained coastal, it was essentially a question of giving the pilot enough information about his position to avoid the dangers of the coast. This need gave rise to numerous maps, so called portolans, with information about ports and straight line directions. They were not very accurate and were often ornamental. They were mostly useful for coastal navigation. These portolans did not take account of the curvature of the Earth and to correct this Mercator proposed a map where meridians are parallel and spaced out so as to preserve the angles. Mercator's projection is still in use today.

In straight line navigation, called dead reckoning (also dead reckoning from deduced), the navigator finds his position from the course taken and the distance sailed from the starting point, whose location is known. The differences of longitude and latitude are then easy to calculate (assuming a plane surface).

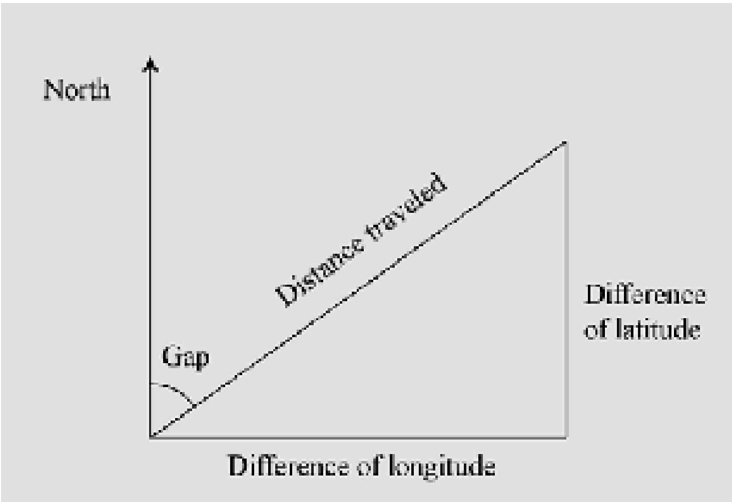
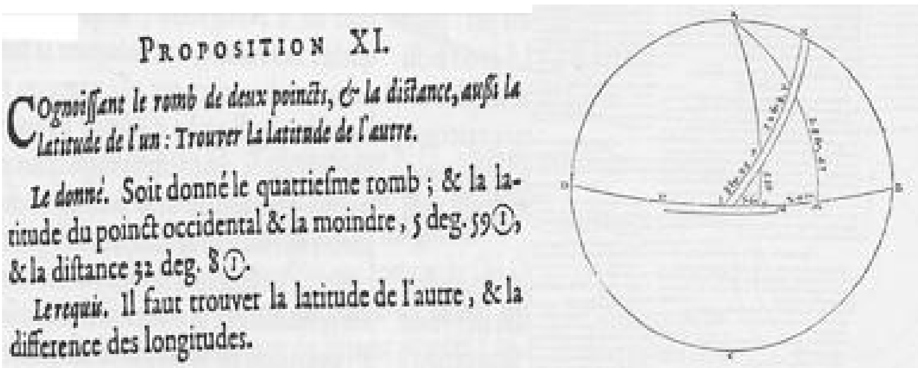


Figure 8: Dead reckoning

THE SEVENTEENTH CENTURY

Simon Stevin, more scientist than sailor, is the author of a work on navigation where the triangle of position appears for the first time, which will later be the foundation of the calculation of location. This is proposition XI from *L'histiodromie*, (1634):



PROPOSITION XI.

Cognoiſſant le romb de deux poincts, & la diſtance, auſſi la Latitude de l'un : Trouver la Latitude de l'autre.

Le donné. Soit donné le quatrieſme romb ; & la latitude du poinct occidental & la moindre, 5 deg. 59①, & la diſtance 32 deg. 8①.

Le requis. Il faut trouver la latitude de l'autre, & la difference des longitudes.

Figure 9: Stevin, S. (1634). De l'Histiodromie ou cours des navires, *Géographie, Livre 4*, trans. A.Girard, Leiden: Elsevier.p163

> Knowing the rhumb between two points, their distance and the latitude of the first one, find the latitude of other one. We give the fourth rhumb, the latitude of the first point, 5° 59', and the distance 32° 8'. Find the latitude of the other one and the difference of longitudes. (pp. 163-164)

See the figure. The fourth rhumb is the fourth direction starting from north towards the east, in divisions of one-eighth of a right-angle, that is bisecting NNE and NE. For 42° 37' of distance, the same table gives 28° 42' of latitude. In fact, according to the latitude, the degrees of longitude do not represent the same distances. (Solution to the end of the article)

During this century graduated paper, called a 'quarter of reduction', (see figure 10) began to be used. If the latitude and the longitude of the starting point are known and the distance sailed on a constant course has been measured, the differences of latitude can be found. For the fourth rhumb, the table of Stevin gives 6^{o} of longitude and 8^{o} 29' of distance from which the longitude is easy to obtain, in plane approximation, from graduated paper.

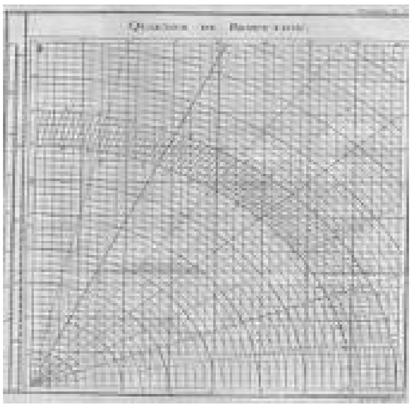


Figure 10: Engraving from Bézout, E. (1781). *Traité de Navigation*, Paris: Pierres (ed).

However, by the end of the century, the problem of determining longitude had still not been resolved.

THE EIGHTEENTH CENTURY, THE CENTURY OF INSTRUMENTS

On 22 October 1707, an English fleet ran aground on rocks close to the Scilly Islands off Cornwall; the cause was due to an inability to calculate its position. This accident led the British Government to propose a prize of £20000 in July, 1714 to anyone who could find an infallible method to determine longitude at sea. The solution lay in having an accurate timepiece. In 1735, the Englishman John Harrison invented the first clock capable of keeping at sea the time of the starting point. Its difference with local time allows one to determine the longitude. At the end of the century, the use of chronometers improved the determination of longitude.

Concerning the measures of angles, the introduction of another instrument, is also important.

We imagined in England a new instrument incomparably more perfect than those we have just talked about. The late Mister Hadley proposed it to the Royal Society of London in 1731; the use has already been introduced in France and it would be relevant if it could be even more common: because this instrument can give the altitude of Celestial bodies within one minute error as I checked it several times. It is a simple portion of circle of 45 degrees: we called it an Octant, because it is the eighth part of the circle, but it is divided into 90 parts, and it is equivalent to a quarter circle because of the property common to the mirrors that are used in its construction.
(Bouguer, (1753) *Nouveau traité de*, Paris: Guérin-Delatour. p.46.)

Why is the Octant equivalent to a quarter of a circle? (see figure11, solution at the end.)

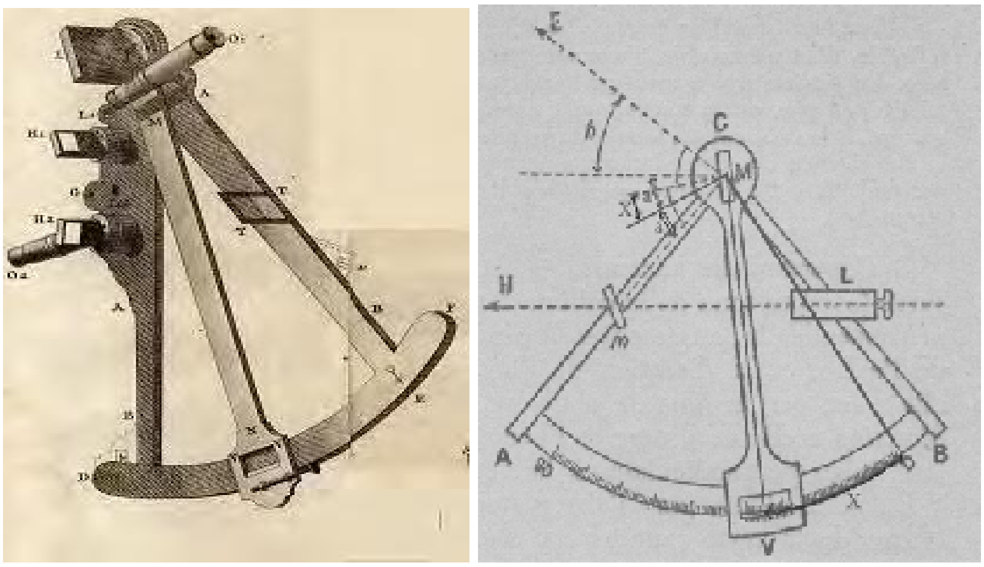


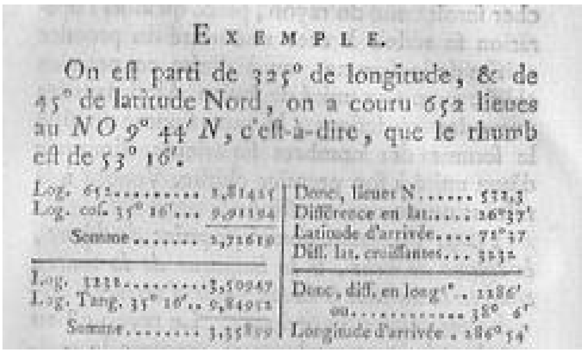
Figure 11: Sextant and octant , engraving is from Lévêque P. (1779). *Guide du Navigateur*, Nantes: Despilly.

By 1780, the octant and sextant (a sixth of a circle) had almost completely eliminated all previous instruments. With the sextant, the precision of the measures allows us to take into consideration the height of the observer, refraction and parallax.

In 1781 appeared the last volume of a mathematics course 'à l'usage des gardes de marine', of *Traité de Navigation* from Etienne Bézout. This remained a standard reference in France for many years.

> Given the starting point, the rhumb line, and the distance, find the latitude and the longitude of the arrival. The radius is for the distance as the cosine of the rhumb is for a fourth term which will be the way made on the north-south line. Reduce to degrees and you will have the change of latitude, thus the latitude of arrival. Look in the table of the increasing latitudes, the difference of the increasing latitudes between arrival and

departure, then do: the radius is for the tangent of the rhumb as the difference of the increasing latitudes is for the difference of longitude. (Pp.101-102)



EXEMPLE.

On eſt parti de 325° de longitude, & de 45° de latitude Nord, on a couru 652 lieues au *NO* 9° 44′ *N*, c'eſt-à-dire, que le rhumb eſt de 53° 16′.

Log. 652	2,81425	Donc, lieues N	532,3
Log. coſ. 35° 16′ ...	9,91194	Différence en lat.	26°37′
Somme	2,72619	Latitude d'arrivée	71°37′
		Diff. lat. croiſſantes ...	3232
Log. 3232	3,50947	Donc, diff. en longit. .	2286′
Log. Tang. 35° 16′ ..	9,84912	ou	38° 6′
Somme	3,35859	Longitude d'arrivée .	286° 54′

Figure 12: Bézout, E. (1781). *Traité de Navigation*, Paris: Pierres (ed), p102

By the way, there is a typographical error, it should be 35°16 and not 53°16.

For the latitude: take R=109, 1°=20 miles and 1'=1/3 mile.

For the longitude: from the table 71°37' gives 6262 and 45° gives 3030 with a difference: 3232.

R/tan35°16' = 3232/(difference of longitude), gives 2286 for this difference, 38°6' on the table.

With the sextant, the precision of the measures allows to take into consideration the height of the observer, the refraction and the parallaxe.

THE NINTEENTH CENTURY

With better educated sailors it became possible to use spherical trigonometry for navigation. To find the latitude and the longitude of a point C of the terrestrial sphere, a method is to resolve the spherical triangle ABC, where A is the north and BC the distance travelled. This is the ‘triangle of position’ (see figure 13).

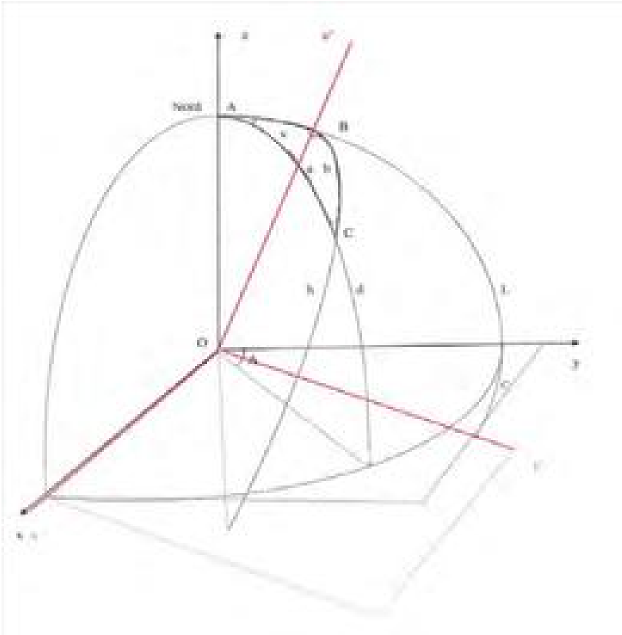


Figure13: Triangle of positio

Such a spherical triangle ABC satisfies the following angular formulae.

sinA.sinb = sina.sinB

sina.cosB = cosb.sinC – sinb.cosA.sinc

cosa = sinbcosAsinc + cosb.cosc

sinh = cosd.cosAcosL + sind.sinL

Show that cosA= (sinh – sind.sinL)/cosd.cosL (Solution at the end.)

In 1837 the American captain Thomas Sumner realized that when he observes a celestial body (the sun) under a certain angle, he is located on a circle, whose representation of an arc on a map can be assimilated to a straight line segment, called the position line.

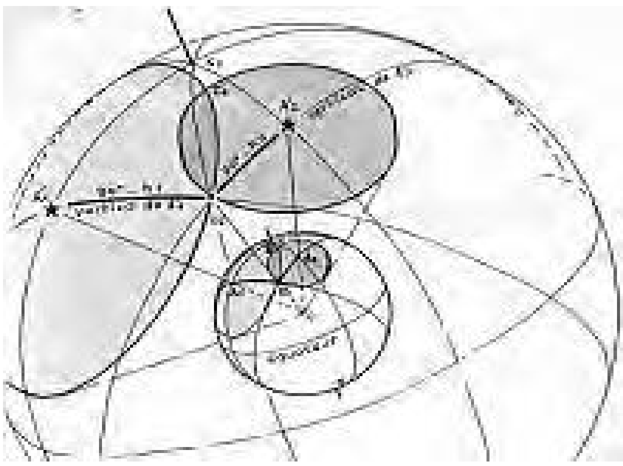


Figure 14: From *Revue Maritime*, (1975) 307, p.102, 1975

When repeating the operation, we can obtain a second segment, and at its intersection with the previous one, the place of the observer. The method was published in *A New and Accurate Method of Finding a Ship's Position at Sea* in 1843 (pp.16-17, see figure 15).

On 17th December, 1837, sea account, a ship having run between 600 and 700 miles without any observation, and being near the land, the latitude by dead reckoning was 51° 37′ N., but supposed liable to error of 10 miles on either side, N. or S.; the altitude of the sun's lower limb, was 12° 02′ at about 10½ A.M., the eye of the observer being 17 feet above the sea; the mean time at Greenwich, by chronometer, was 10ʰ 47ᵐ 13ˢ A.M.

Required, the true bearing of the land: what error of longitude the ship was subject to, by chronometer, for the uncertainty of the latitude: the sun's true azimuth.

dip.	— 4′ 3″	semi-dia.	+	16′ 8″
refra.	— 4′ 23″	px	+	8″
	— 8′ 26″		+	16′ 16″
			—	8′ 26″
		correction of alt. obs'd.	+	8′ 00″

Obs'd alt. ⊙ L. L.		12° 02′
Correction	+	8′
True alt. ⊙'s centre,		12° 10′

1st. The latitude by dead reckoning was 51° 37′ N.; he latitude the next degree *less*, without odd minutes, is 51° N.; and that, the next degree *greater*, is 52° N.

2d and 3d, Find the longitude of these two points, as follows:

⊙'S ALTITUDE 12° 10′.

For the point A in latitude 51° N.

Lat.	51°		N.	- - - - -	sec.	0·20113
Dec.	23	23	S.	- - - - -	sec.	0·03722
Sum.	74	23	nat. cos.	26920		
⊙ Alt.	12	10	nat. sin.	21076		
			diff. - -	5844	log,	3·76671

H.	M.	S.		
1	43	59	from noon = log rising =	4·00506
12	hours.			
10	16	01	app. time at ship.	
—	3	37	equa. time.	
10	12	24	mean time at ship.	
10	47	13	do. by chro.	
	34	49	= 8° 42¼′ west of Greenwich.	

Figure 15: Sumner T.H. (1851). *Finding a ship position at sea*, Boston: Groom & Co, pp. 16-17

After that, he carries out the same calculation for the point of 52° latitude, to obtain his longitude; the two points give a right segment.

In 1875, Marcq de Saint-Hilaire proposed a simpler method to draw a position line, having noticed that this one was to be perpendicular to the direction (the azimuth) of the observed celestial body. By taking an estimated point, the difference between the altitude of the celestial body observed and the one which would be obtained from the estimated point, we obtain the distance between the estimated point and the straight line that can thus be drawn. We can draw a second straight line and obtain the position of the ship on the chart. This method remained in use until the end of the twentieth century.

THE TWENTIETH CENTURY

At the beginning of the century there was an investigation into the use of radio waves for determining position. Broadcasting antennae, with a reach of 200 miles were erected. For example, the antenna of the island of Sein, Brittany, remained in use until 1911. Every antenna transmits a signal on a certain wave length. By directing the receiver, not to have the maximal reception, but on the contrary but to get no reception, one obtains by alignment the direction of the antenna, thus a first straight line. A second radio transmitter will give a second line. The method is called radiogoniometry. For confirmation, a third one will determine, at worst, a triangle. Radio transmission is no longer used for navigation, but other systems are still used, such as Decca.

The `beep-beep' emitted by the Sputnik gave the idea of using such emissions to find the direction of the station. From 1964, the American TRANSIT system could provide a location of a receiver from a satellite. However this system and the others of the same kind used a restrictive number of satellites, and their passages were spaced out too much.

For military purposes, the department of the defence of the United States envisaged from 1968 a system allowing them to localize any point on the earth, all the time and in real time. Its conception dates from 1973, and has been developed around 24 satellites (2 more in reserve) which constantly emit signals allowing the determination of the location of every receiver, as well as the receiver of the administrator of the system. The first satellite was sent into orbit in 1978 and the system has been operational since 1995. The satellites are periodically renewed.

The receiver measures the distances between it and the various satellites from which it receives information. These distances position it on circles whose intersection provides its position. This ‘Global Position System’ (GPS), makes all calculation methods redundant but they are still taught today. For safety?

SOLUTIONS

Example from Pedro de Medina:

The sixth of April, taking the altitude of the sun, the shadow comes to fall to the North, and finds the sun in sixty degrees on the astrolabe, thirty degrees are needed to reach ninety; I close with these thirty degrees, ten degrees of declination which the sun has this day, which are together forty degrees of which I am away from the equator to the North.

Exercise from Simon Stevin:

I look in the table of fourth rhumb and find, for the given latitude, the longitude 6^o and the distance 8^o 29'. I add the given distance (add, because the given latitude is the less, the start), it is 40^o 37'. I look in the table of the fourth rhumb, I find that it agrees with the required latitude of the second point 28^o 42'. By joining the longitude 30^o, among whom the difference with 6^o (first one in the order) we obtain the difference of the required longitude 24^o. The proof is obvious. Conclusion: knowing the rhumb of two points, the distance and the latitude of the one, we found the latitude of other one, which was asked.

Geometry of the sextant:

When the mirror goes from the initial position (measure 0) to the position of measure of altitude of the sun (or the star), it makes an angle X. That also makes the perpendiculars.

With the refraction property:

$$X+a = h+a-X \quad \text{and} \quad 2X = h$$

Calculation of the latitude by Bezout:

R/652= cos35°16'/NS with R = 109 and NS = difference of latitude

If NS = 532.3 with 1° = 20 miles and 1' =1/3 mile, result = 26°37'

Formulae in a spherical triangle:

By writing the coordinates of the point C in the position Ox'y'z' directly then from coordinates in Oxyz and from change of position by rotation of angle C around the axis Ox, the identification gives the three formulae.

The last gives: $\cos A = [\sin h+\cos(d+L)].\sec d.\sec L -1$

Text from Sumner:

In the first part, the author applies to the measure of the altitude of the sun the used corrections. Then, he calculates the secants of the latitude and the declination, the cosine of their sum, the sine of the height to apply the previous formula. A last correction allows to have the time of the boat, thus the longitude, if the latitude is 51°.

The author thanks Chris Weeks for his help in preparing this paper.

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Workshop

A STRONG COLLABORATION BETWEEN PHYSICIANS AND MATHEMATICIANS THROUGH THE XLXTH CENTURY: DOUBLE REFRACTION THEORY

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Université de Nantes

Mathematics and Physics may address the same topics though they employ different methods, experiments on one hand, analysis or geometry on the other. Nevertheless, those two disciplines stimulate one another which usually generate progress.

In order to illustrate this fact, we propose to draw some examples from Augustin Fresnel's wave theory of light, elaborated in 1819.

In 1830, famous mathematician Augustin-Louis Cauchy forecasted elliptical polarization by crystals (phase jump) trying to transcript analytically Fresnel's principles. This phenomenon had not been considered earlier. It was checked by Physicist Jules Jamin, about twenty years later.

Also, in 1832, some mathematical calculations led William Rowan Hamilton to discover conical refraction. A year later, Humphrey Lloyd observed this phenomenon.

However, we shall present an even more surprising interaction between the two sciences, which emerged from crystallography. A whole community of researchers has been animated by the problem of determining the properties of wave surfaces, since the directions of light through crystals can be deduced by Huygens' geometrical construction. Mathematicians invented new tools which led to now famous theorems.

History of Sciences shows the importance to teach physics and mathematics coherently rather than as two separated independent fields. Also researchers from each discipline would take advantage in adapting their language when communicating their discoveries.

Abstracts from Christian Huygens' Treatise on Light 1690, Augustin Fresnel's Memoir on double refraction 1821, William Rowan Hamilton's Memoir on Systems of Rays 1830, Archibald Smith's short paper on Wave Surfaces, will be read and commented.

THEME 5:

CULTURES AND MATHEMATICS

Plenary Lecture

CALENDARS AND CURRENCY – EMBEDDED IN ICELANDIC CULTURE, NATURE, SOCIETY AND LANGUAGE

Kristín Bjarnadóttir
University of Iceland

Iceland was settled from Norway in the 9th century. The settlers brought with them a tradition of a seven-day week. Placed so far from other inhabitancies, they had to create their own week-based calendar. They developed their own currency, based on available commodities, such as fish, which was discarded at the turn of the 20th century with the aid of arithmetic textbooks. Observations of the solar cycle soon revealed errors in the calendar which was cleverly amended. The calendar was later adjusted to the Roman calendar. It remained in common use for secular purposes until the 19th century. Special occasions related to it are still celebrated. Both systems, the currency and the calendar, are embedded in the local language and serve to link generations together in their scope of time, nature and valuables.

INTRODUCTION

Iceland was settled from Norway by Viking raiders, bringing with them Celtic slaves, in the latter half of the 9th century. Icelanders became literary in the 12th century. Once literate, they began to write voluminously, initially to document the laws of the newly-founded Commonwealth (Kristjánsson, 1980, p. 29). The settlers brought Norwegian traditions of sheep and cattle farming, the seven-day week and an empirical lunar calendar. They developed their own calendar of the year, based on their heritage of counting time in weeks, and on their observation of nature. A thirteenth-century law-codex *Grágás* (1980–2000) contains a concise description of a week-based calendar, created in Iceland in the 10th century. The calendar, called *misseri*-calendar or farmers' calendar, was synchronized to the Julian calendar in the 12th century and later to the Gregorian calendar. Remnants of the *misseri*-calendar have survived in the country to present days, and given rise to mid-winter-feasts and the First Day of Summer as a public holiday.

Gradually, Icelanders also made up a system of currency, based on the commodities they had to sell to foreign merchants. Due to Iceland's geographic position, sailing was only possible during summer so the society was isolated. Their trade was mainly made by bartering where the basic trade unit was *fish*, a so-called *valid fish*. This unit became linked to the price of another basic commodity, an ell of woollen cloth, as well as the price of land and salaries of farm workers.

Both systems, the calendar and the currency system, became unique for Iceland and interwoven into the Icelandic culture and language while the Nordic languages diverged apart from each other in the early modern era. In this article, these systems will be explored and explained as an example of how systems could be developed in isolated ancient civilizations.

LITERATURE REVIEW

Ethnomathematics

Ethnomathematics is the study of the relationship between mathematics and culture. The term "ethnomathematics" was introduced by the Brazilian educator and mathematician Ubiratan D'Ambrosio. The subjects of ethnomathematics are the topics of everyday life: In different environments, ethnomathematics differs in terms of *counting time*, *measurements of land and distances*, *systems of taxation*, and arithmetic dealing with the economy such as *trade in the form of barter*, by which trading goods were allocated monetary value.

Researchers in various parts of the world, such as Bill Barton in New Zealand, Eduardo Sebastiani Ferreira in Brasil, and Paulus Gerdes in Mozambique, have made valuable contributions to research of ethnomathematics (Miarka and Viggiani, 2012). Their pioneering work has led the attention of other researchers to their own mathematical cultural heritage. In this paper, the focus is on Icelandic mathematical heritage which developed in relative isolation from the 9th century onwards.

Calendars

Mapping the time – The calendar and its history by E. G. Richards (1998) is a comprehensive overview of the calendar in theory, calendars of the world, calendar conversions and computations of dates of Easter. It mentions briefly the Icelandic Misseri-calendar where the author explains the geographical difficulties in using lunar calendars, practiced by Teutons, including the Angles, Saxons and the Vikings. Therefore, calendars operating according to different principles were developed. In Iceland, resting on the Arctic Circle, the lunar calendar was impossible to use in summer, so they took to counting the weeks instead (Richards, 1998, pp. 203–205).

Svante Janson (2011) has written a scientific analysis of the Icelandic calendar from earliest times. Janson built his work partly on research by physicist Þorkelsson (1923; 1930; 1932; 1936), physicist and science historian Vilhjálmsson (1989; 1990; 1991), as well as on Icelandic medieval sources.

The ancient handbook, Bishop Árnason's (1739; 1838; 1946) *Dactylismus Ecclesiasticus eður Fingra-Rím* is a valuable source of the farmers' calendar. Furthermore, Bjarnadóttir (2010) has published an overview of the pagan calendar.

Currency

Statistics Iceland, the official site of statistics for Iceland, has published Icelandic historical statistics (Jónsson and Magnússon, 1997) where the currency system is explained. Professor emeritus Gísli Gunnarsson (1976; 2002) has written several studies on the currency system, both scientific (1976), and for public education (2002). Arithmetic textbooks and tables, published in the 18th and 19th century, such as Stefánsson (1785), and Briem (1869; 1880; 1898) explained the relations between the various currencies.

THE ICELANDIC CURRENCY SYSTEM

The currency units

Once the Viking loot had been spent, metal for coins was not available. Exchanges with foreign merchants were made by barter. The selling commodities were fish, fat, woolen goods, measured in ells, and walrus teeth, exchanged for linen, grains, wood for boats and larger buildings, etc. All farms were recorded during 1702–1714 (Magnússon and Vídalín, 1980–1990). The records reveal rents paid in butter, cheese or hay, and feeding of livestock: sheep or cattle.

A complicated system of valuing land and goods developed gradually:

> **A hundred** equalled **120 ells** (originally a measure of woollen cloth).
>
> **A hundred** equalled also **240** (valid) **fishes**, that is processed and dried fish, each weighing about 1 kilogramme.
>
> **A hundred** was the equivalent of **a cow**, i.e. a middle-aged, faultless cow in spring, or **six sheep**, woolly and carrying lambs, in spring.
>
> **Farmlands** were also measured by hundreds: An average farm was valued at **20 hundreds**, and it was supposed to support livestock of **20 cows** or **120 sheep** (Jónsson and Magnússon, 1997).

This Icelandic currency system existed from medieval times up to the 20th century. Evaluation of farmlands in hundreds was established in 1096 along with the tithe, and was finally abandoned in 1922. The tithe was a property tax in Iceland and was based on the evaluation of farmlands, while in other countries it was an income tax. Farmlands may have been originally measured by how much livestock they could carry. They were only re-evaluated in case of damage due to floods, erosion etc., or changes in national industries, such as advantages due to access to fishing when fish became the main export commodity after 1200/1300 (Gunnarsson, 2002).

In 1880, the currency of the Danish *króna*, crown, was adopted and arithmetic textbooks emphasized teaching computations by exchange rates. This was done simultaneously with the establishment of the first bank, the National Bank of Iceland, NBI. Its guardian was the reverend and arithmetic textbook author Eiríkur Briem (1846–1929), who was professor of theology by profession.

Two examples from Briem's Arithmetic Textbook

Briem's arithmetic textbook was first published in 1869 and in an expanded version in 1880 with increased emphasis on decimal fractions and the new decimal currency system. I will take examples from Briem's textbook seventh edition of 1898 that my grandfather, born in 1877 and whom I well remember, may have studied. One notices three different units: fishes, pounds of fat, and ells in addition to the hundreds.

> The tithe of 5 [estate] hundreds is 3 **fishes** and proportionally of larger estates. The light-tax is 4 **pound** [sheep] **fat** and each graveyard-tax is 6 **ells**. How high is the income of a church which had tithe from 131.55 **estate hundreds**, and 87 ½ **cash hundreds**, 17 ½ light taxes and 5 ½ graveyard-taxes when each pound of **fat** was worth 35 cents, each fish in tithe 29 ½ pennies, but in graveyard-taxes 29 cents? Answer: 82 crowns, 41 cents. (Briem, 1898, pp. 76–77)

One notices that the price of ells is not given, as everyone should know that each ell was worth 2 fishes. Furthermore one may wonder about the different exchange value of fishes in tithe and in graveyard taxes, and the difference between estate hundreds and cash hundreds. Both point to economic difficulties in using the old currency, and that the time for a new currency had arrived.

Another example concerns salaries; how workers were paid for their work:

> A farmer's hired hands, a man and a woman, made 50 **horses** [i.e. horse-loads] of hay in 4 weeks. The man earned 2 quarters **butter** a week but the woman 25 marks [of butter].
>
> If the food for the man was calculated as 3 **fishes** a day but for the woman as 2 fishes, and each fish was worth 30 cents; each pound of butter 65 cents, and furthermore each horse-load cost 60 cents for hiring land, horses etc., how much would the hay cost? Answer: 156.50 crowns. (Briem, 1898, p. 76)

The reader needed to know that 1 quarter equals 10 pounds and 1 pound contains 2 marks. But the example reveals also the ratio between salaries of men and women who had different tasks in the hay-making.

Both examples bear witness about the commodities that farmers produced and could use for currency. By the turn of the 20th century, the payments probably were through the local merchants or cooperative societies which were established around the country from 1882 onwards. The farmer had an account into which his income was credited. He then took out necessary goods for his family, and so could his workers do for what was left of their share if they had accepted a lamb or feeding of sheep through the winter as part of their payment. The economy was thus basically based on bartering.

THE MISSERI CALENDAR

Introduction

> The construction of calendars, i.e. the counting and recording of time, is an excellent example of ethnomathematics. (D'Ambrosio, 2001, 12)

In 930 CE, at the close of the settlement period in Iceland, a calendar was adopted, counting the year as 52 weeks. Observations of the solar cycle soon revealed errors, which were cleverly amended. The calendar was adjusted to the calendar of the Christian Church in the 12th century. It remained in common use for secular purposes until the 19th century, and special occasions related to it are still celebrated.

The week-based calendar

The settlers of Iceland came from Norway, and they brought slaves from Britain and Ireland. Their common calendar included a seven day week, the days named after their Norse gods (Björnsson, 1990, pp. 71–74; 1993, pp. 18–19, 665–660):

Sunday,*sunnudagur*, the day of the sun
Monday, *mánudagur*, the day of the moon
Tuesday for Tyr, the god of war
Wednesday for Woden, the cunning god
Thursday for Thor, the thunder god
Friday for Freyr / Freyja / Frigg, the god and godesses of love/marrriage
Saturday, *laugardagur*, the day of washing and bathing.

The names related to the pagan gods remain in English and the Nordic languages other than Icelandic, where they were abandoned by the Church in the 12th century for *thridjudagur* (Third Day) for Tuesday, *miðvikudagur* (Mid-week Day) for Wednesday, *fimmtudagur* (Fifth Day) for Thursday and *föstudagur* (Fast Day) for Friday. *Sunnudagur*, *mánudagur* and *laugardagur* have remained intact to this day.

Probably some of the settlers counted the time according to the cycle of the moon, 29.52 days. However, in Iceland, the nights are light from April until late August so the moon can barely be seen. It is also often low at northern latitudes, and the sky is often cloudy. Counting the lunar months in summer was therefore abandoned and counting the summer weeks was adopted instead. Moreover, difficult weather conditions may mean that the moon cannot be seen regularly in wintertime, and in time winter months were standardized at 30 days each (Richards, 1998, p. 204).

A yearly parliamentary gathering was agreed upon in year 930. According to a brief history of Iceland, Í*slendingabók* [*The Book of Icelanders*, *Libellum Islandorum*], written by Ari the Learned in the period 1122–1133, an agreement was reached at the establishment of the parliament on meeting again after 52 weeks or twelve 30-day months and four extra days. The year was to be divided into two terms, *misseri*, the winter-*misseri* to last six months, the summer-*misseri* another six months, and the four extra days were added at midsummer. This system quickly revealed the need for a more reliable system of time-computing. By the 950s it had become clear that the summer "moved back towards the spring", i.e. the summer according to this calendar began earlier and earlier vis-à-vis the natural summer (Benediktsson, 1968, p. 9). This was inconvenient, as the parliamentary gathering had to assemble after the completion of certain necessary farming tasks, such as lamb-births, and before others, such as hay-making, were due to begin. This is recorded in *The Book of Icelanders*. The book exists in manuscripts from 17th century. In this context, Benediktsson's (1968) edition of *The Book of Icelanders* states:

> This was when the wisest men of the country had counted in two misseris 364 days – that is 52 weeks, but twelve thirty-night months and four extra days – then they observed from the motion of the sun that the summer moved back towards the spring; but nobody could tell them that there is one day more in two semesters than can be measured by whole

> weeks, and that was the reason. But there was a man called Þorsteinn Surtur … when they came to the Althing then he sought the remedy … that every seventh summer a week should be added and try how that would work … (Benediktsson, 1968, pp. 9–10)[1]

The error seems to have been realized by Þorsteinn Surtur from an observation of the location of the sunset, which in northern areas moves rapidly clockwise along the horizon before the summer solstice, and subsequently anti-clockwise. According to the source, cited above, Þorsteinn Surtur recommended in year 955 that every seventh year an extra week be inserted, Summer's Extra Week, making the average year 365 days. Below, we see the view to the west direction from Þorsteinn Surtur's farm at 65° N latitude. Only at summer solstice the sun sets right north of Eyrarfjall (Vilhjálmsson, 1990).

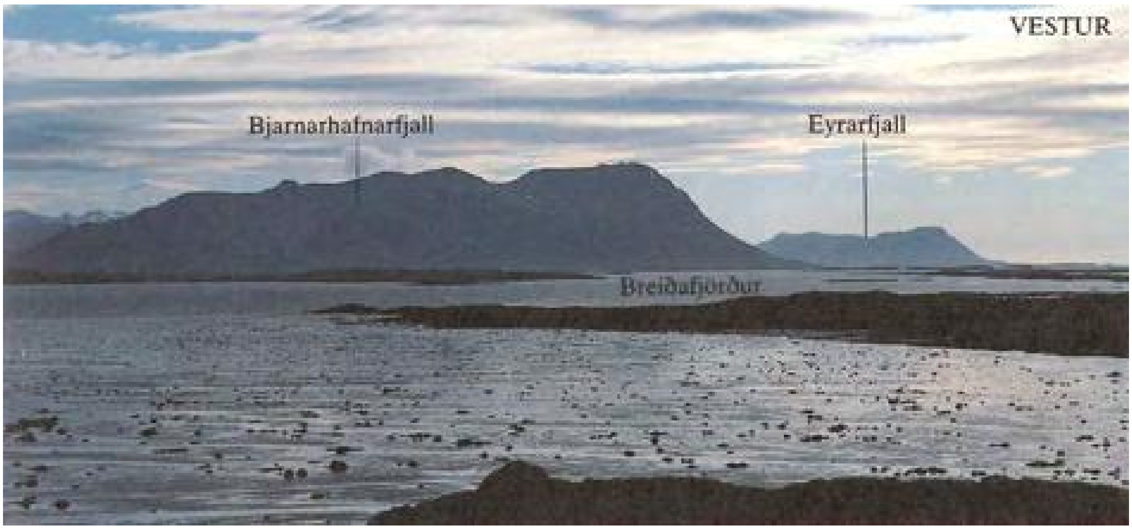

Figure 1: The view to the west direction from Þorsteinn Surtur's farm at 65° N latitude. (Photographer: Grétar Eiríksson)

By the year 1000 the parliament gathered after ten weeks of summer had passed, instead of nine weeks, which illustrates that the eleven missing leap years since year 955 had also caused "a move back to the spring" as earlier, even though slightly slower. In the *Book of Icelanders* it says: "Then it was spoken the previous summer by law, that men should arrive at *Althingi* when ten weeks of summer had passed, but until then it had been a week earlier." (Benediktsson, 1968, p. 15)

Simulation of the sun track

The **axial tilt** of the Earth is the inclination angle of the Earth's rotational axis in relation to its orbital plane around the sun. The axial tilt is currently about 23.44°. The axis remains tilted in the same direction throughout a year; however, as the Earth orbits the Sun, the hemisphere tilted away from the Sun will gradually become tilted towards the Sun, and vice versa. Whichever hemisphere is currently tilted toward the

[1] All Icelandic passages have been translated by the author, KB

Sun, experiences more hours of sunlight each day. In the northern hemisphere, the maximum tilting towards the Sun is at **summer solstice** in June and the minimum at **winter solstice** in December. At **equinoxes** in September and March, the axial tilt does not have effect on the observed sun track.

We will simulate the sun track at 65° North, the latitude of the farm of Þorsteinn Surtur, and for comparison at 42° North, the latitude of Rome. In Figure[2] 2 the altitudes of the sun at 65°N and 42°N through 24 hours are approximated by

Iceland(x) = – (90–65)*cos(2πx/360) and
Rome(x) = – (90–42)*cos(2πx/360)

respectively. The 0°–360° scale on the horizontal axis denotes the direction of the sun on the 360° horizon during 24 hours, while the scale on the vertical axis denotes the altitude in degrees.

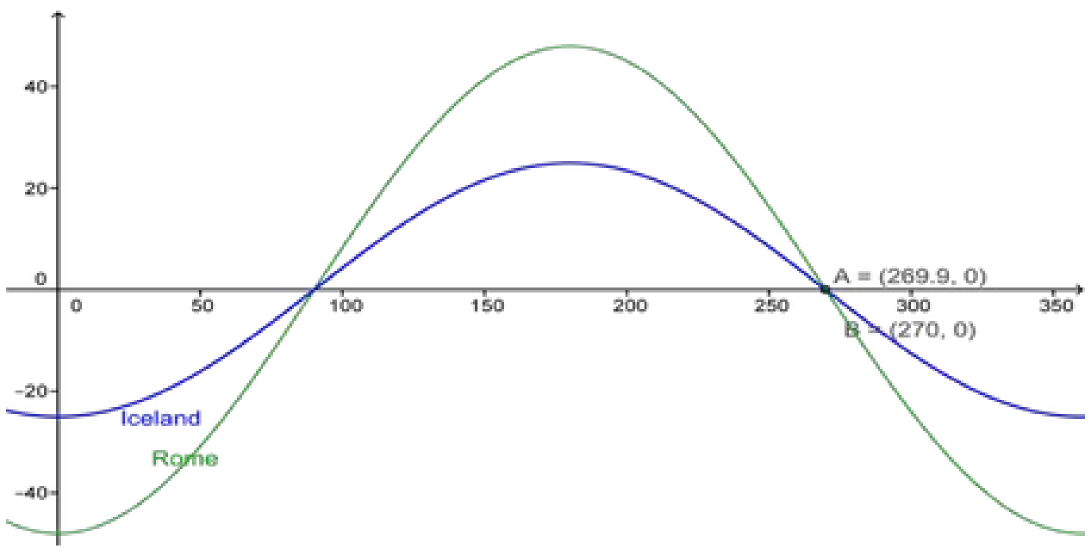

Figure 2: Simulation of the Sun track seen from 65°N and from Rome at 42°N at equinoxes

We notice that the track of the sun is flatter at northern latitudes than closer to the equator. At winter-solstice the sunset at 65°N is at 200° on the horizon. At summer-solstice the sunset is at 340°, see Figures 3 and 4. The range of the sunset is about 140°. For comparison, the approximated sunset in Rome at 42°N at winter-solstice is at 241° on the horizon, and at summer-solstice at 300°. The range in Rome is about 60°.

The altitude of the sun at 65°N at noon is 90°–65° = 25° at the equinoxes. At summer solstice the sun is therefore only 25°–23.4° = 1.6° below the horizon at its lowest position. Since the sun is so close to the horizon at that time, the night is bright enough for reading a book. The official calendar for Iceland does not record darkness in Reykjavík at 64°N from May 19 until July 23 (*Almanak fyrir Ísland*, 2014).

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[2] Figures 2, 3, 4 and 5 are made by the aid of Geogebra-software.

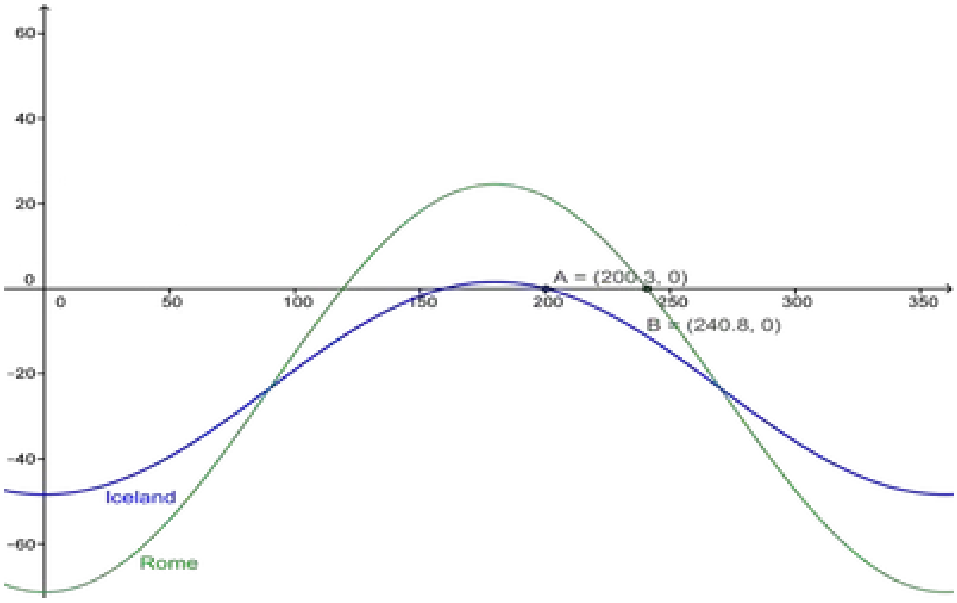


Figure 3: Track of the sun seen from 65°N and from Rome at 42°N at winter solstice when the observed tilt, –23.44°, is subtracted from the formulas for the altitudes of the Sun at equinoxes.

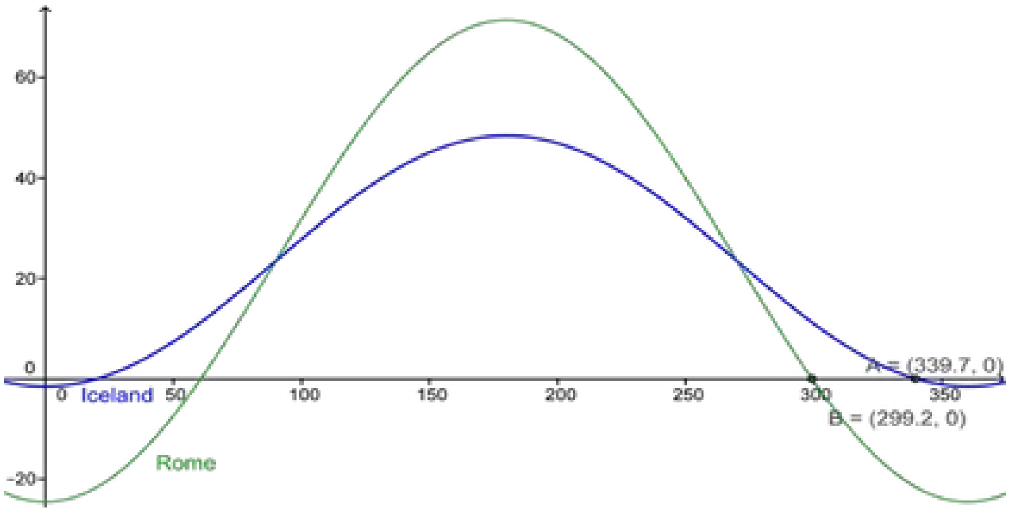


Figure 4: Track of the sun seen from 65°N and from Rome at 42°N at summer solstice when the observed tilt, +23.44°, is added to the formulas for the altitudes of the Sun at equinoxes.

As the track of the sun is flatter at northern latitudes than closer to the equator, the sunsets move more rapidly there along the horizon near the solstices than at places closer to the equator. In early June, when the observed axial tilt is one degree less than maximum, the difference then between the sunset location and the northernmost location sunset on the horizon in Rome is 1.5°, see the coordinates of points B and D on Figure 5, while at 65° latitude the difference is nearly 6°, see coordinates A and C on Figure 5.

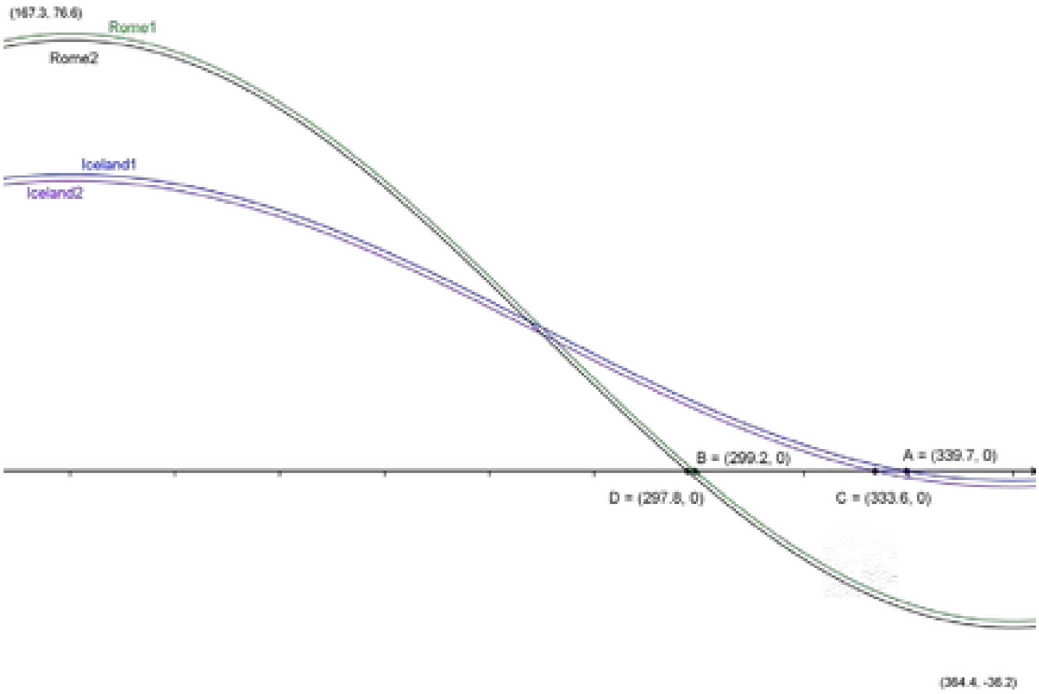


Figure 5. The placements of sunsets at 22.4° observed tilt and at maximum observed tilt, 23.4°, in Rome at 42°N and in Iceland at 65°N.

This may easily be seen from many places on the west coast of Iceland, e.g. in Öskjuhlíð hill in Reykjavík, see Figures 6 and 7.

Figures 6 and 7. The sunset on June 11 2008 at 23:55 on the left, and on June 19 2008 at 24:04 on the right. The pictures are taken from Öskjuhlíð hill in Reykjavík, Iceland, at 64°North. (Photographer: The author)

The sunset moved 2.2° clockwise along the horizon in 8 days in June 2008. The situation in nature cannot be compared computationally to the modelling drawings above, as refractions in the air have to be taken into account. However, the pictures in Figures 1, 6 and 7 may illustrate how Þorsteinn Surtur could discover in year 955 the error in the calendar, adopted in year 930.

Discovery of errors in the Christian calendar

Icelanders agreed to accept the Christian Faith in year 1000. The Christian Church as an institution was established in the late 11th and early 12th century. The Church introduced the Julian calendar with one extra day to the 365 days every fourth year, the leap year. By adding a day to 365 days every fourth year, the average length of the year became 365.25 days, while in reality it is approximately 365.2422 days. The Julian calendar assumed the summer solstice to be on June 21, decided upon in Nikea in 325 CE. In the 12th century, it fell on June 15, due to the addition of six too many leap-year days, which would have been skipped at years 500, 600, 700, 900, 1000 and 1100 according to the correction by the Gregorian calendar.

In the first half of the 12th century, Oddi Helgason (1070/80–1140/50), a farm-worker, made observations of the annual motion of the sun, of which an account is found in the ancient treatise *Odda-tala* [*Oddi's Tale*], contained as a separate treatise in the oldest part of the manuscript *GKS 1812, 4to*, written around 1192 (*A dictionary of Old Norse Prose – Indices*, 1989, p. 471). The Icelandic week-based *Misseri*-calendar was adjusted to the Julian calendar in the early 12th century. Oddi observed the summer solstice and the winter solstice to be earlier than the official date, i.e. on June 15 and December 15 instead of June 21 and December 21. Oddi also explained the curve of the height of the sun, counting the weekly increase and decrease in its height, measuring the rise as 91 sun diameters (Vilhjálmsson, 1991). In the Icelandic calendar treatise, *Rím II*, [*rím*: rhyme, meaning calendar] written in the late 13th century (Beckman and Kålund, 1914–1916, pp.81–178), it says:

> Solstice in summer is four nights before the mass of John the Baptist ... It is so in the middle of the world. Some men say that it is close to a week earlier in Iceland. (Beckman and Kålund, 1914–1916, p. 121)

The Mediterranean [Mid-Earth] Sea was considered the middle of the world. The error of the Julian calendar had thus been discovered in Iceland in the 12th century but was blamed on different latitudes. Better estimates of the year than the Julian calendar presented had already been made. Examples are shown in *Table 1*.

Researcher	Location	Year	Length of the year
?	Babylon	c. 700 BC	365.24579 days
Hippachus	Egypt	150 BC	365.2466 -
?	Mexico (Mayan)	700 AD	365.2420 -
Da Yen	China	724 AD	365.2441 -
Al-Battani	Arabia	900 AD	365.24056 -
Al-Zarqali	Arabia	1270 AD	365.24225 -

***Table 1*: Examples of early estimates of the length of the year (Richards, 1998, p. 33).**

The Julian and Gregorian Calendars

The Gregorian calendar was a reform to correct the discrepancies of the Julian calendar. By 1700, when the Gregorian calendar was accepted in the Danish Realm, eleven days were omitted, November 17–27. Evangelic Lutheran Bishop Jón Árnason (1739; 1838; 1946) published a thorough guide, *Dactylismus Ecclesiasticus eður Fingra-Rím*, to computing the ecclesiastical calendar according to the new style, both by mathematical formulas and by counting on fingers. The calendar of the "farming-year", the *Misseri*-calendar, was attached as a second part.

For both calendars the *Sunday letters*, *dominical letters*, were important. Each day of the year is assigned a letter, called *calendar letter*, A, B, C, D, E, F or G. Each year is then assigned a letter, *dominical letter*, according to the calendar letter of the Sundays that year (Richards, 302–307).

A regular 365-day year begins and ends on the same weekday, which implies that the dominical letters of succeeding years are displaced back one place – except after leap years, when they are displaced two places. February 29 and March 1 have the same calendar letter. The leap years therefore have two dominical letters. Every 4th year was a leap-year and the week counts 7 days; the lowest common multiple of 4 and 7 is 28, so that the sequence of dominical letters, called the Solar Cycle, repeated every 28 years in the Julian calendar. In the Gregorian calendar the leap years were skipped in years 1700, 1800 and 1900 so the length of the Solar Cycle including these years extended to 40 years. *Table 2* shows the reverse order of dominical letters in a sequence of 28 years. The same information on the dominical-letter sequence, arranged on the fingers and palms in Bishop Árnason's *Dactylismus*, is shown on Figure 6. This was convenient for the general public for whom paper was expensive and scarce.

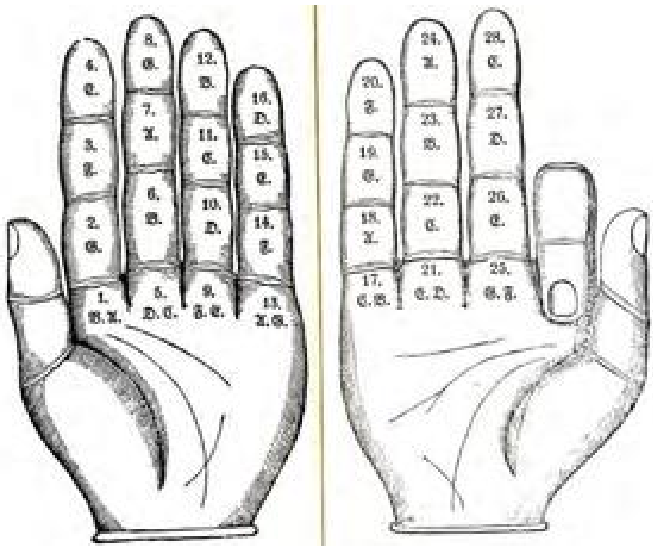


Figure 6. Dominical letters, arranged on the palms and fingers of both hands (Árnason, 1838; 1946, pp. 106–107).

Place # in the Solar Cycle	**Two last digits of the date**	**1600 2000**	**1700 2100**	**1800 2200**	**1900 2300**
1	00	BA	C	E	**G**
2	01 29 57 85	**G**	B	D	F
3	02 30 58 86	F	A	C	E
4	03 31 59 87	E	**G**	B	D
5	04 32 60 88	DC	FE	**AG**	CB
6	05 33 61 89	B	D	F	A
7	06 34 62 90	A	C	E	**G**
8	07 35 63 91	**G**	B	D	F
9	08 36 64 92	FE	**AG**	CB	ED
10	09 37 65 93	D	F	A	C
11	10 38 66 94	C	E	**G**	B
12	11 39 67 95	B	D	F	**A**
13	12 40 68 96	**AG**	CB	ED	GF
14	13 41 69 97	F	A	C	E
15	14 42 70 98	E	**G**	B	D
16	15 43 71 99	D	F	**A**	C
17	16 44 72	CB	ED	GF	BA
18	17 45 73	A	C	E	**G**
19	18 46 74	**G**	B	D	F
20	19 47 75	F	**A**	C	E
21	20 48 76	ED	GF	BA	DC
22	21 49 77	C	E	**G**	B
23	22 50 78	B	D	F	A
24	23 51 79	**A**	C	E	**G**
25	24 52 80	GF	BA	DC	FE
26	25 53 81	E	**G**	B	D
27	26 54 82	D	F	A	C
28	27 55 83	C	E	**G**	B
1	28 56 84	BA	DC	FE	**AG**

***Table 2*. Dominical letters of years 1600 to 2300 according to the Gregorian calendar.**

The larger Solar Cycle is 400 years. The *Dactylismus* finger-rhyme helps to find the number of each year in the 28-year Solar Cycle (Figure 7).

Years 1600, 2000, ... # 1, dominical letters B and A.

Years 1700, 2100, ... # 5, dominical letter C. etc.

The years in-between are counted onwards with modulus 28.

The numbers 1, 5, 9, 13 correct the cycles due to missing leap years at the turns of centuries.

The numbers 4, 1, 2, 3 at the top denote the classes of the centuries within the 400-year cycle.

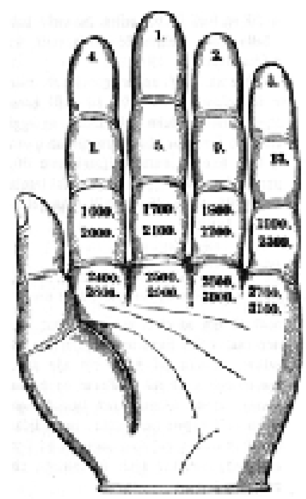


Figure 7. Numbers, denoting classes of centuries within the 400-year cycle (Árnason, 1838; 1946, p. 102).

Adjustments of the *Misseri* calendar to the Roman calendar

By the adjustments to the Julian calendar, the Summer's Extra Week was inserted every sixth year, or every fifth year if two leap years were in-between. The First Summer-Day was the beginning of the secular year. It was to fall on Thursday in the week April 9 to 15. In the late middle ages, April 9 was the beginning of the light-night period in Northern-Iceland, while the date April 15 was the Norwegian official beginning of summer (Þorkelsson, 1930). The First Summer-Day transferred later according to the Gregorian calendar to Thursday in the week April 19–25, and other dates, mid-summer, beginning of winter etc., accordingly (Sæmundsson, 1972).

The Summer's Extra Week is inserted in mid-summer, beginning on Sunday after 13 weeks. The years of Summer's Extra Week are those which begin on Monday, that is when the dominical letter of the year is G, and those which begin on a Sunday, i.e. have the dominical letter A, if it is the year before a leap year. The year that begins on a Sunday and is the year before a leap year is called Rhyme-Spoiler. This happens once in the 28-year Solar Cycle, but is disturbed at 1700, 1800 and 1900 when there are no leap years and the Solar Cycle is 40 years. The Rhyme-Spoiler moves all dates forward one day from Summer's Extra Week until leap-year's day. This means that Summer's Extra Week is always inserted in years of Sunday July 22, or Sunday July 23 when next year is leap-year. In *Table 2*, the dominical letters for years of Summer's Extra Week are marked by bold letters. To take examples of years of Summer's Extra Week, there are 2001, 2007, 2012 and 2018. Year 2023 with dominical letter A brings Rhyme-Spoiler and so does year 2051.

The present pagan-calendar 30-day months

The Icelandic *Misseri*-calendar was based on weeks and they were counted during summer months when the moon could not be seen. There were also thirty-day long months, more in use during winter. The names have been different during the centuries while the high winter months, *Þorri* and *Góa*, have remained since earliest times. The farm-year began in spring. The present system and names are as follows:

Harpa begins on Thursday in April and marks the beginning of summer-*misseri*
Skerpla begins on Saturday in May
Sólmánuður [Sun-month] begins on Monday in June
    Aukanætur, four extra nights, begin on Wednesday in July
Heyannir begins on Sunday in July, at mid-summer
Tvímánuður begins on Tuesday in August
Haustmánuður [Autumn-month] begins on Thursday in September
    Summer's Extra Week every sixth or fifth year
    Veturnætur, two extra [winter]-nights
Gormánuður begins on Saturday in October and marks the beginning of winter-*misseri*
Ýlir begins on Monday in November, ending at *yule*
Mörsugur begins on Wednesday in December
Þorri begins on Friday in January
Góa begins on Sunday in February
Einmánuður begins on Tuesday in March

All weeks of summer-*misseri* begin on Thursdays but weeks of winter-*misseri* begin on Saturdays. Therefore, the two last days of *Haustmánuður* [Autumn-month] are called *veturnætur* [winter-nights] and are outside weeks. The last week of *Einmánuður* lasts only five days.

The names of the high winter months, *Þorri* and *Góa*, were also names of pagan gods. When *Einmánudur* [One-Month or Lone-Month] commences there is one month left until summer begins (Þorkelsson, 1928). The beginning of *Þorri* marks mid-winter and has been an occasion for mid-winter festivities.

Þorri (masculine) begins on Friday in the 13th week of winter (in late January); this was Husbands' Day.

Góa (feminine) begins on Sunday in the 18th week of winter (in late February); this was Wives' or Women's Day.

Einmánudur (masculine) begins on Tuesday in the 22nd week of winter (in late March); this was the Young Men's Day.

Harpa (feminine), the first month of summer, begins on Thursday in April 19–25, First Day of Summer; this was the Young Girls' Day (Björnsson, 1993, pp. 766-783).

The First Day of Summer has been a public holiday in Iceland for centuries. Youth and child-care organizations organize festivities in cooperation with local authorities. Furthermore, international Mother's and Father's Days are not much celebrated in Iceland: rather the first days of *Þorri* and *Góa* (Björnsson, 1993).

Figure 7. Celebration of First Day of Summer. (Photographer unknown)

The First Day of Summer also depends on the dominical letter of the year, see *Table 3*:

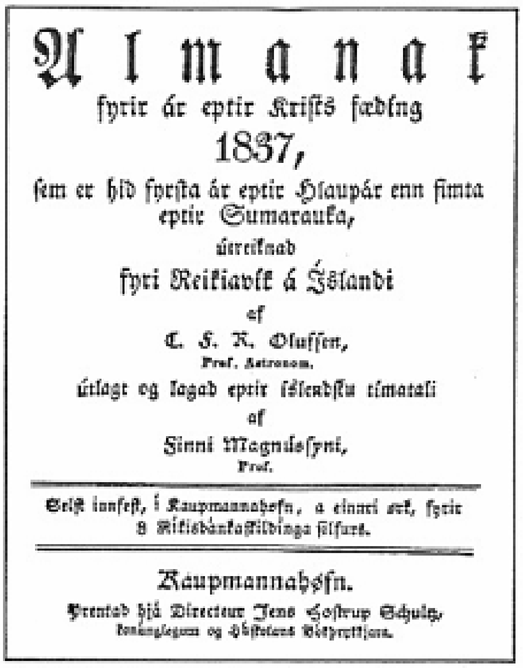
Dominical letter	First Sunday in April	First Day of Summer
A	April 2	April 20
B	April 3	April 21
C	April 4	April 22
D	April 5	April 23
E	April 6	April 24
F	April 7	April 25
G	April 1	April 19

***Table 3*. The dates of First Day of Summer.**

The Icelandic *Almanak*

Before Bishop Árnason's *Dactylismus*, Danish calendars were in use for a few centuries, but these did not meet the needs of Icelanders, most of whom were more familiar with the *misseri*-calendar. The *Dactylismus* therefore must have been the main handbook for Icelanders during the 18th century. Icelandic calendars of the years 1800 to 1836 exist in manuscripts, adjusted to the environment in Northern Iceland, but they were not continuous. In response to these, which were deemed to violate the University of Copenhagen's monopoly on publication of calendars, a calendar in Icelandic, the *Almanac* for Iceland, was first published in 1837 by the University of Copenhagen. The calendars were computed by professors at the University of Copenhagen until 1923, and translated into Icelandic by prominent Icelandic scholars. They added the *misseri* calendar with all its features, to the regular ecclesiastical calendar of the Evangelical Lutheran Church, in addition to local geographical information, such as time of the tide for various places and time of sunrise and sunset in Reykjavík. (Sigurgeirsdóttir, 1969). Figure 8 below shows the cover of the first issue of the *Iceland Almanac.* In translation it says:

Almanac for year after Christ's birth 1837, which is the first year after leap year but the fifth after Summer´s Extra Week, calculated for Reykjavík on Iceland, by C.F.R. Olofsen, Prof. Astronom, translated and adjusted to the Icelandic calendar by Finnur Magnússon Prof. (Sigurgeirsdóttir, 1969).



Almanak
fyrir ár eptir Krists fæding
1837,
sem er hid fyrsta ár eptir Hlaupár enn fimta
eptir Sumarauka,
útreiknad
fyri Reikiavík á Íslandi
af
C. F. R. Olufsen,
Prof. Astronom.
útlagt og lagad eptir íslendsku tímatali
af
Finni Magnússyni,
Prof.

Selst innfest, í Kaupmannahöfn, á einni ark, fyrir 8 Ríkisbánkaskildinga silfurs.

Kaupmannahöfn.
Prentad hjá Directeur Jens Hostrup Schultz,

Figure 6: The cover page of the first issue of the *Iceland Almanac*.

The publication of the *Almanac* was transferred to the University of Iceland in 1917, and from 1923 the computations have been made by Icelandic mathematicians or astronomers (Sigurgeirsdóttir, 1969). The *Almanac* has been sold in 10,000 copies a year to a population of 320,000.

DISCUSSION

One may ask why the *Misseri*-calendar survived. There are several possible answers to that:

- It had been in use for two centuries before the introduction of the Christian calendar.
- It was maintained as a secular calendar by the law, contained in the oldest collection of laws, *Grágás*, registered in a mid-13th century manuscript *Konungsbók.* It is thus rooted in the medieval literary heritage that was studied in Iceland through the centuries.
- Bishop Árnason respected it and made it compatible to the calendar of the church in his 1739 *Dactylismus* as did 19th century scholars, at the establishment of the *Iceland Almanac,* and its later calculators.

- Registrations of births and deaths, done by the Church, were only prescribed from 1746, after the publication of bishop Árnason´s *Dactylismus.* Some birthdates were recorded according to the Misseri-calendar, a certain weekday in a certain week of summer or winter until the 20th century. The following data are from the 1920 national census:
  - Sigurður Jónsson born on Sunday in 12th week of winter 1859
  - Guðlaug Einarsdóttir on the 16th Saturday of summer in 1850

The conversion to the regular calendar sometimes caused confusion and people claimed to have another birthday than had been registered.

We may repeat the question why the pagan Misseri-calendar has survived in a Society of European Christian culture and propose an answer:

In northern latitudes such as in Iceland, the difference between darkness in winter and brightness in summer is extreme. Celebrating mid-winter *Þorri* and First Day of Summer and counting the weeks in-between is a tribute to the light and is intimately related to life in northern nature.

IMPACT ON LANGUAGE

The domestic tradition in currency and calendar has influenced the language in a variety of ways. It for example survives in poetry which has become national property. Until the mid-20th century, children of poor families might be brought up by people outside the family for as small an amount paid from public funds as possible. A well-known verse written by a poet brought up in such circumstances is:

Líf hans var til fárra fiska metið,.	His life was evaluated to only a few fishes.
Furðanlegt hvað strákurinn gat étið.	Amazing how much that lad could eat.
(Arnarson, 1942, p. iii)	

Children in the latter half of the 20th century probably did not realize that “fishes” is here a real currency unit, but all the same they may have felt the resentment and pain of these childhood memories. Official registration reveals that the poet had been placed in a farm at the age of three for 30 fishes from community funds after his father drowned, even if his mother worked at the same farm. Normal payment for a child who could not work was 240 fishes (Þráinsdóttir, 2011).

There are numerous examples of the various currency units referred to in proverbs, phrases and expressions. An example is *komast í álnir*, lit. “reach to ells”, means becoming prosperous. The Icelandic word *fé*, which is of the same origin as the English word “fee”, means both “sheep”, “livestock” in general, and “money”. *Fé*, meaning money, has remained as a natural expression in Icelandic, while its etymological roots are the Latin word *pecūnia*, “money”, and *pecū* in Latin means “a flock of sheep” (Magnússon, 1989).

The calendar was also a source of idiomatic phrases and proverbs. *Að þreyja Þorrann og Góuna*, to endure *Þorri* and *Góa*, the high winter months, means to wait patiently for better times. There are also meteorological sayings, such as "*þurr skyldi Þorri, þeysin Góa, votur Einmánuður, þá mun vel vora*", saying that *Þorri* should be dry, *Góa* windy and *Einmánuður* wet, to expect a good spring. The name of the pagan winter-solstice month, *Ýlir*, including the heathen festival "yule", has survived in the Nordic words *jól*, *jul*, for the Christian feast of Christmas.

CONCLUSIONS

In the earlier days of civilization, the mathematical elements varied greatly from one culture to another, so much that what was called mathematics in one culture would hardly be recognized as such in certain others (Wilder, 1950). D'Ambrosio has suggested an answer to why we should study ethnomathematics:

> It may compatibilize cultural forms – we should incorporate ethnomathematics in such a way that they facilitate the acquisition of knowledge, understanding, and the compatibilization of known and current popular practices. (D'Ambrosio, 1985, p. 70)

The *Misseri*-calendar was successfully compatibilized to the Julian calendar of the Christian world in the 12th century and further successfully compatibilized to the Gregorian calendar in Bishop Jón Árnason's *Dactylismus* in 1739 where the two calendars were printed parallel to each other.

Literature, phrases and proverbs preserve tradition in the language and culture, and link the present with the past by referring to human experiences, independent of time and space. The Reverend Eiríkur Briem was the right person at the right time to compatibilize the ancient currency of ells, fishes and butter to the modern currency of crowns.

At periods of rapid changes it is of great value to make the old compatible to the new in order to link generations together. This leads the attention to present times. Will the 21st century see compatibilization to the world of cyberspace, such as descendants of bit-coins?

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Workshop

THE SOLAR CYCLE AND CALENDARS, CURRENCY AND NUMBERS: RELATIONS TO SOCIETY AND CULTURE

Kristín Bjarnadóttir

University of Iceland

In the workshop, the solar cycle, seen from different latitudes and at different times of the year, will be studied and linked to calendars in use at various times and places, based on the solar cycle.

The solar cycle will be simulated by the use of the free software GeoGebra. Participants are encouraged to bring their own computers and download GeoGebra in order to create their own solar-cycle programs. Such programs are based on trigonometric functions and therefore suit students aged approximately 15 to 18 years.

Furthermore, a choice of currencies will be explored and their origins studied in order to reveal their connections to their original socio-economic environments.

Lastly, numerals, numerical systems and counting in selected languages and societies will be studied with respect to their origins and their connections to societal activities.

Participants are encouraged to contribute examples and ideas to the topics above and other related topics.

Workshop

MATHEMATICAL ACTIVITIES IN THE CLASSROOM ON THE OCCASION OF THE EXHIBITION « ART ET SAVOIR DE L'INDE » DURING THE FESTIVAL EUROPALIA-INDIA (BRUSSELS, OCTOBER 2013 – JANUARY 2014)

Jean Michel Delire

University of Brussels

In the context of the Europalia-India Festival, an exhibition « Art et Savoir de l'Inde » took place, of which mathematics was an essential part. Teachers and future teachers were thus involved in the preparation of math lessons inspired by this Indian context. The construction of the fire altar in the shape of a bird leads to fractional calculations through the manipulation of wooden puzzles, while the construction of a square with poles and a cord asks interesting questions about exactness and proofs. After the description of the implementation of these activities in the classroom, we add some remarks about the advantages teachers and pupils could take from such historical approaches to mathematics, remarks corroborated by the observations already made by our students during their teaching practice.

INTRODUCTION

Every second year, Europalia presents, In Brussels and in Belgium at large, a grand festival about every cultural aspect of an invited country, for instance China in 2010 and Turkey in 2015.

In 2013, India was the invited country and, as specialists of Indian mathematics and astronomy, we took this opportunity to conceive an exhibition on Indian science. With the help of some partners, we decided to develop six aspects of Indian rational thinking : ayurvedic medicine (Pierre-Sylvain Filliozat and Sandra Smets), architecture (Kiran Katara and her students at the University of Brussels : Serge Delire, Nikita Itenberg, Manuel Leon Fanjul, Milosz Martyniak, Lucas Brusco, Jorge Serra, Vincent Guichot), ritual geometry (Jean Michel Delire), astronomy at Jai Singh's time (J.M.Delire), counting systems and operations (François Patte), games of India (Michel Van Langendonckt).

The exhibition, entitled « Art et Savoir de l'Inde », took place at the University of Brussels, accompanied by a cycle of lectures on the different subjects and a two-day conference on Indian games organized by M.Van Langendonckt and J.M.Delire at the Haute Ecole de Bruxelles (HEB) and the University of Brussels.

My wife Béatrice and I, as mathematics teachers, could not conceive of this exhibition without proposing a guided visit to secondary and (end of) primary classes. Naturally, the idea of building classroom activities based on the mathematical part of the exhibition arose from discussions with our colleague and students at the HEB. Robert

Sadin who is teaching at the primary school L'école du petit chemin (Drogenbos, a suburb of Brussels) and a maître de formation pratique at the HEB; and Chaimaa Haddar, Charlotte François, Mariam Ben Messaoud, Jessica Sassatelli, Sara El Yacoubi, Stéphanie Petit, Karim Abboud, Tariq El Boujamai and Rabi Nejjari who are future math teachers in the secundary school.

We built two activities and presented them in different schools: the calculation of the size of the bricks for the fire altar in the shape of a bird of prey and the exact construction of a square with cords and poles.

THE FIRE ALTAR IN THE SHAPE OF A BIRD OF PREY (ŚYENA-CIT)

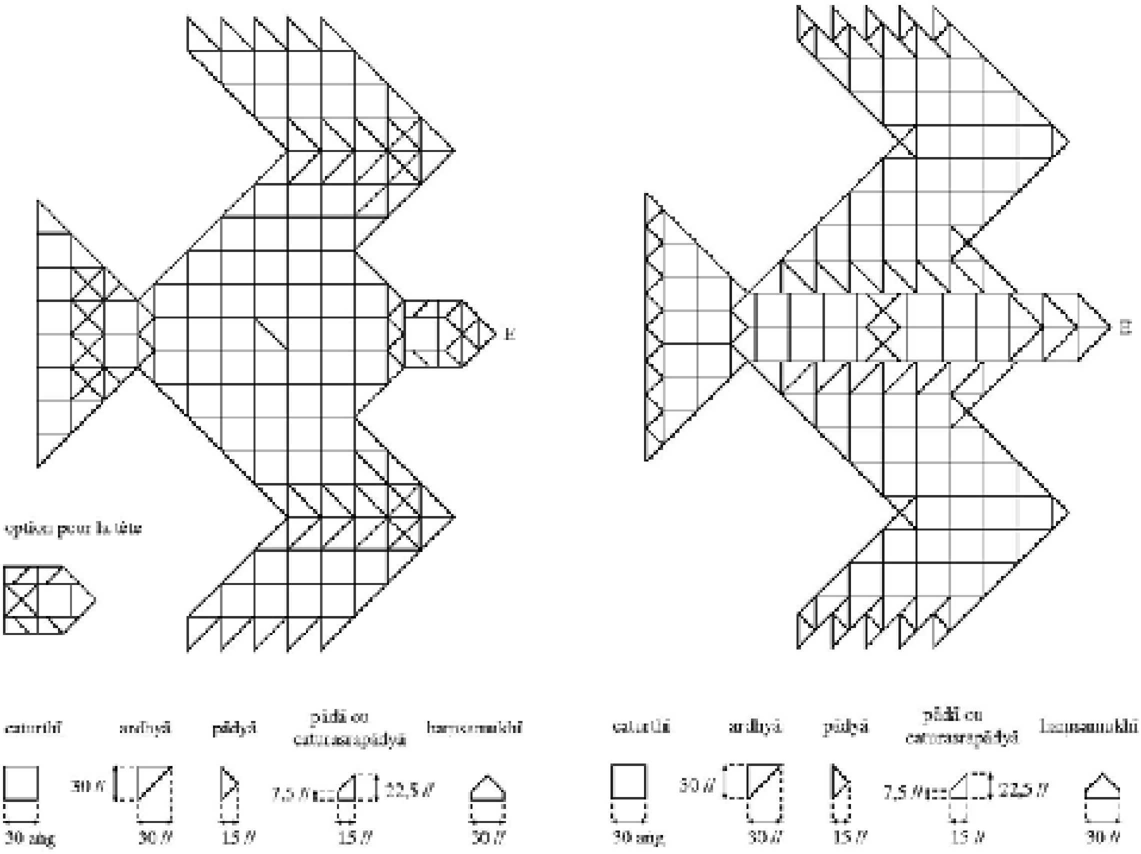


Figure 1 : odd layer **even layer**

Some Indian texts, the *Śulbasūtras*, dating from the last millenium BCE, precisely describe the shapes and sizes altars must have. These altars were prepared to receive the fire that makes the junction between men and gods in the sacrifice. Some were circular, like the *Gārhapatya* representing the world of men. Some were squares, like the *Āhavaniya* representing the world of gods. Others had much more complex shapes, like birds, turtle, a spoked wheel. But all of them had to cover the same area of 7.5 puruṣas², where the puruṣa is the height of the sacrificer with his hands pointing upwards. An altar was made of five layers of 200 bricks each. In the case of the śyena-cit (Figure 1), there are two types of layers (even and odd) and five types of bricks, of which the simplest is a square called caturthī because its side is one fourth (caturtha[1]) of a puruṣa.

Classroom exploitation of the fire altar śyena-cit

For our first mathematical activity, we asked the pupils to calculate the side of this brick with respect to their own puruṣa. Since they do not know the meaning of caturthī, they can not do better than transform 200 bricks of different shapes into an equivalent number of caturthī bricks. For that purpose, they had to count the number of bricks of each type and transform their total area into a number of caturthīs by comparing their respective areas with the caturthī's. To facilitate the procedure, we suggested them to fill the following table :

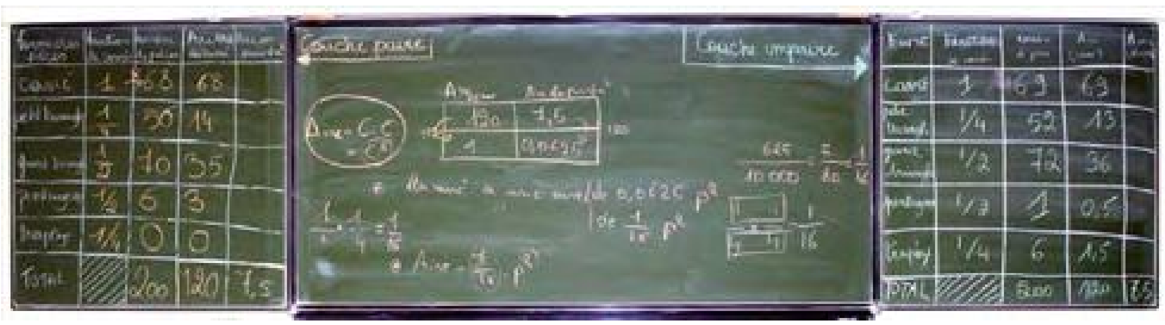
Shapes of the bricks	Fraction of the square	Number of bricks	Total area of the bricks (square = unit)	Total area of the bricks in puruṣa²
Total area				7.5

When the table is completed, we ask the pupils to fill the following table:

Number of squares	Area in puruṣa
1	

Which necessitates dividing the area of 7.5 puruṣas² by the number of caturthī bricks of which total area is identical to the 200 initial bricks.

The activity ends with the calculation of the side length of the caturthī brick, by extracting the square root of the last result. This step could be tricky if the pupils do not like to work with fractions and prefer to work with a calculator.



Organizing and observing the activities

The activities were initiated by a short – 15 minutes – introduction with slides to the Indian context of the sacrifice, the different shapes of the altars and their dimensions. After this was completed with all the pupils together, we placed them in groups of four to six and gave to each group a plastified map of one layer (odd or even) and a puzzle made of 200 wooden bricks. To each pupil we gave a sheet of paper with the tables.

The puzzle was useful for comparing the areas of the different bricks. The pupils usually began by covering a larger brick by two smaller ones, for instance one caturthī by two ardhyās or one ardhyā by two pādyās[2].

Finding that four pādās exactly cover a caturthī, without knowing that pāda means ‘fourth’, was more difficult. One has to unite two pādās so that they form a rectangle half of a caturthī (Figure 2).

Figure 2 : comparing the bricks’ areas

The most difficult comparison was that of the haṃsamukhī (‘goose beak’) brick with the caturthī. One cannot cover a caturthī by two haṃsamukhī to show that its area is half of the square. The only possibility, which is also mathematically interesting, is to notice that two pādās cover a haṃsamukhī, so that its area is equal to $2.\frac{1}{4} = \frac{1}{2}$ of a caturthī. One can also imagine halving one haṃsamukhī along its axis and rearrange it around another haṃsamukhī (Figure 2).

When counting the number of bricks of each type, the pupils used two different strategies, either they pile up the wooden bricks by tens or they distinguish the bricks on the map with different colors. To control their work, they can check that the total number of bricks is 200.

Computing the total areas for every kind of bricks in terms of caturthīs did not present any difficulty. Adding them yields 120.

But the division of the total area, 7.5 puruṣas , by 120 leads to problems when it is made with the help of a calculator. In that case, the result would be 0.0625 puruṣas of which square root, 0.25 puruṣas could be obtained by using 625 = 25 . In the case of fractions, one can replace $\frac{7.5}{120}$ by $\frac{15}{240} = \frac{1}{16}$ whose square root $\frac{1}{4}$ (one caturtha) can be found if one knows how to multiply fractions. This root extraction was the only task the pupils could not complete without our help.

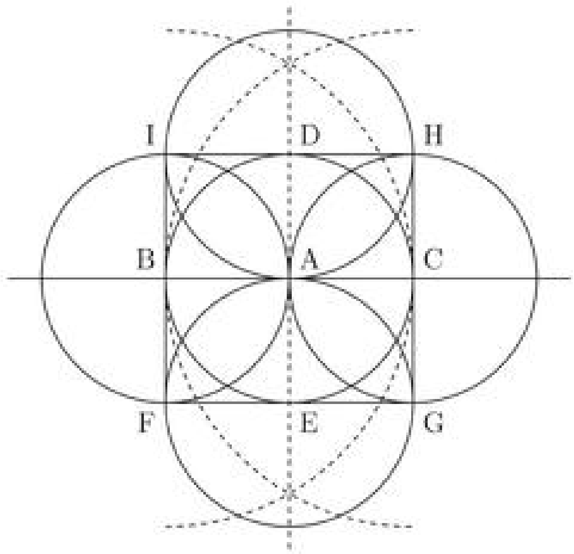
All these tasks were accomplished by the different groups of pupils, before completing the recreational activity of building the puzzle. This necessitates organization due to the number of bricks to be placed and the huge size of the puzzle. With a scale of one-tenth the actual size of the altar, the measure of the wingspan was about 90 centimetres. This forced the pupils to put two tables together and to distribute their roles, one for the head, one for each wing, one for the tail and one for the body. Since they had to follow a single map and to find the bricks they needed in a single box, they often helped each other.

THE CONSTRUCTION OF A SQUARE WITH POLES AND A CORD

In the introduction to the Indian context of the sacrifice, some steps of the actual construction of the altar are visible[3] and one of them shows that, before putting any brick in place, the shape of the bird was outlined with the help of a cord attached to poles driven in the ground.

The role played by the caturthī brick and the fact that some square altars, like the *Āhavaniya*, had to be built attest to the importance the square had for the Indian rituals. The *Śulbasūtras* give a construction of the square with poles and a cord following a procedure we have written in eight steps for pedagogical reasons:

1. Make loops at the two ends of a cord of length equal to the side *of the square.*
2. Drive a pole (A) on the West-East line[4].
3. Hook the two loops of the cord (on A) and draw a circle around.
4. Draw two poles at the West (B) and East (C) extremities of the diameter. At each of them fix a loop and draw a circle.
5. At the points where those two circles meet, draw a straight line which contains a second diameter that is perpendicular to the West-East line.
6. Fix two poles at the North (D) and South (E) extremities of this new diameter.
7. Fix the two loops respectively at the four poles (B, C, D, E) and draw four circles.



8. The vertices *of the square* are to be found at the intersections (F, G, H, I) of these circles.

Classroom exploitation of the construction of a square with poles and a cord

We did not give the entire procedure to the pupils. We did not mention that we wanted to build a square. In the first and last steps, the italicised words were missing. The reason of this omission was pedagogical. We wanted the pupils to guess that the shape they built is a square, but they also had to realize that this square was not perfect due to their imperfect way of tightening the cord. We asked them to prove that the figure aimed by the text is a square by retracing their building steps and find in (some of) them what properties of the square are exhibited.

Organizing and observing the activities

In this activity, we wanted the pupils to work as much as possible exactly as the Indians did. Since it was not possible to drive poles in the ground, we replaced them by drawing pins to be driven into a large cardboard placed on a table or on the ground. We had a cord and pens to draw the circles. Depending on time, we have either asked each of the pupils to complete collectively only one step, or to do the entire construction beforehand and then with the rest of the class complete a common drawing step by step. Generally, the pupils were asked to work in pairs. The use of pens placed in the loops for drawing circles was a source of errors, but this ensured that the final square would not be perfect and serves our purpose of requiring a proof.

The collective way of drawing was less satisfactory because the "step consisting of letting the pupils dive into the problem and try to find by themselves the intented shape was missing"[5]. In the other case, the pupils "were in a scientific process where, after finding the intented shape (induction), they had to recollect all the information from the steps. They then selected the most useful of them for the proof. Finally they wrote the argument in a logical sequence (deduction)".[6] In both cases, we encourage the pupils to express their arguments orally, in order to raise the level of the debate and make it more lively.

Another exploitation of the construction of a square with poles and a cord

We also organized this activity in the primary school of our colleague Robert Sadin. In that case, Robert Sadin had prepared a large box containing leveled clay and asked the pupils to draw the figure with the help of cords and little wooden pegs and markers carefully carved so that the cord can be kept in the same position while drawing. This equipment ensures a correct square shape if the pupils tighten the cords and do not prompt them to prove this fact. This was not our objective in primary school. We rather encouraged the pupils to discover the properties of the circle.

THE OPTIONAL HEAD IN THE ODD LAYER OF THE FIRE ALTAR

One can wonder why another disposition of bricks is given for the odd layer of the fire altar. This is related to the requirement, expressed in the *Śulbasūtras* that no crack between two bricks stands on the crack between the lower bricks. That is the reason why the odd and even layers are different. We seldom had the opportunity to build one layer above the other with the pupils, but Robert Sadin imagined a way to do it easily, by asking a team to build one layer on a plexiglass sheet and move it afterwards above a layer made by another team. Thanks to this experiment we discovered the reason why the *Śulbasūtras* propose an optional disposition for the head. This is left as an exercise for the reader.

HOW CAN SUCH CONTEXTUALIZATIONS OF MATHEMATICS BE OF GREAT BENEFIT TO TEACHERS AND PUPILS?

Before letting Mariam Ben Messaoud and Chaimaa Haddar, who wrote their memoirs about slightly different aspects of this contextualization in the case of Indian mathematics, I want to emphasize three of the many advantages offered by teaching mathematics with the help of history:

1. Mathematics is a human activity.
   Contextualization of mathematics in its history helps pupils to understand that they were invented by people like them over a long period of time and that they are still evolving nowadays. It also shows that the concepts were built to solve problems and not, as in too many mathematics lessons, the problems to illustrate concepts. Since problems vary according to places and times, the concepts are also different.[7] Moreover, taking the history into account motivates pupils who are more interested in history and language, especially if they have the opportunity to read ancient mathematical texts.

2. The origins of mathematics are varied.
   Mathematics are universal but history shows that all civilizations, even remote, made their own contribution. The teacher can bring the pupils' attention to that point by recalling the name and a concise biography of the mathematician who

discovered the application they are studying. For instance, the « generalized Pythagoras' theorem » is now called « al-Kāšī's theorem », because this Arab mathematician of Iranian origin, born at the end of XVth century, stated this theorem with the help of trigonometric functions[8]. To study the ill-named « Pythagoras' theorem », discovered by many civilizations, the teacher can show an application or a proof provided by each of them, and decide to give it a more neutral name as « theorem of the diagonal of the rectangle ».The teacher can also ask the pupils to prepare a short presentation of a mathematician. This has the advantage of involving them and making the pupils of foreign origin, numerous in Brussels and other large European cities, more aware of mathematics, especially if they choose a mathematician from their own country.

3. Mathematics are sensible
   History of mathematics usually leads to a shifting away from the centre and often helps us to give more meaning to the automatisms learned in primary school. For instance, the explanation of the binary couting system used in computers or of the sexagesimal one still present in our watches and necessary for trigonometry throws light on the decimal one.[9] Sara Ozdemir, one of my former students, who prepared an exhibition on the history of ancient numerations with her own pupils, wrote[10] « (…) the pupils less motivated by mathematics were the most interested by the activities and discoveries about the history of counting systems. They have accepted and loved to learn new methods of calculation ». Moreover, the reaction of one of Sara's pupils was[11]: « Mrs Ozdemir helped us to understand everything, so to better assimilate the subject, so to love it and she gave us the desire to learn it. That reminded me a lot of things in mathematics and to explain them myself was for me a challenge and a real fun. »

CONCLUSIONS

Mariam Ben Messaoud[12]: « The positive point was the pupils' motivation to learn differently, to make use of other instruments and to discover other aspects of mathematics. These activities allowed me to remind them of notions like area, rule of three, fractions and geometrical figures. They also induced the pupils to make use of their knowledge to take up a challenge. They had to work together to reach it and this showed them that we always succeed by mutual aid. »

Chaimaa Haddar[13]: « Erasing the name of the quadrilateral [in the exact construction of a square] added to the activity a challenge. Some pupils even considered this exercize as an enigma, which renders the atmosphere of the class very pleasant. About the motivation, one must add that introducing this activity by history gave the opportunity to the pupils who were not especially interested by mathematics to participate as well as the others. So they found a kind of entrance (…) different from those they already knew. »

Using history in a mathematical course offers those who have taken time to keep informed many advantages and few drawbacks except, perhaps, a slight loss of time in the classroom. The question is: would the pupils and the teacher really loose time if they allow history to make their mathematical activity more meaningful?

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[1] One can guess catur is akin to Latin quatuor 'four', -tha being a suffix for fractionnal or ordinal numbers, as -th in four-th. These similarities are due to the appartenance of English, Latin and Sanskrit, the language of the *Śulbasūtras*, to the same linguistic family.

[2] It is linguistically interesting to note that pādyā derives from pāda, which means 'foot' or 'fourth' (animals have four feet).

[3] As in the film *Altar of Fire*, by R.Gardner and J.F.Staal for the Film Study Center at Harvard University, color, 58 min, 1976 (http://www.der.org/films/altar-of-fire.html).

[4] This line is considered as the spinal column of the sacrifice. Its position is usually fixed by a method based on astronomy.

[5] Chaimaa Haddar, *Comment sensibiliser les élèves à la notion de démonstration (en mathématiques et ailleurs) en s'inspirant de son évolution historique*, Memoir presented for the diploma of BA – teacher in mathematics, Haute Ecole de Bruxelles, 2014, p.32.

[6] Chaimaa Haddar, *ibidem.*

[7] M.Wolff, *La bosse des maths est une maladie mentale ?*, La découverte, Paris, 1984, p.32, and J.P.Friedelmeyer, « *L'indispensable histoire des mathématiques* », *Repères IREM*, n°5 (oct.1991), p.25 : « Les constructions mathématiques ont répondu aux problèmes d'alors, dans un contexte de civilisation donné. »

[8] Following Youschkevitch, *Les mathématiques arabes*, Paris, 1976, pp.71 and 136. It was already in Euclid, *Eléments*, II.12 and 13, but without trigonometry, of course.

[9] About an interdisciplinary work on the Cassini's map of France (1666-1715), Xavier Lefort wrote (« *L'histoire de la carte de France de Cassini* », *Repères IREM*, 14 (January 1994), p.25) : « In mathematics, for instance, one must endeavour to find again the ancient techniques ; then the pupils look at many notions in a new light. »

[10] Sara Ozdemir, *Utiliser l'histoire dans l'enseignement des mathématiques les rend plus humaines*, Memoir presented for the diploma of BA – teacher in mathematics, Haute Ecole de Bruxelles, 2009, p.45.

[11] Sara Ozdemir, *ibidem.*

[12] Mariam Ben Messaoud, *Amener une autre vision des mathématiques via l'exploitation de* l'exposition « Art et Savoir de l'Inde », Memoir presented for the diploma of BA – teacher in mathematics, Haute Ecole de Bruxelles, 2014, p.36.

[13] Chaimaa Haddar, *op.cit.*, p.32.

Workshop

A GEOMETRIC PROBLEM ON THREE CIRCLES IN A TRIANGLE (THE MALFATTI PROBLEM) — A THREAD THROUGH JAPANESE, EUROPEAN AND CHINESE MATHEMATICS

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The geometric problem about three circles lying inside a triangle each of which touching two sides of the triangle and the two other circles has come to be known as the Malfatti Problem, although the problem was studied in Japan about thirty years before Malfatti studied it. This problem appeared in the latter part of the eighteenth century in Japan, also in the early part of the nineteenth century in Europe and was studied in the latter part of the nineteenth century in China. Other than just an interesting geometric problem its appeal lies in the different historical and cultural contexts in which the problem was studied during different periods in different countries for different purposes.

1. INTRODUCTION

This is a story about a geometric problem on three circles in a triangle. It appeared in the latter part of the eighteenth century in Japan, also in the early part of the nineteenth century in Europe and was studied in the latter part of the nineteenth century in China. The problem is about three circles lying inside a given triangle, each of which touches two sides of the triangle and the two other circles. It has come to be known as the Malfatti Problem (while the original problem actually asked for three non-overlapping circles of maximal area lying inside a given triangle), although the problem was studied by the Japanese mathematician AJIMA Chokuyen (安島直円 1732-1798) before the Italian mathematician Gianfrancesco MALFATTI (1731-1807) studied it. By itself the problem is just one of many interesting problems in geometry. A more appealing feature is the different historical and cultural contexts in which the problem was studied during different periods in different countries for different purposes.

This story was told in the form of a workshop conducted in the 7th European Summer University on the History and Epistemology in Mathematics Education held in Copenhagen in July of 2014. A main objective is to see how the historical material can be integrated in teaching and learning in the classroom. A worksheet with eight problems (EXERCISE (1) to (8)) can be found in the APPENDIX in the last section with accompanying brief remarks.

Reader who are interested in knowing more details about the Malfatti Problem are invited to consult the paper by Chan and Siu (Chan & Siu, 2012b) and the paper by

Lorenat (Lorenat, 2012) together with the references in their respective bibliographies. For primary historical sources a list is provided below:

(1) 中西聞見錄 *Zhongxi Wenjian Lu* [*Peking Magazine*] (1872 - 1875) [in Chinese], Beijing; thirty-six issues reprinted and collected in four volumes by the Nanjing Antique Bookshop, Nanjing, 1992, available on-line in http:/nla.gov.au/nla.gen-vn514067.

(2) *Annales mathématiques pures et appliquées (de Gergonne)*, available on-line in http://www.numdam.org.

(3) *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, available on-line in http://gdz.sub.uni-goettingen.de/no_cache/dms/load/toc/?IDDOC=238618

(4) K. Chemla, S.C. Guo (郭書春), *Les Neuf Chapitres* (九章算術)*: Le classique mathématique de la Chine ancienne et ses commentaires*, Dunod, Paris, 2004.

(5) Collections of *sangaku*, available on-line in http://www.wasan.jp/index.html.

2. THE MALFATTI PROBLEM

In 1810 the French mathematician Joseph Diaz GERGONNE (1771–1859) established his own mathematics journal officially called the *Annales de mathématiques pures et appliquées* but became more popularly known as *Annales de Gergonne*. This journal, which was the first privately run journal wholly on mathematical topics, was discontinued in 1832 after Gergonne became the Rector of the University of Montpellier. Since Gergonne's mathematical interests were in geometry, this topic figured most prominently in his journal, with many famous mathematicians of the time publishing papers during the twenty-two-year period of its existence. To facilitate a dialogue between the Editor and the readership the journal posed problems regularly besides publishing papers.

In the first volume of *Annales de Gergonne* [Vol.1 (1810-1811), 196] the following problem was posed:

"A un triangle donné quelconque, inscrire trois cercles, de manière que chacun d'eux touche les deux autres et deux côtés du triangle (**) ? […]

(**) Ce problème ne présente aucune difficulté, lorsque le triangle est équilatéral. Jacques Bernoulli l'a résolu pour le triangle isocèle (*Voyez ses œuvres, tome 1, page* 303*, Genève,* 1744); mais sa solution est beaucoup moins simple que ne le comporte ce cas particulier. […] " [1]

Soon after the problem was posed a solution appeared in a later issue of the journal and referred to a letter from a reader, Professor Giorgio BIDONE (1781-1839) in Turin [Vol.1 (1810-1811), 346-347]:

"[…] Les rédacteurs des *Annales* en étaient parvenus à ce point, et ils ne pensaient pas que cette dernière formule fût susceptible de beaucoup de réduction, lorsqu'ils reçurent de M. BIDONE, professeur à l'académie de Turin, la lettre suivante:

Turin, le 12 *mars* 1811. […] *Je prends la liberté, Messieurs, de vous annoncer que ce problème a été résolu par M. MALFATTI, géomètre italien très-distingué. Sa solution est imprimée dans la 1.re partie du tome X des* Mémoires de la société italienne des sciences, *publié en* 1803. […]” [2]

The original problem posed by Malfatti in 1803 asks (Malfatti, 1803):

“Given a right triangular prism of any sort of material, such as marble, how shall three circular cylinders of the same height as the prism and of the greatest possible volume of material be related to one another in the prism and leave over the least possible amount of material? [original text in Italian]”

Malfatti thought that the three non-overlapping circles inside the triangle occupying optimal space would be three “kissing circles”. Actually this is never the solution, but it was only realized with the optimality problem fully settled as late as in 1994 (Zalgaller & Los, 1994; Andreatta & Bezdek & Boroński, 2010)! In our discussion we focus on only the problem of three “kissing circles”, which will be, by abuse of language, also called the Malfatti Problem.

The special case for an equilateral triangle is not hard to solve. (See **EXERCISE (1)**.). There is a clever way to solve this special case. (See **EXERCISE (2)** and **EXERCISE (3)**, the latter offering an interesting “proof without words” in the commentary by LIU Hui (劉徽 *circa* 3rd century) on Problem 16 in Chapter 9 of the ancient Chinese mathematical treatise *Jiu Zhang Suan Shu* [九章算術 The Nine Chapters on the Mathematical Art] compiled between the 2nd century B.C.E. and the 1st century C.E..) From this special case we can already see that the three “kissing circles” do not give an answer to Malfatti’s original problem! (See **EXERCISE (4)**.)

It is interesting to note that the radii r_1, r_2, r_3, of the three “kissing circles” are determined by the three sides of the triangle a, b, c. (See **EXERCISE 5(i)**.) In the aforementioned letter from Bidone the construction by Malfatti reported therein is based on these formulae for r_1, r_2, r_3 in terms of a, b, c. (See **EXERCISE 6**.) This solution obtained by Malfatti through algebraic computation is skillful and interesting but at the same time has its shortcoming of losing sight of the geometric intuition in a purely geometric problem. Some geometers at the time were not satisfied with an algebraic solution and wanted to obtain a synthetic geometric construction. This was first accomplished by the Swiss geometer Jakob STEINER (1796-1863) in 1826. In his paper in the *Crelle’s Journal* (which has a more official name of *Journal für die reine und angewandte Mathematik*) Steiner commented, “In order to show the fruitfulness of the theorems presented in paragraphs (I, II, III) with respect to one suitable example, we enclose both the geometric solution and the generalization of the Malfatti Problem, however without proof.” (Steiner, 1826).

Steiner’s construction is as ingenious as it is mystifying in that it is not at all apparent how he arrived at it! (See **EXERCISE 7**.) The first proof of the construction was given by the Irish mathematician Andrew S. HART in 1857 (Hart, 1857). Another explanation was offered in 1879 by the Danish mathematician Julius Peter Christian PETERSEN (1839-1910) in the second edition of his successful book *Méthodes et théories pour la résolution des problèmes de constructions géométriques avec application a plus de 400 problèmes* on geometric constructions [first edition published in Danish] and in a paper published in 1880 also in the *Crelle's Journal* (Petersen, 1880). The span of some thirty to fifty years between the discovery of the construction to its explanation speaks for the difficulty of the problem.

(I should admit that I cannot yet figure out the underlying theoretical principle embodied in the construction by Steiner, which is probably that of inversion, in which Steiner enjoyed fame in the employment of this powerful geometric notion. I like also to thank Bjarne TOFT of the University of South Denmark who told me about the work of Petersen during the workshop.)

3. THE CHINESE LINE OF THE STORY

We now usher in the Chinese line of the story. In the latter part of the nineteenth century some foreign missionaries, along with spreading Christian faith, worked hard to propagate Western learning in old imperial China through various means, one of which was publishing periodicals. The monthly periodical *Zhongxi Wenjian Lu* [Record of News in China and West] with English title *Peking Magazine*, founded in 1872, announced in the first issue that it adopted the practice and format of newspapers in the Western world in publishing international news and recent happenings in different countries, as well as essays on astronomy, geography and *gewu* [science, literally meaning "investigating things"].

The fifth issue (December, 1872) of this magazine carried the following posed problem:

"有平三角(無論銳直鈍諸角形)內容相切三圓大小不等欲量取三圓之心，其法何若？此題天文館諸生徒皆縮手，四方算學家，有能得其心者，可以其圖寄都中天文館，當送幾何原本一部，且將其圖刊入聞見錄，揚名天下。" [3]

A solution submitted by a reader was published in the eighth issue (March, 1873), followed by a comment by another reader in the twelfth issue (July, 1873) together with an acknowledgement of the error and a further comment by the School of Astronomy and Mathematics of the state-run institution of *Tongwen Guan*. The establishment of *Tongwen Guan* was at first intended as a language school to train interpreters but later developed into a college of Western learning, along with other colleges of similar nature that sprouted in other cities like Shanghai, Guangzhou, Fuzhou, Tianjin, along with the establishment of arsenals, shipyards and naval schools during the period known as the "Self-strengthening Movement" as a result of the fervent and urgent desire of the Chinese government to learn from the West in order to resist foreign exploitation the country went through in the first and second Opium Wars (Chan & Siu, 2012a).

This kind of fervent exchange of academic discussion carried on in public domain was a new phenomenon of the time in China. In 1897 a book on homework assignments by students of *Longcheng Shuyuan* [Academy of the Dragon City], which was a private academy famous for its mathematics curriculum, contained two articles that gave different solutions to the Malfatti Problem with accompanying remarks by the professor. One solution is particularly interesting because it made use of a hyperbola, which is a mathematical object that was totally foreign to Chinese traditional mathematics and was newly introduced in a systematic way only by the mid-nineteenth century.

It is not certain when the Malfatti Problem was first introduced into China. Apparently it was introduced by Westerners into China only two to three decades after the problem became well-known in the West, at a time when the Chinese were just beginning to familiarize themselves with Euclidean geometry, which was not part of their traditional mathematics (Siu, 2011).

It is worth noting, from the active discussion generated around the Malfatti Problem, how enthusiastic the Chinese were in learning mathematics from the Westerners in the late nineteenth century. Let us look a bit more into the historical context. As pointed out by Siu (Siu, in press), "[t]he translation of *Elements* by XU Guang-qi and Matteo RICCI led the way of the first wave of transmission of European science into China, with a second wave (or a wake of the first wave as some historians would see it) and a third wave to follow in the Qing Dynasty, but each in a rather different historical context. The gain of this first wave seemed momentary and passed with the downfall of the Ming Dynasty. Looking back we can see its long-term influence, but at the time this small window which opened onto an amazing outside world was soon closed again, only to be forced open as a wider door two hundred years later by Western gunboats that inflicted upon the ancient nation a century of exploitation and humiliation, thus generating an urgency to know more about and to learn with zest from the Western world. The main features of the three waves of transmission of Western learning into China can be summarized in the prototype slogans of the three epochs. In the late-sixteenth to mid-seventeenth centuries (during the Ming Dynasty) the slogan was: "In order to surpass we must try to understand and to synthesize (欲求超勝必須會通)." In the first part of the eighteenth century (during the Qing Dynasty) the slogan was: "Western learning has its origin in Chinese learning (西學中源)." In the latter part of the nineteenth century (during the Qing Dynasty) the slogan was: "Learn the strong techniques of the '[Western] barbarians' in order to control them (師夷長技以制夷)."

4. THE JAPANESE LINE OF THE STORY

The problem was in fact posed three decades before Malfatti did by the Japanese mathematician AJIMA Chokuyen (安島直円 also known as AJIMA Naonobu, 1732-1798) (Fukagawa & Rothman, 2008). A related problem that asked for the radius of the inscribed circle of the triangle in terms of the radii of the three "kissing circles" was proposed by another Japanese mathematician TAKATADA Shichi on a *sangaku* (算額 mathematical tablet) in the Meiseirinji Temple (明星輪寺) in Ogaki City hung in 1865 (Fukagawa & Rothman, 2008). (See **EXERCISE 5(ii)**.)

A *sangaku* is a wooden tablet with geometric problems written on it together with beautiful drawings, making it a piece of mathematical text as well as a piece of art work. These mathematical tablets, which came into popular existence in the Edo period (江戸時代 1603-1867) in Japan, were dedicated to Shinto shrines and Buddhist temples as religious offerings. About nine hundred such tablets are extant today, but it is believed that at one time there were thousands more than that. They form a body of what one would label as "folk mathematics", as these tablets were from members of all social classes, including professional or amateur mathematicians, students, women and even children. (Fukagawa & Rothman, 2008). Some mathematical texts were written about these many geometric problems on the mathematical tablets hung in shrines and temples, the first of which was by the famed Japanese mathematician FUJITA Sadasuke (藤田貞資 1734-1807) in 1789, with a sequel by his son FUJITA Yoshitoki (藤田嘉言 1772-1828) in 1807.

Japanese mathematics was at one time strongly influenced by Chinese mathematics through the books seized from Korea as a result of an attempted invasion of Korea, with the real objective of invading the Ming Empire of China, during the last decade of the 16th century by the Japanese warlord TOYOTOMI Hideyoshi (豐臣秀吉 1536-1598). At the time Korean mathematics was under the strong influence of Chinese mathematics so that the books transmitted to Japan included two prominent Chinese texts, *Suan Xue Qi Meng* (數學啟蒙 Introduction to the Computational Science) by ZHU Shijie (朱世傑 c.1260-c.1320) of 1299 and *Suan Fa Tong Zhong* (算法統宗 Systematic Treatise on Calculating Methods) by CHENG Dawei (程大位 1533-1606) of 1592. These two texts exerted significant influence in the formation of *wasan* (和算 native Japanese mathematics), which refers to the body of mathematics developed in the Edo period in the historical context of isolation from the West and at the same time of increasing divergence from Chinese mathematics since the mid-17th century with its own elaborate and independent development.

Western mathematics took its root in Japan for a more or less similar reason as it was in China. In July of 1853, when Commodore Matthew Calbraith PERRY (1794-1858) led an American fleet to reach Japan and anchored in Edo Bay (now Bay of Tokyo), the closed door of the country was forced open under military threat. Besides ending the seclusion of Japan this incident also led to the establishment of the Nagasaki Naval Academy and the *Bansho Shirabe-sho* (蕃書調所 literally meaning "Office for the Investigation of Barbarian Books"), both of which were important for instituting systematic study of Western science and mathematics in Japan. With the Meiji Restoration Western learning in Japan was no longer confined to military science for self-defence but was regarded as an integral means for modernization of the country. Foreigners were brought into Japan to teach Western science and mathematics. However, the route to "Westernization" of mathematics education in Japan took a much faster and more drastic turn. The *Gakusei* (Fundamental Code of Education) of Japan in 1872 decreed that *wasan* was not to be taught at school; only Western mathematics was taught (Siu, 2009).

Note the religious and ritualistic aspect and the appreciation of beauty in dedicating geometric theorems to deities in shrines and temples in the form of a *sangaku* as an offering. In contrast, note the "learn from the West to resist the West" aspect of the Chinese line described in the previous section. A question of historical interest would be to study how familiar Chinese mathematicians of the late 19th century were with Japanese mathematics at the time, or would they pay no attention at all to *wasan* of the Edo period, thinking that *wasan* was but a "tributary" of Chinese traditional mathematics (Chan & Siu, 2012b).

5. THE EUROPEAN LINE OF THE STORY AGAIN

We now get back to the European line, which focuses on methodology. Jemma Lorenat gave a detailed discussion in her paper in which she says, "From this perspective we observe efforts towards developing general theories to encompass the approaches for particular problems, the differentiation and competition of geometric methodologies, and the nationalization and internationalization of mathematical communities." (Lorenat, 2012).

According to Lorenat's analysis, before 1826 activities surrounding the Malfatti Problem went on in the French-speaking world. In 1826 Steiner's paper appeared in the *Crelle's Journal*, thereby switching the activities to the German-speaking world from 1826 to the1870s. After the 1870s the focus of activities moved to the English-speaking world, and furthermore from mathematicians to amateurs and educators, thereby enriching the discussion.

In terms of methodology we have seen in the second section how the attention on the solution shifted from algebraic means to geometric means. This point was put quite clearly and explicitly by Christian Felix KLEIN (1849-1925) in a paper of 1892, in which he said, "[…] it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident, and the progress made by the aids of analysis is only a first, though a very important step." (Klein, 1892).

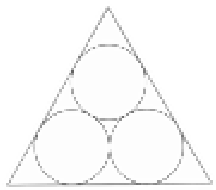
6. FINALE

We have seen in the previous sections the different historical and cultural contexts in which the Malfatti problem was studied during those different periods in different countries for different purposes. In view of the interest and objective of HPM (History and Pedagogy of Mathematics) the natural question to ask is how one can make use of the Malfatti Problem in accord with such interest and objective. (See **EXERCISE (8)**.) The list of exercises made use of in the workshop (see **APPENDIX**) is an attempt in this direction.

In the second volume of *Annales de Gergonne* [Vol.2 (1811-1812), 165] there appeared a letter from a reader, Mr. TÉDENAT, who said, "[…] I think that at least the reflections that I have made on this subject can help the research of those readers who have all the leisure necessary to do it." With this ending note, further investigation of the Malfatti Problem will be left to the enjoyment of those readers who have all this leisure!

7. APPENDIX: EXERCISES FOR THE WORSHOP

(1) Three circles in an equilateral triangle with side of unit length are placed in such a way that each touches the other two as well as two sides of the triangle (see the figure below). Compute the radii of the circles.

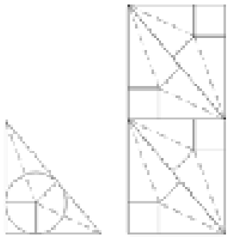


Remark: There are various ways to arrive at the answer $r = \frac{\sqrt{3}-1}{4}$.

(2) The following method offers a quick way to solve the problem in Exercise **(1)**. By symmetry the three circles are of equal radii, so it suffices to compute the radius of one of them. Consider one half of the equilateral triangle. One circle becomes the inscribed circle of the resulting right triangle. It is not hard to compute its radius. Does the same method work equally well with an isosceles triangle?

Remark: Make use of the property of a right triangle to see that $r = \frac{ab}{a+b+c}$, where r is the radius of the circle and a, b, c are the lengths of the three sides of the right triangle with c being the hypotenuse. If the triangle is isosceles but not equilateral, then this computation gives only the radius of the two "kissing circles" touching the base. It requires some more computation to find the radius of the third circle.

(3) There is a clever way to compute the radius of an inscribed circle of a right triangle explained in the commentary by LIU Hui (劉徽 *circa* 3rd century) on Problem 16 in Chapter 9 of the ancient Chinese mathematical treatise *Jiu Zhang Suan Shu* (九章算術 The Nine Chapters on the Mathematical Art) compiled between the 2nd century B.C.E. and the 1st century C.E.. Give a "proof without words" based on the following figure re-constructed from the explanation in the commentary.

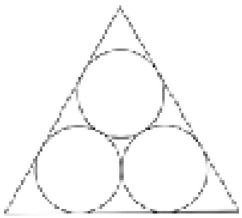


Why would LIU Hui make use of four copies of the right triangle instead of just one or two?

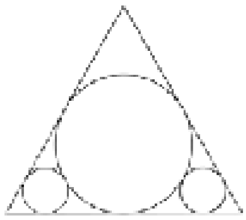
Remark: See an explanation in, for instance, Section 3 of the paper: M.K. Siu, Proof and pedagogy in ancient China: Examples from Liu Hui's Commentary on JIU ZHANG SUAN SHU, *Educational Studies in Mathematics*, 24(1993), 345-357. Using only one copy will require a "flip-over" of some pieces (say that the triangle is coloured red on only one side). Using two copies requires no "flip-over" but needs a bit of argument, while using four copies offers a natural "proof without words" using the method of dissection-and-reassembling.

(4) Compute the total area of the three circles inside an equilateral triangle with side of unit length, when the three circles are placed as shown in

(i)



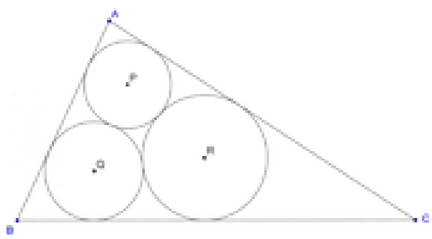
(ii)



Which case gives a larger total area? What does this say about the original problem of Malfatti?

Remark: The total area is 0.3156… in **(i)**, and is 0.3199… in **(ii)**. Actually, **(ii)** yields an optimal answer.

(5) Three circles with respective centres P, Q, R inside a given triangle ΔABC are placed in such a way that each circle touches the other two as well as two sides of the triangle (see the figure below)



(i) Compute the radii r_1, r_2, r_3 of the three circles.

[This is not an easy problem. Various mathematicians in the Western world since the time of Gianfrancesco MALFATTI (1731-1807) gave their answers. The earliest instance was , however, given by the Japanese mathematician AJIMA Chokuyen (安島直円 1732-1798) three decades before Malfatti. Malfatti's formulae (made known posthumously) for r_1, r_2, r_3, the radii of the circles centred at *P, Q, R* repectively, are given by

$$r_1 = \frac{(s-r+IA-IB-IC)r}{2(s-a)}, r_2 = \frac{(s-r+IB-IC-IA)r}{2(s-b)}, r_3 = \frac{(s-r+IC-IA-IB)r}{2(s-c)},$$

where r is the radius and I is the centre of the inscribed circle, and

$s = \frac{a+b+c}{2}$ is the semiperimeter of the triangle.]

(ii) Find the radius r of the inscribed circle of ΔABC in terms of r_1, r_2, r_3.

[This problem was proposed by the Japanese mathematician TAKATADA Shichi on a *sangaku* (算額 mathematical tablet hung in a shrine or a temple as a kind of offering) in the Meiseirinji Temple (明星輪寺) in Ogaki City (大垣市). Its solution appeared in the book *Sanpo Tenzan Tebikigusa* (算法點竄手艸 Algebraic Methods in Geometry) of 1841 by OMURA Kazuhide (大村一秀 1824-1891). The neat formula is

$$r = \frac{2\sqrt{r_1 r_2 r_3}}{\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}-\sqrt{r_1+r_2+r_3}}.]$$

Remark: For **(i)** we try to calculate x_1 (respectively x_2, x_3) which is the distance from A (respectively B, C) to the point of contact of the circle centred at P (respectively Q, R). From x_1, x_2, x_3 we can obtain r_1, r_2, r_3 because

$$r_1 = \frac{x_1 r}{s-a}, r_2 = \frac{x_2 r}{s-b}, r_3 = \frac{x_3 r}{s-c}.$$

It can be shown that x_1, x_2, x_3 satisfy the system of equations

$$x_1 + x_2 + 2b_3\sqrt{x_1 x_2} = c,$$
$$x_2 + x_3 + 2b_1\sqrt{x_2 x_3} = a,$$
$$x_3 + x_1 + 2b_2\sqrt{x_3 x_1} = b,$$

where $b_1 = \sqrt{\frac{s-a}{s}}, b_2 = \sqrt{\frac{s-b}{s}}, b_3 = \sqrt{\frac{s-c}{s}}$.

For a (clever) way to solve for x_1, x_2, x_3 from this system of equations see, for instance, Chapter 3 of the book by Coolidge (Coolidge, 1916). For **(ii)** see the explanation of Problem 4 in Chapter 6 of the book by Fukagawa and Rothman

(Fukagawa & Rothman, 2008). Incidentally, the underlying idea of using area is, in a sense, of a strong Oriental flavour. (Compare with the method in **EXERCISE (3)**.)

(6) According to what Giorgio BIDONE (1781-1839) communicated to Joseph Diaz GERGONNE (1771-1859) the construction given by Gianfrancesco MALFATTI (1731-1807) to his own problem (without proof) is as follows. Verify that it works.

Let the vertices of the triangle be A, B, C. Let I be the centre of the inscribed circle which touches CA, CB at E, D respectively. Produce AB to X such that $BX = CE = CD$. AI intersects the inscribed circle at A' lying between A and I. Further produce AX to Y such that $XY = AA'$. Take Z on AY in the direction of A such that $YZ = IC$. Take W on AZ in the direction of A such that $ZW = IB$. Let M be the midpoint of AW and erect the perpendicular at M to AB, intersecting IA at O. OM is the required radius of the circle touching AB, AC and the two other circles.

Remark: This is a rendering of the formulae given in **EXERCISE (5)(i)** in a geometric language.

(7) The following is a geometric construction to the Malfatti Problem offered by the Swiss mathematician Jakob STEINER (1796-1863) in 1826 (without proof). Verify that it works.

Let the vertices of the triangle be A_1, A_2, A_3. Let I be the centre of the inscribed circle. Inscribe a circle in each of the triangles ΔIA_jA_k. The circles inscribed in ΔIA_iA_j and ΔIA_jA_k have IA_j as one common tangent. Construct the other such common tangent D_jE_j. The circles required are inscribed in the quadrilaterals whose sides are A_iA_j , A_iA_k , D_jE_j , D_kE_k .

Remark: Consult the paper by Hart (Hart, 1857) and the paper by Petersen (Petersen, 1880).

(8) Discuss the different historical and cultural contexts in which the Malfatti problem was studied in those different periods in different countries for different purposes. How can one make use of the Malfatti Problem in accord with the interest and objective of HPM?

Remark: Suggestions and comments from readers will be very much appreciated.

NOTES

1. Given any triangle, inscribe three circles in such a way that each of them touches the other two and two sides of the triangle (**). […]

(**) This problem does not present any difficulty when the triangle is equilateral. Jacques (Jakob) Bernoulli solved it for an isosceles triangle (*see his collected works, volume 1, page 303, Geneva,* 1744); but his solution is much less simple than what the special case involves. […]

2. […] The editors of the *Annales* had thus arrived at this point, and they did not think that this formula was likely of much reduction, when they received the following letter from Mr. BIDONE, professor at the Academy of Turin: *Turin March 12, 1811,* […] *I take the liberty, Sirs, to announce that this problem has been solved by Mr. MALFATTI, a*

very distinguished Italian geometer. His solution is printed in the first part of volume X of Memoirs of the Italian Society of Sciences, published in 1803. […]

3. A plane triangle (acute, right or obtuse) contains three circles of different radii that touch each other. We want to fix the centres of the three circles. What is the method? All students in *Tongwen Guan* retreated from trying this problem. Whoever can solve the problem should send the diagram [of the solution] to the School of Astronomy and Mathematics and would be rewarded with a copy of *Jihe Yuanben* [Chinese translation of Euclid's *Elements*]. The diagram [of the solution] would be published in this magazine so that the author would gain universal fame.

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Workshop

A MATHEMATICAL WALK IN “MUSEUM BOERHAAVE”

Harm Jan Smid

Delft University of Technology

The Dutch National Museum for the History of Science and Medicine (Museum Boerhaave in Leiden) is the home of a large collection of instruments and objects from the history of science and medicine, the oldest dating from the 16th century. It also owns a collection of mathematical instruments and devices, such as an enormous quadrant from 1610, made by Willem Jansz. Blaeu. Museum Boerhaave wants to attract a broad range of visitors and organizes special exhibitions and programs for children of different ages and for school classes. Some years ago it started a project around its mathematical objects, wanting to attract more attention for these objects and to interest children for the history of mathematics and to show them the relations with cultural history and school mathematics.

This resulted in a mathematical walk in the museum. Participants receive a 20-pages tour guide on A3-format which guides them along the different mathematical objects of the permanent exhibition. The tour guide contains not only explanations about the objects, but also interesting facts and stories about the history connected with these objects, assignments that can be made with these objects, maps, plotting paper to work on etc. Since the children are obviously not supposed to work with the original objects, they are for the tour equipped with a suitcase containing (simplified) replica of the original objects, which can be used to do the assignments.

The mathematical walk is intended for schoolchildren around the age of fifteen in the middle classes of secondary school, an age in which it is often difficult to interest them for mathematics. It is hoped that by showing them that mathematics can also be fun and has interesting history gives them a more positive attitude towards mathematics. So far the mathematical walk seems to be a success; the museum is now preparing a mathematical walk for primary school children and a centre where school classes can work on (historically based) problems, including hands on activities.

In the presentation we will give you a short introduction to the Museum Boerhaave and its collection of mathematical objects. We will take you on the mathematical walk through the museum, present the tour guide and the suit case with the replica’s and works on one of the assignments. At the end we will discuss the evaluation on the whole project and the plans and ideas of the museum to develop more educational activities for mathematics around its historical collections.

Oral Presentation

A COURSE TO ADDRESS THE ISSUE OF DIVERSITY

James F. Kiernan

Brooklyn College

Brooklyn College is now offering an upper level CORE course entitled "The Mathematics of Non-Western Civilizations". The course is offered to all students who have completed a number of lower level CORE courses. The original course designed by Jeff Suzuki was based on readings from Katz' Sourcebook (2007). Several instructors in the department have now taught the course.

Since the recent publication of a third of edition Joseph's The Crest of the Peacock (2011), I have successfully used it as the main text in the course. Through the use of this text students become acquainted with concepts such as Eurocentism and Ethnomathematics in addition to learning some elementary mathematics developed in civilizations which have been frequently neglected.

This talk will discuss issues regarding teaching a course on the diversity of mathematical development to a diverse population with diverse abilities. Examples of syllabi, assignments and assessments will be provided. I hope that this presentation will lead to some fruitful discussion of using history of mathematics in your classroom.

Oral Presentation

THE TEUTO-BRAZILIANS OF FRIBURGO – MATHEMATICS TEXTBOOKS AND THE USE OF (NON-METRIC) MEASURE SYSTEMS

Andrea V. Rohrer

Sao Paulo State University

During the 19th century, and until the 1930s, the German immigration in Brazil had brought about eight thousand Germans to live in the state of Sao Paulo. (Kahle, 1937, p. 29; Miranda, 2005; von Simson, 1999, 1997, pp. 63-65)

According to a report from 1852, written by the district administrator of the rural village of Bernkastel (140 km south of Bonn, in Prussia), the production of goods had not increased at the same speed as the growth rate of population. Due to the predictable nourishment difficulties, already by 1846, 633 persons had emigrated either to USA or Brazil (MoZ, 2007).

Meanwhile, a series of intense debates and liberal movements sought to democratize and build a new republic in Brazil, purging all remaining connections to the Portuguese monarchy. Many members of these movements, owners of coffee plantations, had rejected slavery and had started to properly hire laborers, paying them salaries. One of the crucial figures of this liberal movement was senator Nicolau de Campos Vergueiro who, in a partnership with the owner of the Fazenda Sete Quedas, brought, in 1856, 112 German families to work in the fields (Di Francesco, 2007, p. 25; dos Santos Bezerra, 2002, pp. 69ff.; Leite, 2006; Fortes, 2003).

Once the immigrants had evened up their debts with the landowners, they were able to save money and move away; some of them decided to buy land lots in the nearby region and cultivated coffee, potato, beans, etc. (von Simson, 1997) Eight families, among these 112, had settled, between the years 1864 and 1877, in a rural region situated in the southwest of Campinas and north of Indaiatuba. They were immigrants from the German regions of Rheinland-Pfalz and Schleswig-Holstein, and from the canton of Bern in the German Switzerland. In 1879 the district of Friedburg, later called Friburgo, was founded and a school was opened this same year in October (Guebel, 1937, p. 2).

In general, the first German colonies were founded in very remote geographic locations, which had caused a need for opening schools inside the same colony. These schools had been established with German curricula, and were, for a long period of time, the mainstay for the transmission of the German traditions and identity, difficulting the integration process (Schubring, 2003, p. 14).

During this presentation, we would like to present the results of an ethnographic research undertaken in Friburgo, in 2007. In particular, we would like to show that, despite the promulgated laws obligating the teaching in the national language (1920), i.e. Portuguese, and the use of the International System of Units (1872), i.e. the metric system, the school of Friburgo had continued to use mathematics textbooks, edited in German language, which also included conversion tables of non-metric measure systems, until after the 1st World War. As we will see, some non-metric units remain in use, at least orally.

THEME 6:

TOPICS IN THE HISTORY OF MATHEMATICS EDUCATION

Plenary Lecture

NEW APPROACHES AND RESULTS IN THE HISTORY OF TEACHING AND LEARNING MATHEMATICS[1]

Gert Schubring

Instituto de Matemática, Universidade Federal do Rio de Janeiro/ IDM, Universität Bielefeld

Studies on the history of teaching and learning mathematics did not begin in recent times; rather, there were already a number of books and various types of papers published during the 19th century. The work of IMUK since 1908, the forerunner of ICMI, meant a considerable impact for historical investigations. After World War II, pertinent studies were undertaken in ever more countries. Yet, practically all the studies were undertaken within the history of some nation or some culture. They were thus bound to the respective traditions, methodologies and approaches of national educational history.

Meanwhile, the focus has changed to address comparative and international issues in this area of research. At stake is since then to unravel what are general features in the national/cultural developments and what are specific issues and what is the significance of such particular patterns. As particularly revealing have proved three issues of comparative international research:

- *the processes leading to the decisive change of mathematics from a marginal teaching subject to a major discipline, first in secondary schooling;*
- *and, related to these developments, the emergence of Mathematics for All as a program and as a major shift in socio-politics of education;*
- *the role of mathematics in the modernization of various states, in particular during the 19th century, and thus showing the social relevance of mathematics.*

The lecture presents methodological reflections, illustrative historical examples and research perspectives.

I. AN OVERVIEW ON THE DEVELOPMENT OF THE RESEARCH AREA

Research into the history of teaching and learning mathematics does not constitute an entirely new field. In fact, such research can look back to a considerable tradition. In fact, already during the 19th century, numerous pertinent studies have been published; they show the broad area opened by this research field. Best known so far are publications in Germany. One of their foci was the history of mathematics teaching at particular schools. The first such study – to my knowledge - dates of 1843 and assessed the evolution of the mathematics curriculum since the Enlightenment reforms to the Prussian Gymnasium reforms, for the Gymnasium in Arnsberg, a town in Prussian Westphalia (Fisch 1843). It systematized the periods in which mathematics teaching became steadily reinforced since the 1770s.

Another focus was on teaching methods; in 1888, Jänicke published the even today valuable study on methods to teach arithmetic (Jänicke 1888). Maybe the first monograph devoted to this area was the book, of 1887, by Siegmund Günther on the history of mathematics teaching in Germany during the Middle Ages (Günther 1887). And there were even first doctoral dissertations, one on teaching arithmetic (Stoy 1876) and one on mathematics teaching in the German state Saxony (Starke 1897).

Studies in the 19th century did not remain restricted to Germany; another study in book format was by Florian Cajori describing the history of mathematics education in the United States up to the end of the 19th century (Cajori 1890). A book by Christensen on the history of mathematics in Denmark and Norway in the 18th century dealt also with the history of mathematics teaching in these countries (Christensen 1895). In an analogous manner, a book on the history of mathematics in Finland until about 1800 studies mathematics teaching, too (Dahlin 1897).

From the beginning of the 20th century, an intense activity of publishing research studies of considerable scope and ambition reached a first peak – until World War I. Firstly remarkable is that the first doctoral theses defended in the USA in mathematics education were studies on the history of mathematics teaching (Jackson 1906; Stamper 1909). Secondly, there are the embracing studies by Germans, probably nurtured by the historicism mentality dominating there, on various key issues of the history of mathematics teaching:

- the book by Grosse on arithmetic textbooks since the 16th century inaugurated the sub-area of research on schoolbooks (Grosse 1901)
- the book by Pahl on the history of teaching mathematics and the sciences in Germany (Pahl 1913).
- even more ambitious was the approach by Timerding who gave a survey of the history of mathematics teaching, from Egypt and the Greeks to the early 20th century, with special emphasis on the teaching of mathematics in Germany during the 19th century (Timerding 1914)

This new dynamic was also partially due to the initiatives by Felix Klein to reform mathematics teaching, promoted by him as president of IMUK (*Internationale Mathematische Unterrichts-Kommission*), founded in 1908. In the series of German reports for IMUK on the state of mathematics teaching in Germany, there were several pertinent monographs: the study by Schimmack on the evolution of the reform movement (Schimmack 1911), the study by Lorey on the training of mathematics teachers (Lorey 1911) and his study on the mathematics taught at 19th century universities (Lorey 1916). Moreover, the numerous reports on the actual state of mathematics teaching in the various German states constitute today excellent sources for research on this period, in particular the well documented books by Lietzmann on the teaching of geometry and on the teaching of arithmetic (Lietzmann 1912a, 1912b). Examples from other countries are Watson on England (Watson 1909) and Heegard on Denmark (Heegard 1912).

While relatively few researches were published in the Inter-War period, studies intensified after World War II and covered gradually more countries. A first book is of 1945, on geometry teaching in Finland (Nykänen 1945). Jushkevich published in 1947 and 1948 a series of papers on mathematics teaching in Russia, from the 17th to the 19th century (Jushkevich 1947-1948). A book by Prudnikov on Russian mathematics educators in the 18th and 19th centuries followed in 1956 (Prudnikov 1956).

An important impact had two volumes published in 1970 by the NCTM, the mathematics teachers association of the United States, thus giving institutional promotion to this field of study: The first volume was a reader with selections from major documents spanning the period 1831-1959 in the USA (Bidwell and Clason 1970), and the other was the NCTM Yearbook for 1970, edited by Jones and Coxford, with research studies on primary and secondary education in the USA and Canada (Jones and Coxford 1970).

From the 1980s, one remarks a rather continuous flow of publications, regarding ever more countries. Research in the history of mathematics education became now a rapidly developing area. Various trends are now visible:

- There are, on the one hand, more specialized studies for a given country; let me mention Howson's book on mathematics education in England (Howson 1982).
- There are, on the other hand, new attempts to an international history; there is the study by Schubring (1984) who researched the history from Antiquity and of various civilizations according to theoretical categories until Modern Times; Miorim's book (1998) presents the development from Antiquity; for Modern Times, it focuses on Brazil.
- The third and new trend is constituted by methodologically reflected approaches to go beyond the surface of administrative facts and decisions, with the objective to unravel the reality of teaching in school practice. These approaches rely on extensive archival research and on interdisciplinary methodology. I will comment more on these developments of methodology in the next part. A first such study is the book by Schubring (1983[1], 1991[2]), which analysed the reality of the emerging profession of mathematics teachers in Prussia. A following study was done by Siegbert Schmidt who analysed the reality of teacher training for primary schools in a specific region of Prussia (Schmidt 1991). The approach became then applied to the Netherlands where mathematics turned during the 19th century from an unwanted intruder in classical secondary schools to a major discipline in a new school type (Smid 1997).

Particularly noteworthy for this new period is the monumental work: "A History of School Mathematics", presenting studies on North America (Stanic & Kilpatrick 2003).

So far, all these activities were mainly individual initiatives. This state changed decisively with ICME 10, held in Copenhagen in 2004, when the field became internationally institutionalized the first time, as the Topic Study Group 29, on the History of Teaching and Learning Mathematics. In its preparation, a first international bibliography of relevant publications became elaborated, thanks to the cooperation of researchers from many countries. Since 2004, this international structure of a TSG was maintained and continued at each ICME. Moreover, in 2006 was founded the *International Journal for the History of Mathematics Education*, the first journal dedicated to this field of research. The journal is now in its ninth volume.

Without exaggerating one can say that a decisive climax was achieved now in 2014 when was published the first comprehensive *Handbook on the History of Mathematics Education*: it covers a wide spectrum of epochs and civilizations, countries and cultures – in 38 chapters/sections and with 40 authors. It makes not only accessible research published in the language and for readers of an individual country, but the handbook succeeded also in launching research on yet scarcely investigated regions and epochs (Karp & Schubring, eds., 2014). Clearly, the process of elaboration also proved that there are still a lot of open questions.

II. METHODOLOGICAL CHALLENGES

Research into the history of teaching and learning mathematics is confronted with a number of methodological challenges, which traditionally researchers have not been well aware of. Largely, research has had a descriptive character, focussed on data on the surface of historical phenomena and processes.

Modern research in history of education – and in particular history of school - has established, however, new patterns of methodology: it became clear that schools function as a subsystem of the respective society and that it is therefore largely sociological methods, which need to be applied. Thus, sociology of education emerged, which established methodological standards for investigating historical processes in schools. This applied to history of education in general, but research on certain school disciplines use to restrict on sociological methodology only: this is the case mainly for disciplines to which socializing functions are ascribed to, thus mother language, history and religion and where historical research therefore largely makes abstraction of the contents taught. But school disciplines like mathematics need a broader methodological approach, a more interdisciplinary one, clearly conceived of on the basis of social history – but capable to be specific for analysing the development of teaching contents.

The low emphasis on methodology may be caused by what proves to be an illusion: the idea that research into the history of mathematics instruction presents an easy task, that this history is just a collection of facts which are observable without difficulties, and that one only needs to ‘collect’ these facts. This is in particular the view of the history of mathematics instruction as a series of administrative decisions that supposedly were transformed into practice. According to this perspective, the history basically is a history of the curriculum, of the syllabus, managed by centralist authorities. But even when the broad spectrum of historical issues is reduced to the syllabuses, the real problem is whether, and how, centralized decisions were implemented in school practice, and this opens up again the immense range of dimensions relevant to the historical development.

In fact, the history of the teaching and learning of mathematics constitutes an interdisciplinary field of study; the principal disciplines concerned are the history of mathematics and the history of education, but the general history contributes as well. Moreover, sociology is quite essential, in particular sociology of religion.

Realizing the complexity of our field of study, we might even say that it requires an even more complex methodology than the history of mathematics. Clearly, mathematics history is a part of cultural, political and social history, too, but the contents of mathematics and the evolution of its concepts occupy a far more extended domain within mathematics history than within the history of mathematics instruction. Compared with this rather dominant role of mathematical ideas and concepts, the history of teaching and learning mathematics constitutes a social reality within educational systems that needs incomparably more social categories to reveal its dimensions.

Even the entity corresponding to the structured set of mathematical concepts, namely 'school mathematics', is far from being just a derivation or a projection of the 'savoir savant' as Yves Chevallard pretended (Chevallard 1985) – well to the contrary, school mathematics develops as a product of numerous interactions, and even pressures, from and between various sectors of society. But what complicates the research in our field even more is the fact that mathematics never appears in educational systems in an independent way but always functions within structures, which are characterized by a compound of *several* school disciplines. This means that mathematics teaching and learning is always dependent on other factors that it is barely capable of influencing.

Yet, despite this fundamental and structural dependence on a concert of disciplines that in general exhibit no peaceful coexistence, the perhaps most considerable deficiency of the large majority of studies in our field is that they treat mathematics as an *isolated* teaching subject, without regarding relationships, dependencies and hierarchies in the *system* defining school learning.

It should be evident by now that marked progress in research necessitates methodological reflection and refinement. A decisive resort in doing so is presented by comparative issues – not only comparative studies of the history of various school disciplines within a given educational system, but even more importantly comparative studies on the history of mathematics instruction in different states and different cultures. It is quite natural that most research pursued or ongoing is concentrated on the history within a given nation or a given culture as the history of mathematics teaching and learning first and foremost constitutes part of the educational history of that country or culture. But in order not to end up with a collection of separate, isolated histories without interconnections, one has to establish relations between the different national histories and to reveal what is 'general' in them and what constitutes, say, cultural, or social, or political peculiarities of a specific country or culture. Practically all questions in our historical field deserve comparative studies.

The approaches and results from comparative education hence provide an essential tool for an international history of teaching and learning mathematics, in order to grasp national specificities as well as overall and global trends. Of particular methodological importance are qualitative methods, which are also applicable to the study of (historical) documents. Given the primary importance of cultural history, anthropology provides relevant methodological resources as well.

To resume: History of teaching and learning mathematics is an interdisciplinary field, interacting in particular with:

- Sociology of education
- History of education
- History of mathematics, and
- Epistemology: given the importance of the views on mathematics, which use to be specific for cultures and which determine decisively the conceptions of what should be school mathematics

III. KEY ISSUES AND NEW RESULTS

1. Origins of teaching mathematics

In popular literature, a rather idyllic picture is designed of the origins of mathematics: shepherds counting their sheep by means of producing some listings, or farmers counting the cattle. Much folklore has been designed to visualize how early mankind might have counted with fingers or even have registered results. What is reliable, however, is revealed by the famous report of Herodot on the origin of geometry in Egypt – also, there, not as the likewise idyllic history of natural phenomena by which probably unorganized settlers are confronted. Reading Herodot's report carefully, one understands that it deals with the functioning of a professional group – land surveyors – in the service of administration of a state. In fact, extensive research since the 1980s – and thanks to the use of computers then the first time applicable for evaluating an enormous number of documents – has unravelled the origin of mathematics in the somewhat parallel cultures of Mesopotamia. It is due to the innovative research methods of the team Nissen, Englund and Damerow – a historian, an archeologist and a mathematician – that the emergence of sign systems became proven as instigated by the needs of a centralized state administration, from the 4th millennium BCE on. What they were able to show was how a highly differentiated system of signs for object-bound quantities developed into a standardization yielding eventually a system of numbers, even a positional system, the sexagesimal system (Nissen et al. 1993).

More recent research has revealed the origins of mathematics teaching as intimately tied to the emergence of sign systems for the same state administration. And it is highly revealing that in these origins writing and calculating constituted a unity – to the contrary of the later divergence between humanities and sciences. Researchers on the history of writing – a well known specialist is Denise Schmandt-Besserat (see

Schmandt-Besserat 1996) – and researchers on the history of mathematics agree that number and scripture originated together, in the same socio-cultural setting. Eleanor Robson, researcher on Mesopotamian mathematics, formulated the consensus of both sides recently:

> "The temple administrators of Uruk adapted token accounting to their increasingly complex needs by developing the means to record not only quantities but the objects of account as well. Thus numeracy became literate for the first time in world history" (Robson 2008, p. 28).

Writing and calculating was, thus, taught in an intertwined manner:

> "As the production of accounts entailed complex multi-base calculations, trainee scribes had to practice both writing and calculating, and they did so increasingly systematically" (Robson 2008, p. 40).

Archaeological research which had earlier on not given much attention to tokens and calculating tablets has now systematically searched for mathematical tablets and their locations; it was able to even identify buildings which had served as *edubba*, hence as schools for teaching writing and calculating, thus training scribes for the state administration (Robson 2008, p. 98). Christine Proust, in her contribution on Mesopotamia in the Handbook, was able to even reconstruct the structure of the mathematics curriculum in the *edubba*:

Level	Content	Typology	Examples
Elementary	Metrological lists (capacities, weight, surfaces, length) Metrological tables Numerical tables (reciprocals, multiplication, squares) Square and cubic roots	Types I, II and III	See Figures 3-4
Intermediate	Exercises : performing multiplications and reciprocals. Surface and volume calculations	Square-shaped tablets	See Figure 5

Figure, Proust 2014, p. 33 (in Bernard et al. 2014)

Thanks to recent research, one even knows the name of the eldest so far known person practicing mathematics: On a tablet from the palace archives, at the Syrian city of Ebla, dating from about 2350 BCE, one finds at the end: "Nammah wrote the calculation" (Robson 2008, p. 32).

A new study on China confirms this decisive function of state administration for promoting mathematics and for institutionalizing its teaching. Alexej Volkov has presented, in his chapter on China in the Handbook, in particular his research on the "School of Computation". The first such state-run School of Computations was established during the Sui dynasty (581–618) that unified China after a long period of disunity. It is known that the instruction was conducted by one or two "erudites" (*boshi*) and two “teaching assistants” (*zhujiao*) and that the number of students totalled 80. Its full development occurred from the Tang dynasty (618-907). The earliest mention of the school is dated 628; in this year, instructors were hired and students admitted. The age of the students entering the School of Computations ranged from 13 to 18. No information is available about the teaching materials used at that stage, yet one can conjecture which were the textbooks (Volkov 2014, p. 59).

After some interruption, the school opened again in 656, under the supervision of the governmental agency named “Directorate [of Education of] Sons of State” (*Guo zi jian*) – thus, a kind of ministry of education. It was from this year on that the School experienced an important functioning, in the preparation for the exams of admission to the various branches of state administration. In particular, it was in this year that the famous list of mathematical textbooks was established, which served for learning and preparing the exams at this school. This list is known in the literature as the “Ten Classics”, but as Volkov has shown, it were in fact 12 textbooks. One of them is the famous *Jiu Zhang Suan Shu* – the Nine Chapters of Mathematical Procedures, which dates back to about 300 BC and is thus a real rival to Euclid’s Elements. According to the *Tang liu dian* (Six Codes of the Tang [dynasty]) and to the *Jiu Tang shu*, the students of the school were subdivided into two groups each comprising 15 people and instructed by two “erudites” and one “teaching assistant”. The students of the first group studied treatises [1–8], and those of the second one studied treatises [9-10]. These treatises are referred to as “regular program” and “advanced program,” respectively. The study in each program usually lasted 7 years but in exceptional cases could be extended to 9 years. Treatises [11–12] were studied simultaneously with the other treatises in both programs; the time necessary for their study was not specified. The 12 textbooks of the curriculum and the extant mathematical treatises with which they are conventionally identified are shown in Figure 2. Not much is known about the procedures of instruction in the School of Computations; the only element mentioned in the extant sources is “oral explanations” provided by the instructors. There were two kinds of examinations: (1) the quizzes conducted every 10 days and (2) the examination conducted at the end of each year. A lot of details of the examinations are known (Volkov 2014, p. 63).

Table 4.1 Mathematical curriculum of the Tang School of Computations

#	Title	Duration of study	Program[a]
1	*Sunzi* 孫子 (*[Treatise of] Master Sun*)	1 year for two treatises together	Regular
2	*Wu cao* 五曹 (*Five Departments*)		Regular
3	*Jiu zhang* 九章 (*Nine Categories*)	3 years for twotreatises together	Regular
4	*Hai dao* 海島 (*Sea Island*)		Regular
5	*Zhang Qiujian* 張丘建 (*[Treatise of] Zhang Qiujian*)	1 year	Regular
6	*Xiahou Yang* 夏侯陽 (*[Treatise of] Xiahou Yang*)	1 year	Regular
7	*Zhou bi* 周髀 (*Gnomon of the Zhou [Dynasty]*)	1 year for two treatises together	Regular
8	*Wu jing suan* 五經算 (*Computations in the Five Classical Books*)		Regular
9	*Zhui shu* 綴術 (*Procedures of Mending [=Interpolation?]*)[b]	4 years	Advanced
10	*Qi gu* 緝古 (*Continuation [of Traditions] of Ancient [Authors]*)	3 years	Advanced
11	*Ji yi* 記遺 (*Records Left Behind for Posterity*)	Not specified	Compulsory
12	*San deng shu* 三等數 (*Numbers of Three Ranks*)	Not specified	Compulsory

Figure 2: The Twelve Classics (Volkov 2014, p. 61)

2. The way to "Mathematics for All"

As one understands from the origins of mathematics teaching, it became organized by a state for its proper needs of administration and government; the origins and the first developments are therefore due to the needs of *professional* training. Mathematics as element of liberal education became instituted, in contrast, much later and not by state intervention but by initiative of higher social classes, as the first historical example - Greek city states - shows.

One uses not to be aware of a revealing contrast: while mathematics as subject of professional training functioned as a major teaching subject, it used to be taught as a marginal or auxiliary subject within liberal education.

There are therefore two pertinent research questions:

- which dynamics, which movements, which forces effected the change from a minor to a major teaching subject for mathematics within liberal education?
- What effected that the state, hitherto restricted in its actions for education to the needs of professional experts in its service, turned to act for instituting as a major subject of liberal education?

Actually, I have done a lot of research on these questions, in international comparative studies, and I am continuing this research. Basically, it entails the question: how emerged the program "Mathematics for All"? More about it will be discussed in the workshop,[2] and I am restricting here myself to two novel results:

One result concerns the conceptual framework for introducing and realizing the new function for mathematics teaching and learning and its contextualization. Traditionally, the focus in historiography for this question has been on Prussia and its neo-humanist reforms of education from 1810 on. Since Prussia was basically a Protestant country, one is lead to relate this reform conception with Max Weber's famous thesis of Protestant Ethic as the source for the rise of capitalism. Combining Weber's thesis with the Merton thesis according to which it was the Protestant context, which constituted the fundaments for the Scientific Revolution, one would think it to be quite natural that the change of the function of mathematics in secondary schools is due to Protestantism.

At a first glance, one might be confirmed that it was not due to Catholicism. In fact, Christopher Clavius (1537-1612), chief mathematician of the Jesuit order, had proposed an ambitious program, based on conceptions of the Humanism movement, for teaching mathematics in the system of Jesuit colleges becoming established in the Catholic countries in Europe and the Americas. But in the debate, on-going during the sixteenth century, about the certitude provided by mathematics, the Jesuit philosophers claimed that "mathematics does not reach the highest level of certitude, so it is not a science strictly speaking" (Paradinas 2013, p. 167). Thus, in the end, nothing became realized of Clavius's program in the *Ratio Studiorum* of 1599, and mathematics experienced a marginal role in Jesuit teaching (see Paradinas 2013).[1]

On the other hand, the change was not due to Protestantism, neither. Philipp Melanchthon (1497-1560), the principal constructor of a Protestant education system, argued intensely for teaching mathematics at the *Gymnasien* and universities. While mathematics developed there firmly at the universities, continuing what had been initiated during Humanism, mathematics at the *Gymnasien*, however, did for a long time not succeed to overcome a likewise marginal function (see Schubring 2014).

Upon closer scrutiny, one will find, however, that it was one Catholic country where mathematics first achieved this new status. Yet, it was a quite specific Catholic country and not, let us say, a typical Catholic country: it was France. In fact, France was practicing a policy of striving for a Gallican Church – i.e. against an ultramontane obedience to the pope. The Jesuit order had been admitted to France from 1604 only in a somewhat nationalized form; it was Jansenism – an intra-Catholic reform movement in the 17th century –, which urged for a Gallican policy; consequently, it became prosecuted by the Jesuits. A key exponent of Jansenism was the philosopher and theologian Antoine Arnauld (1612-1694). In his seminal textbook *Nouveaux élémens de géométrie* (1667) he was the first to develop a theological argumentation for a primacy of mathematics in education. Jansenism thus turned out to constitute an important source for the Enlightenment movement.[2] Enlightenment in France became closely related with Rationalism, featuring thus a key importance of mathematics in general culture. It was in this context that the state created, from the middle of the 18th century, a net of military schools where mathematics constituted the leading discipline for the formation of military engineers and of officers.

While this functioning during the *Ancien Régime* still occurred within the traditional paradigm, namely the formation of military and technical experts for the needs of the state, the French Revolution effected a fundamental change: the state now assumed an overall responsibility for a public educational system and instituted in this vein mathematics as a major teaching discipline. Most characteristic is the first systematic organization of secondary education in 1802: in this first law (10 December 1802) on organizing a public education system, Latin and mathematics were declared as the two key disciplines.

> "On enseignera essentiellement dans les lycées le latin et les mathématiques" (quoted from Schubring 1984, p. 371).

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[1] The eulogies of Antonella Romano, in her book *La contre-réforme mathématique* (1999), for Clavius's program simply ignore the failure in becoming it accepted by the Jesuit order (see Schubring 2003).

[2] My research on this new kind of establishing a fundamental role of mathematics in education, by Arnauld, and on the role of Jansenism in disseminating this conception is forthcoming in the paper: "From the Few to the Many: On the Emergence of *Mathematics for All*".

3. Bipolarity of mathematics

The *second* result concerns the epistemology of school mathematics. Over extended periods, mathematics had to fight against strong resistance to achieve or to maintain acceptance in secondary schools as a legitimate teaching subject. The fight was with representatives of philology, of teaching classical languages, who claimed superior educational value for these languages. A major resource of mathematics teachers in these fights was to refer to Antiquity, to the inscription above the entrance to Plato's Academy:

ἀγεωμέτρητος μηδεὶς εἰσίτω

Although one doubts today whether such an inscription really existed there, fortunately no philologist then dared to doubt and thus it constituted an argument endowed with a certain power.

One used to think that modern secondary schools – "Realschulen" –, emerging since the 18th century and spreading rapidly during the 19th century, constituted a major backing for maintaining or achieving a strong position for mathematics in teaching. Closer scrutiny shows, however, that the propagators of "modernism", of realist oriented education, had no holistic understanding of education: they started from a principal refusal of classical oriented secondary schools, hence of traditional humanism, and strove to construct a likewise one-sided educational conception, oriented towards utility. And for legitimating this conception, mathematics was used as key argument for claiming utility as goal for the secondary schools.

Some mathematics teachers tried to avoid this embracement by agitators for *Realschulen*; for instance, Carl Friedrich Andreas Jacobi (1795-1855), the teacher who achieved to firmly establish mathematics at the *Landesschule Pforte*, a traditionally extremely humanist Saxon Gymnasium, after it had become Prussian, declared:

> "Mathematics is no modern means of education but a classical one" (Schubring 1985, p. 25).[3]

One has to consider, regarding the epistemology of school mathematics, that mathematics has a *bipolar* character: in view of its logical and foundational abilities, it belongs to the humanities; on the other hand, as correctly expressed by the term 'polytechnic', it enables enormous means of applications. It is hence remarkable that in the original neo-humanist curriculum for Prussia of 1810, the Tralles-Süvern-Plan, applications of mathematics constituted integral elements (Schubring 1991, p. 209).

It is this double-faced nature of mathematics, which constitutes the conceptual challenge for historical analyses: to be aware of this epistemological special character and to use it as a conceptual framework for concrete historical investigations.

4. Function of Mathematics for Modernising society

Usually, the focus for research on the history of mathematics teaching is the major European countries: Germany, France, Italy, England. It is mainly for these countries that one investigates changes and rises in status of mathematics teaching. As a matter of fact, the structures of public education systems became established there – basically during the 19th century – and from there the structure was transmitted or imposed in one or the other variant to cultures and states in other continents. There has been recent research on how this transmission or interaction has occurred, in particular in papers published in the special Issue of the journal ZDM, dedicated to: *Turning Points in the History of Mathematics Teaching – Studies on national Policies* (vol. 44, no. 4, 2012).

As it became clear from various case studies in this special issue, establishing mathematics teaching proves to be an intentional act to modernize the respective society, to contribute to meet the demands with which the state in question is confronted. In fact, it is by actions of the respective Empire or national state, not by some individual's good will or plans of a definite social group, that mathematics becomes ascribed a function within the intended reform process. And in none of these cases, mathematics teaching became imposed from outside – it was upon proper and internally decided strategy to call for this part of knowledge.

Most telling for such ascribed modernizing functions are states suffering profound crises of their traditional modes of existing. It that ZDM issue, three non-European Empires were investigated that reveal the key functions ascribed to mathematics for re-founding the basis of state and society: China, Japan and the Ottoman Empire, all presenting Empires, which used to be regarded as unchallenged powers, but - upon being confronted with Western values or even invasions – their traditional means proved to no longer being able to maintain their status.

The case of China

As a consequence of the Second Opium War of Western powers against China and the devastating defeats (18-18), a faction rose in the government who argued that one could not continue policy in the traditional way as leading immediately into total collapse and that inner reforms were needed. As a first measure, they succeeded in getting founded a *School of Combined Learning*, at first intended as a language school to train interpreters and to thus be able to negotiate with the foreign invaders. It then developed into a college of Western learning, together with other similar colleges in more cities in China. Soon after, the new Foreign Office, also an achievement and basis of the reform faction in the government, proposed the creation of a *School of Astronomy and Mathematics*, thus adopting a reform conception proposed by wider circles in the cultural elite, calling for "self-strengthening" (Chan & Siu 2012, pp.). The opposing conservative faction tried to impede the foundation of this school, in particular since foreigners should be applied there as teachers. It argued that there was no need for teaching mathematics and that one can find experts in China for the necessary technical tasks. When the chief opponent was asked to name such domestic experts, he had to admit he knew no such one. The reformers in the government formulated the basic conviction for reforming China and being able to resist, by adopting Western science and education:

"All Western knowledge is derived from mathematics. Every Westerner of ten years of age or more studies mathematics. If we now wish to adopt Western knowledge, naturally we cannot but learn mathematics" (quoted from Chan & Siu 2012, p. 464).

The case of the Ottoman Empire

The Ottoman Empire, traditionally an expansive power, suffered but defeats during the second half of the 18th century on the Balkans. Reform-minded sultans established engineering and military schools privileging mathematics teaching, from the 1770s. Likewise similar reforms were undertaken by the partially dependent, partially independent governments of Tunisia and Egypt. In Egypt and in Tunisia, after also founding military or engineering schools, schools for general education were established, with a strong function for mathematics. The ruler of Egypt, Muhammed Ali, even sent a number of students to France to study mathematics at the *École polytechnique*. He organized to translate modern mathematics textbooks. And it is characteristic that the there still strong traditional social forces directed their destructions against the new schools, as it happened in the Ottoman Empire in 180x; in Egypt, Muhammed Ali was not strong enough to abolish the traditional system of schools run by the *ulema* and had to face the existence of two parallel systems - the modern state-run system, administered by a Ministry of Education (from 1837) and the traditional schools controlled by the *ulema* (Abdeljaouad 2012).

The case of Japan

In such situations of crisis, the state and its government concentrate on its most urgent needs for maintaining its existence and strength: on engineering and on warfare capacities; and it is thus revealing that the first functions of mathematics for the state are its rationalizing abilities: for technical and military applications. Yet, one remarks that - in a second step – the state becomes the agent for teaching mathematics as a key element of general education. Japan presents a likewise telling example for such two steps: Even before the opening of Japan to the West, by the 1867 Meji fundamental change of policy, the central government had founded a Naval Academy at Nagasaki, in 1855, "where Western mathematics was taught because of its military applications" (quoted from Ueno 2012, p. 476). As part of its profound reforms of society from 1867 on, in particular of abolishing the feudal clan system and their segregated school structures, the new government established a centralized educational system for all, with mathematics as a basic constituent. One remarks here, too, the proper intention of the state to assume responsibility for education by a public education system: practically the first measure was the creation of a Ministry of Education (Matsubara & Kusumoto 1986; Ueno 2012, pp. 477).

5. Connections between cultural values in a society and dominant epistemology of mathematics

The case of teaching mathematics in Italy during the 19th century, and in particular from the national unity after 1860, used to raise bewilderment among non-Italians, while Italian historians used to praise the mathematician Luigi Cremona, the main responsible for the curricular and organizational decisions of 1867:

- Euclid's Elements were introduced as textbook – and not in an adapted version like in England, but in the (translated) original;
- from 1878 on, the status of mathematics teaching in secondary schools weakened ever more – as only such case in a European country.

Research over the last 20 years has clarified already some points to understand this specific development, due to an epistemology extolling pure Greek geometry and aligning this to the dominant classicist Italian culture, but only recently a key structure was revealed by Italian researchers for the international public. While one had always spoken of the *liceo* – and that was understood as *the* secondary school in Italy – it became now clear that secondary school in Italy was divided in two sections – the *ginnasio* and the *liceo* – and that in the *ginnasio* mathematics was a minor subject while it should have been a major one in the following *liceo*. Even so, the meaning of the division is still not clear since – contrary to other countries where such a division means that the first structure is destined also for students not intending to continue to university, thus covering also teaching for non-academic professions – the *ginnasio-liceo* in Italy was conceived of to prepare only for university studies. A look on the timetables for the teaching hours for mathematics already provides first insights and then questions:

School type/grade	1860	1862	1865
Ginnasio I	1		1
Ginnasio II	1		1
Ginnasio III	1		2
Ginnasio IV	3		2
Ginnasio V	3		2
Liceo I	8	6	6
Liceo II	0	2	3
Liceo III	3	3	3

Figure 3: number of weekly teaching hours for mathematics in unified Italy
(no change in the *ginnasio* in 1862)

We look primarily at the time tables for *ginnasio* and for *liceo* from 1860, the beginning of political unity of Italy, until shortly before 1867 and remark at first that in few years the time-tables were changed several times. Actually, this presents a characteristic of the Italian school policy – there was quite few stability in this system. Secondly, we remark a quite low rank of mathematics in the *liceo*. But let us now look at the time-table of 1867, elaborated by a committee of mathematicians, presided by Luigi Cremona:

Gin I	Gin II	Gin III	Gin IV	Gin V	Liceo I	Liceo II	Liceo III
0	0	0	0	5	6	7,5	0

Figure 4: weekly hours for mathematics teaching 1867 (Giacardi & Scoth 2014, p. 211)

This meant a self-decided exclusion of mathematics from the first four years of the secondary school and a concentration in the upper middle part – even in the last grade, important for the final exam, no mathematics should be taught. Looking now at the list of contents ascribed to the teaching in these few grades, one already begins to understand the conception of school mathematics and its epistemology:

G I	G II	G III	G IV	GIN V	LIC I	LIC II	L III
-	-	-	-	Arithm + geom: Euclid 1	Aritm + Alg + Geom: Euclid 2, 3	Algebra, Trigon., Geometry: Euclid 4, 5, 6, 11, 12 and circle, cylinder, cone, sphere accord to Archimedes	-

Figure 5: contents of mathematics teaching in the programme 1867 (ibid., p. 212)

The conception of Cremona had been that only good mathematics can constitute meaningful school mathematics and good mathematics for him was rigorous mathematics, organized deductively. Moreover, the best such mathematics was for him Euclid's geometry. Thus, Euclid's Elements were prescribed as textbook, and since students of the lower grades were not thought of to be mature enough for such a geometry, the teaching of mathematics should begin at only a late stage. In that first grade with mathematics, the last grade of the *ginnasio*, there should be taught also a bit of arithmetic: but only as "rational arithmetic" - a very Italian concept for school mathematics: also arithmetic should be taught in an axiomatic and deductive manner. And remark the overloading in the second grade of the *liceo*: too many weekly hours, too many mathematical topics to be taught!

Cremona believed that the role of the *ginnasio-liceo* was not to give students a mass of knowledge, but rather to provide a method for dealing effectively with problems. In particular, for geometry he suggested following the Euclidean method because 'this is the most appropriate for creating in young minds the habit of inflexible rigour in reasoning'; he exhorted teachers not to contaminate 'the purity of ancient geometry, transforming geometric theorems into algebraic formulas, that is, substituting concrete magnitudes ... for their measurements' (Giacardi & Scoth 2014, p. 212). Likewise, arithmetic was to be taught using the deductive and demonstrative method.

Admittedly, such a mathematical fundamentalism, without any idea for what is achievable in school teaching, could not be maintained for a long time. Soon, the Ministry felt obliged to attenuate somewhat this teaching conception. Also other textbooks than Euclid became allowed, and mathematics was extended to all grades of the *ginnasio*. However, being in general restricted to two weekly hours in all the five grades, it was clearly understood as a minor teaching subject. This week role had the harmful effect that the students upon entering the *ginnasio* were not sufficiently prepared to pass the exams. In fact, students used to fail in the final exam, and since these were central common exams for entire Italy, this soon became a public calamity. From 1878, the Ministry tried to avoid it by simply excluding mathematics as a subject of the final exam (Scarpis 1911, p. 8). Although one tried to find various compromises, the basic message for the students (and their parents) was: mathematics is not of equal status to the humanities.

CONCLUSION

The history of mathematics teaching and learning reveals it as a highly pertinent and rich source for interdisciplinary studies on the role of mathematics in society. Already the double-faced nature of mathematics as a pure science and as an applied science allows to study the functioning of school systems and their evolution along differing needs of societies. One remarks breaks in the legitimation of mathematics as a school discipline, which reveal epistemological dimensions, differing according to cultures, but also religious motivations, which refer to sociology of religion. And one remarks breaks in the status of mathematics teaching due to political breaks. Less studied still is the modernizing function of mathematics for societies endangered politically – like in the case of traditional powers but having become stagnant and thus weak in confrontation with expanding Western powers.

Pertinent factors and dimensions proved to become better and more profoundly understandable by a methodology relying on comparative education. Investigating the history in one country or culture, one will thus be able to discern between what is particular and characteristic for that case and what is revealing general patterns – or what is now called 'local' versus 'global' in history of science.

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[1] I am in general using the term "history of teaching and learning Mathematics", since "history of mathematics education" has a double meaning: history of teaching mathematics and history of the scientific discipline mathematics education, in some languages called, e.g., *didactique des mathématiques*.

[2] In the three-hour workshop in the afternoon, the participants worked on various aspects of the historical reality of mathematics teaching: comparing the contrasting formats of the first syllabi for mathematics teaching: in France (1802) and in Prussia (1810) and deducing differences in the school systems; analysing the official commentary for the 1867 syllabus in Italy and discussing its epistemological implications for school mathematics; commenting upon a rather critical remark on Italian use of rigor in teaching the calculus in secondary schools, in a report by Emanuel Beke in 1914; the French conception of 'history of school disciplines' (Chervel) and its role in the methodology for history of mathematics education.

[3] Die Mathematik [ist] kein modernes, sondern ein antikes Bildungsmittel.

Workshop

GEOMETRY, TEACHING AND PUBLISHING IN THE UNITED STATES IN THE 19TH CENTURY: A STUDY OF THE ADAPTATIONS OF LEGENDRE'S GEOMETRY

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University of Nantes

In the late 1810s, Harvard College sought to reform and modernize its curricula which had failed to keep up with the developing needs of scholars and scientists. To achieve this, Professor John Farrar broke with traditional educational practices informed by English and Scottish contents, pedagogical methods and media of diffusion and sought inspiration from French textbooks and pedagogical methods. For the teaching of geometry, he translated Legendre's Éléments de géométrie, an in-between textbook which broke with the inductive methods of Clairaut but also with traditional Euclidian textbooks such as Scottish Elements of Geometry by John Playfair. However, the introduction of these new practices represented a rupture in learning methods. To minimize this change, American mathematicians who adapted Legendre after Farrar adapted it, deeply altered the French textbook and made it more relevant to local uses. These transformations brought about new and original knowledge - a result of the combination of specific pedagogical needs and tendencies of American textbooks publishing.

INTRODUCTION

In the first half of the nineteenth century, the practice and the diffusion of mathematics within the United States were unprecedented in its transformed. The question of the teaching of geometry in colleges strengthened, as it was introduced as a separate college subject around 1790 (Ackerberg-Hastings, 2000, p. 7). In the early years of American Republic, the teaching of geometry leant on practical geometry treatises inspired by English authors and, later, on English-written versions of Euclid's *Elements* – such as *Elements of Geometry* (1795) by Scottish mathematician John Playfair - as shown in (Karpinski, 1940) and in (Cajori, 1890). For many teachers and educationalists, the teaching of geometry had to match what appeared to be two opposite requirements. On the one hand, it had to train learned minds to rhetoric and deductive reasoning, but on the other hand, it had to avoid useless and time-consuming speculations.

In the 1820s, Adrien-Marie Legendre's *Elements of Geometry* (published in France in 1794) seemed to match these expectations, since it was first translated at Harvard by John Farrar (1819) and then subsequently at West Point Military Academy, by Charles Davies (1828). The two translations became widespread and used in several university courses during the first half of the nineteenth century (Preveraud, 2014, p. 217). After 1840, in addition to these two textbooks reprints, three adaptations of *Legendre's*

Geometry were designed in the United States within three different educational contexts: Elias Loomis (1849), for civil higher education; Francis H. Smith (1867), for Virginia Military Institute and James Thomson (1847), for high schools students. As Davies's (especially the 1834 edition), Loomis's and Thomson's textbooks somehow transformed the original French book, this article will refer to the corpus of the "adaptations" of *Legendre's Geometry*. Although more faithful, Farrar's as well as Davies's 1828 translations can also be called adaptations because they both altered the French book.

Fig 1. Front pages of Farrar's (1819), Davies's (1828), Thomson's (1844), Loomis's (1849) and Smith's (1867) adaptations.

This article aims to analyze American adaptations of *Legendre's Geometry*, relying on a systematic comparison of the continuities and the changes in the successive textbooks contents. After having introduced the American editorial background of the teaching of geometry the adaptations were produced within, the study will then question the reason why five adaptations of the same text were published within only four decades. Thus, the article will proceed to the compared and time-changing analysis of the books with the French original, focusing on several characteristics of geometry textbooks: the arithmetization, the use of the *reductio ad absurdum*, the list of axioms, the statement of the propositions and the proofs. These examples will seek to highlight the combined influence of the targeted readership with the American publishing context upon the writing and the adaptation in the translation process.

TEACHING GEOMETRY IN THE EARLY CENTURY (1800-1819)

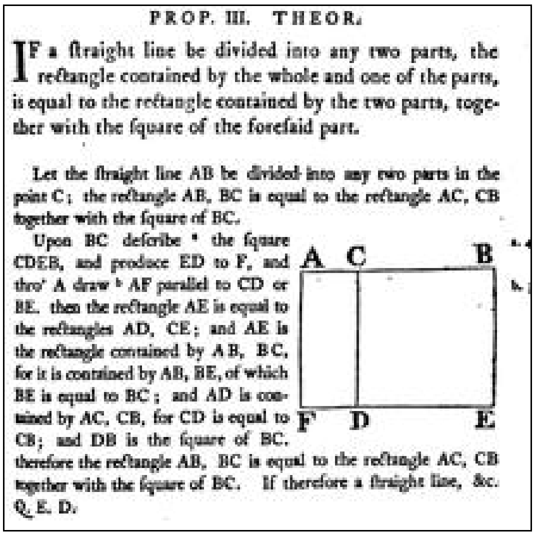
Compendia and Euclidean Scottish textbooks

In the very early years of 19th century America, geometry was taught essentially in colleges [1]. College students were trained mainly with *compendia*, books that covered arithmetic, algebra, surveying, geometry and other subjects related to mathematics. Two famous compendia used in the United States were *Mathematics, Compiled from the Best Authors,* written by Harvard's professor of mathematics Samuel Webber (1759-1810) in 1801, and *A Course of Mathematics*, an English book written by English Charles Hutton for English Military Academy and revised for an American version by Columbia's professor of mathematics Robert Adrain (1775-1843) [2]. The geometry exposed in those compendia was essentially practical. After the definitions, the authors solved problems, giving instructions to complete geometrical constructions [3].

Nevertheless, soon, the compendia in which geometry was too briefly introduced were not appropriate enough to match the requirements of changing curricula in American colleges. Most of them needed geometry not only to perform constructions but, above all, to train students in the art of reasoning more rigorously. American scholars turned to Euclidean geometry, which referred to the Greek book *The Elements.* Two textbooks were predominantly used, as new versions of *Euclid's Elements*, produced in Scotland in the 18th century. *Elements of Euclid*, by Robert Simson, was published in 1756 and offered a restored edition of the previous 16th century Latin versions of Euclid's text. It was almost immediately used in colleges and academies in Scotland and Great Britain because scholars and professors appreciated the logical structure of Euclid, which help students to learn the useful and lifelong skill of reasoning. Throughout the 18th and 19th centuries, Euclidean geometry – taught together with other subjects as Latin, Greek or rhetoric - shouldered the essential role of training up gentlemen at all levels of education from primary schools to university.

Indeed, *The Elements* consisted in an organized arrangement of geometrical propositions, proven through purely deductive reasoning. Each proposition was stated using the definition, the axioms and the previous propositions. In *Euclid's Elements*, most of the solutions of the proposed questions were first laid down and afterwards demonstrated to be true, in order to emphasize the logical process of deductive demonstration. Those so-called "synthetic" demonstrations did not give hints on how the solutions were found, but only on why they were conclusive. For example, to achieve some of his demonstrations, Euclid used *reductio ad absurdum*, a mathematical proof by contradiction, arguing that the denial of an assertion would result in a logical contradiction. *Euclid's Elements* were also known for a specific method of presentation. Each proposition was first stated in the most general way, as the proposition III of Book 2 in *Simson's Geometry*:

> If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the foresaid part (Simson, 1762, p. 45).



PROP. III. THEOR.

IF a ſtraight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the ſquare of the foreſaid part.

Let the ſtraight line AB be divided into any two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB together with the ſquare of BC.

Upon BC deſcribe [a] the ſquare CDEB, and produce ED to F, and thro' A draw [b] AF parallel to CD or BE. then the rectangle AE is equal to the rectangles AD, CE; and AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; and DB is the ſquare of BC. therefore the rectangle AB, BC is equal to the rectangle AC, CB together with the ſquare of BC. If therefore a ſtraight line, &c. Q. E. D.

Fig. 2. Proposition III, Book 2 from *Simson's Elements* (Simson, 1762, p. 45)

This general statement of the proposition, the *protasis*, was then immediately followed by the particular statement of the same proposition related to a particular diagram: "Let the straight line Ab be divided into two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB together with the square of BC" (Fig. 2). Then came the proof and the conclusion. The Euclidean geometry was also a geometry without numbers, and Euclid dealt with magnitudes through proportions.

In 1795, Natural philosopher John Playfair, also from Scotland, revised the labor of Simson in *Elements of Geometry*. He did so because the past editions of Simson's book were deteriorated, especially the illustrations. Moreover, even if he closely followed Simson's structure, he modernized his work, including recent developments of mathematics and appealing to algebra techniques and symbols [4]. Simson's and, later on, particularly Playfair's textbooks, were largely used in American colleges during the 19th century, for the same reason they were used in English colleges. *Playfair's Geometry* was even published in the United States with an American edition in 1806, by Francis Nichols. This American version was printed 39 times between 1806 and 1871.

The success of Legendre in France for the teaching of geometry

At the end of the 18th century, an alternative of *Euclid's Elements* was written in France. *Elements of Geometry* by Adrien-Marie Legendre, first published in 1794, was widely used in French schools throughout the 19th century. The textbook was translated into many languages and used in different countries, such as Italy, Brazil, Greece, Sweden or England (Schubring, 2007).

In France, the publication of *Legendre's Geometry* came after two centuries of criticisms of Euclid. With *Nouveaux éléments de géométrie* (1683) and *Elémens de géométrie* (1741), authors Antoine Arnauld and Alexis Clairaut claimed for a geometry easier to read, in which the propositions were arranged in a more evident order. Designed for the teaching of geometry in 18th century French colleges, Clairaut's textbook emphasized on pedagogical ambitions: "Although geometry is abstract by itself, it is nevertheless admitted that the difficulties [students] have to face come most of the time from the way geometry is taught in ordinary Elements" (Clairaut, 1765, p. i). In Euclidean textbooks, definitions, axioms and propositions, provided a teaching of geometry that Clairaut considered nor meaningful nor interesting for a beginner. As a consequence, the proofs in his textbook were less rigorous, they had to highlight the evidence of the geometrical truth. Mostly because he thought that it was more relevant, for a reader, to understand how the geometrical knowledge was established rather than to be taught unquestionable truths, Clairaut also offered a problematized geometry and methods to solve problems (Barbin, 1991).

But after the French Revolution, the establishment of a general and national system of education reinforced the values of knowledge and reasoning (Schubring, 2007, p. 38). The new structures of French higher education, as *École polytechnique*, for the training of engineers, *École normale de l'an III* for the training of teachers, and *lycées* for the training of future graduate students, all offered very theoretical curricula with high levels of excellence. The very demanding entrance examination for *École polytechnique*, prepared within the *lycées* or the *classes préparatoires* by most of the candidates, required a complete training in reasoning and in advanced mathematics. *Legendre's Geometry* came within the scope of this new orientation for secondary and higher education.

Legendre's Geometry borrowed the method of proof exposed in *Euclid's Elements.* The proofs relied on deductive reasoning, essentially through synthetic order. Legendre also included *reductio ad absurdum* in his demonstrations, as he did to establish the area of a circle (Legendre, 1817, pp. 102-121).

Yet, unlike Euclid, there were very few problems and constructions in his textbook. He also rearranged the order of several properties, especially in book 1, to make it more understandable. He tried to prove some propositions that stood as axioms or postulates in *Euclid's Elements*, which actually could not be proven. This is the case of Euclid's last and fifth axiom concerning the unicity of a line parallel to another drawn from any point. Legendre wrote many proofs of that postulate which all appeared to be erroneous [5]. The demonstration, produced in the 1817 edition, relied on the observation of a particular diagram, whose evidence gave the conviction of the truth of the proposition, according to the French man. Legendre wrote a same kind of proof, which Euclid had always discarded, in order to establish the equality of two right angles in proposition 1 of book 1[6]. Another difference with Euclid was the role of arithmetic and algebra in *Legendre's Geometry*. The French man assimilated magnitudes to numbers in order to perform operations on lines, surfaces and volumes - provided a length unit had been chosen. Thus, unlike Euclid, he gave formula for the area of polygons and the volume of solids. Finally, Legendre removed all the *protasis*, keeping only the particular statement for each proposition.

The soft breaking of John Farrar and the reform of Harvard curriculum

In 1819, Legendre was first translated in the United States by John Farrar (1779-1853), professor of mathematics at Harvard College. John Farrar graduated from Harvard in 1803. He took the chair of mathematics and natural philosophy in 1807. In the 1810s, Harvard's president, John T. Kirkland (1770-1840), a liberal and a reformist, asked Farrar to produce a new series of mathematics textbooks for Harvard curriculum. At that time, Harvard's mathematical studies were pursued with the help of *Webber's Mathematics* and *Playfair's Geometry*. In 1818, probably influenced by the Bostonian open-minded literary, cultural and scientific activity [7], Farrar started writing the translations of eight French textbooks, five of which were dedicated to mathematics, borrowing ideas from Legendre, Lacroix and Bézout [8]. The *Farrar's Legendre's geometry* was the first English language translation of the French textbook ever published in the world.

Legendre was praised by the Harvard's professor. His geometry united "the advantages of modern discoveries and improvements with strictness of the ancient method" (Farrar 1819, p. iii) and its "celebrity" across France and Europe was very well known in America. Introducing Legendre's book for an American audience, Farrar wanted to take his distance from the rigidness of Euclid as he explained to Kirkland:

> There is scarcely anything in which our superiority over the ancients is more manifest and palpable than in mathematics and yet this is almost the only branch of knowledge in which we continued to acknowledge them as our teachers (Farrar, 1817).

Its presentation of Euclidian geometry, using algebraic symbolism and a new arrangement of properties, was perceived by the Harvard's scholar as a good compromise for his teaching between classicism and modernity. Thus, John Farrar's translation of *Legendre's Geometry* was very faithful and introduced a breaking in the teaching of geometry in United States. The author did not introduce major changes, and the small alterations were only concerned with the removal of propositions (Preveraud, 2013a).

DAVIES AND LOOMIS. THE PUBLISHING CONTEXT AND THE COMEBACK TO EUCLID (1828-1849)

With Charles Davies's and Elias Loomis's versions of *Elements of Geometry,* very important changes came to light within the French original.

Charles Davies (1798-1876) was a professor of mathematics at West Point Military Academy and had started publishing a series of textbooks for cadets and more generally higher education students. In 1828, he made use of a previous adaptation of Legendre, published in 1822 by Scottish scientist David Brewster (1781-1868) [9] to publish *Elements of Geometry* [10]. He produced many reprints of his textbooks (Ackerberg-Hastings, 2000, pp. 238-248), notably in 1834. Another American adaptation of Legendre's textbook was intended for college students. It was the publication of *Elements of Geometry* in 1849 by New York University's professor Elias Loomis (1811-1889). The author had spent a few years in Paris and studied natural philosophy and medicine (Newton, 1889-1890, p. 326). Back to the United States, he became a professor of mathematics at New York University at the end of the 1840s. During his position in New York, he started a successful career as an author of mathematics textbooks designed for universities. Both Davies's and Loomis's textbooks were circulated in American higher education, as they each entered at least a dozen colleges curricula as shown in (Cajori, 1890).

Both authors reintroduced many characteristics of Euclid's book that Legendre had removed from his own. First, they added eight original Euclidean axioms to Legendre's first five. Also found in Playfair's and Simson's textbooks, Davies and Loomis added axioms regarding the addition and subtraction of the same magnitude to equal magnitudes. They removed Legendre's proposition I in book I that proved the equality of two right angles, and they changed it into an axiom. They also completely discarded all of Legendre's proofs of the fifth postulate (Davies, 1834, p. 13 & Loomis, 1849, pp. 12-13). Davies judged that the demonstration given by Legendre, in the 1817 edition, was not rigorous enough for a geometry textbook:

> The preceding investigation, being founded on a property which is not deduced from reasoning alone, but discovered by measurements made on a figure constructed accurately, has not the same character of rigorousness with the other demonstrations of elementary geometry. It is given here merely as a simple method of arriving at a conviction of the truth of the proposition. (Davies, 1834, p. 17)

They rewrote many proofs to make them "more Euclidean", including the statements of some propositions that appeared to be closer to Euclid's (or Simson's and Playfair's) than to Legendre's. Thus, Legendre's proposition "Deux triangles sont égaux lorsqu'ils ont un angle égal compris entre deux côtés égaux chacun à chacun" (Legendre, 1817, p. 20) became for Loomis:

> If two triangles have two sides, and the included angle of the one, equal to two sides and the included angle of the other, each to each, the two triangles will be equal, their third side will be equal, and their other angles will be equal, each to each. (Loomis, 1849, p. 17).

A last significant change occurred in Loomis's version with the almost complete abandon of magnitudes arithemization. Symbols +, -, =, >, etc. were removed by the New York University's professor as shown in the following example (Fig. 3).

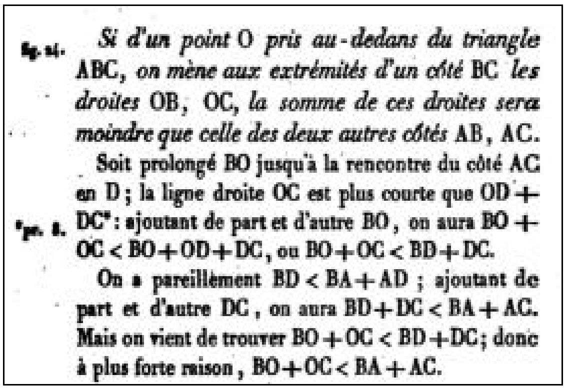
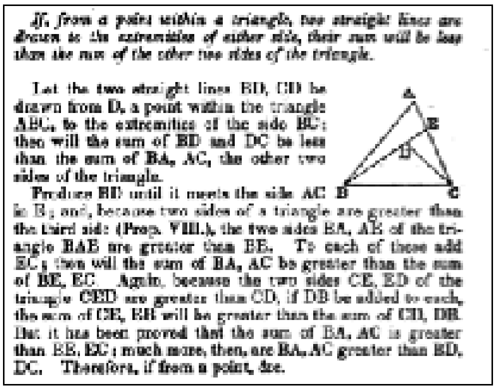


fig. 14. *Si d'un point* O *pris au-dedans du triangle* ABC, *on mène aux extrémités d'un côté* BC *les droites* OB, OC, *la somme de ces droites sera moindre que celle des deux autres côtés* AB, AC.

Soit prolongé BO jusqu'à la rencontre du côté AC en D; la ligne droite OC est plus courte que OD + DC*: ajoutant de part et d'autre BO, on aura BO + OC < BO + OD + DC, ou BO + OC < BD + DC.

*pr. 8.

On a pareillement BD < BA + AD ; ajoutant de part et d'autre DC, on aura BD + DC < BA + AC. Mais on vient de trouver BO + OC < BD + DC; donc à plus forte raison, BO + OC < BA + AC.



If, from a point within a triangle, two straight lines are drawn to the extremities of either side, their sum will be less than the sum of the other two sides of the triangle.

Let the two straight lines BD, CD be drawn from D, a point within the triangle ABC, to the extremities of the side BC; then will the sum of BD and DC be less than the sum of BA, AC, the other two sides of the triangle.

Produce BD until it meets the side AC in E; and, because two sides of a triangle are greater than the third side (Prop. VIII.), the two sides BA, AE of the triangle BAE are greater than BE. To each of these add EC; then will the sum of BA, AC be greater than the sum of BE, EC. Again, because the two sides CE, ED of the triangle CED are greater than CD, if DB be added to each, the sum of CE, EB will be greater than the sum of CD, DB. But it has been proved that the sum of BA, AC is greater than BE, EC; much more, then, are BA, AC greater than BD, DC. Therefore, if from a point, &c.

Fig. 3. Proposition IX, Book 1 in Legendre's *Éléments de géométrie* (left) (Legendre, 1817, p. 12) and in Loomis's (right) (Loomis, 1849, p. 19)

The reason why Davies and Loomis came back to Euclid in their adaptations is to be found in the publishing market context and the local teaching uses. During the first half the nineteenth century, the teaching of geometry in American colleges and schools borrowed mostly from uses and methods that came from England, based on rather strict Euclidean geometry (and also practical geometry textbooks). As a consequence, a large range of textbooks in use and published in America were closer to Playfair's work than to Legendre's. Both authors noticed the gap between the geometry offered in Legendre's and the geometry Americans used to learn and practise, and they decided to fill it. For Loomis and Davies, French original textbooks presentation of mathematics was unsustainable for the horizon of expectation, as defined in (Jauss, 2010), of American readers. Indeed, both authors intended to widely sell their textbooks within the publishing market. In association with Alfred Barnes (1817-1888), an editor from Hartford, Connecticut, Davies published several national series of textbooks, designed for elementary, high school and higher education, based on West Point Davies's first publications, and became a businessman in mathematics publishing (Ackerberg-Hastings, 2000, p. 215).

THOMSON AND SMITH. BACK TO EUCLID VERSUS MODERNIZATION: PEDAGOGIC NEEDS LED TO A COMPROMISE (1844-1867)

In the second half of the century, two other adaptations of *Legendre's Geometry* were published in the United States, but they were not as successful as their predecessors. The first one was intended for a growing but a new audience, and the second for a very specific and small group of readers.

Thomson's (1844) and Smith's (1867) adaptations intended to specific audiences

The 1840s marked the time when geometry started to be taught in high schools (Sinclair, 2008, p. 19). In the mid-century, high schools mainly trained young students for college admission where arithmetic and some algebra were required. From the 1870s, geometry became a requirement in most college admissions. Nevertheless, it had been taught in high schools before then. In 1844, the schoolteacher James Bates

Thomson (1808-1883) wrote another adaptation of *Legendre's Geometry* using Brewster's adaptation, but wrote it for high school readers. He was the co-author of a series of textbooks abridged from Jeremiah Day's series initially published in the 1810s [11].

In 1839, a new Military Academy opened in Lexington, Virginia. The Virginia Military Institute was designed to train military engineers of the South of the United States, as West Point did for the North. As an ex-cadet of West Point, the first superintendent and professor of mathematics, named Francis Henry Smith (1812-1890), organized the Academy explicitly referring to West Point structure, methods and curricula (Wineman, 2006, p. 40) largely relying on French pedagogical methods and textbooks translations (Preveraud, 2013b). He translated Louis Lefébure de Fourcy's *Elements of Trigonometry* (1868) and a new French edition of *Legendre's Geometry*, published in Paris around 1850 by Alphonse Blanchet (1867). When Legendre died in 1833, Blanchet, a mathematics teacher at College Sainte Barbe in Paris, shouldered the publishing of the following editions of the textbook. Starting in 1848, he introduced several changes in the text, and from that moment on he considered himself as a co-author. The main transformation was the introduction of the concept of limits. Blanchet used limits for the writing of the proofs for the measurement of circles and round bodies, whereas Legendre had applied *reductio ad absurdum.*

In the writing of their adaptations, how did Thomson and Smith take into consideration the previous adaptations and the specific needs of their readership? Did they introduce changes to the original, as Davies and Loomis did? Did the pedagogical and publishing context, in which the adaptations were produced, and the type of students they intended to target, also have as consequence a comeback to Euclid?

Thomson and Smith both came back to a Euclidean presentation of geometry, but they did search for a compromise between the virtues of Euclidean textbooks and the advantages of more modern methods - highlighted in this article by the following examples: the status and the role of abstraction in the teaching of geometry and the introduction of analytical tools to facilitate the understanding of some proofs.

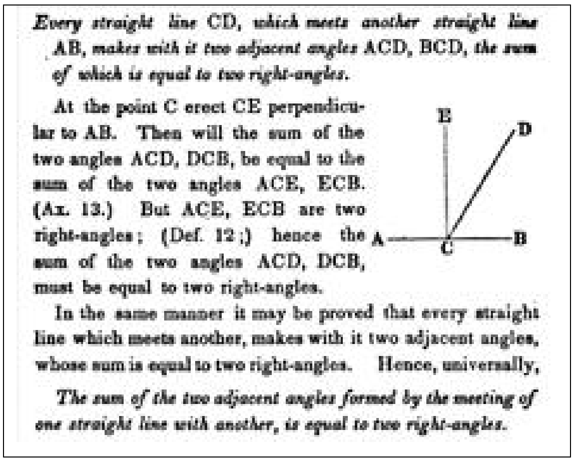
Geometric truths, abstraction and the yardstick of teaching contexts

Unlike Legendre who got rid of the Euclidean *protasis*, Davies re-introduced the general statement of the propositions as he clearly explained in his preface:

> In the original work […], the propositions are not enunciated in general terms, but with reference to, and by the aid, of the particular diagrams used for the demonstrations […]. This method seems to have been adopted to avoid the difficulty which beginners experience in comprehending abstract proposition. But in avoiding this difficulty, and thus lessening, at first, the intellectual labour, the faculty of abstraction, which it is one of the peculiar objects of the study of Geometry to strengthen, remains, to a certain extent, unimproved. (Davies, 1828, p. iii).

According to Davies, Legendre, who had removed the *protasis*, took his distance from one of the virtues of Euclidean geometry, that is to say the work the mind had to achieve in order to lead the learner "to the temple of the truth" (Davies, 1828, p. iv). As a consequence, and because Davies' *Elements of Geometry* were meant to be taught in colleges and academies, it was necessary to change Legendre's way of enunciating the propositions.

In his *Elements of Geometry* designed for high school, Thomson found a compromise between Legendre's and Davies's approaches. He chose to place the *protasis* at the end of the proof as shown in proposition II, book 1 (Fig. 4).



Every straight line CD, which meets another straight line AB, makes with it two adjacent angles ACD, BCD, the sum of which is equal to two right-angles.

At the point C erect CE perpendicular to AB. Then will the sum of the two angles ACD, DCB, be equal to the sum of the two angles ACE, ECB. (Ax. 13.) But ACE, ECB are two right-angles; (Def. 12;) hence the sum of the two angles ACD, DCB, must be equal to two right-angles.

In the same manner it may be proved that every straight line which meets another, makes with it two adjacent angles, whose sum is equal to two right-angles. Hence, universally,

The sum of the two adjacent angles formed by the meeting of one straight line with another, is equal to two right-angles.

Fig. 4. Proposition II, Book 1 in Thomson's *Elements of Geometry* (1844) (Thomson, 1844, p. 21)

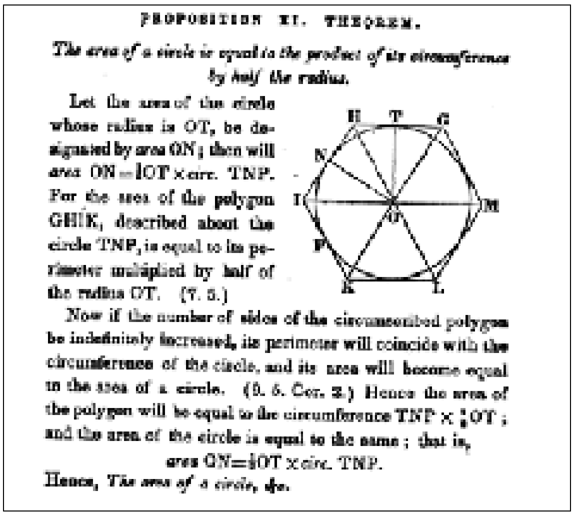
The reason is that he intended his book for high school students, who could experience stronger difficulties to face abstraction than college students might have:

> The principal embarrassment which young minds experience in the study of geometry, arises from the difficulty of comprehending abstract propositions. Legendre has essentially removed this difficulty by enunciating the propositions by the aid of particular diagram which he uses in the demonstration […]. It is found, however, to be inconvenient for scholars to quote a proposition enunciated with reference to a particular diagram […]. To obviate this inconvenience, after the truth of the proposition has been established with respect to the particular diagram in question, the general principle is then deduced, and for the sake of more convenient reference is printed in italics. Thus we begin with a particular case, and arrive at a general conclusion. (Thomson, 1844, p. 6).

In Thomson's approach, the example led the student to the general statement of the proposition, but this statement could not be omitted as conducting the mind to a universal truth, one of the objects of the teaching of geometry.

Introduction of analytical tools for simplification of proofs

Even if Thomson and Smith made Legendre look like Euclid, they openly took their distance from the proofs that relied on *reductio ad absurdum*. Thomson judged them "less satisfying for the mind" than the direct method (Thomson, 1844, p. 222). Indeed, the indirect method was only conclusive provided the conclusion was known before starting the demonstration, and it never gave the path that conducted the mind to the invention of the solution. It also mostly produced long and abstruse proofs for general-interest college or high school students. Consequently, in Thomson and Smith's books, most of the *reductio ad absurdum* proofs were replaced by more modern tools. For example, the proof of the area of a circle was based on the double assertion that the search for area could not be equal to the area of a larger circle nor a smaller one. Thomson changed the nature of the proof, using the limit – without saying the word – of the area of an inscribed polygon whose apothem became closer to the radius of a circle, and whose number of sides "indefinitely increased" (Thomson, 1844, p. 144) (Fig. 5). Noting that those "demonstrations, if not so rigorous as some, had the advantages of being more easily understood than the others" (Thomson, 1844, p. 233), Thomson took into account the needs of his readership in terms of pedagogical methods.



PROPOSITION XI. THEOREM.

The area of a circle is equal to the product of its circumference by half the radius.

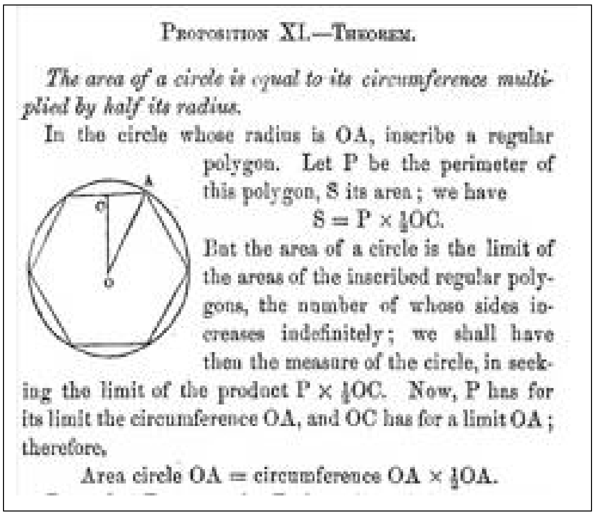
Let the area of the circle whose radius is OT, be designated by area ON; then will area ON = ½OT × circ. TNP. For the area of the polygon GHIK, described about the circle TNP, is equal to its perimeter multiplied by half of the radius OT. (7. 5.)

Now if the number of sides of the circumscribed polygon be indefinitely increased, its perimeter will coincide with the circumference of the circle, and its area will become equal to the area of a circle. (5. 6. Cor. 2.) Hence the area of the polygon will be equal to the circumference TNP × ½OT; and the area of the circle is equal to the same; that is,

area ON = ½OT × circ. TNP.

Hence, *The area of a circle, &c.*

Fig. 5. Area of a circle proof in Thomson's adaptation (Thomson, 1844, p. 144)



PROPOSITION XI.—THEOREM.

The area of a circle is equal to its circumference multiplied by half its radius.

In the circle whose radius is OA, inscribe a regular polygon. Let P be the perimeter of this polygon, S its area; we have

S = P × ½OC.

But the area of a circle is the limit of the areas of the inscribed regular polygons, the number of whose sides increases indefinitely; we shall have then the measure of the circle, in seeking the limit of the product P × ½OC. Now, P has for its limit the circumference OA, and OC has for a limit OA; therefore,

Area circle OA = circumference OA × ½OA.

Fig. 6. Area of a circle proof in Smith's adaptation (Smith, 1867, p. 131)

Years later, Smith used the exact same kind of proof but, then, with the concept of limit (Smith, 1867, p. 131) (Fig. 6). He intended his textbook for future engineers, whose education included differential calculus. The introduction of analytical tools and analytical methods in the demonstration came with the scope of simplifying the reading, the teaching and the learning, considering the needs and the existing knowledge of the readers.

CONCLUSION

One can be surprised by the way American scholars, after Farrar's first translation publication (1819), transformed Legendre's in such a manner that the French original almost disappeared, as in late Davies's or Loomis's versions, while authors still claimed to Legendre's influence. One could argue that this inconsistency can be explained, at least in part, by the international recognition attributed to the French textbook; for an American author, mentioning the name of Legendre on his front cover unquestionably promoted his book sales. Furthermore, it should be correlated to the nineteenth century long-term period of the history of mathematics education in America. Since 1819, decades passed before Davies, Loomis, Thomson and Smiths published their works. If Farrar was searching for a new geometry textbook, matching the reform of Harvard curricula, his successors were rather concerned by pedagogical tools, presentation and contents adapted to the audience they intended to catch and wrote textbooks as close as possible to the standards of publishing to make their diffusion possible. *Legendre's Geometry* adaptations offered mixed mathematics, borrowing from both French and local uses. *Legendre's Geometry* fate was similar to other French textbooks' in the United States (Bourdon's *Eléments d'algèbre* for example), as shown in (Preveraud, 2014, Chapter 5). Therefore, the study of the adaptations of *Legendre's Geometry* provided a relevant and significant example of the way in which French mathematics were transferred and received in America in the 19th century, as well as how teachers and textbooks authors adapted it to reach different audiences. Also involved in the sales of their work, Davies, Thomson and Loomis integrated transformations to adapt to the publishing market. In their textbooks, Legendre's original was the frame of the writing; in its name, its inner structure and organization, its arithmetization of geometry, these features were retained by each of the American authors. In the second half of the 19th century, many other geometry textbooks were also shaped using also *Legendre's Geometry*, and integrated the transformations previously made by Davies and Loomis, as shown in (Preveraud, 2014, p. 282-289). This prolonged and fruitful sedimentation of the French book, in the teaching of geometry, furnishes evidences of the standardization of mathematics practice, teaching and diffusion that would be achieved in that country by the end of the century (Parshall & Rowe, 1994).

NOTES

[1] The teaching of geometry was generalized in high schools not before the 1850s. See (Sinclair, 2008).

[2] See (Preveraud, 2012) for a complete analysis of the textbook and the adaptations Adrain introduced in his American version.

[3] Examples were given in (Preveraud, 2014, p. 94-97).

[4] For an extensive study of Playfair's *Elements of Geometry*, see (Ackerberg-Hastings, 2002). The author analyzed the « styles » Playfair wrote his proofs with.

[5] The most striking proof was published in the 1823 edition. Legendre used an analytical proof relying on the sum of the angles of a triangle. This new demonstration implied a complete reorganization of Book 1.

[6] This was an axiom in *Euclid's Elements*.

[7] In the 1810s, Farrar became a member of the Anthology Club, a Boston literary Society that founded the Bostonian Athenaeum, an independent library where many scientific books were gathered. One of the members, William Tudor (1779-1830), a businessman who had travelled in Europe, had brought back from France mathematics books Farrar had been able to read.

[8] Specific studies were produced, such as for *Lacroix's Algebra* translation in (Pycior, 1989). For a general analysis of Farrar's corpus of French translations, included the upgraded reprints Farrar wrote until 1840, see (Preveraud, 2014, Chapter 2).

[9] Brewster edited the work but did not write the adaptation of Legendre's *Elements*. This task was completed by Scottish scholar and historian Thomas Carlyle (1795-1881). See (Preveraud, 2014, p. 232).

[10] Davies published other translations and adaptations of French textbooks as *Jean-Baptiste Biot's Analytical Geometry* or *Louis P.M. Bourdon's Algebra* (Preveraud, 2014, pp. 253-261 & pp. 269-273).

[11] Day (1773-1867) was the professor of mathematics at Yale College before becoming its president.

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Workshop

WORKSHOP ON NEW APPROACHES AND RESULTS OF RESEARCH INTO THE HISTORY OF MATHEMATICS EDUCATION

Gert Schubring

Universidade Federal do Rio de Janeiro

This workshop will be a sequel to the invited lecture to topic 6: New Approaches and Results in the History of Teaching and Learning Mathematics. We will study various texts, which enable to deepen the questions and issues raised in the plenary lecture. In particular, texts will be used from different methodological approaches, allowing thus to obtain a comparative understanding.

Special emphasis will be given to two issues:

- differences between various countries regarding the processes by which mathematics achieved the status of a major teaching discipline in secondary schools,
- socio-political dimensions in the development and realization of Mathematics for All.

Oral Presentation

MATHEMATICS TEXTBOOKS, IN PORTUGAL: THE CASE OF THE FIRST UNIQUE MANDATORY ALGEBRA TEXTBOOK (1950)

Mária Cristina Almeida

UIED - FCT-UNL/Agrupamento de Escolas de Casquilhos

This paper gives an overview on Portuguese political and educational systems. After, it attempts to provide an understanding on how textbooks were chosen in previous decades. So, it addresses the Portuguese textbook approval systems, in particular the model that emerged in 1947 (unique textbook) and the contemporary published opinion about it. Regarding mathematics, the first unique textbook was approved in 1950; it was the Algebra textbook for the 3rd cycle of Liceus (upper secondary school). This paper discusses the criticism on the Algebra textbook that was published in the journal Mathematics Gazette. In our study we analysed two versions of the first unique Algebra textbook in order to trace changes in content, in this paper we will present part of that analysis. The paper is based mainly on legislation, newspapers, educational magazines and mathematics textbooks.

INTRODUCTION

When we attempt to visualize a schools' discipline past based on its material supports, textbooks are some of the most relevant elements to the study of that past. Choppin (2004) states that "the conception of a textbook is inserted into a specific pedagogical environment and into a regulated context, which, along with the development of national and regional systems, is, most of times, characteristic of educational productions (state editions, approval procedures, freedom of production)" (p.554). As a written document, textbooks are sensitive to national contexts and can be seen, in this case, as probes of the state and structure of mathematical education, its goals and its organization.

In Portugal, studies have shown that textbooks have been the most common resource in classrooms, representing and structuring mathematics school knowledge (APM 1998, Janeiro 2005). Today we are confronted in Portugal with new syllabuses for all grades of elementary and secondary mathematics education. In what concerns secondary mathematics education the syllabus incorporates contents that have not been taught for more than a decade or have never been taught at this level, so teachers are faced with a novel situation requiring teaching new contents. There is the possibility that many teachers experience difficulties and it may be the case that textbooks will come to influence the mathematical knowledge taught in schools. This situation increases the importance of both textbook production and textbook approval.

As regards political control and policies of approval for textbooks Repoussi and Tutiaux-Guillon (2010) developed a typology of approval systems or models, they distinguished five different models: "one single officially approved textbook; several officially approved textbooks; coexistence of officially approved and non-approved textbooks; officially recommended textbooks; and textbooks only produced by private publishers, without official approval" (Repoussi & Tutiaux-Guillon 2010, p.160).

At the present time, in each Portuguese secondary school a group of teachers analyses the certified textbooks available on market and decides on the adoption of the discipline's textbook to be used in the next six years. This period of adoption may be shortened if the syllabus changes. However, during a time legislation forced the adoption of a unique textbook, i.e., all Liceus (public secondary schools) had to use the same textbook per discipline and cycle; this is usually referred as the unique textbook period.

This paper gives an overview on Portuguese political and educational systems. After, it attempts to provide an understanding on how textbooks were chosen in previous decades. So, it addresses the Portuguese textbook approval systems, in particular the model that emerged in 1947 (unique textbook) and the contemporary published opinion about it. Regarding mathematics, the first unique textbook was approved in 1950; it was the Algebra textbook for the 3rd cycle of Liceus (upper secondary school). This paper discusses the criticism on the Algebra textbook that was published in the journal Mathematics Gazette from the point of view of the development of mathematics teachers' professional knowledge. In our study we analysed two versions of the first unique Algebra textbook in order to trace changes in content, in this paper we will present part of that analysis.

SHORT BACKGROUND ABOUT POLITICAL AND EDUCATIONAL SYSTEMS

In 1910, the Portuguese political system became a republic deposing the monarchy. The Constitution of 1933 established the dictatorship of Estado Novo (literally New State) that persisted until 1974. After the end of Second World War social an economic order began to change, we can observe some development and a drive towards strengthening the industry that needed skilled manpower to succeed. The education system needed to adjust to the new reality and the secondary and technical education reforms of 1947 marked the beginning of this accommodation process. By the reform carried out, in 1947, by the Minister of National Education Pires de Lima, the educational system consisted of a mandatory primary cycle (6-9 years old), followed by parallel branches for secondary education: the Liceus and the Technical Schools. The Liceus course encompassed three cycles: 1st (10-11 years old), 2nd (12-14 years old) and 3rd (15-16 years old), this course, especially the last cycle, was oriented to studies at the universities. Technical School studies were oriented to the work market or to pursue studies at the polytechnic institutes.

On April 25th 1974, the Carnation Revolution restored democracy in Portugal. And, in the years following 1974, the structure of the education system gradually began to change. One of the first major alterations was the elimination of the distinction between the two educational tracks that existed for secondary education and the creation of the Unified Secondary Schooling beginning at 7th grade (12 years old) and ending at 11th grade. The immediate result was the alteration of designation, the Liceus and Technical Schools became Secondary Schools. The unification was considered a means to balance educational opportunities for all students. The secondary schooling encompassed two parts: the lower secondary (7th – 9th grade) and the upper secondary. The latter comprehended a 1st cycle (10th – 11th grade) and a 2nd cycle (12th grade). In 1986, the accession of Portugal to the European Union and the society evolution demanded a major reform at all levels of the education system (e.g., structure, methods, contents). At secondary level, the main alteration was that the upper secondary ceased to encompass two cycles. In the present time, secondary schooling (15 -18 years old) is a three years cycle.

SHORT BACKGROUND ABOUT TEXTBOOK APPROVAL SYSTEMS

With regard to textbook approval systems, one could say that from the Estado Novo regime there are two main periods. In 1947 it was established a formal state textbook approval of only one textbook, in which the approval had a prescriptive status, that progressively ended in the first half of the seventies, and is usually referred as the unique textbook period. The 1974 revolution promoted a change in textbooks approval policy and the adoption of textbooks was then and up to now assigned to the teachers of each school.

During the Estado Novo regime, the policies of approval for textbooks produced some controversy. Before 1947, the schoolteachers' council chose the mathematics textbooks to be used for the following year in each Liceu. Books were chosen amongst the ones previously approved by the Minister of Education, the period of approval of a textbook was five years. Then, the disagreements arose mostly within the members of the scientific commissions nominated for textbook assessment and concerned the criteria for textbooks approval (Almeida, 2013).

In what concerns textbooks, the 1947 Pires de Lima reform, on the one hand, established that a disciplines' textbook would be the same for all the Liceus, on the other hand, formalised a new approval system to choose that textbook. After the approval of a textbook, the Ministry of Education would choose the textbook publisher by contract. The aim of the contract was to obtain the lowest sell price for the book. The emergence of this approval system was not peaceful; some teachers argued in favour of the new system and others stated against it, using teachers' bulletins, as well as, the press (Almeida, 2013). We will resume this particular model later on this paper.

As outcome of the 1974 revolution the choice of the textbook becomes the responsibility of teachers and there was an editorial freedom to their design, on the assumption that authors follow the curriculum guidelines, so there were several textbooks available for adoption. In the 1980s, given the increasing number textbooks, the task of analysing and decide for one or another textbook became more complex. Moreover, the extent of time allocated to teachers' assess of the textbooks was short. In the next school year the process had to be repeated because the period of adoption was only one year. From one year to the next, the publishers usually improved the textbooks already on the market, this meant, on the one hand, correct text and scientific mistakes, and, on the other hand, adjust to educational objectives or to the age of the students. The textbook's publishers also developed supplements (teacher's guide, slides, etc.) that began to be distributed to the teachers.

A major change concerning legal mechanisms to control the quality of textbooks was set in 1987. Then, new official procedures and criteria for textbooks assessing and textbooks adoption were established. The assessment would take place every year and be performed by an official nominated committee that approved up to three textbooks by level and by discipline, from which teachers should make their choices. It was also established that the period of adoption of a textbook would be three years at least (Decree-Law 57/87). In 1990, the task of the officially nominated committee changes; at that point, the object of the group's work was to build an evaluation table, that is, criteria for textbook choosing. Next, the Ministry of Education would send the evaluation tool to schools yearly, in order to assist them on textbooks adoption. For secondary education the period of adoption of a textbook does not change (Decree-Law 369/90).

More recently, in 2006, the Ministry of Education, intending to increase the quality of education, introduced a new policy for quality control and textbook assessment. And, so Law 47/2006 identified the evaluation of textbooks as a means of improving their quality. Moreover, it established that the period of adoption of a textbook would be six years and also instituted the system for textbook assessment, approval and adoption. One of the central issues of this new law is the establishment of protocols with universities that will constitute teams to evaluate textbooks in areas of knowledge.

Unique textbook: choosing procedures and contemporary outlooks

In 1947, it was established that textbooks needed to be approved by the National Board of Education, a department of the Ministry of Education and would be the same for all Liceus and private high schools. The textbook could have one volume - with sections, one per year - or more than one volume. Teachers and students would use the book the following five years. During this five-year time, the authors of a unique book could propose, in new editions, amendments they deemed important.

A book approval depended on its consonance to the syllabus, scientific rigor and suitability to support teaching. The process of selecting a unique book began with the opening of a call to which the authors presented their textbooks. Next, the National Board of Education appointed two schoolteachers (jury) to look over the books. Then it would get reports back on what these schoolteachers thought about the books.

Finally, the National Board of Education came to a decision about the book to take, i.e., the unique book. And, announced it officially (Decree-Law n.º 36 508, 17 september 1947).

When a unique textbook was published, the words "Officially approved as unique textbook" on the back cover together with an official stamp and a number, guaranteed the textbook's authenticity.

To choose mathematics textbooks the Ministry of Education appointed two mathematics schoolteachers who worked at the Liceus. The textbooks proposed to a call were individually evaluated by the two appointed school teachers. The evaluator reported on the scientific and pedagogical value of the textbook. In the report he could propose amendments he deemed necessary for the approval or he could consider that the book wasn't worth of approval. Each evaluator had to grade the textbooks accordingly to their scientific and pedagogical value.

At time, in articles published in the daily press, teachers' bulletins and mathematics journals, there was a public discussion about the new textbook approval system. Although the article authors were mainly teachers, we can also see a mathematician's opinion:

> This [book approval] system can relegate to oblivion some good books for the work of students and teachers. (…) it is notorious the huge responsibility of authors and jury, the former in writing the books and the latter in evaluating and approving them, for they are endorsed for five years. (Barros, 1950, p.19, tr. M.A.)

The most mentioned advantages of new textbook approval system were, firstly, the low price of the books, which was officially enforced (Editorial, 1947); secondly, a more uniform preparation of students to take the state exams (Russo, 1956), which was granted by the teachers' obligation to follow the unique textbook content. About the disadvantages that this new system of approval faced, the articles spoke mainly on ones regarding the textbook authors and the jury. The time and effort needed to write a good textbook risking it was not chosen as a unique book (Soares, 1956) along with the knowing that it could only be chosen five years later, at the best, were reasons that kept textbook authors away from the calls (Ataíde, 1956; Soares, 1956). According to Ataíde (1956), gather a jury with enough competence for the job was not easy, given that the important teachers were usually textbook authors and so they were not allowed to take part in textbook evaluation. Indeed, at that time there were few certified teachers.

In terms of this approval system, the first mathematics textbook was approved in 1950, it was an Algebra textbook for the first grade of upper secondary school and it was object of dispute in the mathematics journal Mathematics Gazette.

THE FIRST UNIQUE TEXTBOOK APPROVED FOR THE 3RD CYCLE: ALGEBRA BOOK (1950)

Regarding mathematics textbooks to be used during the 3rd cycle of Liceus, they referred to four topics– Algebra textbook, Geometry textbook, Rational Arithmetic textbook or Trigonometry textbook.

By 1949, the process of approval of an Algebra textbook for the 3rd cycle of Liceus was initiated. António Lopes [1] was the only author to submit an Algebra textbook for the 3rd cycle of Liceus to the first call for the approval of the unique textbook of this topic and his textbook was the first mathematics textbook to be chosen (Almeida, 2013).

The 1947 reform had changed the mathematics syllabus for the 3rd cycle of Liceus (grades 10th and 11th). The infinitesimal calculus, removed from the programs in 1936, was reintroduced. The introduction of derivative study prompted debates about the quality of mathematics terminology in the programs and the unique textbook and also about the ways in which its study should be articulated with the study of limits (Matos, 2014).

On the production of mathematics textbooks, in particular for the 3rd cycle, Sebastião e Silva (1951) [2] argued about the complexity on writing textbooks encompassing the study of infinitesimal analysis. It was his belief that the presentation of infinitesimal analysis should as much as possible match intuition and accuracy. So, he underlined the exposition of this topic as a difficulty authors had to cope in what concerned the development of textbooks for secondary education. Moreover, authors were faced with a novel situation requiring writing about contents that had not been taught for twelve years at this level. He considered that the latter as a cause of the imperfections pointed to the Algebra textbook, which was approved as unique textbook for the 3rd cycle. Silva (1951) mentioned that, while reading the textbook

> one gets the impression that the author tried to reconstruct its mathematical culture and, at the same time, keep up with the deadline to present the textbook to the call - and it is very likely that he, author, already has realized the drawbacks of his hurried decision. (Silva, 1951, p.2, tr. M.A.)

Sebastião e Silva (1951) stated that the huge development yielded in order to clarify the analysis concepts helped matching logic with intuition; however, he points out that some attention has to be paid when trying to bring them together:

> What we immediately observe, in the theory of limits, is that the language becomes more intuitive coming to speak of «limit of a variable» instead of «limit of a succession» or «limit of a function». But, it must be kept in mind that it makes no sense to talk about limit of an independent variable, when, actually, we are addressing function limit in all cases: - functions of a natural variable (succession) or functions of a real variable. (Silva, 1951, p.2, tr. M.A.)

We can perceive from the words of Sebastião e Silva (1951) that the Algebra textbook was object of dispute. The largest discussion on this textbook was published in the

Mathematics Gazette and its author was a mathematician Laureano Barros. According to Barros (1950), the textbook lacked quality, so it should not have been approved as unique; he supported this position by posing the question "how is a teacher supposed to teach well by using books that aren't good?" (p. 19). Barros (1950) criticisms were very sharp, he stated that sometimes it seemed the book had been written by somebody who didn't know what he was talking about, that some definitions weren't accurate and that the exposition of some contents was correct and clear, but the explanation of other was very confusing and totally or partially incorrect. After exposing his critics, who mainly concerned scientific issues, he urged the author to make the necessary amendments.

In his examination of the first unique Algebra textbook Barros (1950) pointed out fourteen imperfections that can be allocated in two types, by the one hand, the deficiencies of scientific nature, on the other hand, the ones of pedagogical kind. In the conclusion of is review he says

> what particularly amazes us is the fact that none of the textbook evaluators had felt the gravity of the errors and defects that we point out (and others that are not mentioned here). And there is no doubt that the jury, collectively, did not feel it; otherwise, it had to propose the appropriate amendments to the Author, as required in the law.
> For the good of Mathematics teaching in our country, let these changes appear soon or, at least, in the next edition of this work. (Barros, 1950, p.24, tr. M.A.)

Barros (1950) comment on the bibliography that he thinks the author of the textbook should have consulted:

> we think, for example, that the [book] Algebra and Analysis Lessons of prof. Bento Caraça (that the author does not mention in the bibliography) could provide almost any material for an elementary exposure, correct and easily accessible, of the limits theory. (Barros, 1950, p.24, tr. M.A.)

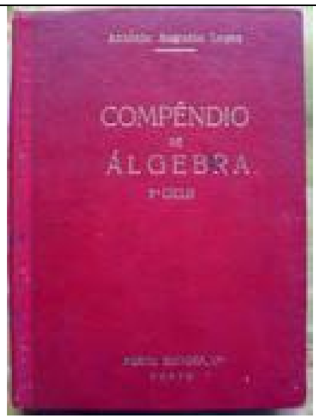
The previous words relate the textbook's quality with its author expertise on the subject and on the subject teaching. This will allow us to use Shulman's (1986) idea of pedagogical content knowledge on our discussion. At the heart of this knowledge is the manner in which subject matter is transformed for teaching. This occurs when the teacher interprets the subject matter, finding different ways to represent it and make it accessible to learners.

We will attempt to illustrate difficulties authors/teachers have to cope when facing a syllabus transition, particularly, when there are changes in the mathematical content to be taught. In this case the author seemed to have experienced problems, especially on the approach of contents that have not been taught for twelve years. We intend to show by evidence – through the text analysis – that the textbook author has enhanced his pedagogical content knowledge in order to improve the textbook content comprehensibility and altered the parts he considered textbook's 'negative' features.

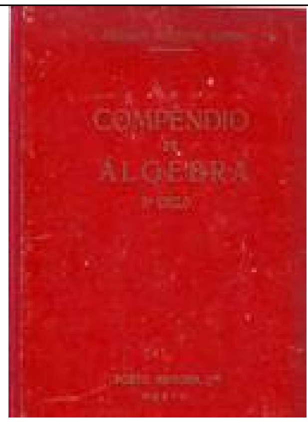
The Algebra textbook (1950): two different versions

Given the criticism to the Algebra textbook and the possibility of amendments to the approved textbook, we conjectured that the author might have made changes to the original content; and, so we searched in the Portugal's National Library and in private collections trying to find different versions of the Algebra textbook.

Using mainly Barros's (1950) observations as a support to trace changes in content we located two different versions of the Algebra textbook – an original and an altered version. We named the original version as Textbook B1 and the altered one as Textbook B2 (Fig. 1).



Textbook **B1**



Textbook **B2**

Fig. 1: Front cover of the two different textbook versions

The two versions have a similar structure: chapter – paragraph – section. Theoretical concepts that the student must learn are the first to be introduced, followed by examples and some exercises (with solutions). Bold is used to highlight the most important parts. Both textbooks include mathematicians' biographies, historical notes and bibliography. From Textbook B1 to Textbook B2, we observe that the bibliography of B2 has two more books than the bibliography of B1: Bento Caraça – Lessons on Algebra and Analysis (vol.2); Léon Brillouin – Mathématiques. Lib. Masson et Cie. Paris, 1947. We consider particularly interesting that the bibliography of textbook B2 includes a book which had been recommended by Barros in his comments about B1.

We analysed Barros (1950) review of the original version of the Algebra textbook, in terms of, the arguments he used to ground his thoughts, and of the suggestions he gave to the author in order to change the content. In the scope of this paper we will present part of the observed changes, focusing on the most criticised chapter – chapter II, *Limits*. On both versions of the textbook, the organization of Chapter II is quite alike, namely the number of paragraphs and their titles, as well as of sections.

As regards paragraph I, entitled *Infinitely large. Infinitely small*, Barros (1950) firstly points out the confusing way as the notions of *infinitely large* (section 1) and of *infinitely small* (section 2) are presented. He continues by referring that the definition of *infinitesimal simultaneous* (section 3) has no content and mentions that following the author tries to correct this however his effort is not successful, also the examples presented contradict what was previously written. Barros considers that the exposition of the topic addressed in section 3 is not at all accurate.

The analysis performed on textbooks B1 and B2 allowed us to trace differences. In *paragraph I,* we identified small changes regarding the definition of *infinitely large* and of *infinitely small* notions and a clearer explanation of these notions. All the examples included in B1 don't appear in B2. There is, in B2, a new section regarding infinitely small that introduces the concept of neighbourhood of zero, which refers "*x* is an infinitely small", i.e. "*x approaches zero" or "the limit of x is zero", that in symbolic language becomes:* "$x \to 0$" or "$\lim x = 0$". In *paragraph I*, the major identified changes occurred in section 3, in which a totally different definition of *infinitesimal simultaneous* is presented and that brought changes to the following content of this section. In B1, the definition of *infinitesimal simultaneous* was as follows:

> Suppose x e y two infinitely small and let ε e δ be two positive numbers, arbitrarily small. We say x e y are *infinitesimal simultaneous*, when to the values of x_n (de x) that verify the inequality, $| x_n | < \varepsilon$ for $n \geq n_1$ correspond values of y_n (de y) in such way the inequality, $| y_n | < \delta$ is verified for $n \geq n_2$. (Lopes, n.d. a), p.47, tr. M.A.)

In B2, the definition of *infinitesimal simultaneous* was as follows:

> Suppose $y = f(x)$ a real-valued function, variable x.
>
> Definition: If to all positive number δ is possible to correspond a positive number ε (variable with δ, i.e., function of δ) by means that the inequality $| y | < \delta$ is verified to all values of x that satisfy the inequality $| x | < \varepsilon$ we say that $y = f(x)$ is an infinitesimal simultaneous with x. (Lopes, n.d. b), p.49, tr. M.A.)

In B2, we identified changes in two other sections, namely, section 4 – *Theorems related to the product of infinitely small* and section 5 - *Theorems related to the sum of infinitely small*. In both sections, as a result of the new definition, there were adjustments on the statement of theorems and on its demonstration.

Concerning paragraph II, entitled *Limit of variables and of functions*, Barros (1950) states that the errors in its content are a repetition of the ones that were manifested in the previous paragraph and names some defects.

One of the identified changes to paragraph II regards the title of a section. In B1, the title of section 6 was *Limit of an independent variable*, and, in B2 it was altered to *Limit of a variable*. In B1, there was an observation concerning the definition of limit and a graphical interpretation to the definition that were removed in B2. The previous mentioned change relates to one of the defects named by Barros. We also identified some other changes related to the defects referred by Barros. In textbook B1, there was a section, entitled *Preliminary Theorems*, which no longer appeared in B2. In the remaining sections, some theorems have a different statement and all proofs are altered.

However, there were changes that do not relate to Barros criticisms, namely, some graphical representations enclosed in B2 (Fig. 2).

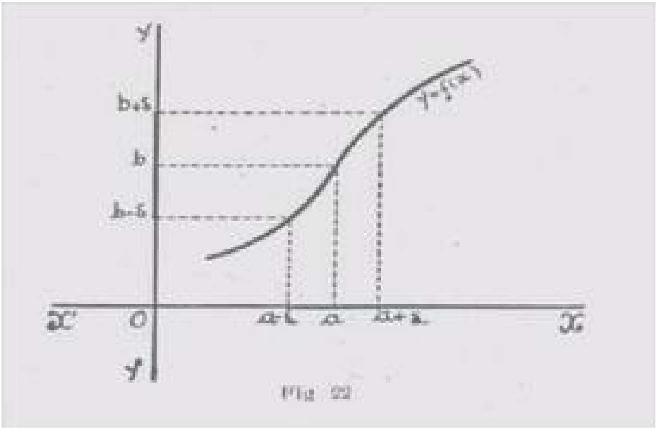


Fig. 2: Graphical representation in the context of the definition of limit of a function at a point (B2, p. 58)

Textbook B1 was text-oriented. The inclusion of images in B2 suggests both, an improvement of the authors' pedagogical content knowledge, and his understanding that the incorporation of visual material in textbooks can greatly enhance students' learning.

In paragraph IV, entitled *Basic notion of continuity of a function*, we traced some differences. The one that stands up occurred in subsection D that addressed the *right-continuous* and the left-continuous notions.

We point out that, in both versions of the textbook, Chapter II ends with an identical historical/biographical note, entitled *A XIXth century mathematician: −Augustin Cauchy*.

CLOSING DISCUSSION

Over the course of time textbooks have been a teaching and learning tool. In Portugal, mathematics teachers rely on the textbook using it as a main source of and tool in teaching. The educational importance of textbooks puts a stress on the quality of textbooks used in schools, which means that the process how these textbooks are approved is of the essence. The policy of textbook approval changed and evolved over time. This text provides an overview on how textbooks were chosen in previous decades. The outline begins in 1947, then it was established a formal state textbook approval of only one textbook, initiating a time that is usually referred as the unique textbook period. The 1974 revolution promoted a change in textbooks approval policy and the choice of textbooks was then and up to now assigned to the teachers of each school. The control of textbooks quality was the main reason of afterward changes in the policy of textbook approval. The most recent change happened in 2006. For the purposes of the study, we focus on the unique textbook period. Here we characterized the unique textbook approval procedures and show the contemporary opinion on this textbook approval system. The most mentioned advantages of this mode were the low price of the textbooks and the guarantee of a uniform preparation to the final exams. On the other hand, the major disadvantage stated was the life span of five years. In which regards mathematics, we centre our attention on the Algebra textbook for the 3rd cycle of Liceus that was the first unique textbook approved and its approval occurred in 1950. The 1947 reform had changed the mathematics syllabus for the 3rd cycle and the infinitesimal calculus, which was out of the program for more than a decade, was reintroduced. There were some criticisms to the above-mentioned textbook that were published in the Mathematics Gazette and teachers' journals. Our analysis of the criticisms on the Algebra Textbook clarified that they concerned mainly to scientific issues. The analysis of two versions of the Algebra textbook showed that some amendments to the original version were carried out. Some of the identified changes are in line with the criticisms made by Laureano Barros, a mathematician. In the scope of this paper, only the chapter referring to the infinitesimal calculus was discussed. Our analysis showed a change in the textbook content that implied a development of the authors' pedagogical content knowledge. This example illustrates the difficulties authors/teachers have to cope when facing a syllabus transition, particularly, when there are changes in the mathematical content to be taught. Today teachers face new mathematics syllabus that, in the case of secondary education, incorporates contents that have not been taught for more than a decade, as well as, new textbooks in line with the new reality. One can say that there are no perfect textbooks, however there are better or worse textbooks. We believe this study, supported on an historical example, can help teachers' understanding on policies of textbook approval in Portugal and on the role that these policies play on the quality of mathematics textbooks.

NOTES

1. António Augusto Lopes (1917-2015) was a certified Liceus teacher since 1941. Later he became a teacher trainer with responsibilities in the initial formation of teachers and a member of the Commission for the reform of the upper cycle of Liceus. He also was an active participant in the Modern Mathematics reform.

2. The mathematician and university teacher José Sebastião e Silva (1914-1972) would become the best known leader of the Modern Mathematics movement in Portugal.

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Oral Presentation

AROUND A BOOK DEDICATED TO CHILDHOOD FRIENDS: *INITIATION MATHÉMATIQUE* BY C.-A. LAISANT

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The purpose of this article is to highlight some of the original thinking found in the book Initiation mathématique *written in 1906 by the French mathematician Charles-Ange Laisant. In this work he offered an innovative approach using varied mathematical ideas for young children, but one can also observe that this work was the result of his lifelong route encompassing his epistemological thinking about the nature of mathematics, his interest in the visualisation of mathematical processes and his convictions about the role of Science in Society. Moreover, it's also interesting to describe the various networks which were at work before and after the publication of this* Initiation.

In 1906, the French mathematician Charles-Ange Laisant (1841-1920) published *Initiation mathématique* his definitive book resulting form a lifetimes work. This work was well recognised at the time and represented the crystallization of its author's goal to propose a new approach for the mathematical education of young children.

It is interesting to look at all the themes that link this book to the varied careers of its author. In fact, by 1906, Laisant's work as a mathematician was almost completed. It even seems quite surprising that this mathematician who followed a highly scientific route should change direction towards the education of children. It is proposed to highlight how this book was actually the result of various scientific experiments and higher mathematical exchanges. It is also intended to show how the book was the result of the production of several mathematical networks and how it enabled Laisant to develop new ones. These two points show how important a book like this was for a mathematician like Laisant during the latter period of his life.

It will be shown how original the role of this book was in the first mathematical education of young children and will highlight its origins and the references that appear in this work. In addition to the teaching tools which are applied to mathematical ideas, the background and implementation of general educational schemes are discussed. Finally it will be explained why Laisant became, following the *Initiation*, a special member of the movement called "new education" at the beginning of the twentieth century.

A BOOK AS A RESULT OF SEVERAL CAREERS

Some biographical facts about Laisant (Auvinet, 2011, 2013) can help us to understand the process that eventually led him to write the *Initiation mathématique*.

Charles-Ange Laisant was born in 1841 near Nantes in France. From 1859, he studied at the École Polytechnique and became a military engineer. He wrote numerous articles first about equipollences and quaternions, then about discrete mathematics (for examples, generalisations of Pascal's triangle or magical squares, Laisant & Arnoux, 1900). He was also an active member of several scientific bodies including the French Mathematical Society for whom he was president in 1888 (Gispert, 1991) and also the French Association for Advancement of Sciences (AFAS, presidency in 1904, Laisant, 1904b). The regional meetings of this association enabled many exchanges with all kinds of people interested by Science (Gispert, 2002).

He was mainly known to be a left-wing deputy from 1876 to 1893, a stormy period that led him to support Général Boulanger in 1889. He eventually came to consider politics as incapable of realising his strong desire to change Society. He subsequently became an anarchist and retained his belief while searching for other ways to realise his plans for popular education.

From 1893, he became a teacher in preparatory class and finally examiner for admissions to the École Polytechnique. Between 1892 and 1897, he published several books of exercises for students of preparatory classes, proposing new themes for preparing entrance exams (Laisant, 1893-1896).

In 1894, he founded the journal *L'Intermédiaire des mathématiciens* with Émile Lemoine (1840-1912), another former student at the École Polytechnique (Pineau, 2006). This original review enabled direct discussions on various mathematical subjects (including mathematical games as "récréations"). Most significantly, he also created in 1899 a true international journal, *L'Enseignement mathématique*, with the Swiss Henri Fehr (1870-1954). This focused for the first time on the progresses made in the teaching of mathematics within secondary schools of all countries (Furinghetti, 2003). This publication was a new medium for exchanges between mathematicians while questions about reforming the mathematical teaching appeared in many countries (see the 1902 reform France for example). These two innovative reviews enlarged his thinking about the mathematical community and the scope of his first work as a mathematician.

The first particularity of the book results through these numerous aspects of Laisant's life. It is a "book outside any curriculum, dedicated to childhood friends" (this is the complete title), written by a well-known mathematician at the time who was used to practicing and teaching higher mathematics. Pedagogy, especially linked to childhood, was a quite recent concern for him with an important political and sociological meaning.

GENESIS OF THE *INITIATION MATHÉMATIQUE*

In 1898, Laisant wrote for the first time a book which dealt with epistemology and included his observations about mathematics. It was entitled *La Mathématique. Philosophie. Enseignement* (2nd edition in 1907, Laisant, 1898). This important work featured some very general and personal ideas about mathematics and their teaching (Auvinet, 2007). He expressly stated that:

> 1° In the current environment, mathematical notions are necessary for everyone;
>
> 2° Anyone of average intelligence is able to acquire these notions, restricted to certain limits [1].

These statements point to the beginning of Laisant's desire to improve mathematical education for children. His goal was to prepare each young child from six years old in order to put him in a good position to receive later (after 12 years) a true mathematical education. This "initiation" was thus the base of a larger scientific culture, required by every citizen in the context of the technological transformations seen at the end of XIXth century.

Other ideas developed in the same work from 1898 were equally important. In addition to the unity of all mathematics, that mathematics is an experimental science. Laisant later explained in his *Initiation* that:

> I consider that all sciences, without exception, are experimental, at least to some extent [...] there is no concept, no idea that could enter our brain without a preliminary contemplation of the outside world and of the facts that the world presents to our observation. [2]

Thus, mathematical notions are the result of an abstract thought from the real objects which surround us. Here, Laisant pointed out that "abstraction" is a simplification. It is not to be explicit for a young child naturally endowed with an instinctive ability for "abstraction". He explained: "we should never try to make him abstractions that he wouldn't make himself" [3].

All these statements and others in *La Mathématique* were consequently fundamental for the processes developed in the *Initiation mathématique*. Here we get another particularity of this book: it is relevant across a wide vision of mathematics. As a result, Laisant was invited to present his educational theories and teaching tools in a series of lectures delivered between 1899 and 1903 at the Institut psycho-physiologique of Paris. These speeches were put together in one volume: *L'Éducation fondée sur la Science* (Laisant, 1904a). The first lectures was entitled "L'Initiation mathématique" (Laisant, 1899). It contained the main principles which should be applied during the mathematical education of children between four and eleven years old. He also strongly criticised teaching methods at the time and explained that this "initiation" remains as essential as the acquisition of reading and writing.

In 1906, the *Initiation mathématique* developed these ideas into a set of 65 lessons plans. As a reflection of its success, it was reprinted (in France) 17 times (Laisant, 1916) and it was well received in Italy too.

It is perhaps of value to state the intended purpose and audience for this work. Laisant explained that this book was not a handbook but more "a guide into the hands of the educator" [4], in particular for mothers. It contained several original educational principles which could be applied to every-day life. As already stated, this work was strongly and explicitly separate from curriculum or dogma and was in contrast with existing exhausting school practices at the time. The "initiation" has also to be presented rationally: it is based on rigorous observations of each child as an individual and the difficulty of the notions taught has to be progressive.

This "initiation" is based on experience and empirical evidence. It takes an physical form with the help of a specific material (cubes and rods for examples). This material was created by Jacques Camescasse (1869-1941), one of Laisant's collaborators and, like him, an Esperantist and a freemason. Laisant explained that the educator has "to bring the first images into child's brain by putting objects in the scope of his senses. The education should be absolutely concrete and only devoted to the contemplation of outside objects" [5]. The result should be to "give the illusion at all levels of education that he [the child] discovers himself the truth that has to be brought into his mind" [6].

Consequently, formalism and symbolism should never precede this stage. For example, the concept of numbers has to be integrated before the introduction of the corresponding notation. It is for this reason that the chapter about digits is not the first lesson (see also fig. 1). Mathematical proofs are therefore prohibited, the word "theorem" is not used instead the child has rather to "feel" things.

This *Initiation* is therefore based on the child's natural curiosity rather than the use of his memory. It attempts to reduce effort and routine. The use of games is actually the only route to learning and the educator has to stop a lesson if any boredom appears. This was why Laisant highlighted experiments such as those of the Swiss educationalist Johann Heinrich Pestalozzi (1746-1827) who was himself inspired by Jean-Jacques Rousseau and who also insisted on giving tangible form to ideas and on the grading of difficulties.

Finally, Laisant summed up:

> We will use entertaining questions as an educational way to attract the child's curiosity and thus effortlessly get to put into his mind the primary mathematical concepts. [7]

From a teaching point of view, Laisant referred to the book *L'Arithmétique du grand-papa* (Macé, 1862) written by Jean Macé (1815-1894), a book which was reissued several times. This tale presented the fundamentals of arithmetic (numeration, the four operations on integers, fractions and decimal fractions, metric system) through a dialogue into which tangible objects were used. Jean Macé was again one of Laisant's friends and founded the League for Education in 1867. The aim of this organisation was to support initiatives for the progress of popular education. However, it was while he taught in eastern France in the 1850's that he wrote several original works dealing with the teaching of history, anatomy and arithmetic.

Laisant referred to many other sources of inspiration (always from a teaching point of view). In addition to Édouard Lucas, whose case will be discussed later on, Laisant mentioned, concerning the teaching of geometry, a professor from the University of Dijon, Charles Méray (1835-1911). His book, *Nouveaux Éléments de géometrie* (Méray, 1874, reissued in 1903), was based on the experimental aspect of geometry. Méray also led several innovative experimentations into the upper primary schools in Dijon from 1876 to 1879.

Carlo Bourlet (1866-1913), another close collaborator of Laisant's and director of the journal *Nouvelles Annales de Mathématiques* like him, was also cited having also taken this experimental approach in his *Cours abrégé de géométrie* (Bourlet, 1906). Both books illustrated this view during the 1902 reform of secondary school in France (Bkouche, 1991). These aspects of the reform were no strangers to some rules in the *Initiation mathématique*.

SOME INNOVATIVE TEACHING TOOLS

Within the lessons of the *Initiation*, Laisant offered some specific principles for reaching the first mathematical fundamentals and even better for approaching more complex concepts such as magic squares, conicals or harmonic divisions.

The most well-used method is through handling actual objects (sticks, beans, matches …) in order to introduce the concept of counting (fig. 1, this section is very close to Macé's work) but also quickly the theory of negative quantities (with stems put in the two directions of the real line).

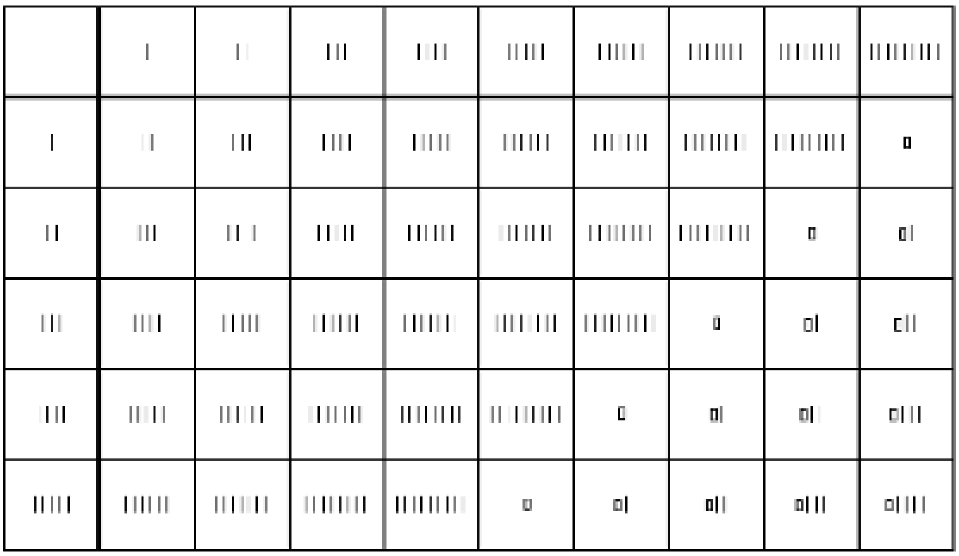

Figure 1: Introducing the addition [8]

The educator can also use graph paper in order to introduce "the concepts of shape, size and position" [9]. André Sainte-Laguë (1882-1950), a professor at the Conservatoire des arts et métiers, also developed these ideas. He was the author of an article "Note sur l'utilisation du papier millimétré" in *L'Enseignement mathématique* (Sainte-Laguë, 1910) where he revealed similar uses (for example, for the sum of odds). He was also the author of *Avec des nombres et des lignes. Récréations mathématiques* in 1937 (Sainte-Laguë, 2001) which took examples from the *Récréations mathématiques* (Lucas, 1882) written par Édouard Lucas in 1882.

Édouard Lucas (1842-1891) was a secondary school teacher and one of Laisant's closest friends. His work on number theory and arithmetic was well-known and influenced Laisant in the 1890's. They both worked on visual ways to present concepts from discrete mathematics. In his *Récréations mathématiques*, he explained to parents how they can help their children to understand mathematics using games and drawings. Some illustrations in the *Initiation* (fig. 3 and 4) are very similar to those proposed by Lucas.

It's also interesting to make a distinction between recreational games (Barbin, 2007) and the numerous visualisations used in the *Initiation*. In his foreword, Laisant differentiated his work on "initiation" and the games exposed in Émile Fourrey's *Curiosités géométriques* in 1907 (Fourrey, 2007) or Lucas' *L'Arithmétique amusante* (Lucas, 1895). Here, there is no theoretical knowledge needed, his goal was not to impart original theories. It is instead humorous and varied but fully ordered questions to be used as teaching tools.

However, the *Initiation mathématique* was linked to these mathematical games that exercised the mind and that were developed in the press at the time. They were both based on visualisation of mathematical processes at different levels. They took their origin from original research (see for example Lucas' *Théorie des nombres*, Lucas, 1891) and were discussed in new mathematical journals like *L'Intermédiaire des mathématiciens* or in the congresses of the AFAS, two places where Laisant was deeply involved and where exchanges fed his thinking. As evidence of this interest, he was also vice president of the Society of Recreational Sciences in 1894 with several of his collaborators (Delannoy, Arnous Rivière for examples).

Laisant also insisted on the importance of drawing and on the use of the compasses in order to learn geometry, believing that by exercising the hand, drawing can be a way to train the mind and to approach mathematical objects like curves. He offered that Carlo Bourlet's courses could be used to study geometry after the "initiation period".

One of the most important principles in the *Initiation* is the use of various visualisations. Below are shown numerous examples including tables for sums and products (with graph paper), visualisations of binomial expansions (fig. 2), of the sum of integers (fig. 3), of the sum of odds, of the sum of the square or the cube of integers or even visualisations of permutations based on Lucas' idea (fig . 4).

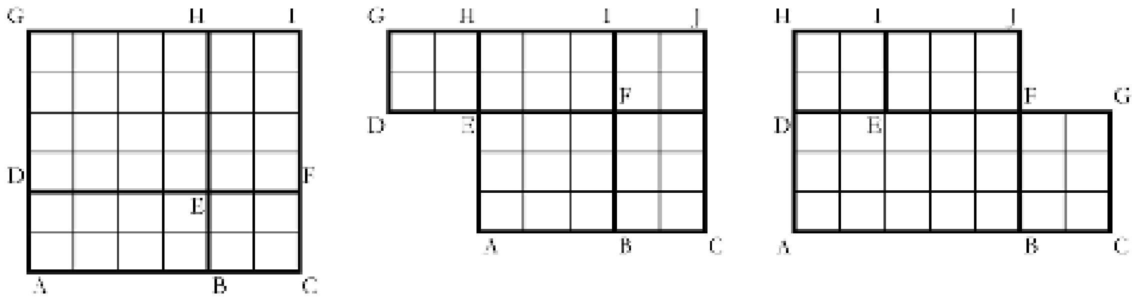

Figure 2: Visualisations of binomial expansions
(*a* + *b*) , (*a* – *b*) and (*a* + *b*)(*a* – *b*) [10]

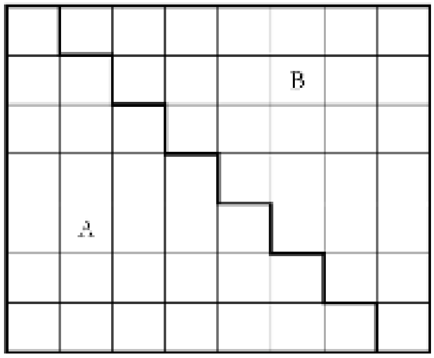

Figure 3: Visualization of the sum of integers [11] (see also Lucas, 1884)

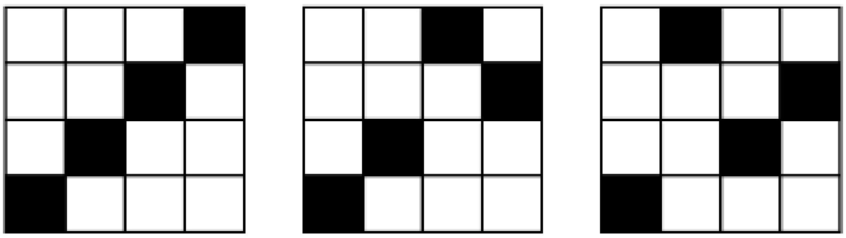

Figure 4: Visualisation of permutations *abcd*, *abdc*, *adbc* [12]

Most of all, the use of curves especially illustrates the concept of function. The first lesson about functions is even entitled "algebra without calculation". One of the corresponding exercises is from Lucas and the question, perhaps proposed after a congress of the AFAS, is:

> Each day, a boat leaves Le Havre to New York and another leaves New York to Le Havre. How many boats will encounter the first during the journey? [13]

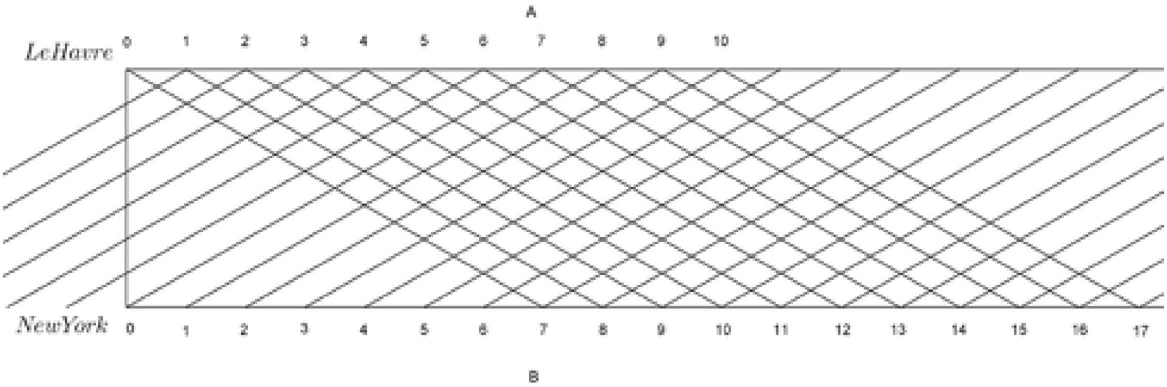

Figure 5: Solving Lucas' problem [14]

Laisant explained that, with his drawing (fig. 5), one can see the solution, without reasoning. He also used many graphics extracted from newspapers and magazines as a help for his "initiation". To sum up, he wanted to "educate by the eyes". A similar idea concluded an article about magical squares written in collaboration with a mathematics amateur, Gabriel Arnoux (1831-1913): "when one can simply say, "See," the proof approaches perfection. One could almost say that the art of exposing is to make diagrams" [15].

Educators may also play on pupils' natural curiosity. That is why Laisant computed the number of digits in 9 to the power 9 to the power 9; he was therefore the first to give the number j_9 = 369 693 100 of the Joyce's sequence. This surprising result was given with several modern remarks about computability and times of calculation in order to arouse the child's interest.

Laisant also used historical references. For example, he recounted the principle of multiplication by gelosia of which he even conceived a method of automating after an exchange with Henri Genaille during a congress of the AFAS. He presented also an ancient proof of Pythagoras' theorem (fig. 6, Fourrey also gave 24 historical proofs). Additionally, Laisant criticised Legendre's presentation in the "Euclidian style" which was very common in the nineteenth century.

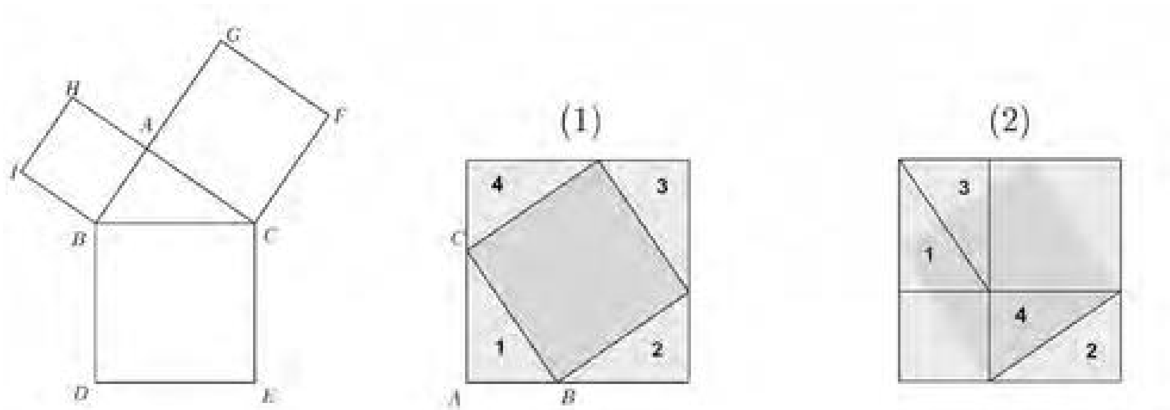

Figure 6: Proof of Pythagoras' theorem [16]

Finally, Laisant insisted on various visualisations of processes and on practical and continual examples to avoid any "artificial abstraction". Many of these themes were already effective in Laisant's previous handbook and even in his pure mathematical works (about equipollences for examples).

WITH THE *INITIATION*, A LAST INVOLVEMENT

With this small book, Laisant wished to establish a collection of similar *Initiations* about other scientific domains. This project implied that he engaged with a new community of authors who shared his points of view about young children's education. It proves also that Laisant always kept in mind a far larger vision of scientific education since his lecture on "L'Initiation à l'étude des sciences physiques" (Laisant, 1901). Camille Flammarion (1842-1925) of the Astronomical French Society like Laisant was the first to join him through the publication of an *Initiation Astronomique* (Flammarion, 1908). The *Collection des initiations scientifiques* included also principles on the teaching of chemistry (1909), mechanics (1909), zoology (1910), botany (1911) or physics (1913) with many reissues for each of them.

Furthermore, Laisant became quickly involved in a journal published from April 1908 to November 1909: *L'École rénovée* (Mole, 2011). This review was founded by the Spanish educationalist Francisco Ferrer (1859-1909), who also created a school in Barcelona named the "Modern School" and who quickly became Laisant's friend and close collaborator (Ferrer, 1962).

The journal contained some anarchists' ideas about education but was in the first place the organ of the International League for the Rational Education of Children, which Ferrer developed with Laisant's help. In this review whose complete title was "Review for developing a modern education plan", one could discuss the main themes from the movement called "the new education", that is to say: a sharp criticism of the traditional education, the children's specific psychology, the development of a liberating education for children and for the whole society, beyond the social origins of each of them (see Laisant's article, "The Purpose of Education" in 1909). These main themes were studied by many libertarian thinkers like Laisant who had arrived to these ideas since his political retreat (Lamandé, 2010).

The journal was a totally independent publication and enabled a broad debate between educational players and thinkers from several countries like the Dutch F. Domela-Nieuwenhuis, the Belgian J. F. Eslander, the Swiss S. E. W. Roorda van Eysinga and Pierre Kropotkine. Many teachers were also present and Laisant established many relationships with them. For Ferrer and for Laisant, the objective was to overthrow the school system and to free children from any dogma (all this is a point of view close to positivism: the child has to reach a rational state). This is also the final message of the *Initiation mathématique*. When Laisant wrote "the solution of the problem of education will determine the future of human society" [17], he highlighted a common interest for all the players of this heterogeneous movement called "the new education": the link between school evolution (in a pedagogical sense) and a social revolution. His final political convictions appeared in his article "The Purpose of Education" in which he described education as "the power to get out of their class [Children's social class], any class, and one day to get rid of the oppression of the ruling classes" [18].

After the execution of Ferrer which Laisant had always condemned, the journal stopped. A second journal was founded in October 1910: *L'École émancipée*. The editorial line was nearly the same as the former one but this new review was attached to the national federation of teachers unions. Many collaborators from *L'École rénovée* were still present in this project: Laisant published several articles between 1910 and 1914. He appeared thus as a militant linked to a larger community, including teachers unions, till the end of his life.

Laisant's desire to bring new guidance to educators about young children's mathematical training relied on new approaches to the first mathematical notions. Thanks to his work on visualisation in discrete mathematics, he succeeded in delivering various processes aimed at reducing off-putting formalism. His modern approach based on the use of graphs and diagrams was the result of many exchanges with mathematicians from various horizons. However the *Initiation mathématique* was also the work of an accomplished mathematician convinced by the positivism, driven by his thinking about the origin and the nature of the mathematics. He became one member of a community which included other famous mathematicians concerned with addressing the problem of education, at the beginning of the twentieth century. Poincaré, Klein or Borel (Borel, 2002) also presented their thoughts on mathematical training following the recent reforms. Experimentation is one of the themes studied by all of them (Gispert, 2013; d'Enfert, 2003).

The *Initiation mathématique* marked a new and last route for Laisant. He subsequently wrote two books entitled *La Barbarie moderne* (Laisant, 1912) and *L'Éducation de demain* (Laisant, 1913) within these he extended his ideas from childhood right through to adulthood. He dealt with a complete and popular education which is lifelong and necessary for anyone in a social sense. He always kept in mind this principle: "mathematical initiation is essential to every child, without distinction of wealth, social status, gender" [19].

NOTES

1. (Laisant, 1898), p. 187.

2. (Laisant, 1899), p. 358.

3. (Laisant, 1899), p. 366.

4. (Laisant, 1916), p. 6.

5. (Laisant, 1899), p. 360.

6. (Laisant, 1899), p. 362.

7. (Laisant, 1916), p. 7.

8. (Laisant, 1916), p. 18.

9. (Laisant, 1899), p. 365

10. (Laisant, 1916), p. 82-83.

First figure: (AB + BC) = DE + BC + AB.DE + EF.FI = AB + 2.AB.BC + BC , that is to say:

$(a + b) = a + 2ab + b$.

2nd figure: AB = area(ACJGDE) – 2.area(BCJI) $\Leftrightarrow$ (AC – BC) = AC + BC – 2.AC. BC, that is to say:

$(a - b) = a - 2ab + b$.

3rd figure: area(ACGD) = AC.AD = (AB + BC).(AB – BC) = area(ABJH) – area(DEIH) = AB – BC , that is to say:

$(a + b)(a - b) = a - b$.

11. (Laisant, 1916), p. 87.

12. (Laisant, 1916), p. 118.

13. (Laisant, 1916), p. 140.

14. (Laisant, 1916), p. 140.

15. (Laisant & Arnoux, 1900), p. 36.

16. (Laisant, 1916), p. 79.

17. As cited in Mole, 2011, p. 20.

18. As cited in Mole, 2011, p. 20.

19. (Laisant, 1916), p. 173.

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Oral Presentation

THE BELGIAN JOURNAL *MATHEMATICA & PAEDAGOGIA* (1953-1974): A FORUM FOR THE NATIONAL AND INTERNATIONAL SCENE IN MATHEMATICS EDUCATION

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In 1953 the Belgian Society of Mathematics Teachers was founded. The Society brought together a few hundred mathematics teachers from both linguistic communities (French and Dutch). It started its own professional journal Mathematica & Paedagogia (M&P). Willy Servais, the Society's first president, became the journal's figurehead. Servais was an open-minded, inspiring personality and an unconditional proponent of international exchange in mathematics education. In the 1950s M&P became an international forum of ideas, in particular for members of CIEAEM. In this paper we present some of the main themes discussed in M&P, with particular focus on the contributions by international scholars. Although the influence of the French structuralist mathematicians was clearly discernible, the pages of M&P were also open to other scholars.

INTRODUCTION

When in 1953, the *Belgian Society of Mathematics Teachers* was founded, it immediately started its own professional journal *Mathematica & Paedagogia* (*M&P*) (Miewis, 2003). From the beginning, the journal was bilingual, with articles in French and in Dutch, serving both linguistic communities in Belgium, although in its early years, most contributions were written in French. The foundation of the *Belgian Society of Mathematics Teachers* is closely related to the activities of CIEAEM, the *International Commission for the Study and Improvement of Mathematics Teaching*. *M&P* ceased publication in 1974 when the Belgian Society was restructured on a linguistic basis into a Flemish and a French-speaking Society of mathematics teachers.

During the fifties and sixties, the main personality of the *Belgian Society of Mathematics Teachers* and its journal *M&P* was Willy Servais (1913-1979) (Vanpaemel, De Bock, & Verschaffel, 2012). Servais was one of the founding members of CIEAEM (Bernet & Jaquet, 1998) and he was quickly convinced of the need to ensure a structural relation between the work of the CIEAEM and the community of Belgian mathematics teachers. To that purpose, Servais took the initiative to found the *Belgian Society of Mathematics Teachers*. He became its first President, a position which he held until 1969. He also served on the editorial board of *M&P* and became the journal's figurehead. Servais was an open-minded, inspiring personality and an unconditional proponent of international exchange in mathematics education. Already in his first *Editorial* for *M&P*, Servais held a strong plea for international cooperation. He wrote: “Mathematics as a truly universal language, has, by its nature, an international vocation; we will open our columns to our colleagues of other countries" (Servais, 1953, p. 4). Besides, the establishment of relations with foreign associations of mathematics teachers, as well as with other international organizations sharing similar goals, were formally included as important objectives of the new Society.

Servais' plea was received favourably by the international math and mathematics education community of that time. Several famous authors submitted contributions. The Belgian journal rapidly became a forum for national and international exchange in mathematics education. Before discussing some key contributions by international scholars and entering the debates of that time, we first explain how the content of *M&P* was structured.

STRUCTURE OF *MATHEMATICA & PAEDAGOGIA*

The first 26 issues of *M&P*, covering the period from 1953 to 1964, had a recurrent structure with clearly distinguishable sections. The journal opened with an *Editorial*, in most cases written by Willy Servais himself. In these Editorials, Servais reflected on pedagogical and other issues, such as the initiatives of the Society or political decisions that had an impact on the teaching of school mathematics.

The first section was devoted to *"Culture mathématique"* (*Mathematical culture*). Main goal of that section was to inform the readership about new developments in mathematics. Most of the articles in that section were written by French and Belgian university professors in (pure) mathematics, including renowned scholars like Jean Dieudonné, Henri Cartan, Gustave Choquet, Paul Libois and Georges Papy. Needless to say that the axiomatic and structural approach, dominant in the French mathematical culture of that time, was reflected in most of these contributions. Because mathematics education is our main point of interest, we will not discuss in depth the purely mathematical articles that appeared in that section.

More interesting in the context of this paper is the second section entitled *Knowledge of the pupils*. In a letter, included in the first issue of *M&P*, Caleb Gattegno stated that *M&P* was the first mathematical journal in the world that reserved some pages to pedagogical research (Gattegno, 1953). Indeed, for Gattegno, Servais, and other CIEAEM members of that time, it was clear that successful teaching of mathematics not only depended on the teacher's expertise in mathematics, but also on his or her knowledge about pupils' development and thinking. In this section of the journal, one can find reports of experimental lessons focusing on pupils' behavior, their correct and incorrect ways of thinking, interpretations of their reasoning, contributions on, what we would call nowadays the "psychology of mathematics education". However, in contrast with contemporary approaches, these contributions, mostly written by mathematics teachers, did not follow a strict experimental methodology. Pupils were observed in the context of real lessons and their individual or collective reactions were described and interpreted.

The third section of *M&P* was called *Instruction* and provided resources for teaching school mathematics (topics for the mathematics classroom, successful didactical approaches, and helpful models and devices). It was the journal's central section, typically covering the majority of its articles. In most *M&P* issues, a fourth section was devoted to *Applications of mathematics* in other sciences. In the context of this paper, however, the subsequent section *Contacts* deserves more attention. It presented the Society's international network of math educators, including Gustave Choquet, Lucienne Félix and André Fouché in France, Caleb Gattegno and Trevor Fletcher in the UK, Félix Fiala and Jean Louis Nicolet in Switzerland, Emma Castelnuovo in Italy, and Friedrich Drenckhahn in Germany. Most of them were early CIEAEM members, Servais' main international network. In *M&P* 5, new contacts with the Dutch associations of mathematics teachers and with the *National Council of Teachers of Mathematics* (US) were announced and their respective journals were introduced to the Belgian community of mathematics teachers. In the *Contacts* section, one can also find reports of eight early CIEAEM meetings (CIEAEM 6, 7, 8, 9, 10, 11, 17 and 18). Because there exist no published Proceedings of these meetings, these reports are one of the rare sources that can give us some insight in how CIEAEM functioned at that time and in the topics being discussed at these meetings.

In the section *Books and journals* mainly international publications in the field were reviewed. In the section *Questions and problems*, problems of official (entrance) exams in Belgium, but occasionally also from other countries, were proposed. Finally, there was an *Administration* section that informed the readership about the Society and its initiatives.

INTERNATIONAL SPOTLIGHTS ON NEW TEACHING AIDS

Several articles by internationally renowned scholars in the field of mathematics education were published in *M&P*. Probably, they were invited to submit a paper by Servais who knew most of them personally from CIEAEM meetings. Until the late 1960s, *M&P* and similar professional journals in other countries, were one of the rare channels to publish work in this field. *L'Enseignement Mathématique*, for a long time the only international journal on mathematical instruction, had become at that time a purely mathematical journal (see Furinghetti, 2009) and *Educational Studies in Mathematics* was only founded in 1968.

A key contributor in the 1950s was Caleb Gattegno (1911-1988), at that time working at the University of London. In *M&P* 3, Gattegno explained his pedagogical approach towards mathematics education in an article on intuition, elaborating on his well-known "pedagogy of situations" (Gattegno, 1954a). According to Gattegno, pedagogical principles should be based on the observation of human learning in many and varied situations. How a student adequately restructures a situation is not a simple response to a stimulus, but a complex interplay of different factors, including affective ones. To find guiding pedagogical principles for the improvement of education, classes should be reshaped into real educational laboratories.

An important topic for math educators in the 1950's was the use of teaching aids. From the very beginning Gattegno showed himself a virulent advocate of the Cuisenaire rods, a set of colored sticks of different lengths that can be used as a didactical tool to discover and to explain various mathematical concepts and their properties. This teaching aid was invented by the Belgian primary school teacher Georges Cuisenaire (1891-1975) and Gattegno became their worldwide ambassador (Gattegno, 1988). In an article that appeared in *M&P* 4, entitled "Colored numbers", Gattegno described the Cuisenaire rods in a lyrical, fairy-tale style:

> Once, there was a primary school teacher (…), who loved his pupils so much that he asked himself what he should do to make the compulsory study of arithmetic look easy to them and give them joy. Where would he find the answer to his question? To consult mathematicians is useless. They do not understand the difficulties children are faced with. Similarly, it does not seem that the aid of psychologists will help us more because their knowledge about what a child can do is much separated from the educational system which determines the child (…). The land, in truth, was a virgin and was missing a brand new idea that would shed new light on the problem. Georges Cuisenaire, primary school teacher in Thuin, did find that idea in the art of music he was always practicing. (Gattegno, 1954b, p. 17)

Gattegno considered the invention by Cuisenaire as the most important contribution in the efforts to find solutions for the problems faced in arithmetic education and he illustrated the rods' various applications. Then, he showed how the rods can also be used in other domains, such as algebra, measurement and geometry. Gattegno considered the rods as a symbiosis between mathematics, educational technology and educational science. From a mathematical point of view, they put the spotlight on relationships and structures. From a psychological point of view, they stimulate intuition and facilitate discoveries. Many other articles in *M&P* came back to the use of Cuisenaire's "colored numbers". These articles often reported about experimental lessons as presented and discussed at national and international conferences during the 1950s.

Teaching aids and models, such as the Cuisenaire rods, were a central topic in the 1950's debates. These didactical tools were basically seen as bridges between intuition and abstraction. In the opening session of CIEAEM 11, summarized in *M&P* 12, the Spanish mathematician Pedro Puig Adam (1900-1960) described their role in mathematics education as follows:

> These tools (…) will not be considered as a set of simple concrete illustrations, as appropriate clothing, to facilitate momentarily an uneasy understanding. For the educator who does not forget the perspective and initial processes of abstraction, these tools are much more, they represent something substantial in their educational function. These tools, structured in the form of models, do not only have the objective to occasionally translate mathematical ideas, but also to suggest them and being at their origin. (…) The old model of a showcase, to be passively completed by the pupils, should make room for multivalent newly designed tools, tools that can be manipulated by the pupils and that in

the meantime induce an activity that creates the knowledge they have to acquire. (Puig Adam, 1957, p. 64)

Another tool, invented by Gattegno for geometry education and discussed from various perspectives, was the geoboard (Vanhamme, 1955). Basically, a geoboard is a wooden plank in the form of a square. The geoboard is subdivided in a network of equal squares, in the centers of which are planted nails. With elastic bracelets, preferably in different colors, one can represent segments, lines, angles and various polygons and discover or illustrate basic geometrical properties. In *M&P* 15, Pedro Puig Adam exemplified the use of the geoboard by a proof of Pick's theorem, a simple formula for calculating the area of a polygon in terms of the number of nails located in the interior and the number of nails on the boundary of that polygon (Puig Adam, 1958). In quite a long article published in *M&P* 19, Gattegno himself explains how individual geoboards of different sizes can be used in basic and more advanced lessons on geometry (Gattegno, 1960).

Another teaching aid that emerged at the math educational scene of the 1950s was the mathematical film. The basic idea was again that intuition should precede logic and proof. Jean Louis Nicolet, a pioneer in the domain of mathematical films, summarized his philosophy quite concisely: "*Logic proves, but does not convince, intuition convinces, but doesn't prove*" (Nicolet, 1954, p. 24). The films by Nicolet, entitled "*animated geometry*", were short, silent hand-animated films presenting simple geometrical situations, but provoking reflection (Gattegno, 2007). Also math educators from other countries produced mathematical films, e.g. Lucien Motard in France and Trevor Fletcher in the UK. Nicolet and Fletcher were active members of CIEAM in the 1950s as well, so it may not surprise that their ideas were disseminated through *M&P* and that their films were already projected and discussed at the Society's first conference in 1954. Fletcher's films illustrated properties of geometrical curves, such as epicyloids and hypocyloids, topics typically not belonging to mathematical programs in the UK at that time. Fletcher showed himself quite ambitious about the potentials of the new medium:

> It is not only a matter of producing films that illustrate the mathematics as it is taught today. By making films, we will create new mathematics, and if the films are of a sufficient quality, they will change the mathematics that will be taught in the future. (Fletcher, 1954, p. 29.

During the 1950s, many other articles referred to new teaching aids in mathematics, several of them addressing the role of the upcoming computers at that time. Because the international interaction in the computer debates was limited, we will not discuss these contributions here.

MODERN MATHEMATICS

At the end of the 1950s and in the 1960s, several articles in *M&P* dealt with new content for school mathematics – sets, relations, logic and structures – and corresponding teaching methods, e.g. Venn and arrow diagrams. As Servais himself was much involved in the international movement towards the reform of mathematics education, the new math became a central focus of the journal. Most of these articles were written by Belgian scholars, with Georges Papy (1920-2011) as their uncontested leader. A main theme was related to the development of new (experimental) programs for the subsequent years in which modern mathematics would be introduced (see, e.g., Papy, 1961, 1962, 1966).

The attention giving in *M&P* to the activities of CIEAEM documents the evolution of the Commission in the years preceding the major reforms in the European countries. The report on CIEAEM 17 (Digne, France, August 1963), convened for the first time under the presidency of Papy (Nachtergaele, 1964), showed a clear shift in the Commission's tradition. The aim of the meeting was "*A reconstruction of mathematics teaching for age 10 to 18*". Under the strong leadership of Papy and his French fellow-thinker André Revuz (1914-2008) the Commission agreed upon a concrete list of topics that pupils should have acquired by the age of 16 (published in *M&P* 25, pp. 86-89, as an annex to Nachtergaele's report) (Bernet & Jaquet, 1998).

But the unanimity within the CIEAEM community was less strong than it initially looked... Already in *M&P* 28 one could become witness of a vehement disagreement between Dieudonné and Choquet, two founding members of CIEAEM (Bernet & Jaquet, 1998), about the most adequate axiom system to be chosen for the teaching of geometry at the secondary level (Revuz, 1965). Dieudonné approached the "Euclidian structure" as a "professional mathematician". Central in this approach were the axioms of a vector space of finite dimension over the field of the real numbers, equipped with a scalar product (Dieudonné, 1964). In contrast, Choquet showed more awareness of the pupils' pedagogical evolution and built an axiom system that tried to reconcile mathematical rigor with the pupils' geometrical intuition (Choquet, 1964). Dieudonné launched a violent attack on this more realistic system proposed by Choquet. In the *Introduction* to his book *Algèbre linéaire et géométrie élémentaire*, Dieudonné argued that Choquet's system demonstrated "*a remarkable ingenuity which shows the great talent of its author, but that he considered as completely useless and even harmful*" (Dieudonné, 1964, p. 17). Revuz tried to reconcile Dieudonné's and Choquet's points of view. He defended Choquet's system as an "intermediate step" between pupils' intuition and the "good" (linear algebra based) system proposed by Dieudonné.

> However, if one believes that geometry is not only a mathematical theory, but also a physical theory, if one thinks that the role of education is not only to know mathematics, but also to learn to mathematize reality, one can think about Choquet's system as an intermediate step, which will not only allow teachers to change their mentality, but perhaps also will enable any student to move easily from the intuitive space to the mathematical theory. (Revuz, 1965, p. 76)

On the initiative of Revuz, the disagreement between Dieudonné and Choquet was officially settled at CIEAEM 19 (Ravenna, Italy, April 1965) with a motion about the role of geometry in the education of 10-18-year old pupils, agreed by all CIEAEM members present (but in the absence of the two disputants). In this motion, also referred to as the "Convention of Ravenna" (Félix, 1985), the special place of geometry in mathematics education was recognized. More concretely, an approach in two stages, inspired by Papy's experiments at the *Centre Belge de Pédagogie de la Mathématique*, was recommended. The Convention was proposed to and solemnly signed by Dieudonné and Choquet at a Seminar organized by the *International Commission on Mathematical Instruction* (ICMI) in Echternach (June, 1965). The "Treaty of Echternach", i.e. the Ravenna Convention with the addition of Dieudonné's and Choquet's signatures, was fully published in *M&P* 28 (pp. 78-81).

Among the rare articles in *M&P* by non-Belgian math educators that were openly in line with the new math approach, we mention a quite long article by the German math educator Hans Georg Steiner (*M&P* 30) on the introduction of the group concept and on computation in groups for 13-14-year olds (Steiner, 1966). Starting with numerous examples of magmas, i.e. sets equipped with a closed binary operation, both in algebraic and geometric contexts, Steiner shows how pupils can work with groups at an elementary level, but meantime opening paths to more general aspects of group theory.

DIFFERENT VOICES: BUNT, FREUDENTHAL AND KRYGOWSKA

Although the new math steam was dominant in the debates that were voiced in *M&P* during the late 1950s and early 1960s, some significant non-Belgian contributors offered a counterbalance. We end this paper by discussing in some detail the contributions by Luke N. H. Bunt and Hans Freudenthal (the Netherlands) and Anna Zofia Krygowska (Poland).

Bunt (1905-1984) was a pioneer in the field of statistics education in the Netherlands. Since 1951 he coordinated a project of the Pedagogical Institute of the University of Utrecht on the teaching of probability and statistics at secondary school departments that prepared pupils for further studies in the social sciences (Zwaneveld, 2000). He was also the author of a textbook on statistics (Bunt, 1956) that was widely used in the Netherlands until the mid-1970s. At the international level, Bunt was a respected scholar too. He was one of the founding members of CIEAEM (Bernet & Jaquet, 1998) and played a main role at the famous Royaumont Seminar (1959), both as an invited speaker and as a co-editor of its proceedings *New Thinking in School Mathematics* (OEEC, 1961), the official report of the Seminar (De Bock & Vanpaemel, 2015).

In *M&P* 17 Bunt reported on the course on probability and statistics he developed with a team of six mathematics teachers and with which he experimented in the alpha streams of Dutch secondary schools (Bunt, 1959). Bunt's didactical approach was rather classical, in sharp contrast with the new math philosophy. First, as mentioned, his target group was different. While most new math protagonists focused on those mathematically gifted students that would become mathematicians or engineers, Bunt aimed at future students in economics, psychology and other social sciences. But he argued that not only these students would profit from such an introductory course, but *all* students because, in the end, every citizen will come in contact with statistical concepts and methods. A second feature of Bunt's approach that contrasted with new math approaches, is his "pragmatism". Bunt deliberately started with *provisional* definitions, definitions that are *not* completely correct from a scientific point of view. E.g., he would at first define the probability of an event as the ratio between the number of favorable and the total number of possibilities. Based on that definition, he would then prove the main calculation rules for probabilities. Later on in his course, when the need arose, Bunt would present a new definition, covering more situations, based on the limit of relative frequencies, and, without further explanation, he would state that "*for probabilities based on this new definition, the previously proven calculation rules remain valid*" (p. 38). As a consequence of his pragmatism, Bunt was able to arrive in a limited number of lessons at the basic ideas of hypotheses testing, assessing the characteristics of a population on the basis of a sample.

The Dutch mathematician and mathematics educator Hans Freudenthal (1905-1990) needs little or no further introduction. Freudenthal was on good personal and professional terms with Servais, although their ideas sometimes diverged (La Bastide-Van Gemert, 2006). In 1958, in the margin of the World Exhibition in Brussels, Freudenthal was invited to give a lecture at the Society's conference. The topic of his talk was the human responsibility of the mathematician (Freudenthal, 1958). The general tenor of Freudenthal's lecture was philosophical rather than practical. Freudenthal stated that the responsibility of the mathematician goes further than creating and transmitting formulas that others can apply. In his view, there is something he called a "mathematical mind".

> … there is a mathematical mind that – I'm convinced of it – will not only determine the character of our relations with the physical world and with the machines we construct, but also our human relations – individual, international and interracial. Rationalization of these relations is a mission we have to fulfill in the remaining years of the XXth century. The rationalization of something is a mathematician-specific activity or at least an activity of the mathematical mind. (Freudenthal, 1958, p. 40)

According to Freudenthal, mathematicians are educators of humanity, even if they dedicate themselves to the most abstract mathematics. They work for a future in which reason will be the regulator of human relations and through their work, they fulfill their human responsibility. Although Freudenthal did not talk about contemporary problems in math education, his invitation at a Society's conference and his collaboration to *M&P* reflects an openness among Belgian mathematics teachers to the ideas of other mathematicians than those belonging to the French structuralist school.

Also Anna Zofia Krygowska (1904-1988), eminent teacher, teacher trainer and scientist in mathematics education, belonged to Servais' network within CIEAEM. Krygowska became active in CIEAEM during the 1950s and soon developed into one of its driving forces. Krygowska became the Commission's vice-president under Papy's presidency (1963-1970) and, in delicate circumstances, after Papy had left the Commission in 1970, she accepted to become president (until 1974). It was the beginning of a new period in the history of CIEAEM in which also Freudenthal assumed a more prominent role and in which "problem-driven education" became the central theme of discussion (Bernet & Jaquet, 1998).

Krygowska wrote no less than four articles for *M&P*, articles that are quite long and thoroughly elaborated. In her first article, published in the section *Knowledge of the pupils*, Krygowska warned for the dangers of formalism and verbalism in the teaching of algebra (Krygowska, 1957). She discussed a large number of systematic errors made by pupils which were not yet an object of systematic study and reflection at that time. Krygowska tried to understand these errors and to unravel the underlying mechanisms. In a second article, Krygowska pointed to a number of misunderstandings in pupils' thinking due to the tension between the formal definition of a geometrical concept and pupils' intuition about that concept, related to its representation and often resulting from a long evolution (Krygowska, 1959). In a third article, Krygowska intervened in the debate about the place of geometry in a unified mathematical framework. Instead of abandoning geometry as an autonomous mathematical discipline, she saw geometry as one of the ways to arrive at a unified mathematics (Krygowska, 1962). Finally, Krygowska held a plea for the need of a strong pedagogical concept for the reform of mathematics education. She stated: *The pedagogical concept of "mathematics for all", adequate to the role of mathematics for integrating the world of today and tomorrow, is still in its infancy*" (Krygowska, 1964, p. 39).

CONCLUSION

During the 1950s and 1960s, the *Belgian Society of Mathematics Teachers* and its journal *M&P* flourished. Due to Servais' dynamism and network within CIEAEM, major scholars of that time, coming from different European countries, contributed to *M&P* and used the journal to express their views about how mathematics teaching and learning could be improved. This paper has demonstrated that *M&P* is a rare and important source for the study of the European history of mathematics education during the 1950s and 1960s, a period in which only few professional journals on math education were available.

NOTE

All translations were made by the authors.

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Oral Presentation

DECIPHERING THE DOODLINGS OF THE "SHOEBOX COLLECTION" OF THE PAUL A.M. DIRAC PAPERS

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Florida State University Libraries Special Collections and Archives hold the complete papers of Paul Adrien Maurice Dirac (1902-1984). In this workshop we shared an overview of the digital scans of 72 documents initially identified for study as part of a project to digitize and preserve the "shoebox papers" in the collection. In particular, we discussed the process of making sense of mathematical problems found on a variety of repurposed paper. During the workshop we focused on the reconstruction of two investigations: (1) work related to solving polynomial equations of degree n = 3, 4, 5, and 6, and (2) a selection of combinatoric problems. Finally, we sought to connect Dirac's doodlings to well-known sources from the history of mathematics and their treatment of similar problems.

INTRODUCTION

Paul Adrien Maurice Dirac was a Professor of Physics at Florida State University from 1972 until his death in 1984 (although different resources report different "start dates"). Among other discoveries, he formulated the famous Dirac equation, which describes the behavior of fermions, and he predicted the existence of antimatter. Dirac shared the Nobel Prize in Physics for 1933 with Erwin Schrödinger, "for the discovery of new productive forms of atomic theory." Dirac is buried in Roselawn Cemetery, in Tallahassee, Florida: "It was his family's wish that he should rest where he left the world" (Pais, 2005, p. 28).

Whereas those familiar with Dirac's work would readily connect him to his famous equation or his shared Nobel Prize with Schrödinger (to whom he referred as "Schröd" in the "shoebox papers"), we approached the collection at FSU with an eye on the lookout for mathematical investigations. The initial motivation to investigate the collection in this way was that Dirac was "wooed…to do a full mathematics degree free of charge" at Cambridge beginning in September 1921. Furthermore, we believed the time period of the identified doodlings is between 1921 and 1923, though due to the scrap paper nature of the papers (that is, Dirac often repurposed paper) used in our study, almost none of the pages are dated.

This paper is organized in the following manner. First, we present a brief biographical sketch of Paul Dirac. Next, we discuss the acquisition of the Dirac Papers at Florida State University (FSU) and describe the current efforts of the FSU Digital Library (FSUDL). Related to this work, we also describe our contribution of identifying sets of related pages for digitization and entry into the FSUDL. Finally, we present several examples of pages from the "shoebox collection" that fall within two broad mathematical investigations: solving polynomial equations and combinatorics.

BRIEF BIOGRAPHICAL SKETCH: PAUL A. M. DIRAC

Paul Adrien Maurice Dirac was born on 8 August 1902 in Bristol, United Kingdom, to Charles Adrien Ladislas Dirac (a Swiss-born French teacher) and Florence Hannah Holten (a library clerk). From 1908 to 1918, Dirac attended Bishop Road Elementary School and the Merchant Venturers' Technical College, which was a secondary school where Dirac's father taught. In 1918 Dirac went on to study engineering at the University of Bristol, and finished a degree there in 1921. Although admitted to study at St. John's College at Cambridge, Dirac was unable to due to lack of sufficient funds to live on. Consequently, he stayed on to study mathematics at the University of Bristol, and he completed a first class honours degree in 1923.

From 1923 to 1926 he studied physics at St. John's College, under the direction of Ralph A. Fowler. While there, he published his dissertation (in Quantum Mechanics) and several technical papers. He also travelled to notable European centres where physics was being investigated, including Copenhagen and Göttingen. Dirac enjoyed a prolific career at Cambridge. In the years 1927 to 1933 he progressed rapidly, being named to numerous research positions. He was elected Fellow at St. John's College of Cambridge University in 1927, Praelector in Mathematical Physics in 1929, Fellow of the Royal Society in 1930, and Lucasian Professor of Mathematics in 1932. In 1933 he was awarded the Nobel Prize for Physics, which he shared with Erwin Schrödinger.

Dirac remained at Cambridge until 1969 (when he reached mandatory retirement age in England). However, from 1933 until 1969 he travelled widely in order to continue his research. Dirac frequently taught and lectured in Europe, Asia, and North America and he held short-term lecturing positions at major universities until his retirement from Cambridge in 1969.

Paul Dirac married Margit ("Manci") Wigner in 1937. Dirac met Margit when she visited her brother, the chemical engineer (and mathematician and physicist) Eugene P. Wigner, at Princeton's Institute for Advanced Studies. Dirac adopted Margit's two children (Judith and Gabriel) from her previous marriage, and they also had two daughters, Mary Elizabeth and Florence Monica.

In 1971, Dirac accepted a physics faculty appointment at Florida State University and the family moved to Tallahassee. Dirac continued to do research and travel until almost the end of his life. He died in Tallahassee on 20 October 1984, and was buried there in Roselawn Cemetary. Margit died in 2002 and was laid to rest with her husband. On 13 November 1995, "a plaque was dedicated in Westminster Abbey commemorating Paul Dirac" (Goddard, 2005, p. ix). Stephen Hawking, who delivered the Dirac Memorial Address in 1995 stated that:

> Dirac has done more than anyone in this century, with the exception of Einstein, to advance physics and change our picture of the universe. He is surely worthy of the memorial in Westminster Abbey. It is just a scandal that it [took] so long. (Hawking, 1995, p. xv)

ACQUISITION OF THE DIRAC PAPERS

The Paul A. M. Dirac Papers are housed in the Special Collections and Archives Division of Florida State University (FSU). The physical collection resides in the Robert Manning Strozier Library on the FSU campus. However, efforts are underway to digitize the full collection of the Dirac Papers. (A brief description of these efforts is given in the next section.)

The Special Collections and Archives Division (SC&A) at FSU acquired the Dirac Papers over a period of 29 years. The first portion of the papers, which were mostly Dirac's work papers and artefacts, were given to SC&A in 1985 by Dirac's widow, Margit ("Manci") Dirac. Further significant portions were given in 1992 and 1997. Finally, between 2012 and 2014, Monica Dirac (the Dirac's only living child) donated the remainder of Dirac's papers – which were mostly of a personal nature – to SC&A.

The Dirac "Papers" are extensive in that there are professional papers, journal articles and their drafts, books, calculation books, lab books, and school work from Dirac's student days, photographs, recordings, awards, and so on. At one point an archivist stated that there were over 190 linear feet of materials, but storage of the collection is currently undergoing modification and this number may no longer be accurate (after September 2014) due to the methods in which the collection is stored.

FSU DIGITAL LIBRARY (FSUDL) EFFORTS

Previous Efforts

The initial work toward processing, digitizing, and preserving the Dirac Papers began in 1999 when SP&A applied for and received a processing grant. This work led to a digitization grant (also in the late 1990s). Unfortunately, the early efforts did not produce high-quality records that documented the work; at the same time, there was a high turnover of administrative personnel within the FSU libraries that also contributed to the lack of sufficient documentation of processing and digitization efforts. The preservation work that is taking place now, which includes reviewing, reordering, and re-boxing materials in the Dirac Papers collection, is somewhat complicated, since idiosyncrasies resulted from previous efforts. Additionally, server failures after the early digitization of some of the materials in the Dirac Collection meant that any previously scanned materials were lost and needed to be digitized again.

Current Efforts

In February 2014, FSU launched the new Florida State University Digital Library, which "provides online access to Florida State University's rich and unique historical collections of photos, pamphlets, maps, manuscripts, and rare books" (Florida State University Libraries, 2014). The digital library uses an open source platform available state-wide and the primary aim of the FSUDL is to provide a central database for FSU's unique digital materials, of which the Dirac Papers is a centrepiece. Since the goal is to facilitate collaboration on mutual use of the Dirac Papers, an important goal of the workshop we conducted at the Seventh European Summer University (ESU-7) was to introduce participants to the collection as a whole and to the example of how we are using the materials that will become part of the FSUDL presence of the Dirac Papers.

"SHOEBOX COLLECTION" EXAMPLES

A portion of the Dirac Papers was originally identified as part of a "shoebox collection," presumably because they arrived at SC&A in shoebox-type containers (or, actually shoeboxes!) from either the Dirac home or Dirac's FSU office. We approached the overall Dirac Papers with an eye on the lookout for mathematical investigations. The initial motivation to investigate the collection in this way was that Dirac was "wooed…back to the lecture theatres in the mathematics department [at Bristol]…to do a full mathematics degree free of charge" at Cambridge beginning in September 1921 (Farmelo, 2009, p.47). This investigation began with the first author reading entries in the former finding aid for SC&A, for clues of which document folders to search. Series 1 ("Family Papers, Student Papers, and Photographs") held the most promise for finding items that we were interested in. According to the former finding aid, Series 1 included "materials associated with [Dirac's] student days at the Bishop Road primary school, the Merchant Venturers' secondary school, the University of Bristol (1918-1921), and at St. John's College at Cambridge University (from 1921 through the awarding of his doctorate in 1926)." Of particular interest were the boxes (and corresponding folders) with dates from 1921 until 1926, as items focused on pure mathematics were more likely to appear during these years because of the influence of the course work that Dirac took when he first arrived at Cambridge. The materials in the following folders were searched first:

Box 10

Folder 6: Merchant Venturers' School (Later, The Cotham School), 1918-1931(?); Physical Description: 4 items

Folder 9: Merchant Venturers' Technical College, Bristol (Calculations: Apparently Text Exercises), 1914-?; Physical Description: 171 items

Box 11

Folder 4: Exercise Book: 'Functions of a Complex Variable'. (Inserted second title: 'Surface Waves'), 1920(?) (21 miscellaneous leaves inserted, about 40 pages); Physical Description: 1 item

Box 12

Folder 1: Calculations, 1920(?); (62 leaves); Physical Description: 5 items

Folder 2: Calculations. Booklet from 3rd year Electrical Technology, 25 Oct 1920 - 8 Feb 1921; others, 1920(?); Physical Description: 2 items

Folder 3: Calculations. Includes Drawings, 1920(?); (38 leaves); Physical Description: 26 items

Folder 4: Calculations. No Context, 1920(?); Physical Description: 10 items

Folder 5: Calculations. School Related Items, 1909-1933; Physical Description: 25 items

In the workshop, we shared an overview of the digital scans of 72 documents initially identified from the folders described above, which was part of the project to digitize and preserve the "shoebox papers" in the collection. The examples discussed in our workshop came from Box 12 of Series 1, and are part of the reconstruction of two investigations found in what we called Dirac's "doodlings" (that is, mathematics problems that seemed to have been worked out on whatever paper Dirac could find): (1) a systematic approach for solving polynomial equations of degree $n = 3, 4, 5$, and 6, and (2) a selection of combinatoric problems, for which we are currently trying to determine the underlying purpose.

First Investigation: Solving Polynomial Equations

The first example presented to participants is the first example that appears on document 3 of item 1e of folder 12, or document "FSUDIRAC_12_1e_0003" [1] (Figure 1).

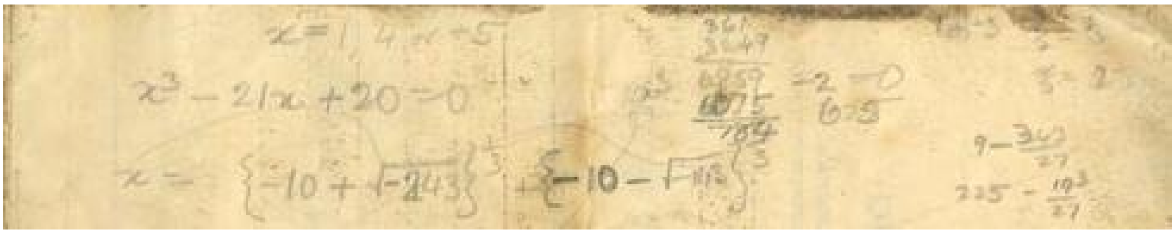

Figure 1: First example found on document FSUDIRAC_12_1e_0003.

In the exercise, it appears that Dirac first lists the three solutions of a desired cubic equation. Then, with no intervening mathematical work, he produces the equation $x^3 - 21x + 20 = 0$. (For the purposes of this paper, we will refer to this as Eq. 1.) Finally, the solution that appears is in the form one may obtain if Cardano's "cubic formula" was employed. That is, for the equation, $x^3 + qx + r = 0$, one solution is given by:

$$= \left(-\frac{r}{2} + \sqrt{\left(-\frac{r}{2}\right)^2 + \left(\frac{q}{3}\right)^3}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \sqrt{\left(-\frac{r}{2}\right)^2 + \left(\frac{q}{3}\right)^3}\right)^{\frac{1}{3}}.$$

Thus, for the equation $x^3 - 21x + 20 = 0$, this solution process would yield:

$$- \left(-\frac{20}{2} + \sqrt{\left(-\frac{20}{2}\right)^2 + \left(\frac{-21}{3}\right)^3}\right)^{\frac{1}{3}} + \left(-\frac{20}{2} - \sqrt{\left(-\frac{20}{2}\right)^2 - \left(\frac{-21}{3}\right)^3}\right)^{\frac{1}{3}}$$

$$x = \left(-10+\sqrt{(-10)^2+(-7)^3}\right)^{\frac{1}{3}} + \left(-10-\sqrt{(-10)^2+(-7)^3}\right)^{\frac{1}{3}}$$

$$x = \left(-10+\sqrt{100+(-343)}\right)^{\frac{1}{3}} + \left(-10-\sqrt{100+(-343)}\right)^{\frac{1}{3}}$$

$$x = \left(-10+\sqrt{-243}\right)^{\frac{1}{3}} + \left(-10-\sqrt{-243}\right)^{\frac{1}{3}}.$$

The final value is indeed the form of the solution given on FSUDIRAC_12_1e_0003. The remainder of the work associated with solving Eq. 1 appears to be an incomplete derivation of the remaining two roots to the equation (Figure 2).

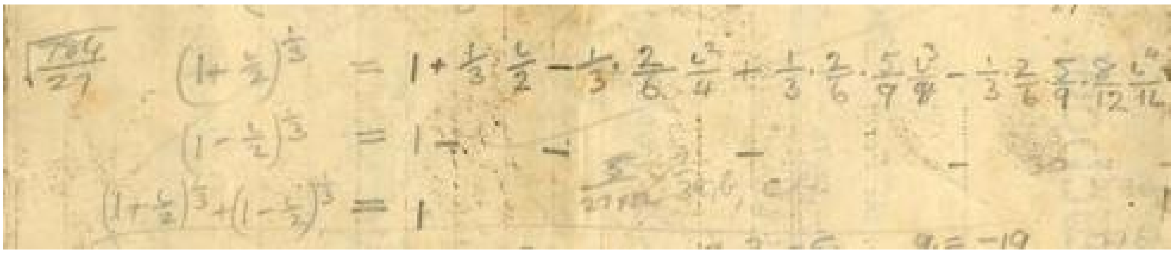

Figure 2: Determining the additional roots to Eq. 1 of FSUDIRAC_12_1e_0003.

The remainder of the document FSUDIRAC_12_1e_0003 contains two additional cubic equations and their solutions; or rather, three integer solutions, the resulting cubic equation, and some form of a Cardano's "cubic formula" solution, along with work toward the remaining two solutions similar to that which is seen in Figure 2. As we discussed with participants in the workshop, we do not know the context for why or how the solutions to cubic equations of the form $x^3 + qx + r = 0$ were of interest to Dirac, particularly since this document was undated. The examples were written on the back of a former student report card form from the school where Dirac's father taught (and, that Dirac also attended, "Merchant Venturers' College"), though this does not help identify a precise date. Some workshop participants offered the idea that perhaps this was an example of Dirac planning for tutoring or instruction of some sort that he participated in during his time at either the University of Bristol where he studied both engineering (degree in 1921) and mathematics (where he was awarded first class honours in 1923). However, there is no evidence that Dirac taught in any capacity in the years leading up to 1923. It is possible that Dirac was investigating mathematics of personal interest to him – and for which he would have possibly doodled about on random sheets of paper – given his talent for mathematics. Dirac stated:

> I consider myself very lucky in having been able to attend [Merchant Venturers']. ... I was rushed through the lower forms, and was introduced at an especially early age to the basis of mathematics, physics and chemistry in the higher forms. In mathematics I was studying from books which mostly were ahead of the rest of the class. (Dirac, 1980)

Thus, if Dirac's penchant for solving cubic equations was not motivated by the need to plan instruction, perhaps it was just for personal interest in either revisiting Cardano's solution methods or comparing them to more modern conceptions. The latter makes sense, as there are numerous examples of such equations and their solutions within the documents that we targeted in our investigation. Furthermore, there are examples of degree 4, 5, and 6, as well as attempts to generalize solutions (see Figure 3 for an example of degree 4).

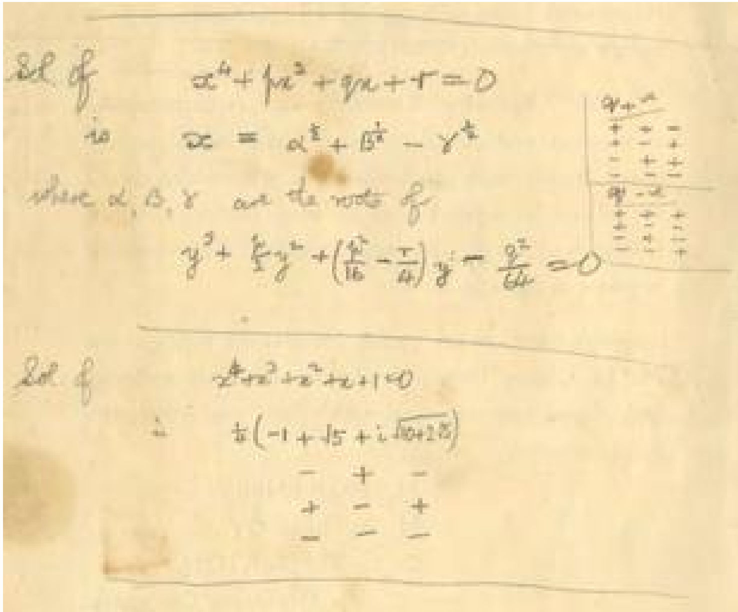

Figure 3: Final two examples of FSUDIRAC_12_1d_0004.

We received helpful advice for how to connect the particular polynomial equation examples to content that Dirac would have been exposed. For example, Bjarne Toft suggested that we try to determine the textbooks that Dirac would have used while attending the University of Bristol. Toft believed that a popular algebra text (*Theory of Algebraic Equations*, by Julius Petersen) would have contained content similar to what we found in the "shoebox collection."

Second Investigation: Combinatorics

The second investigation for which we found numerous related pages we have categorized as "combinatorics." The pages that we identified appeared to have lists of permutations of the letters *a, b, c, d,* and *e* (see the excerpt from FSUDIRAC_12_1a_0018 in Figure 4), as well as evidence of Dirac trying to make sense of patterns of these permutations (see Figure 5), often with tables or other organizational means.

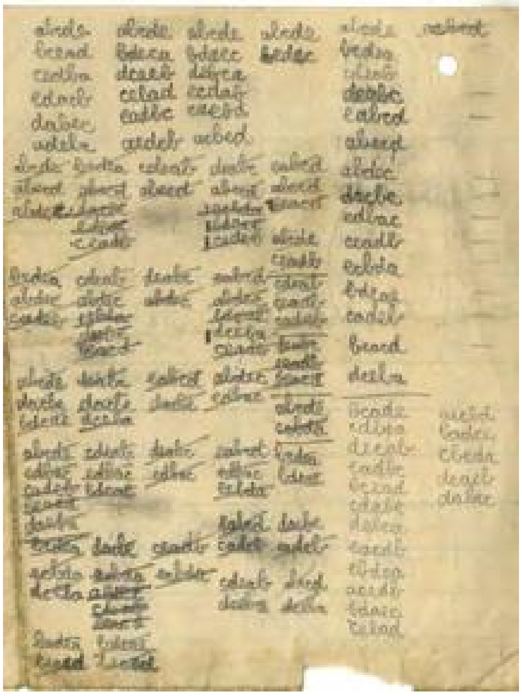

Figure 4: Excerpt from FSUDIRAC_12_1a_0018.

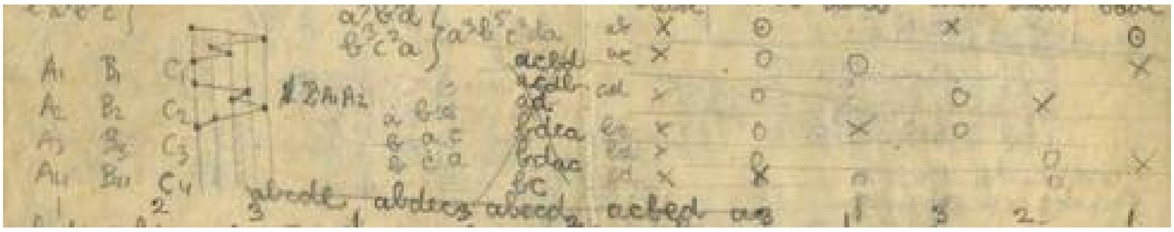

Figure 5: Excerpt from FSUDIRAC_12_1e_0021.

Unfortunately, we had less time to spend on the combinatorics investigation with the workshop participants. The handout we distributed during the workshop prompted Annie Michel-Pajus to provide us with a resource from Leibniz, which may prove helpful with our analysis of what Dirac may have been thinking about. The resource, *Mathesis universalis* (1694-1695) [GM VII, 53-76], shows permutations of powers of *a, b, c, d,* and *e,* as well as the different forms of various degree.

NEXT STEPS

While we intend on following up on the lead from Annie Michel-Pajus while we continue our analysis of what Dirac was attempting in these various mathematical doodlings, we also anticipate that there are connections between two investigations (solving polynomial equations and combinatoric patterns). For example, the excerpt from FSUDIRAC_12_1d_0011 (Figure 6) appears to display such a connection.

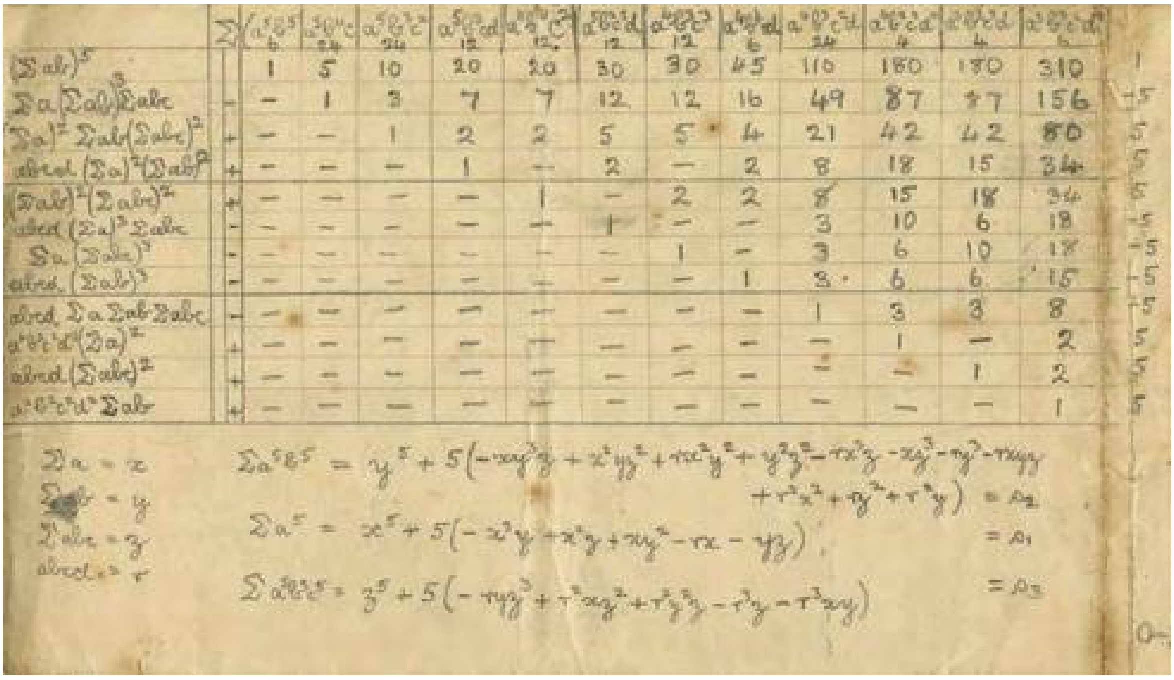

Figure 6: Excerpt from FSUDIRAC_12_1d_0011.

The table in Figure 6 displays some subset of variable combinations (each term of 10^{th} degree), and below the table, a collection of fifth degree equations. Additional documents contain similar versions of the same equations that appear in Figure 6. For example, in FSUDIRAC_12_1d_0012 (Figure 7), we see the same equations, though in a slightly different order and now set equal to different expressions.

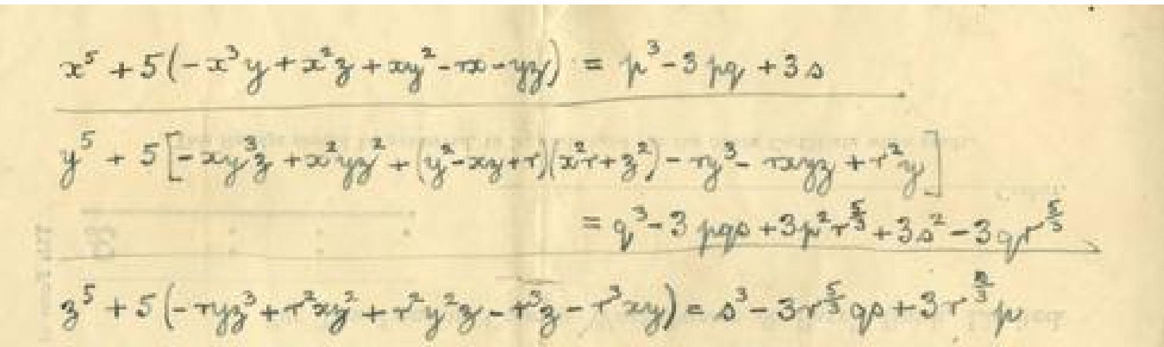

Figure 7: Excerpt from FSUDIRAC_12_1d_0012.

Again, since each of the 72 documents we identified do not contain dates or labels (by Dirac's hand), and the pages were not bound together but in a mostly-random collection, it is difficult to determine in which order the documents FSUDIRAC_12_1d_0011 and FSUDIRAC_12_1d_0012 were produced. This is further complicated by the method of writing: mathematical expressions squished together wherever there was room on a page, often written in 180-degree orientations of each other.

We hope that conference papers such as this one for ESU-7 will generate interest in the mathematical doodlings that we have identified and that will soon be made available as the first project in the Dirac Papers of the FSUDL, and will prompt readers to contact us to contribute to the work. Such collaboration is critical, as we do not possess the mathematical expertise needed to determine the best way analyse Dirac's doodlings. In closing, we welcome partners into this exciting project.

NOTES

1. All of the images ("FSUDIRAC…") are used by permission of the SC&A of the Florida State University Libraries. We are grateful to Kathleen McCormick and Krystal Thomas of SC&A for their assistance and support of this project.

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Oral Presentation

REFLECTIONS ABOUT CHANGING THE TEACHING OF GEOMETRY IN GRADUATION

Gerard Grimberg

Universidade Federal de Rio de Janeiro

In Brazil, and the federal University of Rio de Janeiro in particular, the courses of license split the teaching of the geometry in two, on the one hand Euclidean and non-Euclidean geometry presenting an axiomatic and synthetic approach (in the tradition respectively of Hilbert and Boliai), on the other hand an analytical approach, primarily affin euclidean space definite starting from a normalized vector space. This dichotomy is found also in secondary education where one teaches one year the geometry according to implicit axiomatic and the cases of congruences and similarities of the triangles, and another year analytical geometry.

This division of course has historical reasons which go up with the opposition between synthetic and analytical geometry in the 19th century until the work of Klein and Poincaré, opposition which is found in the handbooks of the 19th century when synthetic geometry and analytical geometry constituted two separate disciplines.

To change this practice a reflection on two levels is required. 1) theoretical: how to present the essential notions of the two points of view so as to constitute only one discipline, 2) didactic: how to determine a unifying element which makes it possible to decline the concepts and their applications.

With regard to the theory, we think of finding a solution in the presentation of the model of affin space checking the axioms of incidence and order, and define a norm checking the axioms of congruence. Another aspect of the vector calculus related to affin space closely connected is to allow barycentric calculation and to deal with by this skew the problems of incidence and to introduce the concept of convexity. The barycentric co-ordinates are besides a first example of homogeneous coordinates, true bridge worm the projective geometry.

At the didactic level, the essential leading element is the traditional presentation of the problems as those treated in the small book of Coxeter (Geometry revisited), or more recently few problems of Geometry revealed of Berger, all with the perspective defended today by Daniel Perrin in various writing and courses. The perspective is not only to solve this problems in one way but also, that is the most important, to present a multiplicity of solutions for every problem. This multiplicity shows that we are not dividing mathematics into little bits (synthetic, analytic, vectorial) but we are teaching only one thing, mathematics.

Texts presented ind this workshop:

Brazilian mathematic programs of the secondary level.

Brazilian cursus of geometry at the federal University of Rio de Janeiro.

Some French and English texts books in 19th century.

Text of the French Kahane French Commission (2000) de Daniel Perrin.

Few texts of Daniel Perrin (in his site)

Few classic problems which any future teacher cannot ignore, and didactical interest of these problems.

Oral Presentation

NEW MATH - A COMPLETE FAILURE?

Tanja Hamann

University of Hildesheim

As in many European countries “New Math” has been implemented in German primary schools and has been abolished only few years after. Nowadays New Math is said to have been a mostly complete failure. In order to work out why this is the case one needs to describe the events, developments and difficulties that prove as crucial influences on the process of the reform. Among the most relevant sources are schoolbooks, which can be viewed – within the bounds of possibility – as a source for what happened in the classrooms. Therefore it is the aim of the workshop to try out textbooks and additional materials in order to analyse their didactical ideas, possible chances as well as possible difficulties for young pupils handling them. The textbooks that are to be worked on are Wir lernen Mathematik by Walter Neunzig & Peter Sorger (eds. 1968 and 1971, additional material: Logic Blocks) and alef by Heinrich Bauersfeld et. al. (eds. 1969 and 1975, additional material: matema Begriffsspiel), for both titles the focus is on material for 1st grade (6-7 years old). The books are in German but as one idea of the protagonists of the reform was to enable learning mathematics independently from children’s language abilities knowledge of German is not required to work with the materials.

Oral Presentation

MATHEMATICS IN T.G. MASARYK'S JOURNAL *ATHENEUM*

Karel Lepka

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Tomáš Garrigue Masaryk was the first president of Czechoslovakia. He was a professor at Charles University in Prague, his branch was sociology. This paper tries to find his relations to the mathematics. He regarded mathematics as an important part of education and he has good knowledge of it. He liked especially probability, this branch he used in his philosophical papers. As a publisher of Athenaeum revue, he contributed to publishing mathematical articles in a journal with a wide range of academic readership.

INTRODUCTION

T.G. Masaryk (1850-1937) is best known as the first president of Czechoslovakia. Before embarking upon a career of a politician, however, he pursued academic career in Vienna and in Prague. He was the founder of the journal *Atheneum*, whose authors (and readers) were active in various branches of research. We will show how the interdisciplinary character of this journal and Masaryk's personality had influence on the well-known episode in the Czech national revival of the 19th century, the so-called "manuscripts dispute".

The "manuscripts dispute" can briefly be described as follows: within the efforts to re-establish Czech as a language, two old manuscripts suddenly emerged, Manuscript from Dvůr Králové and Manuscript from Zelená Hora. Their discovery would have helped in establishing the Czech language as a comparatively old one. A significant part of the Czech academic public, however, did not believe that the manuscripts were as old as it was suggested, and tried to disprove their originality. The echoes of the debate can be seen also on the pages of *Atheneum*. Since Masaryk did not believe that the manuscripts were original, many articles claiming that the manuscripts were fakes were published in the journal. The dispute took a decisive turn when statistics was used to decide the dispute that was primarily based in linguistics and literary studies.

This article brings together the manuscripts dispute and Masaryk's involvement therein as the Editor-in-Chief of the journal in a novel way. It touches upon Masaryk's knowledge of mathematics and puts in due context the contributions of two Czech mathematicians, August Seydler and Matyáš Lerch, to *Atheneum*.

CZECH NATIONAL REVIVAL

The Kingdom of Bohemia had won a significant position within the Holy Roman Empire of the German Nation. The King of Bohemia was one of the seven Prince-

Electors and some of the kings played an important role also in the history of the Empire. Let us mention King Charles IV, who made the Lands of the Bohemian Crown the basis of his power. Bohemia also played an important role in the establishment of the central European composite monarchy in the 16th century, especially during the reign of Rudolf II, who made Prague his seat and thus also the capita of the whole state. Tycho Brahe and Johannes Kepler were invited to work in Prague in this period.

After the defeat of the Bohemian Revolt in 1620 and the forced re-Catholicization, the Lands of the Bohemian Crown became a mere province and their influence diminished. This was also a result of emigration of the non-Catholic elite (e. g. Comenius). At the end of 18th and beginning of 19th century, Czechs strived to improve the position of their country during the so-called national revival. Some of the people active in promoting the use of Czech language, especially those around Václav Hanka, thought that they would help the nation by creating manuscripts proving the high standards of the Czech language already in the Middle Ages. Their effort was made significantly easier by the fact that the publication of old texts did not require censorship, and thus the Czech readers could read for themselves how brave their ancestors had been and how they dealt with any enemy from the Saxons to the Polans. It is thus not surprising that any doubt about the manuscripts resulted in uncritical resistance. Masaryk, however, spent his youth abroad and in addition, his wife was an American (Charlotte Garrigue), and therefor he was able to set himself free from the small problems and advocate the use of critical methods, as was then the habit to the west of our border.

In mid-19th century, Czech mathematics also began to flourish. Periodicals in Czech began to be published, as well as textbooks for elementary schools and high schools. The Union of Czech Mathematicians, founded in 1862, guaranteed the quality of the mathematics textbooks and periodicals and also, through one of the journals, organized a corresponding competition for high school students, which I have elsewhere called "k. und k. mathematical olympiad". During this time, with the help of the personality of Masaryk, a curious connection between mathematics and the humanities occurred. The present article explores this story.

BIOGRAPHY OF MASARYK

Masaryk was born on 7th March 1850 in Hodonín, a small town in Southern Moravia. His father worked as a coach driver, his mother as a cook. Masaryk attended basic school in Čejkovice and *Realschule* (middle school) in Hustopeče. For a short time he was employed as an auxiliary teacher, he also trained to be a locksmith and blacksmith. These activities did not satisfy him and he wanted to continue his studies. He started to study at German classical gymnasium in Brno. His parents were not rich, so they could not support TGM during his studies. Therefore Masaryk gave remedial lessons to children of rich people. The first Masaryk´s supporter was the police chief Anton le Monier, with whom he left for Vienna in 1869. After Monier´s death, the banker Rudolf Schlessinger supported him in studies. Masaryk passed final exam at a classical *Gymnasium* in Vienna in 1879 and started to study at the university. In 1876 he passed doctoral exam, in 1879 he became *Privat Dozent*. During his stay in Leipzig he met the American young lady Charlotte Garrigue, whom he married in 1878. From this time on, he added the surname of his wife to his own.

Fig. 1 TGM at the age of 27

Shortly after Masaryk became Privat Dozent in Vienna, the university in Prague (then called Charles-Ferdinand University) was divided (1882) into two independent parts, namely the Czech and the German part. Every professor had to decide in which school he would teach. Thanks to this fact some professorships were vacant and there was the possibility for TGM to be appointed professor. Masaryk used this opportunity: in 1882, he was appointed extraordinary professor and in 1897 ordinary professor. Masaryk was not satisfied with the provincial atmosphere in Bohemia and tried to change this situation. One of his important acts in this respect was the founding of the journal *Athenaeum*.

Masaryk was also engaged in politics. Together with Karel Kramář and Josef Kaizl he formulated new political trend called realism. He was elected to the Imperial Council and to the Provincial Council in 1891. Two years later he resigned from both positions, partly for family reasons. In 1900 he founded Realistic Party and as a representative of this party, he was elected again to the Imperial Council in 1907 and was its member from 1907 to 1914. At the end of 1914 he left the country and started fighting Austro-Hungarian Empire from abroad. The activity was successful, and on

28th October 1918 Czechoslovakia was established. Masaryk was elected the first president of Czechoslovakia and he held this position till 1935, when he abdicated because of his poor health. Masaryk died on 14th of September 1937.

TGM was also involved in the development of Czech university education system. He accentuated the necessity of competition in the scientific work. In this connection, he pointed out the need for competition for the only Czech university, which was Charles University in Prague. The question was in which town the new university should be established. Masaryk, after corresponding with a majority of the scientific public, suggested the capital of Moravia, the city of Brno, as the seat of the new university and in 1912, he proposed founding this university in the Imperial Council. Although he attached a petition with more than 7 000 signatures from Czech, Moravian, and Slovak municipalities, his activity was not successful. The second university was only founded on 28th of January 1919 and named after him [1].

ATHENAEUM

The first issue of Athenaeum was published on 15th October 1883. In the editorial, we can read: "There is an urgent need for a critical scientific journal. A journal in which we could write about scientific work in this country as well as abroad is missing, which is the reason behind the intention to start publishing the journal *Athenaeum*. Consideration, news and critical articles will be published in the journal. *Athenaeum* will pay attention to all branches of research from theology, law, medicine to technical sciences." The publishers made efforts to fulfil their intents and they were successful. It is true that, especially in volumes 3 and 4, the journal paid attention mostly to the so-called *manuscripts dispute*, but apart from this matter, we can also find papers from other parts of science there.

TGM was firmly convinced that the Manuscripts were counterfeited. It is thus not surprising that *Athenaeum* served as a platform for those opposing the originality of the manuscripts, who published their opinions on the falsity of the manuscripts on the pages of this journal. This fact, however, had unfavourable consequences for *Athenaeum*. The publisher, J. Otta, refused to publish the journal from then on, and thus from volume 6, TGM became the publisher of the journal *Athenaeum.* The journal was printed by Masaryk's brother Ludvk in his printing house in Hustopeče. Also the editor in chief changed: the journal was taken over by Masaryk's colleague J. Keizl.

Despite these difficulties, *Athenaeum* was published for several years after this debate, although its jubilee tenth volume was also its last one. *Athenaeum* influenced the development of critical thinking and brought a number of articles from different branches of science, economy, and politics. As mathematical articles were published in every volume (either the articles mentioned below or especially reviews of our and international publications), we can say that TGM indirectly positively influenced Czech mathematics.

MATYÁŠ LERCH

Fig. 2 Matyáš Lerch

Matyáš Lerch (1860–1922) was a significant Czech mathematician, professor at University in Freiburg in Switzerland (1896–1905), Czech Technical University in Brno (1906–1920), and the founder of the Mathematical Institute at the newly established Masaryk University in Brno. After finishing his university studies he was awarded scholarship by the state and spent one year studying in Berlin, where he met leading German mathematicians. After returning from his stay in Germany, Lerch decided to make Czech scientific society familiar with the contemporary knowledge concerning number sets. He published an article (Lerch, 1886) in *Athenaeum* [2]. In this article Lerch explained pure arithmetic theory of establishing various types number set based on natural numbers. Lerch followed the works of Bolzano, Abel, Cauchy, Weierstrass, Cantor, Dedekind, and also Kronecker, whose influence on Lerch was really strong.

Lerch based his theory on the set of natural numbers N (including zero). He defined equivalency of two arranged couples of number $(a,b)\sim(c,d)$ if $a+d=b+c$. He called the set of all equivalent couples *differenta*. He then gives the definitions of basic arithmetic operation (addition, subtraction, multiplication and division). The sum of two differentas $D_1=(a,b)$ and $D_2=(c,d)$ is also a differenta $(a+c,b+d)$. The subtraction of two differentas is based on the theorem, that for every two differentas D_1 and D_2, there exists just one different D which fulfils the equation $D_2=D_1+D$. Product of two differentas D_1 and D_2 is defined as the differenta $(ac+bd,bc+ad)$. Division of two different is based on finding the different D which fulfils the equation $D_2=D_1\cdot D$. Unlike in the case of subtraction, the differenta D does not exist for every differenta D_1, D_2, which means that division is not closed. In the end, Lerch identified term differenta with the term integer.

In a similar way he established rational number [3], i. e. he gave the definition of the equivalency of two fractions, namely that $\frac{a}{b}=\frac{c}{d}$ if $ad = bc$. He of course eliminated fractions of the sort $\frac{a}{0}$ and $\frac{0}{0}$ [4]. For introducing irrational numbers, Lerch decided to use Cantor's way through limits of the infinite series of rational numbers. Lerch probably chose this method in order to maintain logical unity of the article. As well as in the previous cases he divided the set of the infinite series into equivalent groups; a particular group is called konvergenta. There are two kinds of konvergenta. The first includes constant series numbers a, which represents rational number a. The second kind, which does not contain constant series, represents the irrational numbers. We can learn from other Lerch's papers that he also knew Dedekind cut theory well.

Lerch's article is important, because it is the first mathematical article written in Czech in which number sets are constructed on the basis of natural numbers. Lerch used rather complicated symbolism: he used German letters to denote the differentas, but the article is nevertheless understandable. There is a question why Lerch published the article in *Athenaeum*. One reason could be that Lerch and Masaryk taught at the same university and it is more than probable that they knew each other, although there is no evidence about the depth of their relationship. The second reason could be the fact they both attended foreign universities.

As a sideways remark, let us mention that Lerch also published the theorem about which Runge told him during his stay in Berlin. If the *konvergent* a includes serie $\sum_{\nu=1}^{\infty} \frac{A_\nu}{B\nu}$ where $A_\nu, B_\nu ınZ$ and fulfil $a - a_\nu = \delta_\nu \leq \frac{1}{B_\nu^n + \varepsilon}$, where $\varepsilon > 0$, then a is not a root of an algebraic equation of degree n.

MANUSCRIPTS

On 16th September 1817, a manuscript was found in Dvůr Králové, allegedly dated as a 13th century manuscript (shortly RK, Rukopis královédvorský, i. e. the Manuscript from Dvůr Králové). In November 1818, another manuscript was sent to the National museum, dated as a 9th century manuscript (shortly RZ, Rukopis zelenohorský, i. e. the manuscript from Zelená Hora). Except for those two manuscripts another alleged old literary memorabilia were found in those times. Majority of these discoveries were connected with the persona of Václav Hanka [5].

Fig. 3. Page 6 from Manuscript of Dvůr Králové

All these manuscript proved that Czech language was very rich and that it was possible to write excellent literally works in Czech language. Czech patriots were enthusiastic about these manuscripts and regarded them as a treasure of old Czech literature. Only a small number of scientists doubted the authenticity of the manuscripts, but their number was increasing as time passed. However, in 1886, Jan Gebauer [6] published the article "Requirements of further examinations of RKZ" in *Athenaeum*, which started the dispute in which not only the experts were divided. We can say that half of scientists defended the originality of the manuscripts, while the other half fought against it. Masaryk led the opponents and *Athenaeum* started to publish articles in which scientists of various branches proved the falsity of the manuscripts. Although the controversy had to do with literature, mathematics interfered with this dispute as well.

AUGUST SEYDLER

August Seydler (1849–1891) was a Czech mathematician, physicist and, above all, astronomer. During his studies at Charles University in Prague he met leading Czech scientists and he attended lectures on physics by Ernst Mach. Mach was fond of Seydler and secured him a scholarship of 100 gold coins for two years. After finishing his university studies Seydler started to teach at the University. He was appointed *Privat Dozent* in 1872 and extraordinary professor in 1881. Since 1869 he also worked in the Prague observatory and he is the founder of the Astronomical Institute. His wife was Anna Weyrová, a sister of the well-known Czech mathematicians, the brothers Emil and Eduard Weyr.

Fig. 4 August Seydler

Seydler published paper (Seydler, 1886), in which he tried to prove that the manuscripts are counterfeits. Although *Athenaeum* published many articles proving the contemporary (19th century) origin of manuscripts, they were all written from the point of view of the humanities. Seydler's article looks like from a completely different world, because it refers to mathematics, especially to probability theory. His way of thinking can be explained by the following example. We suppose that in some article we can find some falsehoods, indecent words or some mistakes which were caused by the low acquaintance of the author. While the reader can be confused and unable to decide whether such faults had been caused by chance or whether it was the intention of author, scientists, especially mathematicians proceed in another way. They count such mistakes and try to find the probability whether the author did this or that mistake consciously. When the probability of the casual mistake is p, then the probability that the author made all the n mistakes by chance is $P=(p)^n$. We can bet $1:\frac{1}{P}$ that at least one of the mistakes was caused consciously.

Seydler decided to study the so-called coincidences; that is the fact that some words can only be found in RK and in pieces written by Hanka. He used the well-known Bernoulli formula

$$P=\binom{c}{a}x^a(1-x)^b,$$

where a is the number of incorrect forms of word or phrase and b is the number of correct ones. Then $c=a+b$ and $x=\frac{a}{c}$. For the coincidences he used the formula

$$R=\frac{\binom{p}{m}\binom{c-m}{a-m}\binom{\gamma-m}{\alpha-m}}{\binom{c}{a}\binom{\gamma}{\alpha}}.$$

Here again a or α is the number of incorrect forms, b or β the number of correct forms, and $c = a + b$ or $\gamma = \alpha + \beta$. Words from the manuscripts are denoted with Latin letters, words from Hanka papers are denoted by Greek letters, p is sum of investigated words, and m is number of common strangeness.

Seydler investigated two hypotheses. According to the first one, all the strange phenomena (words and grammar) in manuscripts are accidental and according to the second one, the correspondence between these strange phenomena in the manuscripts and the ones in papers from the beginning of 19th century (before finding manuscripts) are accidental. Result of his work follows:

1) We can bet at least 3 000 millions to one that all the strange phenomena are not accidental.
2) We can bet at least 13 millions to one that the correspondence between the strange phenomena in manuscripts and Hanka's works is not accidental
3) We can bet at least 8 millions to one that the correspondence between the two strange phenomena in RK and Jungmann's [7] poems which were published before finding of the manuscripts, are accidental.
4) We can bet at least 100 trillion (10^{12}) to one that the correspondence between RK and papers published before finding manuscripts are accidental.

Although Seydler did not directly say that the manuscripts were false, every reader of his paper must come to a clear conclusion – the manuscripts are fakes. One weakness of his paper is evident: Seydler relied heavily on Gebauer's view for old Czech language and considered it as absolutely right. Defenders of manuscripts discovered this weakness and started to disprove his results, see for example (Ivanov, 1969). Nowadays linguists regard the majority of coincidences false and Seydler's proof lost a great deal of its persuasiveness. In any case, it is the first attempt to use mathematics in the research of the Czech language. Further it also proves that Masaryk believed in the force of an exact science, which mathematics is, and embraced its help in his fight. In addition probability theory was not widely spread in Czech countries in those times and we can say Czech mathematicians are only starting to get acquainted with it. Approximately at the same time, articles by A. Pánek were published, in which the author introduces probability theory to Czech mathematicians (Pánek, n.d. a; n.d. b).

MASARYK AND MATHEMATICS

So far, we have not sought Masaryk's direct contribution to the progress in mathematics. Now the time is ripe for it. As we said, Masaryk's research belongs primarily to sociology and philosophy. There is, however, evidence that he had good knowledge of mathematics (and physics), because he emphasized philosophical meaning of mathematics. He also wrote several reviews of books on mathematics and physics for the journal *Atheneum*. From the point of view of his research specialization, theory of probability was the most important part of mathematics for him. This area of mathematics does not only describe static relationships between quantities, but allows us also to search for the causes of known phenomena, predict possible outcomes, and formulate hypotheses. He studied the theory of probability thoroughly, as the title of his inaugural lecture at the university proves: *Probability and Hume scepticism* (Masaryk, 1883). In that lecture, we can learn about Masaryk's view on mathematics. He writes that scientists are seeking for probability of its finding. Various branches of science have various level of it, and mathematics has the leading role, because no scepticism can disturb that kind of knowledge. On the other hand, the sciences struggle to be as exact as mathematics is.

Masaryk saw the closest relations between mathematics and logic. We must note that logic was then not regarded a part of mathematics, but a part of philosophy, the one which deals with the way of deducing conclusion and which studies objective conditions of thinking. Sceptics deny casual connection between causation and consequence, and according to them, this can only be caused by the opinion, not by the intellect. As Masaryk briefly sums up Hume philosophy, opinion and intellect excludes themselves. Only mathematics is an exception, because only such science can determine causation and consequence through thinking.

Hume considered only three parts of mathematics, namely algebra, arithmetic and geometry. But there is still one branch of mathematics, which explores the connection between causation and consequence – probability. The origin of this field is connected with gambling, but after the determination its basic laws, probability started to be used in the insurance industry, banking and so on. Some mathematicians started to disprove Hume sceptic theses. It is understandable that Masaryk as a philosopher was interested in probability and there is evidence he knew this branch well.

CONCLUSION

I have brought to the readers' attention the first instance of using mathematics to solve a problem in literature in the Czech lands and the involvement of T.G. Masaryk in this important dispute of the national revival. It is probable that the (in general overlooked) fact that Masaryk knew mathematics played a role in this. To start with, Masaryk did not count on mathematics when he became involved in the manuscripts dispute, but Seydler's article was in the range of Masaryk's interpretation of what mathematics is.

Founding of the journal Atheneum was one of the successful outcomes of Masaryk's efforts. Similarly, Lerch had been in Berlin for one year and his aim was similar to Masaryk's, namely to elevate Czech mathematics to a higher level. That could have been the reason why Lerch published his papers in *Atheneum*, and not in the *Journal for the Cultivation of Mathematics and Physics.*

Such use of mathematics in a literary dispute was the first of its kind in the Lands of Bohemian Crown and it remained a singular and an outsanding one. Possibly, it was the first instance of using mathematics for such a problem.

NOTES

1. From 1960 to 1990, for political reason, the universtity in Brno was not called Masaryk University, but bore the name of the excellant Czech scientist Jan Evangelista Purkyně.

2. Foundantions of the pure arithmetic theory of quantities.

3. Lerch uses term signed number, it means $\pm$. Lerch wanted to emphasize that integers are signed, it follows from the fact, that Czech word číslo (number) means only positive number.

4. Lerch uses instead Czech word zlomek (fraction) term divisanta.

5. Václav Hanka (1791–1861), Czech writer, poet, linguist, editor of old Czech literary memorabilities, Privat Dozent at the University and so on. A leading person of Czech national revival. He is suspected from counterfaiting of some memories.

6. Jan Gebauer (1838–1907), significant Czech linguist, regarded to be one of the most important people in this branch.

7. Josef Jungmann (1773–1847), Czech philogist and writer, author of the modern Czech lexicon. Some scientists suspect him of participating in counterfaiting the manuscripts.

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Oral Presentation

TEACHING MATHEMATICS IN MINING ACADEMIES: AN OVERVIEW AT THE END OF THE 18TH CENTURY

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At the end of the 18th century, many mining academies were created, mostly in the German speaking countries, to improve the overall scientific education of mining technicians and officials. While historians of science have so far dedicated studies to chemistry and mineralogy, mathematics has been vastly overlooked. It is usually assumed that mathematical sciences were taught and used uniformly and at an elementary level in the different Bergakademien of Freiberg, Schemnitz, Clausthal, etc.

In this talk, I intend to show that there was in fact a great variety of approaches in mathematics teaching in these institutions, ranging from academic Gelehrsamkeit to the practical Brauchbarkeit of mining engineers. Some of the sources used are the archive of the various Bergakademien as well as the textbooks published by mathematics professors, but also manuscripts, handwritten lectures and test reports.

The example of the Freiberg mining academy, where mathematics teaching reached a very high level at the end of the 18th century, proves that these disciplines were not intrinsically inferior, but only different from the university lectures of that time. The subterranean geometry (Markscheidekunst) will then be used as an example to illustrate the originality of this scientific and technical teaching tradition.

Oral Presentation

MATHEMATICAL LESSONS IN A NEWSPAPER OF PORTO IN 1853 (PRIMARY EDUCATION)

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The Industrial Association of Porto was founded in 1849. Industrial education was one of its first priorities, having created the Industrial School of Porto in 1852. In 1853, António Luís Soares, a professor of this school, published several arithmetic lessons for primary education in the Newspaper of the association - Lecture of the arithmetic elements in the Association's primary school class. These lessons were addressed to teachers who should reproduce them to their students. The author includes several considerations about the importance of propagating basic math instruction, either for industry or for the trade workers, two important activities for the city. In this paper, we will present these classes and show how this text may be used in the classroom (primary source in the history of mathematics).

INTRODUCTION

> *As vias férreas são a escola primária da indústria, a escola primária popular é a via férrea de toda a civilização*. (Castilho, 1864)
>
> [The railways are the industry's primary school; the popular primary school is the railroad of the whole civilization]

The *Associação Industrial Portuense* (Industrial Association of Porto) was founded on May 3, 1849 (approved by royal law only on August 26, 1852) in the city of Porto (Portugal), still subsisting today under the name *Associação Empresarial de Portugal* (Portuguese Entrepreneurial Association).

> "(...) The Industrial Association of Porto aims to develop the national industry – instructing the industrial classes and particularly the industrial workers in elementary arithmetic, geometry, drawing, and mechanical, chemical and physical arts; and especially in the study of machines, equipment and processes, which are being successively invented or perfected so that the Portuguese industry can be placed at the same level as the most advanced nations (...)."
>
> (Statutes of the Industrial Association of Porto; Chapter II, Article 4, 1852)

In fact, the industrial education was one of the first priorities of the association, having created the Industrial School of Porto almost immediately (December 30, 1852). Even before that, on December 6 of that year, a course on «reading and writing» began, which was attended by 117 students. This group included 25 individuals who attended these classes in order to propagate this knowledge to several villages around the city of Porto. António Luís Soares (1805 - 1875), a professor of the Polytechnic Academy of Porto since 1836 and of the Industrial School of Porto since 1852, was intended to address these teachers in order to "present some works on the teaching of arithmetics". Just a few could "attend the invitation" and so the alternative was to publish such works in the Newspaper of the Industrial Association of Porto - Lecture of the arithmetic's elements in the Association's primary school class (*Da exposição dos elementos da arithmetica na aula de instrucção primaria da Associação*). This newspaper usually published news about the association as well as others concerned with the city and the surrounding region (they were mostly about industrial and commercial events but there were also others about political and cultural activities).

Thus, arithmetic lessons for primary education were published in several numbers (usually once a month, from April to December of 1853). These were addressed to teachers who should reproduce them to their students.

These arithmetic classes were divided into sections, namely: 1. Formation of the numbers; 2. The first arithmetic operations. After these, several tables of units' conversions were also published (linear, area, capacity and weight).

As an introduction to these classes, António Luís Soares did a brief but interesting analysis of how primary teaching was in Portugal (particularly in mathematics), as well as several considerations about the importance of propagating basic math instruction, either for industry or for the trade affairs workers, two very important activities for the economy of the city of Porto at that time.

In the next pages, we will present António Luís Soares and a brief history of the superior schools in Porto. We will also reproduce some parts of these arithmetic classes and show how this publication may be used in the classroom (primary source in the history of mathematics) and how it may be interesting from the perspective of consolidating very basic notions of the discipline of mathematics.

ANTÓNIO LUÍS SOARES, A PROFESSOR IN TWO SCHOOLS (THE HISTORY OF SUPERIOR SCHOOLS IN PORTO)

The Royal Academy of Navy and Trade Affairs of the City of Porto was the first academy in the city of Porto. It was created in 1803 (laws from February 9 and July 20) and had three mathematical years, classes of Rational and Moral Philosophy, Trade Affairs, Drawing (this class existed in the city since the year 1779), the English Language and the French Language. Note that the three years of mathematics were a copy of what was practiced in the Royal Academy of Navy of Lisbon.

This academy had a Master of Navigation, responsible for the practical teaching of navigation; this master was subordinated to the professor of the third mathematical year and was, somehow, the continuation of a Nautical class established in the city of Porto in 1762. This Nautical class was created to provide officers required for the two war ships that the city used to protect the goods that were destined to Brazil (former

Portuguese colony). This class, as well as the two ships, were paid for by the city itself through city taxes and its customs. This class was under the supervision of the General Company for Agriculture of the Upper Douro Vineyards, a monopolistic company created by the Marquis of Pombal, with great privileges in the Douro Valley and the city of Porto. In 1878, this company started to fund this Nautical class through the dividends of their shareholders. This financing scheme would be continued during the creation of The Royal Academy of Navy and Trade Affairs of the City of Porto in 1803, which made it a unique case in the higher education in Portugal at that time. In fact, the civilian character of this academy was present since its inception and its goal was linked to what was important to the city of Porto: forming good sailors and good merchants was fundamental to the city's economy, which was focused on commercial activities with Brazil and northern Europe.

Finally, note that the scientific production of this academy was quite poor but the implementation of upper level mathematical classes (as is the case of the differential and integral calculus) was an important step in Porto, somehow facilitating the possibility of establishing a school with polytechnic characteristics in the city.

In 1837, the city of Porto witnessed this academy of navy and trade affairs becoming the Polytechnic Academy of Porto, a school dedicated to industrial sciences (law of January 13, 1837). Its creation brought about a new paradigm for the higher education that existed in the city, highlighting the three engineering courses (Mining Engineering, Construction Engineering, Bridges and Railroads Engineering) that were then implemented. Despite the importance of the engineering courses, the Polytechnic Academy also presented some courses that were not typical in a polytechnic school, but, certainly, originated in its predecessor - for example, the course of Navy Officers, Ship Pilots and Trades Affairs people.

In addition to the new courses, it should be noted that there was a significant increase in the number of mathematical and scientific disciplines, established with the creation of the new academy; the eleven disciplines then implemented were divided into four sections: Mathematics (5), Rational Philosophy (4), Drawing (1) and Trade Affairs (1). However, this new academy, unlike its predecessor, was administered and funded directly by the Ministry of the Kingdom (central government in Lisbon), *i.e.*, the city of Porto lost control of its academy. It should be also noted that in 1837 the socio-economic context of Porto was substantially different because Brazil had become independent in 1822. By this time, the Liberal Passos Manuel was in power (who reformed many areas like, for instance, the higher education) and Portugal was in a process of some industrialization and in an upgrade of the communications' network.

During its existence, the polytechnic academy gradually evolved into a university level institution, originating a lack of proper education for mid-level employees (especially for the industry). So, the city of Porto, through the Industrial Association of Porto, created a new school in 1852: *Escola Industrial do Porto* (the Industrial School of Porto). It is important to note that the Industrial Association of Porto was formed by influential individuals of the city, with important connections with the economy of the city. In June of 1853 (*Jornal da Associação Industrial Portuense*, 20, June 1, 1853, p. 307), this association was composed by 192 artisans/artifices, 36 manufacturers, 174 traders, 48 goldsmiths, 32 landowners, 30 medical doctors and chemists, 84 public servants, 5 farmers, and 7 militaries (total: 608 members). In this manner, the city once again controlled an important school that could be managed in such a way as to fulfil the effective needs of the city. When this school was created, there was only one single graduation divided in two types: *Habilitação simples* (simple graduation) and *Habilitação plena* (full graduation). After a few years, in 1864, the school changed its name to *Instituto Industrial do Porto* (Industrial Institute of Porto), with a more complete educative offer: factory director, overseer of public works, overseer of machines (steam engines), overseer of mines, telegrapher, master of public works and master of chemistry. There was also a factory worker graduation that was preparatory for all of the other courses.

The Polytechnic Academy of Porto and the Industrial School of Porto were connected in various ways like, for instance, many professors taught simultaneously at both schools and both shared the same building until 1933 (about 80 years…). There were also proposals to fuse both into one single institution: in 1864 by José Maria de Abreu (a political proposal from Lisbon) and in 1882 by Rodrigues de Freitas (professor from Porto); however, there was a major Reform of the Polytechnic Academy of Porto in 1885 (in 1884 the Academy received the most important Portuguese mathematician at that time: Gomes Teixeira) and, from that point, the difference between the two institutions was definitely settled: one more theoretical (high level studies) and the other more technical (intermediate studies). One of the most important characteristics that both schools shared was the fact that both were created to attend the effective necessities from the city (sailors, trade men, engineers, industrial workers,…) and both were funded by the initiative of private institutions of the city.

António Luís Soares (Porto, 1805 - 1875) was a professor of the Polytechnic Academy of Porto since 1836 (First discipline: Arithmetic, Elementary Geometry, Trigonometry and Elementary Algebra) and a professor of The Industrial School of Porto since 1852 (Arithmetic, Algebra and Geometry). There is a lack of information about this professor and, except the text that we present here, there is only a reference (Scipião, pp. 9-10) to another text (*Exposição dos elementos de Aritmética para uso dos estudantes do Colégio de Santa Bárbara na cidade de Pelotas*, Pelotas, Brazil, 1849), an arithmetic textbook published in Brazil; no copy of this text has been found (Scipião says that António Soares was in Brazil between 1847 and 1851, but no justifications for his stay are pointed out).

LECTURE OF THE ARITHMETIC ELEMENTS IN THE ASSOCIATION'S PRIMARY SCHOOL CLASS

The Industrial School of Porto was created in 1852 (December 30) by the Industrial Association of Porto. Even before that, although primary education was not their main focus, on December 6 of that year, a course on «reading and writing» opened, which was attended by 117 students (many of them were destined to the Industrial School). This group included 25 individuals who attended these classes in order to propagate this knowledge to several villages around the city of Porto. António Luís Soares was intended to address these 25 "teachers" in order to present some works on the teaching of arithmetic. Just a few could "attend the invitation" and so the alternative was to publish such works in the Newspaper of the Industrial Association of Porto, a biweekly newspaper whose first issue came out on August 15, 1852 (fig. 1).


JORNAL DA ASSOCIAÇÃO INDUSTRIAL PORTUENSE.

NUMERO 16. SEXTA FEIRA 1 DE ABRIL. ANNO 1853.

Figure 1. Header of the Newspaper of the Industrial Association of Porto (number 16, April 1, 1853).

Thus, arithmetic lessons for primary education were published in several numbers (usually once a month, from April to December of 1853). These were addressed to teachers who should reproduce them to their students. The text published could be divided in two parts: one that should be replicated to students and another (presented in italics) that was dedicated to professors. The italic part aimed to provide guidance to future professors (for example: how many times should the exercise be repeated; what should the student know at the end of each lesson). These Arithmetic classes were divided into sections, namely:

- Formation of the numbers (in which a study of the metric system is included and compared with the usual Portuguese measures);
- The first arithmetic operations (only addition and multiplication).

After these, several tables of units' conversions were also published (linear, area, capacity and weight). See the following table highlighting the structure of these classes (table 1).

<table>
<tr><th>Date</th><th>Number
(pages)</th><th>Sections</th><th>Lessons</th><th>Observations</th></tr>
<tr><td rowspan="2">April, 1</td><td rowspan="2">16
(pp. 244-248)</td><td rowspan="7">Section 1:
Formation of the numbers</td><td>Introductory observations
1. numeration system</td><td></td></tr>
<tr><td>2. spoken numbering</td><td></td></tr>
<tr><td rowspan="2">May, 1</td><td rowspan="2">18
(pp. 277-281)</td><td>[3.] written numbering</td><td></td></tr>
<tr><td>4. observations about quantities</td><td></td></tr>
<tr><td rowspan="2">June, 1</td><td rowspan="2">20
(pp. 307-312)</td><td>5. metric system (units, multiples and submultiples)</td><td></td></tr>
<tr><td>6. metric system (written abbreviations)</td><td></td></tr>
<tr><td rowspan="2">July, 1</td><td rowspan="2">22
(pp. 339-345)</td><td>7. the “big” numbers; roman numerals</td><td></td></tr>
<tr><td rowspan="2">Section 2:
The first arithmetic operations</td><td>1. addition</td><td></td></tr>
<tr><td rowspan="2">July, 31</td><td rowspan="2">24
(pp. 374-383)</td><td>addition (cont.)
[2.] multiplication</td><td>Why did he not publish anything about subtraction and division?</td></tr>
<tr><td colspan="2" rowspan="2">Tables of units conversions</td><td></td></tr>
<tr><td>August, 1</td><td>1 (T2)
(pp. 2-3)</td><td>Just the continuation of tables from the previous text.</td></tr>
<tr><td>December, 1</td><td>9 (T2)
(pp. 138-141)</td><td colspan="2">How to convert the ancient Portuguese units to the metric system and vice versa</td><td>At the end of this text the indication “To be continued” appears, but this was the last text in this newspaper by António Soares.</td></tr>
</table>

Table 1. Structure of the classes.

António Luís Soares, as an introduction to these classes, did a brief but interesting analysis of how the primary teaching in Portugal was (particularly in mathematics), as well as several considerations about the importance of propagating basic math instruction, either for industry or for the trade affairs workers, two very important activities for the economy of the city of Porto at that time.

In the next pages, we will explore some aspects of these classes in detail. The first section is denominated "Formation of the numbers"; in the first lesson the author explains how to increase and decrease a set of objects and to compare two small sets of objects (table 2).

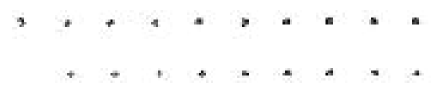
	Which set is bigger? It is easy to see that it is the upper "line" (it is easy to compare small sets of objects… just align all of the objects, one set above the other…)

Table 2. Example from Lesson 1 ("Formation of the numbers").

In the second lesson, the spoken numbering until 10 was taught (table 3).

Um e um fazem dous. Dous e um fazem tres. &c. &c.	One and One does two; Two and One does three; …

Table 3. Example from Lesson 2 ("Formation of the numbers").

In the same lesson, he taught the numbers between 10 and 10000 (table 4).

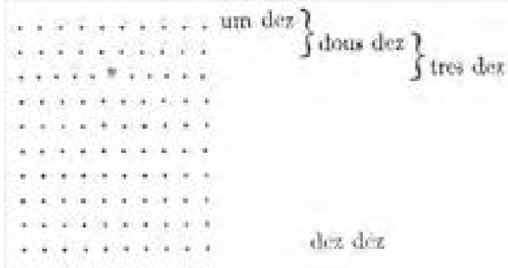
	One ten two tens three tens … ten tens (= hundred)

Table 4. Second example from Lesson 2 ("Formation of the numbers").

At this stage, the author did not use words like twenty, thirty, forty, etc; he also did not use the words eleven, twelve, thirteen, fourteen, etc. For example, 14 is one ten and four; 47 is four tens and seven; 30 is three tens; 328 is three hundreds, two tens and eight. Note that, until now, the author did not introduce the writing digits (1, ..., 9) nor the digit zero (0).

In the third lesson, the writing digits (symbols to use in writing) were presented, from 1 to 9. After that, the author said that it is possible to write the numbers above nine without creating anymore new symbols (he did remark that if each number had its own symbol, it would be impossible to remember them all…). There is only the need for the value in each digit to vary depending on its position (table 5).

um dez e um 1 1	dous dez e um 2 1	One ten and one	1	1
um dez e dous 1 2	dous dez e dous 2 2	One ten and two	1	2
um dez e tres 1 3	dous dez e tres 2 3	One ten and three	1	3
&c.		…	…	
nove dez e nove 9 9		Nine ten and Nine	9	9

Table 5. Example from Lesson 3 ("Formation of the numbers").

At this point, the digit 0 was presented (used when there was a "gap" in the "units", to distinguish, for example, the numbers 2 and 20) and a generalization of this construction for numbers up to 1000. The author also remarked that each digit has two (possibly) different values: a proper or depending on the form (7) and a local or relative to its position (7, 70, 700, …). For example, the digit 7 represents 7 hundreds in the number 723; the digit 7 represents 7 tens in the number 273. The author said that this arrangement was "an ingenious agreement, the best abbreviation that was possible to think".

In the fourth lesson, the author made several observations about quantities. He observed that the numbers ("a number is a set of things with the same name") learned are useful to count, for instance, men and trees. But what about a big quantity of grains of wheat? It was impossible and inutile to count each single grain… There are things that we cannot measure the quantity using only numbers; so, it is necessary to have another type of measurement units for dry volumes (grains and beans, for instance), surfaces (lands and fields), liquid volumes (milk, wine), weights (reference to the scale of two arms), time and money (coins). At this point, the author only did a first introduction to this subject and a reinforcement of the need of other measurement units for everyday life/industry/commerce (the numbering system alone is not sufficient…).

The metric system is formally presented in lesson number 5 (units, multiples and submultiples). The author explained that, historically, the first measurement units were, naturally, the *Palmo* (hand) and the *Pé* (foot), but this kind of units had proven to be difficult to work with and a source of problems and errors over time… Afterwards, he made reference to France and to the difficulties in the implementation of the metric system. But he thought that the context in 1853 was different:

> "However, there was hope that this time was possible to introduce the metric system in the plenitude. The artisans did not fight against this system anymore, because they are familiarized with the new units by the visits of the academic/theoretic people to their shops."
>
> (*Jornal da Associação Industrial Portuense*, 20, June 1, 1853, pp. 307-308)

The next step, in his opinion, was to propagate these units in the common retail trade, because it would facilitate commercial transitions in everyday life. Then, he presented some common old Portuguese units in such a way as to highlight two major problems: first of all, it is difficult to memorize all of the relations between them. For example, he presented the following units: length units – 1 *braça* = 2 *vara*; 1 *vara* = 5 *palmo*; 1 *palmo* = 3 *pollegada*!!!!!; weight units – 1 *quintal* = 4 *arroba*; 1 *arroba* = 32 *arratel*; 1 *arratel* = 16 *onça*; 1 *onça* = 8 *oitava*!!!! It is also very difficult to operate with them (for instance, what is the relation between *quintal* and *oitava*?).

Afterwards, the author finally presented some metric units: the meter (linear); the are (surface; 100 square meters; note that the square meter is too small to measure fields…); the liter (capacity) and the gram (weight). Then, the multiples of these units were presented («Deca» means 10 primitive units, «Hecto» means 100 primitive units and «Kilo» means 1000 primitive units) and submultiples («Deci» means 1/10 primitive units, «Centi» means 1/100 primitive units and «Milli» means 1/1000 primitive units). The author then stated an important warning: for surfaces (1 m^2 = 100 deci-m^2) and capacities (1 m^3 = 1000 deci-m^3) we must be very careful when working with multiples and submultiples.

Lesson number 6 was the continuation of the presentation of the metric system. He taught the written abbreviations and presented various tables comparing the old Portuguese units with the "new" metric system (units for big lengths, small lengths, agricultural, small surfaces, liquid volumes, dry volumes, solid volumes like wood, weights and small weights).

In lesson number 7, the author returned to the numbering system, expanding what he had already taught regarding the "big" numbers (thousands, millions, billions, trillions,…). On the other hand, it is only at this point that he teaches the formal name of some numbers like 11 (eleven instead of ten and one), 12, 13, 14, 15, 20 (twenty instead of two tens), 30, 40, 50, 60, 70, 80 and 90.

Curiously, at the end of this lesson, he presented the Roman numerals (using the history of mathematics in the classroom…). As an introduction to this subject, he said that the digits that were presented in these classes did not come from antiquity and many people used the alphabetic letters to write numbers, namely the Greeks and the Romans. The author explained the roman numerals because they were still in use at the time (ancient coins, opening date of buildings and to numerate items in texts) and it is also a positional system (the same symbols could represent different numbers; for example, IV is different from VI)

The second section is denominated "The first arithmetic operations". In the first lesson of this new section, the author explained addition, beginning with a table with all sums up to 10 (fig. 2).

	2	3	4	5	6	7	8
2	4	5	6	7	8	9	10
3	5	6	7	8	9	10	
4	6	7	8	9	10		
5	7	8	9	10			
6	8	9	10				
7	9	10					
8	10						

Figure 2. Addition table.

Afterwards, he taught how to add small numbers (up to 10). Next, to explain how to add bigger numbers, he presented a scheme resembling an abacus (though he did not use this designation) with an example: 326 plus 172 (table 6).

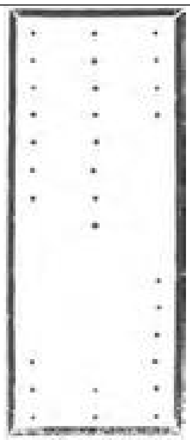
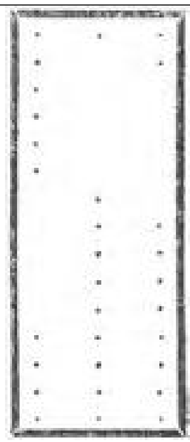
	326		Total: 498 (just count the dots in each column)

Table 6. Example from Lesson 1 ("The first arithmetic operations").

Of course, after this example, the author explained what we should do when the sum of two digits in a column is bigger than 9. Afterwards, he presented another example, but now with no abacus scheme (table 7).

7 8,5 6 4. 3.2 8 9.5 7 3 4 6.8 7.3 4.5 2 2 5 8 3 7 9	First right column: 4 + 5 + 8 + 2 = 19. And 1 (') goes to the next column. Second right column: 6 + 9 + 6 + 5 **+ 1** = 27. And 2 (' ') goes to the next column. …

Table 7. Second example from Lesson 1 ("The first arithmetic operations").

Afterwards, he presented a very pedagogical example to emphasize the fact that we should always do the columns from right to left and not the other way around (table 8).

354 213 401 ——— 968	In this first example is indifferent.	632 763 528 852 ——— 26	But not in this second example… If we start on the left column we need to erase the 6 that will be corrected with the value that goes from the mid column (**16**).

Table 8. Third example from Lesson 1 ("The first arithmetic operations").

Then, he noted that adding numbers with decimal parts is basically the same and presented several examples (highlighting that this is not more difficult than adding integers). The decimal points just need to be vertically aligned and the method is exactly the same. In fact, this is an important advantage from the metric system: it is easier to work (in this case, add) with the sub-units of the metric system than with the old Portuguese subunits. And to emphasize this point of view, he presented a very difficult example (table 9) using linear units (note that: 1 p(olegada) = 12 l(inha); 1 P(almo) = 8 p; 1 B(raça) = 10 P).

B. P. p. l. 30 3 7 5 12 8 2 11 ——————— 13 2 2 4	5 l. + 11 l. = 16 l. = 1 p. + **4 l.** 7 p. + 2 p. + 1 p. = 10 p. = 1 P. + **2 p**. 3 P. + 8 P. + 1 P. =12 P. = 1 B. + **2 P**. 30 B. + 12 B. + 1 B. = **43 B** (there is a typo).

Table 9. Fourth example from Lesson 1 ("The first arithmetic operations").

For the students' homework, the author suggested more examples with the old Portuguese units, even more complicated, in order to convince everyone that it was confusing to work with the old units and it was necessary and easier to adopt the «new» metric system.

The second lesson of this section concerned multiplication that was presented as a particular case of addition, in which all of the numbers to add are equal. First of all, he presented all of the products between two numbers lower than 10, explaining how to construct the following multiplication table (fig. 3).

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

Figure 3. Multiplication table.

Afterwards, he explained how to multiply bigger numbers recurring to an example (table 10).

Supponhamos que se quer juntar 4 vezes o numero 237, se escrevessemos como para achar a somma ter-se-hia .... 237 Mas devemos resumir a escripta, e vê-se que seria bastante assentar 237, e indicar por baixo o numero de vezes que era necessario escrever este numero, assim .... 237 / 4 Ora pela columna das adições sabemos que a somma é de 4 vezes 7 unidades .... 28 4 3 dezenas .... 12 4 2 centenas 8	237+237+237+237 = 237x4 7+7+7+7 = 7x4 units = 28 units 3+3+3+3 = 3x4 tens = 12 tens 2+2+2+2 = 2x4 hundreds = 8 hundreds So, the final result is 28 + 120 + 800 = 948

Table 10. Example from Lesson 2 ("The first arithmetic operations").

After this first example, he provided other examples, including some numbers with zeros because, in that situation, there are simplifications. Unfortunately, the example that he gave has several typos and, consequently, it was not very clear for the reader what he meant (table 11).

Multiplicando. . 376864 Multiplicador, . 27020 Prod. por 20. .... 753728 » 7000. .... 2638048 » 20000. .... 753728 1018286528	Intermediated lines are not aligned properly; A zero is missing at the end of the final result.

Table 11. Second example from Lesson 2 ("The first arithmetic operations").

This passage is an example of several typos in this text that, probably, caused many difficulties for the readers who intended to replicate these classes. Similarly to other places of this text, he also suggested that the students should practice a lot by doing many exercises (in this case, many multiplications).

Once again, the author pointed out that there is no problem in working with numbers with decimal parts; it is not more difficult than the multiplication of integers – we just need to be careful with the position of the decimal point.

Finally, these classes ended with the presentation of several tables connecting the old Portuguese units with the metric system. These tables were very complete and covered all important economic areas of that time: length (big and small); area (big and small); dry and liquid volumes; weight (big and small, including specific tables for chemical and pharmaceutical drugs, gold, silver and diamond; to see the difficulties of operating with the ancient units note that 1 *Marco* is equal to 1152 *Oitava* if it is silver and 1 *Marco* is 768 *Oitava* if it is gold). See, as an example, fig. 4.

PEZOS

Systema metrico								
Centigram. 4,97969618	Grão							
Grammas 1,195127083	24	Seropulo						
Grammas 3,58538125	72	3	Oitava					
Grammas 28,68305	576	24	8	Onça				
Grammas 458,9288	9216	384	128	16	Arratel			
Kilogrammas 14,6857216	294912	12288	4096	512	32	Arroba		
Kilogrammas 58,7428864	1179648	49152	16384	2048	128	4	Quintal	
Kilogrammas 793,02896640	15925248	663552	221184	27648	1728	54	13,5	Tonellada

Figure 4. Table connecting the old Portuguese weight units with the metric system.

Later that year (December), António Luís Soares published similar tables in this newspaper again, under the title "How to convert the ancient Portuguese units to the metric system and vice versa". This new text was now dedicated to industrial and trade workers and should be used as a guide table by those professionals. In fact, all of the texts published by this author in the Industrial Association's Newspaper had a professional teaching agenda, *i.e.*, to instruct the low and middle working classes.

APPLICATION IN THE CLASSROOM

There are various possibilities for using this ancient publication in the classroom:

- Put older students teaching arithmetics to younger students;
- Give small parts of the text to the students for interpretation and to compare the «old» mathematics and units with the present time;
- Ask to write a similar text (classes of mathematics in a public newspaper) but adapted to current times;
- Research work, as it could be an interesting interdisciplinary theme: History (Portuguese history, scientific history, implementation of the metric system;…); Chemistry/Physics (the measurement units, the industrial applications of these disciplines); Biology (some units used in agriculture) and Portuguese language (reading and interpretation of a text, comparing the old language with the current one).

Note that this text is in Portuguese, a fact that constrains its utilization to natives of this language (for instance, Portugal and Brazil). However, this is a good example that it is possible to find educative and attractive texts in many different contexts. This text is particularly interesting because it is strongly linked with the history of the city of Porto and it is very rich in contents and examples. In fact, with the study of this text, it is possible to:

- Review some basic arithmetic's notions;
- Understand and emphasize the importance of our decimal numbering system (connection with the metric system), showing that it is a good system to operate and easy to work with;
- Recall some basic properties of the decimal system (always omnipresent in everyday math classes, but not always noted): positional (the value of each digit depends on its position), economical (just needs ten digits/symbols) and the importance and significance of the digit zero in a number (how to distinguish 202 from 2002 without the zero?);
- Emphasize the difficulty of doing basic calculations in other eras (when the technologies were too rudimental);
- Understand that the metric system was a major breakthrough that came to facilitate measurements and calculations (avoiding many errors and complications in calculations).

CONCLUSION

The classes presented here are intimately connected with the socio-economic context of the city of Porto: an industrial and commercial city and the second city of the country. On the other hand, these classes were sponsored by an Industrial Association, which explains the fact that they were classes with a very practical goal (teaching the basic arithmetic always with the aim of using it in the industrial/commercial trade). It is also relevant to note that the metric system was implemented officially in Portugal in 1852 (the first attempt was in 1814 but with no success), so it was a very important and new subject when these classes were published. All of these factors explain the (excessive?) focus on the metric system and its relation with the old Portuguese units. However, this text has a very rich content in basic arithmetic and could be applied in the classroom (primary source in the history of mathematics) in various ways, for example, as an interesting tool for consolidating very basic notions of the discipline of mathematics.

ACKNOWLEDGEMENTS

This work was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese

Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project PEst-OE/MAT/UI4106/2014.

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Oral Presentation

THE BEGINNING OF ALGEBRA IN SPANISH IN THE SIXTEENTH CENTURY: MARCO AUREL'S ARITHMETICA ALGEBRATICA

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The book Arithmetica Algebratica, published in Valencia in 1552, is the first printed book on algebra written in Spanish. However its author, Marco Aurel, had not Spanish as his mother tongue, he was a German residing in Spain.

A German working in Spain and writing in Spanish in the sixteenth Century is not a so strange fact. When Marco Aurel's book is published, the King of Spain, Carlos I, being also heir of the House of Hausburg and the House of Valois-Burgundy, has become, as Karl V, the Emperor of the Holy Roman Empire, and the Archduke of Austria, and is ruling then over German territories. This is the cultural and political context in which the book is published, and these are the features that make singular Marco Aurel's book. But the reason why we are specially interested in this book is the fact that it is the only book using the German cossic signs written in Spanish, and published in Spain. Algebras written or published in Spain during the sixteenth Century were rethorical or, when syncopated, they used Italian abbreviations.

Marco Aurel's book gives us the only opportunity to study in a Spanish text the way in which the use of cossic signs shapes algebraic reasonings.

To carry out this study we have examined the intertextual relation of Marco Aurel's book with other books on Algebra, that dialogue with it, by referring explicitly or implicitly to it, by being its contemporaries in Spain, or by being the result of readings of Marco Aurel's book.

Oral Presentation

QUAND LES « MATHEMATIQUES MODERNES » QUESTIONNENT LES METHODES PEDAGOGIQUES DANS L'ENSEIGNEMENT SCONDAIRE (1904-1910)

Hervé Renaud

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*En 1907, la Revue de Philosophie publie une série d'articles critiquant vivement un livre du mathématicien Hermann Laurent et la logique symbolique défendue par le philosophe Louis Couturat. Leur auteur, Charles Lucas de Pesloüan, polytechnicien, y adopte le ton et la méthode utilisés par Henri Poincaré lors de la controverse qui l'a opposé à Couturat et à Bertrand Russell en 1905 et 1906 à propos de la logique et des mathématiques. Pesloüan va cependant plus loin, développant une critique politique et sociale et situant le débat sur le terrain de l'enseignement secondaire. Ces articles révèlent des craintes liées au contexte de réformes de l'enseignement secondaire et aux questionnements pédagogiques que posent les apports théoriques survenus en mathématiques durant la deuxième moitié du XIX*e *siècle.*

INTRODUCTION

En 1907, Charles Lucas de Pesloüan (1878-1952), ancien élève de l'École Polytechnique, fait paraître dans la *Revue de Philosophie* une série de quatre articles sous le titre « Sur les fondements de l'arithmétique et autres questions connexes ». Il y critique violemment des livres du philosophe Louis Couturat (1868-1914) et du mathématicien Hermann Laurent (1841-1908). Sous-titrés « Lettres à M. N***, professeur de mathématiques au collège de X*** », ils situent d'emblée le débat sur le terrain de l'enseignement secondaire.

Ces articles furent publiés en marge de la controverse qui opposa en 1905 et 1906 le mathématicien Henri Poincaré (1854-1912) à Couturat et au philosophe et logicien britannique Bertrand Russell (1872-1970) à propos de la logique et des mathématiques. A travers l'étude de ces textes de Pesloüan, repris et complétés dans un livre paru en 1909, *Les Systèmes logiques et la logistique. Étude sur l'Enseignement et les Enseignements des mathématiques modernes*, cet article s'intéresse aux liens que Pesloüan tisse entre logique, « mathématiques modernes » de ce début de XXe siècle et enseignement secondaire. Ces liens mettent en lumière des craintes exprimées par différents acteurs du système éducatif français durant cette période marquée par la question des fondements en mathématiques et par des réformes majeures de l'enseignement secondaire, en particulier celle de 1902. Certaines de ces craintes sont imaginaires, conséquences du contexte politique et social. D'autres reposent sur de réels enseignements de ces « mathématiques modernes ». Nous tenterons de les mettre en évidence, à la lumière de la lecture des *Conférences du Musée pédagogique* de 1904 et d'articles de revues.

DES TEXTES EN MARGE DE LA CONTROVERSE POINCARÉ–RUSSELL, COUTURAT

Le premier article de Pesloüan est une critique du livre de Laurent, *Sur les principes fondamentaux de la théorie des nombres et de la géométrie*. Les trois suivants forment un ensemble centré sur la critique de deux livres de Couturat : *L'algèbre de la logique* et *Les principes des mathématiques*, publiés en 1905. Le premier est un ouvrage d'initiation aux théories élémentaires de la logique. Le second reprend, avec quelques modifications, une série d'articles parus dans la *Revue de métaphysique et de morale* en 1904 et 1905. Ce sont ces articles, compte rendu du livre de Russell paru en 1903, *The Principles of Mathematics*, qui ont entraîné la réaction de Poincaré et provoqué la controverse évoquée précédemment, le mathématicien réfutant les positions de Russell et Couturat qui affirmaient construire les mathématiques sur des bases logiques.

Quelques brèves indications sur la logique au début du XX^e^ siècle [1]

Au début du XIX^e^ siècle, la logique est encore essentiellement la logique des syllogismes forgée par Aristote, non pas une science mais un outil de raisonnement. On peut considérer que le premier vrai réformateur de la logique au XIX^e^ siècle est le britannique Georges Boole.

Formé aux mathématiques, il applique à la logique les méthodes algébriques. Il reprend au mathématicien et logicien britannique Augustus De Morgan l'idée d'univers du discours qu'il note 1, 0 désignant « rien », et utilise les symboles opératoires +, ×, – pour élaborer son algèbre. Raisonnant en termes de classes (d'ensembles, en langage actuel), il note par exemple x la classe des hommes, $1 - x$ désignant la classe des objets qui ne sont pas des hommes. Ainsi, l'un des principes fondamentaux de la logique, le principe de non-contradiction qui exprime qu'on ne peut pas penser simultanément une chose et son contraire s'écrit : $x(1 - x) = 0$, conséquence de ce qu'il nomme « la loi fondamentale de la pensée » : $x \ = x$.

Le mathématicien allemand Ernst Schröder parachève l'algèbre logique de Boole dans les trois volumes de ses *Vorlesungen über die Algebra der Logik* parus entre 1890 et 1895. Couturat situe son livre, *L'Algèbre de la Logique*, dans cette lignée Boole-Schröder. Ce petit volume d'une centaine de pages est conçu comme un livre d'initiation destiné au public français car, en France, philosophes et mathématiciens sont restés à l'écart de ce mouvement de renouvellement de la logique.

Le livre se présente comme un calcul formel, une algèbre qui peut s'interpréter en logique. Cependant, à la fin de son livre, Couturat précise qu'il n'a traité que de la logique classique, restant dans le domaine circonscrit par Aristote. Il n'a pas traité, annonce-t-il, de la logique des relations qu'il qualifie de véritable logique des mathématiques et dans laquelle ces dernières trouvent leurs concepts et leurs principes fondamentaux.

En effet, la majeure partie des raisonnements fondés sur les relations ne relèvent pas de la syllogistique [2]. De Morgan est considéré comme le fondateur de cette logique des relations. Il a mis en évidence les propriétés de réflexivité, symétrie, transitivité, asymétrie. Elle est ensuite développée par le philosophe américain Charles Sanders Peirce qui invente les quantificateurs et les tables de vérité.

C'est dans cette logique des relations que va se développer le projet logiciste de reconstruction logique des mathématiques. Le mathématicien et logicien allemand Gottlob Frege conçoit le projet de reconstruire l'arithmétique sur des bases logiques. Russell se propose, lui, de reconstruire toutes les mathématiques sur des bases purement logiques [3]. A cette fin, il adopte et développe le langage symbolique inventé par le mathématicien italien Giuseppe Peano [4]. Couturat désigne cette logique symbolique sous le nom de logistique.

La crise des fondements en mathématiques

Ces projets de reconstruction des mathématiques se situent à un moment de l'histoire des mathématiques parfois qualifié de « crise des fondements ». Si l'expression n'était pas employée par les mathématiciens du début du XX^{e} siècle, ils s'accordaient cependant sur des bouleversements apportés par quelques découvertes majeures au siècle précédent : les géométries non-euclidiennes par Lobatchevski en 1829 ainsi que les fonctions continues sans dérivées par Weierstrass en 1872. Ces découvertes remettaient en cause les notions intuitives de l'espace géométrique et de la continuité.

A ces découvertes il faut ajouter le mouvement d'arithmétisation de l'analyse qui débute dans les années 1820 sous l'impulsion de Bolzano et Cauchy (Boniface, 2002). La volonté d'établir l'analyse sur des bases rigoureuses amènera aux constructions des nombres irrationnels dans les années 1870, à celle des nombres relatifs, des fractions et des nombres entiers. C'est bien d'ailleurs dans ce mouvement de reconstruction de l'analyse sur des bases purement arithmétiques que Couturat situe l'origine du projet de reconstruction logique des mathématiques.

Un autre événement majeur du dernier tiers du XIX^{e} siècle est l'émergence de la théorie des ensembles avec les travaux de Cantor et Dedekind. Pour ses contemporains, c'est cependant le nom de Cantor qui, en France, reste attaché à cette théorie. Aux mathématiciens français elle apparaît dès le départ comme ambivalente. Les premiers mémoires de Cantor sont traduits en français en 1883. Charles Hermite et Poincaré s'opposent à certains travaux de Cantor (Dugac, 1984), qualifiés de philosophiques. Mais Jules Tannery (1848-1910) signale la même année l'importance de la notion d'ensembles dérivés. Il donne des éléments de la théorie des ensembles en 1886 dans son livre, *Introduction à la théorie des fonctions d'une variable*, écrit selon lui d'après des leçons faites à l'École Normale Supérieure. Très rapidement, les mathématiciens français Poincaré, Borel, Baire, Lebesgue vont s'emparer de cette théorie pour faire progresser l'analyse (Gispert, 1995) tout en continuant à se méfier des symboles de l'infini de Cantor. Cette théorie, devenue au début du XX^{e} siècle la base de la théorie des fonctions, ne dispose pas encore d'une axiomatique solide et a le défaut d'engendrer des paradoxes.

Ces mathématiques fondées sur la seule idée de nombre et utilisant la théorie des ensembles semblent être celles que Pesloüan qualifie de « modernes » dans ses textes.

La controverse Poincaré–Russell, Couturat [5]

C'est donc à la suite de la publication des articles de Couturat dans la *Revue de Métaphysique et de Morale* qu'éclate en 1905 cette controverse. Poincaré va faire paraître, dans la même revue, des critiques qui donneront lieu à une série d'échanges entre les divers protagonistes. Le ton est polémique. Par exemple, sur le modèle du piano logique, une machine destinée à effectuer des calculs logiques, inventée par l'économiste et logicien britannique Stanley Jevons, Poincaré imagine

> […] une machine où l'on introduirait les axiomes par un bout pendant qu'on recueillerait les théorèmes à l'autre bout, comme cette machine légendaire de Chicago, où les porcs entrent vivants et d'où ils sortent transformés en jambons et saucisses (Poincaré 1905, pp. 824).

Ton polémique qu'on ne trouve pas dans les textes de Russell mais que saura lui aussi manier Couturat, dénonçant par exemple « cette clarté vulgaire qu'on nomme l'intuition, et que M. Poincaré prise si fort » (Couturat 1906a, pp. 209).

L'intuition est en effet au cœur de la controverse. Selon l'affirmation de Couturat lors de sa leçon inaugurale au Collège de France en 1905, la logique dit comment « l'on doit penser si l'on veut penser normalement et correctement » (Couturat 1906b, pp. 320). La réponse de Poincaré s'adresse à ceux qui pensent que « les mathématiques sont entièrement réductibles à la logique et que l'intuition n'y joue aucun rôle » (Poincaré 1905, pp. 815), thèse qu'il récuse. Les raisonnements mathématiques ne peuvent, selon lui, se réduire à un algorithme de calcul logique. Il accuse les logisticiens d'avoir simplement déplacé les appels à l'intuition en remplaçant les axiomes par des postulats qui n'ont pas l'évidence des axiomes. Il doute en outre que leur théorie ne repose que sur neuf notions indéfinissables et vingt propositions indémontrables, écrivant : « je crois que si c'était moi qui avais compté, j'en aurais trouvé quelques unes de plus » (Poincaré, 1905, pp. 829).

A ces reproches, il faut ajouter des critiques plus fondamentales que Poincaré adresse aux logisticiens. Il les accuse notamment de se livrer à des pétitions de principe comme, par exemple, dans la définition du nombre 1 donnée par Burali-Forti :

$$1 = \iota\, \mathbf{T}' \{ \mathrm{Ko} \frown (u, h) \ni (u \,\varepsilon\, \mathrm{Un}) \}$$

Sans donner la signification de cette assertion [6], il écrit :

> J'entends trop mal le Péanien pour oser risquer une critique, mais je crains bien que cette définition ne contienne une pétition de principe, attendu que j'aperçois 1 en chiffre dans le premier membre et Un en toutes lettres dans le second (Poincaré, 1905, pp. 823).

L'INTERVENTION DE LUCAS DE PESLOÜAN

Cousin de Maurice Barrès, sorti en 1899 de l'École Polytechnique, ingénieur dans les Chemins de Fer puis dans les Mines, Lucas de Pesloüan n'est ni mathématicien, ni philosophe [7]. Ami de Charles Péguy qui s'est engagé contre la réforme de 1902 et a exprimé à cette occasion son rejet de la modernité [8], Pesloüan signale une première fois son intérêt pour les questions d'enseignement en 1903 par un article intitulé « Sur la nécessité du postulat d'Euclide » dans la revue *L'Enseignement mathématique.* Les articles qu'il écrits à la suite la controverse entre Poincaré, Russell et Couturat sont publiés dans la *Revue de Philosophie*, revue fondée en 1900 par l'abbé Peillaube, professeur de philosophie à l'Institut Catholique de Paris [9].

Tout comme la *Revue de Métaphysique et de Morale*, cette revue propose de faire entrer en collaboration scientifiques et philosophes. Mais son objectif annoncé est de promouvoir une philosophie thomiste, anti-moderniste et, en particulier, anti-positiviste.

Des articles scénarisés

Les articles se présentent sous forme de lettres écrites par un ancien ingénieur à un professeur de mathématiques d'un collège de province. A la suite d'une remarque de l'inspecteur, l'ingénieur met à profit un séjour à Paris pour rechercher des ouvrages qui permettront au destinataire des lettres de découvrir des méthodes modernes d'enseignement des mathématiques.

Lors d'une première visite chez un libraire il choisit le livre de Laurent. Par la suite il rencontre un étudiant en philosophie, partisan convaincu de la logistique, qui lui conseille les livres de Couturat. Il retrouve aussi à Paris un ami, ingénieur retraité, farouche opposant de la logistique à laquelle il oppose Pascal.

Les lettres sont les récits de ces rencontres et les commentaires de sa lecture des trois livres. L'étudiant y apparaît un peu ridicule, parfois proche du fanatisme. L'ingénieur retraité a, avec le rédacteur des lettres (qui représente bien évidemment l'auteur), une grande proximité intellectuelle mais fait preuve d'opinions tranchées que Pesloüan-narrateur ne reprend pas toutes à son compte.

Une méthode empruntée à Poincaré : ton polémique et critiques fondamentales

Pesloüan, comme Poincaré, adopte un ton polémique qu'il tire vers l'ironie et reprend certaines des critiques fondamentales formulées par le mathématicien. Sa première cible est le livre de Laurent.

Laurent est un ancien élève de l'École Polytechnique. Il y devient répétiteur puis examinateur et publie à partir de 1862 un nombre important de traités concernant principalement l'analyse et le calcul des probabilités. Le livre paru en 1902, *Sur les principes fondamentaux de la théorie des nombres et de la géométrie*, dont le titre résume bien le propos, est revendiqué par l'auteur comme un livre d'enseignement, et même un livre d'enseignement de l'arithmétique pour de jeunes enfants, à la condition d'admettre un certain nombre des théorèmes démontrés.

Dans sa présentation, Laurent déclare renoncer à la « doctrine des axiomes très évidents », un « axiome [étant] une proposition que l'atavisme, l'éducation, l'autorité du maître, nous ont fait accepter comme vraie sans examen » (Laurent, 1902, pp. 2). Il propose d'y substituer le plus petit nombre « d'hypothèses plausibles » et construit son arithmétique sur six hypothèses.

Ce renoncement à la doctrine des axiomes que Laurent fait remonter à Pascal est la critique majeure de Pesloüan. Reprenant l'idée de Poincaré sur le décompte des principes fondamentaux de la logistique, Pesloüan traque les hypothèses dissimulées dans le texte. L'exemple qui suit est en outre révélateur du ton adopté par Pesloüan.

L'arithmétique de Laurent commence par la définition de l'égalité de deux objets :

> Deux objets égaux sont donc deux objets jouissant d'une même propriété énoncée ou sous-entendue et dont on laisse de côté les autres propriétés.
>
> Il n'y a là rien de conventionnel de ma part, et je ne fais que préciser une idée que nous nous faisons tous de l'égalité. L'égalité est une propriété toute relative, et qui dépend du point de vue auquel on se place. Ainsi un cheval est l'égal d'une poule, quand, faisant abstraction de toutes leurs autres propriétés, on les considère l'un et l'autre comme des animaux.
>
> Si des objets A et B sont égaux à un objet C, A et B sont égaux entre eux, par définition. (Laurent, 1902, pp. 4)

L'objection de Pesloüan est la suivante :

> Cette définition est à mon avis une hypothèse : en effet, si le cheval est l'égal de la baleine en tant que mammifère, et la baleine l'égal de l'éponge en tant qu'habitant de la mer, il n'en résulte pas que le cheval soit d'une de ces deux façons l'égal de l'éponge. Cette hypothèse signifie qu'on peut, sur A, B et C, faire le même travail d'abstraction. (Pesloüan, 1907a, pp. 376).

Cette définition qui permet à Pesloüan de tourner le texte en dérision lui donne aussi l'occasion d'exprimer une critique fondamentale : Laurent a défini l'égalité de deux objets jouissant d'une même propriété. Il faut donc que l'égalité de A et B puisse se faire dans ce cadre, ce qui n'est pas *a priori* évident pour trois objets.

Un autre type de critique que Pesloüan reprend à Poincaré est l'accusation de pratiquer des pétitions de principe. Il signale ainsi dans le texte de Laurent chaque occurrence d'un nombre avant que Laurent n'ait donné la définition du nombre entier.

Nous retrouvons cette méthode critique dans les lettres sur la logistique. Les définitions par postulats de la logistique ne lui conviennent pas plus que les hypothèses plausibles de Laurent : les unes comme les autres éloignent la science des réalités sensibles.

Mais la critique de la logistique passe progressivement d'une critique scientifique à une critique sociale et politique des textes de Couturat.

La crainte d'un monde logisticien ?

La critique sociale est en fait sous-jacente dans le sous-titre des articles. Les lettres sont adressées à un professeur de collège. Un passage du livre que Pesloüan publiera en 1909 nous renseigne sur sa perception des collèges [10] :

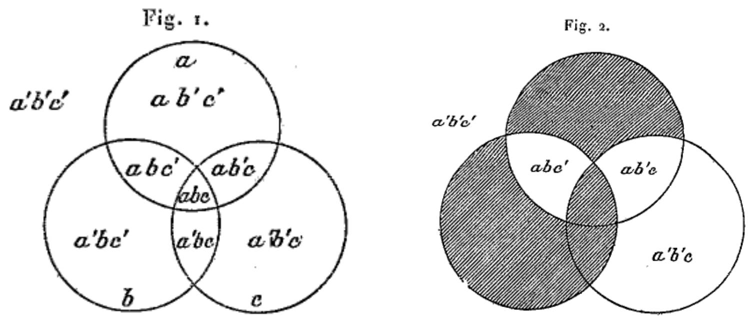
> De l'architecture d'un temple grec à celle d'un musée ou d'un collège, *le collège départemental de toutes les connaissances*, telle est la dégradation de l'image qu'on peut se faire des hiérarchies scientifiques (Pesloüan, 1909, pp.28).

La critique sociale et politique se développe dans les lettres consacrées à la logistique. L'exemple du problème de Venn traité par Couturat dans *L'algèbre de la Logique* l'illustre parfaitement. L'énoncé en est le suivant :

> Les membres du Conseil d'administration d'une société financière sont, soit des obligataires, soit des actionnaires (mais pas les deux). Or tous les obligataires en font partie. Que faut-il en conclure ? (Couturat, 1980, pp. 73) [11].

Couturat résout le problème par le calcul puis avec les « schèmes géométriques de Venn », donnant les figures suivantes, qui nous semblent à présent bien familières :

Figure 1 : Diagrammes illustrant la résolution du problème de Venn (Couturat, 1980, pp. 76,77)



Si *a* désigne l'ensemble des membres du conseil, *b* celui des obligataires et *c* celui des actionnaires, l'intersection de deux ensembles se traduisant par un produit, la réunion par une somme et le complémentaire de *a* étant noté *a'*, etc, on obtient : $abc + ab'c' + a'bc + a'bc' = 0$, ou, de façon équivalente, en utilisant le complémentaire : $a'b'c' + abc' + ab'c + a'b'c = 1$. De ces relations, Couturat déduit seize causes et seize conséquences. Par exemple, la 11^{e} cause est la suivante : $abc' + ab'c + a'b'c = 1$, ce qui équivaut à : $(b = c')(c' < a)$, $c' < a$ se traduisant en langage ensembliste actuel par $c' \subset a$.

Pesloüan détourne le problème de Venn, qui devient sous sa plume :

> Dans une ville est une mutualité dont les membres sont, soit des libéraux, soit des socialistes (mais aucun n'étant les deux à la fois). Or tous les libéraux en font partie. Que faut-il en conclure ? (Pesloüan, 1907a, pp. 497).

Il traduit par exemple la 11e cause par : « les libéraux ne sont pas socialistes et les non-socialistes, c'est-à-dire les libéraux, sont mutualistes » (Pesloüan, 1907a, pp. 500). Il nous faut nous reporter au contexte politique de l'époque : socialistes et libéraux sont opposés, les catholiques conservateurs, public visé par la *Revue de Philosophie*, étant eux-mêmes opposés à ces deux courants de pensée. Nous pouvons imaginer que l'énumération sur trois pages de conclusions semblables, sur un mode ironique, devait réjouir un certain nombre de lecteurs.

La dernière lettre offre un autre exemple de ce type de critique. S'appuyant sur l'article d'un juriste paru dans la *Revue de Métaphysique et de Morale* qui se prononce pour l'enseignement de la logistique en Faculté de Droit, l'ingénieur retraité dénonce l'ambition logisticienne : « Les logisticiens veulent juger, légiférer puis gouverner » (Pesloüan, 1907b, pp. 204) afin de créer une société logisticienne où « dans les écoles, on enseignera aux enfants le parler nouveau par signes et par figures » (Pesloüan, 1907b, pp. 205). A ces craintes il faut cependant opposer l'attitude plus modérée d'un narrateur dubitatif qui conclut à la manière de Voltaire :

> Nous voici peu avancés en ce qui concerne notre enseignement : je crains bien qu'il ne nous faille continuer comme nous faisions autrefois, et, pour cultiver notre jardin, conserver nos vieux outils. (Pesloüan, 1907b, pp. 206)

Les réactions aux articles de Pesloüan

Ces textes semblent n'avoir suscité aucune réaction en France, ce qui ne sera pas le cas du livre [12]. Ce dernier reprend les articles, précédés d'une longue préface et suivis de neuf appendices. Dans la préface, l'auteur affirme ne s'intéresser qu'à l'enseignement supérieur délivré dans les facultés, contrairement à ses articles qui évoquaient presque exclusivement l'enseignement secondaire. De ce déplacement de son centre d'intérêt, il ne dit rien. Les articles ne subissent que quelques modifications, toutefois révélatrices [13]. Les appendices mélangent texte purement mathématiques (il donne les démonstrations de la non-dérivabilité des fonctions de Riemann et de Weierstrass), textes de vulgarisation (sur les géométries non-euclidiennes et sur la théorie des ensembles), et textes de mathématiciens (Fourier, Weierstrass), etc.

Une recension du livre paraît en 1909 dans la *Revue de Philosophie*, signée des initiales H.P. [14]. Reprochant au texte des longueurs et certains passages qu'il qualifie de nuageux, l'article lui adresse cependant de sincères éloges. En effet, il juge que Pesloüan a découvert les nombreux postulats implicitement admis par Laurent et donné de la logistique des « indications […] qui suffisent même […] à nous ôter tout désir de la connaître d'avantage » (H.P., 1909, pp. 456).

Tel n'est pas l'avis de Tannery qui est sous-directeur de l'École Normale Supérieure, responsable des études scientifiques. Plus que ses travaux scientifiques, ce sont ses fonctions institutionnelles et son rôle de formateur qui lui font jouer un rôle privilégié dans la communauté mathématique française (Gispert, 1991 et Zerner, 1994). Il a été mis en cause à deux reprises dans le livre de Pesloüan. La première, pour son livre

Introduction à la théorie des fonctions d'une variable, accusé de proposer une présentation trop abstraite de l'analyse, encombrée d'une foule de détails dont les élèves ne comprennent ni le sens ni la portée. Plus loin Tannery est accusé de participer à une mystification du public, lui faisant croire à tort que, dans l'enseignement des facultés, tout l'effort est porté sur la pratique.

Dans sa recension, Tannery place à son tour le débat sur le terrain de l'enseignement secondaire. Il reproche à Pesloüan de laisser croire que le livre de Laurent où ceux de Couturat aient pu avoir la moindre influence sur l'enseignement, affirmant que « ce n'est pas dans les lycées français que les professeurs passeront leur temps à cette subtile analyse [des principes] » (Tannery, 1912, pp. 78) [15].

« MATHEMATIQUES MODERNES » ET ENSEIGNEMENT SECONDAIRE

Les articles de Pesloüan ne sont pas que des textes de circonstance provoqués par la parution des livres de Laurent, Couturat et par la controverse avec Poincaré. D'autres raisons liées à l'enseignement secondaire justifient plus probablement leur publication et leur positionnement qui hésite entre critique outrancière et conservatisme modéré. Les craintes qu'ils expriment permettent de mieux comprendre le contexte dans lequel ils ont été écrits.

Une peur fantasmée : un monde logisticien

Rien ne permet d'étayer les craintes d'une survenue de ce monde logisticien redouté par l'ingénieur retraité mais Pesloüan l'évoque car il existe en effet un lectorat réceptif à ces craintes.

La réforme de 1902 de l'enseignement secondaire, d'inspiration positiviste, a augmenté significativement la part des sciences dans les programmes, et placé l'enseignement scientifique, notamment les mathématiques, sous le signe du concret (Belhoste, 1995). Cette orientation nouvelle a suscité des rejets dans certains milieux, d'autant plus qu'accède ensuite au pouvoir une majorité dominée par les radicaux et les radicaux-socialistes. Le ministre de l'Instruction Publique, Jospeh Chaumié, fait voter une loi destinée à combattre l'enseignement congréganiste, loi qui touche aussi les institutions privées laïques. Pour certains, c'est une remise en cause de la Loi Falloux qui organisait depuis 1850 la liberté d'enseignement. Voici par exemple ce qu'écrit Adolphe Legorju, Secrétaire Général de la Société Nationale d'Éducation de Lyon pour qui il existe une relation étroite entre la loi Chaumié et la mise en application des nouveaux programmes dans l'enseignement secondaire :

> Après un demi-siècle de monopole suivi d'un demi-siècle de liberté, voici que le droit d'enseigner est à nouveau remis en question. Après un demi-siècle d'éducation libérale, voici que l'utilitarisme semble prévaloir comme principe d'éducation. On se demande si la Révolution s'est faite au nom de la Liberté ou au nom du Socialisme. (Legorju, 1905, pp. 51)

L'une et l'autre participent, selon Legorju, d'une volonté d'assujettissement de l'individu à la société : pour leurs promoteurs, l'enfant appartient d'abord à la société. Parmi les inspirateurs de cette politique, il cite le philosophe, logicien et économiste britannique John Stuart Mill, le parrain de Russell. Mill nous ramène ainsi à Russell et Couturat.

Les propos polémiques de Legorju qui ne craint pas de manier l'amalgame sont à situer dans le contexte des violents débats politiques de l'époque sur les questions d'enseignement. Destinés à mobiliser l'opinion, ils n'hésitent pas à attiser les craintes de leurs partisans. Les articles de Pesloüan se font ainsi l'écho de la peur que les réformes en cours dans le système d'éducation ne préparent de profonds changements sociétaux.

Une réalité : des tentatives « d'enseignement logique » des mathématiques dans le secondaire

Dans son livre Pesloüan distingue logistique et système logique. Il appelle système logique un système de reconstruction logique des mathématiques. Le livre de Laurent en est sans doute pour lui l'exemple le plus caricatural mais, à un degré moindre, la construction de l'analyse sur le nombre entier proposée par Tannery dans son *Introduction à la théorie des fonctions d'une variable* en est un autre. Et si, rien ne permet de supposer l'existence d'une quelconque tentative d'enseignement de la logistique, il n'en est pas de même avec les systèmes logiques. C'est bien ce que semble nous indiquer l'extrait suivant du discours de Louis Liard.

Le philosophe Louis Liard, vice-recteur de l'Académie de Paris, est l'un des principaux instigateurs de la réforme de 1902. Les programmes de mathématiques qui ont été élaborés sans consultation des enseignants vont faire l'objet de critiques de la part de ces derniers (Belhoste 1995). Des conférences sont alors organisées pour les professeurs des lycées parisiens. Dans son discours d'ouverture aux conférences Liard déclare :

> On dit que depuis une vingtaine d'années les mathématiques subissent une crise d'idéalisme transcendantal. […] Mais ce qui est à sa place dans l'enseignement supérieur ne l'est pas dans l'enseignement secondaire. Or on m'assure que là, sous l'influence des plus hautes spéculations, il s'est introduit, depuis quelques années, des façons qui ne seraient pas sans péril. Ne perdons pas de vue que, dans nos classes, il s'agit de former, non des candidats à la section de géométrie de l'Académie des Sciences, mais des esprits clairs, voyant juste, raisonnant juste. (Liard, 1904, pp. VIII).

En 1906, Raoul Bricard, rédacteur de la revue *Nouvelles Annales de Mathématiques* et examinateur d'admission à l'École Polytechnique, fait écho aux propos de Liard:

> L'enseignement des mathématiques élémentaires subit en ce moment une crise profonde. Les progrès de la philosophie des mathématiques ont bouleversé les idées traditionnelles sur les fondements de la Science, et l'influence des doctrines nouvelles se fait de plus en plus sentir sur les méthodes d'exposition. C'est ainsi que la notion d'ensemble […] tend à

devenir, didactiquement comme philosophiquement, le fondement même des mathématiques. (Bricard, 1906, pp. 511)

Des professeurs du secondaire ont sans conteste introduit dans leur enseignement des façons de faire, des définitions puisées dans un de ces systèmes logiques dont parlait Pesloüan. Cet état de fait, sous-entendu par Liard et pointé par Bricard, permet peut-être de mieux comprendre certains passages de la conférence de Poincaré à qui Liard passait la parole en 1904. Le sujet de la conférence était : « Les définitions générales en mathématiques ». Poincaré y critiquait notamment certaines définitions ne figurant alors dans aucun programme du secondaire.

En arithmétique, Poincaré prenait l'exemple des fractions. Enseignées à l'école primaire à l'aide des partages, une fraction devenait, disait-il, à l'École Normale Supérieure, un ensemble de deux nombres entiers séparés par un trait horizontal sur lesquels on définissait par convention des opérations. Il concluait, à propos des définitions possibles d'une fraction : « Quant aux définitions plus subtiles, à celles qui sont purement arithmétiques, il faut les abandonner à l'enseignement supérieur, s'il en veut » (Poincaré, 1904, pp. 15). Cet exemple de la définition des fractions choisi par Poincaré correspond à un sujet fréquemment discuté à cette période. Dans son livre d'arithmétique pour la classe de mathématiques élémentaires, Tannery avait proposé, en complément de la définition concrète des fractions, cette même définition purement arithmétique (Tannery, 1894) qu'il enseignait à l'École Normale Supérieure. En 1899, Robert de Montessus, professeur de mathématiques dans un collège des Jésuites, proposait lui aussi une introduction logique des fractions dans la revue *L'Enseignement mathématique* (Montessus, 1899). S'il ne la destinait pas à tous les élèves de mathématiques élémentaires, il pensait profitable d'en faire un complément de cours pour les meilleurs élèves et conseillait leur introduction en deuxième année de mathématiques élémentaires. Dans la même revue, en 1904, un professeur de Genève, C. Cailler, proposait de définir les fractions comme des opérateurs, méthode qu'il disait avoir utilisée dans le secondaire (Cailler, 1904). Et c'est à l'occasion de la recension d'un ouvrage qui proposait une définition purement arithmétique des fractions que Bricard évoquait cette crise de l'enseignement des mathématiques élémentaires.

Une autre question d'arithmétique fréquemment discutée durant cette période est celle de la définition des nombres irrationnels. Poincaré indiquait qu'il fallait définir les nombres incommensurables en partant des longueurs. Dans le même livre d'arithmétique, Tannery définissait les irrationnels par la méthode des coupures de Dedekind. Le chapitre était signalé comme hors programme mais, selon l'auteur, les efforts nécessaires pour comprendre cette construction n'étaient pas « au-dessus de la portée d'un bon élève de la classe de mathématiques élémentaires » (Tannery, 1894, pp. VII). Le *Traité d'Arithmétique* de Eugène Humbert, publié un an auparavant, destiné lui aussi aux élèves de mathématiques élémentaires, définissait déjà les irrationnels comme limites de suites convergentes (Humbert, 1893). C'est aussi l'exemple de la définition des irrationnels que prendra Couturat lorsqu'il écrira :

> Il semble qu'on se soit trop hâté (par un souci de rigueur logique qui fait honneur à la conscience des professeurs) d'introduire dans l'enseignement élémentaire des lycées des théories un peu trop abstraites et subtiles. (Couturat, 1916, pp. 880) [16]

Et c'est toujours à propos des irrationnels que Émile Picard rappellera la critique suivante faite à Tannery :

> On a parfois reproché à Tannery d'avoir exercé une mauvaise influence sur l'enseignement des mathématiques spéciales. Certes, il a pu arriver qu'un normalien, à sa sortie de l'École, ait voulu montrer les abîmes que cachait la notion de nombre incommensurable à de jeunes lycéens. (Picard, 1925, pp. XVIII)

Enfin, quittant le champ de l'arithmétique mais restant dans celui de l'arithmétisation de l'analyse, nous terminerons par la définition de l'intégrale définie à laquelle Poincaré consacre un passage de sa conférence. Selon lui, elle doit être définie comme une surface. Comparant les définitions des mathématiciens de l'époque à celles de « leurs pères », il écrit :

> Et alors, pour définir une intégrale, nous prenons toutes sortes de précautions : nous distinguons les fonctions continues de celles qui sont discontinues, celles qui ont des dérivées de celles qui n'en ont pas. Tout cela a sa place dans l'enseignement des Facultés ; tout cela serait détestable dans les lycées. (Poincaré, 1904, pp. 23)

Or, dès la fin des années 1880, des manuels destinés aux élèves de la classe de mathématiques spéciales donnaient la définition de Riemann de l'intégrale définie [17]. En réaction contre ces introductions, le programme du concours d'admission à l'École Polytechnique précisait en 1895 que la notion d'intégrale définie devait être fondée sur la notion d'aire. Cette notion sera supprimée au concours de 1897. À la suite de cette suppression, Maurice Fouché, professeur au collège Sainte Barbe, proposera, dans les *Nouvelles Annales de Mathématiques*, une définition de l'intégrale définie basée sur la considération des ensembles en s'appuyant sur les travaux de Tannery sur les incommensurables (Fouché, 1896).

Nous avons beaucoup évoqué Tannery. Son influence sur les professeurs de lycée était considérable en raison de sa position institutionnelle [18]. Ses textes, et en particulier son arithmétique, sont fréquemment cités par des professeurs du secondaire. Mais comment discerner les signes de cet « enseignement logique » dans la pratique même de ces enseignants ? Les rapports d'inspection, bien que souvent peu diserts sur le déroulement d'une leçon, font partie des sources qui peuvent nous y aider. Ainsi, lorsqu'on lit que tel professeur « a l'esprit philosophique », que l'inspecteur craint que le cours d'un autre « ne soit un peu trop savant », ou qu'un troisième ne propose « des démonstrations trop philosophiques » on peut se demander s'il n'est pas question d'un de ces « enseignements logiques » [19].

CONCLUSION

Faut-il croire Tannery quand il écrit que le livre de Laurent n'a pas eu d'influence sur l'enseignement secondaire ? Sa position institutionnelle donne un indiscutable crédit à ses propos. La critique de ce « système logique » par Pesloüan est à replacer, comme celle de la logistique, dans le contexte politique et social de l'époque marquée par d'importantes réformes du système éducatif. Mais, en citant le nom de Tannery, Pesloüan a désigné l'un des acteurs majeurs de la formation des professeurs de lycée. Diffuseur de théories nouvelles, particulièrement celles issues de l'école allemande [20], on peut légitimement supposer que de jeunes normaliens aient eu envie, comme le rappelait Picard, de faire partager à leurs élèves quelques uns des principes qu'il leur enseignait. Cependant, dans la formation des professeurs, d'autres jeunes mathématiciens rompus à ces « mathématiques modernes » intervenaient. Citons notamment Émile Borel, maître de conférences à l'École Normale Supérieure et Jacques Hadamard professeur adjoint à la Sorbonne.

Enfin, suivant en cela Pesloüan, cet article s'occupe essentiellement d'arithmétique. Le sujet est débattu, nous l'avons vu, dans les revues de l'époque, mais le nombre d'articles consacrés à l'arithmétique reste très inférieur à celui des articles se rapportant à la géométrie. Ceci est particulièrement vrai après les modifications de programmes de 1905 et les expériences pédagogiques inspirées par la géométrie de Méray [21].

C'est bien tout le champ de l'enseignement des mathématiques qui est touché par une volonté de modernisation qui trouve en partie son origine dans les découvertes théoriques du XIX^{e} siècle. Les réactions de Pesloüan à cette modernité, réelle ou fantasmée, ne sont qu'un exemple de ces oppositions qui mériteraient d'être mieux connues.

NOTES

[1] Pour ce paragraphe, voir en particulier : Rivenc et Rouilhan 1992, Vernant 2003, Belna 2005.

[2] Par exemple, le raisonnement : Si « F est le fils de P » alors « P est le père de F » ne peut être réduit à une succession de syllogismes.

[3] Dans l'arithmétique de Russell, un nombre cardinal est une classe de classes équivalentes, c'est-à-dire entre lesquelles on peut établir une relation biuniforme (une bijection en termes actuels), 0 étant la classe qui comprend la seule classe nulle (l'ensemble vide).

[4] Peano ne se situait pas dans le projet logiciste. Il avait inventé ce langage symbolique afin de réécrire les mathématiques sans utiliser les langues traditionnelles. Un exemple d'emploi de ce langage est donné plus loin.

[5] Cette controverse, son histoire, les problèmes philosophiques et mathématiques qu'elle soulève, a suscité et suscite encore de nombreuses études. Voir par exemple les actes du colloque *Louis Couturat... de Leibniz à Russell...* (1983), Heinzmann (1994), Schmid (2006), Le but de ce paragraphe est simplement de signaler, parmi les points essentiels de la controverse, ceux qui fourniront la trame de la critique de Pesloüan.

[6] Dans sa réponse Couturat en donne ce qu'il appelle une traduction verbale: « 1 est le type d'ordre des classes ordonnées dont le nombre cardinal est un » (Couturat, 1906b, pp. 227).

[7] Pesloüan poursuivra cependant des recherches mathématiques en dehors de tout cadre institutionnel, faisant publier en 1920 deux notes d'Analyse mathématique dans les *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*.

[8] On pourra à ce sujet consulter le *Bulletin de l'Amitié Charles Péguy*, n° 45. Dans le chapitre III de son livre où il critique la « suprématie » des mathématiques dans l'enseignement, en particulier celui de l'histoire, Pesloüan dépeint son travail comme une contribution aux idées de Péguy. A propos de Barrès, signalons qu'il participera en 1929 à la publication du livre *: Le dernier projet littéraire de Maurice Barrès, Descartes et la princesse Élisabeth*, Paris, Champion.

[9] Rappelons qu'à cette époque les établissements secondaires sous la tutelle de l'Église Catholique regroupent environ 90 000 élèves, contre 86 000 dans les collèges et lycées de l'enseignement public et 9 000 pour les établissements de « l'enseignement libre » laïque.

[10] En ce début du XXe siècle, le collège public est un établissement secondaire qui peut dispenser les mêmes enseignements que le lycée mais, plus souvent, une partie seulement de ces enseignements. A la différence du lycée il est financièrement à la charge des communes pour les locaux et le traitement des personnels mais reste cependant sous l'autorité de l'état.

[11] Cette seconde édition du texte de Couturat par la Librairie Scientifique et Technique Albert Blanchard, en 1980, est une reprise intégrale du texte de la première édition.

[12] La revue américaine *The Journal of Philosophy, Psychology and Scientific Methods* propose une brève recension des articles en janvier 1908. Elle donne aussi un compte rendu du livre. Le philosophe italien Benedetto Croce signale le livre dans sa revue *La Critica*, en 1910. Le philosophe Georges Sorel, qui avait recommandé le livre à Croce, le cite aussi dans son article « Vues sur les problèmes de la Philosophie » paru dans la *Revue de Métaphysique et de Morale* en 1910.

[13] Par exemple, dans la dernière lettre, il insère un long paragraphe dans lequel il soutient que la logistique a de fortes attaches avec des courants de pensée de « l'École » (comprendre l'École Normale Supérieure).

[14] Il est fréquent de trouver dans diverses revues de l'époque les brèves recensions signées des initiales de l'auteur. Aucun collaborateur de la *Revue de Philosophie* n'ayant ces initiales, on ne peut éviter de penser à Henri Poincaré, d'autant plus que certains passages sont proches par le ton et les propos des articles de Poincaré sur la logistique. On ne connaît cependant pas d'autre recension de la main de Poincaré.

[15] Le texte cité ici est extrait de l'ouvrage posthume de Tannery, *Science et Philsophie*, qui réunit ses écrits concernant la philosophie des science, publiés dans diverses revues.

[16] Il s'agit d'un article posthume de Couturat. Nous ne disposons pas de plus d'information sur la date à laquelle il a été écrit.

[17] Citons notamment : Niewenglowski, B. (1889). *Cours d'Algèbre*, Paris, Armand Colin et Pruvost, E., Pieron, D. (1893). *Leçons d'Algèbre*, Paris, Paul Dupont (Pruvost était Inspecteur Général de l'Instruction publique et Piéron Inspecteur de l'Académie de Paris).

[18] En 1903, au moins 1/3 des professeurs de lycée étaient d'anciens élèves de tannery.

[19] Ces quelques exemples ont été obtenus en consultant une dizaine de dossiers de carrière pris parmi ceux d'anciens élèves de l'École Normale Supérieure ayant obtenu leur agrégation entre 1884 et 1902. Sources : Archives Nationales, respectivement dossiers : F/17/24520, F/17/24512 et F/17/24899

[20] Quand Laurent répond à une critique de Tannery sur sa définition du nombre, il écrit qu'il souhaiterait un jour faire une critique des « méthodes allemandes » (Laurent, 1898).

[21] Au sujet de l'enseignement de la géométrie au tournant du XX^{e} siècle on pourra notamment consulter Bkouche (1991), Belhoste, Gispert et Hulin (1996), d'Enfert et Gispert (2010). On pourra aussi lire, par exemple, l'article de Marotte (1907), Les Nouveaux Éléments de Géométrie de M. C. Méray, *La Revue de l'Enseignement des Sciences*, 10, 359-367, dans lequel Marotte qualifie Méray de logicien et son livre d'une arithmétisation de la géométrie qu'il apparente aux travaux de Hilbert.

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Oral Presentation

SPECIALIZATION AND GENERALIZATION OF THE MATHEMATICAL CONCEPT BY MR. INATSUGI SEIICHI IN JAPAN

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I describe the nature of the "life arithmetic" which was a mainstream in the arithmetic education of Japan in the 1930s. Mr. Shigeo Katagiri said about it that it was the ideal arithmetic education as an integrated part of the "life" and "mathematical. "On the other hand, Mr. Tsutomu Okano said about it that it was to train a attitude of "arithmetic view and arithmetic ideal" by using a "typical life" material. However, both men are evaluating Mr. Inatsugi Seiichi. This point is very interesting for us. However, although both of the above mentioned men cite the discussion of Mr. Inatsugi, Mr. Katagiri was positioned as the "true of life arithmetic." And, Mr. Okano was positioned as the highest reached point of the educational content research to oppose the black covered textbook, which lead to the academic mathematics starting from the amount. Therefore, I want to clarify the flow from the arithmetic reform movement of children centricity to the formation process green covered textbook by considering the claim of Mr. Inatsugi in the 1920s and 1930s. As the result, the next thing became clear. Mr. Inatsugi insisted that the mathematical contents of geometry and algebra were introduced into the elementary school by the influence of mathematics teaching movement. On the other hand, he was opposed to the arithmetic education of the children living center that does not live up to the system of the subject. Mr. Inatsugi had the goal to connect mathematics to the arithmetic according to the logical system of arithmetic. He was not only to induct the arithmetic theory from the life experience of the children. He was able to justify the teaching contents of the mathematics and advanced arithmetic by including a further generalization of the theory which is generalized once. In this regard, Mr. Katagiri was regarded as integration "life" and "mathematical principle" about the claim of "specialization and generalization of mathematical ideas" by Mr. Inatsugi. By consideration of this, I found that the integration which was tried by Mr. Inatsugi was not a "living" and "mathematical principle." The integration he tried was the learning along the lines of the subject to teach the "children centricity" and "higher mathematics." Also, Mr. Okano overlooked this attempt by Mr. Inatsugi for he was not watching a claim of arithmetic education of the children centricity. I say about what was claimed "specialization and generalization of mathematical ideas", why at this time. In 1926, "Elementary School Ordinance Enforcement Regulations" was amended in Japan. The geometry and the algebra were introduced in the high school. The languages which are "an experiment and survey is used" and the "It is made to get used to handling of a chart, a table of compound interest, etc." was added in the elementary school. At the National Council of school teacher in1929, the intention of the new national textbook compiled to replace the black covered textbook had been shown from the Ministry of Education in Japan. The opinion of Mr. Inatsugi was an arithmetic educational theory aiming at integration a children centricity and the mathematics which thought the arithmetic theoretical system as important.

THEME 7:

HISTORY OF MATHEMATICS IN THE NORDIC COUNTRIES

Plenary Lecture

JULIUS PETERSEN AND JAMES JOSEPH SYLVESTER - THE EMERGENCE OF GRAPH THEORY

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Mathematics in Denmark was for centuries a rather sad story. Denmark is without world famous people in mathematics, unlike in physics and astronomy with Tycho Brahe, Ole Rømer, H.C. Ørsted and Niels Bohr, to mention the most famous. We would however like to think that things have changed and that mathematics in Denmark now does rather well. This being so, when did it change? It is of course difficult to point to a single year, but if we have to, then a good suggestion would be 1871. That year saw the appointment of two young friends at Copenhagen University and the Polytecnical School, Hjeronymus Georg Zeuthen and Julius Petersen. In their days Zeuthen was the clear number one , but today Petersen is probably the best known of the two. Petersen's claim to fame rests on his development of and contributions to two fields: elementary plane geometry and the theory of graphs.

The story of how the theory of graphs emerged is an interesting piece of history of mathematics, involving James Joseph Sylvester, who visited Sweden and Denmark in 1889. This led to Sylvester's collaboration with Petersen and to Petersen's famous paper "Die Theorie der regulären graphs" in Acta Mathematica in 1891. Petersen used the English word "graph" in his otherwise German language paper, because "graph" is an English word that he learned from its inventor Sylvester (who by the way is also responsible for mathematical words like matrix, discriminant, and many more). The story of Petersen and Sylvester, put into a broad framework, will be the topic of this lecture.

Workshop

JULES HOUËL (1823-86): A FRENCH MATHEMATICIAN WELL CONNECTED TO THE NORDIC MATHEMATICIANS IN THE SECOND PART OF THE XIXTH CENTURY

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The French mathematician and astronom Jules Houël (1823-86) taught especially real and complex analysis in the Bordeaux Science Faculty from 1859 until 1884. He was known as polyglott and a brilliant computer; he diffused the ideas of noneuclidean geometries into France. He founded and co-chaired – from 1870 until 1883, with G. Darboux - the “Bulletin des sciences mathématiques et astronomiques”, which should inform French scientifics of European mathematics and astronomy. So Houël was connected to many European mathematicians. The study of the Houël's remaining correspondences shows two geographic poles: Italy (Battaglini, Bellavitis, Beltrami, d'Ovidio, Forti, ...) and Scandinavian (C.A. Bjerknes, Dillner, Lie, Mittag-Leffler, Zeuthen). With Italian mathematicians, Houël discussed especially noneuclidean geometries and their diffusion and with Nordic ones complex analysis, elliptic functions and also Abel's work and life. The second point will be the object of the present workshop. The most important of Houël's Nordic correspondents – in term of durée and quantity – was J. Diller, assistant professor in Uppsala University and editor of the Journal “Tidskrift för mathematik och physik”. But no letters between both has been found; we only know about it from other correspondents. Dillner is an important person as the “supervisor” of Mittag-Leffler's Ph.D and encouraged him to write to Houël. The correspondence between Mittag-Leffler and Houël lasted from July 1872 until may 1883. It is really interesting because Houël is the first non Scandinavian mathematician whom Mittag-Leffler wrote to so we can follow the genesis of his ideas and many issues are covered. Obviously, the starting point is mathematics and more precisely complex analysis. Repeatedly, Mittag-Leffler and Houël discussed the theory of functions of one complex variable and elliptic functions; they discussed also the ways of teaching them. The organization of mathematics education in Europe and especially in France and Germany is a recurrent topic. Finally, the mathematics journals are omnipresent in the correspondence. Houël corresponded with S. Lie, a collaborator; in the early 80s, Lie informed him of the publication of the N.H. Abel's biography in Norwegian by C.A. Bjerknes; the old Houël decided to translate it into French in order to diffuse the mathematical ideas and the life of that great genius. Houël asked Bjerknes help for translating it. They corresponded from beginning of 1882 until May 1885; after May 1885, Houël could not work anymore: he died in June 1886. So that translation is the last work of Houël. Houël asked that his name not to be written on the publication: he found his work not enough successful... We will provide letters translated into English from Houël to Mittag-Leffler, to Lie, to Bjerknes and from Mittag-Leffler, Bjerknes to Houël in order to present those strong connections between Houël and Nordic mathematicians.

Oral Presentation

E.G. BJÖRLING'S WORK WITH CONVERGENCE OF INFINITE FUNCTION SERIES IN VIEW OF PROSE AND CALCULUS

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In this paper we consider the distinction between the terms prose and calculus which originates from the philosophy of mathematics. We suggest that prose and calculus are useful historiographical tools in order to get a better understanding of historical texts in mathematics. This is concretized by means of an investigation of the Swedish mathematician E.G. Björling's contribution to the development of Cauchy's sum theorem from the mid 19th century. On the basis of the terms prose and calculus Björling's own distinction between "convergence for every value of x" and "convergence for every given value of x" is considered in a historiographical point of view.

INTRODUCTION

Sometimes historical texts in mathematics are interpreted in various ways. A typical example is *Cauchy's sum theorem,* first formulated in 1821, which has been debated both by contemporary mathematicians to Cauchy and by modern interpreters. The contemporary mathematicians Abel (1826), Stokes (1847) and Seidel (1848) came up with exceptions and corrections to theorem. Meanwhile, modern interpreters have struggled with what Cauchy really meant with his convergence condition, but also what he meant with concepts such as functions, continuity and infinitesimal quantities (see for instance Domar, 1987, Grattan-Guinness, 1986, Laugwitz, 1980, and Spalt, 2002). It may be tempting to use the modern distinction between pointwise and uniform convergence in order to interpret Cauchy's convergence condition. But of course that would not be a proper or historically correct way of understanding how Cauchy was *thinking* since pointwise and uniform convergence are based on a modern conceptual framework which was not available for Cauchy.

An issue that has been debated within the community of history of mathematics is the legitimacy, historiography and methodology of historical research in mathematics. Grattan-Guinness (2004) has come up with a distinction between "history" and "heritage", as two different approaches of considering historical texts in mathematics. The history approach deals with what happened in the past regardless of the modern situation. Meanwhile, the heritage approach is focused on the contemporary situation and the way of getting to the contemporary situation. According to Grattan-Guinness each approach is just as legitimate as the other but conflating history with heritage can lead to anachronistic readings of historical texts in mathematics.

In this paper we introduce the terms *prose* and *calculus* as a way to specify the *historiographical* perspective of Grattan-Guinness' distinction between history and heritage. The terms prose and calculus originate from the field of philosophy of mathematics. Some philosophers consider mathematics from two different levels; a prose level, which refers to the language of mathematics and a calculus level, which refers to mathematics as a symbolic system.

The aim of this paper is to consider the distinction between the terms prose and calculus as a historiographical tool in order to better understand historical texts in mathematics. This has been applied on the Swedish mathematician E.G. Björling's (1808-1872) work with convergence of infinite function series during the mid 19th century. Björling lived during a time period when mathematics underwent a major change from being connected to the intuition of time and space to becoming purely abstract (see for instance Jahnke, 1993). On the basis of his own distinction between "convergence for every value of x" and "convergence for every *given* value of x", where the latter is a stronger condition than the former, Björling formulated his own version of *Cauchy's sum theorem* in an article from 1846. Due to our investigation, it seems that Björling was developing his mathematics at a prose level in the sence that he was expressing his convergence conditions in ordinary language and by means of specific examples of infinite function series. But he had trouble expressing his convergence conditions within a mathematical system (or a calculus) of well-defined terms.

The intention in this paper is also to consider the wider question why historical texts in mathematics can be interpreted in various ways. This question is also concretized by considering Björling's work with convergence of infinite function series. In 1853 Björling returned to his distinction between "convergence for every value of x" and "convergence for every *given* value of x". This time, inspired by Cauchy, Björling was interpreting his own expression "for every value of x" to include $= 1 - \frac{1}{2n}$, concluding that when we require x to depend in such a way on n, then we are using the stronger of his convergence conditions. Here Björling was clearly inspired by Cauchy who earlier the same year had used that if an equality holds "always" it must hold for $x = 1/n$. However, it is unclear what expressions such that $x = 1 - \frac{1}{2n}$ and $x = 1/n$ really meant to Cauchy and Björling. In modern time, these expressions have been interpreted differently by several historians of mathematics, see for instance Katz & Katz (2011), Giusti (1984), Laugwitz (1980) and Bråting (2007). In terms of prose and calculus, we suggest that one reason why these expressions can be interpreted in several ways lies in a conflation between what was prose and what was calculus. In this particular example it seems that historical prose has been interpreted in modern terms in the light of a modern calculus. This is exemplified with Laugwitz' (1980) interpretation of $x = 1/n$ as an infinitesimal quantity generated by the sequence $1/n$ within Schmieden and Laugwitz' (1958) (modern) theory of non-standard analysis.

PROSE AND CALCULUS

Within the field of mathematics' philosophy there are different opinions of how to view elementary arithmetical relations such as 5+7=12, especially when it comes to the so called necessity of 5+7=12. Some philosophers (for instance Putnam and Quine) claim that 5+7=12 is a proposition whose opposite is unintelligible. Meanwhile, some other philosophers (for instance Wittgenstein) claim that it is not a proposition (which is true or false) but rather a *calculus*: no experiment can refute that 5+7=12, if we get 13 electrons when we count 5 and 7 electrons we have not added them. In other words, if we get 13 electrons we must have applied another rule than our usual addition rule. On the other hand, mathematics has a *prose* that among other things enables us to apply the *calculus*. Perhaps one can say that the prose provides mathematics meaning in applications. In this way, one can (in applications) consider 5+7=12 as a proposition. Within Wittgenstein's philosophy it is essential that mathematics has a prose level, where we speak mathematics in our ordinary language, and a calculus level, which can be considered as a system of rules. When we for instance say "there is a triangle whose angle sum is greater than 180 degrees" is a true proposition, we mean that within a calculus which is based on non-Euclidean geometry there exists such triangles.

In connection to an essay regarding the origin of symbolic mathematics Stenlund (2014) considers Wittgenstein's view of the distinction between prose and calculus. In his middle period, Wittgenstein spoke of mathematics as consisting of symbolic systems, games or calculi that are autonomous systems determined by rules for the operation and transformation of expressions. According to Stenlund, the separation of calculus and prose was an important part of Wittgenstein's conceptual investigation. And it was a difficult part since we are not always aware of the ways in which the use of 'prose-expressions' in a mathematical system affects our understanding of the system as a whole (Stenlund, 2014, p. 58). Wittgenstein repeatedly warned about the "*prose accompanying the calculus*". In *Philosophical Grammar*, he writes:

> If you want to know what the expression "continuity of a function" means, look at the proof of continuity [of functions]; that will show what it proves. Don't look at the result as it is expressed in prose, or in the Russellian notation, which is simply a translation of the prose expression; but fix your attention on the calculation actually going on in the proof. The verbal expression of the allegedly proved proposition is in most cases misleading, because it conceals the real purport of the proof, which can be seen with full clarity only in the proof itself (PG, p. 369-370).

In the beginning of the 1930th Wittgenstein had a strict way of considering mathematics as a pure calculus. He writes:

> Mathematics consists entirely of calculations. In mathematics *everything* is algorithm, and *nothing* is meaning, even when it doesn't look like that because we seem to be using *words* to talk *about* mathematical things (PG, 1974, p. 468).

However, in the mid 1930th Wittgenstein's thinking changed on this point when he entered a more anthropological point of view. The use of mathematical signs in applications outside mathematics was contributing to the meaning of mathematical signs (Stenlund, 2014, p. 61). In *Remarks on the foundations of mathematics,* written in 1942, Wittgenstein argues:

> It is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics (RFM, p. 257).

According to Stenlund, the calculus/prose distinction was still important in Wittgenstein's later work, though in a less dogmatic sense.

However, it should be pointed out that we do not believe that all mathematics can be distinguished as either prose or calculus, but we think that in a historiographical perspective it can be helpful to discuss old mathematical texts by means of the aspects of prose and calculus.

Prose and calculus as historiographical tools

We believe that the terms *prose* and *calculus* can be useful in the historiography of mathematics and in connection with how historians of mathematics are studying and interpreting old mathematical texts. Sometimes historical texts in mathematics are interpreted in different ways. A reasonable question is how the same mathematical text can give rise to several different interpretations. Should not a mathematical text be expressed precise enough that it cannot be interpreted in different ways? Ivor Grattan-Guinness (2004) has discussed this issue on the basis of his own distinction between *history* and *heritage*, which can be seen as two different approaches of interpreting old mathematical texts. We believe that Grattan-Guinness' distinction between history and heritage, seen from a *historiographical* perspective, can be specified by means of discussing mathematics from a prose and calculus perspective. Let us first consider Grattan-Guinness' distinction between history and heritage and refer to the terms prose and calculus in order to specify the picture of Grattan-Guinness' distinction historiographically.

The history approach focuses on what happened in the past, regardless of the modern situation. In order to study a specific mathematical theory, definition, theorem, concept etc., the history approach focuses on the details of its development, its prehistory, the chronology of progress and its impact in the years immediately following. Grattan-Guinness claims:

> History addresses the question "what happened in the past?" and gives descriptions; maybe it also attempts explanations of some kinds, in order to answer the corresponding "why?" question. History should also address the dual questions "what did not happen in the past?" (Grattan-Guinness, 2004, p. 164).

In the citation right above, we would like to specify what it really means to "give descriptions" and "explanations of some kinds".

We believe that one of the main purposes of studying old mathematical texts with a history approach is to find out *what calculus* the historical mathematicians were using and in *which context* that calculus was used. Consequently, Grattan-Guinness' question cited above "What happened in the past?" may be specified with the following two questions:

- "What *calculus* did they use?" and
- "What *prose* did they use?".

These two questions are also helpful in connection with Grattan-Guinness' suggestions regarding the history approach that we cited above: "give descriptions" and "explanations of some kinds". Perhaps these two questions could be specified in the following way: "give descriptions and explanations by means of discussing which prose and calculus that was used during a specific time period".

Let us now turn to Grattan-Guinness' *heritage approach*. According to Grattan-Guinness the heritage approach focuses on the modern form of the notion studied with attention to the course of its development. Grattan-Guinness points out that heritage often refers to the impact of a certain mathematical notion (a mathematical theory, definition, theorem, concept, etc.) upon later work. He writes:

> Heritage addresses the question "how did we get here?" and often the answer reads like "the royal road to me." The modern notions are inserted into the notion [studied] when appropriate (Grattan-Guinness, 2004, p. 164).

Grattan-Guinness claims that the heritage approach is just as legitimate as the history approach, but he points out that conflating history with heritage can lead to an anachronistic reading of historical texts. Grattan-Guinness writes:

> The confusion of the two kinds of activity is not legitimate, either taking heritage to be history (frequently the mathematicians' view – and historians' sometimes!) or taking history to be heritage (the occasional burst of excess enthusiasm by a historian); indeed, such conflations may well mess up both categories, especially the historical record (Grattan-Guinness, 2004, p. 165).

We do not fully agree with Grattan-Guinness, at least not when heritage is viewed by means of the terms prose and calculus. We believe that the heritage approach can mean two things; that one reads in a modern calculus into an old calculus, alternatively, that one interprets historical prose in modern terms in the light of a modern calculus. We believe that the result of both these two approaches is that heritage alone often ends up as anachronistic interpretations of historical texts. That is, we do not fully agree with Grattan-Guinness that the problem with studies of historical mathematical texts lies in a conflation between history and heritage, but rather that the problem consists of a conflation between what was prose and what was calculus (which will be concreticized in the next Section of this paper).

E.G. BJÖRLING'S WORK WITH CONVERGENCE OF INFINITE FUNCTION SERIES

Emanuel Gabriel Björling (1808-1872) got his doctoral degree at Uppsala University in 1830. He was a prominent author of textbooks in mathematics with great influence on Swedish school mathematics (Gårding, 1998, p. 16). Björling was an active reader of Cauchy's mathematical texts. Among other things he was interested in the definitions of the functions x^y and $\log_\beta x$, which was a subject for a correspondence with Cauchy. One of the results of his contact with Cauchy was Björling's concern to clarify fundamental concepts in mathematical analysis, such as functions, continuity and convergence.

During the mid 19th century Björling was involved in the development of the famous *Cauchy's sum theorem.* Björling's contribution has been considered earlier by among others Grattan-Guinness (1986), Domar, (1987), Gårding (1998), Bråting (2007) and Katz & Katz (2011). Grattan-Guinness (1986) has claimed that Cauchy must have read some of Björling's work on the subject and has even suggested that Cauchy was inspired by Björling when he was modifying his own theorem in 1853.

In this Section a brief summary of the development of *Cauchy's sum theorem* will be considered before Björling's work with convergence of infinite function series will be presented and discussed in terms of prose and calculus.

Cauchy's sum theorem

In 1821 Cauchy claimed that a convergent series of real valued continuous functions is a continuous function.[i] However, the proof was relatively imprecise which has led to different interpretations of the formulation of the theorem. Contemporary mathematicians to Cauchy criticized *Cauchy's sum theorem* and came up with both exceptions (Abel, 1826) and corrections (Stokes, 1847, Seidel, 1848). For instance, Abel (1826, p. 316) showed that the trigonometric series

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots \qquad (1)$$

was an exception to Cauchy's 1821 theorem, since although (1) was a convergent series of real valued functions, the sum was discontinuous at $x = (2k+1)\pi$, for each integer k. In (Cauchy, 1853, p. 33), more than 30 years after his first mentioning of the theorem, Cauchy modified the theorem by adding the word "always" (toujours) to his convergence condition.

Not only contemporary mathematicians to Cauchy have had problems of interpreting Cauchy's 1821 theorem. Also modern interpreters have come up with different interpretations of the meaning of the theorem, especially the convergence condition but also concepts such as functions, continuity and infinitesimal quantities have been debated (which will be considered later in this paper).

Björling's convergence conditions interpreted with prose and calculus

In an article from 1846 Björling tried to explain and prove *Cauchy's sum theorem* on the basis of his own distinction between

- "convergence for every value of x" and
- "convergence for every *given* value of x",

where the former is a stronger convergence condition than the latter. [ii] Björling formulates his version of the sum theorem as follows:

> If a series of real-valued terms $f_1(x), f_2(x), f_3(x), \ldots$ is convergent for every real value of x from x_0 up to and including X, and in addition its particular terms are continuous functions of x between the given limits; then the sum $f_1(x) + f_2(x) + f_3(x) + \cdots$ necessarily has to be a continuous function of x between the given limits (Björling, 1846, p. 21). [iii]

Apparently, Björling applies the stronger condition "convergence for every value of x" in his theorem. In a footnote in the same 1846 article, Björling tries to explain the difference between his two convergence conditions by considering the two function series

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots \qquad (2)$$

and

$$\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \cdots \qquad (3)$$

He begins to discuss series (2) above, which is a similar series to Abel's exception, that is series (1) mentioned above. Björling concludes that the sum of series (2) is not continuous in the vicinity of $2k\pi$, k integer. For instance, within the limits 0 and 2π the sum of this series is $\frac{\pi - x}{2}$, but for $x = 0$ and $x = 2\pi$ the sum is 0 (Björling, 1846, p. 66). But, according to Björling, this does not affect the validity of his version of the sum theorem since series (2) only converges for "every *given* value of x" within 0 and 2π but indeed not *"...for every value of x from one limit up to the other"* (Björling, 1846, p. 66). That is, for Björling series (2) is not an exception to his version of the sum theorem since it does not satisfy the condition "convergence for every value of x" in the vicinity of $2k\pi$, k integer. One can note that Björling does not define his two convergence criteria, he rather describes them on the basis of the function series (1) and (2).

In order to further clarify his convergence conditions, Björling introduces the distinction

- "next to" and
- "indefinitely close to".

He argues:

> […] if a series of continuous functions is convergent for every value of x "next to" a certain value X, it does by no means guarantee that it converges also for x-values "indefinitely close to" X (Björling, 1846, p. 66).

Björling returns to series (2) and claims that this series only converge "next to" $x = 0$ and $x = 2\pi$, but it does *not* converge "indefinitely close to" $x = 0$ and $x = 2\pi$.

To further explain the distinction between "next to" and "indefinitely close to" Björling considers series (3) above. Series (3) differs from (2) since it, in Björling's terminology, converges for every given value of x within the limits $x = 0$ and $x = 2\pi$, but is *divergent* at $x = 0$ and $x = 2\pi$. Björling argues:

> […] it is quite obvious that a series divergent for x=X may converge for x-values "next to" X, but it can certainly not converge "indefinitely close to X" (Björling, 1846, p. 66).

Björling points out that for a series such as (2), which converges also for the x-values 0 and 2π, it is less obvious that it does not converge for x-values "indefinitely close to" 0 and 2π. He draws the conclusion that it is especially for excluding series such as (2) that his version of the theorem can be useful, that is, for a series which is convergent "for every *given* value of x".

It seems that the distinction between "for every value of x" and "for every *given* value of x" means that the latter notion does not include x-values "indefinitely close to" a particular point. However, Björling's explanation of his two distinctions is far from satisfactory. The main reason for this is that he does not show *why* a certain series converges only for "every *given* value of x" or only for "x-values next to" a specific x. He gives examples of series that only converge "for every *given* value of x", that is, these series only satisfy the weaker of his convergence conditions. But he does not explain how one can determine which convergence condition an arbitrary series satisfies.

In terms of prose and calculus, one can perhaps argue that one reason why Björling was unable to explain his convergence conditions properly was that he developed his convergence conditions at a prose level in the sense that he was expressing his conditions by means of "ordinary language" and specific examples of infinite functions series. Further evidence of this claim can be found in other parts of Björling's work. For instance, Björling considers a function in the following way:

> *F(x)* denotes an analytical expression which contains *x* (a real variable), that is a function of *x* […] (Björling, 1852, p. 171).

Hence, for Björling every expression containing a variable x is a function. In 1852 Björling considers $\frac{x}{|x|}$ as a function which is two-valued at $x = 0$, but has no derivative at $x = 0$ since the graph of the first derivative function jumps at $x = 0$. Meanwhile, according to Björling, the function $|x|$ exists at $x = 0$ and attains the two values ± 1 (Björling, 1852, pp. 173-177). These examples reveal that Björling had trouble to anchor his concepts in a corresponding mathematical system (or calculus), but instead was working more or less at a prose level.

Seven years later, in 1853, Björling returns to his sum theorem. Earlier the same year Cauchy had modified his 1821 theorem by adding the term "always" (as mentioned above) to his earlier convergence condition. Björling verifies that Cauchy's modified theorem is correct and claims that his own version from 1846 is in complete agreement with Cauchy's modified theorem. Björling provides further examples of series that only converge "for every *given* value of x", but what is more interesting, this time Björling expresses an "x-value indefinitely close to" a specific point as a sequence that depends on n, such as $x = 1/n$. That is, Björling gives a mathematical explanation of his convergence conditions which he was unable to do in 1846.

In his 1853 paper, Björling uses the expression $x = 1 - \frac{1}{2n}$ as an input value to show that the function series

$$x^{2n}(1-x) + x^{2n+2}(1-x) + x^{2n+4}(1-x) + \cdots \qquad (4)$$

does not "converge for every value of positive $x < 1$" and therefore is not an exception to *Cauchy's sum theorem* (Björling, 1853, pp. 147-151).[iv] At this point Björling probably is inspired by Cauchy since the latter earlier the same year had used that if an equality holds "always" it must hold for $x = 1/n$. (According to Grattan-Guinness (1986), Cauchy in his turn was inspired by Cauchy when he came up with his stronger convergence condition "always convergent".)

However, it is unclear what expressions such as $x = 1/n$ and $x = 1 - \frac{1}{2n}$ meant to Cauchy and Björling. In 1821 Cauchy defined an infinitesimal as a variable which becomes zero in the limit.[v] On the basis of Björling's 1853 paper one can draw the conclusion that when we require x to depend on n we are using Björling's stronger convergence condition "convergence for every value of x". In (Bråting, 2007) this is interpreted as, for Björling, x-values "indefinitely close to" a particular point are allowed to move in the sense that we for all n are allowed to choose a new x. However, Björling was never able to connect the variables n and x in expressions such as the modern term $f_n(x)$; he only "quantified" over the variable x. As Grattan-Guinness (2000) points out, during the 19th century there was a problem to distinguish between "for all x there is a y such that..." from "there is a y such that for all x..." (Grattan-Guinness, 2000, p. 70).

In modern time the expression $x = 1/n$ has been interpreted in various ways. For instance, Giusti (1984) claims that $x = 1/n$ should be viewed as an ordinary sequence having 0 as a limit. Menawhile, Laugwitz (1987) emphasizes that this expression should be seen as an infinitesimal quantity generated by the sequence $1/n$. Katz and Katz (2011) distinguish between two different continua; the standard Archimedean continuum and a continuum enriched with infinitesimals called the Bernoulli continuum. They argue that the expression $x = 1/n$ belongs to the latter continuum and state that Björling's condition "convergence for every value of x" refers to x-values from the Bernoulli continuum and "convergence for every *given* value of x" refers to x-values from the Archimedean continuum (Katz & Katz, 2011, pp. 432-433).

A justified question might be how it is possible that the same mathematical text can give rise to several different interpretations? Apparently one possible answer is that in modern terminology we can give many different versions of Cauchy's and Björling's mathematics. But when it comes to the question how Cauchy and Björling really was *thinking* we have to put the modern mathematical terms aside, which sometimes can be difficult. It is easy to unintendently interpret historical mathematics at a calculus level even in those cases when the mathematics was developed more or less at a prose level. In such cases we believe that it can be helpful to take into account the questions “What was the prose?” and “What was the calculus?”, respectively. The distinction between history and heritage is focusing between two different approaches of doing historical research. However, the prose and calculus distinction can specify Grattan-Guinness distinction since it can be used as a tool to better understand what it means to do historical research in mathematics.

Perhaps one can argue that historical prose sometimes has been interpreted in the light of a (modern) calculus or, at least, the terms prose and calculus have been conflated. The different interpretations of $x = 1/n$ is a concrete and typical example of this problem. For instance, Laugwitz’ (1987) interpretation of $x = 1/n$ as an infinitesimal quantity generated by the sequence $1/n$ is connected to a modern theory (or calculus) based on a non-standard analysis developed by Schmieden and Laugwitz (1958).[vi] Within Schmieden and Laugwitz’ non-standard analysis it is possible to establish *Cauchy’s sum theorem* utilizing a weaker convergence condition than uniform convergence, yet sufficient for the theorem to be true (Palmgren, 2007, pp. 171–172). In fact, Palmgren proves that this weaker convergence condition not only is sufficient, but also a necessary condition for the theorem to be true (see Palmgren, 2007, p. 171). That is, one obtains a more precise version of *Cauchy’s sum theorem* in non-standard analysis than in our modern standard analysis. However, such an interpretation is not proper in a historical perspective. As mentioned above, Schmieden and Laugwitz’ theory are based on a modern calculus that, for instance, presupposes a modern function concept which was not available to Cauchy and Björling (remember that Björling defined a function as “an analytical expression which contains a variable x” which was discussed above). With such an imprecise way of defining functions as Björling’s it seems unlikely that a weaker convergence concept than uniform convergence can guarantee continuity in the limit.[vii]

Öberg (2011) considers the development of *Cauchy’s sum theorem* in his doctoral thesis and points out that in 1821 there was no system of defined mathematical terms available to make a precise mathematical sense of Cauchy’s theorem:

> […] we do not know what Cauchy had “in mind” originally, and perhaps a good answer is that he did not know himself, since he lacked a system of well-defined mathematical terms (functions, etc.) to express an intuition he had about the convergence of function series. He was working in an excessive way on the level of mathematical “prose,” although this was something he certainly wanted to avoid. But mathematics at the time suffered from a confusion about some of the most fundamental terms in mathematics and in particular of a sensible way of setting up a relation between them. Although there is not one single right way of setting these things up, there was in 1821 no system of defined mathematical terms available to make a precise mathematical sense of Cauchy’s ”theorem” (Öberg, 2011, pp. 131-132).

Öberg’s observation gives a reasonable explanation of why Björling in 1846 was unable to express his convergence conditions by means of well-defined mathematical terms. There was simply no proper mathematical system, or calculus, available. When Björling in 1853 uses the expression $x = 1/n$ in order to express and clarify his convergence conditions from 1846 he implicitly says that "this is what I meant in 1846", which gives further evidence to the historiographical claim given in this paper; that Björling was developing his mathematics mostly at a prose level.

CONCLUDING REMARKS

In this paper the distinction between prose and calculus has been considered as a historiographical tool in order to better understand historical texts in mathematics. Especially, this has been applied on Björling's work regarding convergence conditions for infinite function series in connection to *Cauchy's sum theorem*. Due to our investigation, it seems that Björling in 1846 had an intuition of what he meant with his convergence conditions but was not able to express this intuition within a well-defined mathematical system. Stenlund (2014) points out that historically there was no sharp division between pure and applied mathematics, which reflected itself in the language of mathematics, in the prose of mathematics (Stenlund, 2014, p. 46). Stenlund exemplifies this with the word 'quantity'; "[…] does it derive its meaning from its use in physics, or does it have its meaning from the way it is operated within the mathematical calculus?" (Stenlund, 2014, p. 58).

In terms of prose and calculus, Björling was developing his mathematics at a prose level in the sense that he begun to express his convergence conditions more or less in ordinary language and by means of specific examples of infinite function series. After some years, influenced by Cauchy, he found a way to mathematically explain his convergence conditions by interpreting "infinitely close to" to include $x = 1 - \frac{1}{2n}$, concluding that when we require x to depend in such a way on n, then we are using the stronger of his convergence conditions ("convergence for every value of x"). It is interesting that Björling 1853 must interpret his own convergence condition from 1846 and in a way argue that "this is what I meant". It seems that he was trying to find a suitable way to mathematically express his convergence conditions afterwards. But the prose may be misleading, as we cited earlier in this paper: "[…] we are not aware of the ways in which the use of 'prose-expressions' in a mathematical system affects our understanding of the system as a whole" (Stenlund, 2014, p. 46).

But even though Björling in 1853 had found a way to express his convergence condition in mathematical terms it was not a mathematically satisfying explanation of what he meant with his convergence condition. As Öberg (2011) points out this may depend on that there in 1821 was no system of defined mathematical terms available to make a precise mathematical sense of *Cauchy's sum theorem*.

In this paper we have also used the terms prose and calculus in order to answer the question why historical texts in mathematics can be interpreted in different ways. In terms of prose and calculus, one answer to that question is that modern interpreters sometimes (unintendedly) conflate the prose level and the calculus level. Here that argument is mainly based on the example of the various modern interpretations of Björling's and Cauchy's expressions $x = 1/n$ and $x = 1 - \frac{1}{2n}$. We have come to the conclusion that sometimes modern historians of mathematics interpret historical prose in modern terms in the light of a modern calculus as, for instance, Laugwitz' interpretation of the expression $x = 1/n$ as an infinitesimal quantity generated by the sequence $x = 1/n$. In this particular case historical prose has been interpreted in the light of Schmieden and Laugwitz' own non-standard analysis which for instance presupposes the modern function concept.

In terms of Grattan-Guinness' distinction between history and heritage, which has been discussed in this paper, one may argue that Laugwitz had mixed up history with heritage, which according to Grattan-Guinness is not legitimate. But perhaps one could also state that Laugwitz was using a heritage approach in the sense that ”modern notions have been inserted into the notions studied when appropriate” which within Grattan-Guinness' theory would be perfectly legitimate. Here one has to take the interpreter's intention into account; was the intention to understand how Cauchy and Björling was thinking or was the intention to interpret historical mathematics within a modern theoretical framework? We believe that the result of both these two approaches is that heritage alone often ends up as anachronistic interpretations of historical texts. That is, we do not fully agree with Grattan-Guinness that the methodological problem with studies of historical mathematical texts lies in a conflation between history and heritage. We do rather believe that the problem consists of a conflation between what was prose and what was calculus.

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[i] Lorsque les différens termes de la série $u_0 + u_1 + u_2 + \cdots$ sont des fonctions d'une même variable x, continues par rapport à cette variable dans le voisinage d'une valeur particulière pour laquelle la série est convergente, la somme s de la série est aussi, dans le voisinage de cette valeur particulière, fonction continue de x (Cauchy, 1821, pp. 131-132).

[ii] Björling wrote two articles on this subject, one in 1846 which was written in Swedish and a similar article in 1847 written in Latin. In this paper we will refer to the former.

[iii] In his 1846 paper, Björling gives a proof of his own version of *Cauchy's sum theorem* which has been translated into English in (Bråting, 2007).

[iv] In (Bråting, 2007) a detailed survey of how Björling uses $x = 1 - \frac{1}{2n}$ to show that series (4) does not not "converge for every value of $x < 1$" is given.

[v] Lorsque les valeurs numériques successives d'une même variable décroissent indéfiniment, de manière á s'abaisser au dessous de tout nombre donné, cette variable devient ce qu'on nomme un *infiniment petit* ou une quantité *infiniment petit.* Une variable de cette espéce a zéro pour limite (Cauchy, 1821, p. 19).

[vi] Schmieden and Laugwitz (1958) developed the so called Ω-calculus, which is a more intuitive alternative to Robinson's (1966) non-standard analysis.

[vii] This is considered in detail in Bråting (2007).

POSTER SESSIONS

Poster Session – Theme 1

A HERMENEUTIC APPROACH TO HISTORY AND EPISTEMOLOGY IN MATHEMATICS EDUCATION: THE CASE OF PROBABILITY

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The paper presents a theoretical framework for a hermeneutic approach based on the interpretative line proposed by Bagni (2009). Our purpose is to show how this approach may serve as the basis of work with historical sources in teachers education. Its advantage is that it enables to present also teachers positions incommensurate with current mathematical discourse, thus relaxing the need for coherence imposed by an epistemological approach to learning. As an example of the approach, we look at the history of probability.

In his work *Interpretazione e didattica della matematica: una prospettiva ermeneutica* (Bagni, 2009), the Author suggests a shift from epistemology to hermeneutics in mathematics education. Indeed, contrasting Jahnke's hermeneutic approach (Jahnke et al., 2000) - which is Gadamer inspired and needs to see hermeneutics having an episodic character in the epistemological "stream" - Bagni's approach is Rorty inspired and allows to see hermeneutics in antithesis to epistemology. This approach authorizes to propose a radical pragmatic-hermeneutical approach to mathematics education. Although we use here historical sources to explain the research of interpretation tools, the subject works to each kind of text.

As investigative tool Bagni proposes an adaptation of Peirce's semiotics. At the basis of this semiotic approach we find the semiotic triangle but, from a global point of view Peirce's semiosis is a potentially unlimited process leading to the progressive construction of the meaning of a dynamic object. Bagni shows how the initial sign, which allows to start the semiotic chain, is comparable to an initial attitude (habit), and what Peirce calls "the final logical interpretant", can be seen as a mental "effect" (habit change) (Bagni, 2009, p. 212). We can explain this in the following matter. Facing the historical source, the subject is obligate to investigate the beliefs that induced the Author to formulate the sentences. She/he will do this according to her/his current beliefs and this may produce an awareness of the absence of an adequate knowledge, necessary for the interpretation; this allows to start the semiotic chain of meaning construction. The "habit change" would be a new, more meaningful attitude to face the text. The semiotic chain can be repeated using other sources until the subject judges the new attitude adequate to face a didactical processing of problems which treats the matter. This explains why the approach is able to produce and interpret changes in teachers' beliefs (Goldsmith et al., 2014) about mathematics, even when those changes involve radical reorganization of their system of reference: the reorganization is interpreted and measured by the acquired ability and don't' refer to real, objective values.

Probability is a meaningful example in this sense; its history has experienced at least two major epistemological ruptures, which teachers don't always seem to be aware of. The first, with Buffon (1777), involves a shift in focus from discrete situations common in classical treatments to continuous ones with a concomitant shift in operational tools, namely, from arithmetic to geometric tools. The second, with Kolmogorov (1933) and his axiomatization, which completely changes point of view and leaves so aside the question of the nature of probability. Furthermore, the asking of an answer of the last question, which can be accomplished starting from irreconcilable philosophical and epistemological assumptions (Cera, 1990), was an obstacle for the construction of mathematical theory of probability and can be seen as an epistemological obstacle (Bachelard, 1938). We suppose also that the history of probability provides a good example of epistemological ruptures arising from a cultural substrate and we are convinced that teachers' difficulties with probability, beyond those arising from an inadequate mathematical background (Stohl, 2005), may be ascribed to obstacles in the interpretation of probability concepts. Our treatment may thus contribute to the debate concerning the theory of epistemological obstacles as it appears in the work of Guy Brouseau (Perrin-Glorian, 1994) and in Luis Radford's Cultural Semiotics (D'Amore, Radford & Bagni, 2006).

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THE CONTRIBUTIONS OF FUNDS OF KNOWLEDGE AND CULTURALLY RELEVANT PEDAGOGY AS METHODOLOGIES FOR THE DEVELOPMENT OF SOCIOCULTURAL PERSPECTIVE OF HISTORY OF MATHEMATICS IN MATHEMATICS CLASSROOMS

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This study is grounded in the Sociocultural Perspective of History of Mathematics, Funds of Knowledge (FoK), and Culturally Relevant Pedagogy (CRP) theories. It was conducted with the purpose of seeking contributions to activities based on gaining insight into parts of students' culture, specifically, their FoK. The other purpose is to understand the role of History of Mathematics (HM) that can help teachers to comprehend students' questioning and reasoning about mathematics. The population was composed of 72 students from two classes in a first year of a technical course in a public technical high school in Ouro Preto in the state of Minas Gerais, Brazil. The researcher collected information that could answer the research question: What are some of possible contributions that activities based on students' funds of knowledge and anchored in sociocultural perspective of History of Mathematics can bring to teaching and learning functions through the use of Culturally Relevant Pedagogy approach? Two questionnaires, two focus groups, field notes, interviews and informal conversations with participants, and three documental records containing mathematics activities related to functions content were used. HM was applied in both implicit and explicit ways, which served as an orientation guide so that the researcher-teacher could develop the proposed activities by applying the FoK of participants, which helped in the analyses of the students way of represent and/or write functions concepts. We highlight the use of History of Mathematics in high school context in explicit and implicit ways. The implicit way let the teacher-researcher guide some activities and understand some of students' answers. On the other hand, the explicit way was used as problems taken from history to be worked out by the students.It was found that the acquisition of mathematical knowledge and algebraic symbolic language in the classroom is related to students' cultural experiences. This approach allowed us to use some propositions of CRP, which is defined as a critical pedagogy that is committed to collectivity and is based on a tripod composed by critical awareness, cultural competence, and academic success. For data collection, analysis, and interpretation of qualitative and quantitative data, a mixed methods study QUAN + QUAL and content analysis were used. Data were collected and analyzed concurrently in all phases of the study. Thereafter, the results were analyzed, discussed, and interpreted in order to be addressed as part of the research. The interpretation of the results showed that the majority of participants

learned and improved their knowledge in relation to symbolic algebraic notation by highlighting the importance of rhetoric stage of algebra in order to understand symbolism and academic development of symbolic algebraic language. Besides that, we drew attention to the fact that History of Mathematics used in explicit way cannot be applied as a teaching methodology for high school teachers in all mathematical content. However, it can be used in an implicit way to help teachers to understand students' reasoning even though the sociocultural context is very important in this understanding.

Poster Session – Theme 1

DESIGN RESEARCH ON INTEGRATING THE HISTORY OF MATHEMATICS INTO TEACHING AND APPLICATION

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INTRODUCTION

One common phenomenon: "High Evaluation, Low Application", always appears in mathematics teachers' comments and reflections on (IHT) (Wang,X. &Zhang,X., 2005) Therefore, this paper proposes a model of design research on IHT that may be useful in the conduction of HPM research, and show an example of application of design research on IHT.

WHAT IS DESIGN RESEARCH ON IHT?

Based on studies about design research (Bannan-Ritland 2003; Brown, 1992; Cobb, 2001; Yishay, 2010), the author proposes a model of design research on IHT (as shown in Figure 1). Using design research of IHT circularly processes five steps: Investigation & Preparation, Development & Design, Implementation & Operation, Analysis & Evaluation, Popularization & Application to produce reliable IHT instruction design, and to promote the development of theory and practice on IHT.

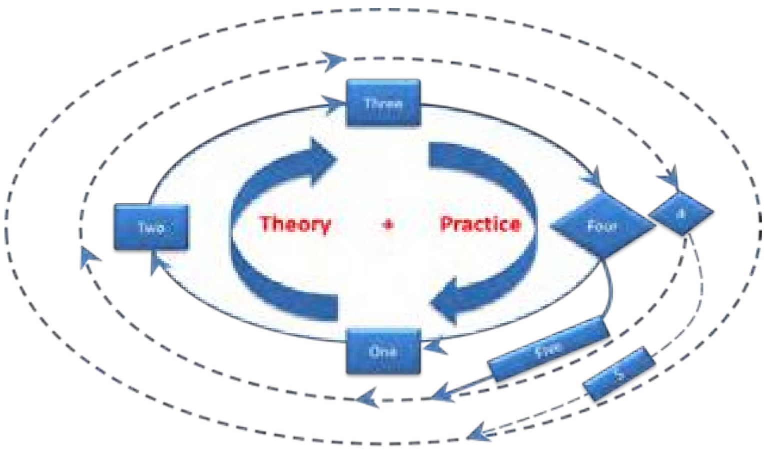

Figure 1: The five steps of HPM's design research

EXAMPLE ON USING IHT DESIGN RESEARCH

Many teachers complained that it was very difficult to teach mathematical induction (MI) and students cannot understanding the two steps meaning. Therefore, teachers will use teaching strategy of IHT by using design research on IHT. Based on the five steps of Figure 1, I will show how to use design research on IHT in the following.

Preparation & Investigation: the researchers started to search some materials on MI as many as possible when they received SOS from the teachers. Researchers will cooperate with historian of mathematics to obtain the original materials about MI, and receive some suggestions. Then, researchers well know that development of MI in history: a) Inductive reasoning in the period of Rhetoric Algebra; b) Recursive reasoning in the period of Abbreviated Algebra; c) Recursive reasoning in the period of Symbolic Algebra; d) Formalization after the Peano axioms (Katz, 2008). Finally, researchers discussed with the teachers, and decided that they would use the reconstruction to integrate the historical materials into the design of lesson.

Design & Development: Based on the preliminary design framework, the researchers and teachers develop the framework into the figure 2. At that time, they would choose some historical materials or problems to design the detailed lesson plan. The problems in Figure 2 includes towers of Hanoi, wrong pattern on natural numbers and Fermat's wrong conjecture, L-shaped tiles, Domino cards, and exercise problems on MI.

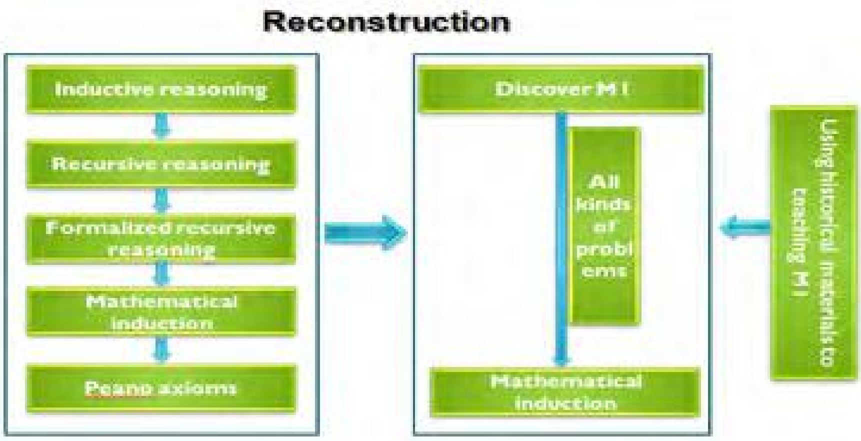

Figure 2: The framework of instruction design of mathematical induction

Operation & implement: Researchers and teachers work together to implement the design. During the process of teaching, Researchers cared about how to modify the historical materials to be adaptable for the students' cognition. The purpose of this way is to improve the quality of teaching.

Analysis & Evaluation: After the discussion and analysis the effect of teaching, researcher found some problems- wasting time in explaining the reasoning, the design cannot adequately use the historical materials. L-shaped tile problem is too difficult for students. Meanwhile, they thought about how to solve the

Application & Promotion: After repeating the four same steps, the design had been modified to the Figure 3. Most importantly, teachers and researchers agreed that the design can be recommended to other instructors for their checking after completing the analysis and evaluation. Of course, the new instructor will process the same procedures when they use and text the design. Therefore, the whole process can provide some evidence to support the theories of integrating history of mathematics into teaching practice, and promote the development of theories and practices on IHT.

In general, this example is to illustrate the application of IMT design research although the process is succinctly introduced, and much detailed information will be shown in another paper.

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Poster Session – Theme 2

PROGRESSIONS AND SERIES IN ANCIENT CHINA

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In this poster we present a brief chronology of the series numerical progressions and Chinese mathematics. The arithmetic and geometric progressions appear in China in a book called 九章算術 or Jiuzhang suanshu (Chu Chang Suan Shu) or Nine Chapters on the Mathematical Art, written approximately around 200 BC (Some historians assigned a date between 100 BC and 50 a. C.), and which over time mathematicians were adding various comments. In Chapter 3, named Cui fen or Distribution by Proportion, there are problems whose solution involves the use of arithmetic and geometric progressions.

Zhang Qiujian (also known as Chang Ch'iu-Chin or Chang Ch'iu-chien) wrote a book called Zhang Qiujian suanjing (Zhang Qiujian's Mathematical Manual) between 468 d. C. and 486 d. C. (Some historians date Zhang Qiujian 100 years before) consisting of 98 problems divided into three chapters. In this work solves and computes the sum of arithmetic progressions. Zhu Shijie also known as Chu Shih-Chieh was born around 1260 near Beijing, China. It is known that he wrote two works considered as surprising. The first named Suan xue qi meng (Introduction to mathematical studies) published in 1299 and came to be used as a textbook of mathematics in Japan (printed in 1658) and Korea (printed in 1660) deals with polynomial equations and polynomial algebra, areas, volumes, rule of three and a method equivalent to Gaussian elimination. The second book published in 1303 is Siyuan yujian (True Reflections of the Four Unknowns) and it includes the famous Pascal's Triangle to the eighth power. He solves polynomials with 1, 2, 3 and 4 unknowns. It also features 288 problems divided into three volumes with 24 chapters. He presented formulas of sums of integers like

$$1+4+10+20+\ldots+\frac{n(n+1)(n+2)}{6}=\frac{n(n+1)(n+2)(n+3)}{24}$$

among others, and also provided the sum of series of the kind $1+4+9+16+25+36+\ldots$, $1+5+14+30+55+91+\ldots$

Yang Hui was born in 1238 in Qiantang (present Hangzhou, China). In 1261 Yang wrote the Xiangjie jiuzhang suanfa (Detailed analysis of the mathematical rules in the Nine Chapters and Their reclassifications). Yang also gave the Pascal Triangle's scheme to the sixth row and also gave formulas for the sum of series like

$$1+3+6+10+\ldots+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6}$$

and the sum of the squares of the natural numbers between m^2 and $(m + n)^2$.

POSTER SESSION – THEME 2

GENETIC APPROACH TO TEACHING DERIVATIVE: A TEACHING EXPERIMENT ON THE GEOMETRIC INTERPRETATION OF DERIVATIVE IN SENIOR HIGH SCHOOL

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Abstract: The problem of the tangent line is one of the most important problems which lead to the birth of calculus. Through a questionnaire survey conducted to 201 students, we concluded that there are historical parallelism between the students' understanding and that of the ancient Greek mathematicians. On the base of historical and epistemological analysis of the concepts of derivative, we design a teaching instruction by integrating history of the birth of calculus, such as problems of light reflection, curve movement. Based on the reconstructed history, The Cyclotomic Rule by Liuhui is introduced to construct a bridge connecting the static and dynamic concept of the tangent, enabling students to pass from the finity to infinity naturally and successfully. It is revealed through interview and a questionnaire survey that the genetic approach to teaching derivative in conducible to better understanding of the concepts of derivative.

INTRODUCTION

Calculus is the main mathematical subject taught in both senior high school and university. However, the teaching of the concept of calculus is universally difficult. In In high school, the derivative is taught in the twelfth grade. Three aspects are emphasized: 1) the concept of the derivative itself and how it is calculated; 2) its geometric representation; 3) applications. Students often know how to compute the derivative, but know nothing about its meaning in nature. Geometrically, the derivative may be interpreted as the slope of a curve at a point. The problem of the tangent line is one of the most important problems which lead to the birth of calculus. This teaching experiment is a part of action research on derivative, focusing on the teaching of the geometric interpretation of derivative from the HPM perspective.

RESEARCH QUESTION

The research questions are: 1) How do senior high school students understand the concept of tangent? 2) Can the integration of mathematics history contribute to the learning of the concept of tangent? 3) Can the integration of mathematics history improve students' understanding of the mathematical idea of replacing curves by straight lines?

RESEARCH METHOD

A historical and epistemological analysis of calculus is a way to reveal some possible sources of students' difficulties as well as an inspiration in the design of activities for students. Otto Toeplitz first summarized and elucidated calculus in terms of an organic evolution of ideas beginning with the discoveries of Greek scholars and developing through the centuries in his book< The Calculus: A Genetic Approach>. The genetic approach to teaching and learning is that a subject is studied only after one has been motivated enough to do so, and learned only at the right time in one's mental development.

Historical analysis and epistemological analysis

Many mathematicians have done a lot of work to descript the tangent. Greek mathematician Euclid speaked of a line that touches the curve at a point on it and is such that no other line can be drawn from the point and between the curve and the original line, that is, it will cut the curve. Apollonius and Archimedes used the same definition that the tangent is a line which touches a conic section or spiral at just one point. Fermat developed a general technique for determining the tangents of a curve by using his method of adequality in the 1630s. Torricelli and Roberval developed a method for finding the tangent by using instantaneous velocity. Leibniz defined the tangent line as the line through a pair of infinitely close points on the curve. L'hospital defined the tangent as an extension line of one side of a inscribed polygon. We can conclude that in ancient mathematics, the tangent touches the curve at one point, lies entirely on one side of the curve without crossing it and in modern mathematics, the tangent line to a curve is a straight line representing the limiting position of the secants.

To grasp the starting points of students' understanding about the concept of tangent line, we carried out a questionnaire survey by using these two questions.

Question 1: Is the straight line y=1 the tangent line of the curve $y = \sin x$? why?

Question2: Is the straight line y=0 the tangent line of the curve $y = x^3$? why?

	Response	Number	Reason
Q1	Yes	117	the curve lies on one side of the straight line
	No or no response	84	there are more than 1 common points
Q2	Yes	48	there is only 1 common point
	No or no response	153	the curve doesn't lie on one side of the straight line

Table 1: results of the questionnaire survey

Through the questionnaire survey, we can conclude that there are historical parallelism between the students' understanding and that of the ancient Greek mathematicians. We should help students to understand that whether the tangent lies on one side of the curve is not the criterion, to motivate them to seek the new definition of tangent line.

Teaching experiment

The question of finding the tangent line to a graph was one of the central questions leading to the development of calculus in the 17th century. Three questions from real life, namely, light reflection, instantaneous speed, the slope of an arch bridge, have motivated the research of tangent problems in 17th century. We ask students to think about the definition of the tangent of a circle and a conic section, then to explore the new definition of tangent.

The teaching project of the geometric representation of derivative has a distinctive feature that it deeply connects to the Cyclotomic Method, to calculus the area of a circl presented in the famous Chinese book of mathematics *The Nine Chapters on the Mathematical Art.* "*Multiply one side of a hexagon by the radius (of its circumcircle), then multiply this by three, to yield the area of a dodecagon; if we cut a hexagon into a dodecagon, multiply its side by its radius, then again multiply by six, we get the area of a 24-gon; the finer we cut, the smaller the loss with respect to the area of circle, thus with further cut after cut, the area of the resulting polygon will coincide and become one with the circle; there will be no loss*". Two examples can motivate students to explore the connection between tangent and derivative and understand the transition from the algebraic representation to the geometric representation. In the case of the ellipse, the construction of the tangent rested on the theorem that the tangent at the point of tangency forms equal angels with the two focal radii drawn from the point of tangency. After we magnifies the ellipse 100 diameters, students can find the curve and the straight line is overlapped, which is the mathematical thinking of replacing curves by straight lines.

RESULTS AND REFLECTIONS

We conducted interviews with 8 students and carried out a questionnaire survey in two 12-grade classes comprising 105 students. They all think the class was interesting. There were no difficulties in understanding the concept of tangent and the mathematical idea of replacing curves by straight lines. Almost all students agreed that the integration of the Cyclotomic Method can help them understand the concept of tangent and the process of magnify the ellipse can improve their comprehension on the mathematical thinking of replacing curves by straight lines.

We think the genetic approach to teaching and learning is that a subject studied only after one has been motivated enough and learned only at the right time in one’s mental development. The historical and epistemological analysis can help us recognize the starting point of the students’ comprehension of the concepts of tangent. The integration of mathematical history should reconstruct the natural way of developing a concept from realistic problem situation.

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Poster Session – Theme 5

PLIMPTON 322: A VIDEO DOCUMENTARY TO MOTIVATE STUDENTS TO STUDY MATHEMATICS

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I shall provide an introduction to the 33-minute film Plimpton 322: The Ancient Roots of Modern Mathematics, which was produced to motivate college and high school students – especially minority students – to pursue mathematics. The mathematics that drives our modern world owes its origins to ancient cultures in the Middle East, Asia and Africa. Set against a backdrop of today's New York City, the film explores the extent of our debt to this tradition. Along the way we meet up close some precious and revealing ancient artefacts that now have their homes in New York, most of all a controversial cuneiform tablet from Mesopotamia known as Plimpton 322. We witness ancient mathematical ideas still playing crucial roles in 21st century society and technology.

This film celebrates the diversity underlying our mathematical culture. Teaching at a large, urban, multiethnic university, I have found it useful in encouraging students in courses ranging from mathematics for liberal arts students to history of mathematics. The film incorporates brief introductions to two mathematical topics, positional number notation and Pythagorean triples, which can be developed in the classroom. The purpose of my presentation is to bring this freely available resource to the attention of educators and to discuss with anyone interested its use in the classroom. As well as a poster, I shall display a trailer and excerpt from the film on a laptop computer or tablet.

The film is at http://faculty.baruch.cuny.edu/lkirby .

Poster Session – Theme 6

MATHEMATICS, EDUCATION AND WAR

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In this work we present some ideas related to a PhD project started in 2011 where we try to problematize some relations between mathematics, education and war. More precisely we have been trying to understand the modifications in the field of mathematics education in the context of Cold War. Here we try to problematize the enigmatic phrase of Jean Dieudonné: "Euclid must go!". For a long period in the history of mankind the Euclidian geometry played an important role not only in the warfare, but at the schools and in science in general. So, why, in a context where the "Western" headed by United States seemed to be losing the conflict capitalism versus communism, Euclid must go? Tracing some relations between the development of disciplines as Operational Research, Game Theory, Linear Programing, the emergence of computer sciences and the structuralist mathematics proposed by the Bourbaki group, we shall enlighten the so called "modern mathematics movement" and its relation with the Cold War. Apparently there is no explanation available for the demise of Euclidian geometry and the predominance of "New Math" solely in terms of mathematics neither in terms of society neither of mathematics education, then we need to look for fields of human activities and its relations to explain this event. Based on a post-structuralist theoretical approach, mainly on the works of the "second" Wittgenstein, Jacques Derrida and Michel Foucault, we try to develop a grammatical deconstructive therapy of mathematics, education and war in the specified context. We believe that such kind of historic-philosophical problematization could help teachers to understand the relations between the field of mathematics and other fields of human activity which may help them to show to students the role of mathematics in different contexts of socio-cultural practices.

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