Sevilla
14-21
July / julio
1996

8 th International Congress on Mathematical Education
Selected Lectures

8º Congreso Internacional de Educación Matemática
Selección de Conferencias

Edited by / Editado por

Claudi Alsina
José María Alvarez
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S.A.E.M. THALES

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CONTENTS
ÍNDICE

PREFACE / *PREFACIO*

This book comprises selected lectures from the 8th International Congress on Mathematical Education (ICME-8), held in Sevilla, Spain, from July 14 to 21, 1996. There were some 56 "regular lectures" given during ICME-8 and 33 of them are included in the present volume, with authors coming from 19 different countries. (We do not take into account here the four "plenary lectures" presented at ICME-8, which are to be found in the book of Proceedings of ICME-8.)

As was the case with previous ICMEs, the Executive Committee of the International Commission on Mathematical Instruction (ICMI) appointed an International Programme Committee (IPC) for ICME-8, responsible for setting up the structure of the scientific programme and for inviting the main speakers and the organizers of various components of the programme. One of the first decisions of the IPC was to maintain the practice reintroduced at the previous ICME congress (ICME-7, Québec, 1992) of having an important number of lectures, in addition to the customary plenary lectures or the mini-lecture presentations to small groups. While adhering to the principle that an essential aspect of the quadrennial ICME congresses is the opportunity they provide for face-to-face debate and discussion among mathematics educators, either in the context of Working Groups, Topic Groups, Poster Presentations or informal gatherings, the IPC felt the "traditional lecture", in spite of all its well-recognized deficiencies, still provides an excellent medium for putting participants in contact with key issues in mathematics education.

The ICME-8 IPC therefore reserved six one-hour slots on the programme for the regular lectures, about 10 presentations being scheduled simultaneously each time. Invitations were issued to a selection of the best theoreticians, researchers and practitioners in the field around the world. Speakers were chosen according to their professional quality, theircommunication abilities, the selection of topics and the levels of education the IPC wished to cover. The written versions of thirty-three of these presentations make up this volume.

The reader should be aware that the set of lectures we offer here is in many ways biased. It is a proper subset of the lectures presented at ICME-8, which were themselves only one component in the scientific programme. We make pretence that it gives a complete picture neither of the field of mathematics education nor of the ICME-8 programme as a whole. The reader should consult the ICME-8 Proceedings to see how the lecture topics complement the topics treated in other programme strands: the plenary lectures, the Working Groups, the Topics Groups, the reports of ongoing work by the official Study Groups of ICMI, the reports of the ICMI Studies, etc. Here, however, the reader will find written versions of some fine talks well worth reading and reflecting upon.

As was the case with ICME-7, the Sevilla congress results in the production of two books, one being the Proceedings of ICME-8 themselves and the other, the present volume of selected lectures. A difficulty in the production of such a book is that for many authors, English (the "lingua franca" in mathematics education at the international level) is a second, if not a third, language. Although all efforts have been made to insure a high level of correction in the written language, it was the policy not to change authors' style, staying as close as possible to the original versions.

The editors wish to acknowledge the contribution of a number of people to the production of this volume. First, and above all, we extend our thanks and gratitude to the authors themselves: by investing the extra effort, in addition to the preparation of their oral presentation, necessary for their presentation to be brought in a written format, they have made this volume possible. We also wish to thank all the Local Institutions and sponsors, which made all their best to contribute to the success of ICME 8.

Bernard R. Hodgson
for the Editing Committee:
Claudi Alsina, José Mª Alvarez Falcón, Bernard R. Hodgson, Colette Laborde, Antonio Pérez Jiménez.

LAS CONCEPCIONES SOCIOEPISTEMOLÓGICAS DE FRÉCHET EN SUS INVESTIGACIONES SOBRE LA TEORÍA DE LOS ESPACIOS ABSTRACTOS Y LA TOPOLOGÍA GENERAL

Luis Carlos Arboleda A.

Luis Cornelio Recalde C.[1]

1. Introducción

La forma como el científico se representa su proyecto investigativo es algo que llama constantemente el interés en los estudios sociales sobre la ciencia. Uno de los aspectos importantes del análisis sobre las condiciones socioculturales de la producción de las teorías, es la explicación del sistema de valores y conceptos del matemático creador y, muy particularmente, de los procedimientos prácticos y conceptuales que pudieron haberlo conducido a ciertos resultados empleando determinadas estrategias.

Con el término de **representación** nos referimos, de manera aproximada, a las formas en las que se revisten los objetos de los mundos matemáticos en el ámbito de la conciencia, por efecto de la intermediación de *doble vía* de la práctica matemática de los individuos con un sistema de valores y creencias (sociales, culturales, religiosas, etc.). Que algunos matemáticos mantengan fervientemente que el único objetivo de la ciencia -como decía Jacobi refiriéndose a la teoría de números- es 'la búsqueda del honor del espíritu humano'[2], o que otros crean que incluso las nociones más simples como los números enteros no son intuiciones puras o juicios sintéticos a *priori,* sino 'conceptos primigenios que surgieron de complicadas nociones humanas'[3], unas y otras son representaciones

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2 Dieudonné, J., **Pour l'honneur de l'esprit humain. Les mathématiques aujourd'hui.** Hachette, Paris, 1987.

3 Fréchet, M., **Les mathématiques et le concret.** P.U.F., Paris, 1955; idea expresada en el capitulo sobre los *Orígenes de las nociones matemáticas,* al cual volveremos páginas más adelante.

orgánicamente relacionadas con estrategias cognitivas e investigativas contrapuestas. Uno y otro tipo de enfoques son construcciones articuladas estrechamente por configuraciones sociales, como puede comprobarse a través de una indagación cuidadosa de carácter histórico y cultural.

El propósito de la presente comunicación es aprovechar algunos materiales documentales de la investigación socio-histórica que adelantan los autores sobre la teoría de los Espacios Abstractos del matemático francés Maurice Fréchet (1878-1973)[4], con el fin de analizar algunas de las representaciones del científico sobre su obra. Especialmente nos interesa destacar aquellas apreciaciones y valoraciones filosóficas de Fréchet sobre la relevancia para la creación matemática, de los procesos de generalización fundamentados en lo concreto de las modelizaciones abstractas con soporte empírico.

Una adecuada comprensión sobre este problema puede contribuir a explicar a matemáticos, filósofos, historiadores, sociólogos y educadores matemáticos, la naturaleza y función de los factores del contexto social y cultural en la formación de pensamiento matemático. En particular en el campo de la educación matemática, es sabido que la **transposición didáctica** tiene como propósito explicar los mecanismos que permiten el paso de un objeto de saber a un objeto de enseñanza. En el estudio de caso de la noción de *distancia* de Fréchet,[5] se señala una vía fecunda para explorar estrategias didácticas viables sobre esta noción, las cuales tienen en cuenta aquellas representaciones del matemático que son asociadas con el pensamiento creador, y no solamente las ideas acabadas que se establecen por acuerdos consensuales entre la comunidad matemática. Tales acuerdos constituyen el llamado saber erudito (*savoir savant*) que es, por naturaleza, despersonalizado. Una de esas representaciones consisitió, según el mismo Fréchet, en generalizar en los espacios abstractos una idea muy *simple*: que una *cualidad* (la proximidad o vecindad en torno a un elemento) se puede expresar por un *número* (la noción de *distancia*).

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4 Arboleda L.C, y L.C. Recalde. **Matemáticas y experiencia: La generalización de la noción de espacio abstracto en Maurice Fréchet.** Proyecto financiado por Colciencias y la Universidad del Valle. 1996-1998

5 Chevallard Y., y Joshua M.-A., Un exemple d'analyse de la transposition didactique: la notion de distance, **Recherches en didactique des mathématiques,** vol. 3, 2, 1982

2.Fréchet y la Matemática de las Formas.

En lo que sigue nos limitaremos a analizar las ideas de Fréchet sobre su producción matemática a partir de dos ejes fundamentales que, como veremos, se presentan íntimamente ligados al observador histórico:

i) En relación con el programa filosófico de orientación leibniziana que articula su obra, y

ii)En relación con sus concepciones sobre el origen fundamentalmente empírico de la producción matemática en general, y de la suya en particular.

Estas ideas de Fréchet se encuentran reunidas básicamente en dos publicaciones suyas: la **Noticia** sobre sus trabajos científicos [6], y **Las Matemáticas y lo Concreto** [7]. La primera es un ensayo sobre su contribución matemática durante el período 1904-1928. En este trabajo Fréchet explica lo específico y original de la misma a los evaluadores del concurso para ser admitido como miembro correspondiente de la **Academia de Ciencias de París**. La segunda es una recopilación de artículos divulgativos y pedagógicos elaborados a lo largo de su carrera como investigador y profesor. Entre ellos, hay dos particularmente útiles para los propósitos de la presente comunicación en cuanto abordan temas relacionados con sus concepciones empiristas: *Desaxiomatización de la ciencia y Orígenes de las nociones matemáticas.* Este último artículo contiene, como aspecto a destacar, una reseña del debate que mantuvo Fréchet con algunos colegas y filósofos contemporáneos suyos (Enriques, Gonseth, Bernays, Lukasiewisz, Wavre y Kérékjarto), en las **Entretiens de Zürich,** realizadas entre el 6 y el 9 de diciembre de 1938, alrededor del tema de los fundamentos y el método de las ciencias matemáticas.

En la **Noticia**, Fréchet escoge el siguiente epígrafe de Leibniz para explicarle a sus colegas de la Academia, la naturaleza del programa filosófico que articula y soporta su contribución en diversos campos (análisis funcional, análisis general, análisis clásico, geometría, probabilidad y matemáticas aplicadas):

> *Quienes prefieren avanzar en los detalles de las ciencias deprecian las investigaciones abstractas y generales, y quienes profundizan en los principios, entran raramente en particularidades. En cuanto a mí, le doy igual importancia a lo uno y a lo otro, porque he descubierto que el análisis de los principios permite el avance en las invenciones particulares.*

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[6] Fréchet, Maurice. **Notice sur les travaux scientifiques.** Paris. Hermann, 1933.
[7] Fréchet, Maurice. **Les mathématiques et le concret.** Paris. PUF, 1955.

Fréchet explica enseguida las razones por las cuales encuentra que su obra se enmarca en el programa filosófico leibniziano sobre las matemáticas. En primer lugar, se destaca el interés de ambos en orientar las investigaciones dentro de estrategias de generalización específicas; Leibniz en el análisis infinitesimal de **R**, y Fréchet en el análisis general de espacios de naturaleza cualquiera. Los resultados obtenidos por Fréchet durante el período 1904-1928 al aplicar este enfoque de generalización a distintas teorías matemáticas (topología general, análisis funcional, teoría de funciones, etc.), fueron sistematizados en su célebre obra de 1928 sobre los Espacios Abstractos[8].

El análisis de los principios que se preservan en las generalizaciones hace parte de un programa filosófico con una cobertura que va más allá del campo matemático. La idea de Leibniz era elaborar una especie de alfabeto del pensamiento humano, de tal suerte que a partir de él se pudiera inferir y discutir sobre cualquier aspecto, en lo que denominó **charasteristica generalis** (característica universal). Buscaba crear una **Mathesis universalis** con el propósito de registrar en el pensamiento simbólico aquello que la percepción nos permite percibir, ya que "todo lo que la imaginación empírica abarca a partir de las figuras, lo deriva el cálculo de signos por una demostración inequívoca; e incluso conduce a otras consecuencias a las que la facultad de imaginar no puede llegar"[9].

Fréchet comparte el propósito central del programa leibniziano de desarrollar esa **matemáticas de las formas** que permitiera registrar en el pensamiento simbólico aquello que la percepción nos permite observar[10]. La idea de una matemática universal articula sus estrategias investigativas en cuanto a la construcción de un lenguaje matemático integrador de una red conceptual de teorías especializadas, y que en los años 1920 se centraban en el Análisis General de los Espacios Abstractos. Pero también se haya presente en actividades que Fréchet emprendió sistemáticamente como proyecto de vida, en tanto ciudadano y como intelectual.

Recordemos que Fréchet fue durante muchos años, promotor y organizador de la **Unión Universal de Esperanto,** lengua en la cual

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8 Fréchet, Maurice. **Les espaces abstraits et leur théorie considérée comme introduction à l'Analyse Générale.** París. Gauthier-Villars, 1928.

9 Leibniz citado en Granger, Guilles-Gaston. **Formes, opérations, objets**. Paris. Mathesis, Vrin. 1994; capitulo "Philosophie et Mathématique Leibniziennes", p.p. 216.

10 Leibniz citado en Granger, **op. cit.,** p.216.

publicó interesantes resultados matemáticos. Si bien esto implicaba reducir el impacto científico esperado en la circulación de tales investigaciones, Fréchet quería así convencer a sus colegas de que era posible y deseable escoger una lengua "ordinaria" muy general y tal vez más apropiada que otras para comunicar enunciados matemáticos del nivel "universal" de los suyos.

Como sabemos,[11] la creación del cálculo leibniziano reposa sobre algunos principios lógico-filosóficos: la **combinatoria** (en la cual procedimientos como la diferenciación se establecen al margen de consideraciones infinitesimales y en relación con las propiedades del triángulo característico de Pascal); la ley de **continuidad** (principio metafísico que permite la extensión de propiedades de lo finito a lo infinito; permite entender, por ejemplo, cómo la tangente conserva en el límite las propiedades de la secante a través de la relación invariante entre la subtangente y la ordenada); y el conocimiento de los objetos matemático a través de su representación en el **universo simbólico**. En relación con este último principio, la solución más general posible de un problema, por el ejemplo el de la tangente, pasa por encontrar un algoritmo en el cual se expresen simbólicamente todos los pasos implicados en el proceso de generalización.

En el contexto de este programa filosófico se entiende mejor la manera en que Leibniz introdujo las conocidas técnicas y simbolismos del cálculo, como las notaciones "∫" y "**d**" para la integración y la diferenciación, entendidas éstas en tanto operaciones recíprocas. También sabemos que precisamente la fuerza notacional del cálculo de Leibniz jugó un papel determinante en la aceptación de su enfoque con relación al de Newton. Entre otras cosas porque siendo para Leibniz los símbolos "∫" y "**d**" muy próximos a nuestra idea actual de operadores, se llegaba fácilmente a resultados fecundos en el análisis. Por su parte, la engorrosa metodología newtoniana de las fluxiones y la notación de punto, complicaba inútilmente los procesos. En fin, mientras que el enfoque de Newton estaba referido a variaciones infinitesimales en el tiempo, el de Leibniz trataba variables más generales en las cuales, no sólo el tiempo podía ser tomado como un caso particular, sino que su manejo era independiente de consideraciones infinitesimales.

No deja de llamar la atención que en cierta medida algunos de estos principios lógico-filosóficos se encuentran subyacentes a la creación

[11] Ver, por ejemplo, Granger, **op cit**, y Belaval, Yvon. **Leibniz critique de Descartes.** París, Gallimard. 1960; capitulo "La géométrie algébrique et le calcul infinitésimal"

matemática más general y abstracta de Fréchet. Así como se los halla contribuyendo y validando al campo teórico del cálculo infinitesimal, también es posible reconocerles esta misma función dentro del proyecto de establecer el análisis general, entendido como el estudio de las correspondencias entre variables de naturaleza cualquiera. En la obra de Fréchet se realiza el propósito leibniziano de encontrar en el análisis más general y absoluto de los principios (la matemática de las formas), la explicación más fecunda de los objetos particulares a la cual se refiere el ya citado epígrafe de la **Noticia**.

3. La diferencial de Fréchet y la generalización de principios útiles y sencillos.

Una excelente ilustración de la filiación de sus investigaciones sobre el análisis en los espacios abstractos con el programa de Leibniz, es la llamada 'diferencial de Fréchet', o diferencial de una función definida sobre un espacio particular de funciones, con la bien conocida aplicación al caso de una transformación definida entre partes de dos espacios lineales normados. Refiriéndose a los procedimientos y criterios que utilizó en esta extensión de la diferencial a los espacios abstractos, Fréchet recalca que inicialmente su interés se orientó en la dirección de generalizar aquel principio 'útil y necesario' que originó la noción de diferencial en el cálculo infinitesimal; esto es, que la diferencial dF es la función más **simple** con respecto a la variación Δx de la variable, y más **aproximada** a la variación ΔF de la función. [12]

En su noticia necrológica de 1977 sobre Fréchet, Szolem Mandelbrojt escribió:[13]

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[12] En su conferencia de 1925 en Berna sobre la *Désaxiomatisation de la science* (**IN: Mathématique et le Concret, op. cit.**, pp. 8-9), Fréchet afirma que con esta definición había querido retornar a la concepción de los inicios del cálculo diferencial, y darle a la diferencial un uso más intuitivo y riguroso.

[13] *Notice nécrologique sur M.F.. par Szolem Mandelbrojt,* le 19 novembre 1977. **IN:** Dossier Administratif, Fonds Fréchet, Académie des Sciences, Paris. En un trabajo posterior analizaremos con detenimiento los procedimientos de Volterra y Fréchet en relación con dos 'estilos' diferentes de generalizar la diferencial. Anotemos por el momento como hecho interesante, que Mandelbrojt encontró en su Noticia razones para establecer la conexión entre las generalizaciones de Fréchet y la búsqueda consecuente de un ideal leibniziano que habria caracterizado las distintas facetas intelectuales, matemáticas y humanas de su vida.

En el Análisis funcional concebido por Vito Volterra (actualmente se le da ese nombre a otra rama más general de las matemáticas), se estudian las funciones cuya variable es una línea o una función, llamadas 'funcionales' por Hadamard. Al explorar ese nuevo dominio Volterra empleó el método del paso de lo finito a lo infinito, el cual no es siempre riguroso en esta disciplina.

Y destacando entre otras contribuciones suyas al análisis funcional la introducción de la diferencial abstracta de Fréchet, Mandelbrojt agregó enseguida lo siguiente:

Fréchet comenzó sus investigaciones a este respecto utilizando un método directo, general y muy riguroso. Sustituyó la definición de la diferencial de una función de línea por una definición valedera en casos mucho más amplios.

En efecto, trabajando en un espacio de funciones de línea, Vito Volterra había introducido en 1887 las nociones de derivada y diferencial. Volterra empieza analizando la variación de $F(f(x))$, con pequeños cambios de la función f en la vecindad de un punto particular x. Volterra define la derivada de F en relación a f en el punto x, como la funcional $F'(f \quad, x)$, la cual es continua en cada variable bajo ciertas condiciones

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14 V. Volterra, *Sopra le funzioni che dipendono da altre funzioni*, **Rend. R. Accad. dei Lincei,** ser. 4, vol. 3 (1887); tres notas respectivamente en las pp. 97-105, 141-146, 153-158.

15 La evolución de los principales resultados sobre la diferencial hasta alcanzar su estado canónico, se encuentra expuesta con todo detalle en: Taylor, A.E., *A Study of Maurice Fréchet III: Fréchet as Analyst, 1909-1930*, **Arch. for History of Exact Sciences,** vol. 37, 1987; pp. 25-76. El estudio de la cuestión desde la perspectiva italiana -particularmente interesante por la argumentación en contra de la polémica explicación dada por J. Dieudonné a la contribución de Volterra a los inicios del Análisis funcional-, se encuentra, entre otros, en Fichera, G., *Vito Volterra and the birth of functional analysis.* **IN**: Pier, J.-P. (ed.), **Development of Mathematics, 1900-1950**, Birkhäuser, Berlin, 1994; pp. 171-183. En una próxima publicación los autores del presente trabajo harán un balance sobre estas diversas interpretaciones, en el marco de su investigación sobre las ideas de Fréchet en relación con el problema de las generalizaciones matemáticas a partir de la experiencia.

restrictivas, obteniendo a partir de allí la expresión:

$$\lim_{\varepsilon \to 0} \frac{F(f+\varepsilon\, g)-F(f)}{\varepsilon} = \int_a^b F'(f,x)\, g(x)dx$$

donde f y g están definidas sobre [*a*, *b*]. La expresión ε g(*x*) es llamada la variación de *f*(*x*), y será denotada por δ f. De lo anterior se sigue que:

$$F(f+\varepsilon\, g)-F(g) = \varepsilon \int_a^b F'(f,x)\, g(x)dx + \rho$$

donde ρ es tal que $\rho/\varepsilon \to 0$ cuando $\varepsilon \to 0$.

A la expresión $\delta F = \varepsilon \int_a^b F'(f,x)\, g(x)dx$

Volterra la llamará 'primera variación' de *F*.

En 1902, Hadamard muestra que la definición de Volterra es demasiado restrictiva, pues no puede extenderse a casos más generales de funcionales en los cuales él mismo estaba interesado. Propone someter la definición a hipótesis más generales, para ampliar su campo de incidencia y posibilitar así que a tales funcionales se les pudiesen aplicar los métodos del cálculo infinitesimal. En particular Hadamard recomienda que se tengan en cuenta las funcionales *F* cuya variación es una funcional lineal de la variación de *f*.

En 1912, Fréchet, retoma las ideas de Hadamard con respecto a la diferencial abstracta como parte de su programa de extender los principios fundamentales del cálculo diferencial al cálculo funcional. Es una comunicación enviada a Volterra el 30 de julio de 1913, Fréchet explica el método empleado por él para obtener tal generalización: [16]

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16 Nurzia, L, *La Corrispondenza tra Vito Volterra e Maurice Fréchet sui Fondamenti dell'Analisis Funzionali.* **Revista di storia della scienza,** vol. 4, 3, 1987; pp. 391-416. En este trabajo se publican por primera vez apartes de la correspondencia de Fréchet a Volterra que aportan información histórica importante para valorar las contribuciones de estos autores a los comienzos del análisis funcional, así como para comprender las diferencias de 'estilos cognitivos', motivaciones y representaciones –particularmente de Fréchet- sobre la naturaleza de los conceptos y del razonamiento matemático

(...) La idea que constituye la sustancia de mi artículo y que me parece original, consiste en adoptar el punto de vista de Hadamard [recordemos que Hadamard había propuesto atender a la variación de f] pero aplicándolo no a la variación de f (x) sino a su crecimiento (...)

Más adelante, en 1914, Fréchet escribe:

Me incliné pues a tratar de retomar la antigua definición, generalmente olvidada hoy en día: la diferencial es la parte principal del crecimiento de la función cuando el crecimiento de la variable se considera infinitamente pequeño.

Un paso decisivo para llegar a la generalización del tipo que buscaba Fréchet, consistió en apoyarse en una definición previa de la diferencial total de una función de varias variables. En este aspecto, Fréchet incorpora un nuevo elemento teórico en la definición de la diferencial de una función $f(x, y)$, al hacer depender la diferencial de la existencia de una función homogénea y de primer grado con relación a los crecimientos Δx, Δy. Se dice que la función $f(x, y)$ tiene una diferencial en (x, y), si existe una función lineal homogénea
$A\,\Delta x + B\,\Delta y$, tal que:

$$\frac{1}{\Delta}\Big(f\,(x+\Delta\,x,\ y+\Delta y)-f\,(x,y)-A\,\Delta\,x-B\Delta\,y\Big)=\varepsilon$$

donde $\varepsilon \rightarrow 0$, cuando $\Delta \rightarrow 0$. La función lineal homogénea $[A\,\Delta x + B\,\Delta y]$ es la diferencial de f.

La definición para el caso de la funcional $F\,(f)$ que propone Fréchet, se 'vislumbra' ya en la expresión anterior:

$F(f)$ es diferenciable en f si existe ε y una funcional $\delta F(f; g)$, lineal en relación a g, tal que:

$$F(f+g)-F(f)-d(f;g)=\varepsilon\,M(g)$$

donde, $M(g) = max\,|g(x)|$ en $[a, b]$ y $\varepsilon \rightarrow 0$, cuando $M(g) \rightarrow 0$.

Pero Fréchet no se detiene en este nivel de generalización de la diferencial. En 1925 aplica su definición a una clase de espacios que él había introducido años atrás y que se habían revelado extremamente importantes en teorías de naturaleza diferente: los 'espacios distanciados vectoriales' o espacios normados lineales:

Sean E_1 , E_2 espacios normados lineales. Si $M = f(m)$ es una transformación del punto m de E_1 en el punto M de E_2, y si m_o es un punto interior del conjunto, donde está definido $F(m)$, se dice que la transformación es diferenciable en m_o si existe una transformación lineal Ψ de E_1 en E_2 tal que:

$$F(m_o + \Delta m) - F(m) - \Psi\Delta(m) = \varepsilon \, ||\Delta m \, U,||$$

donde U es un vector unitario variable en E_2 y $\varepsilon \to 0$, cuando $||\Delta m|| \to 0$.

Observemos la expresión simbólica simple del algoritmo; principio lógico que era característica común a los procesos de generalización y abstracción en el análisis matemático de los siglos XIX y XX. Así mismo se puede comprobar la intervención -en este proceso de definición de la diferencial abstracta de Fréchet-, de otros de principios filosóficos como, la ley de los homogéneos (simbolización y manipulación teórica distinta entre variables y cantidades respectivamente comparables), y la aceptación de que a nivel de lo general y abstracto –por simetría con lo que ocurre en el mundo de lo concreto-, también se pueden establecer relaciones y manipulaciones teóricas entre objetos con un cierto grado de vaguedad, aproximación e indeterminismo.

4. Las ideas de Fréchet sobre la relación de las Matemáticas con la Experiencia.

Además de los mencionados principios lógico-filosóficos de factura leibniziana, el campo de representación de la creación matemática de Fréchet nos parece que está articulado por sus concepciones sobre el origen empírico de las nociones más generales y abstractas. Resulta interesante estudiar la cuestión de si existe una cierta recurrencia entre estos dos órdenes de representación en el sistema de valores y conceptos matemáticos de Fréchet. Es decir, preguntarse hasta qué punto cabe hablar de las concepciones socioepistemológicas de Fréchet como factor movilizador de sus investigaciones sobre la teoría de los espacios abstractos y la topología general. Este es un asunto que los autores abordarán en otros trabajos. Por ahora limitémonos a considerar las concepciones de Fréchet sobre la naturaleza de los conceptos y el pensamiento matemático.

Las nociones fundamentales de todas las ramas matemáticas son construidas a partir de la experiencia.[17] La experiencia conduce a la abstracción, y la abstracción sustituye y enriquece la experiencia. El Análisis General se fundamenta en ideas que a través de la experiencia de matemáticos en diferentes campos teóricos particulares se revelaron **simples, útiles y fecundas**, pero que sin embargo antes de Fréchet nadie había examinado en forma tan sistemática ni en toda su generalidad. Fréchet reconoció tales ideas "candidatas a generalización", a través de su propia experiencia y la de su entorno intelectual. Escrutó en las distintas teorías cuáles eran las condiciones necesarias y suficientes en las que radicaba su éxito; separó los aspectos accesorios de los fundamentales, y les imprimió a los fundamentales su forma a la vez simple y fecundidad conceptual. Por último validó sus generalizaciones de nuevo a través de la experiencia, y comprobó que permitían desarrollar nuevos campos teóricos y experimentales, al mismo tiempo que perfeccionaban los ya existentes tanto en el nivel conceptual como técnico. Veamos las fases del procedimiento de doble vía entre abstracción y realidad concreta que explica -de acuerdo con Fréchet-, la dinámica del pensamiento matemático.

La primera etapa de generalización a partir de lo concreto se opera con la **síntesis inductiva**. Esta tiene que ver con ese a menudo largo proceso que consiste en articular en un mismo campo teórico las versiones particulares que tiene la teoría general en campos restringidos y hasta entonces inconexos. Muchas veces la generalización matemática se produce mediante procesos de iteración. Una teoría o modelo abstracto que es reconocido como válido en la experiencia dentro de espacios restringidos, se extiende entonces a campos generales, incluso a espacios de naturaleza cualquiera.

En sus investigaciones científicas Fréchet utilizó corrientemente este procedimiento de generación inductiva sobre nociones **simples y útiles.** Como tratamos de demostrarlo en nuestro proyecto, en ello habría podido influir, aparte de ciertos principios lógico-filosóficos, la experiencia de utilización **exitosa** de este método en los trabajos de análisis funcional que hacia comienzos del siglo XX ya había hecho célebres a matemáticos como Ascoli, Volterra, Arzelà e Hilbert, y a su propio maestro Hadamard. Fréchet tenía la convicción profunda de que las generalizaciones se hacen poco a poco, respondiendo a necesidades del campo teórico, en la

[17] Fréchet, M., *Entretiens de Zürich,* **IN: Les Mathématiques et le concret, op. cit.,** pp. 11-51.

medida que ellas son reconocidas como tales por los propios investigadores.

La segunda etapa corresponde a los procedimientos clásicos de la axiomatización de la teoría. A través de esta fase se busca la organización racional del campo teórico, la reconstrucción del objeto y la reproducción del mismo campo. Esta fase es la que permite reemplazar experiencias directas demasiado dispendiosas o simplemente alejadas del alcance de nuestras posibilidades empíricas, por experiencias indirectas que se apoyan en proposiciones de la teoría deductiva. En la tercera fase se trata de constatar que la teoría abstracta así generalizada y axiomatizada, preserva las propiedades fundamentales de la red de teorías particulares asociadas con la teoría abstracta.

En este punto es necesario hacer una acotación sobre las apreciaciones de Fréchet con respecto a los peligros del formalismo excesivo. En varios de sus trabajos se refiere a la situación de autonomización del pensamiento abstracto, en virtud de la cual, en la medida que se extienden los procesos de generalización, la comunidad matemática se encierra en ella misma, desarrollándose cada vez más los conceptos y métodos dentro de un mundo matemático hermético[18]. Las teorías progresivamente se vuelven más difíciles de relacionar y analizar de acuerdo con categorías familiares al mundo de la experiencia y de la vida cotidiana. Como dice Sal Restivo[19], el mundo matemático va dejando atrás los mundos de las representaciones y de los paisajes de cosas identificables, para reducirse a un mundo de símbolos y notaciones autoreferenciado.

Fréchet observa esta situación (el comienzo de la hegemonía de programa de Hilbert), y alerta sobre el hecho de que las matemáticas no se pueden reducir a su parte deductiva, so pena de convertirlas en un juego del entendimiento sin ningún alcance práctico. Llega incluso a afirmar que, no obstante todas las ventajas que se le reconocían al método axiomático, "sería interesante identificar igualmente una

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18 Por ejemplo, en su biografía del matemático alsaciano Louis Arbogast (1759-1803) (**IN: Revue du Mois,** 1920, pp. 337-362), Fréchet se apoya en las políticas educativas por las cuales Arbogast abogó en la Convención Nacional, para denunciar el dogmatismo del logicismo puro que estaba en boga entonces sobre todo entre los jóvenes investigadores. Decía que si bien tal dogmatismo podía ser cómodo, correspondía a un programa de trabajo estrecho. Había pues, necesidad de luchar por desterrar esas concepciones de las mentes de los candidatos a la agregación impara quienes la comodidad de la lógica de exposición era lo portante.

19 Restivo, Sal. **The social construction of mathematics.** Zentralblatt der mathematik, 1988.

construcción científica basada sobre principios diferentes e incluso opuestos" (in [4]; **op.cit.**, pp. 3). Su propuesta parece aplicarse en todo caso a aquellas teorías "que hubieran ya alcanzado un alto grado de abstracción".

Dentro de la concepción de Fréchet sobre la relación matemática y experiencia, se pone en cuestión la creencia generalizada de la autonomía de las matemáticas con respecto a la realidad. Aún en el trabajo matemático más abstracto interviene inevitablemente -con un efecto epistémico determinado sobre la teoría- algún tipo de representación de la experiencia. Fréchet constata que la orientación de las ciencias matemáticas, el sentido hacia donde se producen sus progresos, no está condicionada solamente por necesidades internas (organización, sistematización, simplificación de resultados de tales transformaciones lógicas). También son motivadas por demandas externas, por problemas concretos planteados por la naturaleza y la técnica.

5. La estrategia comunicativa de Fréchet sobre sus ideas sobre las matemáticas.

Enfrentado como estaba en el período de entreguerras, a un contexto intelectual en el cual se comenzaban a valorar más los enfoques estructuralistas, formalistas y axiomáticos, Fréchet se vio conducido a poner en práctica una verdadera estrategia de divulgación de las concepciones socio-epistémicas, con orientación empírica, en las cuales reposaban sus propias investigaciones. Las ideas que hemos venido comentando en este trabajo, están articuladas en función de una *retórica persuasiva* que a todas luces utiliza Fréchet para hacer comprender a sus diferentes interlocutores (académicos, comunidad matemática en general, profesores, distintos usuarios científicos y no científicos), la construcción, naturaleza y función de sus teorías más abstractas y generales. El gusto que siempre manifestó por el uso de citaciones de autores consagrados en la literatura matemática y en la historia de las ciencias, se convierte en muchos casos en un instrumento privilegiado para reforzar la argumentación persuasiva sobre sus concepciones socioepistemológicas o pedagógicas.

Conviene traer a colación en esta parte final de la exposición uno de los tantos textos y citaciones en los cuales acostumbraba Fréchet sustentar la defensa de sus delicadas argumentaciones. En el escrito es el cual propendía claramente por la **Desaxiomatización de la ciencia**[20]

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20 Fréchet, **op. cit.**, p. 10.

TEACHING AND LEARNING ELEMENTARY ANALYSIS

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I. INTRODUCTION

It is not easy for students to enter the conceptual field of analysis.Thanks to the educational research carried out in this area during the last years (Tall, 1991), today we better understand the real nature of students' difficulties and we are aware of the general failure of standard teaching strategies. As a consequence, all over the world, new curricula are developed, trying to find the way for a meaningful and accessible entrance into this conceptual field (Artigue & Ervynck, 1992). More intuitive and experimental approaches relying on the use of new technologies seem to be widely privileged now. What are their potential and their limits? What can we learn from the experience of countries where such approaches were established some years ago? In this text, I would like to address these important issues.

Firstly, I shall try to synthesize the main results obtained by didactical research in this area. I do not pretend to be exhaustive but just try to give a personal view of the present state of the art. Then, I shall analyse teaching practices and their evolution, by referring to one particular case : the case of French secondary curricula which, in my opinion, illustrates the general tendency, fairly well. Finally, I shall come to the potential and limits of the new approaches, as one can evaluate them from the French experience.

II. STUDENTS' DIFFICULTIES WITH THE CONCEPTUAL FIELD OF ANALYSIS

Didactical research in this area has evidenced the existence of strong and resistant difficulties. They have different origins but they tightly intervene and reinforce mutually, in a kind of complex network. Nevertheless, in order to facilitate the synthesis, I have chosen to group them according to three categories, which are the followings :

• Difficulties linked to the mathematical complexity of the basic objects of the field : real numbers, functions, sequences, objects which are still in a construction phase when the official teaching of analysis begins.

• Difficulties linked to the conceptualization of the notion of limit at the core of the field, and to its technical mastery.

• Difficulties linked to the necessary breach with algebraic thinking.

II.1. DIFFICULTIES LINKED TO THE BASIC OBJECTS OF THE FIELD:

We cannot consider that the basic objects of analysis are new notions for students when they enter the field. In France, for instance, irrational numbers, linear and affine functions are introduced at grades 8 and 9 and, at grade 10, the notion of function becomes a central notion. Nevertheless, we cannot say that these objects are yet stabilized; on the contrary, analysis will play an essential role in their conceptualization and maturation.

Real Numbers

Various pieces of research tend to show that conceptions developed by students about real numbers are not really adequate. The distinction between the different categories of numbers remains fuzzy and strongly dependent from their semiotic representations (Munyazikwiye, 1995). Moreover, the increasing and uncontrolled use of pocket calculators tends to reinforce the assimilation real number / decimal number.

At this level of schooling, real numbers are algebraic objects. Real order is recognized as a dense order but, depending from the context, students can conciliate this property with the existence of numbers just before or just after a given number : for instance, 0.999... is often said to be the number just before 1 ; more than 40% students entering French universities consider that, if two numbers A and B satisfy the condition : $\forall n>0\ |A-B|<{}^{1}/_{n}$, they are not necessarily equal, just very close, in some sense successors. The association between real numbers and the real line also lacks coherence. Even if a priori students accept the principle of a one to one correspondence between R and the line, they are not necessarily convinced that such or such precise number has a place on the line (Castela, 1996).

Functions

As far as functions are concerned, the situation is even more complex and it is difficult to summarize in a few words the huge amount of existing research results. I will only mention some main categories of difficulties, which, once more, do not act independently.

• *Difficulties in identifying what really a function is and in considering sequences as functions.*

It is well known that the criteria used by students in order to check functionality are at variance with the formal definition of the notion, even for students who are able to reproduce this formal definition (Vinner & Dreyfus, 1989). These criteria depend more from typical examples taken as prototypes and from associations such as the association : function - formula or the association function - curve. So, the same object may be considered as a function or not, depending from the form of its semiotic representation : for instance, the function f defined by f(x)=2 is not recognized as a function because the given algebraic expression does not depend from x, but is considered as a function if given through its graphic representation as it is represented by a straight line. Such phenomena led researchers to differentiate between what they call « concept definition » and what they call « concept image » (Tall &Vinner, 1981).

• *Difficulties in going beyond a process conception of functions and being able to link flexibly the process and the object dimension of this concept, and develop with respect to it a proceptual view (Tall & Thomas, 1991).*

Research clearly shows the qualitative gap existing between these two levels of conceptualization : process level and object level (Sfard, 1992) (Dubinsky & Harel, 1992). One can trace it, for instance, in the difficulties students often meet at considering as equal functions defined by two equivalent processes, or at working with functions defined by a general property. Mathematical work in analysis becomes very difficult if students can only rely on a process view of functions, as they have to engage them as objects, in more complex processes (such as integration, differentiation...) and as they also have to consider not only particular objects but classes of functions defined by specific properties : continuous, C1, Riemann integrable ...functions.

• *Difficulties in linking the different semiotic registers which allow us to represent and work out functions.*

These difficulties have been extensively analyzed (Romberg, Carpenter, Fennema, 1994), both those related to the translation from one semiotic register to another one, especially from the graphical register to the algebraic one, and those related to the use of information referring to different notions within a given register : function and its derivative, in the graphical register, for example. Research has also evidenced the poor sensitivity of usual teaching to these difficulties and nicely explained how these practices tend to reinforce difficulties by the way they treat graphical representations and the low status they give to graphical reasoning.

• *Difficulties in going beyond numerical and algebraic modes of thinking.*

This category of difficulties is less mentioned in the literature, perhaps because students are rarely given the responsibility of the kind of thinking they have to develop in problem solving. Nevertheless, it is an essential one as analysis is, since Euler at least and his famous book : «Introductio in analysin infinitorum», a mathematical field organized around the notion of function, around functional thinking. Current research in France (Pihoué, 1996) tends to prove that, when entering grade 11, students who have been exposed to functions during three years at least, do not really see the interest and economy of functional thinking. For a great majority, it remains a simple matter of didactic contract.

II.2. DIFFICULTIES LINKED TO THE CONCEPT OF LIMIT

Difficulties students must face are not reduced to this first category. Those associated to the conceptualization of the notion of limit are well documented in the research literature (Cornu, 1991). As regards this specific domain, it has to be noticed that several researchers refer to the notion of epistemological obstacle introduced by the philosopher Gaston Bachelard (Sierpinska, 1988), (Schneider, 1991). According to Bachelard, scientific knowledge is not built in a continuous process but results from the rejection of previous forms of knowledge, which constitute «epistemological obstacles ». So these authors make the hypothesis that some learning difficulties, especially the more resistant ones, result from forms of knowledge, which have been, for a time, coherent and efficient in social or scholar contexts met by students. In other words, they focus on difficulties which can be expressed as resulting from coherent and locally efficient forms of knowledge, appearing both in the historical development of the concept and in its current learning, even if they do not take exactly the same form, due to evident cultural differences.

As far as limits are concerned, the different authors seem to agree on the following epistemological obstacles at least :

• the common sense of the word « limit » which induces resistant conceptions of the limit as a barrier or as the last term of a process, or tends to restrict convergence to monotonous convergence,

• the over-generalization of properties of finite processes to infinite processes, according to the continuity principle stated by Leibniz,

• the strength of a geometry of forms which prevent from clearly identifying the objects involved in the limit process and their underlying topology. This makes difficult the subtle game between the numerical and geometrical settings at play in the limit process.

The strength of such an obstacle is attested by the difficulties encountered by students and even graduate students when they are asked the following unfamiliar question : why the same method : cutting a sphere into small slices and approximating it by the corresponding pile of small cylinders, then taking the limit, gives a correct value for the volume of the sphere and an incorrect value for its area ? As the pile of cylinders has the sphere as evident geometrical limit, most of them do not understand why the different magnitudes associated to the pile of cylinders do not necessarily converge towards the corresponding magnitudes for the sphere !

In the research literature about limits, the identification of epistemological obstacles plays an important role, but students' difficulties cannot be reduced to those relevant of this particular category. The concept of limit, as the concept of function, has two facets : a process facet and an object facet and the ability to cope efficiently with these two facets requires cognitive processes whose complexity and difficulty are well known, now. This fact contributes to explain why students, all over the world, have so strong difficulties in identifying 0.999... and 1. The first semiotic representation 0.999... is obviously of a process type and the second of an object type. Equaling the two imposes not to be trapped by these semiotic characteristics, it imposes to be able to see beyond the infinite process described by 0.999..., the number created by this process and detached from it.

Another important category of difficulties arises from the characteristics of the formal definition of the concept, at least in the standard analysis which is taught nowadays : its logical complexity and the fact that it requires to reverse the direction of the function process which goes from the variable x to the value of the function f(x). But, beyond these formal characteristics, there is one essential point : between an intuitive conception of limits and a formal conception, there is a major qualitative gap. The formal concept of limit is a concept which partially breaks with previous conceptions of the same notion. Its role as unifying concept, as foundational concept is as important, perhaps more important than its productive role in problem solving. We meet there an epistemological dimension of the concept whose didactic transposition is not evident. In fact, some re-searchers such as A.Robert (Robert & Robinet, 1996), have got the conviction that it needs specific mediations, at a meta-level. Some relation-ship could certainly be made with the ideas developed by Vigotsky about the formation of scientific concepts.

II.3. DIFFICULTIES LINKED TO THE NECESSARY BREACH WITH ALGEBRAIC THINKING.

Mathematical activity in analysis, strongly relies on algebraic skills and competencies but, at the same time, entrance in « analysis thinking » requires to take some distance from algebraic thinking (Legrand, 1993). The breach between algebraic thinking and analytic thinking has various different dimensions but I shall limit to some essential points.

Firstly, in order to enter analysis thinking and be efficient in it, one has to develop another vision of equality, to develop and master new techniques for proving equalities. Note that a similar change was evidenced by didactical research in the transition from numerical thinking to algebraic thinking. Briefly speaking, in algebra, in order to prove that two expressions a(x) and b(x) are equal, the standard strategy is the following: to transform one or the two of them by successive equivalencies, up to obtain two obviously equivalent expressions, or to transform their difference (resp. quotient) up to obtain 0 (resp. 1). In analysis, of course if one does not restrict analysis to its algebraic part, this strategy is often out of range or at least not the most economic one, as we do not know the objects of analysis as we know the algebraic ones and as we often work with local properties. We have to develop a vision of equality linked to local « infinite proximity », that is to say linked to the fact that if : $\forall \varepsilon > 0 \ d(A,B) < \varepsilon$, for an adequate distance d, A=B. As a consequence, inequalities are taking in analysis a predominant role over equalities and reasoning locally by sufficient conditions on algebraic expressions becomes a fundamental mode of reasoning.

For instance, if you have to prove that there exists a neighborhood of x_0 such that : $a(x)<b(x)$, you do not try to solve this inequality as you would certainly do in algebra. You transform it by introducing successive expressions : $a_1(x)$, $a_2(x)$... $a_n(x)$, and by reducing if necessary the initial neighborhood such as locally : $a(x) < a_1(x) < \ldots < a_n(x)$, up to get the evidence that for some neighborhood of x_0, you can assure : $a_n(x) < b(x)$. Each step of the process can require difficult choices : you have to accept to loose information on a(x), but not too much as you want to stay locally under b(x) and you have to combine these choices with a subtle game on neighborhoods.

Looking at these changes, at the underlying increase in technical difficulty, helps us to understand better the distance which separates the ability to articulate the formal definition of the limit, even to illustrate it by examples and counter-examples, by graphical representations, from the ability to technically master this definition, that is to say to be able to use it as an operational tool in problem solving and proofs.

In order to close this section, I would like to stress another dimension of this breach between ancient modes of thinking and analytical thinking. Entering the world of analysis requires also to reconstruct objects which were familiar, but in other worlds. The notion of tangent provides us with a typical example of such necessary reconstruction. As showh by (Castela, 1995), the educational system, in France at least, is not sensitive to this problem and this poor sensitivity has evident negative effects.

III. THE EVOLUTION OF FRENCH SECONDARY CURRICULA

III.1. THE 1902 REFORM : A PRAGMATIC AND ALGEBRAIC APPROACH TO ANALYSIS

As in many other countries, analysis appeared in the classical secondary curriculum at the beginning of this century with the 1902 reform (Artigue, 1996). This was a successful introduction, supported by the most eminent mathematicians of the time : Poincaré, Borel, Hadamard... not to quote the others, as attested by the ICMI study devoted to these questions ten years after (Beke, 1914).

For the mathematicians involved in the reform process, taught analysis had to be rigorous, free from any kind of metaphysics (thus from any kind of infinitesimals), but, at the same time, it had to be accessible to students and useful, both for mathematics and physical sciences. The following quotations from a famous conference given by H.Poincaré on mathematical definitions (Poincaré, 1904) and from the report of the ICMI study illustrate these positions :

> « No doubt, it is difficult for a teacher to teach something which does not satisfy him entirely, but the satisfaction of the teacher is not the unique goal of teaching : one has at first to take care of what is the mind of the student and what one wants it to become » (Poincaré)

> « Our main duty is to introduce the notions of differential and integral calculus in an intuitive way, by starting from geometrical and mechanical considerations, and gradually rise to the necessary abstraction. All our affirmations have to be true, but we do not have to target the whole truth. » (Beke)

These mathematicians were convinced that it was possible to develop a curriculum in analysis coherent with these principles without major difficulties and, with regard to the notion of limit, one can read in the final report of the ICME study :

> « The notion of limit is so present in secondary teaching and even at beginning levels (unlimited decimal fractions, area of the circle, logarithm, geometrical series...) that its general definition would not be likely to occasion any difficulty. »

What was taught in fact, at that time, was standard calculus, but we have to be aware that the increase in power and rigour offered by this algebraic analysis for solving classical problems at secondary level was so evident that the interest of this new calculus could not be denied.

III.2. THE SIXTIES' REFORMS AND THE NEW MATH REFORM: TOWARDS A FORMAL APPROACH OF ANALYSIS

The curriculum remained stable until the beginning of the sixties. At that time, structuralism was becoming dominant. Mathematicians had discovered the power of algebraic structures and foundation issues were taking a predominant role. In France, it was the golden age for the Bourbaki Group created in 1937, with the aim of renewing the universitary course of differential and integral calculus. Traditional teaching of analysis was then seen as an obsolete object, unable to cope with the central ideas of the field. Renovation entered the secondary curriculum in 1960, introducing a conception of analysis less empirical and pragmatic, with more emphasis on fundamental concepts and their structural dimension. At the same moment, quantifiers were officially introduced as well as elements of set theory and algebraic structures. The formal definition of the limit was explicitly mentioned in the syllabus. This was a real renovation, reinforced in 1965, but not a revolution. A careful look at textbooks shows that it was in fact a transition period, that newness introduced did not bowl over the ancient organization.

The new math reform definitively turned the page at the beginning of the seventies. Analysis was not its main part but teaching of analysis was deeply influenced by the spirit of the reform. It became essentially formal and theoretical, focusing more on definitions and foundation issues than on problem solving. It was rejected soon after within the wave of global rejection of the new math reform.

III.3. THE REJECTION OF THE NEW MATH REFORM AND THE INTRODUCTION OF INTUITIVE AND EXPERIMENTAL APPROACHES

The last important reform directly resulted from the rejection of the new math reform and took place in 1982. It was influenced by the reflection and experimental work undertaken in the IREMs. It was supported by the vision of mathematics as a science historically and culturally produced and, as such, dependent from historical and cultural contexts, which was becoming predominant. It tried to find a more adequate equilibrium between the mathematical coherence of the field and students' cognitive development. It tried also to find a better equilibrium between the « tool » and « object » dimensions of analysis, according to R.Douady (Douady, 1986), that is to say between the internal development and structuration of the fundamental concepts of the field and their use as tools for solving problems internal or external to mathematics. The proposals of the commission interIREM Analyse published in 1981 reflected these ambitions. They were the followings :

• to modify the relationships between theory and applications and organize the syllabus around the solving of problems rich enough and epistemologically representative of the field, a field being now considered, according to J.Dieudonné, as a field where « approximating, majoring, minoring » were core processes,

• to find better equilibrium between qualitative analysis and quantitative analysis, by giving more importance to quantitative problems, thanks to the use of calculators,

• to give particular importance to typical and simple examples which could then serve as a reference, and to avoid any early interest in pathological cases,

• to theorize only when necessary with reduced levels of formalization accessible to students,

• and last but not least, to develop a constructivist approach to teaching.

These proposals were directly reflected in the new curriculum, proposed in 1982 and the general strategy used in order to introduce the different key notions clearly illustrates these positions:

- explore typical simple behaviors, both numerically and graphically, with the help of calculators for the numerical part,
- use these explorations in order to produce quantitative definitions adapted to the most simple cases and work them out,
- by introducing more complex cases, let students become aware of the limitations of the first approach and introduce general and qualitative definitions.

For instance, the notion of derivative was introduced through the notion of first order expansion by exploring numerically and graphically the local behavior (at 0) of typical functions allowing majorations of the form : $|f(x+h)-f(x)-f'(x)h| \leq Mh^2$. Later on, functions which did not allow such simple majorations were introduced and led to the general definition of a first order expansion.

Formalization was strongly reduced. Only one formal definition was introduced for the limit in 0 and teachers were explicitly asked not to use it extensively. This formal definition whose role was vanishing, disappeared in the curriculum adjustment which took place three years later and the paragraph of the syllabus devoted to limits was modestly renamed «language of limits ».

Mathematical activity was organized around problem solving : optimization problems, approximation of numbers and functions, modelisation of discrete and continuous variations... The notions of derivative, and above all of derivative function, essential tool for solving these problems, became the central notion. The logical order : limits - continuity - derivatives was thus broken : a minimal intuitive language of limits was introduced for supporting the introduction of the derivative, then the derivative function was the central piece of the edifice, the notion of continuity nearly disappeared, all the more as, with the definition chosen for the limit, every function having a limit at a point in its definition domain was necessarily continuous.

The influence of analysis was already evident at grade 10, one year before its official introduction, as attested by the following excerpt of the syllabus :

« Themes for activities :
1. Majoration and minoration of a function on an interval
2. Research for extremum in optimization problems
3. Rate of variation - inequalities such as : $|f(x)-f(y)| < M|x-y|$. Geometrical interpretation
4. Use of variations of functions in order to solve equations f(x)=b and inequalities. »

This was an ambitious curriculum. It tried to make alive the epistemological value of analysis as a field where approximation played a central role and to organize the progressive entrance of students in it, both at conceptual and technical levels. In this introduction, links between algebraic, numerical and graphical representations and techniques played an essential role, as explicitly stressed in the syllabus. Analysis taught was not a formal one, it was approached in an intuitive and experimental way, and there was a desire not to limit it to algebraic practices. This syllabus was then modified in 1985, 1990, 1993, in order to better fit the increasing democratization of high schools but the spirit remained the same, at least as expressed in the syllabus. So we can measure today the long term effects of nearly 15 years of such intuitive and experimental approaches.

IV. SOME POTENTIALS AND LIMITS OF THESE APPROACHES

Firstly, there are some evident positive outcomes. I would like to only mention three of them which, in my opinion, are specially important :

• Such an approach made the field accessible to any category of students, up to a certain point, and this positive outcome cannot be considered as unimportant, above all if one takes into account that the great majority (about 70%) of what we call each « age class » of the population enters high school and is taught analysis,

• Very soon, students are in contact with important problems at the core of the field, such as optimization and approximation problems ; analysis is not reduced to its algebraic part, and according to the syllabus, textbooks try to give importance to the numerical and graphical dimensions of both concepts and techniques,

• Calculators and even graphic calculators are regularly used by students. They help to make viable the numerical and graphical approaches encouraged in the syllabus.

In spite of these evident positive outcomes, I am far from thinking that we found some ideal way towards analysis. Some very important issues remained unsolved and new problems are emerging. Once more, I would like to focus on some of them.

• *The limits of the help provided by calculators and the issues linked to their integration*

Calculators and even graphic calculators are widely spread out in France, as mentioned above. In 1981, a decision of the Ministry of Education allowed them to be freely used in assessments at secondary level and this is still the case today. Students for instance can take the « Baccalauréat » with a graphic calculator or even a TI92. This decision was taken in order to foster the institutional integration of such technological means. But, even now, calculators are mostly considered as private students' tools. Recent research (Trouche, 1996) shows the negative effect of such an uncontrolled use on the conceptions developed by students, about concepts such as the concept of limit, about numerical approximations or graphical representations. An effective learning of analysis with graphic calculators requires the development of specific competencies, of specific knowledge. This fact is not easily recognized by the educational system which remains reluctant to devote time to such specific learning.

•The difficulties of viability of the « approximation » dimension of analysis

Recent evolution has also put to the fore the difficulties encountered by the educational system with the « approximation » dimension of analysis. As stressed above, developing this dimension requires to take some distance from usual modes of thinking and relies on difficult techniques whose learning is a long term process. Teachers encounter evident difficulties at organizing and preserving an « ecological niche » for such mathematical practices, all the more as they cannot avoid the competition between approximation techniques and algebraic techniques, which look easier (Artigue, 1993).

• The difficulties of viability of teaching through rich and significant problems.

New curriculum wanted to organize analysis approach around rich and significant problem solving activities. We note an increasing gap between these ambitions which are still explicit in the syllabus and the content of textbooks. What one can find in most recent textbooks tends more towards an unstructured accumulation of problems with a limited scope whose solving is so much decomposed in sub-questions that students can hardly understand their global coherence. Such an evolution clearly shows the strength of didactic transposition processes which shape and condition the real curriculum (Chevallard, 1985). In the educational world too, as far as assimilation remains possible, accommodation is not the rule.

• The difficulties resulting from the increasing lack of structuration

Once more, these difficulties are evidenced by recent textbooks. Status of objects, notions, assertions remains fuzzy. Formal definitions have been banished, replaced by descriptive sentences expressed in

« natural language », these being a priori thought more accessible and intuitive. In fact, these sentences only have the appearance of natural language : they have nothing to do with the vernacular language spoken by students. They do not support an operational control of practices. Moreover, as quantifiers are generally situated partly at the beginning of the sentence, partly at the end, they do not help students to become sensitive to the complex game quantifiers play in the corresponding definitions. Theorems are accepted on the base of a few explorations and not necessarily labeled as such. At reading textbooks, one has the uncomfortable feeling that the coherence induced by the logical constraints of knowledge has progressively fainted without being replaced by another evident coherence.

For many of our students, what we are developing, beyond the standard algebraic part of analysis, is perhaps more a world of « pottering about » than the mathematical world we wanted to begin to make alive.

V. SOME CONCLUDING REMARKS

Didactical research clearly attests that it is not easy for students to enter the conceptual field of analysis, if analysis is not reduced to its algebraic part, if entrance in this conceptual field aims at developing the modes of thinking and the techniques which are now considered as fundamental in it. Secondary education has been facing this problem for about one century. At the beginning of the century, analysis entered the general secondary curriculum, and this introduction, which provided teachers with very efficient tools for solving classical problems, both in mathematics and physical sciences, was highly appreciated. With the reform undertaken in the early sixties, new ambitions entered the secondary curriculum in analysis : roughly speaking, analysis taught took its autonomy from algebra and the object dimension emerged from the tool dimension. Soon after, the new math reform imposed a formal vision where foundation issues tended to become predominant. This formal vision was soon rejected and, in the early eighties, a new organization around problems and techniques at the core of the field emerged ; experimental and intuitive approaches were encouraged. These intuitive and experimental approaches to analysis progressively imposed themselves and today, they appear as the only reasonable entrance gate, all the more as analysis teaching is no longer limited to some mathematical or social elite. But, we have to confess that they did not succeed in making teaching and learning analysis miraculously easy and satisfactory. They helped to solve some problems but, in the long range, if they are not carefully controlled, they tend to generate some unavoidable problems. The necessity to better control these approaches is evidenced by the evolution of the didactic transposition process along the last fifteen years in France and the cognitive effects this evolution produces. Current didactical research in the field, in France, is addressing these crucial issues, both at high school level and at the transition form high school to university.

Analysis of this particular didactic transposition process also evidences the difficulties raised by the exploitation of didactical results or local successful experimentations, for undertaking substantial and global actions on the educational system. For this purpose, epistemological and cognitive approaches which have been predominant in the field and essentially used, up to now, are obviously insufficient. We have to integrate approaches to didactical research which allow us to better take into account the role played by institutional and cultural constraints in both learning and teaching processes.

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INNOVACIÓN EDUCATIVA: UN RETO PROFESIONAL

Luis Balbuena

La innovación educativa puede ser definida, descrita e interpretada desde múltiples ópticas. Como en el proceso educativo intervienen investigadores, administraciones públicas, maestros, padres, alumnos, etc., cada uno puede tener, y de hecho lo tiene, un modo de entender y de hablar de la innovación en el área de la educación. Y por si esta complejidad fuese poca, en la literatura dedicada al tema existe un conjunto de palabras tales como reforma, cambio, renovación, investigación, mejora, innovación, etc., de significados parecidos y sin que exista aún un acuerdo generalizado sobre cuáles son o deben ser los límites de unas y de otras.

Por eso considero importante que trate de delimitar lo mejor que me sea posible cuál es el entorno educativo en el que centraré mis reflexiones.

Si se ha leído con atención el extenso programa de este 8º ICME, se habrá podido comprobar que esta preocupación que les intento transmitir es la parte central de varias actividades. Significa que estamos ante un problema que ocupa la atención de los que nos movemos en el mundo de la Educación Matemática, aunque quizás preocupe más a los que cada día hemos de practicar con grupos de alumnos más o menos extensos. Quizás seamos también los que tengamos más difuso nuestro rol en el área de la Educación Matemática.

El tema es abordado directa o indirectamente en varias conferencias regulares, es el eje específico de un grupo de trabajo y seguro que planeará en muchas más de las actividades que contiene el programa.

Pero empecemos a precisar los conceptos.

Parece que hay un cierto consenso en considerar la Reforma educativa como un cambio en el sistema a gran escala. Un cambio que afecta, no sólo al curriculum de las distintas disciplinas y a las formas de enseñar, de evaluar, etc., sino que puede introducir cambios estructurales sobre lo ya existente. Así, por ejemplo, en España, se acomete en estos momentos una reforma del sistema educativo que modifica la estructura existente hasta hoy de manera sustancial. Introduce dos segmentos absolutamente novedosos: Educación Infantil (de tres a cinco años), y la Enseñanza Secundaria Obligatoria (de 12 a 16 años), con lo que la obligatoriedad se amplía de los catorce años actuales a los dieciséis. Por otra parte, reduce la enseñanza primaria de los ocho años que tiene ahora a seis y también el actual Bachillerato (incluyendo el Curso de Orientación Universitaria) pasará de cuatro a dos años.

Es, por tanto, una reforma que como indica Sack (1981) "es una forma especial de cambio, que implica una estrategia planificada para la modificación de aspectos del sistema de educación de un país, con arreglo a un sistema de necesidades, de resultados específicos, de medios y de métodos adecuados".

La reforma educativa va generalmente ligada a políticas y programas gubernamentales con las que se intenta dar algún rumbo a todo el país. Los gobernantes son conscientes de que ningún cambio profundo puede realizarse en una sociedad si no se reforma y se orienta el sistema educativo hacia esos objetivos. Nuestros gobiernos, en España, orientan el sistema educativo hacia la nueva situación del país: hacia la democracia y hacia la plena integración con Europa.

Queda claro, por tanto, cuál es concepto de reforma aplicado a la educación.

<u>La investigación en Educación Matemática</u> es otro término que conviene clarificar, aunque parece que hay un cierto consenso en considerar a la investigación en esta área como aquella que se realiza siguiendo los métodos propios de un proceso investigador. Dice G. Vázquez (1987) "... deben formularse dos requisitos para dar validez a la investigación en esta área: primero, que sea científica y segundo, que sea pedagógica, esto es, adecuada a la naturaleza de nuestro objeto de estudio: la educación como resultado y como proceso ".

D.J. Fox en su "Modelo del proceso de investigación" habla de 17 etapas divididas en tres partes:

Primera parte: diseño del plan de investigación (trece etapas).

Etapa 1.- Idea o necesidad impulsora y área problemática.
Etapa 2.- Examen inicial de la bibliografía.
Etapa 3.- Definición del problema concreto de la investigación.
Etapa 4.- Estimación del éxito potencial de la investigación planteada.

Etapa 5.- Segundo examen de la bibliografía.
Etapa 6.- Selección del enfoque de la investigación.
Etapa 7.- Formulación de las hipótesis de la investigación.
Etapa 8.- Selección de los métodos y técnicas de recogida de datos.
Etapa 9. - Selección y elaboración de los instrumentos de recogida de datos.
Etapa 10. - Diseño del plan de análisis de datos.
Etapa 11. - Diseño del plan de recogida de datos.
Etapa 12. - Identificación de la población y de la muestra a utilizar.
Etapa 13.- Estudios pilotos del enfoque, métodos e instrumentos de recogida de datos y del plan de análisis de recogida de datos.

Segunda parte: ejecución del plan de investigación (tres etapas).

Etapa 14. - Ejecución del plan de recogida de datos.
Etapa 15. - Ejecución del plan de análisis de datos.
Etapa 16. - Preparación de los informes de la investigación.

Tercera parte: aplicación de resultados (una etapa).

Etapa 17. - Difusión de los resultados y propuesta de medidas de actuacion.

Sobre la investigación en Educación Matemática creo que se ha estudiado y teorizado suficientemente. Los criterios de calidad están establecidos y más o menos aceptados por todos, pero creo que es la práctica, esa tercera parte que apunta Fox, la que, en la mayoría de los casos, determina la validez y la viabilidad de cualquier investigación por muy sesuda y rigurosa que sea.

Generalmente se piensa que la labor del investigador acaba con la elaboración de un informe y su presentación ante un tribunal cuando se trata de una tesis doctoral. Opino, sin embargo, que es responsabilidad del investigador, no sólo la difusión de su estudio, sino también procurar que sus deducciones y conclusiones lleguen a producir los cambios y mejoras que investigó, máxime, cuando la mayor parte de las veces, su investigación ha sido financiada con dinero público.

Por otra parte, nadie pone en duda la trascendencia y las aportaciones de la investigación en Educación Matemática para la mejora del aprendizaje y de la enseñanza de esta disciplina. Esto es un axioma.

No voy a incidir más en este aspecto que considero del máximo interés, porque quizás me desvíe demasiado de mi objetivo aunque pienso que es necesario que reflexionemos acerca del papel que juega o puede jugar la investigación en los cambios que se producen en el sistema educativo. Resulta cuando menos preocupante, la frecuencia con que los cambios que se proponen (desde las administraciones, sobre todo), no se basen en investigaciones contrastadas y no deja de causar cierta inquietud y perplejidad que, a los que enseñamos día a día en el aula, no nos lleguen tampoco ni ideas ni resultados producto de investigaciones que nos ayuden a enseñar mejor o a conseguir que nuestros alumnos aprendan con más intensidad y eficacia.

Y llego así al tercer concepto que quisiera intentar precisar y delimitar:

Innovación educativa.

No existe demasiada literatura sobre este concepto y ello, quizás, debido a lo complicado que resulta marcar con claridad sus límites y los criterios de calidad y eficacia que debe conllevar toda innovación educativa. Hay incluso quien piensa que se trata de una distinción (entre investigación e innovación), interesada y artificiosa. Pero también he escuchado a investigadores, como el Prof. Rico (Universidad de Granada), hablar de la necesidad de aceptar ambas concepciones como perfectamente diferenciadas, coexistentes y potenciables.

Empezaré exponiendo algunas opiniones sobre la innovación educativa que puedan orientar al profesor "de aula" a la hora de intentar clarificar su rol innovador.

Según T. González y J.M. Escudero (1987) "... suele emplearse el término innovación para referirse a cambios a menor escala o más concretos ". Desde hace años otros autores hablan de la innovación como cambios deliberados que pretenden dar al sistema educativo una mayor eficacia para el cumplimiento de sus objetivos.

Así pues, parece que se quiere reservar el término "reforma" para aquellos cambios cuantitativamente amplios; el término "investigación" (en nuestro caso, en Educación Matemática), para cambios cualitativamente más profundos y se reserva la "innovación" para cambios cualitativa y cuantitativamente menores. Esa ambigüedad nos sitúa ante una idea difícil de delimitar, para la que no existe una medida adecuada que permita precisar cuándo se rebasan los límites de la innovación para situarnos en otra cosa.

Debemos, no obstante, intentar establecer criterios y conceptualizaciones que acerquen la innovación a sus protagonistas que son (y en esto parece que no existe demasiada controversia), los profesores y profesoras que cada día desarrollan su labor profesional en centros educativos, ante grupos de estudiantes cuya correcta formación depende, en gran medida, de las actitudes y de las aptitudes de sus profesores.

Esa tendencia de reservar la innovación para lo que podríamos llamar "la práctica educativa" es la visión que, sobre el tema, sostienen J.M. Sancho y otros (1992) que hablan de "procesos deliberados y sistemáticos" que intentan producir cambios en la práctica educativa. También M.V. García (1995) indica que "... para algunas personas la innovación es algo cotidiano, algo propio del quehacer profesional, vinculado a su preocupación por la educación y por el aprendizaje de sus alumnos". Creo que esta visión del concepto nos sitúa sobre la pista de lo que muchos entendemos por innovación educativa, de qué es lo que cabe en él y qué tipo de trabajos y actividades se pueden considerar como innovadores.

Surge, pues, una pregunta clave: ¿pueden establecerse principios que permitan identificar procesos innovadores en Educación Matemática?

Intentaré aportar algunas señas de identidad sobre lo que considero como innovación educativa.

Como punto de arranque, considero la innovación aplicada a nuestro campo, como aquellas experiencias que suponen acciones prácticas y sistemáticas por medio de las cuales se intenta introducir y promover ciertos cambios tanto en la forma de aprender y de enseñar matemáticas, como para conseguir actitudes más positivas en torno a nuestra disciplina. Este último aspecto tiene, para mi, una gran importancia. Se trata de conseguir que los estudiantes se acerquen a las matemáticas de una forma distinta a como suele hacerse, que se superen ciertos tabúes e ideas preconcebidas, que la vean y la consideren como a una amiga. Es posible.

La innovación no siempre está motivada por algún grado de insatisfacción, bien ante el sistema educativo en su conjunto o bien ante la práctica cotidiana. El profesor debe considerar la innovación como algo propio de su quehacer y con la cual puede mejorar su práctica. Ahora bien, la alteración que supone pasar de una situación inicial a otra final diferente, no debe hacerse sin una programación que clarifique al detalle el por qué, el para qué, el cómo y el cuándo se hace. El objeto de nuestra innovación es tan delicado y preciso que nos debe obligar a meditar y reflexionar sobre el proceso de cambio que deseamos realizar.

Pero ahí no debe acabar la innovación. En ningún caso debe plantearse el cambio por el cambio. Es necesaria la evaluación de lo realizado. Él por qué, el para qué y el cómo, han de analizarse y comprobar si los objetivos propuestos con la innovación se han cubierto o no. Se debe verificar si el producto final es mejor que el del punto de partida. En resumen, en toda innovación educativa, ha de existir una programación y una evaluación. Sobre todo una evaluación, pues aunque pueda parecer paradójico, existe poca "cultura evaluativa" entre los profesores. En pocas ocasiones realizamos una evaluación de nuestro propio trabajo y esta es una de las condiciones básicas para garantizar la calidad de cualquier trabajo innovador. Debemos aplicarla a nuestra labor como algo habitual y como una especie de control de calidad de nuestro trabajo.

Otra característica que, a mi juicio, debe contener la innovación para que pueda concedérsele la categoría de eficaz, es que sea transferible. Que se pueda repetir. Que pueda ser aplicada por el mismo profesor o por otro en distintas circunstancias. Así, las innovaciones que un profesor decida realizar no quedarán restringidas sólo a los alumnos de su grupo, sino que deben ser conocidas, aplicadas y desarrolladas, si así se deseara, por otros también.

Por otra parte, la innovación debe ser proyectada y desarrollada tratando de ajustarse al máximo al método de trabajo que requiere una investigación. Es evidente que existen múltiples limitaciones para que una innovación, hecha desde el aula, pueda seguir estrictamente los pasos que requiere una investigación. El profesor en ejercicio está sometido a muchas restricciones: de medios, de tiempo, de formación, etc. , que, en general, le impiden realizar investigaciones siguiendo todas y cada una de las pautas que éstas requieren. El importante rastreo bibliográfico, que Fox señala en dos etapas, por ejemplo, es una seria limitación. Pero aún cuando todas las etapas no puedan realizarse en toda su extensión, el espíritu que anime al innovador, a aquel profesor que pretenda introducir algún cambio en su práctica docente, debe ser el mismo que anima al que quiere hacer una profunda investigación: ha de seguir un proceso totalmente deliberado y sistemático.

La innovación debe actuar directa e inmediatamente sobre el sistema educativo. Esta es una de las características que permite reconocer una innovación, pues ésta se plantea y diseña para producir efectos inmediatos. Es parte de su grandeza y también de su peligro. El profesor desea experimentar un nuevo material didáctico (manipulable o no), que le permita mejorar la introducción de un determinado contenido;

o bien quiere conseguir desarrollar mejor ciertas capacidades del alumno (abstracción, generalización,...) a través de contenidos nuevos o de contenidos del curriculum pero enfocados de otra manera o desea, en fin, cambiar la actitud de sus alumnos hacia las matemáticas proporcionándoles ideas y actividades más creativas y acordes con sus capacidades y con su formación. En cualquiera de los casos, el profesor innovador no se plantea obtener resultados a medio o a largo plazo, como suele ocurrir con la investigación, sino que desea obtener resultados (buenos, malos o neutros) de forma inmediata. Es que, además, esos resultados actúan como estímulo para proseguir mejorando e innovando. Creo conveniente clarificar que no es la innovación un factor discriminante entre el buen y el mal profesor. Un profesor con inquietudes innovadoras, realizadas con las condiciones de calidad, suele corresponder a un buen profesor. Sin embargo, el recíproco no siempre es cierto.

Tras estas caracterizaciones de la innovación, demos un simbólico paseo por la realidad.

Desgraciadamente el sistema educativo, casi nunca, procura las mejores condiciones para que el "profesor de aula" se sienta estimulado a realizar labores innovadoras. En general, lo hace o ha de hacerlo "a pesar del sistema", derrochando grandes dotes de profesionalidad y utilizando parte o gran parte de su tiempo libre, en el supuesto de que disponga de él, pues cada vez es más escaso...

El profesor innovador ha de superar, por tanto, diversas situaciones que le inducen a la desmoralización. Entre otras:

- La formación. Generalmente tenemos poco tiempo para la formación permanente, sobre todo en los aspectos docentes: leer revistas, libros, asistir a cursos, a congresos, etc. . Nada de esto suele ser fácil. Unas veces por falta de recursos económicos, otras porque la estructura del sistema (horas de clase, burocracia, permisos oficiales,...) lo impiden. Por otra parte, tampoco suele enseñarse el cómo innovar. Es algo que tenemos que aprender con nuestros medios y si tenemos interés por hacerlo. (Obsérvese que no he hecho referencia a la formación científica porque doy por supuesto que se tiene la suficiente e incluso más).

- El encorsetamiento de los sistemas educativos. Aunque en este aspecto hay una cierta superación, aún los sistemas educativos son demasiado rígidos. Se marcan no sólo los contenidos, sino que muchas veces se señalan las metodologías. Y si no lo hace la administración responsable, lo hacen las editoriales a través de sus libros de texto que, en general, marcan y obligan a seguir pautas.

- Falta de incentivación. Es una de las más graves deficiencias de los sistemas educativos. En casi todos los trabajos (sobre todo en los promovidos por la empresa privada), el trabajador tiene unos incentivos que le estimulan a procurar hacerlo cada vez mejor. En nuestra profesión esos estímulos sólo se encuentran en la profesionalidad de cada uno, pues las administraciones educativas no suelen ofrecer más que el sueldo, que no siempre está acorde con la labor que desarrolla el docente. Y cuando hablo de incentivos, no me refiero sólo a los de tipo económico. Existen formas de reconocer la labor y el esfuerzo del docente que no pasan necesariamente por elevar sus emolumentos.

Si bien para el profesor universitario la actividad investigadora forma parte de lo que podría ser su propia definición, en el profesor no universitario no sólo no toma parte de su definición, sino que ni siquiera es tenida en cuenta de forma adecuada entre sus méritos profesionales. En numerosas convocatorias de las que suele hacer la administración educativa para seleccionar profesores que han de desarrollar labores que requieren cierta especialización en temas didácticos, los trabajos realizados por un profesor aspirante relacionados con la innovación o con aportaciones al sistema educativo (publicaciones, artículos en revistas especializadas, comunicaciones o ponencias en congresos e incluso una tesis doctoral), son considerados méritos "no preferentes" y baremados con ínfimas puntuaciones. Es evidente que esta falta de reconocimiento conlleva una general inhibición del profesorado a la hora de plantearse proyectos de innovación y no digamos ya, de investigación.

Un reconocimiento formal de esta parcela de nuestro trabajo traería como consecuencia una transformación de nuestro perfil profesional. Posiblemente la subliminar y solapada "carrera docente" actual, basada en la consecución de la máxima titulación y de conseguir un máximo de certificados de cursos y más cursos, daría paso a otra basada más en las aportaciones que cada cual sea capaz de hacer al sistema educativo, con lo enriquecedor que eso podría ser para todos.

- Es evidente que no está cerrada la lista de limitaciones (soledad, incomunicación, dificultad para publicar resultados, burocratización cada vez mayor del sistema, y otras tal vez de tipo más particular de una zona, país o estrictamente personales). Sin embargo, creo que a pesar de todas esas limitaciones (que nos pueden afectar en mayor o menor medida), nuestra profesionalidad, es decir, el amor hacia nuestro trabajo y el convencimiento de que estamos desarrollando una labor de gran trascendencia para la sociedad, nos debe obligar a estar por encima de esas posibles trabas y hemos de convertir la innovación permanente en una de las características definitorias de nuestra profesión. Esa actitud de constante búsqueda de mejoras es buena para el sistema, es buena para nosotros, los profesores, porque nos obliga a reflexionar sobre nuestro quehacer, pero, sobre todo, es buena para nuestros alumnos porque entre otros efectos positivos, es un buen referente de profesionalidad que perdurará.

Existen infinitos campos en los que aquel que lo desee, puede innovar. En este sentido hemos de tratar de superar la falta de autoestima que suele tener el profesor de aula; el maestro que día a día se comunica con sus alumnos en el grandioso recinto de su aula. Ese hecho ya es en sí, sumamente importante. Pero es que, además, nuestros ensayos, nuestras innovaciones, por muy sencillas que nos parezcan, tienen el mérito de ser realizadas sin que nadie nos obligue a ello. Tendemos a pensar que nuestras experiencias no tienen valor y, en general, ni nos preocupamos de comunicarlas y mucho menos nos esforzamos en escribir acerca de ellas para transmitirlas, bien a través de congresos, bien por medio de revistas o por cualquier otro sistema. Sin embargo, tengo la convicción de que esa tendencia va a cambiar. Los profesores de aula hemos de ser conscientes de que nuestros problemas los tendremos que resolver nosotros mismos, ayudándonos y comunicándonos unos con otros. No podemos seguir esperando a que mi problema cotidiano sobre cómo estimular a un alumno para que comprenda y ame las matemáticas me sea resuelto ni se sabe cuándo, ni se sabe por quién, ni se sabe desde dónde. Nosotros también tenemos resultados que aportar en la Educación Matemática. Nosotros también debemos aportar resultados a la Educación Matemática. En ese sentido es esperanzador constatar la gran cantidad de comunicaciones presentadas por profesores de aula.

Por otra parte, los sistemas educativos se están reformando en muchos países. Es un deseo en todos, que el sistema que resulte sea bueno, el mejor posible y creo que una condición "sine qua non" para que un sistema educativo sea considerado bueno es que consiga que su profesorado se sienta con deseos de innovar, que se sienta apoyado y estimulado para ello y que se pongan a su disposición medios para poder hacerlo. Por eso hay un rayo de esperanza para que, si esto se comprende, podamos tener en el futuro mejores sistemas educativos y una mejora de nuestro rol profesional.

En definitiva, como se deduce de lo anterior, la innovación educativa necesita de la contribución de todos para conseguir una sistematización, una fundamentación teórica, la legitimación conceptual, clarificar su contenido, diseñar y consensuar unos criterios de calidad que garanticen la utilidad, la generalización y, sobre todo, que incida en la práctica escolar. Es necesaria la innovación permanente para afianzar y ampliar nuestro prestigio ante la sociedad. Nos queda, pues, tarea por hacer.

Como no quisiera que mi planteamiento fuera estrictamente teórico o que quedase en el terreno de los deseos, quisiera explicarles sucintamente una actividad que venimos desarrollando en mi centro de trabajo, que es un Instituto de Bachillerato (enseñanza Secundaria) con unos 850 alumnos y alumnas de edades comprendidas entre los 14 y los 18 años. Es, concretamente, el I.B. "Viera y Clavijo" de la ciudad de La Laguna - Isla de Tenerife - Canarias - España. Los ocho profesores que impartimos matemáticas en el centro formamos el Seminario de la asignatura y un equipo de trabajo compactado.

La actividad en cuestión la titulamos "Semana de Matemáticas" y la venimos desarrollando desde hace varios años. Cada edición nos permite ampliar y mejorar la anterior. El título de la actividad puede resultar engañoso porque el trabajo se desarrolla prácticamente durante todo el curso escolar. Moviliza en torno a las matemáticas a un buen número de alumnos del centro debido, principalmente, a que procuramos ofrecer un variado conjunto de actividades en las que poder participar. Esquemáticamente incluye:

- -Concurso del cartel anunciador (se convoca un mes después de empezar el curso).
- -Concurso de "Fotografía y Matemáticas" (también se convoca en noviembre).
- -Liguilla matemática "¡Yo qué sé!" (Los alumnos se organizan en equipos para participar en las cuatro sesiones de que consta).
- -Revista "¡Yo qué sé!" (Se edita paralela a la liguilla publicando trabajos diversos sobre matemáticas).
- -Talleres impartidos por los alumnos.
- -Show matemático.
- -Conferencias Ilusiones ópticas, La medida del tiempo, Las matemáticas ¿para qué?, Las celosías, Los números y la numerología: lo serio y lo menos serio,...
- -Visitas pedagógicas (museo de la Ciencia y el Cosmos; Centro provincial de meteorología; planetario; observatorio astronómico; ...).

-Concurso de trabajos científicos.
-Construir aparatos y sencillas máquinas.
-Exposición de trabajos realizados por equipos durante el curso (celosías, la medida del tiempo, los cuarenta principales, ilusiones ópticas, simetrías,...).
-Realización de alguna actividad mascota: icosaedro de doce metros de diámetro, superficies regladas con rectas directrices de diez metros, laberinto gigante de 222 nudos y 40 vértices cambiables.
-Exposición de material manipulable; incluye un amplio conjunto de actividades que se realizan con material manipulable, elaborado, en muchos casos, por los propios alumnos: puzzles planos y espaciales, demostraciones visuales, puentes de Köningberg, mundo de los espejos, grafos, aparatos de Galton, la cicloide, la torre de Hanoi, la mesa de las celosías y las teselaciones,...

Es una actividad que entre otros muchos efectos positivos, permite presentar otra cara de las matemáticas y ser un punto de atracción para muchos alumnos.

Además de la actividad brevemente relatada, solicitamos a la autoridad educativa poder implantar una asignatura optativa para los alumnos de 2º y 3º (15 - 16 años) que hemos titulado "Taller para re-crear matemáticas" y que nos fue concedida. Este taller, con dos horas semanales, nos permite dar "rienda suelta" a la creatividad matemática toda vez que la programación tiene ese aspecto como uno de sus objetivos. La innovación está presente de manera casi permanente ya que intentamos desarrollar actividades que complementen la formación algoritmizada que se les ofrece en las clases ordinarias. Así que se profundiza en procesos que contengan grandes dosis de abstracción, creatividad, intuición matemática, estrategias de resolución de problemas matemáticos y no matemáticos, procesos lógicos, acercamiento a la historia de las matemáticas y de las ciencias en general, etc. En síntesis, desarrollar capacidades más que conocimientos estrictamente matemáticos.

Como conclusión quisiera animar a todos a innovar. Esta actitud no debe ser una excepción sino la regla general. Necesitamos las aportaciones de todos para que los profesores de aula encontremos y definamos claramente nuestro rol en el área de la Educación Matemática.

Debemos ser reflexivos y creadores de nuestro propio trabajo. Creo que es una forma honrada y directa tanto de ejercer la profesión como de dignificar y prestigiar al profesor ante la sociedad.

No quiero terminar sin hacer una mención expresa a Gonzalo Sánchez Vázquez, al maestro y amigo que convalece. Él nos ha enseñado el camino cuando dice que la Educación Matemática la debemos construir entre todos. En ocasiones nos ha hablado de la necesidad de formar equipos de trabajo en los que participemos no sólo los profesores de aula (a los que gusta llamar "profesores de a pie"), sino también investigadores, psicólogos, pedagogos y todos cuantos puedan y tengan algo que decir en esta área a la que él ha dedicado parte de su vida y de su sabiduría.

Muchas gracias.

DRAWING INSTRUMENTS: HISTORICAL AND DIDACTICAL ISSUES

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Introduction

The aim of this paper is to present some cultural artefacts, originally developed as drawing instruments, and to analyse them from two different, yet related, perspectives, namely their status in the historical development of geometry and a possible use in didactics of mathematics at secondary and tertiary level. As for the first problem, we shall base our arguments on well known results in the history of mathematics, whom the interested reader is referred to. As for the second problem, we shall hint at some recent and ongoing research studies in didactics of mathematics and in the psychology of mathematics education.

The following discussion is mainly drawn from the results of a fifteen years long research project in didactics of mathematics in secondary school, i. e. the project *Mathematical Machines*, that has been developed in cooperation by an academic researcher (the author of this presentation) and a group of secondary school teachers. Among them the contribution of Annalisa Martinez, Marcello Pergola and Carla Zanoli has to be especially acknowledged: they did the historical research and built more than one hundred and fifty models of historical relevance and used them systematically in the classroom (grades 9-13). The whole collection is temporarily kept in the laboratory of mathematics of their school (Liceo Scientifico 'A. Tassoni' at Modena) and is waiting to be moved to a more suitable site.

The context of the teaching experiments on *mathematical machines* has two characteristic features (Bartolini Bussi & Pergola 1996):

(a) the presence of manifold teaching aids, among which physical large size models (either statical or dynamical), the so - called mathematical machines, that give the name to the whole project.;

(b) the recourse, under teacher's guidance, to selected historical sources,

1) to contextualise problems, 2) to describe the functioning of machines, 3) to approach the problem of the historical development of mathematics in a broad perspective.

This context, in spite of the presence of intrusive physical models is highly mathematised: it allows students to go through the history of some mathematical theories from the very starting of their existence as geometrical objects.

In the following section we shall present some historical sheets that contextualise some of the models available in the mathematical laboratory. They refer to two topics, i. e. conics and geometrical transformations. In the other section we shall discuss the didactical use of these models by presenting shortly a teaching experiment on pantographs.

Drawing Instruments: Historical Issues

Descartes' Curve Drawers

It is well known that the Greek approach to geometrical construction relied upon the use of straightedge and compasses. The theory of conic sections was three dimensional, whence the name of *solid problems* for the problems which required conics to be solved. Surely other mechanical instruments had been built, like curve drawers (e.g. Nicomedes' conchoid) or mean finders (e.g. Erathostenes' mesolabon). Mechanical ways of generating conics too, on the base of their 3d-definition were known for centuries, as the enclosed figure 1 shows: it is a parabolic compass, drawn by Leonardo da Vinci. But their use was not accepted in scholarly geometry.

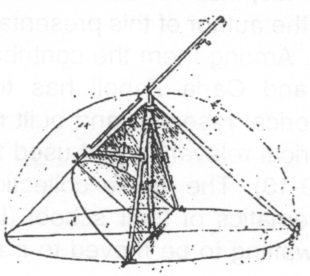

figure 1

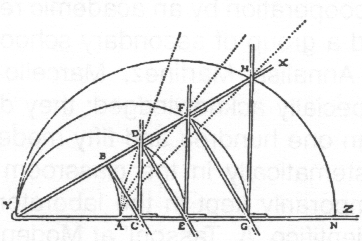

figure 2

We can contrast this position with Descartes' one. In the Géométrie he started his program of refounding geometry, that would have produced modern algebraic geometry, by introducing, in a very interlaced way, the so-called Cartesian geometry, i.e. the method of describing curves by means of measuring numbers, with respect to two lines, and the recourse to mechanical generation of curves, by means of movements whose relation admits exact determination (figure 2).

Consider the lines AB, AD, AF and so forth, which we may suppose to be described by means of the instrument YZ. This instrument consists of several rulers hinged together in such a way that YZ being placed along the line AN the angle XYZ can be increased or decreased in size, and when its sides are together the points B,C,D,E,F,G,H, all coincide with A; but as the size of the angle is increased, the ruler BC, fastened at right angles to XY at the point B, pushes towards Z the ruler CD which slides along YZ always at right angles. In like manner, CD pushes DE which slides along YX always parallel to BC; DE pushes EF; EF pushes FG; FG pushes GH and so on. Thus we may imagine an infinity of rulers, each pushing another, half of them making equal angles with YX and the rest with YZ.

Now as the angle XYZ is increased the point B describes the curve AB, which is a circle; while the intersections of the other rulers, namely the points D,F,H describe other curves, AD, AF, AH, of which the latter are more complex than the first and this more complex than the circle. Nevertheless I see no reason why the description of the first cannot be conceived as clearly and distinctly as that of the circle, or at least as that of the conic sections; or why the second, third or any other that can be thus described, cannot be as clearly conceived of as that of the first: and therefore I see no reason why they should not be used in the same way in the solution of geometric problems.

(Descartes, *La Géométrie*, 1637)

This very short quotation shows that in Descartes' geometry the status of mechanical devices such as curve drawers is theoretical. The existence of a curve drawer that allows one to determine exactly movements is the criterion to accept the product curve inside geometry.

It is important to recall that Descartes poses the problem of algebraic curves but does not solve it. Actually the identification of the set of plane algebraic curves with the set of curves that can be drawn (at least locally) by a linkage is solved only two centuries later by Kempe, as we shall see in the following (see also Bos 1981).

In the laboratory several models of curve drawers are available, dating back to either the classical age or to the systematic studies of post-cartesian age. Most of them can be simulated by computer (e.g. Cabri-Géomètre software) in order to draw the curve as a locus of points.

Approach to Geometric Transformations in the Seventeenth Century

The figures 3 and 4 from Dürer, show drawing activity with the help of instruments.

In the first figure the painter uses the picture plane as a 'window' from which to look at the space of the room. In the second figure the painter uses an instrument to obtain, point by point, a correct perspective drawing of a lute. A thread, weighted to keep it taut, is stretched from a point (the position of the eye) to a point of the lute: it passes through a point of the window that is later marked by means of two adjustable threads (that function as coordinate lines in the window); then the thread is moved out of the way and the 'little door' carrying the drawing paper is swung round into the picture plane, so that the position of the point (the intersection of the two threads) can be marked in the drawing.

figure 3

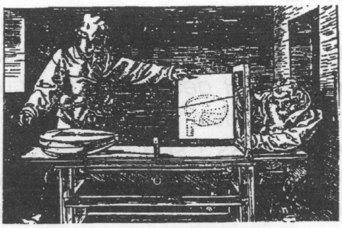

figure 4

In the mathematical laboratory a large size model of the first Dürer's perspectograph is available together with several models of pairs of perspective planes, made out of wood and/or plexiglas, with corresponding points joined by weighted threads. Some pairs of planes are movable up to the complete superimposition of each other. During the movement the eye's point (if any) changes, but the projection is conserved, according to Stevin's statement:

If the picture plane rotates round the ground line and if the observer rotates in the same sense round his own foot so as to be parallel to the plane, the perspective will not be troubled and will be kept also when the picture plane is turned over on the horizontal plane.

(Stevin,, Oeuvres Mathematiques, augmentees par A. Girard, Leyde, 1634)

This set of models embodies the birth of the projective approach to geometrical transformations in the seventeenth century. This approach, even if grounded on the tradition of Greek geometry (e. g. Euclid and Apollonius) represents a change for several reasons, e. g.:

(1) while in Greek geometry the attention was focused on figures, conceived as isolated realities, in the seventeenth century the attention is shifted to the whole plane (or space);

(2) while in Greek mathematics infinite existed only potentially, in the seventeenth century infinite is conceived also in actuality;

(3) while in Greek culture the separation between geometry and practical knowledge was emphasised, in the seventeenth century geometry develops in a dialectical relationship with other fields of knowledge, not only offering but also adopting methods, e. g. algebra (from commercial arithmetic), drawing (from architecture, technology - e. g. sundials and astrolabes - and art - the theory of perspective).

Some of these reasons seem to be understandable inside mathematics, but actually they depend on inside as well as outside factors, as some historians point out (Raymond 1979, Kline 1972).

The introduction of ideal points in the plane is the mathematical counterpart of the introduction of vanishing points in the picture plane, as centres of the pencils of lines obtained by projecting pencil of parallel lines: the vanishing points on the horizon of the picture plane can be conceived as external representations of the ideal points, which are at infinite distance from the observer. Besides the introduction of ideal points allows the unified treatment of conics, according to the general need of finding general methods that apply to individual cases (i. e. combinatoric reasoning), that is typical of the whole development of science in the seventeenth century (Raymond 1979).

The work of mathematicians such as Desargues has to be contextualised in this complex cultural space. Desargues brings from the practical tradition to the scholarly tradition of geometry two crucial elements:

(1) the concern with problems as being three-dimensional, as the practical tradition deals with the real world and not with diagrams on paper;

(2) the concept of projection from object to image.

However Desargues' concept of projection is not the same as in earlier texts on perspective: in Renaissance usage, objects rendered in per-spective are usually said to be 'degraded': as the emphasis is upon what has been changed by the projection (dimensionality and shape), the essentially symmetrical relationship between object and image is lost. The important original contribution of Desargues seems to have been the concept of invariance (Field & Gray 1987).

The Pantograph of Scheiner

The instrument we are describing now is the so - called Scheiner's pantograph, used in the seventeenth century either to make a scale copy of a drawing (figure 5) or as an aid to drawing in perspective (figure 6).

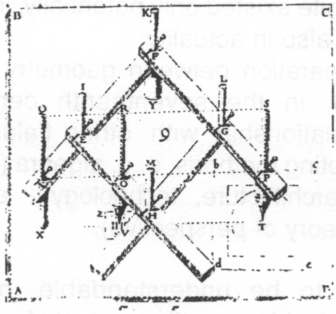

figure 5

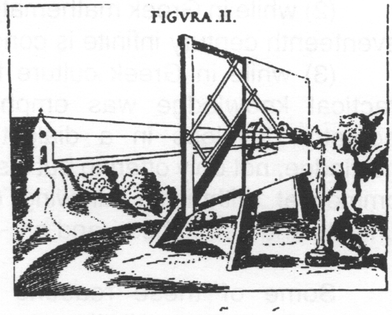

figure 6

The second use is so described by Scheiner (1631) himself:

The object, to be seen, sends its own image or visible species - called intentional by philosophers - to the eye through air or other diaphanous body, in pyramid's shape, whose base is the very object and the vertex is in the centre of our eye. This pyramid, wherever it is mathematically cut, has always on the section surface the lively and right image or portrait of the object. When we draw distant bodies, we cannot physically touch them with the pointer and extract immediately from them the copy by pen; hence, if we portray their species represented on the section of the segment of the visual pyramid, that we can touch mathematically with the pointer, as it is close to us, we shall make in the same time a copy very similar to the very object; as in optics there is a proposition, credited by everybody as true, that if two things are similar to another, they are similar to each other. Hence, as both the object and the image formed by us with the instrument are similar to the visible species, they are similar to each other.[..] As the image to be touched by the pointer is not real, but is only the intentional species of the object on the surface of the segment of the visual cone, and the copy formed by the pen must be real and physical, the plane on which we work must be partly real and physical and partly rational and mathematical.

Two copies of the Scheiner's Pantographs are available in the Laboratory. The first is designed to make a scale copy of a drawing. The second is for perspective drawing. The rational part is on the left and the physical part (where a sheet of paper can be stuck) on the right.

From Scheiner's Pantographs to Linkages

The Scheiner's pantograph is one of the early documented linkages used for realising geometrical transformations. If we use the linkage to make a scale copy of a drawing, we allude to a plane similarity that transforms the drawing into a smaller, equal size, or larger drawing. The use of the Scheiner's pantograph in perspective drawing alludes in modern terms to a composition of application, the first from the object to its 'intentional' species and the second from the intentional species to the physical copy made on the paper.

The study of such linkages arises from practical purposes, such as perspective drawing, but the attitude towards mathematical activity of the mathematicians of the seventeenth century is deeply contrasting with the attitude of ancient mathematicians. The introduction of linkages for theoretical purposes is reconsidered also in the nineteenth century. We can quote from Kœnigs (1897) the historical reconstruction of this theory:

The theory of linkages dates back to 1864. There is no doubt that such articulated systems have been used also before: maybe some passionate and precise investigator can track them down in remote antiquity. [...] When in 1631 Scheiner published for the first time the description of his pantograph, he surely did not know the general concepts that his small instrument contained in embryo; we can actually state that he could not know them, because these concepts are linked to the abstract theory of transformations, a theory peculiar of our century, which gives a unitary imprint to all the fulfilled progresses.

The merit of Peaucellier, of Kempe, of Hart, of Lipkine was not so much to have been able to trace some special curves by means of linkages as to have seen there a technique to realise real geometrical transformations. This is the very generality of the theory of linkages

It is well known the device called the parallelogram of Watt: it is a device which aims at describing, in an approximate way a line segment by means of the pole of a piston. Peaucellier in 1864 found a rigorous solution by means of a simple linkage. [...] Sylvester was very interested in this discovery and engaged to disseminate and to extend it. His intervention made the linkages very popular in England, where they were extensively studied by Hart, Clifford, Roberts, Cayley and Kempe.

(Kœnigs 1897)

The studies of the quoted geometers concern mainly the mechanical description of individual algebraic curves and individual birational transformation. But later Kempe proves that every algebraic curve can be traced by means of an articulated system, and that every birational transformation can be realised by means of an articulated system (Lebesgue 1950).

Later the study of linkages stops being interesting for pure mathematicians, but becomes a fundamental part of the Theory of Machines and Mechanisms and Robotics (Bartolini Bussi & Pergola 1996).

In the laboratory there are more than a dozen elementary linkages that realise geometrical transformations together with dozens of composed linkages that combine elementary ones to give mechanical proofs of theorems on composition of transformations

Drawing Instruments: Didactical Issues.
The Small Group Study of the Pantograph of Sylvester

In the previous section we have presented some sets of instruments available in the laboratory, by means of their historical contextualisation. In this section we shall briefly describe how they are actually used in mathematics lessons. The small group study of these models, historically contextualised by the teacher, defines the activity in the mathematical laboratory.

Small group study of a model is realised by means of a task, given by the teacher: the students are not completely free to manipulate the model, but they are given a precise list of questions to be answered. For instance, the study of a pantograph of Sylvester is done by means of a list of questions to be answered in writing: in particular, we shall focus on two questions (further details in Bartolini Bussi 1993, Bartolini Bussi & Pergola 1996):

A specimen of the pantograph, see figure 7, is given to a small group of students (11th grade) together with a list of questions. We shall focus on only two of them.

QUESTIONS

1. Represent the mechanism with a schematic figure and describe it to somebody who has to build a similar one on the base of only your description.

4. Are there some geometrical properties which are related to all the configurations of the mechanism? Try to prove your statements.

The first question is a typical communication task. It is well studied in didactics of mathematics (Brousseau 1986). The mathematical task of describing the geometrical structure of the pantograph is inserted into the social task of communicating it to an interlocutor. The small group acts on the mechanism with the aim of producing a written message, including a figure: the decoder, in this case a fictitious one, is supposed to gain the information, by means of which he has to reconstruct the building action. As the students have not handled the pantograph before, they do not know anything about it: so, also the elementary features of the polygons which constitute the schematic figure (a parallelogram and two similar isosceles triangles) are to be discovered, by means of handling and even measuring, to be sure that some bars have the same length and some angles have the same width. Yet, even if the students are allowed to measure, they are given this physical object in a school setting, in a mathematics laboratory, where a typical situation of speech communication is given. This very setting inhibits the recourse to a practice-oriented message, such as the one that could have been used in some out-of-school setting. The students immediately realise that coding essential points with letters is crucial: in small group they can point at the mechanism ostensively and speak of 'this point', 'that angle', and so on, but, as a student says, if we write so, he (i. e. the fictitious interlocutor) will understand nothing. Hence, coding points by letter is given a new sense: it is no more a social rule of the contract between the teacher and students, based on the tradition of all the school books ; rather, it is a true need of communication.

The fourth question requires three different processes, described in (Bartolini Bussi 1993): (a) guessing and stating a conjecture; (b) looking for a proof; (c) writing down a proof.

The conjecture is produced with the help of the teacher. It is a difficult conjecture as it concerns objects that are not directly observable in the mechanism, namely the line segment OP and OP'

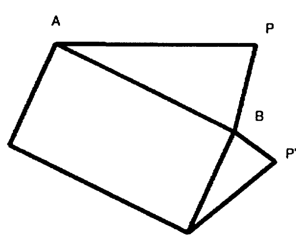

Figure 7

CONJECTURE
OP = OP'
POP' is constant

it means that, during the deformation of the mechanism, the point P' is always the image of P under a rotation around O. The rotation angle is PAB = BCP'.

The process of looking for a proof is quite interesting. It could be described as a continuous exploration of a metaphorical space that contains 'known' facts and statements to be proved. Among known facts there are not only results from geometrical theory that has been already studied by students but also empirical information from the configurations of the mechanism. The pieces of available knowledge are used without investigating whether they are empirical information or theoretical statements. So, for instance (see details and further examples in Bartolini Bussi 1993, but the same structure has been found in different experiments carried out with graduate students too), we can observe the following path in the process of proof search:

A: 'OP = OP' and the angles POP' are equal for all the configurations of the linkages' (it is the conjecture to be proved and it has been stated on the base of empirical observation);

B: 'OP = OP' because we have proved that the triangles OAP and OCP' are congruent';

C: 'As the angle POP' is constant and the triangle POP' is isosceles, all the triangles of the infinitely many configurations of the linkage are similar'.

In the step C two information of different nature are treated as if they had the same status: actually *'POP' is constant'* is based on empirical evidence, while *'POP' is isosceles'* is based on an already produced proof. Later, during small group work, the statement *'POP' is constant'* will be proved independently, i. e. connected to the available knowledge by means of theorems on the angles.

In the following process of writing down the proof, the time order of exploration has to be changed. According to the standard script of proofs, it is necessary to build a path from the geometrical properties of the model of the physical mechanism to the statement to be proved. The conjecture is the thesis and cannot be used any more as available information. An implicit or explicit separation between known facts (either axioms, or theorems or inference rules) and facts to be proved (thesis) will be introduced.

However the shift from looking for a proof to writing down a proof is not simple and some traces of the old time organisation are still evident in the final text produced by the students:

Thesis. The angle POP' is constant.
The angle POP' is constant as the triangles POP' obtained by means of the deformations of the mechanism are always similar, whatever the position of P and P'.

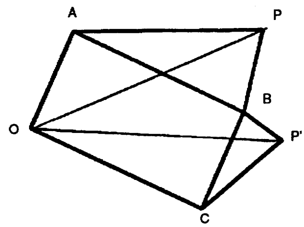

Figure 8

In fact OP=OP', because the triangles OCP' and OAP are congruent, as CP'=OA,CO=AP and OCP'=OAP (BCO=OAB and P'CB=BAP)

The above triangles are also similar to a third triangle PBP', because, as the triangles BCP' and BAP are similar, it follows:
BP' : BP = CP' : CO
and the angle P'BP = OCP' as (posing that CP'B = CBP' = α and CBA =β) we have
PBP' = 360 - ($2\alpha+\beta$)

OCP' = 360 - (2a+β).

This is true because prolonging the line BC from the side of C the angle supplementary to BCP' is equal to 2α and the angle supplementary to BCD is equal to β as two contiguous angles of a parallelogram are always supplementary

Discussion

The introduction of mathematical machines in the classroom at secondary school level seems to answer to two related problems of didactics of mathematics

1) the socio-cultural construction of consciousness;

2) the construction of a pragmatic basis for proof.

The first issue is related to the first part of this paper. The small group work is historically contextualised by the teacher by means of wide historical introductions, like the ones that have been presented, and put in this way in a broader perspective. The importance of this historical perspective extends well beyond the students' discovery that similar problems existed a long time ago. In the course of a historical study, what is in the foreground is the process of constructing meaning and the idea that this process is not individual but collective (Otte & Seeger 1994).

The second issue is related to the second part of this paper. During small group work, the students are confronted with the global process of production of 'new' theorems: they are put in the situation of exploring, making and testing conjectures, and devising their own proofs. They are working on physical objects, to be transformed into geometrical objects; they are coping with the complex epistemological relationship between deductive reasoning on the one hand and its application on the other (Hanna & Jahnke 1993). The deep connections between the exploring phase, the conjecturing phase and the proving phase that have been only hinted at in this paper are the focus of some research studies that are now in progress on the early approach to mathematical proof in the field of experience of sunshadows (Boero, Garuti & Mariotti 1996, Boero, Garuti, Lemut & Mariotti 1996) or in the context of Cabri-geometry (Laborde, personal communication). In the studies, the standard habit of asking the students to understand and repeat proofs of statements supplied by the teacher is upset. In all the cases the standard paper-and-pencil context for geometrical proofs is substituted by a dynamic context (i. e., linkages, sunshadows and Cabri-geometry), that encourages exploration and the statement of conjectures. Further research studies seem to be necessary to ascertain whether this not episodic experience could provoke effects on proof construction in more traditional paper-and-pencil contexts and outside geometry as well.

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BASIC IMAGERY AND UNDERSTANDINGS FOR MATHEMATICAL CONCEPTS

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Basic Imagery and Understandings for Mathematical Concepts

1. Despite immense progress in the field of mathematics didactics there are still a lot of mathematics educators as well as teachers who adhere to a rather narrow picture of their subject, namely consisting on the whole of abstract relations between abstract objects and some calculation. For them, intuitive, vivid, enactive or application oriented ways of doing mathematics do not belong to true mathematics, but are mere approaches. The advantage of this picture is that the contents can be identified exactly, and can easily be made accessible to presentations in textbooks, as well as to empirical research on how students handle them, or to (so-called) intelligent tutorial systems.

Yet, this picture is not a suitable foundation for teaching and learning mathematics (neither for doing or applying mathematics), as in it the category of **meaning** is ignored and hence the constitution of meaning is not a matter of education. — But in every thinking or learning process the individual assigns some meaning to some notion, situation or circumstances, and teachers, in particular mathematics teachers, have to take into account these processes of assignment.

Closely connected to the difficulties in recognizing and controlling the students' learning processes is the problem of matching concepts in the realms of mathematics, epistemology and psychology (which I will call 'mathematical', 'epistemological' and 'psychological concepts' respectively). — The conception of basic imagery and understandings (BIU) offers a didactical frame for this matching problem. In German mathematics education this conception has a long tradition. Rudolf vom Hofe (1995) investigated its history and found a lot of variants in the last 200 years, most of them tackling the matching problem by designing ideal normative mathematical concepts in the epistemological mode (vom Hofe names them "basic ideas") serving as models for the students' formation of concepts in the cognitive mode (which he names "individual images").

It seemed to be natural to all those educators to found their conceptions on an analysis of subject matter and to include their rich teaching experience as an empirical background. Thus they were much closer to their students than many mathematics professors at the universities or teachers at the Gymnasiums (in former times with the top 10 % of each age-group) who taught (and often still do teach) mathematics — maybe in an elementarized, but still — in a rarely modified manner as a pure discipline. On the other hand, those educators, too, did often not care for what really happens in the students' brains, and, furthermore, in spite of their good ideas, their efforts had only little success.

But one must admit that only since the 1970s has there been reasonable technology for thoroughly studying classroom actions, namely video recordings. Of course, even with this technology one does still not know how cognition 'really' works. Neither the mathematical formalization of thinking processes, nor the definition of man as an information processing being similar to a computer (Simon 1969) brought about much new insight in to **human** cognition. But based on the talents of video technology we learned a lot about communicative and social interaction in the classroom, in particular, how mathematical meaning is implicitly and explicitly negotiated between the participants (cf. Bishop 1985).

Due to the constructivist and connectionist roots of their theories, some cognitive scientists underestimate, ignore or deny a dominant influence of the teacher and, consequently, of the subject matter on the students' learning processes. — In fact, during painstaking examinations of videotaped and transcribed micro situations, middle and long term effects can easily get out of sight. If one concentrates on social and communicational characteristics of a situation, the subject matter tends to play only a minor role. And comparing students' deviating verbal and non-verbal manifestations with teachers' obvious original intentions may support severe doubts in the efficacy (or even possibility) of extraneously deter-mined learning processes. — These tendencies are supported by the researchers' aim to overcome the old theories because of their meager success.

On the other hand, careful re-analyses of classroom situations under **subject matter aspects** often lead to plausible recasts or improvements (as well as to verifications) of former interpretations based on interaction-theoretical grounds. So to me it sounds unreasonable to exclude these aspects when exploring such a situation. As I pointed out before, in German mathematics didactics, for a lot of mathematical concepts there are well known elaborated teaching routines. Whether a teacher relies on such a routine or not: From the words, diagrams etc. that she or he uses, from her or his rejection or acceptance of students' answers etc., the observer frequently can disclose the teacher's own imagery and understandings about the mathematical concept in question. (Throughout this talk, the notion of mathematical concept includes theorems, mathematical structures, procedures etc.) — Surprisingly often, the teacher's own imagery and understandings seem to be inadequate, or at least the teacher evokes inadequate imagery and understandings among the students. — These problems can be tackled didactically with the help of the conception of BIU, which is meant to be a theoretical and practical frame for a normative, descriptive and constructive treatment of concept formation processes.

A radical constructivist would argue that there is no adequacy or in adequacy of imagery and understandings. Here we have reached a point of discourse where there might be no agreement. For me, **adequacy** of concepts or adequacy of imagery and understandings is a useful and important didactical category. Of course, adequacy cannot be proved like a mathematical theorem. Whether a student's concept is adequate even cannot be stated uniquely, neither in the prescriptive, nor in the descriptive mode. But there are strong hints in either mode: If a student's statements about, and actions with a mathematical concept sound plausible and seem to be successful to her or his own common sense as well as to experts, we would concede some adequacy (for more details cf. E.J. Davis 1978).

From a didactical point of view, it is not crucial whether teachers actually 'teach' their students or whether they only **stimulate** their students' concept formation processes. Good 'teaching' always contains stimulating the students' own activities.

2. By the adjective 'basic' there are expressed several essential characteristics of the conception of BIU:

— It includes a tendency of **epistemological homogenity and obligation** how mathematical concepts should be understood.

— **Psychologically** speaking, it indicates that students' individual concepts normally are, and in the teaching processes the epistemological concepts should be, **anchored in** the students' **worlds of experience.**

— With respect to **subject matter** it stresses the importance of **fundamental ideas** (in the sense of Halmos' elements, 1981, or Schreiber's universal ideas, 1983) guiding the study of any mathematical discipline.

Epistemological homogeneity: This tendency seems to be in contradiction with modern pedagogical and didactical paradigms like "the students should create their own mathematics", or "the students have to find their individual ways in solving mathematical problems" etc. In fact, teaching does not mean telling (Campbell & Dawson 1995), but it means stimulating students' cognitive activities, negotiating mathematical meaning in the classroom etc.

But this way of conceiving the teaching-learning process does not entail any obligation for the teacher to tolerate or even to support inadequate individual concepts; on the contrary: it makes the teacher's task much more difficult. She or he must be provided with a good theoretical and practical competency in mathematics, mathematics applications, epistemology, pedagogy, psychology, social sciences etc., in order to

— develop her or his own view of the epistemological kernel (which must not be identified with a mathematical definition) of some mathematical concept which the students shall acquire,
— perceive the students' actual individual concepts as truly as possible and to judge their adequacy,
— help the students, if necessary, to improve or to correct their individual concepts into adequate ones near the epistemological kernel,
— possibly learn by the students and improve her or his own individual concepts.

This task imparts a predominant role in the teaching-learning process to the teacher's own imagery and understandings and to their transposition into didactical action. For example, if for the calculation of π a circle is approximated by a sequence of polygons and the teacher uses a phrase like "in this sequence the polygons have more and more vertices, and finally they turn into the circle", the students' formation of an adequate concept of limit is obstructed.

The epistemological kernel of a concept corresponds to a commonly shared socio-psychological kernel. Such a socially constituted kernel is an important prerequisite for the construction of individual argumentation and its introduction again into classroom interaction (cf. Krummheuer 1989). — It is obvious that this commonly shared kernel should be as extensive as possible, which, again, gives the teacher a central position in the teaching-learning process.

Anchoring mathematical concepts in the students' worlds of experience: Even working mathematicians need some real world frame for doing mathematics ("we consider ...", "if x runs through the real line ..." etc.; cf. Kaput, 1979, and many others). All the more do students need such frames so that they can constitute meaning with the subject matter they are about to learn (Davis & McKnight, 1980, Johnson, 1987, Fischbein, 1987, 1989, Dörfler 1996 etc.). As such frames do not belong to the epistemological concepts, the teacher is rather free when constructing real world situations where basic imagery and understandings can be unfolded.

These situations need not be absolutely realistic; on the contrary, by alienating them with the help of fairy-tale traits and concentrating on the essence they can be turned into metaphors with their explanatory power. One can take human beings, animals, things, which are more or less anthropomorphized and more or less mathematized. These participants in the situation have to act somehow, following some arbitrary rules, pursuing some arbitrary plans, obeying arbitrarily physical and other natural laws, or not.

For a lesson about the integral as area function for a given function I designed the following situation: The x-axis is a hard-surface road; to the north of this road (in the coordinate system) there is a uniformly wet swamp which is bounded by the road and the graph of the function in question. A vehicle drives on the road in the positive (eastern) direction, with an arm perpendicular to the road which is sufficiently long to reach all parts of the swamp during the trip. With the help of this arm the water is **absorbed** uniformly from the swamp (on the basis of some uninteresting technology) and collected in a cylindrical jar. Thus, at any moment the level of the water in the jar is a linear measure of that part of the area which has already been passed by the vehicle.

If the vehicle reaches a position where the function is negative, the metaphor has to be extended: To the south of the road there is a desert which is bounded by the road and the negative parts of the graph and which has to be **watered** uniformly by the vehicle. For this purpose the vehicle has a second arm perpendicular to the road which is sufficiently long to reach all parts of the desert during the trip. Again, the exact mechanism is not interesting; the only important thing is that the level of the water in the jar drops proportionally with the desert area passed.

Of course, this metaphor contains a lot of technical and didactical problems which have to be considered thoroughly: — What happens if the jar is full (empty) and there is still swamp (desert) area to be drained (watered)? — Draining the swamp and watering the desert have to be accomplished with the same velocity of flow (whatever this physical notion means). — In principle, one needs a new coordinate system for the function of the water level (the integral function). — When the vehicle makes a half turn and then drives in the negative (western) direction, the two arms change their positions, and now the desert has to be to the north and the swamp has to be to the south of the road (in accordance with the mathematical changing of positive and negative area). — But if the starting point of the vehicle is finally made a variable, the efficiency of the metaphor comes to an end.

Every metaphor has its limitations (cf. Presmeg 1994), but this is no drawback. The one which I just described should make plausible

— continuous measurement,

— the transfer from area measurement into linear measurement and

— the concept of negative area.

It thus appeals to common sense, and if the teacher wants the students to maintain their common sense, it is a must to emphasize the limitations of any metaphor.

Situations which are appropriate for mathematics teaching rarely come along by themselves. Genuine mathematics applications are often not suited for supporting concept formation, as they are frequently overloaded with alien problems. At the same time the teacher should not evoke the impression that some artificial situation, designed for the use in mathematics teaching, would be an example for genuine mathematics applications. Sometimes, this coincidence can happen, but usually it does not; and students with common sense realize the artificial character of such a situation.

Fundamental ideas for mathematical disciplines (in an epistemological and psychological sense): Basic imagery and understandings are not only meant as a peg on which to hang some mathematical content, but they shall lay the foundations for further meaningful interpretations of concepts within a mathematical discipline.

3. The notions of imagery and understanding stand for two fundamental psychological constructs. There exists an extensive literature about them. Different authors have different definitions, most of them not very concise. A lot of contemporary cognitive scientists disregard these two constructs anyway, as they escape hard empirical research and do not fit a computer related view of intelligence. — But it is just these — seen behaviouristically — shortcomings, their vagueness and flexibility, which turn these constructs into suitable means for analyzing (and promoting) such complex didactical objects as human teaching-learning processes.

Imagery can be grasped as: mental, often visual (but also auditory, olfactory, tactile, gustatory and kinesthetic; cf. Sheehan 1972) representations of some object, situation, action etc. having their sensory foundations in the long term memory and being activated in **conscious** processes. A person activating some imagery has already some meaning, some intentions in mind and organizes these processes according to these intentions (Bosshardt 1981). — Imagery is closely related to intuitions, but its objects are more concrete, and meaning plays a more important role.

The objects of imagery (and understandings) can be given in different modes, namely analogous or propositional. I don't want to resume the cognitive scientists' quarrel in the 1970s about the interrelations between these two modes or about their separate existence as ways of thinking. In my opinion both are valuable means for analyzing imagery and understandings in teaching-learning processes.

Apparently, imagery is more closely connected to the analogous mode, and understandings are more closely connected to the propositional mode of thinking. But it is difficult for a person to activate some imagery without propositional elements, in particular in didactical situations, as in these situations verbalization is **the** fundamental means for a participant to communicate either with others or with her- or himself (this communication with oneself being a transposition of a social situation to one's mind which is typical for teaching-learning processes). On the other hand, there can be no process of understanding without recurring to any plausible imagery and to analogous elements.

Obviously, thinking in the analogous mode can be stimulated by analogous means like pictures, diagrams etc. (with a lot of limitations; cf. Presmeg 1994), and the propositional mode can rather be stimulated by propositional means like verbal communication. In the age of paper and pencil and of books, analogously given objects frequently are of a visual, static nature, and the learners have to undertake some effort to make these 'objects' plausible, meaningful, vivid imagery matching their worlds of experience. In the nearest future the use of multi-media in schools (in the western world) possibly will relieve the students from these efforts.

Whether multi-media will be conducive to the students' learning processes, is not yet settled: The students' inclinations and abilities to undertake efforts to generate mathematical concepts could be undermined. — This problem is complementary to the following classical one, related to the use of visualizations (diagrams, icons etc.): Among educators there is a naive belief that visualizations do facilitate the students' learning processes. But as, for example, Schipper (1982) showed with primary graders, many visualizations are not self-explanatory at all, but they are subject matter which has to be acquired for its own sake, on the one hand, and in relation to the visualized contents, on the other hand. — As a matter of course, visualizations can be successful didactical means, but not because they would reduce necessary effort, but because they demand more effort and give hints how to direct and structure this surplus effort and thus make it effective.

There are didactical situations, as well as mathematical concepts, as well as students, for which respectivily one of the modes is more suitable. For teaching and learning mathematics it is important that there has to be a permanent transformation between the two modes. Maybe geometry can be treated predominantly in the analogous mode, and algebra in the propositional mode; maybe the teacher is even able to take into consideration the preferences of single students. But in principle, both modes must be present.

Taking into account the widespread propositional appearance of mathematics teaching, in particular on the secondary level, there is need of an increased use of the analogous mode all over the world. — By stressing the students' anchoring of their individual concepts in their worlds of experience, the conception of BIU lays some accent on the analogous mode, as a prophylactic counterweight to the preponderance of the propositional mode in the upper mathematical curriculum.

The psychological construct of understanding is still more complicated and non-uniform. For didactical reasons the following aspects are relevant:

(1) One can understand people, their actions, situations, the motives or the aims of the participants (practical knowledge of human nature, **common sense**).

(2) One can understand utterances **medially** and **formally** (e.g., if they are made loud enough and in a language one knows).

(3) One can understand the **content** of a **message** made by someone (understand what this someone **means** by a certain communication, text, phrase, word, symbol, drawing etc.).

(4) One can understand technical matters, working principles of gadgets, mathematical structures, procedures etc. **(expertise).**

At first glance, aspect (4) seems to be most suitable for the conception of BIU. But it becomes immediately clear that each of these aspects is important for the learning of mathematics and has to play an essential role in the conception, in particular (3). This aspect is a classical psychological paradigm, but the general opinion about it has changed: Today, one does not believe anymore that it consists just of finding some objective meaning of given signs, but that the receiver of a message tends to and has to embed the message in some context and, in doing so, tries to reconstruct its meaning (cf. Engelkamp 1984), thus getting near aspect (1).

It goes without saying that there is no understanding (3) without (2): the sender and the receiver of a message have to have a common language, not only in a direct, but also in a figurative sense: As Clark & Carlson (1981) put it, there has to be a "common ground", which, again, refers to aspect (1). — In school teaching, and in particular in mathematics teaching, the common ground of teachers and students is often rather thin, if existing at all. — But, extending the common ground does not only mean that the students have to be better instructed so that they make the teachers ground their own. Rather the teacher must engage in the students, attach importance to them (and not only to the subject matter), understand them as human beings (again, aspect (1)), and try to reconstruct or to anticipate their ways of thinking.

By following the conception of BIU, to some extent the teacher is forced to do so, and furthermore, her or his expertise can be promoted. But this way of teaching and learning demands much more effort for both parts, in comparison with the ususal way, where teachers, in good harmony with the students, are satisfied with students' instrumental understanding (in the sense of the late Richard Skemp 1976).

In the following example, the teacher (resp. the researcher) did not quite understand the student's ways of thinking. It was originally described by Malle (1988) and re-analyzed by vom Hofe (1998): In order to develop the concept of negative numbers, Ingo, the student, was given the following situation: "In the evening the temperature is 5 degrees (Celsius) below zero. During the night a warm wind moves inland, and the temperature rises **by** 12 degrees. — What is the temperature next morning?" Ingo answers correctly: "7 degrees", but in the dialogue with the interviewer, he shows inadequate imagery. When he sketches the situation, he asks whether he must draw three thermometers, and later he explains that at midnight the temperature went up to +12 degrees, and in the morning it dropped to +7 degrees.

Malle gives well known and, of course, correct explanations for Ingo's obvious inadequate dealing with the situation: Ingo is not able to identify the elements which are important for solving the problem, but invents additional information and tells fairy tales, and he does not distinguish between the starting and the final state (i.e. the starting and final temperature, represented on the thermometer), on the one hand, and the change between the states (the rise of the temperature), on the other hand.

In his careful re-analysis, vom Hofe shows that the problem lies in Ingo's imagery about the physical situation, which is no suitable basis for the formation of the mathematical concept. Whereas the interviewer expects Ingo to focus on the changes of the mercury column (as a direct model of the number line), Ingo imagines two masses of air, a cold and a warm one, which mix and result in a third mass with average temperature. Therefore he needs **three** thermometers, and in the night the temperature does not rise **by** 12 degrees, but **up to** 12 degrees, and goes down again in the morning. The idea of mixing air masses is, physically speaking, not at all inept, but it merely does not fit the mathematics that the interviewer has in his mind. For Ingo, there are two states of temperature which result in a third one, the weighed arithmetical mean, and not one state which changes into another.

Granted that every human being tends continually to conceive, or to make and to keep her or his environment meaningful and sensible, one must admit that usual mathematics teaching in large parts has a contra-productive effect. — The striving for "constancy of meaning" (Hörmann, 1976) is in my opinion a characteristic trait of humans which, for example, is largely ignored in Piaget's biologistic theory of equilibration.

Mathematics teaching, too, is such an environment which humans who are in touch with it try to make meaningful and sensible. As an **extreme** example, (in a famous French movie from 1984) in a physics lesson in Paris the absent-minded student from Algeria understands "le thé au harem d'Archimède", when he hears "le théorème d'Archimède" (which means: "tea time in Archimedes's harem" instead of "the theorem of Archimedes"). Even if we omit such extreme cases, it still seems to be rather normal all over the world that students tend to develop their own non-conforming imagery and understandings, which, however, often remain implicit.

Fischbein (1989) calls them "tacit models" and characterizes them as simple, concrete, practical, behavioural, robust, autonomous and narrowing. Their robustness results from their simplicity, their anchoring in the students' worlds of experience, and their short term success with convenient applications (see the example of Ingo and the temperature). Inadequate tacit models come into being because of lack of adequate basic imagery and understandings, which in their turn would also be concrete, practical, etc., successful and therefore robust, and not narrowing, but capable of expansion.

So the conception of BIU includes the strategy of occupying the students' frames with adequate basic imagery and understandings from the beginning, i.e. to give them the possibility and to enable them to develop such imagery and understandings by themselves.

Nevertheless, students will still generate a lot of inadequate tacit models, and teachers must be able to recognize them and to help the students to settle them. In this, again, the teachers can be supported by the theoretical and practical frame of the conception of BIU, thus using the constructive aspect of the conception (as vom Hofe, 1995, puts it).

Fischbein (1989), like many other educators and cognitive scientists, recommends that the students should undertake meta-cognitive analyses in order to discover and eliminate the defects in their frames. — I couldn't find evidence in the literature that students would be able to successfully analyze their own (wrong) thinking without massive interventions by the teacher or by some interviewer. According to my own experience with young people in all grades, they are overstrained if they reflect reflexively about their own reflections.

Indeed, in many classroom situations there can be found actions of understanding on a meta-level; for example, if students recall how they solved a certain problem, or if they try to find out the teacher's intentions, instead of trying to understand the contents of her or his statements. But, in general, this kind of understanding (aspect (1)) is not explicitly reflected by the students.

One essential trait of every didactical situation is (or should be) that the participants strive for understanding the contents of some message given, verbally or non-verbally, by the teacher, students, the textbook etc. (aspect (3)), with the underlying aim that the students shall acquire expertise (aspect (4)). Whereas aspect (3) stresses the **processes** of under-standing, aspect (4) stands for the **products** of these processes. The products are not only results, but at the same time they are starting points for new processes, and each understanding process starts on the ground of some already existing understanding.

In mathematics teaching, both aspects of understanding ((3) and (4)) deal with the same objects: the messages, seen ideally, deal with mathematical concepts, about which the students shall acquire expertise. — In the humanities and in the social sciences, as well as (in an indirect way) in mathematics, this expertise again often refers to social situations (in a wide sense) and thus is in parts identical with aspect (1). — So, finally, in normal teaching-learning processes all the aspects of understanding discussed here belong together and are essential for success.

In my view, there is no understanding without imagery, and no imagery without understanding. With the notion of imagery there are stressed the analogous mode, roots in everyday lives, intuitions etc., — whereas with the notion of understanding there is laid some accent on the propositional mode, on subject matter, on predicates etc., — but not only do both notions appear together, they have a large domain of essence in common.

4. There can be identified roughly four types of BIU for use in mathematics teaching in the primary and secondary grades:

A. More or less **global** BIU, especially for the formation of the concept of number and for elementary arithmetic: multiplication as repeated addition; division as partitioning (splitting up; 'Aufteilen') or distributing (sharing out; 'Verteilen'); fractions as quantities or as operators, negative numbers as states or as operators, the machine model for operators, the little-people metaphor for running through an algorithm. Basic imagery and understandings are not bound to primitive, non-quantifiable actions (in the sense of intuitive understanding according to Herscovics & Bergeron, 1983), and their formation is not a kind of mathematical propedeutics or pre-mathematics, but — in my opinion — genuine mathematics (just without calculus with symbols). They would be useful in the upper secondary grades as well, for example, with the concept of limit and infinitesimal thinking as a whole.

B. More or less **local** BIU, e.g. the arithmetical mean, the internal rate of return of an investment, the circumcircle of a regular polygon.

C. BIU for **extra-mathematical** concepts, situations, procedures (from physics, economics, everyday lives etc.), which are to be used in mathematics teaching (example: Ingo and the temperature).

D. BIU for **conventions,** e.g. the meaning of symbols, or of diagrams. Example: The teacher tries to explain subtraction with the help of the following situation: "Mother baked six cakes for her daughter's birthday; the dog Schnucki ate four of them. How many are there left?" She draws 6 circles on the blackboard and crosses out four of them (each with one line), hoping to support visually the understanding of the problem 6–4=2: But Ralph, a learning disabled child, wonders why the teacher halves the marbles (Mann 1991).

It goes without saying that the prototypes, metaphors, metonymies (Presmeg 1994) used for BIU should not obscure the concepts they refer to, as in the following example:

Euclidean geometry: As a preparation for proving the existence and uniqueness of the incircle of a triangle, the teacher asks the students: "Imagine a cone with several balls of ice cream intersecting each other physically, and a plane section through the cone containing its axis of rotation. Do so three times, by identifying the tip of the cone with the three vertices of the triangle one after another." — A more suitable imagery would be to stick a small circle near one vertex between its adjacent edges. When the circle is blown up like a two-dimensional balloon, it moves away from that vertex, still touching the two edges, until it meets the third edge and thus reaches its final position as incircle. In dissociation from the pure Euclidean way of doing geometry, this metaphor makes use of kinematic and continuous physical phenomena from the students' worlds of experience.

Of course, mathematical concepts should not be falsified, as is the case with the concept of circle in Papert's (1980) original idea of **turtle geometry.** The children shall draw a circle by programming the turtle to do a straight motion of length 1, then to do a right turn of the amount 1, and to repeat these two actions 360 times, i.e. by drawing a fuzzy regular polygon with 360 vertices. In fact, the result looks like a circle, but the way in which it was produced belongs to a concept which is essentially different from the Euclidean circle. It's true that every line on the computer screen is a sequence of squares; but this is not the point, as students with some experience with paper and pencil as well as with computer screens will recognize the shortcomings of any realization of geometric forms and will be able to idealize these forms, if at least the underlying activities are appropriate. — But the procedure for making a Logo circle is not appropriate for Euclidean geometry. — Furthermore, I doubt that the Logo geometry is a good preparation for differential geometry, — eventually it is a helpful model for someone who already has the concept of mathematical limit at her or his disposal, whereas it is likely to be a mental obstacle for someone who is still on her or his way of acquiring this concept, let alone for primary graders.

Transformation geometry: When in the late 1960s and early 1970s transformation geometry was pushed into the mathematics curriculum, it was assumed that real motions of real objects could serve as BIU. In fact, the students accepted these BIU willingly and transferred them easily into continuous motions of point sets in the plane. But the crucial point was the abstraction of the motions, which the students in general did not manage to perform. Their BIU of transformations grounded on motions were so robust that the good advice to focus their attention to the starting and final positions of the geometric forms or to the plane as a whole remained useless, because, for example, the notions of starting and final position, again, evoked imagery about motions (cf. Bender 1982).

Thus, mathematicians and mathematics educators failed to establish in the curriculum the full algebraization of geometry by transformation groups, and up to today geometric transformations are not treated as objects on their own, but only as means to investigate geometric forms. The idea of embodying Piaget's groupings of thinking schemes in geometric transformation groups proved to be too naive.

By the way, there are reasonable didactical applications of continuous motions, e.g. in good old **congruence geometry** by Euclid and Hilbert: Two geometric forms are conguent if they can be moved to each other in a way that both exactly cover one another. In German there is a synonym for the word 'kongruent' which is due to this reciprocal covering (='decken'), namely 'deckungsgleich' ('gleich'= 'equal'). In congruence geometry, different from transformation geometry, the specific form of these motions is not essential at all. So the students need not, cannot and, in fact, do not memorize them, and motions are not likely to turn into mental obstacles against viewing congruence as an interrelation between two stationary geometrical forms.

For **functional reasoning** in geometry and other mathematical disciplines, like calculus, there is needed a different, and slightly more abstract, concept of motion: What happens in the range of a function, if one 'walks' around in its domain? Example: The area function assigns to each triangle of the Euclidean plane its area. Starting with one triangle, one changes one of its vertices, and one observes, how the area changes. — The metaphorical character of this situation is obvious: There is a space (the domain, i.e. the set of all triangles) and someone or something (the variables) who 'walks' around; and the motions of this someone or some thing are transferred by some mechanism, like an abstract pantograph, into another space (the range, i.e. the positive real numbers).

One more example, where 'dynamic' imagery and understandings seem to be not helpful for basic concept formation, is the **concept of sequence and limit:** Many students have the wrong idea that a mathematical sequence would possess a last element or that one could at least reach such an element (whatever the notion of reaching should mean). — The ground for this misconception is often laid in mathematics teaching itself, e.g. when determining the number π by an approximation:

The students consider a sequence of polygons which have more and more vertices until they finally turn into a circle. Even if the teacher carefully avoids such wrong diction, the students still can easily get the impression that the circle would be the last element of that sequence: Firstly, because of optical reasons, and secondly, because the aim of the lesson is to determine a limit by a sequence of elements, and all the activities evoke this impression, whether or not the teacher expresses verbally that the limit cannot be reached. Even if the students accept that it is impossible to reach it, they tend to ground the impossibility on limited time and limited arithmetic of human or electronic calculators.

Another example, which is dealt with in the curriculum even earlier, is the decimal fractions of **rational numbers**. The students prove, e.g., that $^1/_3 = 0{,}333...$, The teacher states that the equality holds if there are infinitely many digits '3', and finally, in Germany, there is written $^1/_3 = 0{,}\widehat{3}$ as abbreviation. By this notion the double nature of the concept of limit is expressed. The symbol 'lim...' stands for a request to run through a process and, at the same time, for the result of this process.

For an **algebraic** term, like a+b , this double nature (to be a request for some activity and to be the result of this activity) is well known and useful, but it fails when the activity includes some infinite process. So the students are right when they refuse to accept the correctness of the equality $0{,}\widehat{3} = {}^1/_3$ and all the more $0{,}\widehat{9} = 1$. They take the dynamic part of the double nature of limit seriously (because this part, grounded on the didactical principle of supporting 'dynamical' thinking, is always stressed), and they correctly deny that running through that infinite process, for which an expression like $0,\widehat{9}$ stands, will result in the limit. Fischbein (1989) observed that students even deny the symmetry of the equality sign, as they accept $^1/_3 = 0,\widehat{3}$, because this expression can be read from left to right, and the digits on the right can be written down one after the other, whereas they refuse $0,\widehat{3} = {}^1/_3$, because one can never have on the left side all the needed ingredients to produce the result, one never comes to an end and one is not able to say " = $^1/_3$ ".

5. In all times, all over the world, mathematics educators reflected and still do reflect on basic imagery and understandings for mathematical concepts, though they usually do not name them like that and possibly have different or no conceptual frames. There is still missing a theory unifying the relevant disciplines 'mathematics', 'epistemology' and 'psychology'. The work of vom Hofe and my work is one attempt. But the realization in didactical and teaching practice is at least as important as the theory. Which basic imagery and understandings do we think to be adequate? How can we support the students generating adequate basic imagery and understandings? Which inadequate basic imagery and understandings can occur? How are they caused? How can they be improved or corrected? — In my opinion, these are fundamental questions of mathematics education.

6. In the discussion after the talk I was asked, what the conception of BIU had to do with learning. — This question was a harsh criticism, as in my opinion I hadn't talked about anything but what the conception of BIU has to do with learning.

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TRANSFORMING MATHEMATICS INSTRUCTION IN EVERY ELEMENTARY CLASSROOM: USING RESEARCH AS A BASIS FOR EFFECTIVE SCHOOL PRACTICE

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Project IMPACT (Increasing the Mathematical Power of All Children and Teachers) is a National Science Foundation-funded collaborative venture that joins university researchers at the University of Maryland at College Park with school district coordinators and teachers in Montgomery County Public Schools, Maryland, USA. The purpose of the project was to design, implement, and evaluate a teacher enhancement model that sought to enhance student understanding and to support teacher change in predominantly minority urban schools in the United States. The emphasis was on teaching mathematics for understanding, focusing on problem solving and concept development. The model intended for instruction was consistent with a constructivist perspective on mathematical learning, emphasizing interaction and collaboration. The project accessed intact mathematics faculties within schools. Evaluation addressed both student achievement and teacher change. This research was conducted in demographically diverse elementary schools, enrolling children of ages 5 through 10 or 11, in low socio-economic communities in Maryland, just outside Washington, D.C. The area served by the schools was urban, but not what is termed "inner-city" in the United States. This longitudinal project began in December, 1989, and concludes in February, 1997.

Two Perspectives to be Merged

Two perspectives influenced this project. First, the school district wanted to address student mathematics achievement in racially diverse, low socio-economic schools. The district was attempting to implement a new kindergarten through eighth-grade mathematics curriculum based on the NCTM *Curriculum and Evaluation Standards for School Mathematics.* They recognized the need for a more focused and supportive teacher enhancement model, because prior professional development efforts had led to only limited student achievement. Further, student achievement data indicated persistent and continuing differential performance by student racial ethnicity. As a university researcher, I was interested in investigating the use of research as the basis for transforming school practice in elementary mathematics. A study was designed that would implement and evaluate a teacher enhancement model based on constructivist principles and existing research on the teaching and learning of elementary mathematics. This project was to involve *all* elementary mathematics teachers in the participating schools and to be evaluated in terms of student achievement, not just in terms of teacher beliefs.

This project presumed a constructivist theory of knowledge, and this provided the theoretical basis for Project IMPACT. In particular, two basic principles were: (a) learners actively construct knowledge through interaction with their surroundings and experiences, and (b) learners interpret these occurrences based on prior knowledge and their rendering of their observations and actions (Noddings, 1990).

Assumptions for Project IMPACT

In 1989-90, when this Project was initiated, there was "no 'constructivist teaching model' out there waiting to be implemented" (Pirie & Kieren, 1992, p. 506). But there were researchers who had hypothesized characteristics of instructional approaches that would be compatible with a constructivist perspective. Further, research studies on addition, subtraction and place value offered evidence that when teachers had knowledge of children's thinking and focused on fostering understanding through problem solving, increased student achievement and teacher change resulted (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb, Wood, & Yackel, 1991; Cobb, Yackel, Wood, Wheatley, & Merkel, 1988).

There were two important assumptions in Project IMPACT. One assumption addressed application of constructivist theory to all children. The other assumption addressed school-based reform.

First, it was assumed that all children can understand and construct mathematical meaning. This meant that the Project had to implement a policy of expecting and fostering the mathematical understanding of each child, not a policy of remediation. Therefore, the Project had to address teachers' pedagogical and mathematical content knowledge, encouraging instructional change and decision making to support children's construction of knowledge. Further, the intention of the Project was to move beyond equal opportunity for students to educational justice in terms of treatment and outcomes.

Second, this Project assumed that the critical unit for change in mathematics instruction is the school. Therefore, the study's design did not rely on teachers to volunteer for enhancement nor did it scatter the enhancement across partial faculties in many schools. The Project accessed intact mathematics faculties in the participating schools. This was done because it was believed that programmatic and instructional reform could not depend on isolated, heroic teachers who were left to work independently. Generally, when given sufficient notice, teachers viewed the expected summer in-service program with either acceptance or eagerness.

School Selection

The 21 predominantly minority elementary schools in the district were ranked on the following criteria: percentage of minority students enrolled; percentage of families receiving free or reduced-fee breakfast, lunch, or both; percentage of low scores on the Grade 3 statewide assessment; and percentage of students categorized as below grade level on the school system's mathematics curriculum assessment upon leaving Grade 3. Third-grade achievement was used as a measure of academic tradition because the project was initially funded only for kindergarten through third-grade implementation.

The principals of the six highest ranking schools in these categories were invited to join the Project. Of the six schools initially identified, two schools decided against participation and the next two schools on the listing agreed to participate. These six cooperating schools were matched in demographic pairs and identified as either a treatment site, participating in the teacher enhancement, or a comparable site, receiving no enhancement from the Project. Schools had to commit to Project IMPACT prior to identification of treatment or comparable-site status. The assignment of project status was determined by three coin tosses, conducted in the presence of the school principals or their designees. Designation of treatment or comparable-site status was the choice of the winning schools. Both treatment and comparable-site schools had to permit classroom observations in the fall and the spring and student assessment in the winter and spring. Following the coin toss, two of the principals requested treatment status; one principal requested that his school be a comparable site. These schools participated in a longitudinal study from June, 1990, through June, 1993. There were two years of implementation support at each grade beginning with the kindergarten and first grade in 1990, adding the next grade in each successive year.

Three of the six cooperating schools were primary schools, enrolling children through Grade 3. The other three schools were elementary schools, enrolling children through Grade 5. In 1993, the school district funded the Project's summer in-service program for the primary grade teachers in the comparable-site schools. At the same time, the teacher enhancement model was implemented in Grades 4 and 5 across all Project schools. At that time, two more schools became involved in the Project. These were the fourth- through sixth-grade schools that enrolled the children who had come from one of the three original primary schools participating in the Project. There were no comparable-site schools for the fourth and fifth grade. The Project was implemented at the fourth and fifth grade from June, 1993, through June, 1995.

The Project schools enrolled a diverse student population. The 9% Asian population were primarily children from Cambodia and Vietnam with other children from the nations of Southwest Asia and the Middle East. Another 26% of the children were coded White; these were children from the United States, with some children from Eastern Europe. This category included children who were not coded as Black, Asian or Hispanic by a caregiver. The Hispanic children constituted 30% of the student population. They were primarily from Central America, with some children from the northern countries of South America. The remaining 35% of the children were coded Black. These included African-American children from the United States and children from Africa, Haiti, and islands in the Caribbean. In each of the five implementation years, approximately 500 children formed a grade-level cohort across the schools. Of these, 60% of the children received free or reduced-fee breakfast, lunch, or both meals at schools, and 25% of the children were enrolled in English classes for speakers of other languages.

The Teacher Enhancement Model

The Project IMPACT teacher enhancement model involved (a) a summer in-service program for all teachers of mathematics, (b) an on-site mathematics specialist in each school, (c) manipulative materials for each classroom, and (d) teacher planning and instructional problem solving during a common grade-level planning time each week. This report only addresses the summer in-service program.

The summer program was grade related, involving all of the kindergarten and first grade teachers in the participating schools during the first summer and teachers of subsequent grade levels participating by grade in the following summers. Each summer in-service addressed:

(a) adult-level mathematics content; (b) teaching mathematics for understanding, including questioning, use of manipulative materials, and integration of mathematical topics; (c) research on children's learning of those mathematics topics that were deemed critical to the grade-level focus as well as research addressing a constructivist theory of learning; and (d) teaching mathe-matics in culturally diverse classrooms. The summer in-service program accessed a summer school program for children, providing teachers with an opportunity to begin instructional change with a small group of children without all the demands associated with academic year instruction. The in-service program also included time to plan for the coming academic year.

Consider one approach that was used to address research on children's learning. Videotapes of teaching and videotapes of children solving mathematics problems were used. These two kinds of tapes served different purposes. The teaching tapes were to illustrate and provoke discussion about how "constructivist teaching" might look. In particular, the sessions addressed the teacher's role as a facilitator of learning, the crucial nature of teacher questioning, and the meaning of student responses. The children's problem-solving tapes provided a context for discussing re-search. After watching one of these tapes, teachers struggled to characterize how the understanding of a particular child might be interpreted. Further, the teachers would discuss what approaches they might use or what issues they might want to address if that child was in their class. The reason for using videotapes was to focus the teachers' attention on children's learning, whether thinking about examining mathematical meaning, considering instructional approaches, or defining curricular emphases. Thus, in this Project, the intention was to make a research perspective a critical basis for making decisions.

Evaluation and Results

Project IMPACT sought to evaluate the effectiveness of its efforts by examining student achievement and by collecting data regarding teacher change. Surveys of teacher beliefs and confidence were administered in order to characterize the rationale that might be guiding teachers' actions. Classroom observations provided information regarding the teachers' actual conduct. Student achievement was evaluated by assessments administered twice each year, using both student interviews and written tests.

Teacher Change

Two surveys were constructed to examine the teachers' rationale and perspectives. One survey characterized teachers' beliefs about mathematics, mathematical pedagogy, and equity; the second survey estimated teachers' confidence for implementing instructional reform.

The primary teachers assumed a more constructivist perspective regarding how children learn mathematics and how mathematics should be taught than did the upper grade teachers. Further, the primary teachers' perspectives of equity in mathematics education favored supporting every child's access to challenging mathematics whereas the upper grade teachers were more likely to base instructional decisions on an evaluation of children's existing skills and assessed needs. There was no difference between upper and lower grade teachers' change in beliefs regarding (a) the relationship between skill and understanding or problem-solving instruction, (b) the way mathematics instruction should be sequenced, or (c) the nature of mathematics. Within each of these, the primary and upper grade teachers made comparable shifts toward a more reform-based perspective.

Although the confidence level of each group of teachers changed in a direction indicating more willingness to interact with mathematics and to implement instructional approaches supporting reform, this change was more pronounced among the primary teachers. Both sets of teachers made a strong shift evidencing more confidence for interacting with mathematics over the course of the academic year. As the reality of classroom practice became evident, primary teachers became slightly more confident about their ability to implement a reform perspective whereas the upper grade teachers became slightly less confident, but this difference was not statistically significant.

Classroom instruction was observed in order to characterize the growth and changes evidenced within the teachers. If a teacher was still in the classroom of an IMPACT school after two years of implementation, the status of instruction at the end of those two years is characterized. If a teacher left an IMPACT school prior to two years of implementation, the evaluation of her instructional change was fixed at the level evidenced when she was last observed at an IMPACT site. Therefore, the characterizations that follow present a distribution skewed towards less change for the upper grade teachers as none of the fifth grade teachers had had two years of supported implementation. Finally, each of the following frequency estimates are truly only educated predictions, subject to further analysis and interpretation of the classroom observation data.

About 10% of the primary teachers (5 out of 52) and 17% of the upper grade teachers (7 out of 41) made no real change in their instruction. Another 17% of the primary teachers (9 out of 52) and 24% of the upper grade teachers (10 out of 41) moved considerably beyond routinized practice and direct instruction. These teachers used manipulatives and small-group activities. They asked how problems were solved and accepted different strategies, but they did not pursue the meaning of student explanations. They generally did not use know how to use an incorrect answer as an instructional probe, typically ignoring incorrect answers or telling the errant child the correct response. About 19% of the primary teachers (10 out of 52) and 37% of the upper-grade teachers (15 out of 41) evidenced instructional changes consistent with a constructivist perspective. These teachers sought the input of their colleagues and asked questions when manipulative materials were in use to keep a mathematical focus. In these classrooms, it was not uncommon to hear a teacher probing the reasoning behind a child's response, but these teachers generally did not relate strategies as a way to highlight mathematical meaning. In about 54% of the primary classrooms (28 out of 52) and 22% of the upper grade classrooms (9 out of 41), instruction was supportive of children's construction of knowledge and attentive to mathematics. These teachers made links between mathematical topics and frequently asked questions to focus the children's attention towards mathematical generalizations or abstractions. These teachers often used questioning to address the similarities or differences between offered strategies. When students responded incorrectly, these teachers tried to ask questions that might cause the children to reexamine their procedure or reasoning.

Student Achievement

The data for grades 4 and 5 are still being analyzed, so this report is limited to the kindergarten through third-grade data. The students were assessed at the middle of the school year and at the end of the school year each year, using both a scripted problem-solving interview and, in Grades 2 through 5, a written test. The assessments were administered in one of six languages. The assessment addressed numerical and computational skill, whole number and place-value concepts, problem solving and reasoning, geometric properties and relationships, and rational number.

Student data indicated that there was no significant difference in total mean achievement through first grade. However, there was a significant difference in mean achievement favoring the children in the treatment classrooms in second and third grade. Other significant effects were attributable to race and English language proficiency. To better understand the source of the effects, the items in the assessment were subsequently categorized by mathematical topic and were re-analyzed.

This analysis revealed no significant difference in numerical and computational skill over the four-year data set. A significant difference in geometry achievement was noted in the first grade favoring the treatment group; this persisted in each successive grade. Similarly, there was no difference in problem solving and reasoning through the middle of first grade, however, a significant difference became evident favoring the treatment group at the end of first grade. This difference was due to increased performance on graphing and word problem items, and it persisted through third grade. The significant difference in achievement on whole number and place-value concepts became evident at the middle of second grade and continued in the next year's data. Finally, the rational number achievement data showed a significant difference favoring the treatment children at the middle of second grade and again throughout third grade.

Conclusion

Project IMPACT's influence on student achievement was not immediate. It seemed to become evident first in the least abstract mathematics represented in the curriculum and then increase in breadth over time. The beneficial effect of focusing on conceptual understanding and problem solving became more evident and persistent as the level of the mathematical abstraction became more pronounced.

Instructional change is not easy. It is demanding, threatening, and risky. The evidence of Project IMPACT is that change in urban centers can happen when individual teachers are not left alone to accomplish it in isolation. It is easier to attempt change in an atmosphere of support. It is easier to succeed when people work together for a common goal relevant to their needs.

Finally, IMPACT addressed both the mathematical and pedagogical knowledge of teachers in order to support decision making. Both components are critical. Research on children's learning provides an important lens for teacher enhancement, but so does mathematical content.

Project IMPACT demonstrates the potential of applying research to instructional decision making across whole schools. It has done so in the reality of public schools that do not have a tradition of strong mathematics achievement. If the potential and problems associated with reform are to be understood, if the implications of reform are to be recognized, and if the vision of mathematical power for all is to be realized, then reform cannot just be carried out in idealized settings.

We, as mathematics educators, must accept the challenge to expand and to maintain our commitment, not to just selected classrooms, but to educational change across schools.

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ACCOUNTING FOR MATHEMATICAL LEARNING IN THE SOCIAL CONTEXT OF THE CLASSROOM

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This paper focuses on the issue of accounting for students' mathematical learning as it occurs in the social context of the classroom. In the opening section of the paper, I first clarify why this is a significant issue for myself and my colleagues and develop criteria for classroom analyses that are relevant to our purposes. In the second part of the paper, I outline the interpretive framework that we currently use by presenting a sample analysis. In the final section, I reflect on this analysis to address four more general issues. These concern the contributions of the type illustrated by the sample analysis, the relationship between instructional design and classroom-based research, the role of symbols and other tools in mathematical learning, and the relation between individual students' mathematical activity and communal classroom processes.

Social Context and Developmental Research

In recent years, there has been a shift away from theoretical perspectives that focus on individual, isolated learners and towards those that bring to the fore the socially- and culturally-situated nature of mathematical activity (e.g., Bishop, 1988; Nickson, 1992; Nunes, 1992). Analyses conducted from this latter viewpoint continue to be vitally concerned with the process of mathematical development. However, in contrast to purely psychological perspectives, individual students' mathematical interpretations, solutions, explanations, and justifications are seen not only as individual acts, but simultaneously as acts of participating in collective or communal classroom processes. Viewed in this way, mathematical learning is seen to be necessarily situated in social context. It should be acknowledged that this paradigm encompasses a range of theoretical positions that include various versions of constructivism, sociocultural theory, and sociolinguistic theory. Comparing and

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contrasting these alternatives is beyond the scope of this paper, and I will instead focus directly on the version of social constructivism to which I and my colleagues subscribe.

From the social constructivist perspective, the challenge of accounting for learning in social context involves analyzing both 1) the evolution of the communal practices in which students participate, and 2) the development of individual students' mathematical understandings as they participate in and contribute to the evolution of these classroom practices. Consequently, from this point of view, a first criterion when accounting for learning in social context is that such analyses should *focus on the mathematical development of both individual students and of the classroom communities in which they participate.*

Stated in this way, the rationale for this first criterion is primarily theoretical and reflects a particular view of the relation between individual activity and communal practices. In considering other criteria, I attempt to ground the issue of accounting for mathematical learning in social context in my own and my colleague's classroom-based activity of collaborating with teachers to design learning environments for students. In this work, we draw on the theory of Realistic Mathematics Education developed at the Freudenthal Institute when developing sequences of instructional activities for students. In following this approach, we initially conduct an anticipatory thought experiment in which we envision how students' mathematical learning might proceed as an instructional sequence is enacted in the classroom (Gravemeijer, 1994). These thought experiments involve conjectures about both 1) students' possible learning trajectories, and 2) the means of supporting, organizing, and guiding that development. These conjectures are then continually tested and modified as we engage in classroom-based research and attempt to make sense of what is actually happening as the instructional activities are realized in interaction between a teacher and his or her students in the classroom. It is here that the issue of accounting for students' mathematical learning in social context gains pragmatic force in that the ways in which we look at communal classroom practices and at individual students' activity profoundly influences the instructional decisions that we make when we experiment in classrooms. Given our agenda as mathematics educators who conduct classroom-based developmental research, a second criterion is therefore that analyses of mathematical learning in social context should *feed back to inform the ongoing process of instructional development.*

In addition to conducting ongoing analyses of classroom events on a daily basis, we also video-record all classroom lessons so that we can conduct retrospective analyses of teaching experiments that typically last several months. The time frame of these analyses gives rise to further challenges in that analyses that locate students' mathematical activity in social context often deal with only a few lessons, or perhaps focus on just a few minutes within one lesson. The issue that I and my colleagues have been struggling with is therefore that of stepping back from and coming to grips with what transpires in a classroom not during a ten-minute episode but over, say, a three-month time period. A third criterion that arises when conducting developmental research is therefore that analyses should document the mathematical learning of both the classroom community and of individual students over extended periods of time.

Interpretive Framework

The interpretive framework that has emerged from our attempts to analyze classroom events while engaging in developmental research involves the coordination of social and psychological perspectives (see Figure 1). The social perspective is an interactionist perspective on collective or communal classroom processes (Bauersfeld, Krummheuer, & Voigt, 1988) and the psychological perspective is a constructivist perspective on individual students' and the teacher's interpretations and actions as they participate in and contribute to the development of these communal practices (cf. Cobb & Yackel, 1996). The entries in the column headed "social perspective" - social norms,

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematica activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

Figure 1. An Interpretive Framework for Analyzing Mathematical Activity in Social Context.

sociomathematical norms, and classroom mathematical practices - refer to aspects of the classroom microculture that we have found it useful to differentiate given our research agenda. The corresponding entries in the column headed "psychological perspective" refer to what, for want of better terminology, might be called their psychological correlates.

I will give an extended example taken from a year-long teaching experiment to illustrate how analyses can be organized in terms of the framework. This experiment was conducted in a first-grade classroom with six- and seven-year-old students and focused on the development of core quantitative concepts. A series of three individual interviews conducted with all 18 students at the beginning, middle, and end of the school year indicated that the experiment was reasonably successful in that the students made significant progress (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, in press). The data collected in the course of the teaching experiment therefore constitute an appropriate setting in which to explore ways of accounting for students' mathematical development as it occurs in the social context of the classroom. In the following paragraphs, I summarize this analysis by briefly outlining the social and sociomathematical norms established by the classroom community and then considering the classroom mathematical practices in more detail.

Social norms. The first step in the analysis of the first-grade teaching experiment involved documenting the social norms to delineate the classroom participation structure. This participation structure proved to be relatively stable by the midpoint of the school year and can be summarized as follows:

1) Students were obliged to explain and justify their reasoning.
2) Students were obliged to listen to and to attempt to understand others' explanations.
3) Students were obliged to indicate non-understanding and, if possible, to ask the explainer clarifying questions.
4) Students were obliged to indicate when they considered solutions invalid, and to explain the reasons for their judgment.
5) The teacher was obliged to comment on or redescribe students' contributions, sometimes by notating their reasoning.

As shown in Figure 1, we take the psychological correlates of the social norms to be students' beliefs about their own roles, others' roles, and the general nature of mathematical activity in school. We therefore conjecture that in guiding the renegotiation of social norms, teachers are simultaneously supporting students' reorganization of these beliefs. This conjecture, it should be noted, is open to empirical investigation.

Sociomathematical norms. It is apparent from the list of social norms given above that such norms are not specific to mathematics, but apply to any subject matter area including science or social studies classes as well as to mathematics classes. The second aspect of the classroom microculture that we differentiate focuses on normative features of students' mathematical activity (Yackel & Cobb, 1996). With regard to the analysis of the first-grade classroom, one sociomathematical norm that emerged was that of what counted as an acceptable mathematical explanation. In the most general terms, acceptable explanations in this classroom had to be interpretable by other members of the classroom community as descriptions of actions on numerical entities (cf. Sfard, 1994). A second sociomathematical norm that emerged concerned what counted as a different mathematical explanation in this classroom. It appeared that solutions to additive tasks were judged as different if they involved either 1) different quantitative interpretations (e.g., the task "14 cookies are in the cookie jar and I take 6 out. How many cookies are in the jar now?" interpreted as 6+_=14 rather than 14-6), or 2) difference in calculational processes such that numerical entities were decomposed and recomposed in different ways (e.g., a solution in which a student reasoned 14-4=10, 10-2=8 would be judged as different to that of another student who reasoned, 7+7=14, 14-7=7, 7-1=6). Significantly, by the midpoint of the school year, various counting methods that would be judged as different by researchers (e.g., counting all versus counting on) were not judged as different in this classroom, but were all simply described as counting. This observation highlights the claim that what counts as a different explanation can differ markedly from one classroom to another, and that these differences can profoundly influence the mathematical understandings that students develop.

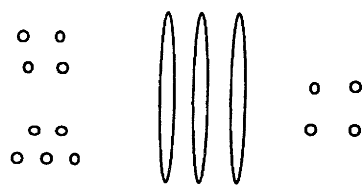
A third sociomathematical norm that emerged concerns what counted as an *insightful mathematical solution*. It is important to clarify that, by the midpoint of the school year, the teacher responded differentially to students' contributions, and that in doing so, she indicated that she particularly valued what she and the students called grouping solutions. This appeared to be an important facet of her effectiveness in supporting her students' learning in that it enabled them to become aware of more sophisticated forms of mathematical reasoning. This, in turn, made it possi-ble for their problem solving efforts to have a sense of directionality (cf. Voigt, 1995). In accomplishing this, however, the teacher continued to accept and actively solicited counting solutions from students who she judged were not yet able to develop grouping solutions. In doing so, she actively managed the tension between proactively supporting the evolution of classroom mathematical practices and ensuring that all students had a way to participate in those practices.

As is shown in Figure 1, we take students' specifically mathematical beliefs and values to be the psychological correlates of the sociomathematical norms. We therefore conjecture that in guiding the renegotiation of these norms, teachers are simultaneously supporting students' reorganization of the beliefs and values that constitute what might be called their mathematical dispositions. Once again, this conjecture is open to empirical investigation.

Classroom mathematical practices. The third aspect of the interpretive framework concerns the mathematical practices established by the classroom community and their psychological correlates, individual students' specifically mathematical interpretations and actions. The objective when analyzing the evolution of classroom mathematical practices is to trace the mathematical development of the classroom community against the backdrop of the social and sociomathematical norms. For illustrative purposes, I will focus on one short instructional sequence called the Candy Shop that was enacted during 12 lessons midway through the school year. The instructional intent of this sequence was to support students' initial construction and coordination of units of ten and of one. The teacher first introduced the anchoring scenario by developing a narrative with her students about a character called Mrs. Wright who owned a candy shop. In the course of these initial discussions, the teacher and students established the convention of packing candies into rolls of ten.

The first mathematical practice identified was that of counting by tens and ones to evaluate collections of candies. For example, in one of the first instructional activities, the teacher gave the students bags of lose unifix cubes and asked them to act as packers in the candy shop. Before they began, however, the students estimated how many rolls of ten they thought they would make. This served to orient them to enumerate the candies/unifix cubes as they packed them. In a subsequent instructional activity, the teacher used an overhead projector to show the students a pictured collection of rolls and individual candies and asked them to figure out how many candies there were in all. In both of these instructional activities and in others, solutions in which students first counted rolls by ten and then individual candies by one became routine and beyond need of justification. As this first mathematical practice illustrates, a practice does not necessarily correspond to a particular type of instructional activity but can instead cut across several activities. It is also important to clarify that the emergence of a practice typically involves a process of negotiation. For example, several students participated in the first instructional activity described above by counting rolls by ten, 10, 20, 30, ..., 80, but then said that they had made 80 rolls. Other students challenged them, arguing that they had made eight rolls that contained 80 candies. The actual process by which the practice of counting by tens and ones emerged can be documented by conducting detailed microanalyses of this and other specific episodes.

A second mathematical practice emerged as the teacher and students continued to discuss solutions to tasks involving pictured collections of rolls and candies. The graphic the teacher used when presenting one task is shown below. The reasoning of the first student who gave an explanation, Chris, proved difficult for the teacher and other students to follow. However, he appeared to mentally group ten individual candies together.



T: How did you figure it out, Chris?

Chris: Well, I knew there was 13 pieces not counting the rolls, all those pieces that are loose,

T: OK.

Chris: and then those three rolls make 30 and if you go up and I got past ten, and I got to 13, so I got past 30, and then I knew if you added ten and three, and I used up two of those, I mean three of those (points towards the screen from his position sitting on the floor). You have 30, and you add the ten, you used up the ten on the 30 and then you had three left and that made 43.

Chris' subsequent clarifications indicate that when he spoke of "using up a ten on the 30," he probably meant that if he counted ten more from 30 he would complete the 30s decade and have 40, and that three more would be 43.

The teacher asked Chris to repeat his explanation and then began to redescribe his solution, possibly to verify her interpretation with him.

T: Chris, you said this was 30 (writes 30 beneath the rolls). Then you have five here (circles a group of five candies).

Chris: Yes.

T: Then you had (circles a group of four candies).

Chris: Four, and then that one over there made five. (T circles the candy he points to.) So that's ten. I used up that 30 right there, I used up that 30 with ten, you see 30 is a whole entire ten almost, it's not really a whole entire ten—after 39 comes 40 and that used up the ten.

T: So there's the 30 that he used (points). Now does everybody see the ten that he used? He had five there, and then you saw he had four and one more made another five. So did you add the five and five to make ten?

Chris: Yes.

T: So then you had 30 plus ten and that got you up to 40.

Chris: Yes.

T: And then you still had these three more (circles the group of three candies) made 43.

Chris: Yes.

In the course of this exchange, the teacher accommodated to Chris' way of speaking, saying "there's the 30 that he used." However, in doing so, she assumed that Chris was referring to the three picture rolls when he in fact seemed to be referring to the 30s decade. Thus, there was a subtle difference in their individual interpretations. Nonetheless, they appeared to communicate effectively as the exchange continued. To take account of such differences, I and my colleagues speak of interpretations being taken-as-shared rather than shared.

As the discussion continued, several other students indicated that they did not understand Chris' reasoning and he gave a series of increasingly articulate explanations. In the remainder of this lesson and in subsequent lessons, the act of mentally grouping ten pictured candies gradually emerged as an established classroom mathematical practice. The teacher for her part indicated that such solutions were particularly valued. For example, after several students had explained how they had figured out how many candies there were in a picture showing eight rolls together with groups of six, four, and three individual candies, she asked, "Is there another way that you could group to figure out 93?"

Ben: (Walks to the screen.) I think it's 93 because I took this six (points to the group of six individual candies) and I broke it up and I took one away and I put it with the four (points to the group of four individual candies) to make five and five, to make ten, and I knew that was 80, so it would be 90, and then 93.

As Ben described his solution, the teacher indicated that she particularly valued it by writing arithmetical sentences to record his reasoning. In addition, the protracted discussion of Chris' solution had also implicitly served to legitimize solutions of this type.

The third mathematical practice identified during the Candy Shop sequence emerged when the teacher introduced a new type of instructional activity in which the students generated different partitionings of a given collection of candies. The teacher explained that Mrs. Wright was interrupted as she packed candies into rolls.

Teacher:What if Mrs. Wright had 43 pieces of candy, and she is working on packing them into rolls. What are different ways that she might have 43 pieces of candy, how many rolls and how many pieces might she have? Sarah, what's one way she might find it?

The students, as a group, were able to generate the various possibilities with little apparent difficulty.

Sarah:	Four rolls and three pieces.
Elizabeth:	43 pieces.
Kendra:	She might have two rolls and 23 pieces.
Darren:	She could have three rolls, 12 pieces, I mean 13 pieces.
Linda:	One roll and 33 pieces.

The teacher for her part recorded each of their suggestions on the whiteboard as shown in Figure 2. Previously, the students had evaluated pictured collections of candies. In contrast, the teacher now drew pictured collections to record the results of their reasoning as they generated alternative partitionings.

At this point in the exchange, one of the students, Karen, volunteered, "Well see, we've done all the ways." She then went on to explain how the configurations the teacher had drawn could be arranged in order. Most of the students seemed to take Karen's purpose for ordering the configurations as self-evident, and a second student proposed an alternative scheme for numbering the pictures. The discussion during the remainder of the session then focused on the merits of different ways of organizing and labeling the configurations.

Summary. This necessarily brief account of the mathematical practices that emerged during the Candy Shop sequence can be summarized as follows:

1. Counting by tens and ones to evaluate collections of candies.
2. Grouping ten candies mentally when evaluating collections.
3. Generating alternative partitionings of a given collection of candies.

We, as observers, can see in this sequence of mathematical practices the initial emergence of the invariance of quantity under certain transformations. In this regard, it can reasonably be argued that the learning of the classroom community was mathematically significant. As a further point, it is important to stress that an analysis of mathematical practices

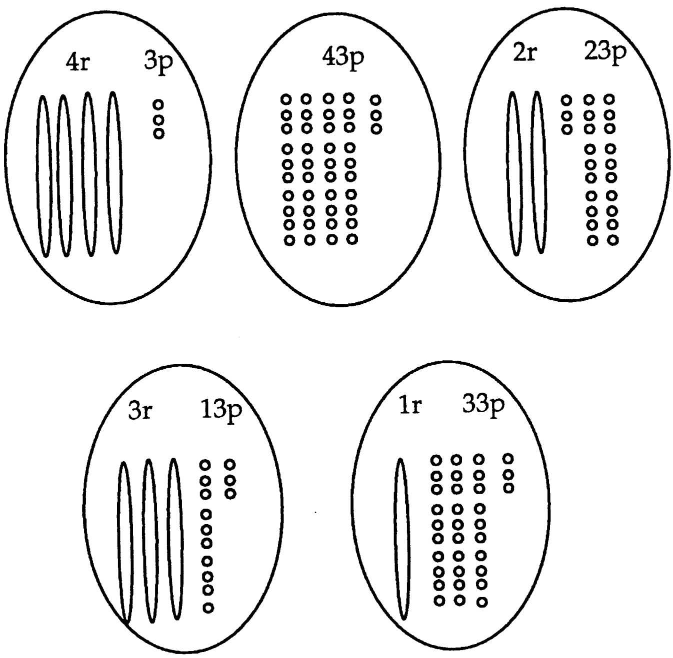


Figure 2. The Teacher's Record of the Students' Responses to the Task Involving 43 Candies.

does not merely involve listing a sequence of activities, methods, or strategies. The analysis also has to sketch the collective developmental route the classroom community has taken by indicating how one practice might have emerged from previously established practices. For example, consider again Ben's reasoning when he evaluated a collection of eight rolls and 13 candies.

> I think it's 93 because I took this six (points) and I broke it up and I took one away and. I put it with the four (points) to make five and five, to make ten, and I knew that was 80, so it would be 90, and then 93.

Here, in reasoning "80, so it would be 90, and then 93," Ben in effect established nine units of ten and three units of one as an alternative to 93 organized as eight tens and 13 ones. This, of course, is not to say that he consciously related these alternative partitionings. Instead, the relationship was implicit in his activity as he participated in the second mathematical practice. As the third mathematical practice emerged, what was previously implicit in students' activity became an explicit topic of conversation. This example is paradigmatic in that it illustrates that an analysis of classroom mathematical practices should account of the process of the classroom community's learning.

Reflections

My purpose in the final section of this paper is to step back from the analysis of the candy shop sequence to address several more general issues. The first of these concerns the contribution of analyses that delineate sequences mathematical practices. To this end, imagine that, at the end of the school year, we had interviewed not only the students in the teaching experiment classroom but also those from another first-grade classroom in the same school. I am sure that if we shuffled the video-recordings of these interviews, most viewers could almost unerringly identify the classroom from which each student came. It is precisely this contrast between the mathematical reasoning of the two groups of students that is accounted for in terms of participation in the differing mathematical practices established in the two classrooms.

To continue the thought experiment, suppose that we now focus only on the students in the teaching experiment classroom. The contrast is now between the activity of individual students in the same classroom community and it is here that qualitative differences in their reasoning come to the fore even as they participate in the same practices. In my view, psychological analyses of the individual students' diverse ways of participating in these practices are needed in order to account for these qualitative differences. An analysis of this type, when coordinated with an analysis of communal practices, documents the process of individual students' mathematical development as they participate in and contribute to the evolution of the classroom mathematical practices.

The second more general point is to note that, in documenting the evolving mathematical practices, we have in effect documented the Candy Shop sequence as it was realized in interaction in the classroom. However, participating in these practices also constituted the immediate social situation in which the students' mathematical learning occurred. As a consequence, the analysis also documents the evolving social situation of their mathematical development. These interrelations between 1) an analysis of classroom mathematical practices, 2) the instructional sequences as realized in interaction, and 3) the evolving social situation of the students' mathematical development is encouraging in that it brings together the two general aspects of developmental research -- instructional development and classroom-based research. Analyses of classroom mathematical practices might therefore make it possible to develop a common language in which to talk both about instructional design and about individual and collective mathematical development in the classroom.

The third more general point concerns the role of symbols and other tools in mathematical learning. It should be apparent from the sample analysis that ways of symbolizing do not stand apart from classroom mathematical practices but are instead integral aspects of both these practices and the activity of the students who participate in them. For example, participation in the second mathematical practice involved reasoning with pictured collections of candies. This observation in turn implies that the ways of symbolizing established in the teaching-experiment classroom profoundly influenced both the mathematical understandings the students developed and the process by which they developed them. It is in fact possible to trace the evolution of the ways of symbolizing:

signified$_1$	—	signifier$_1$	—	signifier$_2$	—	signifier$_3$
candies		unifix cubes		pictured collections		verbal enumerations recorded as 3r 13p etc.

In Walkerdine's (1988) terms, one can speak of a chain of signification emerging as the mathematical practices evolved. Walkerdine notes that succeeding signifiers may initially be established as substitutes for preceding terms, with the assumption that the sense of those terms is preserved through the links of the chain. For example, pictured collections were initially introduced as substitutes for collections of candies/unifix cubes. However, Walkerdine goes on to argue that the original sign combination (i.e., candies/unifix cubes) is not merely concealed behind succeeding signifiers. Instead, the meaning of this sign combination evolves as the chain is constituted. Walkerdine's fundamental contention is that a sign combination that originates in a particular practice slides under succee-ding signifiers that originate in other practices motivated by different concerns and interests. In the case of the Candy Shop sequence, for example, the meaning of the candies/unifix cubes sign combination was initially constituted within a narrative about Mrs. Wright's candy shop. The concerns and interests in this instance were those of a simulated buying and selling activity. Later, the concerns and interests were primarily mathe-matical and involved structuring collections of candies in different ways. When the students participated in the third mathematical practice, rolls of ten candies instantiated units of ten of some type, and the activity of packing candies by making bars of ten unifix cubes had been displaced by that of mentally creating and decomposing such units. In a very real sense, they were no longer the same candies that the students had acted with as when they participated in the first mathematical practice. An analysis of the chain of signification that was constituted as the classroom mathe-matical practices evolved accounts for this change of meaning and thus for the underlying process of mathematization in sociolinguistic terms. It should be stressed that, in such an account, the symbols themselves do not have any particular magic. Instead, the focus is on ways of symbolizing -- on symbols as integral aspects of individual and collective activity rather than as separate entities that stand apart from thought and reasoning.

The final more general point concerns the relationship between individual students' mathematical activity and communal classroom practices implicit in the sample analysis. To put the matter as succinctly as possible, we take this relationship to be reflexive. This is an extremely strong relationship and does not merely mean that individual activity and communal practices are interdependent. Instead, it implies that one literally does not exist without the other (cf. Mehan & Wood, 1975). I would therefore question an account that spoke of classroom mathematical practices first being established, and then somehow causing students to reorganize their mathematical understandings. Similarly, I would question an account that spoke of students first reorganizing their understandings and then contributing to the establishment of new practices. The theoretical position inherent in the interpretive framework and in the sample analysis is one that focuses on both individual students' activity and on the social worlds in which they participate without attempting to derive one from the other. From this point of view, individual students are seen to contribute to the evolution of classroom mathematical practices as they reorganize their mathematical understandings. Conversely, their participation in those practices is seen to both enable and constrain the ways in which they reorganize their understandings.

In this analytical approach, the process of coordinating psychological and social analyses is not merely a matter of somehow pasting a conventional psychological analysis on to a separate social analysis. Instead, when conducting a psychological analysis, one analyzes individual students' activity as they participate in the practices of the classroom community. Further, when conducting a social analysis, one focuses on communal practices that are continually generated by and do not exist apart from the activities of the participating individuals. The coordination at the heart of the interpretive framework is therefore not that between individuals and a community viewed as separate, sharply defined entities. Instead, the coordination is between different ways of looking at and making sense of what is going on in classrooms. What, from one perspective, is seen as a single classroom community is, from another, seen as a number of interacting individuals actively interpreting each others' actions. Thus, the central coordination is between our own ways of interpreting classroom events. Whitson (in press) clarifies this point when he suggests that we think of ourselves as viewing human processes in the classroom, with the realization that these processes can be described in either social or psychological terms. Throughout this paper, I have attempted to illustrate that both of these perspectives are relevant to the concerns and interests of mathematics educators who engage in developmental research. The interpretive framework I have outlined represents one way of coordinating these perspectives that is rooted in classroom-based research of this type.

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CONCEPTUALIZING THE PROFESSIONAL DEVELOPMENT OF TEACHERS

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Introduction

Most mathematics educators are involved with teacher education in some way. The teachers may be at the preservice or inservice level and they may be oriented toward elementary, middle, or secondary schools. Regardless of their status or level of teaching most teachers participate in some form of teacher education designed to promote "better teaching" however we define that term. As such, teachers are learners. Brown and Borko's (1992) review of research on how teachers learn to teach mathematics emphasizes that teachers are rational and cognizing agents with all of the implications that entails. As with any learner, we can consider the conceptual development of the teacher as that development pertains to the different domains of knowledge acquired. The nature of those domains is complex as suggested by Lappan and Theule-Lubienski (1994) and Bromme (1994). Even a casual consideration of Shulman's (1986) notion of pedagogical content knowledge reveals the complexity of teachers' knowing as various elements of epistemology, psychology, mathematics, philosophy, and pedagogy are woven together by the teacher, implicit as that weaving process may be.

The notion of professionalism is likewise complex. Romberg (1988) identified four aspects he considered central to the concept of professionalism: (a) an accumulation of knowledge that sets the "professional" apart from others, (b) use of that knowledge when making occupational decisions, (c) membership in an organization that performs an indispensable public service and which has elements of self-regulation and autonomy, and (d) the presence of indicators that allow for change within the profession. Romberg (1988) and Noddings (1992) have argued that the presence of these conditions are frequently lacking in the professional lives of teachers. Noddings (1992) concludes that teachers generally fall short of professional status given the lack of prestige associated with teaching, the fact that teachers labor in isolation, and that they lack collegiality necessary for a rich professional life. An underlying factor in this lack of professional status is the question of legitimacy of the knowledge perceived necessary to become an accomplished teacher.

Simply put, is that knowledge commonsensical or the product of disciplined inquiry? I submit that it is both based on the following argument.

The professional development of teachers can be thought of in many ways. One could, for example, think of professional development in terms of the teacher's acquisition of knowledge and skills that lead to a particular model of teaching perhaps developed through disciplined inquiry. Good, Grouws, and Ebmeier's (1983) model of active mathematics teaching comes immediately to mind. Alternately, we can conceptualize professional development as moving toward some idealized notion of teaching as defined by recognized expert teachers. While we can debate the relative merits of different models of teaching, it is undeniable that some are grounded in knowledge derived from research. Most definitions of reform-oriented teaching assume teachers have acquired the kinds of knowledge suggested by Shulman (1986), Lappan and Theule-Lubienski (1994), and Bromme (1994). The commonsensical part is the artistic translation of this knowledge into action, an artistry that is dependent on the teacher's propensity to monitor his/her own actions and to generate alternatives for dealing with conceived constraints. These alternatives can be enhanced by the knowledge gained from studying teachers and classrooms.

Teaching ought to be about adapting to and taking advantage of students' cognitive structures while maintaining high expectations regarding what the student should eventually achieve. The adaptation to which I refer is a function of the teacher's ability to be a reflective being, to attend to various circumstances, and to sort out what constraints exist (and don't exist), and to envision ways of dealing with those constraints. Consequently, I think of professional development along a continuum in which a teacher acquires the ability to monitor his/her actions in accordance with the circumstances in which he or she is teaching or learning. The development involves the teacher's flexibility of thinking and adaptability when reacting to various constraints. Attention to context is an integral part of this conceptualization of development because it leads to questions of why an activity was effective in one situation and not in another or why some students seem to develop intellectually and others do not. The question then arises of how we can conceptualize such development among teachers.

Reflective Teaching and Mathematics

Presently the notion of reflective teaching and the value derived from the act of reflection has considerable currency. There are many factors that have contributed to this. Schön's (1983, 1987) seminal work on the reflective practitioner is foremost among them. His notions of reflecting-in-action and reflecting-on-action not only honor a practitioner's tacit knowledge but highlight the practitioner's ingenuity in translating knowledge into action via reflection. To watch a world class athlete or artist perform is to remind ourselves of the apparent ease with which the individual performs. "He/she makes it look so easy." is often the conclusion reached by observers. It is that kind of ease that marks the accomplished teacher as well--an ease punctuated by reflecting-in-action and adapting to a given context. I maintain that reflection and the recognition of constraints should be a central component to any teacher education program that desires to educate teachers to become adaptive agents in the classroom.

It is worth noting that a conception of teaching rooted in adaptation is consistent with recent developments in the philosophy of mathematics. Lakatos' (1976) classic work *Proofs and Refutations* emphasizes the proposition that mathematics, at least in part, consists of a dialogue among individuals in a societal context. His emphasis on dialogue embodies a characteristic of master teaching, namely, taking a student's intellectual hand and guiding his/her intellectual development. Davis and Hersh's (1981) descriptions of mathematical experiences reinforce the notion that mathematics is a function of human experience and is a far more problematic subject than often considered by the general populace. Nearly 20 years ago Tymoczko (1979) suggested the presence of a Kuhnian paradigm shift in mathematics in which the traditional notions of intuition and logic were being supplemented by the aid of computer explorations created by humankind. Witness the proof of the four-color map theorem. In his Quebec address Tymoczko (1994) drew the following conclusion.

> Educators ignore humanistic mathematics to their peril. Without it, educators may teach students to compute and to solve, just as they can teach students to read and to write. But without it, educators can't teach students to love, to appreciate, or even to understand mathematics. (p. 339)

This apparent shift in the notion of what constitutes mathematics has not gone unnoticed in the field of mathematics education. Ernest (1991) has taken arguments about what constitutes mathematics posited by various philosophers of mathematics and discussed the relevance of these arguments for mathematics education. While we can question whether these shifts in mathematical foundations have fueled reform in the teaching of mathematics or, as Tymoczko (1986) suggests, the relationship is the other way around, what seems clear is the consistency between thinking of mathematics as a human endeavor and the notion that teaching is an exercise in adaptation.

The question arises, then, as to what theoretical constructs can guide our thinking about teaching so conceived?

The Theoretical Challenge

As a profession we have come to accept that what a teacher believes about mathematics and the teaching of mathematics is integrally related to the quality of mathematics being taught in the classroom. (See Thompson, 1992.) We know less about how those beliefs are structured and the extent to which they are permeable in the face of tensions rooted in potentially conflicting evidence. Much of this permeability requires attention to context.

We have always honored context as an important contributor to understanding mathematics. For example, it is one thing to know the Pythagorean Theorem in the sense of finding the length of the missing side of a right triangle given the lengths of the two remaining sides. It is another matter to understand the Pythagorean Theorem as a special case of Pappus' Theorem or to understand the consequences of considering the existence of the Pythagorean Theorem in the hyperbolic plane. Similarly, our students learn that the graphs of the equations $y = (1/2)x + 3$ and $y = -2x - 1$ are perpendicular but often fail to realize that the significance of this finding is rooted in the properties of the rectangular coordinate plane. Should the corresponding x and y axes intersect at a 60º angle, what then would we observe about the interesting graphs?

I use these examples to emphasize the importance of mathematical context in teaching mathematics. We all come to believe certain things about mathematics and about teaching mathematics. To what extent are our beliefs dependent on the contexts in which we have come to hold those beliefs? Why is it that we believe the Pythagorean Theorem or any other mathematical proposition? Because someone told us it is so? Because we formally deduced it? How do we come to believe anything? These questions are relevant as we examine the influence of the contexts on what we believe and on how those beliefs are held and, concomitantly, how they might be modified.

If we honor mathematics as a human endeavor then we must see the teaching and learning of mathematics as context bound, recognizing and dealing with constraints in the classroom, and generating alternatives for addressing those constraints. The challenge theoretically speaking is to identify and apply theoretical precepts that enable us to conceptualize a teacher's ability to be adaptive and attend to contexts. I will posit several perspectives that I think have particular merit in conceptualizing teachers' professional development so conceived.

Relevant Theoretical Constructs

I give considerable credence to the notion of self autonomy for I would claim that a person's autonomy is critical to being a reflective and adaptive individual. I do not mean autonomous in a boorish sort of way but rather autonomous in that the individual is capable of integrating a variety of perspectives into a cohesive set of beliefs to which the individual is committed. Bauersfeld (1988) speaks of the importance of the social dimension as an influencing factor on how we organize ourselves. To him, patterns of interaction are a product of social interactions, forever shaping our behavior in light of the implicit obligations we encounter as social beings. No less is true of teachers and students in classrooms. The issue arises whether the individual is capable of monitoring those interactions thereby having the ability to shape them. Von Glasersfeld's (1991) notion of reflection as the ability of an individual to "step out of the stream of direct experience, to re-present a chunk of it, and to look at it as though it were direct experience, while remaining aware of the fact that it is not" (p. 47) is particularly relevant to this discussion. For von Glasersfeld, reflection is a critical ingredient for an individual's ability to re-present the schemes that guide his/her thinking and actions. Dewey (1933) suggests that reflection involves a state of doubt and the act of searching for resolution. For Dewey, reflection is an explicit act, similar in many ways to Schön's (1983) notion of reflecting-in-action which necessitates an awareness of the action taken. This awareness is quite important for it emphasizes the individual's attention to context.

Green (1971) differentiates teaching and indoctrination, the former being a matter of creating contexts in which an individual comes to know in a personal and rational way, the latter being a matter of accumulating information verified solely by an external being. We see this difference being played out daily in classrooms around the world. On the one hand, we see teachers who invite students to explore, to reason, and to conclude that propositions are true by virtue of reasoning. On the other hand, we encounter teachers who emphasize students learning proclamations whose verification rests with them or the textbooks. This latter case is antithetical to reform in the teaching of mathematics. The distinction that Green makes involves how one positions himself or herself relative to an external authority. If mathematics is seen as a subject that is handed down, first from professors of mathematics to teachers of mathematics and then from teachers of mathematics to the students in our schools, then mathematics is cast as a subject devoid of human invention. It relies on some external authority to verify truth, that authority being external to the student. In contrast, a teacher who encourages students to learn and own their mathematics fosters a humanistic view of mathematics as it is the individual who builds mathematical knowledge, not an external being who transmits it. Much of this contrast has been couched in the language of constructivism and appropriately so. My interest here is to focus on the way that an individual comes to know and believe. Rokeach's (1960) analysis of the open and closed mind is another means of characterizing an externally (or not) oriented person. The closed minded person denies context as a mitigating factor as their judgments are absolute. To remind ourselves of the existence of this absoluteness, we have only to recall the number of times our preservice teachers have implored us to tell them the right way to teach mathematics. In contrast the open-minded person sees shades of gray depending on the contexts in which judgments are made. For example, a beginning teacher shared with me her experience in using cooperative learning groups, a technique she initially abandoned. She saw value in using learning groups and was analytical regarding the difficulties she encountered in her failed attempt. She was planning to use them again.

Attention to context differentiates teaching from indoctrination, the notion of open minded from close minded, and, to a great extent whether constructs are permeable (Kelly, 1955) or not. The basis for schemes developed by Perry (1970) and Belenky, Clinchy, Goldberger, and Tarule (1986) are rooted in the extent to which individuals come to know based on their own reasoning processes versus relying on external sources for their knowledge. Perry's (1970) analysis of the intellectual development of male Harvard students resulted in a scheme consisting of nine positions which can be clustered into four basic categories: dualism, multiplism, relativism, and commitment. Briefly, dualism involves seeing the world in absolute terms, truth being defined by an external being. The stage of multiplism consists of recognizing that various opinions exist but truth is still defined by an external authority which has yet to reveal which of the various positions have legitimacy. Brown, Cooney, and Jones (1990) noted various studies that revealed teachers often communicate a dualistic or multiplistic orientation toward the teaching of mathematics. Relativism is the ability to assess the merits of various perspectives which are recognized as not necessarily being of equal merit. Attention to context is an integral part of this analysis. Finally, Perry's stage of commitment involves not only analyzing the merits of various perspectives but making a personal commitment to one of the positions. A teacher who asks students to consider and analyze various ways of representing data, to analyze the value of each representation, and to make a case for the most appropriate representation would be encouraging this more sophisticated way of knowing.

Belenky, et al., (1986) use the metaphor of voice to describe different ways women come to know. Oversimplified, these positions are described briefly below.

a. Silence--one who perceives herself as having no voice
b. Received knower--one who listens to others often at the expen se of suppressing her own voice
c. Subjective knower--one who listens to the voice of inner self and seeks self-identify
d. Procedural knower--one who applies the voice of reason in separate or connected ways,
e. Constructed knower--one who integrates many voices including her own voice

The Belenky, et al. (1986) scheme is not stage like as is the Perry scheme but it does characterize different ways that women and, I submit, others as well, come to know.

A critical aspect of both the Perry and Belenky et al. schemes is a transition from relying on external authority as the primary determiner of truth to valuing one's own voice and critically incorporating voices of others into one's belief systems. This transition point, no doubt evolutionary in nature, is a critical aspect of the professional growth of teachers as it marks the point at which a teacher values his/her own voice while integrating the voices of others. In Perry's scheme, this transition occurs in the movement from the position of true Multiplism Coordinate to the more evaluative stages of Relativism and Commitment Foreseen. Perry (1970) describes this transition as a "radical reperception of all knowledge as contextual and relativistic" (p. 109) rather than "assimilate the new, in one way or another, to the fundamental dualistic structure with which they began" (p. 109). For Belenky, et al., (1986) this shift occurs in the subjective knowledge position in which "an externally oriented perspective on knowledge and truth eventuates in a new conception of truth as personal, private, and subjectively known or intuited" (p. 54). These transitions are important for they signal the emergence of the teacher's ability to be analytical and attentive to context--both prerequisites for reflection as defined by Dewey (1933) and others.

Creating a Theoretical Perspective

Over the past five years we have studied preservice secondary teachers as they progress through their program at the University of Georgia and into their first year of teaching. More recently, this research has been conducted within a National Science Foundation supported project entitled Research and Development Initiatives Applied to Teacher Education (RADIATE) directed by Patricia Wilson and myself. We have found considerable merit in using the theoretical constructs previously described to characterize teachers' professional development. Some preservice teachers' orientations reflect Perry's (1970) dualistic or early multiplistic stages in that they seem unable or unwilling to reflect on the possible interpretations of teacher education activities. Henry's orientation (Cooney, Shealy, & Arvold, in press) seems to fit this category. Henry was convinced that he knew the right way to teach when he began his formal studies in mathematics education. He opposed the use of technology, a position that he maintained throughout his teacher education program. He was steadfast in his belief that good teaching consisted of effective methods of telling students what they needed to know. Throughout his teacher education program he felt uneasy with the methods of teaching being used and suggested. He was discouraged, almost to the point of withdrawing from the program. But Henry's student teaching assignment was with a traditional teacher who conducted a very teacher-centered classroom. This affirmed Henry's belief about teaching. He was critical of his teacher education program and felt that the University instructors were off the mark and failed to appreciate what teaching was really like in the schools. His first year of teaching consisted of teaching lower level classes. Observations of his classes revealed a very teacher-centered classroom. Henry wanted to teach higher level students so that he could *really* teach mathematics. (See Cooney, et al., in press.)

Harriet (Cooney & Wilson, 1995) was another student who seemed to gain little from her teacher education program. Her mathematics education courses placed considerable emphasis on reflecting and analyzing both mathematical and pedagogical situations. Harriet, too, had definite ideas about teaching, gained mostly from her mother, a highly respected middle school mathematics teacher. Over the 15 month period in which she took most of her courses in mathematics education, Harriet was resistant to activities that called for reflection or problem posing. Her orientation toward mathematics was decidedly arithmetical and computational. She cared deeply about helping students, an attitude that translated into helping students acquire basic skills. Harriet was not unhappy about her teacher education program as much as she just seemed oblivious to it. Observational data collected during her first year of teaching were consistent with interview data gathered during her preservice program in that she conveyed a caring attitude toward the students. But the mathematics she taught in her first year of teaching was limited and computational and was taught in a teacher-centered classroom which also characterized her student teaching. (See Cooney & Wilson, 1995.)

Although Henry was critical of his teacher education program, Harriet, in general, was not. She was pleased with her the field experiences and enjoyed exposure to topics such as alternative assessment (although she used none of the ideas in her teaching) and of the opportunity to share ideas with her peers (although she seemed not to incorporate the thinking of others into her own belief structures). Like Henry, her beliefs reflected Perry's notions of dualism or early stages of multiplism.

Nancy (Shealy, 1994/1995; Cooney, et al., in press), like Harriet, was influenced by family members to become a teacher. Nancy was oriented toward doing what others expected of her. Consensuality was very important to her. She saw creativity in others but not in herself. For the most part, Nancy's knowledge about teaching stemmed from the voices of others--her peers and her professors. She accepted diversity of views but saw herself as a spectator rather than a contributor to that diversity. Differences among her peers and professors were seen as misinterpretations rather than profound disagreements. She wanted students to like and respect her. In many ways, her way of knowing reflects what Belenky, et al., (1986) call a "received knower." We have only limited information about Nancy's first year of teaching, but we do know that she experienced considerable difficulty during her first few months of teaching and that she was quite discouraged. She did not seem to command the respect or love that she had hoped would come from the students. Although Nancy was sensitive to what others said and in that sense was reflective, it was not the kind of reflection defined by Dewey (1933), Schön (1983), or von Glasersfeld (1991). Although Nancy's orientation was not dualistic, neither was it relativistic. (See, Cooney, et al., in press.)

Despite their differences, Henry, Harriet, and Nancy have much in common. Shealy (1994/1995) used the term *naive idealist* to characterize Nancy. He saw in Nancy a certain naiveté borne out her penchant for consensuality. Nancy was not resistant to new ideas. Indeed, she willingly received ideas from others suggesting an "idealist" nature to her orientation toward teaching mathematics. In contrast, Henry and Harriet were resistant to new ideas, Henry more openly so. They were absolute in their beliefs about teaching and generally failed to take context into consideration with respect to their beliefs. In this sense they were *isolationists.* (See Cooney, Shealy, & Arvold, in press.) It is difficult to see how an isolationist could become a reflective individual. On the other hand, it is possible that the naive idealist could move into a reflective mode. For example, there is reason to believe that a naive idealist like Nancy could eventually become a reflective practitioner when she realizes that her own voice is part of the choir of voices contributing to her beliefs about teaching.

Some preservice teachers were quite reflective. Greg (Shealy, 1994/1995), for example, held the core belief that his role of teacher was to help students prepare for life. This was a permeable belief. Initially, Greg saw technology as counterproductive to students developing their reasoning skills. As he progressed through his teacher education program, which involved extensive use of technology, he modified this belief and eventually became committed to the use of technology as a means of teaching mathematics. Greg's cohesive set of beliefs were based on the core belief that he would help prepare students for life. Greg had a relativistic orientation toward various situations as evidenced by his attention to the context in which he was teaching or learning mathematics. Other preservice teachers e.g., Sally (Cooney, et al., in press) and Kyle (Cooney & Wilson, 1995) were also reflective and generally analytical. They made connections but fell short of resolving conflicts involving beliefs about mathe-matics or the teaching of mathematics. For example, Kyle felt it quite important to demonstrate to students the importance of how mathematics could be applied to real world situations yet he felt that he missed some of the basics in calculus because his calculus professor spent a considerable amount of time solving application problems. Sally, who initially seemed to be a received knower, later developed a more sophisticated conception of teaching as she reflected on her experiences as a student and as she encountered new ideas about teaching. Yet there were contradictions that Sally never resolved perhaps because she was not committed to becoming a teacher as evidenced by the fact that she decided not to teach following her graduation in mathematics education. Kyle, too, was reflective and attended to context as evidenced by statements such as, "I would use cooperative learning, but not all the time. Don't make all your examples from the book. You can tell a lot about what they know about a problem by the mathematical terminology they're using." Kyle generally connected his experiences to his core belief that mathematics should be made interesting for students by enabling them to see connections between mathematics and the real world. Yet, Kyle felt tension between the importance he placed on basic skills and his orientation toward real world connections, a tension never fully resolved during his teacher education program. He seemed to relish and prosper from reflective activities in a way Harriet did not. Yet, neither did he develop a set of beliefs into a coherent whole as Greg had done. (See Cooney & Wilson, 1995.)

Cooney, et al. (in press) suggest that there are two kinds of connectedness that characterize preservice teachers. First, there is connectedness as demonstrated by Sally and Kyle in which connections are made through reflective activity and attention to context but tensions are not resolved. We describe this as *naive connectionism*. On the other hand, a student such as Greg not only made connections but was able to take various positions and weave them into a coherent set of beliefs. This permeability of beliefs represents a more sophisticated notion of connectedness which we call *reflective connectionism*. These various positions are also reflected in our research with other preservice teachers who have participated in project RADIATE.

Although we have less information about inservice teachers, they, too, reflect these different positions. Some are resolute that they are not interested in trying various methods of teaching mathematics such as cooperative groups, technology or alternative assessment. The reasons are varied and some with reasonable justification. But often they are rooted in their unwillingness to leave the security of traditional teaching. Ellen, for example, restrained from trying different ways of teaching for fear of loosing control of what the students would learn. Observations of her teaching revealed a very teacher-centered classroom, a position from which she was very reluctant to deviate. Ellen was an isolationist of sorts. Yet other inservice teachers dismiss reasons such as time constraints and lack of support and willingly venture into the unknown, confident that good things will happen. Usually these adventurous teachers are analytical and attentive to the context in which a particular teaching technique is more likely to be effective. David, for example, continually tried different ways of assessing students' understanding, monitoring his own assessment practices in the process. David fits the mold of a reflective connectionist.

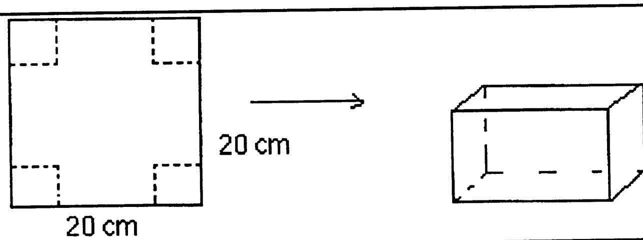
Implications for Practice

During the instructional phase of project RADIATE we placed considerable emphasis on encouraging preservice teachers to reflect on various pedagogical and mathematical situations. Much of this material is based on the notion of integrating content and pedagogy as developed by

Cooney, Brown, Dossey, Schrage, and Wittmann (1996). In one scenario, Ms. Lopez presents "the biggest box problem" to her high school algebra students.

THE BIGGEST BOX PROBLEM

What size square cut from the corners of the original square maximizes the volume of the figure formed by folding the figure into a box without a top?

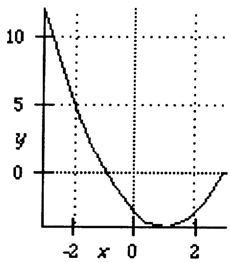


The students are asked to solve the problem using graphing calculators and spreadsheets. But when one student wants to know the "exact" answer, Ms. Lopez is not sure how to find that answer without using calculus which the students have not studied. As she fumbles mathematically, students become impatient and problems arise. The question then becomes one of having the preservice teachers not only solve the mathematical problem in various ways and extend it to related problems, e.g., what size cut would maximize the volume if no material was wasted, but consider the various options for handling the troublesome classroom situation.

In another situation teachers are asked to classify various representations of functions in much the way that Kelly (1955) advocates in his use of the imp grid technique. The activity below is but a sample of the materials presented in Cooney, Brown, Dossey, Schrage, and Wittmann (1996). The purpose of the activity is to consider various contexts in which functions appear and to focus on ways that functions are classified. This and other card sort activities are designed to promote recognition of the importance of classification in mathematics in general, functions in particular, and to consider how the activity can be adapted for use with other content areas in school mathematics.

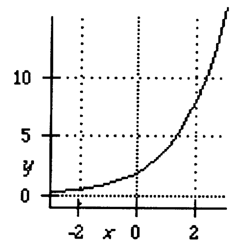
THE CARD SORT PROBLEM

Consider the following six representations of functions. Group them into two or three piles using whatever criteria you wish.



Fred is considering which size pizza is the better buy. He wonders what happens to the area of the circular pizza when the diameter of the pizza is doubled.

The 1990 census shows that Central City has a population of 40,000 people. Recent studies indicate that over the past 20 years the population has steadily increased at an annual rate of 2%. Social scientists predict Central City will experience this same growth rate over the next 20 years. Ms. Callahan has been asked to predict Central City's population for each of the next 15 years.

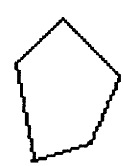


How many different groupings did you identify and what criteria did you use?

The following open-ended question was developed and used by the RADIATE project.

THE PENTAGON PROBLEM

Consider the pentagon drawn on a sheet of regular typing paper. What is the largest pentagon having the same shape that can be drawn on the paper?



There are a variety of ways of solving the pentagon problem ranging from using dilations to using an overhead projector or even a copy machine to enlarge the figure. We ask our teachers to solve the problem and then create and analyze contexts in which the problem could be used with students. In short, they are asked to consider both the mathematical and pedagogical contexts of the problem. In all cases, convincing arguments must be given as to why the solution figure is similar to the original pentagon and why it is believed that the solution figure is the largest possible figure.

In project RADIATE preservice teachers use e-mail to communicate their reflections and analyses with the instructor and with each other. They are asked what influenced them to become mathematics teachers and what factors contributed to what they believe about mathematics. For example, they are asked whether they plan to become a mathematics teacher because of the mathematics or because of the desire to work with students. In another context, the teachers are asked to consider which of the following types of people would best fit their notion of being a mathematics teacher.

newscaster	orchestra conductor	physician	missionary	gardener
engineer	social worker	entertainer	coach	

The issue is not what they pick but the rationale for their selection. The task, which serves both an instructional goal and as a source of data for research, is designed to encourage teachers to reflect on their beliefs about mathematics and teaching and to consider the implications of their biographies of becoming a mathematics teacher for the way that they might eventually teach mathematics.

A Concluding Thought

Properly conceived, teacher education ought not to be a random activity. It should take advantage of what the teacher brings to the enterprise just as we hope teachers consider what their students bring to the classroom. In part, this necessitates listening to what teachers tell us and basing teacher education on their perceived needs. Still, this cannot be the whole of teacher education as some teachers experience motivated blindness just as we do as teacher educators. That is, we sometimes lack the perception of what the real problems are because we are caught up in the particulars of a situation and fail to see the big picture.

I would claim that teacher education ought to be about providing visions of what teaching mathematics can be like in the idealized sense, of providing contexts in which teachers can define their vision and acquire the knowledge and skills to move toward that vision, and of enhancing their ability to be reflective and adaptive agents in the classroom so that they can monitor their own progress. With respect to this last point, I have argued that it is important for us to understand the struggles and tensions teachers experience as they strive to be adaptive agents and to understand what constitutes progress during that process. Such knowledge provides us a basis for conceptualizing both research and development activities in teacher education. More importantly, it provides a basis for allowing teachers to realize their potential and for them moving toward the type of professionalism of which Romberg (1988) and Noddings (1992) speak. Our tensions and struggles as teacher educators ought to focus on ways we can enable teachers to accomplish this end. For then, we will have moved teacher education beyond being activity bound and toward being a field of disciplined inquiry. What better way to spend our professional lives than finding means of enriching the professional lives of others and concomitantly enhancing our own as well?

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ETHNOMATHEMATICS: WHERE DOES IT COME FROM? AND WHERE DOES IT GO?

Ubiratan D'Ambrosio

My early thoughts on Ethnomathematics.

Mathematics, as a form of knowledge, is subordinated to the general behavior of the human being. Hence Mathematics results from the cummulative responses of the individuals and the communities to the drives towards *survival* and *transcendence.* These cummulative responses build into *culture.* In this process, individual actions are the result of different behaviors: *sensorial, intuitive* (and *instinctive), emotional* and *rational.* And individuals interact through *communication,* understood in the broad sense. Thus a body of knowledge is *generated,* which is *intellectualy* and *socially organized,* and is *diffused.* I try to understand knowledge, hence Mathematics, as the result of all these categories.

Particularly important are the relations between Mathematics and Society. I have been concerned with this issue for a long time. Five ICMEs ago, in the Third International Congress of Mathematical Education, in Karlsruhe in 1976, I was invited to prepare a paper reflecting the state of the art and to organize a discussion on the objectives and goals of Mathematics Education. In those years I had accumulated experience of work in Brazil and most Latin American countries, in the USA, in Europe and in a number of African countries. In this process, a broader conception of mathematics started to take shape in my mind. The paper, entitled "Why Teach Mathematics?", although published in several languages by UNESCO[1], did not draw attention. There I presented the basic ideas that led to the Program Ethnomathematics, mainly focusing on a socio-cultural critique of Western Mathematics. The name Ethnomathematics had not yet occurred to me.

Soon after I started to use the word Ethnomathematics. It looked adequate to me after learning about important works on Ethnomusicology,

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1 Ubiratan D'Ambrosio: "Overall Goals and Objectives for Mathematics Education Why Teach Mathematics?", *New Trends in Mathematics Teaching IV,* UNESCO- ICMI, Paris, 1979; pp. 180-196.

on Ethnobotanics, on Ethnohistory, on Ethnopsychiatry and on Ethnomethodology. These "ethnodisciplines" have much to do with work done by anthropologists and undeniably in my early work on Ethnomathematics I was very close to Anthropology and Ethnography. The name was explicitly used, in the broader sense I attribute to it nowadays, in the plenary talk I gave in the Fifth International Congress of Mathematics Education, ICME 5, in Adelaide, Australia, in 1984[2]. This broader sense challenges the usual way of understanding knowledge linearly, focusing on each of these categories which built into knowledge isolatedly, thus creating areas of study known as cognitive science [generation], epistemology [intellectual organization], history [social organization] and education [diffusion]. Also the drives which move materially and spiritually our species [survival and transcendence] and the dimensions of the responses to these drives [sensorial, intuitive/instinctive, emotional, rational] are usually studied as isolated categories. I see all of these categories of study as interdependent and interrelated. Thus my adoption of a holistic approach for understanding the human species. [3]

Ethnomathematics has only more recently been considered an area of Mathematics and Mathematics Education.[4] But since antiquity we recognize concerns similar to those that we label nowadays Ethnomathematics. Herodotus says: "If the river carried away any portion of a man's lot, he appeared before the king [Sesostris], and related what had happened; upon which the king sent persons to examine, and determine by measurement the exact extent of the loss; and thenceforth only such a rent was demanded of him as was proportionate to the reduced size of his land. From this practice, I think, geometry first came to be known in Egypt, whence it passed into Greece. The sun-dial, however, and the gnomon with the division of the day into twelve parts, were received by the Greeks from the Babylonian." [5] Most probably those persons whom

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2 A short summary of the paper was published in the *Proceedings of the Fifth International Congress of Mathematical Education,* ed. Marjorie Carss, Birhäuser, Boston, 1986; pp. 1-6. The full paper was published as Socio-cultural bases for Mathematics education, UNICAMP, Campinas, 1985.

3 For a very brief and introductory discussion of the holistic approach see Ubiratan D'Ambrosio: "Uniting Reality and Action: A holistic approach to Mathematics Education" in *Teaching Teachers. Teaching Students,* eds. Lynn A. Steen and Donald J. Albers, Birkhäuser, Boston, 1981; pp. 33-42.

4 It appears Zentralblatt/Mathematics Reviews Subject Classification of 1991 as History: 01A07.

5 *The History of Herodotus,* translated by George Rawlinson, Book II, 109, Great Books of the Western World, 1994; p.70.

the king sent to examine the lands and determine the measurements were the "mathematicians" of ancient Egypt. We find similar considerations in Maya accounts.[6]

In every culture we recognize practices of measurement, of quantification, of observation, of classification, of inference. These practices were applied by distinguished members of the society for a variety of purposes, such as to record events and facts [scribes], to administer society [public officers], to change the face of the land through buildings and works [architects], to predict events and possibly to interfere with their course [diviners, magicians, astrologers, healers]. The corpora of knowledge that those distinguished members of society knew have many of the characteristics of what we call nowadays Mathematics. But they would never call themselves mathematicians nor Mathematics what they practiced. Much the less Ethnomathematics!

Throughout the history of mankind every culture has recorded, in different ways, reports of travellers who have seen or heard about ways of coping with reality and explainig facts and phenomena which are different from their own. The encounters of cultures are, evidently, responsible for the dynamics of cultural changes.

Ethnomathematics as a corpus of knowledge incorporate [Western] Mathematics.

Although I was led to use Ethnomathematics by similarity with other "ethnodisciplines", I soon recognized that it should be viewed as the recognition of different styles, forms and modes of thought [***tics***] aiming at explaining and dealing with reality [***mathema***], which were developed in different natural and cultural environments [***ethno***].[7] These developments

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6 See Michael Closs (ed.): *Native American Mathematics,* University of Texas Press, Austin, 1986.

7 The search of a name to express all these ideas led me to commit *un abus d'étymologie* and use the Greek roots *ethno, mathema* and *tics* (a modification of *techné*). See my article "Ethnomathematics: A Research Program on the History and Philosophy of Mathematics with Pedagogical Implications." *Notices of the American Mathematical Society,* December 1992, vol. 39, n°10; pp. 1183-1185. Ethnomathematics is a reseach program, delineated in the name itself. Of course, it is supported by a broader view of History. Hence the discussions (Letters to the Editor) which appeared in subsequent issues of the Notices. Of course, it is of fundamental importance for this program to look for mathematical ideas in different cultures.

are not immune to cultural dynamics and reveal inumerous contributions from other cultures and run throughout history.

A corpus of knowledge results from a complex of needs and interests, of experiences and memories, of simbols and representations. The intangible process of imaginative thought which underlies the acquisition of knowledge distinguishes the human species from all other living creatures. The quest of men and women about themselves and the other, about nature and the cosmos, gives them their special dignity and the feeling of truth. The efforts to present images of truth in forms that will delight the mind and senses of the beholder gives meaning to humanity. Distortions in presenting these images lead to preposterousness, to arrogance and arrogation and to hegemony.

Western Mathematics does not escape from these considerations. Besides looking into the mathema of different cultures, my research program would naturally include an analysis of how these issues are seen in the development of Western Mathematics. Thus Ethnomathematics is also a research program in the History and Philosophy of Western Mathematics.

Western Mathematics developed out of the Mediterranean environment, hence they belong to the ensemble of behaviors of the Mediterranean cultures. This assertion is generally accepted. Even Herodotus comes in support of these remarks. But, for obvious reasons, those in power are zealous on reaffirming the intellectual hegemony of the West[8] . The strongest argument using to support this intellectual hegemony is Science. Hence the seemingly unchallengeable position of Mathematics.

The non-recognition of the process of cultural dynamics has lead to distorted views of history. The turning point was, undobtedly, the expansion of the West beginning in the last quarter of the XV century, which opened up the colonial enterprise.

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8 There is a widely spread refusal to recognize this obviousness. I invite those reluctant in accepting my claim that the reasons are obvious to look into the qualitative changes in the state of the world in this century.

The colonial statute and a vision of history.

The colonial statute strongly relied on the strategies of conversion, which had been the main characteristic of both Christian and Islamic expansion, instead of allowing the flow of the process of cultural dynamics. This was particularly important in the evolution of mathematical ideas and in defining a style of mathematics education which was grounded in the ideals of the Enlightment, and acquired a firm standing in the XIX and XX centuries, clearly as a response to the major objectives of the colonial empires.

In the process of conquest it was decisive to remove historical and intellectual knowledge of the conquered, with the consequent elimination of their intellectual thrust and pride. To attain these goals, the strategy of religious conversion was efficacious. New religions brought to the new lands new conceptions of space [permanence] and of time [fluidity], which are the most relevant categories in the foundations of mathematical knowledge.

The era of political colonialism came to an end after the World War II. Since then there was the expectation of the emergence of a new protocol in economic and cultural relations. The Third World was a proposal for a new order.[9] Although the expectations are not yet realized, the process is irreversible and as a consequence there is evidence of the emergence, in just about every field of human activity, of new forms and styles of explanations, of understanding and of practices. Regretably, the former colonial mind are as yet reluctant in recognizing different styles of knowing, freed from the colonial biases.

A new historicity.

A new historicity, hitherto ignored and even repressed, now emerges and is increasingly accepted and adopted as a guide for action. It challenges truisms and social and cultural behaviors, taken as normal in modern civilization.

These studies combine the skills of the archaeologist, anthropologist, ethnographer, the conventional historian, the specialist in the disciplines,

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9 The phrase "Third World" was coined by the Algerian writer Frantz Fanon in his book *Les damnés de la Terre,* Maspero, Paris, 1961, in writing about the newly-emergent nations after WW II. There was then an optimistic mood of hope that the era of damned peoples and cultures was approching its end.

and all this make up for a typical interdisciplinarian approach. The focuses include combining collection of data from tangible materials and from oral traditions, analysis of behavior, comparative studies and cultural dynamics. History thus gains a new breadth, for the concept of sources has to be largely amplified and the chronology entirely revised in order to include developments which followed different, in many cases unrelated, strands.

How could Mathematics stay unaffected by the opening of these new directions of critical inquiry? Thus the inevitability of Ethnomathematics. An excellent account of the trajectory of Ethnomathematics is given by Paulus Gerdes as Chapter 24 of the *International Handbook of Mathematics Education.*[10]

Ethnomathematics may be, and indeed it is, tolerated, even taught, admired and practiced in some academic environments. It is sometimes looked upon as a fashion. And regarded as politically correct. Indeed it is. But there is a cultural arrogance intrinsic to these views. There is a general acceptance and praise of the fact that some cultures show achievements that match -- even if minimally -- results of Western Mathematics, which continues to be the paragon of rationality.

No one would dare to challenge the fact that Western Mathematics is the paragon of rationality. Much of the research in Ethnomathematics today has been directed to identify results and practices that resembles Western Mathematics in non-Western cultures, and to analyze these results and practices by Western instruments. I entirely accept the need of this research. They are absolutely necessary and constitute the most accepted, attractive and indeed the most developed strand of Ethnomathematics research.

Different styles of knowing.

We now have the elements for going deeper into understanding other styles of cognition. The key issue, which sometimes is not explicit in most developments of Ethnomathematics, is the recognition that the Mathematical development of other cultures follow different tracks of intellectual inquiry, different concepts of truth, different sets of values,

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10 *International Handbook of Mathematics Education* , 2 volumes, eds. Alan J. Bishop et al., Kluwer Academic Publishers, Dordrecht, 1996.

different visions of the self, of the other, of mankind, of nature and the planet, and of the cosmos. These visions come all together and can not be considered isolated from each other. These visions build into the behavior of each human being and of societies and are inseparable from the history of each human being and of each society.[11] Civilization, as a category of historical analysis, is the result of this. This was the proposal implicit in Spengler's historiography and more explicit in the historiography of the Annales.

The frameworks of modern society, its science and technology, its religion and the arts, its political organization and philosophical schemes, all sprang out of the Mediterranean. From some elements an entire corpus of knowledge results.

Every culture reveals mythological attributes to facts of reality. The formalisms which derive from the mythological attributes generate distinct corpora of knowledge. Nowadays, all the discourse about indigeneous development refer to these broad aspects of knowledge.

There is a general acceptance that cultures are relative in the sense that something that is true or good in one cultural context may be false or bad in another. But this carries with it the adoption of standards of the outsider. Cultural relativism is indeed ethnocentric. And uses Mathematics as a standard, hence a demonstrative tool.

Much of the arguments are based on the claim of the universality of Mathematics, unique among all cultural manifestations. Of course, Mathematical knowledge is the same in Rome or Lapland or Amazonia, the same as myths, music and hot-dogs. But how about "producing" and "consuming" mathematics, myths, music and hot-dogs? All these cultural manifestations have to do with people.

What do the Lapps and Yanomami have to say about hot-dogs, music and myths? This is a good and respectable question. But if one puts similar questions about Mathematics, the answer is simply "this does not make sense". The claim is that Lapps and Yanomami do not have the intellectual tools to discuss Mathematics. To them we just say: learn Mathematics (of course, Western Mathematics).

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[11] For a presentation of these issues, with special emphasis in the Navajo culture, see James F. Hamill: *Ethno-Logic. The Anthropology of Human Reasoning*, University of Illinois Press, Urbana, 1990.

These were key points in the conquest and the colonial process. Chroniclers have reported and identified all the contradictions resulting from the way the encounter was handled. To exhibit a "succesful" colonized has always been an important evidence of the magnanimity of the colonizers and the adequacy of their pedagogy. No one denies that some Lapp and Yanomami may even receive a Fields Medal! The undeniable possibilities of these results leave unresolved, indeed it masquerades, the real issue.

A theoretical approach and recent developments.

Current historiography is inadequate to deal with Ethnomathematics. The chroniclers of the conquest and colonization are not recognized sources in the History of Mathematics. Hence Ethnomathematics has been, and to a large extent continues to be, treated as curiosity, the same as ethnoreligions are seen as obscurantism and ethnomedicines as superstitions.

A theory of knowledge can be built on five main strands:

1. Description of *ad hoc* knowledge and recognition of methods.

2. How do methods give rise to theories?

3. How do theories lay the ground for invention?

4. What are the socio-political frameworks of knowledge?

5. How do we think? or How is knowledge generated?

A theory of Ethnomathematics should also focus in these five strands. These are not isolated strands. They twine and weave as in a fabric. The metaphor is specially appropriate for Ethnomathematics. How these strands come together in such an indissoluble way in Ethnomathematics is well illustrated in chapters 24 [by Paulus Gerdes], 26 [by Bill Barton] and 35 [by Ole Skovsmose and Lene Nielsen] of the *International Handbook of Mathematics Education* (see Note 9).

Much of the current research in Ethnomathematics has been directed to strand 1. This has also been the main objective of ethnography, psychology, and cultural and social anthropology. There is an increasing literature on these fields supported by the way peoples perform mathematically in different cultural environments in all the parts of the world. We have today a good amount of published research in these directions, which support Ethnomathematics, sometimes not explicitly. And a good amount of research on Ethnomathemastics following this strand.[12]

Of course, the pedagogical implications of these researchers is evident. From mere awareness of Ethnomathematics in different cultural environments to a deeper knowledge of these Ethnomathematics, it is undeniable that the classroom benefits benefits in several ways from it. We may think of a a practical and instrumental objective of Mathematics in Schools. Ethnomathematics relates to everyday uses in the more immediate cultural environment.[13] Political awareness and steps towards full citizenship, avoiding inequities, is more easily achieved, since Ethnomathematics is intrinsically critical, as a result of its dynamics. It has not the characteristics of frozen knowledge which prevails in most of academic mathematics.[14] But probably the most important is to restore cultural awareness and esteem of groups that have been subjected to iequities and discriminations which characterizes the colonial times, but still persits nowadays. This comes in the direction of strand 4.[15]

The arrogance of the dominator finds in Mathematics [academic, as practiced in the traditional school] a perfect ally. The interview with Paulo Freire, presented in this same congress, is very explicit on this. Freire reports that in his school days he never looked on Mathematics as accessible to him. This was regarded as something for individuals who were more like gods!

My early motivation for these thoughts came initially from artisan-ship (basketry in the Amazon), craftsmanship (boat builders in the Amazon basin), buildings (mosques in West Africa) and other cultural manifestations. This invited a reflexion on how these manifestations have, implicit in them, mathematical ideas and contents, which normally are not recognized,

[12] See Chapter 24, #4 [An Overview of Ethnomathematics Literatures] by Paulus Gerdes; op.cit. in Note 10.

[13] See the important book by Marilyn Frankenstein: *Relearning Mathematics. A different Third R: Radical Mathematics,* Free Association Books, London, 1989.

[14] The book of Marilyn Frankenstein, cited in note 13, is particularly concerned with this. See also the activities of the group "Political Dimensions of Mathematics Education", which was established and met for the first time in London, 1990.

[15] See my chapter in the NCTM Yearbook 1997.

while equivalente cultural manifestations of Western civilization support the relations between mathematics and society. We always talk about cathedrals, paintings and the medieval conceptions of God as decisive in the elaboration of what would be later called non-euclidean geometries. These facts are as common in other cultures and determine their developments of modes of explanation and of the more immediate needs of dealing with their environment [*mathema*]. Of course, this follows different paths in different cultural and natural environments. The same can be said about the techniques of registering space [cartography, maps], one of the most influential factors in the development of Western Mathematics. Corresponding techniques, with the same objectives, are present in every culture. Hence, they have determined specific developments of their "mathematical" ideas.[16] This is the typical approach to strand 1.

Strands 2. and 3. are as yet very incipient. Recent work by Samuel Lopez Bello with the Guarany Kayowáa, in Mato Grosso, of Chateubriand Amancio Nunes, with the Kaingang, in Southern Brazil, of Pedro Paulo Scandiuzzi, among the Kamaiurá nation in the Xingú region, aims at identifying the epistemological foundations of their Ethnomathematics.[17]

It is sure that a better knowledge about the generation and organization of knowledge in different cultural environments will shed light into the difficult strand 5. There is no point in attempting to understand the human mind by looking into the behavior of the dominant culture. Indeed this is preposterous. The theoretical understanding of Ethnomathematics and its history are essential elements for understanding both the human mind and the dynamics of cultural encounters.

The pedagogical benefits of the Ethnomathematics approach are easily recognized. There is considerable research supporting situated cognition and building-up on self-esteem. Both are intrinsic to the Ethnomathematics pedagogy. There has been much writing and several different proposals about this. Again, a good source of current developments is Chapter 24, by Paulus Gerdes, in the *International Handbook of Mathematics Education* (see note 12).

I see Ethnomathematics as a thriving field of research and of pedagogical practice.

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16 The recent paper of Marcia Ascher touches this aspect of Ethnomathematics: "Models and Maps from the Marshall Islands: A Case in Ethnomathematics", *Historia Mathematica* 22, 1995; pp. 347-370. .

17 Some of these researches have been reported in the NEWSLETTER of the ISGEm. This periodical publication is, to my knowledge, the best source of new developments in Ethnomathematics.

SOME ASPECTS OF THE UNVERSITY MATHEMATICS CURRICULUM FOR ENGINEERS

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1. Introduction

The main objective of Hanoi University of technology is to train engineers for the country. The undergraduate education in our university is for 5 academic years, one year being composed of 2 semesters. The period of the first 2 academic years is for general education, the remaining period is for speciality education in engineering. There are 28 specialities in the university, including the speciality of applied mathematics.

Traditionally the Core Curriculum in mathematics for engineers includes :
- Calculus
- Linear algebra
- Probability and statistics
- Numerical methods

These courses are given as compulsory courses in the period of general education. In the remaining period, the departments offer specialized mathematics courses which are closely connected to the specific fields as Fourier analysis, special functions, equations of mathematical physics, differential geometry, approximation theory (splines, wavelets, ...), combinatorial optimization.

Over the years mathematics curriculum in our university have gone through a lot of changes. Due to various reasons, less probability and statistics, less numerical methods were being taught. Since 1988 the reform of curriculum in mathematics in the primary and secondary schools was carried out. Therefore we have to change our mathematics curriculum for providing a smooth transition from schools, to review the aims of mathematics education, the principles of implementation of mathematics curriculum in order to meet the requirement of our society in the coming century.

In this talk, I would like to expose what has been proposed for the core curriculum in mathematics for engineers at my university and for the mathematics curriculum of the Department of Applied Mathematics.

2. Role of mathematics in engineering education.

Throughout history it is well known that mathematics play the role of the key to engineering development and engineering development stimulates the development of mathematics .

In the engineering curriculum, the role of mathematics is :
- to provide a training in logical thinking
- to develop a good level of knowledge (basic concept, basic ideas, basic techniques) and skills, of mathematics literacy
- to provide a tool for the derivation of quantitative information about natural systems, a tool for learning across the disciplines, a modeling tool for other sciences.

We notice in the last decades the strong development of abstract activities of engineers : modeling, design of numerical models, using computers. However engineers are not mathematicians. Therefore mathematics education for engineering students is aimed not to train mathematical rigour but train the following abilities :

- to develop mathematical thinking, to train mathematical reasoning
- to understand elementary mathematical models of engineering and to solve them by computers
- to formulate mathematically problems arising in engineering and to solve them by using mathematical ideas and techniques.

As the rate of technological change accelerates in the recent time, some engineering knowledges become outdated very quick. So the engineers must be able to learn independently for adapting successfully to the changes in demands upon them, to change their subjest of work. During the years of their professional lives, the engineers have to pass a number periods of technological updating for understanding new concepts and mastering new techniques. For this reason, our curriculum should be aimed to provide a training for change and not an education for life. We have to recognise and promote the role of mathematics in the continuing education of engineers in collaboration with industry.

As the specialisation in engineering increases due to the sophistication of technology, the engineers must be able to work with others, to work in team, in which engineers can collaborate, communicate easily with specialists of various domains.

3. Core Curriculum in Mathematics for engineers.
3.1 Incorporation of discrete mathematics in the Core Curriculum.

As I mentioned in the introduction, traditionally the Core Curriculum in mathematics for engineers in our university includes calculus, linear algebra, probability and statistics, numerical methods. It's clear that differential and integral calculus occupies a very important place in the mathematics curriculum in the last year of secondary school and the first year of university. Calculus provides methodology and techniques ne-cessary for the study of functions and e abstract tools which are fundamental for the further study of higher mathematics. Moreover, calculus provides the foundations for many applications of mathematics in other sciences and engineering. All these applications are based on continuous mathematical models.

With the widespread use of computers and the strong development of informatics in the last decades, the interest of discrete mathe-matics increases rapidly. Computers are discrete machines. Discrete mathe-matics play to the role of mathematical foundation of informatics. Moreover, the application of computers has stimulated the use and development of discrete mathematical models in many disciplines. So far the course of discrete mathematics is offered to the students of the third year of the department of informatics. Because computers are very efficient computing tools for all branches of sciences and technology, the students of all other departments have to learn some subjects of informatics, but they have no opportunity to learn discrete mathematics. For this reason, we would like to incorporate discrete mathematics in the core curriculum in mathematics for engineers. We prefer to develop a new mathematical course which includes algebraic structures, discrete mathematics and linear algebra. This course is given to all students of engineering of our university in the period of general education. So the new core curriculum in mathematics for engineers in our university now includes :

- calculus
- algebra (including discrete mathematics)
- probability and statistics
- numerical methods

3.2 Teaching of calculus.

We observe that engineers graduated from our university are not competent in mathematical modeling. When they investigate real problems arising in engineering, they are confused to identify the quantities which participate to the problem, to establish various relations between them, to find necessary and sufficient conditions so that the problem is well posed. The main reason of this fact is that the courses of calculus offered in our university are too abstract. In the lectures the mathematics teachers try to define precisely basic concepts of calculus, to expose main ideas, main methods, main results relative to these notions. Due to the pressure of time and the lack of knowledges of applications of calculus in various domains of applied sciences of professors, these applications are considered as auxiliary in the lectures of calculus. Our students are able to find the derivative or the integral of a wide class of functions, to integrate some classes of differential equation, but only very few students understand deeply the practical origin of derivative or integral and can apply these notions in the investigation of real problems in physics, mechanics, engineering. In the courses of calculus in our university, the demand of mathematical rigour in some fundamental aspect like in the construction of real number should be decreased, but the importance of calculus in various areas of applied sciences and and technology should be increased. To train engineers for the future, it is necessary to provide them the knowledges on the interpenetration between mathematics and other branches of sciences and technology, on the industrial dimension of mathematics, on the development of mathematical tools and the application of these tools in the resolution of mathematical models. The control of students knowleges should include some miniprojects which integrate the following aspects : modeling, numerical resolution of mathematical models by computers, discussion on received results. The project work should be included as a component of the mathematics curriculum. The project work contributes :

- to develop autonomy, initiative and personal work of students.
- to enhance the work in team
- to extend cultural perspectives of mathematics, of mathematics application, to promote the relevance of mathematics to industrial needs.

3.3 Core Curriculum in Calculus.

Calculus I. *Simple variable calculus*

* Sequences
* Functions : concept of mapping, function, graphical representation
* Differential calculus : definition and rules of differentiation, derivative of inverse function, higher derivatives, Taylor and Mac Laurin expansion, approximation and asymptotic behaviour of functions
* Integral calculus : indefinite integral and definite integral, fundamental theorems and standard techniques of integration. Engineering applications. Improper integrals

Calculus II. *Multivariable calculus*

* Function of n (≥ 2) variables : representation, partial derivative, total differential, change of variables, implicit function theorem, Taylor expansion, extrema, Lagrange multipliers.
* Multiple integral : double integral, triple integral, change of variables
* Line integral, Green's theorem, surface integral, Stoke's theorem and Gauss theorem, with physical signifiance

Calculus III. *Infinite series. Ordinary differential equations (ODE)*

* Infinite series : convergence tests, absolute convergence, uniform convergence of series of functions, power series
* Fourier series
* ODE : first order differential equation, second order linear diferential equations with constant coeffcients, systems of differential equations.

3.4 Core Curriculum in Algebra

Set and mappings.

* Mathematical logic : propositions, connectives, truth-tables, rules, of inference, quantifiers, introduction to Boolean algebra, reasoning by recurrence
* Set : operations on sets, relation to Boolean algebra.
* Binary relations : equivalence, order

* Mapping : injection, surjection, bijection.

Combinatorics and graph.

* Counting techniques, the product rule, inclusion-exclusion
* Arrangenent, permutation, combination
* Graphs, directed, undirected, trees

Algebraic structure

* Group
* Ring
* Field

Linear space and transformations

* Space,linear independence, bases, subspaces, scalar product, Euclidean norm
* Linear transformation, matrix representation, change of basis, orthogonal transformation

Matrix algebra and system of linear equations

* Matrix representation of system of linear equations, solution of system of linear equations, by elimination method.
* Matrix algebra, inversion, rank
* Determinants
* Decomposition methods. Consistency, uniqueness of solution

The eigenvalue problem

* Algebraic methods for determining eigenvalues and eigenvectors
* Reduction a matrix to diagonal and Jordan form
* Quadratic form

3.5 Core Curriculum in Probability and Statistics.

Probability and basic laws

* Random events
* Definition of probability. Frequentist and combinatorial approaches
* Conditional probability

Random variable and probability distribution

* One-dimensional random variables. Basic distributions (binomial, Poisson, exponential, normal, ...)
* Multidimentional random variables. Conditional distributions, inpendences, linear combinations of random variables
The law of large numbers and the central limit theorem

Statistical treatment.

* Classical treatment : point estimation of parameters, unbiased estimator, consistent estimator, the maximum likelihood method, umbiased estimation of expectation and variance, estimating parameters of specific distributions, the confidence intervals for the mean and variance, for the parameters of the above distributions
* Bayesian treament

Estimation

* Point estimation
* Interval of confidence

Hypothesis testing

Linear regression

3.6 Core Curriculum in Numerical Methods

* Errors
* Numerical solution of algebraic and transcendantal equation : iteration, chord, bissection, Newton method, order of convergence
* Numerical solution of system of linear equations, method of iteration
*Polynomial interpolation
*Numerical solution of ordinary differential equation : Euler method, Runge-Kutta method.

4.Structure of the courses of Mathematics of the Department of Applied Mathematics

Applied mathematics is concerned with the development of mathematics models and techniques with applications to the resolution of problems arising in other sciences as physics, mechanics, computer sciences, in engineering. Computational techniques are of great interest to Applied Mathematics

The objective of the Department of Applied Mathematics is to train mathematics engineers. They have to master :
- mathematical aspects of modeling : differential and stochastic models
- scientific, economic computing and informatics
- some disciplines in applied areas : physics, mechanics, ...

The mathematics courses offered in the period of speciality education for Applied Mathematics are aimed to provide knowledges of higher level of abstraction, of deeper level of significance in mathematical modeling, in solving these models by mathematical tools, by computer. The courses comprises four areas of study : basic concepts, mathematical methods, computing and some domains of applied sciences.

The list of courses offered in the third and fourth years is as follows :

- Advanced discrete mathematics
- Data structure and algorithm
- Optimization
- Combinatorial algorithms
- Theory of programmation
- Functional analysis
- Equations of mathematical physics
- Advanced numerical analysis
- Finite difference and finite element methods
- Mathematical statistics and data analysis
- Time series
- System and control
- Economical models
- Signal processing
- Fluid dynamics
- Mechanics of deformable media

In the first semester of the fifth year, a reasonable flexibility in content is allowed. The last semester is for the memoir writing.

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GEOMETRY IN PARAMETER SPACES
A standard geometrization process

A. Douady

1. Introduction

Since the beginning of the 20th century – I could say following Hilbert– geometry has proved to be an extremely efficient way to tackle many problems in mathematics or other sciences. In many cases, it is not in Euclidean geometry in dimension 2 or 3 that one ends up, but in geometry in some space adapted to the problem –sometimes a space of parameters constructed ad hoc.

In very general terms, the strategy in this geometrization process, i.e. in the transfer from the initial framework to a geometrical framework, can be described as follows:

The initial problem is formulated as to find a configuration (in the given situation) satisfying certain requirements. One looks at the set of all configurations of that nature. One then identifies this set with some subset E of a space in which one is used to work geometrically (typically $\mathbf{R}^n$); the set E is then a *parameter space* for the configurations considered. The problem then becomes to construct a point in E with certain geometrical properties.

In this talk, we give three examples of implementation of this strategy:

1) A property of circle intersections
2) Possibility of reversing a line without letting it be tangent to a given curve;
3) Existence polynomials of degree 4 with prescribed critical values.

In the first two examples, the initial framework is already geometrical, but transfer to another geometrical framework is necessary (or at least useful) to solve the problem. In other words, we have to provide an already geometrical problem with a new geometrization.

In the third example, the initial problem is algebraic. It could be easily formulated in geometrical terms, but the geometric representation we use is different from the obvious one.

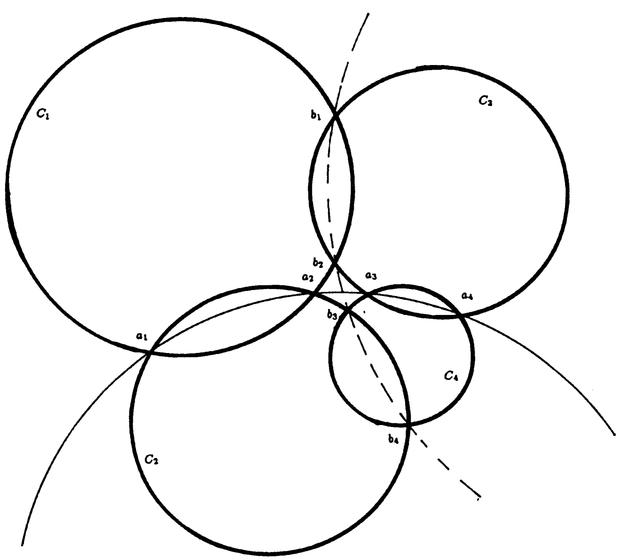
In each of these examples, the parameter space E is a part of $\mathbf{R}^3$ or $\mathbf{R}^2$, but it is easy to imagine that it is not always so. For instance, the set of possible orbits of a planet has dimension 5; in robotics, one has to deal with the set of all possible positions of a solid. This is a 6 dimensional manifold, which can be imbedded in $\mathbf{R}^9$.

We shall first give the statement of the problems in their initial framework, to allow the reader to investigate various ways to tackle them before looking at the one we propose.

2. The Problems

2.1 Circle intersections.

Let C_1, C_2, C_3, C_4 be 4 circles in the plane. We suppose that:
C_1 and C_2 intersect in a_1, a_2 ;
C_3 and C_4 intersect in a_3, a_4 ;
C_1 and C_3 intersect in b_1, b_2 ;
C_2 and C_4 intersect in b_3, b_4 (Fig. 1).

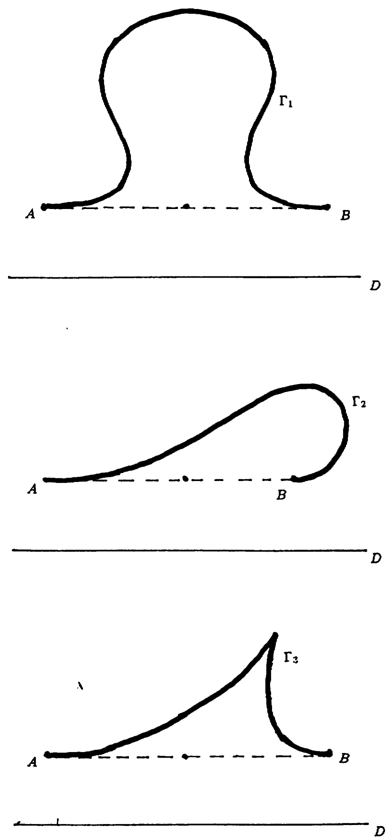


[Fig. 1]

Prove that, if a_1, a_2, a_3, a_4 are on a circle or a line, then b_1, b_2, b_3, b_4 are on a circle or a line.

2.2 Reversing a line (*a problem suggested by David Epstein*)

This problem has 3 versions. Consider in the plane an arc of curve Γ which is one of Γ_1, Γ_2, Γ_3 drawn in Fig. 2, and a straight line D not intersecting Γ. Is it possible to move D continuously and to take it back to its initial position with orientation reversed, without letting it to be tangent to Γ at any time during the movement ?



[Fig. 2]

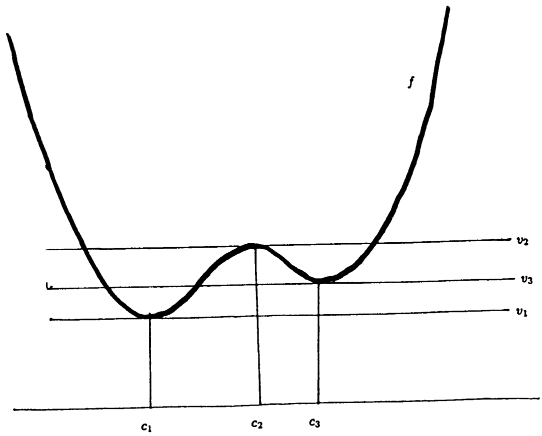
Hint: The answer is not the same for the 3 curves: there are 2 yes and 1 no, or the other was around. Before you look at section 4, try to make guesses.

2.3 Polynomial with prescribed critical values.

Let f be a monic polynomial of degree 4 with coefficients in **R**:

$$f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

("monic" means $a_4 = 1$). We suppose that the derivative f' has 3 real roots $c_1 < c_2 < c_3$ (critical points). Then the critical value $v_i = f(c_i)$ satisfy $v_2 > v_1$ and $v_2 > v_3$ (Fig. 3).



[Fig. 3]

Question 1: *Given real numbers v_1, v_2, v_3 such that $v_2 > v_1$ and $v_2 > v_3$, can one chose a_0, a_1, a_2, a_3, so that the critical values of f are v_1, v_2, v_3 ?*

If we translate f horizontally, i.e. replace it by $x \to f\ (x - b)$, the critical values are unchanged.

Question 2: *Is a monic degree 4 polynomial uniquely determined up to translation by its critical values v_1, v_2, v_3 ?*

This problem arises naturally for arbitrary degree d. There is a classical proof involving complex analysis. It was proposed as a challenge to get a proof staying in the real framework. P. Sentenac and I proposed a way to do so, I present here the case of degree 4, the simplest non trivial one.

3. Travel to the country of circles
3.0. Preliminary considerations

We are considering the first problem, that on circle intersections. This is a problem on circles: supposing that a_1, a_2, a_3, a_4 are on a circle, we are looking for a circle passing through b_1, b_2, b_3, b_4 (or at least trying to prove the existence of such a circle).

According to our general strategy, we have to look at the set C of all circles in the plane.

3.1. First attempt

A circle is determined by its center and its radius. Let us denote by $C_{a,b,r}$ the circle of radius r whose center has coordinates (a,b). Representing the circle $C_{a,b,r}$ by the point (a,b,r) of **R**3, we identify the set C with the half space $\mathbf{R}^2 \times \mathbf{R}_+$ (the points on the frontier plane represent point circles).

This is very natural, but it leads to some trouble. Remember we are looking for a circle passing through 4 points. Given a point $P = (x,y)$ in the plane $\mathbf{R}^2$, the set $C(P)$ of circles passing through P is represented by the cone

$$\{(a,b,r) \mid r = d((x,y),(a,b))\}$$

The set $C(P,Q)$ of circles passing through two points P and Q is a branch of hyperbola. For 4 points, we are looking at the intersection of two hyperbolas in $\mathbf{R}^3$, this is very complicated, and we soon get drowned.

So we have to look for a more clever representation of the set C.

3.2. More algebraically

The equation of the circle $C_{a,b,r}$ is $(x - a)^2 + (y - b)^2 = r^2$, i. e.

$$(*) \qquad x^2 + y^2 - 2ax - 2by + c = 0$$

where $c = a^2 + b^2 - r^2$. Any equation of the form (*) describes a circle, provided $c \leq a^2+b^2$ (accepting point circles). Representing $C_{a,b,r}$ by the point (a,b,c) in $\mathbf{R}^3$, we identify C with the region exterior to the paraboloid Σ of equation $c=a^2+b^2$. Points on the paraboloid represent point circles; points in the region inside represent imaginary circles with no real points.

What are the goods and the odds of this new representation?

Disadvantages: This seems less natural. Moreover we get a region in $\mathbf{R}^3$ which is harder to describe.

Advantage: The set $C(P)$ is now a plane, namely that of equation (*) where *a,b* and *c* are the variables, x and y being considered as constant.

This plane touches the paraboloid Σ at the point representing the point circle $\{P\}$, otherwise it lies in region exterior to Σ. Given two points P and Q, the set $C(P,Q)$ becomes the intersection of two planes, i.e. a straight line (lying entirely in the region exterior to Σ).

3.3. Solving the problem

We use the second representation and identify C with the region E in $\mathbf{R}^3$ exterior to Σ. A circle C in $\mathbf{R}^2$ is represented by a point in E that we denote also by C. Given two points C and C' in $\mathbf{R}^3$, we denote by Dc,c' the line passing through C and C'.

To say that a_1, a_2, a_3, a_4 are on a circle G means that there is a point $\Gamma \in$ E which lies both on the line Dc_1,c_2 and on Dc_3,c_4, i.e. that these two lines meet. It implies that the points c_1, c_2, c_3, c_4 are in a plane, and therefore that the lines Dc_1,c_3 and Dc_2,c_4 meet or are parallel. If they meet in a point Γ', this point represent a circle passing through b_1, b_2, b_3, b_4.

Remains to see that, if the lines mentioned are parallel, the points b_i are on a line. There are several ways to show that, we will propose a limit argument.

3.4. Limit argument

Suppose that the lines Dc_1c_3 and Dc_2,c_4 are parallel in a plane H. Move slightly the point C_4 in the plane H to a point $C_4(\varepsilon)$ so that the two lines intersect. Then we get 4 points b_1, b_2, $b_3(\varepsilon)$, $b_4(\varepsilon)$ on a circle. When e tends to 0, the points $b_3(\varepsilon)$ and $b_4(\varepsilon)$ respectively, while b_1 and b_2 remain fixed. Therefore b_1, b_2, b_3, b_4 are on a circle or on a line.

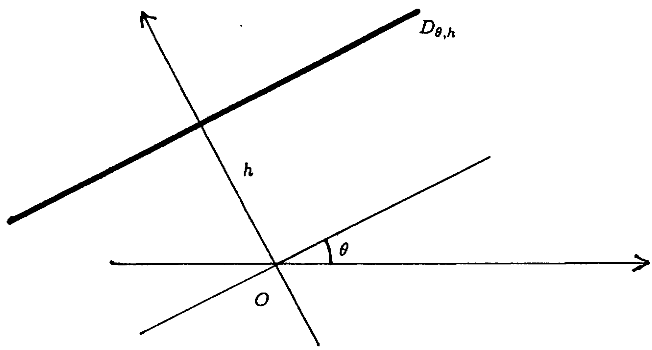
In the same way we can prove the converse. Applying it to $a1,\ldots,a4$, we get that if these points are on a line, the lines Dc_1,c_2 and Dc_3,c_4 are parallel, thus the points $C_1,\ldots,C_4$ are in a plane. We have seen that this implies that $b_1,\ldots,b_4$ are on a circle or on a line.

This finishes the proof of the result.

4. Movement of a line

4.1. An unexpected Moebius strip

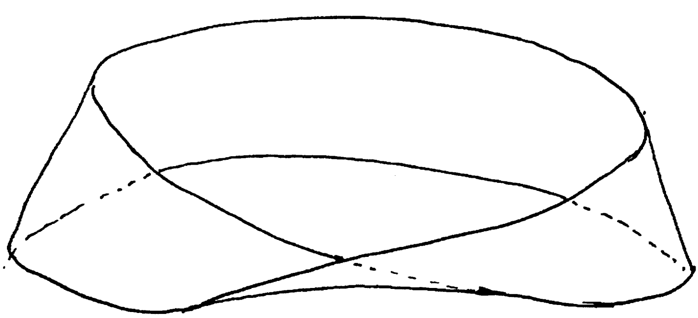
The problem suggested by D. Epstein is a problem about lines in the plane $\mathbf{R}^2$, so we have to look at the space D of lines in the plane.



[Fig. 4]

Given θ and h in **R**, let $D_{\theta,h}$ denote the line making an angle θ with the first axis Ox, and such that the projection H of O on it has abscissa h on the axis making angle $\theta + \pi/2$ with Ox (Fig. 4). Any line in the plane can be written as $D_{\theta,h}$, and this in several ways: given (θ,h) in $\mathbf{R}^2$, the pairs (θ',h') such that $D_{\theta',h'} = D_{\theta,h}$ are the $(\theta+2k\pi,h)$ and the $(\theta + (2k+1)\pi, -h)$. So D can be viewed as the quotient of $\mathbf{R}^2$ by the equivalence relation defined this way, or if you prefer as the strip $[0, \pi] \times \mathbf{R}$ with $(0, h)$ and $(\pi, -h)$ identified.

Note that **R** is homeomorphic to the open interval $]-1, 1[$ (and we can choose a homeomorphism ϕ so that $\phi(-h) = -\phi(h)$), so the strip $[0, \pi] \times \mathbf{R}$ is homeomorphic to $[0, \pi] \times]-1, 1[$, and finally D can be identified to $[0, \pi]\times[-1,1[$ with $(0,y)$ and $(\pi, -y)$ identified, i.e. to a Moebius strip with its boundary curve removed. (Fig. 5)



[Fig. 5]

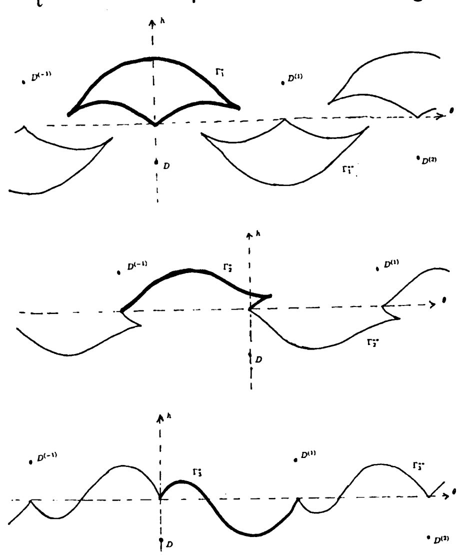
Here I have a scruple, I feel I am cheating a little bit: D has been defined only as a set. But the statement of the problem implicitely involves a topology on D, since it mentions a continuous movement of a line, i.e. of a point in D. To be rigourous, we should define the natural topology on D, and show that the identification above is a homeomorphism. I don't want to develop on this theme here. We shall admit that saying that we have a continuous movement $t \to D_t$ parametrized by $[0,1]$ means that $D_t = D_{\theta(t),h(t)}$ with $\theta(t)$ and $h(t)$ depending continuously on t, and that the requirement that D_t comes back to its initial position with orientation reversed means that $\theta(1) = \theta(0) + (2k + 1)\pi$ for some k, and $h(1) = -h(0)$.

So the question is whether it is possible to have such a movement avoiding the forbidden set: the set of lines tangent to Γ. What we have to do now is to describe this forbidden set in the 3 cases $\Gamma = \Gamma_1, \Gamma_2, \Gamma_3$.

4.2. Description of the forbidden set

Say we take as origin the midpoint O between the extremities A and B of Γ, and the axis Ox to be the line AB.

Let $M(s)$ be a point which ranges over Γ when s ranges from 0 to 1. Write the tangent to Γ as $D_{\theta^*(s),h^*(s)}$ with θ^* and h^* continuous. Then $(\theta^*(s),h^*(s))$ describes in $\mathbf{R}^2$ an arc Γ^*, and the forbidden set in $\mathbf{R}^2$ is the union Γ^{**} of copies of Γ^* translated by $(\theta,h) \to (\theta + k\pi, (-1)^k h)$ for $k \in \mathbf{Z}$. The sets Γ^*_ι and Γ^{**}_ι have the aspect drawned in Fig. 6.



[Fig. 6]

The question is then whether it is possible to join the point D_0 to some $D0^{(k)}$ with k odd without meeting $\Gamma\iota^{**}$. It is clear on the picture that this is possible for $\iota = 1$ and impossible for $\iota = 2$ and $\iota = 3$.

4.3. Proofs

Let us try to transform this visual evidence into a proof. We start by Γ_3, for which it is the easiest. In this case θ^* is monotonous and ranges from 0 to π, so it is a homeomorphism $[0,1] \to [0,\pi]$. Therefore $\Gamma_3{}^*$ is the graph of a continuous function $h : [0,\pi] \to \mathbf{R}$ with $\eta(0) = \eta(\pi) = 0$, and $\Gamma^{**}{}_3$ is the graph of the same function extended to $\mathbf{R}$ by $\eta(\theta+\kappa\pi) = (-1)\kappa\eta(\theta)$. Now, supposing we have q and h continuous $[0,1] \to \mathbf{R}$ with $\theta(1) = \theta(0) + (2\kappa+1)\pi$ and $h(1) = -h(0)$, then $h(f) - \eta(\theta(t))$ is continuous and takes opposite values at 0 and 1, so it must vanish somewhere. This means that there is a $t \in [0,1]$ so that $(\theta(t), h(t)) \in \Gamma^*{}_3$. q.e.d.

Let us come to $\Gamma 2$. The important feature is that $\theta^*(s)$ varies continuously from 0 to $-\pi$ (it would work also with any $k\pi$, $k \neq 0$ in $\mathbf{Z}$). We shall use only this fact; maybe we could get a slightly simpler proof using other particularities of the situation, I think it is not worth trying.

We extend θ^* and h^* to $\mathbf{R}$ by $\theta^*(s+k)=\theta^*(s)-k\pi$ and $h^*(s+k)=(-1)^k h^*(s)$. The set Γ^{**} is then the image of $\mathbf{R}$ by $s \to (\theta^*(s),h^*(s))$. The functions $s \to \theta^*(s) + \pi s$ and h^* are continuous and periodic, thus bounded.

By hypothesis $D = D_{\theta(0),h(0)}$ does not intersect Γ, and we can suppose that $D_{\theta(0),h}$ does not intersect Γ for $h \leq 0$. We extend $t \to (\theta(t),h(t))$ by

$$\begin{aligned}(\theta(t),h(t)) &= (\theta(0),h(0) + t) && \text{for } t \leq 0;\\ &= (\theta(1),h(1) + t - 1) && \text{for } t \geq 1.\end{aligned}$$

The result then follows from the following lemma (applied to $(-\theta^*,h^*,-\theta,h)$:

Crossing lemma.- Let $s \to (x1(s),y1(s))$ and $t \to (x_2(s),y_2(s))$ be two continuous maps $\mathbf{R} \to \mathbf{R}^2$. Suppose that the functions y_1 and x_2 are bounded, and that x_1 and y_2 range from $-\infty$ to $+\infty$. Then the two path cross, i.e. there is a pair (s,t) such that $(x_1(s),y_1(s)) = (x_2(t),y_2(t))$.

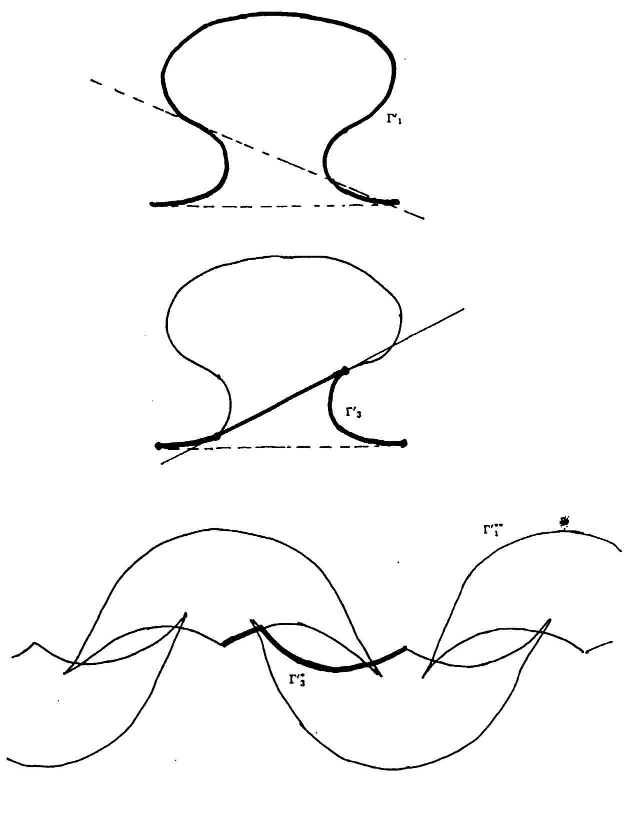
This lemma is classical in topology. The usual proof involves the notion of index of a loop around a point (number of turns).

4.4. The case of Γ_1

For the Ω-shaped curve Γ_1, the movement is possible. We can just exhibit it in the initial framework:

As the picture is drawned, the line AB and the two inflexion tangent limit a triangle below AB. Move D parallel to itself so that it passes through some point in this triangle, and then just rotate it by half a turn.

Remark: If we would take a curve Γ'_1 looking like Γ_1 but with the inflexion tangent intersecting Γ again, the result would be different: then there would be a double tangent L, and we could extract from $\Gamma'_1 \cup L$ a curve Γ'_3 looking like Γ_3 (but containing a line segment), and the answer would be NO as for Γ_3. (Fig. 7)



[Fig. 7]

5. The problem on degree 4 polynomials
5.1. Natural tackling of the problem

The set of monic polynomials of degree 4 is naturally identified to $\mathbf{R}^4$ via $f \to (a_0,\dots,a_3)$. In this set, polynomials having 3 distinct real critical points form an open set defined by a complicated inequation.

But we are interested only by polynomials up to horizontal translation: we could restrict to *centered* polynomials, i.e. those with $a_3 = 0$. Indeed each class modulo horizontal translation contains a unique centered polynomial. The centered monic polynomials of degree 4 form a space $\mathbf{R}^3$, in which those with 3 distinct critical points form the open set Ω defined by $a^3{}_2 - 27a^2{}_1 > 0$.

We want to show that the map $f \to (v_1,v_2,v_3)$ from Ω to $\{(v_1,v_2,v_3) | v_2 > v_1, v_2, v_3\}$ is bijective. This is the natural way of tackling the problem, but we shall proceed in a slightly different way.

5.2. First transformation

We first draw our attention on the invariance of the problem by vertical translation. Indeed, if we have a polynomial f such that v_2-v_1 and v_2-v_3 have the prescribed values, then it is not difficult to adjust f by adding a constant so that v_1, v_2, v_3 have the prescribed values.

So we look at f up to vertical translation, i.e. we look at

$$f' = 4x^3 + 3a_3x^2 + 2a_2x + a_1$$

The critical points are the zeroes of f', we require that there are 3 distinct real ones $c_1 < c_2 < c_3$. Then v_2-v_1 and v_2-v_3 can be interpreted as areas:

$$v_2 - v_1 = A_1 = \int_{c_1}^{c_2} f'$$

$$v_2 - v_3 = A_2 = \int_{c_2}^{c_3} |f'|$$

So we can formulate the problem in the equivalent form:

Question: *Given* A_1 and A_2 both positive, does there exist a polynomial $g = 4x^3 + b_2x^2 + b_1x + b_0$ with 3 real roots $c_1 < c_2 < c_3$ such that $\int_{c_1}^{c_2} g = A_1$, $\int_{c_2}^{c_3} = -A_2$? Is such a polynomial unique up to horizontal translation?

5.3. Second transformation

We now take into account the invariance by horizontal translation. Note that $g = f'$ is given by

$$g(x) = 4(x - c_1)(x - c_2)(x - c_3)$$

Take as unknown
$l_1 = c_2 - c_1 > 0$
$l_2 = c_3 - c_2 > 0.$

These numbers determine f' up to horizontal translation. So they allow to compute A_1 and A_2, and we can look at the map

$$\Phi : (l_1,l_2) \to (A_1,A_2)$$

from $\mathbf{R}^2{}_+$ too itself. The question is now:
Question: *Is* Φ *a bijection ? a homeomorphism ?*

5.4. Study of the map Φ

Lemma 1 (homogeneity) .- If Δ is a half line from O, so is $\Phi(\Delta)$.
Proof: If $\sim l_1 = \lambda l_1$ and $\sim l_2 = \lambda l_2$, the graph of ~g is obtained from that of g by $x \to \lambda x$, $y \to \lambda^3 y$. So $\sim A_1 = \lambda^4 A_1$, $\sim A_2 = \lambda^4 A_2$.

Lemma 2.- If you increase l_1 and decrease l_2 so that $l_1 + l_2$ is unchanged, then A_1 increases and A_2 decreases.

Proof: Take $\sim c_1 = c_1$, $\sim c_3 = c_3$ and $\sim c_2 = c_2$, and define ~g from $\sim c_2$, $\sim c_2$, $\sim c_3$. Then ~g – g is a polynomial of degree 2 vanishing at c_1 and c_2, so it has a constant sign on $]c_1,c_3[$. This sign is it is > 0, because it is at $\sim c_2$.
Then $\sim A_1 = \int{\sim c_2 c_1} \sim g > \int c_2 c_1 \sim g > A_1$
and $\sim A_2 = \int c_3 \sim c_2 |\sim g| < \int c_3 c_2 |\sim g| < A_2$.

5.5. What for d > 4 ?

For $d > 4$, the problem can be treated along the same lines, but we must make use of more sophisticated notions.

We define in the same way a map Φ from $\mathbf{R}^{d-2}{}_+$ to itself. Starting from a point inside $\mathbf{R}^{d-2}{}_+$, we can look at the linear map tangent to Φ at this point. By an argument similar to that of Lemma 2, we can see that this linear map is an isomorphism. Therefore Φ is a local homeomorphism.

Using homogeneity and extension to the boundary, we see that Φ is a proper map (i.e. the inverse image of a compact set is compact). Therefore it is a finite covering map, and since $\mathbf{R}^{d-2}{}_{+}$ is simply connected, Φ is a homeomorphism.

Conclusion

In the 3 examples treated here, we consider the set of objects (circles, lines, polynomials) of interest for the problem. This is an abstract set; in order to be able to work in it, we have to make a chart, i.e. to define a correspondance with a set of points in a space with already a geommetrical structure, or with a numerical set (typically a part of $\mathbf{R}^n$).

The choice of the chart is the crucial point. It is where one can show one's skill. It is not always the most natural choice which is the most efficient, as we have seen in examples 1 and 3. Sometimes, the correspondance is not bijective, as in example 2: the set of lines can be identified with a quotient of $\mathbf{R}^2$ homeomorphic to a Moebius strip, but the work actually takes place in its covering space $\mathbf{R}^2$.

In these examples, the work is rather easy after a proper geometrization has taken place. Certainly it is not always so, but such a process is often a very good first step.

SOCIAL CONSTRUCTIVISM AS A PHILOSOPHY OF MATHEMATICS

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Introduction

My aim in this paper is two-fold. First to give a written report of my conference lecture at the International Congress on Mathematical Education in Seville. Second, to fill in a little the sketch of my current work in developing a social constructivist philosophy of mathematics which I presented there, briefly pointing to some the implications for mathematics education. But initially I need to spell out the problematic present position in the philosophy of mathematics and the need for a new approach.

Much of modern epistemology has understood knowledge to be made up of knowledge claims with ironclad warrants and justifications (Kant 1950, Moore 1959, Klein 1981). Thus, Ayer (1946), for example, claims that empirical knowledge of the world can attain certainty, and that the truths of mathematics and logic are both certain and necessary. However there is a sceptical tradition in epistemology stretching back to the presocratic philosophers and strongly present in contemporary philosophy which regards all knowledge as fallible and based on revisable foundations (Bernstein 1983, Everitt and Fisher 1995, Wittgenstein 1953, Rorty 1979, Rosen 1989). In the philosophy of science there has similarly been a shift in the leading views of knowledge away from infallibility and certainty (Feyerabend 1975, Kuhn 1970, Popper 1959). Likewise, in the sociology of knowledge and social studies of science there is a consensus that far from being necessary, scientific and mathematical knowledge, and indeed all forms of knowledge, are contingent social constructions (Bloor 1991, Fuller 1988, Latour 1993, Lyotard 1984).

Traditionally, mathematical knowledge has been understood as universal and absolute knowledge, whose epistemological status sets it above all other forms of knowledge. The traditional foundationalist schools of formalism, logicism and intuitionism sought to establish the absolute validity of mathematical knowledge. Although modern philosophy of mathematics has in part moved away from this dogma, it is still very influential, and needs to be critiqued. So I wish to begin by summarising some of the arguments against absolutism, as this position has been termed (Ernest 1991).

My argument is that the claim of the absolute validity for mathematical knowledge cannot be sustained. The primary basis for this claim is that mathematical knowledge rests on certain and necessary proofs. But proof in mathematics assumes the truth, correctness, or consistency of an underlying axiom set, and of logical rules and axioms or postulates. The truth of this basis cannot be established on pain of creating a vicious circle (Lakatos 1962). Overall the correctness or consistency of mathematical theories and truths cannot be established in non-trivial cases (Gödel 1931).

Thus mathematical proof can be taken as absolutely correct only if certain unjustified assumptions made. First, it must be assumed that absolute standards of rigour are attained. But there are no grounds for assuming this (Tymoczko 1986). Second, it must be assumed that any proof can be made perfectly rigorous. But virtually all accepted mathematical proofs are informal proofs, and there are no grounds for assuming that such a transformation can be made (Lakatos 1978). Third, it must be assumed that the checking of rigorous proofs for correctness is possible. But checking is already deeply problematic, and the further formalising of informal proofs will lengthen them and make checking practically impossible (MacKenzie 1993)

A final but inescapably telling argument will suffice to show that absolute rigour is an unattainable ideal. The argument is well-known. Mathematical proof as an epistemological warrant depends on the assumed safety of axiomatic systems and proof in mathematics. But Gödel's (1931) second incompleteness theorem means that consistency and hence establishing the correctness and safety of mathematical systems is indemonstrable. We can never be sure mathematics theories are safe, and hence we cannot claim their correctness, let alone their necessity or certainty. These arguments are necessarily compressed here, but are treated fully elsewhere (e.g., Ernest 1991, 1997). So the claim of absolute validity for mathematical knowledge is unjustified.

The past two decades has seen a growing acceptance of the weakness of absolutist accounts of mathematical knowledge and of the impossibility in establishing knowledge claims absolutely. In particular the 'maverick' tradition, to use Kitcher and Aspray's (1988) phrase, in the philosophy of mathematics questions the absolute status of mathematical knowledge and suggest that a reconceptualisation of philosophy of mathematics is needed (Davis and Hersh 1980, Lakatos 1976, Tymoczko 1986, Kitcher 1984, Ernest 1997). The main claim of the 'maverick' tradition is that mathe-matical knowledge is fallible. In addition, the narrow academic focus of the philosophy of mathematics on foundationist epistemology or on platonistic ontology to the exclusion of the history and practice of mathematics, is viewed by many as misguided. However there is still a heated controversy over whether the acceptance of mathematical knowledge is at root a social process, or whether proofs and hence the justification of mathematical knowledge are based on reason and logic alone, no matter how imperfectly these ideals are realised in actuality.

WHAT IS FALLIBILISM?

The term 'fallibilism' is ambiguous, and leads to some confusion, so it is useful to distinguish three versions. First of all, there is what might be termed fallibilism1 which asserts that mistakes occur in mathematics because humans make mistakes. Fallibilism1 is trivial because clearly human beings are fallible, i.e., make mistakes, and all philosophies of mathematics would accept this. So this version is discarded with no further ado.

Secondly, there is fallibilism2 which claims that mathematical knowledge is or may be, of itself, false. Two possible versions of this might be distinguished. The first subcase is the claim that all mathematical knowledge is or may turn out to be false. This is easily rejected because it is absurd to say that 2+2=4 is or may be absolutely false. The second weaker subcase is the claim that some mathematical knowledge is or may turn out to be false. To support this claim it is enough to find one falsehood in mathematics, or stronger, one contradiction. Gödel's Theorem means we cannot eliminate this possibility. However the implication of this version of fallibilism2 is that absolute true/false judgements can be made about ma-thematical knowledge, i.e. there is absolute truth, but mathematics fails or may fail to attain it. This version of fallibilism is thus absolutist.

Third, there is fallibilism3 which claims that mathematics is a relative, contingent, historical construct. This version denies the assumed absolutism of fallibilism2. According to fallibilism3 absolute judgements with regard to truth or falsity, correctness or incorrectness cannot be made. This is because the criteria and definitions of these concepts themselves vary with time, context, and never attain a final state. There are no absolutes concerning truth, correctness, certainty, necessity, and hence however good and well founded mathematical knowledge is or becomes it can never attain perfection, and no absolute or perfect criteria exist either.

Fallibilism3 is the position of social constructivism, which claims that the concepts, definitions, and rules of mathematics were invented and evolved over millennia, including rules of truth and proof. Thus mathematical knowledge is based on contingency, due to its historical development and the inevitable impact of external forces on the resourcing and direction of mathematics. But is also based on the deliberate choices and en deavours of mathematicians, elaborated through extensive reasoning. Both contingencies and choices are at work in mathematics, so it cannot be claimed that the overall development is either necessary or arbitrary. Much of mathematics follows by logical necessity from its assumptions and adopted rules of reasoning, just as moves do in the game of chess. Once a set of axioms and rules has been chosen (e.g., Peano's axioms or those of group theory), many unexpected results await the research mathe-matician. This does not contradict fallibilism3 for none of the rules of reasoning and logic in mathematics are themselves absolute. Mathematics consists of language games with deeply entrenched rules and patterns that are very stable and enduring, but which always remain open to the possibility of change, and in the long term, do change. And as they change, so does the range of possible discovered within a mathematical system.

Social constructivism and fallibilism3 reject absolutism, which involves the following three sub-theses (Harré and Krausz 1996). First of all, there is the thesis of universalism, which asserts that all knowing beings at all times and in all cultures would agree on truth and on mathematical knowledge. It may immediately be noted that this is false if 'do' is put for 'would', for groups such as intuitionists and classical mathematicians already disagree fundamentally on what is legitimate mathematical knowledge. But a more general problem for the thesis of universalism is the question of how would we know if it were true. Since people's knowledge and beliefs must be transformed to validate it, and this cannot be done universally, it must remain an indemonstrable article of faith. As such the thesis is rejected by social constructivism.

Second, there is the thesis of objectivism, which asserts that truth depends on objective reality, not views of persons or groups. This raises the problem of privileged access to 'objective mathematical reality', i.e., the 'god's-eye view' of mathematicians into the universe of mathematics. This thesis is unsatisfactory because it is not what mathematicians intuit which is taken as objective truth, but rather what they prove that is regarded as true. Thus how mathematical truth 'depends' on a mathematical reality that plays no part in its justification is unclear.

Third, there is the thesis of foundationalism, which asserts that there is a unique permanent foundation for knowledge. This foundation has not been identified historically. Foundationalist philosophies of mathematics (Logicism, Formalism, Intuitionism) have all failed, as I argue above. Furthermore, the idea that a basic foundation for mathematical knowledge exists leads to a vicious cycle, because no basic set of assumptions can ever ultimately be dispensed with (Lakatos 1962).

Lakatos' contribution

The philosopher of mathematics who has contributed most to the maverick tradition is Imre Lakatos. He is responsible for both the negative thesis (the rejection of absolutism by fallibilism3) and the positive thesis (philosophy of mathematics needs to be reconceptualised to include the history and methodology of mathematics) of the maverick tradition. He argued, as above, for a fallibilist epistemology of mathematics on the ground that any attempt to find a perfectly secure basis leads to infinite regress, and mathematical knowledge cannot be given a final, fully rigorous form. As he put it "Why not honestly admit mathematical fallibility, and try to defend the dignity of fallible knowledge from cynical scepticism" (Lakatos 1962: 184)

It should be mentioned that my interpretation of Lakatos is controversial. The editors of Lakatos (1976) claim that he had or would relinquish fallibilism3. He undoubtedly did change his position over time, and my account is of early Lakatos, where he unequivocally states that although formalisation of theories and logic increases rigour "one had to pay for each step which increased rigour in deduction by the introduction of a new and fallible translation." (Lakatos 1978: 90). Lakatos would not support social constructivism as I describe it here. He believed mathematics is fallible3 but wholly rational and not at root based on social agreement or conversation.

Lakatos's (1976) best known contribution is his logic of mathematical discovery (LMD) or the method of proofs and refutations, which is a methodology of mathematics with three functions. First there is the epistemological function: to account for the genesis and justification of mathematical knowledge naturalistically, as part of his fallibilism3 (concerning mathematical knowledge in the timeless present). Second, there is the historical function: to provide a theory of the historical development of mathematics (concerning mathematical knowledge in the past). Third, there is the methodological function: to account for the methodology of practising mathematicians (concerning mathematical knowledge in the future).

Lakatos's LMD is a cyclic theory of knowledge creation in mathematics with a dialectical form, which may be represented as follows.

Given a mathematical problem (or set of problems) and an informal mathematical theory, an initial step in the genesis of new knowledge is the proposal of a conjecture. The method of proofs and refutations is applied to this conjecture, and an informal proof of the conjecture is constructed, and then subjected to criticism, leading to an informal refutation. In response to this refutation, the conjecture and possibly also the informal theory and the original problem(s) are modified or changed (new problems may very well be raised), in a new synthesis, completing the cycle. This is illustrated schematically in Table 1, which shows one complete step, and the beginning of the next step in the cycle.

Table 1: Cyclic Form of Lakatos' Logic of Mathematical Discovery

STAGE	CONTEXT	COMPONENTS OF CYCLE
Some Stage in Process	Problem Set	Conjecture
	Informal Theory	Informal proof of conjecture
		Informal refutation of conjecture
Next Stage	New Problem Set	New Conjecture
	New Informal Theory	

Although Lakatos (1976) is explicit about the role of the conjectures, proofs and refutations in this cycle, the part played by problems and informal theory are often implicit in his account. He does stress the role of problems in the development of mathematics in some places, such as at the end of his dialogue where the teacher states that "a scientific inquiry 'begins and ends with problems'" (Lakatos 1963-4: 336, quoting Popper).

Wittgenstein's contribution

In looking for guidance on how to develop a broader, more inclusive social philosophy of mathematics, a unique source and inspiration is Wittgenstein (1953, 1956), who proposed a revolutionary naturalistic and fallibilist social philosophy of mathematics, which to this day remains under-appreciated under-developed. (My interpretation of Wittgenstein is also personal, and likely to be controversial).

Wittgenstein's philosophy is based on his key concepts of 'language games': how we use language co-ordinated with our actions and embedded in and inseparably a part of 'forms of life': which are our historico-cultural practices. "The term 'language-game' is meant to bring into prominence the fact that speaking of language is part of an activity, or of a form of life" (Wittgenstein 1953: 11). "Mathematics teaches you, not just the answer to a question, but a whole language-game with questions and answers." (Wittgenstein 1956: 381)

His naturalism gives priority to existing mathematical practice, as Maddy (1990) concurs. "You don't make a decision: you simply do a certain thing. It is a question of a certain practice." (Wittgenstein 1976: 237). His philosophy is fallibilist, because he grounds certainty in the accepted (but always revisable) rules of language games (Rorty 1979). "The ma-thematician is not a discoverer: he is an inventor." (Wittgenstein 1956: 111). "To accept a proposition as ... certain means to use it as a grammatical rule: this removes uncertainty from it." (Wittgenstein 1956: 170) He argues that proof serves to justify mathematical knowledge through persuasion, not by its inherent logical necessity. "In a demonstration we get agreement with someone." (Wittgenstein 1965: 62). Thus Wittgenstein puts forward what can legitimately be termed a social constructivist philosophy of ma-thematics. He challenges foundationalism, rejects the universally adopted prescriptive approach of his day, and demands the reconceptualization of the philosophy of mathematics so as to be descriptive of practice.

RECONCEPTUALIZING THE PHILOSOPHY OF MATHEMATICS

As I have indicated, traditional philosophy of mathematics seeks to reconstruct mathematics in a vain foundationalist quest for certainty. But this goal is inappropriate, as a number of philosophers of mathematics agree: "To confuse description and programme - to confuse 'is' with 'ought to be' or 'should be' - is just as harmful in the philosophy of mathematics as elsewhere." (Körner 1960: 12), and "the job of the philosopher of ma-thematics is to describe and explain mathematics, not to reform it." (Maddy 1990: 28). Lakatos, in a characteristically witty and forceful way which paraphrases Kant indicates the direction that a reconceptualised philosophy of mathematics should follow. "The history of mathematics, lacking the guidance of philosophy has become blind, while the philosophy of ma-thematics turning its back on the...history of mathematics, has become empty" (1976: 2).

Building on these and other suggestions it might be expected that an adequate philosophy of mathematics should account for a number of aspects of mathematics including the following:

1. **Epistemology**: Mathematical knowledge; its character, genesis and justification, with special attention to the role of proof

2. **Theories**: Mathematical theories, both constructive and structural: their character and development, and issues of appraisal and evaluation

3. **Ontology**: The objects of mathematics: their character, origins and relationship with the language of mathematics, the issue of Platonism

4. **Methodology and History**: Mathematical practice: its character, and the mathematical activities of mathematicians, in the present and past

5. **Applications and Values**: Applications of mathematics; its relationship with science, technology, other areas of knowledge and values

6. **Individual Knowledge and Learning**: The learning of mathematics: its character and role in the onward transmission of mathematical knowledge, and in the creativity of individual mathe-maticians (Ernest 1997)

Items 1 and 3 include the traditional epistemological and ontological focuses of the philosophy of mathematics, broadened to add a concern with the genesis of mathematical knowledge and objects of mathematics, as well as with language. Item 2 adds a concern with the form that mathematical knowledge usually takes: mathematical theories. Items 4 and 5 go beyond the traditional boundaries by admitting the applications of mathematics and human mathematical practice as legitimate philosophical concerns, as well as its relations with other areas of human knowledge and values. Item 6 adds a concern with how mathematics is transmitted onwards from one generation to the next, and in particular, how it is learnt by individuals, and the dialectical relation between individuals and existing knowledge in creativity.

The legitimacy of these extended concerns arises from the need to consider the relationship between mathematics and its corporeal agents, i.e., human beings. They are required to accommodate what on the face of it is the simple and clear task of the philosophy of mathematics, namely to give an account of mathematics.

SOCIAL CONSTRUCTIVISM

Social constructivism is proposed as a philosophy of mathematics building on the ideas elaborated above with the aim of potentially or possibly addressing these six aspects or dimensions of an enlarged philosophy of mathematics. Social constructivism is based first of all on Lakatos' Logic of Mathematical Discovery for negotiation and acceptance of mathematical knowledge, concepts and proofs.

The second source of social constructivism is Wittgenstein's notions of 'language game' and 'forms of life'. Thus mathematical knowledge is taken to rest on socially situated linguistic practices, including shared rules, meanings and conventions, i.e. on both tacit and explicit knowledge and symbolic practices.

A central element of social constructivism is the reinterpretation of objectivity as social and intersubjective. Following Bloor (1984), Fuller (1988), Harding (1986) and others objective knowledge is understood as social, cultural, public and collective knowledge, and not as personal, private or individual belief, nor as external, absolute or otherwise extra-human.

A novel central feature of social constructivism is that it adopts conversation as the basic underpinning representational form for its epistemology. Thus this position views mathematics as basically linguistic, textual and semiotic, but embedded in the social world of human interaction.

Conversation

Beyond the metaphor of the 'great conversation' for philosophy and the history of ideas used by Michael Oakshott, conversation is taken as a basic epistemological form by Rorty (1979), Harré (1983), Shotter (1993), Gergen (1985), and many others.

> If, however, we think of "rational certainty" as a matter of victory in argument rather than of relation to an object known, we shall look toward our interlocutors rather than to our faculties for the explanation of the phenomenon. If we think of our certainty about the Pythagorean Theorem as our confidence, based on experience with arguments on such matters, that nobody will find an objection to the premises from which we infer it, then we shall not seek to explain it by the relation of reason to triangularity. Our certainty will be a matter of conversation between persons, rather than an interaction with nonhuman reality. (Rorty 1979)

So what is conversation? The original form is naturally interpersonal conversation, which consists of persons exchanging of speech, based on shared experiences, understandings, values, respect, etc. That is, it is language-games situated in human forms of life. Two secondary forms of conversation are derived from this. First, there is intrapersonal conversation, i.e., thought as constituted and formed by conversation. According to this view (verbal) thinking is originally internalised conversation with an imagined other (Vygotsky 1978, Mead 1934, see Figure 2 below). Second, there is cultural conversation, which is an extended version, consisting of the creation and exchange of texts in permanent (i.e., enduringly embodied) form. Indeed, it can be said that the reading of any text is dialogical, with the reader interrogating it and creating answers from it.

These three forms of conversation are social in manifestation (interpersonal and cultural), or in origin (intrapersonal). Conversation has an underlying dialogical form of ebb and flow, comprising the alternation of voices in assertion and counter assertion. Conversation is the source of feedback, in the form of acceptance, elaboration, reaction, criticism and correction essential for all human knowledge and learning. Thus the different conversational roles include the following two forms, which occur in each of the three forms, but originate in the interpersonal:

1. The role of proponent or friendly listener following a line of thinking or a thought experiment sympathetically, for understanding (Peirce, Rotman)

2. The role of critic, in which an argument is examined for weaknesses and flaws.

Taking conversation as epistemologically basic re-grounds mathematical knowledge in physically-embodied, socially-situated acts of human knowing and communication. It rejects the cartesian dualism of mind versus body, and knowledge versus the world. It acknowledges that there are multiple valid voices and perspectives on knowledge, which has significant ethical implications (cf. Habermas 1981)

The conversational nature of mathematics

My claim is that mathematical text is conversational, for the following reasons. Mathematics is primarily a symbolic activity, using written inscription and language to create, record and justify its knowledge (Rotman 1988, 1993). Viewed semiotically as comprising texts, mathematics is conversational for it addresses a reader. "In all cases the word is orientated towards an addressee" (Volosinov 1973: 85)

Analysis of mathematical texts, proofs, etc., reveals the verb forms to be in indicative and imperative moods. The indicative mood is used to make statements, claims and assertions describing the future outcomes of thought experiments which the reader can perform or accept. (Peirce, cited in Rotman 1993: 76) The imperatives are shared injunctions, orders or instructions issued by the writer to the reader. (Rotman 1988, 1993). Thus mathematical texts comprise specific assertions and imperatives directed by the writer to the reader. The reader of mathematical text is therefore either the agent of the mathematician-author's will, whose response is an imagined or actual action, or a critic seeking to make a critical response. In all cases the mathematical text is conversational.

The conversational structure of mathematical concepts

Dialogical / conversational processes also underpin a substantial class of modern mathematical concepts as the underlying meanings or possible interpretations.

Table 2: Dialogical concepts in Mathematics

TOPIC	DIALECTICAL CONCEPT
Analysis	ε -δ definitions of the limit: for each value...there is...
Constructivist Logic	Interpretation of quantifiers: $\forall x \exists y$... "You choose x, and I show how to construct y"
Recursion theory	Arithmetical Hierarchy - $\forall\exists\forall\exists\forall\exists$...
Set theory	Diagonal argument:for any enumeration, omitted element
Set theory	Game-theoretic version of Axiom of Choice
Game Theory	Alternation of moves by opponents
Number theory	J. Conway's game theoretic foundations of number
Statistics	Hypothesis testing (H_0 versus H_A)
Probability	Analysis of wagers, betting games

Table 2 illustrates the wide occurrence of dialogical or conversational concepts in mathematics, through which the formal interplay between persons, or a back and forth movement of choices is structured into the concepts themselves. (For more details see Ernest 1994).

Origins and basis of proof

In Ancient Greece proof developed from of cultural practice of disputation, i.e. conversation (Struik 1967). The term 'dialectic' derived from verb 'to discuss'. (Cornford 1935). So the origins of proof may be said to be conversational. In modern Proof Theory many developments also treat proofs as if part of a dialogue. For example, according to Heyting (1956), in intuitionistic mathematics every assertion is a promise to provide a proof. Such claims are valid only if the opponent convinced. Natural Deduction techniques likewise build proofs based on sets of assumptions or hypotheses agreed by both a proposer and an opposer. In the method of semantic tableaux, the version or proof constructed is an explicit attempt to refute the claim or story as put forward by another in dialogue. Lorenzen's conversational proof method is likewise based on two disputants(Roberts 1992). One tries to maintain a thesis over other's objections, and the connectives used have explicit conversational meanings. Overall, it can be said that both the beginnings of logic and proof and modern developments confirm that mathematical proof is at root dialectical, based in human conversation and persuasion.

Acceptance of mathematical knowledge is conversational

A widespread but controversial view is that the acceptance of mathematical knowledge and proof is social act, a conversational act, as illustrated in the quote from Rorty, given above. Thus "A proof becomes a proof after the social act of 'accepting it as a proof'. This is as true of mathematics as it is of physics, linguistics and biology." (Manin). The structure of a proof is a means to this epistemological end of persuading others, and ultimately the mathematical community, to accept it as a warrant for a theorem. Furthermore, the acceptance of a proof depends on largely tacit criteria and informed professional judgement, just as a teacher's decision to accept mathematical answers from a student depends on professional judgement. In both cases such judgements are based on criteria including the rhetorical style of the proposed item of mathematical knowledge, not just on rigid and explicit logical rules of correctness.

The acceptance of mathematical knowledge depends on dual roles developed and internalised through conversation. These are, first of all, the role of proposer of would-be new knowledge, or analogously, of a sympathetic reader or listener. The second role is that of a critical reader or listener, a reviewer, assessor, or gatekeeper. These roles are deployed in the generalised logic of mathematical discovery, which is the proposed conversational mechanism for acceptance or modification of mathematical knowledge. This is illustrated in Table 3.

Table 3: The Generalised Logic of Mathematical Discovery

<table>
<tr><td colspan="2">SCIENTIFIC CONTEXT for Stage n Background scientific and epistemological context: problems, concepts, methods, informal theories, proof criteria and paradigms, and meta-mathematical views.</td></tr>
<tr><td colspan="2">THESIS Stage n (i) Proposal of new/revised conjecture, proof, solution or theory.</td></tr>
<tr><td colspan="2">ANTITHESIS Stage n (ii) Dialectical and evaluative response to the proposal:</td></tr>
<tr><td>Critical Response
Counterexample, counter-argument, refutation,criticism of proposal</td><td>Acceptance Response
Acceptance of proposal. Suggested extension of proposal.</td></tr>
<tr><td colspan="2">SYNTHESIS Stage n (iii) Re-evaluation and modification of the proposal:</td></tr>
<tr><td>Local Restructuring Modified proposals: new conjecture, proof, problem-solution, problems or theory.</td><td>Globlal Restructuring of Context: changed problems, concepts methods, informal theories. Changed proof paradigms, criteria, meta-mathematical views.</td></tr>
<tr><td colspan="2">OUTCOME Stage n+1 (i)
Accepted or rejected proposal, or revised scientific and epistemological context.</td></tr>
</table>

The generalised logic of mathematical discovery is so-termed because it is a generalisation of Lakatos's logic of mathematical discovery (Table 1) to overcome the criticism that it does not describe the full generality of mathematical knowledge developments, including 'mathematical revolutions' (Gillies 1992).

It is clear that conversation and dialectical processes play a key role in the generalised logic of mathematical discovery shown in Table 3. The underlying logic is dialectical, and that this underpins the genesis and warranting of mathematical knowledge. In this process mathematical proofs and other proposals are offered to the appropriate mathematical community as part of a continuing dialogue. They are addressed to an audience, and are tendered in the expectation of a response, drawing upon tacit and professional knowledge. Critical scrutiny of a proof by the mathematical community leads to either (a) criticism, requiring development and improvement (concerning the context of discovery), or (b) acceptance as a knowledge warrant (concerning the context of justification). The same conversational logic of mathematical discovery is at work in both cases. There can be no proofs in mathematics which are above critical scrutiny and this logic, no matter how rigorous. Thus mathematical proof has not only evolved from dialogical form, but its very function in the mathematical community as an epistemological warrant for items of mathematical knowledge requires the deployment of that form.

However, a word of caution is needed. Although mathematics is claimed to be at root conversational, it is also the discipline par excellence which hides its dialogical nature under its monological appearance, and which has expunged the traces of multiple voices and of human authorship behind a rhetoric of objectivity and impersonality. This is why the claimed conversational nature of mathematics might seem surprising: it is the exact opposite of the traditional absolutist view of mathematics as disembodied and superhuman, critiqued above.

See figure next page

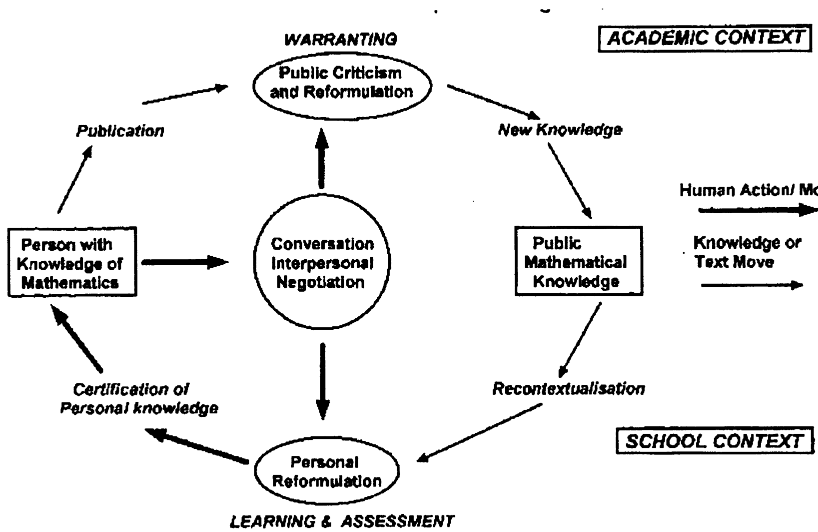

Figure 1: *The Cyclic Mechanism for the Social Construction of Mathematical Knowledge*

Figure 1 summarises the social construction of mathematical knowledge in the contexts of research and schooling, and shows how they are interrelated in an overall cyclic mechanism, and the role of conversation and negotiation in each. In the context of research mathematics, individuals use their personal knowledge both to construct mathematical knowledge claims (possibly jointly with others), and participate in the dialogical process of criticism and warranting of others' mathematical knowledge claims. In the context of mathematics education individuals use their personal knowledge to direct and control mathematics learning conversations both to present mathematical knowledge to learners directly or indirectly (i.e. teaching), and participate in the process of warranting and criticism of others' mathematical knowledge claims or performances (i.e. the assessment of learning). Ultimately, individuals emerge from this process with their personal knowledge warranted or certified, and may therefore be able to participate in these conversations as teachers or mathematicians, after further professional preparation.

As figure 1 illustrates, the mechanism for the social construction of mathematical knowledge has the form of a cycle. What travels for part of the cycle is embodied mathematical knowledge. This is represented publicly by mathematicians as a text, and after possible modification is then approved, if it passes muster, and then becomes part of the pool of accepted knowledge representations. Selections from this pool are recontextualised into the school context where they are offered to learners. Learners appropriate and internalise this knowledge, with a greater or lesser degree of personal reformulation (see Figure 2 below). Mathematical knowledge is now embodied in the skills and dispositions of individuals, i.e., as personal knowledge, and when these individuals are certified as knowledgeable, the cycle is completed. For potentially they can now participate in the construction of new mathematical knowledge, as research mathematicians, or can help the development of the personal knowledge of others, as teachers of mathematics.

The cyclic pattern of the social construction of mathematical knowledge finds a parallel in Harré's (1983) social model of mind (also referred to as Vygotskian space, in Shotter 1991). This also sees thought or knowledge cycling as it alternates between public and private manifestations, and between individual and collective social locations. This is illustrated in Figure 2.

Figure 2: Vygotskian space - Harré's social model of mind

	INDIVIDUAL	SOCIAL LOCATION	COLLECTIVE
Public Manifestation	Public & Individual Mind/Knowledge	Conventionalisation → (2)	Public & Collective Mind/Knowledge
	Publication ↑ (1)		↓ Appropriation (3)
Private Manifestation	Private & individual Mind/knowledge	←Transformation(4)	Private & Collective Mind/Knowledge

The cycle in Figure 2 is orientated differently from Figure 1, so the analogy is clarified by stating that the process of publication (marked 1) corresponds to the assessment and warranting of individual knowledge and learning, whereas the process of conventionalisation publication (marked 2) corresponds to the assessment of a new contribution to public knowledge. The process of appropriation (marked 3) corresponds to the recontextualisation of accepted mathematical knowledge into the context of schooling, and its offering to learners. The process of transformation (marked 4) corresponds to learner appropriating and assuming ownership of the knowledge as individual and personal knowledge.

Clearly this model has strong parallels and potentially strong implications for the teaching and learning of mathematics. There are many more aspects of the social constructivist theory which there is not space to report her, but which were indicated in the talk. These include an extended model of mathematical knowledge based on Kitcher (1984) and Kuhn (1970) incorporating both tacit and explicit knowledge elements. There is the important role of rhetoric in both research mathematics and school mathematics. There is the parallel with socially situated views of learning which are becoming more widely accepted in psychology and mathematics education. These are all treated in Ernest (1997), the source from which this article is a condensed selection.

Conclusion

In conclusion, it might be claimed that the novelty of social constructivism is to realise both that mathematical knowledge is necessary, stable and autonomous, but that this co-exists with its contingent, fallibilist3, and historically shifting character. As Vico said, with regard to geometry: the only things we can know completely are those we have made (because there is nowhere else to look than within our own construction).

In addition, social constructivism links both the learning of mathematics and research in mathematics in an overall scheme in which knowledge travels either embodied in a person or in a text, and the processes of formation and warranting in the two contexts are parallel. Social constructivism provides an explanation of how mathematics and logic seem irrefutably certain, yet are contingent, historical creations. The full account also offers explanations for how the objects of mathematics are cultural fictions emerging from the use of mathematical language and symbolism, yet seem so solid; and how mathematics is so unreasonably effective in providing the conceptual foundations of our scientific theories about the world. The central explanatory concept is emergence: the evolutionary history of culture and the individual, and the shaping role of conversation. However adequate arguments and explanations go beyond what I can offer here. This talk is based on my current book Ernest (1997), but other relevant publications are Ernest (1991, 1992, 1994).

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THE RUSSIAN STANDARDS: PROBLEMS AND DECISIONS

Victor Firsov

Alongside with space rockets and Bolschoy ballet, School Mathematics could be referred to the category of "sacred cows" of modern post-soviet Russia: it is accepted to be proud of them and not to criticize them. It is obviously no more than national myth that had its basis in achievements of the Past and that does not require today's confirmations. Typical for periods of social crisis the nostalgia on the Past promotes the creation of similar myths. At the same time mythological consciousness is dangerous for corresponding areas of human activity because it idealizes the Past and does not promote their development with taking into account the realities and the needs of modern Russia.

The critical analysis allows to see that some of past advantages of Russian School Mathematics are disappeared. Other ones have lost their significance in new system of school coordinates. Third ones have turned to the denying. In result the brilliant picture at one time has grown dull. The superiority myth becomes similar to an old mirror showing the image appreciably more attractive in comparison with the reality.

Large volume and high theoretical level of Mathematics courses were considered always as traditional advantages of Russian School. The question if it is necessary to each student, sounds increasingly stronger in modern conditions of greater freedom and greater openness of school to social demands. The opponents put forward the requirement of resolute humanitarization of school. They challenge the necessity of exceedingly ambitious goals of school mathematical education. They also ascertain significant break between the advanced requirements of the Mathematics Curriculum and the real achievements of the majority of the schoolchildren.

The officious soviet propaganda ignored break mentioned by the practice of overestimating minimum positive marks (writing "3" and keeping in mind "2"). Actually the data of researches of the Institute into Content and Methods of Education, USSR Academy of Pedagogical Sciences on check of mathematical preparation of the schoolchildren [1]* shows that from 30 up to 50 % of the pupils did not take possession of

*The numbers in parenthesis refer to the list of references at the end of the text.

basic Mathematics skills that are minimal necessary for continuing of education. There is no basis to hope that the today's situation is appreciably improved: the Curriculum and the textbooks have not practically changed, and, say, the indicators of health of the children fall down even in comparison with unsatisfactory ones of the last Soviet years [2].

Critics of Russian School Mathematics condemn excessive, by their opinion, enthusiasm for the formal goals of education to the detriment of real ones. Moreover in a part of achievement of the real goals of Mathematics Education the priority is given back to development of the technical apparatus instead of direct application of Mathematics to the solution of real practical problems.

The results of international comparative research like IAEP and TIMSS are indicative in this relation despite all their ambiguity. So, the Soviet Union looked not bad by results of IAEP [3]. The more detailed analysis [4] of test groups' results has compelled to change such evaluation. There was found out that Soviet students did not surpass practically the pupils from other countries in fulfilment of the tests requiring the understanding of Mathematics concepts and its practical application. Significant advantage was formed because of higher results on development of the technical part of Mathematics. Thus, by the price of serious expenditures of educational time and overloading of the children by mathematical technics it becomes possible to reach comparable results on those components of Mathematics that are most important from positions of General Education. We shall agree that even the opportunity of discussion of similar conclusion puts a question on efficiency of Soviet model of School Mathematics Education.

This model was developed for specific historic conditions of the period of Soviet industrialization in 30-ties. The model oriented to preparation of the future engineers at the system of high technical institutions. Till now each pupil in Russia is trained the Mathematics as if her (his) hot desire to receive the engineering diploma (and Soviet diploma of 30-ties namely) is known beforehand.

It effects the selection of School Mathematics content and style. It explains the orientation of Soviet School Mathematics to development of refined technical apparatus of transformations of algebraic and transcendent expressions and of algorithms for precise solution of the equations and inequalities appearing archaic today. It leads to the orientation of a subject to development of continuous Mathematics and to suppression of discrete Mathematics. Just mention the absence of such topics as combinatorics, data analysis and probability, elements of mathematical logic in modern Mathematics Curriculum for Russian School [5].

It is possible to continue the critical analysis of School Mathematics content. However, even if to take into account disputability and ambiguity of questions discussed above, we are nevertheless compelled to ascertain the necessity of Curriculum reform and it's reduction to greater conformity with the needs of our times.

The analysis of usual practice of teaching and learning Mathematics leads to similar conclusions [6]. It's orientation to needs of strong highly motivated learner (future student of university or technical institute) and neglect of interests of low-learners are the serious lacks. The aspects of School Mathematics as communications development and cooperative learning that are necessary for low-learners are developing unsatisfactorily. Assessment system imposes the penalties for non-achievement by the student of a level of the advanced requirements. Assessment system works under the circuit of "subtraction" that is opposite to cumulative assessment.

Unfortunately, the Russian Mathematics community does not welcome to discuss these questions. Powerful lobby of higher schools' professors and of Math educators works under the slogans "to increase the number of academic hours to study Mathematics at school" (one of the greatest in the world), "to raise the level of mathematical preparation of students" (meaning the future students and completely ignoring a consequence for other following to corresponding actions), "to keep high traditions of National Mathematics Education" (understanding it as the necessity of preservation of Soviet model). Honestly, there is hidden disrespect to the children (their rights not to love Mathematics are rejected resolutely) and to the Mathematics (educational and cultural values of School Mathematics for each pupil of school are declared on words whereas the course is designed actually according to the needs of the best pupils only).

Obviously it becomes impossible to carry out the contradictory social order within the framework of unique model of School Mathematics Education. The decisions conducting to a greater variety of Mathematics Education were found in conditions of Soviet school yet.

The development at schools since mid-30-ties of the system of voluntary mathematical circles and olympiads for students has become the most efficient of them. The names of such outstanding mathematicians and popularizators of Mathematics as I.Gelfand, I.Perelman and I.Yaglom are connected with the movement. Just this system has caused a large part of our mathematical elite. Just to it our country is obliged by outstanding results shown by the Soviet participants at International Mathematical Olympiads.

The schools with advanced studies of Mathematics at the high secondary stage are more known in the world. The names of A.Kolmogorov, A.Lyapunov and S.Shwarzburd are connected to its origin. Arising since 1959 as more systematized variant of mathematical circles, passing through period of official disallowance and fight for the existence, they have received "the rights of citizenship" to the end of Soviet epoch. Such schools become today the most popular model for “streaming” at high secondary school not only in a direction of Mathematics already.

Unfortunately significant overloading of pupils and teachers alongside with unreasoned state policy in the field of teacher's salaries have resulted to practical liquidation of mathematical circles as of a mass movement today. It is not enough completely to have the opportunities of diversity connected with streaming at high secondary school only.

The slogan of diversity as bases of organic development of school has an exclusive importance for modern Russia. Construction of a democratic civil society and market economical relations require freedom and pluralism and exclude the presence of "only right decisions". On the other hand, the positive achievement of Russian school that are answering to the realities of today's life should be protected from unreasonable innovations. Thus, new school appropriate to principles of unity and diversity [7] should come on change of extremely unified Soviet school that brought up the "small screws" of big state machine (according to metaphor of J.Stalin). These complementary and cooperating principles are called to ensure the evolutionary character of school’s development just as heredity and variability provide biological evolution.

The development of new Russian School requires the creation of new generation of normative documents ensuring the goals. New Law of Russian Federation on Education named these documents as Educational Standards. The law provides that Educational Standards should determine “the compulsory minimum of the contents of Education”, “the requirements to the levels of preparation of students graduated the stages of school” and “maximal allowable volume of study load of children” [8].

We shall notice at once that pedagogical public of Russia has apprehended extremely sensitively the term "standards" borrowed from American experience. For people in our country the word "standard" associates with compulsory uniformity of the Soviet times.

Really the goals of the standards' implementation in USA and in Russia look opposite diametrically. It seems that the authors of the project of National Standards [9] aspired to come to common understanding of the goals and objectives of School Mathematics Education in conditions of a superfluous variety of American schools. The similar problem does not arise at all in Russian school. Opposite, we here need more de-standardization of Education.

The collision of various interests and positions accompanies with the whole short history of creation of Russian Educational Standards. We shall designate basic conceptual "bifurcation points" where the positions of the parties differ essentially.

First and probably main distinction consists in understanding of the basic purpose of the Standards. The supporters of one position see it in maintenance of unity of Russian school [10]. Their opponents consider the maintenance of unity of school as one of a list of necessary conditions only. They specify priority purpose of the Standards as being the instrument of school's development [11].

The marked divergence has no scholastic character. It is directly connected with opposition of conservative guarding tendencies to the movement for reforming of Russian School that was described above. It is relevant to notice that this conflict has obvious political colour in today's Russia. If to finish up the first considered position to its logical end, the best decision will appear as return to the unified typical Curriculum of Soviet School, and complete refusal from a variety of the educational programs as a consequence.

Searches of ways of realization of a contrary position result to a new point of opposition. The conceptual divergences concern here to the ways, with the help of which future Standards would determine the directions of school's development.

Traditional approach (perceived often as only possible) supposes the setting of educational goals through the system of Educational Standards. In this case the standards are called to determine directly the major elements of Educational System, imposing them to school.

The opponents of the traditional approach remark that anybody cannot apply for knowledge what school should be. The achievement of public consensus in a question on the goals of Education looks perfect Utopia. In this context in Russia they always recollect the words from the song of outstanding poet and dissident A.Galich:

Do not fear of plague, do not fear of prison,
Do not fear of stench and of hell.
But fear of person only that could tell:
"I know as it should be!"

Within the framework of the alternative approach standards take a role similar to the role of the legislative restrictions in a lawful state: the standards specify what is forbidden to do. Thus, there are the conditions for realization of a democratic principle "it is allowed everything that it is not forbidden."

This principle means real freedom for schools, and apparent paradox only consists that freedom is provided by means of the interdictions. The standards become the expression of the similar interdictions concerning the content of Education at School. With their help the principle of "allowance" receives non-declarative mechanism of its realization.

Thus, in the alternative approach the standards set "a field of freedom" in a choice of the content of Education; freedom results in occurrence of various educational programs; the variety of the programs provides an opportunity of their choice by the parents and children. Through this choice the society would influence schools showing them people's educational needs and preferences. These needs and preferences will find reflection in the contents of the next variants of the standards, and so on... The democratic interaction of society and school will come true through the standards by such way.

Certainly, the serious discussions arise in the relation of the content and of the forms of standards presentation. These discussions go often around of some basic questions.

Whether It is necessary to adjust the content of Education that should be taught or which should be acquired by each student? Here the authors of the alternative projects took the same position: it should be both standards of "teaching" and standards of "learning". However, the different circuits of the forms of standards representation are selected with own advantages and lacks.

Should the standards be oriented on today or on prospect? In the first case the standards will easily enter School, but will fix those minuses which we aspire to take away from School. In the second case the standards will allow to specify urgent directions of development of School (for example, so necessary unload of compulsory educational content). At the same time such standards appear poorly connected to the textbooks available of schools. It is necessary to admit that this question did not find its satisfactory solution yet.

How to co-ordinate correctly federal and regional interests? How to distribute the responsibility for standards execution in appropriate way? How to alter the corresponding systems of assessment and control?

The search of the answers on these and similar questions continues. It is necessary to notice that in the field of Mathematics Education the alternative projects of the standards appear rather alike. Obviously it is connected to circumstance that in the field of School Mathematics the whole necessary way was passed much earlier in comparison with other subjects. The standards of School Mathematical Education in understanding as above entered into Soviet School already in 1982. Now the specification of the earlier entered standards comes true unless. Thus, our final conclusion is that existing projects of Standards could not ensure the required reform of Russian School Mathematics. The genuine reform is still ahead yet.

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HIGHLIGHTS AND SHADOWS OF CURRENT JAPANESE NATIONAL CURRICULUM OF MATHEMATICS FOR SECONDARY SCHOOLS

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§1. Introduction

Firstly, focusing on SHS (Senior High School) mathematics curriculum, we describe the underlying philosophy, basic strategy, courses and their contents of the current Japanese national curriculum, which is called the Course of Study (CS). Actually, the author was involved in organizing this CS as a member of Council for Educational Curricula. Reports on initial experiences of implementation are mentioned.

Secondly, as a newly arising difficulty in mathematics education in Japan, we analyze recent tendency of Japanese youths' disinclination for mathematics as well as for science and technology, which has become a matter of national concern. It is our belief that the underlying idea and methods of the current CS will be effective also to overcome this crisis of Japanese mathematics education.

While some points of our discussion are specific to Japan, we think that the other points are of international interests and concern, at least, in some countries.

The paper is composed of 7 sections. In §2, we sketch the Japanese educational system, the process of text book authorization and the process of revision of CS. The second half of §2 is devoted to a description of the necessity of reform and the chronic issues of Japanese mathematics education which the current CS is intended to meet, particularly, the polarization of students, and backwardness in school use of computers.

In §3, we state our philosophy that the purpose of mathematics education (at the secondary level) is to cultivate students' mathematical intelligence with the two foci targets; fostering ML (Mathematical Literacy) and enhancing (MT (Mathematical Thinking Power). In order to be compatible with Japanese social tendency that dislikes apparent differentiations and, at the same time, in order to increase flexibility of curriculum, we have adopted an original structure, the so-called Core and Options curriculum (COM) structure as explained also in §3.

§4 gives a brief description of courses and topics of the current CS to be taught as the core or as optional modules. In fact, the program as a whole is a restricted realization of COM.

Reports on initial experiences of implementation of the current CS are mentioned in §5.

§6 is a brief analysis of Japanese crisis of mathematics education mentioned above. Serious symptoms are students' (even élite students') lowering of scholarship and running away from deep thought in mathematics. Possible causes of this crisis and related sociological backgrounds are mentioned, including the demographic factor that the number of Japanese youths is constantly decreasing.

In §7, this paper is concluded by a description of directions of our future efforts to improve mathematics education, including suggestions for the coming revisions of CS.

§2. Educational System in Japan

The Japanese school system is similar to the American one. Namely, its main body is of the 6-3-3-4 structure, while it is preceded by Kindergarten and is followed by the graduate school. Statistics for 1995 are:

school	years of study	type	enrollment ratio
Kindergarten (K)	1 – 3	volunteer	63.2 %
Elementary school (ES)	6	compulsory	99.99 %
Junior high school (JHS)	3	compulsory	99.99 %
Senior high school (SHS)	3	volunteer	96.7 %
University (Colleges) (UG)	4 (2)	volunteer	32.1 % (13.1 %)
Graduate school (GS)	2 + 3	volunteer	9.0 %

Remark

In Japan, Jyuku, (informal learning institute) is flourishing. Approximately, 20 % of pupils of elementary schools and 50% of students of junior high schools take evening and / or weekend courses in Jyukus for extra learning.

2.1. Course of Study

The contents of teaching in Japanese schools from Kindergarten to SHS are strongly controlled by the Course of Study (CS) issued by Monbusho (Ministry of Education, Science and Culture). In recent years, CS has been totally reorganized approximately every ten years. Actually, the current CS was announced in 1989, while previous revisions of CS had been made in 1978 (post-modernization), 1970 (modernization), 1969, 1955, · · · . Textbooks for school use must be written according to the CS and have to undergo authorization by Monbusho.

Usually, the formal process of revision of the CS starts with forming the Council of Education Curricula (CEC) and with an inquiry from the minister of Monbusho to the Council, which are often preceded by the minister's receipt of recommendations from a superior council, the Central Council of Education. After the answer of CEC is submitted, working committees for each school subject are formed and work out the new CS, which is then announced and put in force by Monbusho. Then publishers create textbooks in accordance with the new CS and submit them for authorization by Monbusho.

Thus, it takes several years to begin school teaching according to the new CS after its announcement. Actually, the current CS was announced in 1989, and firstly implemented 1992 with ES, 1993 in with JHS and in 1994 with SHS.

2.2. Recognized issues

The following issues were recognized as basic problems to be met by the new CS, when we started our informal and then formal preparation for organization of the current CS in 1980's.

1) *The nearly saturated high advancement rate to SHS,* which reaches 95% or over. This implies lowering and diversity of scholarships among the majority of students.

2) Practically and concretely, the mathematics part of the new CS has to meet *the polarization of students* into the following two groups;

• Students of better aptitude who are bound for university education, professional career and who need strength in mathematics.

• Majority of students who need mathematical literacy to live and work as intellectual citizens.

3) *Retarded state of use of computers* in mathematics teaching and learning has to be recovered. In spite of popularity of computers in Japanese society (workplace as well as home), introduction of computers in school education had been difficult without formal approval for it by CS.

Remarks

1) Generally speaking, the Japanese mathematics teachers are good in regard to the academic background. All of them are graduates of 4 years universities or higher.

2) The entrance examination to prestigious universities, good SHS, and excellent private JHS is very competitive. It exerts a strong driving force for students to work hard as well as distortion of sound learning of mathematics.

§3. Guiding Principles and Strategies

Our basic philosophy and strategies in organizing the current CS are as follows. Incidentally, in addition to publication of papers in references, these principles and strategies were orally presented by the author to the international community of mathematics education on occasions like ICME–5, Adelaide (1984), UCSMP & NCTM Conferences, Chicago (1988), ICME–6, Budapest (1988), ICMI Sessions in ICM'90, Kyoto (1990).

3.1. A historical view point

We claim that the current progress of mathematical sciences, where the new applied mathematics plays the core role, should be recognized as the fourth peak in the history of mathematics to follow the preceding three, namely, the birth of Euclidian geometry, the discovery of infinitesimal calculus and formation of abstract mathematics. Furthermore, the coming fourth peak of mathematics shares basic features with the second peak in the sense that they are both characterized by a vivid expansion of concepts and methods and by rich applications. Mathematics education must reflect these trends of mathematics.

3.2. The purpose of mathematics education

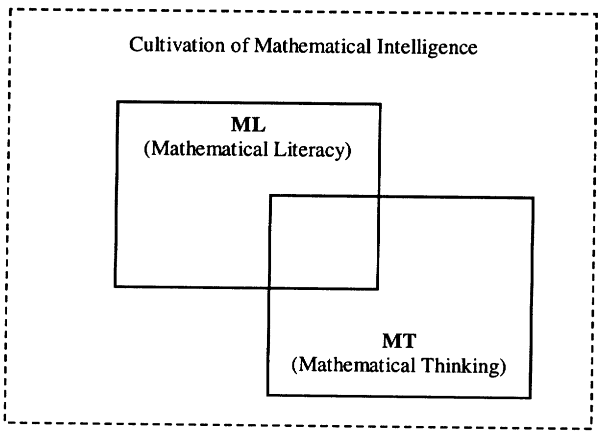
We have reviewed the purpose of mathematics education at SHS level and reached the assertion as follows. Particularly, as for SHS, the purpose of mathematics education is to cultivate mathematical intelligence of the students through the two-foci targets: namely, by fostering their

ML = mathematical literacy and *MT = mathematical thinking power.*

These two targets should be pursued in an appropriate balance depending on individual schools, classes or individual students and in view of the students' intended career and aptitudes.

To be a little more specific, we claim that

ML = mathematical competence of intellectual citizens
= mathematics for intelligent users,
MT = mathematical potentiality for future career.



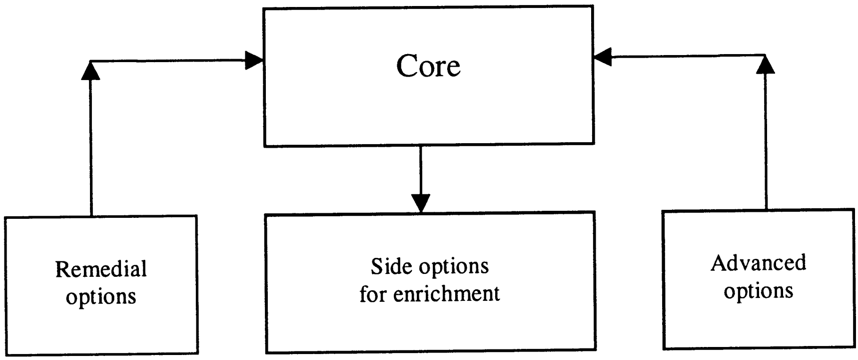
3.3. Flexibility curriculum

As a fundamental strategy, we have adopted the curriculum structure of *Core and optional modules (COM),* which was originally proposed by Prof. F. Terada.
Original structure of COM

Conceptually, the mathematics curriculum of the core-options structure is designed as follows.

- Math curriculum = { core } $\cup$ { option modules }
- Core = standard math-competence of intellectual citizens
- Option = { Remedial options } $\cup$ { advanced options }

– Core-Options diagram –



Intended merits of COM

1. Flexibility of curriculum to meet diversity of students without appa rent differentiation.
2. More chances of self-adjustment.
3. Compatible with the two foci goals, ML and MT.
4. Compatible with introduction of computers and calculators into school mathematics.

§4. Current Courses of SHS Mathematics

The current SHS in accordance with 1989-CS is a restricted realization of COM, and is actually composed of courses as described below.

4.1. Courses

Course	Requirement	Standard Units	Structure	Year
Math I	compulsory	4	Omnibus	1st
Math II	selective	3	Omnibus	2nd
Math III	selective	3	Omnibus	3rd
Math A	selective	2	Option modules	1st
Math B	selective	2	Option modules	2nd
Match C	selective	2	Option modules	3rd

Remarks

As mentioned above, this program of the current CS can be viewed as a restricted realization of COM curriculum. Such restriction was necessary to avoid the impact of drastic change, to be feasible in the customary way of closed class teaching, to be practically compatible with the entrance examination to universities and finally to be subject to control by Monbusho through CS.

According to the current CS, we regard that the core for general students is

$$\{MathI\} \cup \{MathII\},$$

while the core for science and technology bound students is

$$\{MathI\} \cup \{MathII\} \cup \{MathIII\}.$$

The option modules are contained in Math A, Math B, Math C. If a student takes one of these courses, he/she is normally required to learn two of the modules of his / her choice among four modules belonging to each course. On the other hand, Math I, Math II and Math III are of integrated omnibus structure: If a student takes one of these courses, he/she must learn all topics in it.

It is emphasized that Math C should be learned with due use of computers and from the view point of applied mathematics.

The arrangement of topics through the whole program is not very systematic. Some of necessary knowledge to learn a topic or module may have to be prepared "on the spot" when needs come up.

4.2. Contents of courses

Math I	Math A
1) quadratic functions	1) numbers and expressions
2) figures and measurements, simple trigonometry included	2) plane geometry
3) number of elements and cases combinatorics	3) numerical sequence
4) probability	4) computation and computers

Math II	Math B
1) various functions e^x, $\log x$, trigonometric functions	1) vectors
2) figures and equations straight lines and circles	2) complex numbers and complex plane
3) change of values of functions simple differentiation and integration	3) probability distributions
	4) algorithm and computers

Math III	Math C
1) functions and their limit 2) differential calculus 3) integral calculus	1) matrices and linear computation 2) various curves including conics and polar coordinates 3) numerical computation 4) statistical processing

§5. Report on Initial Implementation

Students to graduate from SHS in 1997 March are the first runners who have studied according to the current CS. Hence, reports only on the initial experiences of implementation of CS are available now.

5.1. Statistics of selective courses and modules

1) Selective courses are shown below by percentages of students who take them. For instance, since Math I is compulsory, its percentage is 100 %, while Math C is taken only by 19.8 % of students who are mostly bound for science and technology.

Math I	Math II	Math III	Math A	Math B	Math C
100 %	86.1 %	21.4 %	71.6 %	43.9 %	19.8 %

2) *Selective modules* taught are shown by percentages of schools which offer teaching of them. Statistics for Math C is not available yet in 1996.

Math A

Number and expression	Plane geometry
91.4 %	25.6 %

Numerical sequence	Computation and computer
81.9 %	6.4 %

Math B

Vector	Complex number and computer plane
76 %	70 %

Probability distribution	Algorithm and computer
25.0 %	3.2 %

5.2. Reports and comments

The following is a part of various reports and comments with appreciation or criticism from initial experiences of the CS.

1) *Text books:* Reliable but somewhat conservative text books have been created to match the new CS.

2) *Response of students:* In many schools, Math I matches the majority of students.

3) *Appreciation by teachers:* The flexible structure of the new CS is stimulating for enthusiastic or active teachers, while the non-systematic arrangement of topics is not liked by some conservative teachers who are mostly senior.

4) *Entrance examinations:* Notwithstanding the difficulty caused by the increased flexibility of curriculum, the Center of University Entrance Examination has decided to pose a full menu examination including all modules of the selective subjects Math A and Math B.

On the other hand, however, most of the prestigious national universities are not very cooperative with the new CS and have flatly designated two modules to be tested from each of Math A, Math B, Math C, in particular, excluding the computer related modules. This exerts a strong negative influence on SHS, and makes it difficult for SHS to implement the CS as it was intended.

5) *Computers:* Computer-related modules are rarely offered to teach a SHS, although many SHS recognize importance of these modules and the hardware environments are not poor nowadays, probably because teachers are not well prepared, and, as mentioned above and because of exclusion of these modules from the range of mathematics problems at the individual entrance examination to many universities. Nevertheless, a number of conscientions universities have announced to include computer modules to the range of their examinations.

6) *Response of university professors:* Except for those who are particularly concerned with the secondary education since before, university mathematics professors feel that the current CS is too non-systematic to lose the characteristic nature of mathematics, and is inconvenient practically in setting problems of entrance examination. Some of them hate the increased teaching burden caused by the acceptance of students with diverse backgrounds which is a consequence of new CS.

On the other hand, because of the recent issues which they seriously experience in teaching mathematics of the university level, a number of university professors express their support of the underlying idea of the new CS.

§6. Crisis of Mathematics Education

In Japan, youth's disinclination for study of science and technology as well as for study of mathematics has become a serious matter of concern. As for mathematics, on July 2, 1994, presidents of academic societies of mathematics and mathematics education made a public appeal, "Crisis of Mathematical Education in Japan" in order to call for measures and efforts to meet this difficulty. Furthermore, the ICMI national committee of Japan chaired by Prof. S. Iitaka sent a letter on July 15, 1995 to Professor A. Arima, President of the Central Council of Education, requesting to incorporate necessary measures to resolve this serious issue when the Council works out their recommendation to the minister of Monbusho.

6.1. Some symptoms

Particularly in regard to mathematics, the following symptoms of the crisis are observed:

1) *Lowering of scholarship of university students,* remarkable even with engineering students of prestigious universities.

2) *Lowering of quality of learning* is a matter of concern. Children and students opt to avoid mathematical problems which require deeper thinking. Even when they earn some good scores in mathematics, children are poor with verbal problems and students hate problems involving proof.

3) *Decay of popularity of mathematics among élite students* should be noted, although the decrease in number of SHS students who take selective mathematics courses is not drastic yet.

In suffering from students' falling away, mathematics is not alone. Japanese youths with better aptitude prefer, as their majors, fields of literary or social sciences to science and technology, although nobody denies the importance of science and technology in the coming era.

At SHS, physics is learned only by 20% of students, while physics was taken by 80% of students some decades ago. As reasons for this tendency, we may refer:

- Decay in youths of traditional cultural backgrounds like diligence and perseverance.

- Youths dislike sustained efforts which S&T (Science and Technology) requires.

- Freshmen's wish to relax and enjoy after competitive entrance examinations is difficult to realice with S&T.

- Scientists and engineers are not well paid in comparison with bankers and commercial managers. They are unhappy in companies when their knowledge becomes obsolete.

- Facilities and learning environments of S&T in universities are poor.

- Curriculum for science at the secondary level may be too stiff to kill students' fondness of science.

6.2. Demographic factor

The number of 18 year - old in Japan decreases sharply after the peak in 1992.

Number of 18 year - old youths. (Unit = 10,000)

1980	1985	1986	1990	1991	1992	1993	1994	1995	1996
158	156	185	201	204	205	198	186	177	172
1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
167	161	153	151	150	147	141	137	133	130

This decrease in size of this age group has favourable as well as bad influences on school education. For instance, the number of children in a classroom may be reduced. The competition at the entrance examination will be relaxed except for those who aim at very prestigious schools and universities. On the other hand, however, new employment of young and active teachers will become difficult, because the educational budget depends monotonously on the number of pupils and students. Furthermore, universities of medium level may exclude mathematics from their entrance examination in order to attract more applicants who do not like mathematics, and thus discourage mathematics learning.

6.3. Appeal by presidents of societies

As mentioned above, the presidents of the following societies signed the Appeal *"Crisis of Mathematics Education"*: Mathematical Society of Japan (S. Iitaka), Japanese Society of Mathematical Education (T. Uetake), Japan Society for Industrial and Applied Mathematics (H. Fujita), Society of Mathematical Education (K. Yokochi).

Items Suggested in the Appeal are:

1. More school hours, particularly in Junior high schools, must be alloted to mathematics.
2. Ample mathematical literacy must be fostered in all students through enjoyable (not painstaking) approaches.
3. Study of coherent curricula of mathematics with an integral view over all grades and school levels is needed.
4. Active and enjoyable learning of mathematics by students should be promoted with active use of computers.
5. More mathematics teachers with a lively mathematical sense must be brought up and employed.
6. The weight of mathematics at entrance examinations should not be reduced, while improvement of its problem-setting must be done increasingly.

§7. Required Further Efforts

In the coming reorganization of CS, which is to be started during 1996 by forming the Council of Educational Curricula, we have to try to achieve the following:

1. Pursuit of the objectives of the current CS with necessary amendments of the content.
2. More active use of computers and graphic calculators into each topic in mathematics.
3. Review of JHS mathematics, which used to be too uniform with little selective components. We have to re-organize the curriculum so that even in this period of compulsory education, we can foster germs of various abilities of individual students, and on the other hand, we do not force the majority of students to learn too sophisticated topics beyond their aptitude.
4. Re-examination of teaching topics to meet the 5-days (per week) school system.
5. Organize curriculum with ML-MT targets which is properly applicable to the group of stronger (even gifted) students as well as the majority of students, provided that their teachers are qualified and well-prepared.

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ISSUES IN ASSESSMENT: A NEVER ENDING STORY

Peter Galbraith

Let us begin by agreeing that winds of change have been blowing for some time in the world of mathematics assessment and that these winds vary in strength from gentle zephyrs to potential hurricanes. Let us recognise that shifts in emphasis have occurred and are occurring; for example a move away from sole reliance on formal testing; the introduction of so-called coursework assessment involving project work and other activities of an open nature; an increase in the number of countries, states and territories that involve teachers actively in assessment activities; and an increased consciousness that assessment should form an integral part of teaching/leaming transactions.

Simultaneously let us acknowledge, in apparent conflict with some of the above, moves towards system wide 'accountability' at various levels of schooling through the agency of national or state mandated tests of mathematical 'ability' or 'competence'. Rather obviously the different 'winds' have sources in belief systems that themselves vary in what they value most.

It seems of doubtful benefit in a paper such as this to discuss the detail of various assessment instruments or methods, existing or developing, or the efficacy or efficiency with which such instruments might be administered, for inevitably any given choice will be outside the current interests of some countries and systems, and hence by definition peripheral to some potential readers. There is an expanding source of reference material that may be consulted with respect to such developments, for example (Niss, 1993(a) and (b); Romberg, 1992; Gifford and O'Connor, 1992; Leder, 1992; Grouws, 1992; as well as a variety of National Reports).

So it is my intention to assume that these or similar, actions, circumstances, methods, and contexts exist in some form in all countries and to attempt to address at a more general level issues and questions that are consequently posed for the mathematical education community.

An international perspective

Recognising that circumstances vary between countries, there do seem to be changes in social, political and educational climates that form a broad backdrop to the assessment saga on a sufficiently global basis as to affect almost all of us.
Acknowledging that any selection is subjective, I will nominate five that seem to me to be illustrative of such movements.

1. Educational opportunities and requirements are being broadened globally - this may involve more children completing primary school, more children undertaking secondary education, more students proceeding to tertiary studies, or changes in the composition of schooling due to policies of positive discrimination on behalf of population sub-groups.

A consequence is a continuing decline in the so-called 'unskilled population', an increase in the level of formal qualifications demanded by employers, and at the global level a developing international trade in qualifications and courses which may be expected to increase through expanded use of the Internet.

Within such societal shifts, assessment is seen as a means of exercising control (Wolf, 1996) through which we find manifested:

- accountability as a public demand particularly in industrialised countries
- monitoring of standards through national and international agencies for social and economic development
- increased linking of teaching with work experience
- increased concern on the part of students, teachers, parents and employers for education to provide an adequate basis for occupational selection.

These moves have locked into the pre-existing interest in international comparisons (FIMS; SIMS, TIMSS) (Robitaille, 1992) involving testing across perceived common standards and supported by more general tracking of societal indicators over time by agencies such as the World Bank and UNESCO.

Nationally 'standards' have become prey to political processes that result in something like the following:

> "Standards must be raised so we need assessment based on standards; hence transfer power from schools to central agencies and professional test providers."

Since reliability as a public perception is paramount, it is easier to sacrifice validity, and a way to muzzle resulting critique is to freeze teachers and the mathematics community out of the process. Now it becomes a question of who decides what quality is, and who controls it! - a political question rather than an educational one.

2. There are re-definitions of the assessment agenda that use common terms but with different meanings. For example various 'deconstructed' meanings are given to key words and what these imply for practice. A word such as Accountability can be construed respectively as, following Wolf (1996):

(i) being able to *explain* (account for) actions; or
(ii) being responsible *for* some actions; or
(iii) being responsible *to* someone for actions taken.

These interpretations have substantially different meanings in practice where for example (1) provides for considerable teacher autonomy while (3) can mean the withholding of funds unless prescribed standards and methods (as specified by major stakeholders) are adhered to.

When defining such standards associated with criterion based assessment the construct of transparency must be added to the traditional constructs of validity and reliability suitably re-defined. Transparency refers in particular to criterion targets that aim to ensure that students can see clearly where to go. The more transparent the criteria the better the student should be able to target learning. However following the maxim that 'every silver lining has a cloud' explicit clarification of what is to be achieved also opens the door to challenge, especially in litigatious societies.

3. Issues of 'equality' continue to add unresolved questions to the assessment debate. Some learners, for example through culture and background, are better equipped to engage in 'assessment games' than others so that opportunities are unequal even though public procedures have an air of objective legitimacy. The traditional use of mathematics as a filter (based on some version of faculty psychology), through which mathematics performance continues to be viewed as a proxy for general ability, continues. Given the acceptance of this view society then feels empowered to use assessment as a means of disciplining students, teachers and institutions as argued by Niss (1993(a)) in terms of the carrot/stick approach to assessment. In places where external examinations continue as vestiges of colonial power, Ridgeway and

Passey (1992), the impact of associated social and political beliefs are as strong (or stronger) than pressures for reform deriving from research-based educational outcomes or generative professional wisdom.

4. Conceptions of ability continue to represent a pervasive fundamental conflict among issues yet to be resolved. Beliefs that innate intelligence and aptitude are the determiners of mathematical success continue to exist and sustain fatalistic approaches to mathematics performance. Such beliefs effectively excuse individuals on the one hand from trying as students, and on the other from trying as teachers. That fatalism is alive and well in my own country amongst both teachers and the public was demonstrated all too cleariy through two research studies (Galbraith and Chant, 1990, 1993).

The other aspect of ability attribution that continues as an issue involves the domain specific versus domain general explanation of the development of human talent (Brown and Campione, 1992). If intelligence as the ability to leam is innate, then it does not matter much what kind of testing is used to assess its level, providing it meets conventional notions of reliability and validity - as capitalized by commercial producers of educational tests. If, however, performance can be enhanced by careful teaching then it matters a great deal how the standards are set, how the learning goals are made transparent and explicit, and how the assessment program is matched to the specific leaming goals. Related issues expounded by Brown and Campione (1992) include the amount of guidance provided, rather than the number of trials needed for leaming to appear (a metacognitive rather than a behaviourist position), and the focus of assessment on the effectiveness of current learning rather than on the fruits of past learning. The paper presents one of the more powerful theoretical supports for a dynamic interaction between assessment and instruction, both for learning potential, and for the linking of assessment and instruction as an alliance against the re-ification of test scores into fixed cognitive entities. This theoretical debate continues to underlie activities that at another level appear to be concerned with practice.

5. Forms of reporting progress have increasingly become a focus for innovation. Zarinnia and Romberg (1992) provide seven categories within which assessment outcomes might be reported. In referring to a range of positive or negative side-effects associated with each they draw attention to advantages and distortions potentially deriving from new assessment directions. For example while focussing on process may present mathematics as powerful and active, it may also cause fragmentation in which various processes become ends in themselves rather than as means to an holistic end -no less a fragmentation than has occurred with

a focus on content. Similar analyses in a variety of countries point firmly away from reliance on standardized testing to the use of contextualized evidence obtained from tasks incorporated in regular instruction. This in turn sets an agenda for the re-professionalization of the teaching community in which beliefs about the nature of authentic mathematical activity is a target for change, and a basis for developing exemplars descriptive of quality as a pre-cursor to the development of ways of promoting inter-judge agreement on the quality of student work. These continuing manifestations of the nature/nurture debate impact in various ways on all efforts to change the emphasis in assessment towards learning in context.

A question of interests

The preceding section has sampled some of the events and movements currently engaging the field of Mathematics Education. It is assumed that through our pedagogical and assessment practices we hope to provide for the development of a liberated person in the sense described by Siegel (1980) as one who is "free from the unwarranted control of unjustified beliefs, insupportable attitudes, and the paucity of abilities which can prevent that person from completely taking charge of his or her life". The search for liberty inevitably calls into question the "interests" that are served by concepts and procedures used to control and label the capacities of individuals.

The question of "interests" as it has been applied in Education generally is associated with the 'critical theory' movement from which it has drawn its conceptual basis (Habermas, 1971; Gibson, 1986; Carr and Kemmis, 1986). Three levels of interest have been articulated, viz technical, practical, and emancipatory, and much has been written concerning these in the general teacher education literature, see for example van Manen (1977) and particularly Zeichner and his associates (Zeichner and Liston, 1987; Tabachnik and Zeichner, 1984; Zeichner, 1993).

Technical interests refer to the interest in gaining knowledge for the purpose of efficient and effective application in controlling the environment defined in a broad sense. This may be taken to include the attainment of prescribed educational objectives requiring technical skill, and mastery approaches to learning and assessment provide an exemplar for the achievement of technical interests in mathematics. Technical interests in the eyes of critical scholars are not necessarily bad, however they do not represent the only kind of knowledge to be sought.

Practical interests refer to a conception of action that involves explicating and clarifying assumptions behind alternative actions and evaluating the ultimate consequences of the actions. Actions and decisions are linked to value positions and those who are actors in the system must consider the worth of the various alternatives.

This type of knowledge is concerned with interpretive understanding. However, the subjective meanings embedded in aspects of this knowledge are controlled (limited) by the context in which it is enacted, and hence govern the extent of what can be achieved. Reflection plays a significant role in developing the scope of *practical interests* which might encompass, for example, acceptance of a range of assessment forms as necessary to evaluate different types of valued mathematical activity.

Emancipatory interests refer to issues of justice, equity, and fulfilment, and can only be served by a 'critical approach' that identifies the restrictions referred to above and reveals how they may be eliminated. Thus they offer an awareness of how aims, purposes and possibilities have been repressed and distorted, and what actions are required to eliminate sources of inadequacy or frustration.

It is emphasised that it is distortions that offend this latter viewpoint not the pursuit of other interests as such. For example technical interests may facilitate emancipation through the provision of mathematical power to learners, but they may also disempower through culturally inappropriate methods of assessing and thence accrediting competence.

It is my purpose now to use these conceptions of interests as a lens through which to view contemporary issues in assessment such as were sampled in the preceding section.

Let us take for example the term *accountability* as it might be interpreted within an Education system. If we define this term as meaning being able to explain actions (i.e. account for) we give freedom for teachers and schools to argue for or defend a variety of assessment practices based on corresponding value positions – practical interests at least are provided for. If we define the term *as being responsible for* some action we again locate the power and responsibility with those who design the learning and the assessment. Decisions made must be defensible in terms of the context of operations. If however the term is defined to mean being responsible to someone (or some organization), then there is a major shift in the location of power. This can involve a requirement to carry out a particular educational program with non-compliance punishable by withdrawal of funding. Interests served in this situation are likely to be narrowly technical. Those concerned with practical or emancipatory interests are likewise outraged by this position which is often compounded by their exclusion from the design and rationale of assessment procedures. The former's outrage is essentially intellectual, concerned for the quality of mathematics mandated by the procedures; the latter's outrage extends to encompass the political - concern for students and educational professionals, at the narrowly technical expertise they are offered, and concern that educational decisions have been usurped by ideological expediency.

The point at issue for the profession is that of involvement. At the outset terms like accountability tend to be used as buzzwords in as yet uncrystallised forms. Emancipatory interests would demand that the profession become involved at the outset, while there may still be time to influence the definition ultimately adopted for implementation, and to fight where necessary to overthrow imposed structures and definitions deemed educationally or morally unsound or restrictive.

It is, however, possible for developments generated from a concern for practical or even emancipatory interests to become re-ified in a way that consigns them to technical rationality. Zarinnia and Romberg (1992) sound a timely warning in alerting that the elevation of process categories of performance assessment, introduced for the purpose of increasing the mathematical power of all students, can lead to fossilisation just as restrictive and mind-numbing as content based categories have been deemed to be, e.g. 'working through' or 'writing reasons' can become separate ends in themselves rather than parts of a greater whole.

Perhaps nowhere is conflict of 'interest' more of a continuing issue than in the development of assessment instruments, and in particular those associated with the newer emphases of investigatory work, problem solving, and modelling. However they are otherwise presented, arguments ultimately centre around the location of power. When power has traditionally resided with universities, state or national examining bodies, or test manufacturers, the implications are substantial both in intellectual and monetary terms.

Coursework assessment represents a substantial transfer of authority from external experts to schools and the teaching profession. With the opportunity of taking into account the school context and associated societal factors there is a potential shift in focus away from purely technical interests. Indeed some authors (e.g. Keitel, 1993) have suggested how the opportunity exists to address also wider issues of justice and emancipatory concerns through developing such foci. Not surprisingly some of the most vigorous territorial battles are fought in this domain for the potential power shift is substantial since at least two 'interests' battles must be fought here.

(1) to facilitate a change in 'beliefs' about what constitutes authentic mathematical activity, when past conceptions have been moulded by narrowly defined technical interests.

(2) to support the profession in its transformation into a community of practice: for example through the provision of training in the development of inteijudge reliability in assessing project work - an attribute requiring the same type of professional knowledge and judgment that is currently used in assessing the worth of doctoral theses by the academic community.

Hoge and Colardarci (1989) leave no doubt as to where they believe opposition to such transfer of power lies. Following their survey showing that teachers were, by and large, very good judges of their students' performance, they go on to nominate groups who continue to reject teacher judgments as valid assessment data in the interests of maintaining control over assessment practices.

It is possible to go on to identify conflicting interests in almost every area of the assessment debate. The question is what this does for the advancement of knowledge and process, for awareness and critique at one level can serve to harden and re-ify opposition and defences at another, or can remain simply as critique. One possible path to progress is to adopt a dialectical strategy, that is to specify potential conflicts of interest and subject them to informed rational debate, rather than polarize and entrench opposing viewpoints through verbal attack or to strive for an uneasy but superficial peace. To question rather than condemn is also consistent with the Habermasian approach exemplified in his promotion of the Ideal Speech Situation (Habermas, 1971; Gibson, 1986).
The ideal speech situation (ISS) requires that each individual communication possess four qualities. It should be

(a) comprehensible - e.g. made in a shared language
(b) true, i.e. matching what we perceive as reality
(c) correct, i.e. legitimate within the context of the topic
(d) sincere

Furthermore, the ISS requires that all speakers (or communicators) have equal rights to dispute, assert and question and, by inference, have equal access to relevant knowledge. Put differently, this requires that an ISS be free from domination and, given this condition and genuine goodwill, progress can be pursued by rational argument.

It is not however difficult to identify impediments to this ideal in our debates on assessment, e.g. unequal 'rights of speech' afforded to political voices versus educational voices, administrators versus teachers, academic mathematicians versus professional educators, parent demands versus teacher ideals, competency advocates versus wider mathematical adherents.

How might such an approach work? A simple beginning would involve selecting issues of significance and for each posing questions that direct debate to the intellectual content and power relations represented by identified altemative interests made public. The following examples illustrate the dialectic properties of this approach.

Interests of teachers in developing understanding	vs	Interests of students in achieving grades
Interests of administrators in placating parents	vs	Interests of teachers in using innovative assessment
Interests of schools in meeting league table pass levels	vs	Interests of weaker students abandoned to enable others to reach that level
Interests of subject department heads in controlling assessment	vs	Interests of teachers in developing best practice
Interests of Universities in maintaining power over school mathematics	vs	Interests of schools in increasing professional autonomy
Interests of employers in prescribing 'basics'	vs	Interests of schools in achieving 'fundamentals'
Interests of Testing Services in selling instruments	vs	Interests of teachers in designing assessment
Interests of the State in competency measures	vs	Interests of the school in quality measures

Such a listing is not intended to imply a "good" versus "bad" characterisation. The purpose in establishing a dialectic is to create an agenda and a forum through which interests may be disclosed and addressed that are often present but not made public. While there is no assurance that this will always work, public declarations of position and the relentless exposure of assumptions, enhance the possibility of progress and render a little less likely the use of vicarious means to avoid confronting issues and the use of managerial methods to overthrow educational goals (Bates, 1984).

This section has been written bearing in mind comments of Marshall and Thompson (1994) that their review of six books on assessment had revealed a plethora of issues and approaches, many criticisms of current practices, but few suggestions for progress. The classification of interests has been introduced as a set of 'spatial coordinates' together with a mechanism (ISS) that provides a structured means for advancing debate on contentious issues - given however a genuine desire to achieve progress on the part of all parties.

A question of scale

Another way of approaching issues is to estimate the extent and type of their impact. Kaput and Thompson (1994) used nautical metaphors to describe the perceived profundity of impact of various aspects of computer technology on mathematics learning.

Borrowing their terminology and adding minor changes of emphasis we obtain the following classification.

Surface wave:	represents procedures and impacts at the level of the individual classroom and school.
Swell:	larger scale and more pervasive so there is impact at the local system level.
Tidal wave:	generated beyond local frames of reference and requiring timescales with larger orders of magnitude.
Sea level change:	fundamental reshaping, analogous to a change in sea level due to global warming.

Using this broad classification we may now assign a range of assessment issues -clearly this contains a substantial element of subjective judgment. At the level of *surface waves* we assign variations of test procedures including the development of contextualized assessment materials such as school-based projects and investigations, and the use of data collection alternatives such as interviews and innovative writing tasks.

At the *swell* level we have the re-conceptualization of concepts like reliability and validity (to encompass notions of transferability and trustworthiness), moves to accept the veracity of and enhance the quality of teacher judgments, and moves to incorporate a range of non-test assessment in certification procedures.

At the *tidal wave* level we look for more pervasive change identifiable beyond the boundary of a local system and might include respectively, the profound change wrought when a formal assessment system is redesigned on the basis that teaching and assessment form an integrated whole; the transforming action of a new epistemological stance (a current example would be constructivism); and the changes wrought as a consequence of some countries' responses to international studies of mathe-matics achievement.

Although tidal influences are profound there may be regions that remain isolated from their effects and for a *sea-level* change we look for uni-directional forces of massive impact and irresistible momentum. It would be a very wise or very foolish individual who felt confident in pronouncing definitively on such issues, so I will merely suggest three which seem at this time to be potential candidates. They are respectively the continuing development of mathematical software and associated technology, the communication properties and potential of the Internet, and an internationalization of notions of competence and standards as a consequence of a developing international trade in qualifications and courses.

This section has attempted to address another difficulty perceived in the assessment debate. The issues we consider and the books we write tend to be important but undifferentiated: better methods of testing computation are considered together with ways to improve the quality of feedback on problem solving, together with the merits and demerits of national and international testing etc etc. The metaphor seems to be J.J. Thomson's plum pudding model of the atom with issues scattered around like currents and raisins in a mass of material labelled assessment.

The nautical metaphor suggests an approach closer to the Rutherford-Bohr model by separating assessment issues into 'energy levels' on the basis of their perceived location and pervasiveness. This is a crude device but it is one way of structuring issues into family groups that may then, if desired, be examined using another set of dimensions such as 'interests' served, or other agreed criteria.

Towards a theory of assessment

As noted above, in their review of six books on assessment, Marshall and Thompson (1994) remark on the feature, common to them all, of identifying what is wrong with current assessment practices. Niss (1993) refers to 'conflicting interests, divergent aims, and unintended or undesired side-effects' characteristic of assessment modes and practices and observes that 'difficulties involved in devising and employing effective, harmonious assessment modes, free from serious internal and external problems seem to be fundamental and universal in nature'. Webb (1992) in summarizing a similar range of developments and concerns calls for a 'specific theory of mathematics assessment within a general theory of educational assessment'.

What are issues relevant to the development of a theory of mathematics assessment as a home within which to locate and extend the plethora of current issues, actions, needs, and beliefs. Three or four years ago I raised the role of basic assumptions about the nature of knowledge in influencing the direction and purpose of mathematics assessment (Galbraith, 1993) and it is not the intention to repeat those arguments here. However nothing has happened in the intervening period to lessen the conviction that basic belief systems are the ultimate determiners of positions (causes), which at another level appear in the form of arguments that refer to the specifics of assessment processes and products (symptoms). To the extent that they underpin the discussion that follows I have summarized basic assumptions of the two approaches to 'knowledge' and 'reality' described by Lincoln and Guba (1990) as characterising respectively the positivist and naturalistic (constructivist) paradigm (see below).

Axioms about	**Positivist Paradigm**	**Naturalistic Paradigm**
Nature of reality (ontology)	Single, tangible and 'out there'	Multiple, constructed and holistic
Relationship of Knower to Known	Knower and known are independent	Knower and known are interactive
Possibility of gereralisation	True and context-free generalisations are possible	Only time ad context-bound working hypotheses are possible
Role of values	Inquiry is value-free (fact-value independence)	Inquiry is value bound (fact-value interdependence)

Four philosophical 'realities' relevant to respective ontological positions are identified.

(1) *Objective reality* - a tangible reality exists and given enough time and sound principles of investigation the reality can be converged upon (even though individual studies may be only approximations).

(2) *Perceived reality* - a reality exists but it cannot be known fully but only through a spectrum of perceptions. Reality for any individual or group is at most a partial portrait of the whole and capable of different interpretations when considered from different viewpoints.

Positions (1) and (2) agree on the existence of a "real" reality but differ in what they consider to be knowable about that reality.

(3) *Constructed reality* - defined terms such as "problem", "learner", "examination system" evoke different meanings within individuals. None exist in forms other than those constructed by persons who 'recognise' the term in question. Such constructed realities may indeed match 'tangible entities' quite closely. For example a group of individuals from out of town may agree closely on the scope of actions displayed by a policeman on traffic duty. However this represents a consensus among the observers rather than the discovery of a single true reality. Truth by consensus is the approach consistent with this ontological position.

(4) *Created reality* - effectively means that there is no reality at ah until it is perceived as a consequence of some action. If stranded by a rockfall on one side of a deep chasm over which the single possibility of escape resides in a damaged footbridge, two possible futures exist side by side. Creation of the reality of survival or doom depends upon the act of testing the bridge as a means of escape.

Positions (3) and (4) share basic assumptions about the nature of 'reality' - viz that it doesn't exist until it is constructed by an actor (3) or is created by a participant (4).

Now with respect to discussions about the principles and practices of assessment we find representations of all four positions. Belief in ability as inherent intellectual endowment independent of context or task continues to support test construction with its corresponding emphasis on reliability of measures as the attribute of paramount importance, a manifestation of position (1). Position (2) attracts adherents who value a range of assessment types as a better means of converging on that elusive concept 'mathematical ability' that nevertheless exists in some absolute form. Non traditional assessment tasks may be utilized but are only to be trusted if set and marked centrally and administered under conditions that in some sense may be described as standard.

Position (3) is the province of those who admit and indeed rejoice in the possibility of equivalent but different contributions to the assessment of quality. A pattern of performance that includes subjective measures of quality, and consensus rating of judges, is characteristic of this position. Verbal statements of criteria and standards must be translated for application across a range of conditions and scholarly consensus represents the only viable basis for agreement. Project work, problem solving, modelling and other forms of extended assessment are valued within this viewpoint as is more conventional testing under certain circumstances. These circumstances however are approved in terms of the purpose of the test (e.g. to check facility with a calculator or assess the level of algebraic skill) and not in terms of some objective measure of mathematical ability supposedly measured by the test score. Project work et al within this view is best marked by the professional most familiar with the students' work (i.e. the teacher) with the latter's expert judgment a consequence of the application of principles characterizing a community of practice and accredited by that community.

Position (4) supports the radical possibility of creating new 'realities' of assessment. In fact this is not as radical as it sounds depending upon the circumstance. An actual example comes to mind of a subject head who introduced a policy that no assessment would be approved that could not be conducted within the time and format of ordinary class lessons. This replaced a scheme whereby schooling was suspended at intervals for the 'exam week'. The new 'reality' was a commitment to the integration of teaching and assessment in the fullest sense, a carefully crafted alternative to the 'objective' model previously in force.

And so to the question of a theory of mathematics assessment. Any theory must rest upon some agreed ontological position and its epistemological consequences. It can safely be asserted that no theory of assessment can be built on a positivist ontology -not because of questions of right and wrong but simply because so many in the mathematics community reject this philosophical stance. If such an approach were to be attempted it would be rejected by large sections of the community and hence could not function as an effective theory. This then brings us to the ontology of the naturalistic (constructivist) paradigm and the situation here is not clear either. By its very nature the constructivist position cannot claim universal interpretations and truths that are a positivist legacy, and must depend upon consensus for progress to be achieved. Here a difficulty emerges because of the many viewpoints, and in the words of Niss (1993) 'conflicting interests, divergent aims, and unintended or undesired side-effects'. Yackel, Cobb and Wood (1992) make an important point when discussing the role of the teacher in a constructivist classroom. In pointing out that teacher and student roles are not symmetric they remind us that 'it is the teacher who expresses hisiher institutionalised authority in action to help students develop an understanding of what is, and what is not, an acceptable mathematical argument'.

And here is our dilemma. Yackel et al are referring to the learning of mathematics: a disciplined body of knowledge for which agreed norms and practices have been established and accepted by its community of scholars. We have as yet no such agreed precepts and principles for assessing the quality of that closely related yet distinct property of mathematics performance. Indeed we are in the process of trying to establish them.

It seems that we are collectively engaged in an activity much more profound than that which occurs in many classrooms allegedly run according to constructivist principles. It is ironic that in many such classrooms there is encouragement to interpret, construct, and re-construct mathematics; indeed to treat everything (except one) as a candidate for constructivist action. "You (the student) are encouraged to question, interpret and reconstruct everything except the method by which I am teaching you."

The teacher in constructivist mode is still in control in an ultimate sense supported by the impressive edifice of mathematical knowledge from which to draw (while granting that many enriching uncertainties occur in probing new directions of content and process).

I suggest that a major reason why a theory of assessment is so elusive is that the whole mathematics community is engaged in an exercise in construction more profound than that occurring in any classroom. It is endeavouring to construct an edifice of assessment while lacking disciplinary consensus on what particular building materials should go into the cbnstruction. That is, it does not have an established agreed knowledge base to which to appeal - for that is part of the construction in progress. We are attempting to build in a few short years structures parallel to that which took centuries to construct, test, and re-construct for the discipline of mathematics.

To use an analogy from the History of Mathematics to describe our progress towards a theory of assessment, we have been growing through our 'Babylonian and Egyptian' periods. The pragmatic needs of assessment have seen us engineer and refine instruments and methods for application in a variety of practical circumstances, and doing the practical job required has been a sufficient end. We are now attempting to enter our 'Greek' period, to set out practices within a theoretical framework that will simultaneously provide a quality test for our methods, and a springboard for further development.

It is imperative that we retain all parties and viewpoints in the dialogue if some consensual agreement is to be achieved, and for this we need to adopt a global view that transcends our own particular national or local context. Withdrawal from debate and discussion will be as detrimental to the ultimate purpose as a student refusing to participate in the classroom construction of knowledge.

If I treat discussions involving external examinations as irrelevant on the grounds (true) that these are not part of my State's assessment practices, how am I different from a student insisting that every mathematical topic of study should be of personal relevance to him/her? If I reject project work out of hand as valid assessment what distinguishes my approach from a student who refuses to participate in a group? If I, attempt to suppress the analysis of power relationships implicit in an assessment program because I prefer things as they are, how can I require students to engage in mathematical activity beyond their preference for say basic computation?

In other words the kind of involvement we as teachers require of our students as participants in knowledge. construction, we need to require of ourselves in building a foundation on which a theory of assessment might be erected. Perhaps indicative of our pre-theory phase is the existence side by side of procedures that on examination appear to rest upon quite different and conflicting assumptions. Assessment in tertiary mathematics embraces both common timed examinations for large classes and the PhD model of research assessment. The first is commonly argued for on the basis that everyone should be tested under common conditions to enable relative ability to be revealed - a positivist position. Suggestions that PhD students should research the same topic or be subject to uniform conditions are too ludicrous to be even thought about. The PhD model embraces the constructivist position recognizing that quality has a variety of manifestations and may be assessed by the consensual agreement of experts in the field. A theory of mathematics assessment must address such issues among the others.

It has been argued that our community of practice yet lacks an agreed ontology on which to construct a comprehensive theory of assessment. In seeking this goal it is useful to consider issues within frameworks structured around questions of 'space' and 'time'.

The Habermasian dimensions of technical, practical and emancipatory interests provide one means of examining assessment in relation to its purpose and impact across 'space', i.e. with respect to individuals, groups, and societies.

Nautical metaphors provide a means for separating issues in terms of their perceived impact and hence a means of re-organising priorities and expectations. Such frameworks aim to facilitate analysis and discussion at a level beyond the mechanics of particular assessment methods.

Ultimately the way we agree to view 'reality' or 'genuine mathematics' or 'true mathematical ability' will be definitive as a basis for a theory of assessment. Present evidence suggests that we are in the process of a giant exercise in social constructivism within which premature closure will inhibit the ultimate achievement of a theory. Within this activity two approaches threaten to inhibit our collective progress. These are respectively withdrawal from debate on the grounds of incommensurability of ideas, and comfortable living within a restricted community of like minds.

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A SOJOURN IN MATHEMATICS EDUCATION

Solomon Garfunkel

This is a personal article. It relates one person's passage into the professional world of mathematics education. It is a story that I tell here, not so much to enhance my own cult of personality, but to give some historical developments in a context I know and hopefully (some day) will come to understand. As with all personal stories it's somewhat difficult to know where to begin. But for want of a better alternative, I'll start with my first full-time job.

I received my Ph.D. in mathematical logic from the University of Wisconsin, Madison in 1967 and at the ripe old age of 24 began my academic career as an instructor at Cornell. On my first day, the chairman of the mathematics department, Alex Rosenberg, invited me in for a chat. Alex was in many ways the ideal chair of a math department. He treated everyone alike, from instructor to full professor. His rabbinical demeanor certainly helped.

While I can't remember his exact words, his fatherly speech went something like, "I hope that you are a good teacher. We want good teaching here at Cornell.

But if you think for one second that anything you do in front of the classroom will have anything to do with whether you eventually get tenure here, you are mistaken. I don't say that this is how things should be; but this is how things are. So, if you ever have to make a choice between working on your research or preparing for a course, and what you care about is staying here, then work on your research."

I was shocked, truly shocked. I saw myself as an adequate researcher, but I knew how much I wanted to teach. I went home and wrote a handwritten (olden days) letter to John Gardner, then the Secretary of Health, Education, and Welfare saying in effect that the gods must be crazy. To my surprise he wrote back. Essentially he told me to work hard, get tenure, and then try to destroy the system from within. And more or less, with many twists and turns, that is what I've tried to do.

I spent three years at Cornell as instructor and assistant professor and then moved to the University of Connecticut at Storrs. At Cornell I had met and become friendly with a biochemist who went to work at M.I.T. at the same time I moved to Connecticut. He was working at a place called the Education Research Center, a sort of curriculum development lab where people had funny titles like senior research scientist. ERC was directed by Jerrold Zacharias.

For those of you unfamiliar with that name, Zach, as everyone called him, was a research physicist of great renown who had pioneered work in physics education in the post-Sputnik era. NSF was rumored to measure grants in Zachs (supposedly 250K dollars per).

At any rate, when I went to visit my friend the biochemist, he introduced me to the people working on mathematics curriculum development. I should note that at that time (the early 70's) almost all NSF grants were at the undergraduate level. So the math group was working on calculus reform, although they didn't call it by that name. The nature of the reform was applications. There was after all a strong bias towards physics and physical applications at ERC. Their overriding philosophy was that for the vast majority of students the only practical way to motivate their study of calculus was to show them early on what calculus was used for. Believe it or not, this was considered an extremely radical position back then.

As I watched people create bubble accelerometers, make wine from grape juice, and slide cars down inclined planes, I was hooked. After all, I was a logician. My training had been as pure as the driven snow. This stuff was brave new world. And more importantly, I recognized the truth behind the basic premise, namely that we could not teach all students with the same assumptions we used for majors. So, I began commuting one day a week from UConn to M.I.T. and working with the math group. I should note that there was a fair amount of pedagogical experimentation as well - programmed learning, mastery learning, hands-on group activities, etc.

At any rate, I brought it all to UConn, running experimental section after experimental section. To be honest, this work was not greatly appreciated.

To be fair, they hired a research logician and it was becoming increasingly clear they got a naive, but energetic math education novice. After surviving a rather bitter tenure fight, I found myself devoting all of my time to curriculum development. And a funny thing happened. I started to have some ideas, to write them up as proposals, to talk to funders, and to have some of those proposals funded. I was a grantee.

In 1976 I co-authored the Undergraduate Mathematics and its Applications (UMAP) grant proposal to the National Science Foundation (NSF). This grant had two goals. The first was to produce, through a user/developer network, a body of modules which taught specific mathematical topics through their contemporary applications. The second was to create an organization which could continue that effort after federal funding was terminated. After all, it was clear that there was no fixed body of modules that could cover all of the applications and models. Moreover, the network of authors, reviewers, field-testers, and users was as much about staff development as materials development - and the need for staff development doesn't go away. I called this mythical organization, COMAP, the Consortium for Mathematics and its Applications.

For the next four years COMAP remained a mental construct, while work on module development proceeded apace. In December of 1980 we formally incorporated and applied for tax-exempt status, which was granted. I need to back up one step. In 1976, soon after NSF funded the UMAP project, the math group at ERC moved to the Education Development Center (EDC). The reasons for that move had mostly to do with Zach's relationship to M.I.T. (which I was much too junior to know about at the time). However, EDC was a friendly home, with many exciting curriculum projects going on in math and science, and a staff of people connected to several Boston area colleges. The UMAP project flourished at EDC and we were able to produce a good many modules and to found the quarterly UMAP Journal.

Fast forward to the fall of 1981. I had a sabbatical coming up and UConn was still quite unhappy with my decision to focus my efforts on math education.

To be fair, I doubt that I could still do any research in logic by that time.

At any rate, I decided to spend the year in Boston and see if COMAP could become more than a paper corporation; for while we were incorporated, all of our funds and administration went through EDC.

In October of 1981 we wrote the proposal to the Annenberg/CPB Project which became our first television course - For All Practical Purposes. For those of you who may not remember, 1981 was the year that the NSF science education budget went almost to zero as Ronald Reagan's administration took hold. While there were still funds in the UMAP grant, the future for NSF funding was incredibly bleak. The cutbacks were so severe that essentially all of the staff at NSF had left or been reassigned. For COMAP's future, the Annenberg moneys were a lifesaver. we received co-funding from the Carnegie Corporation of New York and the Alfred P. Sloan Foundation. After a fair amount of contract negotiations, the actual award was made at the end of summer 1982.

UConn of course had asked me about my plans well before. But the announcement that our proposal was successful had come in by that time and I requested a leave without pay. It was quickly granted. I suspect that they were just as happy not to have me on campus, my replacement was cheaper, and they didn't want to lose the line. By February of 1983, with the Annenberg/CPB grant in hand and the last year of the UMAP grant funds we decided to leave the nest.

We rented office space in Lexington, MA, bought some accounting software and took the independence plunge.

Those first few years were a true struggle to survive. We learned the hard way about taxes, social security, retirement plans, auditors, etc. We also had a crash course in television production. But to be honest, it was great fun. COMAP was a labor of love in the best sense. Then came 1984. In '84 moneys started to come back to the NSF science education budget. But interestingly, the funds were pretty much all earmarked at K-12 education.

This was hardly an accident. It was a direct result of "A Nation At Risk", the first of a series of reports detailing the deficits in the U.S. educational system and student performance.

Well, what could be more natural than HiMAP, a module project designed for high school teachers? By the time the HiMAP project was funded, the theorem was proven. COMAP could and would work. For All Practical Purposes was a great success. The accompanying text sold almost 50,000 copies in its first three year edition. We soon received two more Annenberg/CPB grants - Against All Odds and In Simplest Terms, telecourses in statistics and college algebra.

A number of additional NSF grants followed including GeoMAP, HistoMAP, and most importantly ARISE.

The ARISE project represents a culmination of our efforts on the high school scene. We have produced many sets of modular materials at the secondary level for both teachers and students. But ARISE is meant to be a comprehensive curriculum, grades 9-11 (and some day 12, NSF willing). We have always been on the outside looking in, telling instructors what they could do, and showing them with exciting pieces of curriculum. But ARISE is truly different. NSF has said to us, ok you win, we believe. Now go and write a complete high school curriculum. Teachers want to know what to do Monday morning - tell them.

This has been our most complicated project to date. Five years have gone into the effort, with a writing team of over twenty high school teachers (and a few of us college types, including Henry Pollak). This fall, our first ARISE text will be on the market. It's both exciting and scary. After all, we've changed the content, the pedagogy, the technology, and the assessment. It is extremely hard to know how the community will react. In some ways, though, that is the joy of working at COMAP. We push the edge of the envelope. We create new materials for new courses. So far we have been extraordinarily lucky. Our materials have been well received and helped to generate a revenue stream which keeps us in business.

But no one can time markets, and some ideas may simply not work or may be too far ahead of the curve. That's all right; someone has to be there stirring the pot, trying new things.

The real joy of COMAP is the people I work with. I honestly believe that they are the finest mathematicians and educators in the country. And they are truly too numerous to mention. From Joe Malkevitch who believed in the idea of COMAP and helped to found the organization with his ideas and his writings to David Moore, who worked on his book as he worked on the programs for Against All Odds to Joe Blatt who was our executive producer on the telecourses, to Frank Giordano who has helped make the Mathematical Contest in Modeling a fixture on our college campuses, to Landy Godbold who has worked tirelessly as project director for ARISE, I have had the privilege and honor to be their colleague.

Moreover, COMAP was the right idea at the right time. There is simply an enormous amount of talent and expertise in american classrooms. There are untold numbers of faculty at all levels ready and willing to write, review, field-test and try out new materials because they are truly dedicated to their students. COMAP opened up an avenue for the expression of that talent. COMAP is proof that some ideas are simply better than the people that have them.

ON CULTURE AND MATHEMATICS EDUCATION IN (SOUTHERN) AFRICA

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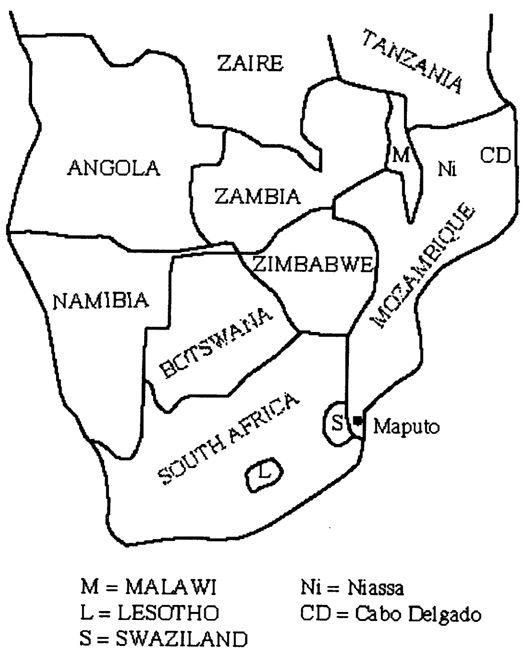
To introduce the theme of culture and mathematics education in a multicultural African context, I should like to present a recent testimony by Salimo Saide, one of the mathematics teacher graduates from Mozambique's 'Universidade Pedagógica'.



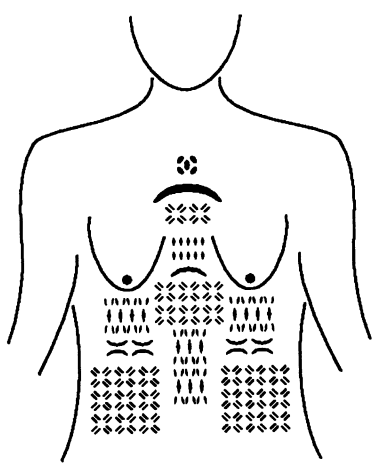
Map of Southern Africa

Testimony by Salimo Saide

"I was born June 20, 1965 in Lichinga, capital of the northern Niassa

Province (see Map of Southern Africa). There I went to primary and secondary school. From 1985 to 1987 I took part in a teacher education program. From 1987 to 1991 I taught Mathematics and Physics at the secondary school of Pemba, capital of the Cabo Delgado Province. In that province I coordinated the local Mathematics Olympiads. In 1991 I came to the capital Maputo in the south to continue my studies and in 1996 I concluded my 'Licenciatura' in Mathematics and Physics at the 'Universidade Pedagógica'.

In 1977 I had the opportunity to read a book written by the priest Yohana, entitled "Wa'yaowe", that means "We the Yao people". It opened a whole new horizon for me. I was lucky to be able to read and write Yao – in school only Portuguese is taught. During my whole youth I loved to read more in Yao, but there did not exist any opportunity. When I came to the national capital Maputo to continue my studies, I thought my dream had died. However, when I took part first in a voluntary "circle of interest" on mathematical elements in African cultures and then in the optional course "Ethnomathematics and Education" my dream started to live again. I found a strong link between mathematics and the art of my grandparents. My participation let me feel returning to my land, let me remember my grandmother, her decorated mats and baskets, and her beautiful "nembo" – tattoos (see Figure 1) and pot decorations (see Figure 2). The idea "caught" me and during my holidays I made three field trips to Niassa to study the geometry of ceramic pot ornamentation. Now after having finished my university program, I hope to return definitively to my land, to con-



Example of a “nembo” tattoo
Figure 1

tinue my research and to teach mathematics integrating the "nembo" of the Yao people into it." [1]

It was not easy for Salimo to realize his fieldwork. Sometimes it took him various encounters on several successive days to win the confidence of the old female pot makers, as they did not understood easily why a young man, speaking with the accent of someone educated in the cities, could be interested in their nowadays downgraded and disappearing female art and craft of pot decoration; why would he be interested to see their tattoos when the churches, both Christian and Islamic, have been combating tattooing so strongly? However, once he won their confidence, they were happy to speak about their craft and art, and about how they learned it, to discuss with the student alternative ways of reviving, of valuing their symbolic language, their knowledge, wisdom, and creativity. For instance, it was suggested to decorate 'capulanas' – square woven cloths worn by the women around their middle – with ceramic "nembo" and T-shirts with tattoo "nembo".

Through their fieldwork, students like Salimo came also to grasp with 'out-of-school' learning processes: from individual learning by imitating and copying or by trying and experimenting, learning through guidance from older family members or friends, to collective learning environments among children from the same age group. Through their fieldwork, they understood better how important social and cultural factors like interpersonal relationships, family context, opportunity, needs, and motivation are in stimulating or blocking what, how, and what for something like a craft is learned. The results of making decorated pots, baskets, etc. may seem similar; the learning processes involved may be very distinct. This field experience reinforced the students' reflection on what school mathematics education might learn from 'out-of-school' learning processes.

Teachers like Salimo – who as students voluntarily took part in 'clubs' and optional courses related to culture and mathematics education – return, well motivated, to their home provinces, determined to work as mathematics teachers in such a way that it is both useful for their people and dignifying to its cultural heritage.

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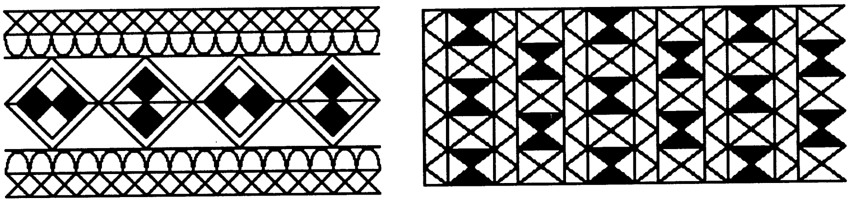
1 S. Saide, Sobre a ornamentação geométrica de panelas de barro por mulheres Yao (Província de Niassa), in: P. Gerdes (Ed.), *Geometria e Artesanato em Niassa e Nampula*, UP, Maputo (in press)



Two examples of "nembo" strip decorations on pots
Figure 2

On Culture, Education, and the Development of Africa

Well-known African politicians, historians, scientists, and educators alike have lately stressed the importance of cultural factors for Africa's development in general and education in particular.

The South Commission, led by the former President of Tanzania, J. Nyerere, criticizes in its report, *"The challenge to the South"*, [2] development strategies that minimize cultural factors, as such strategies only provoke indifference, alienation and social discord. Development strategies "have often failed to utilize the enormous reserves of traditional wisdom and of creativity and enterprise in the countries of the Third World" (p.46). Instead, the cultural wellsprings of the South should feed the process of development.

In the book *"Educate or Perish: Africa's Impasse and Prospects"* [3] the historian J. Ki-Zerbo explains that today's existing African educational system is generally still rather "unadapted and elitist" and "favors foreign consumption without generating a culture that is both compatible with the original civilization and truly promising". "Too often, schools alienate children from their social environment" (p.69). What is necessary is "a new educational style which focuses the whole variety of approaches on the objective of Africanization, especially by integrating the natural and cultural environments into the educational process, along with productive work..." (p.91, italics pg).

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2 J. Nyerere (Ed.), *The challenge to the South – Report of the South Commission*, Oxford University Press, Oxford, 1990, 325 pp.

3 J. Ki-Zerbo, Educate or Perish: Africa's Impasse and Prospects, UNESCO-UNICEF, Dakar / Abidjan, 1990, 109 pp.

In *"African Thoughts on the Prospects of Education for All"* [4] African peoples' cultural identities (including the awareness of these identities), are seen as the springboard of their development effort (p.10). It is stressed that Africa needs culture-oriented education, that would ensure the survival of African cultures, if it emphasized originality of thought and encouraged the virtue of creativity. Scientific appreciation of African cultural elements and experience is considered to be "one sure way of getting Africans to see science as a means of understanding their cultures and as a tool to serve and advance their cultures" (p.23).

On Mathematics, Education, Culture, and History

African countries face the problem of low 'levels of attainment' in mathematics education. Relatively few students pursue university programs with a strong mathematical component. Math anxiety is widespread. Many children (and teachers too!) experience mathematics as a rather strange and useless subject, imported from outside Africa; as something that exists only in schools.

In his editorial "Mathematics and Africa",[5] T. Isoun, former Vice-Chancellor of Rivers State University of Science and Technology in Nigeria and first editor of "Discovery and Innovation" – the scientific journal of the African Academy of Sciences –, remarks "The way mathematics is taught in primary and secondary schools in Africa, appears to turn off many bright young men and women" (p.6). Therefore, there is "need for mathematicians in Africa to write textbooks to reflect our cultural background, and ensure that mathematics is firmly grounded within our environment" (p.6). The cultural-scientific heritage of the continent may constitute a source of inspiration. Surely "any inspiration we gather from the African past must be used to enable the peoples of Africa to participate in the generation, advancement and use of the scientific and technological enterprise. Mathematics, as a cultural, artistic, humanistic, as well as scientific and technological enterprise, will serve as a key to the full participation of the continent" (p.4). The African Academy of Sciences launched its program on science and culture in 1992. The African Mathematical Union Commission for the History of Mathematics in Africa (AMUCHMA), formed in 1986, stimulates research in the history of mathematics in Africa and promotes the dissemination of research findings[6] in African universities

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4 *African Thoughts on the Prospects of Education for All,* UNESCO-UNICEF, Dakar / Abidjan, 1990, 193 pp.

5 T. Isoun, Mathematics and Africa, *Discovery and Innovation,* African Academy of Sciences (P.O.Box 14798), Nairobi, 1992, Vol.4, No.1, 4-6

6 For an overview, see P. Gerdes, On Mathematics in the History of Sub-Saharan Africa, *Historia Mathematica,* New York, 1994, Vol.21, 345-376

and teacher education colleges.[7] Present chairman of the African Mathematical Union Commission on Mathematics Education (AMUCME), M. El Tom from Sudan, singled in a recent report to the Third World Academy of Sciences[8] ethnomathematical-educational research and experimentation as a noteworthy exception and necessary activity in a context where most African countries attempt to imitate major curriculum reforms in the West. [9]

An ethnomathematical research perspective [10]

Defined as the cultural anthropology of mathematics and mathematics education, Ethnomathematics is a relatively new field of interest.[11] The view of Mathematics as "culture-free", as an "universal, basically aprioristic form of knowledge" has been dominant internationally. And a reductionist tendency tended to dominate mathematics education, implying on culture-free cognition models.[12]

At the end of the 1970s and beginning of the 1980s, there developed internationally a growing awareness among mathematicians of the societal and cultural aspects of mathematics and mathematical education. It is in that period that U. D'Ambrosio (Brazil) proposed his ethnomathematical program as a methodology to track and analyze the processes of generation, transmission, diffusion and institutionalization of (mathe-matical) knowledge in diverse cultural systems. In his view one should compatibilize cultural forms, i.e. "...the mathematics in schools shall be such that it facilitates knowledge, understanding, incorporation and compatibilization of known and current popular practices into the curriculum.

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7 Readers interested in receiving the AMUCHMA Newsletter may send their request to the author.

8 M. El Tom, The present Status of Mathematics and Mathematics Education in Africa, (unpublished) report presented to the Third World Academy of Sciences, Triest, 1995

9 Readers interested in receiving the AMUCME Newsletter may contact AMUCME Secretary, Prof. Cyril Julie, Department of Didactics, University of the Western Cape, Private Bag X17, Belville 7535, South Africa

10 For an overview see P. Gerdes, Ethnomathematics and mathematics education: an overview, in: A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, C. Laborda (Eds.), *International Handbook on Mathematics Education*, Kluwer, Dordrecht, 1996 (909-943)

11 Analyzing the contents of "Tanbih al-albab" by the Maghrebian mathematician Ibn al-Banna (1256-1321), A. Djebbar considers several parts of it as related to ethnomathematics. See A. Djebbar, *Mathematics in the medieval Maghreb*, AMUCHMA Newsletter, Maputo, 1995, No. 15, 3-42 (p.19)

12 Cf. U. D'Ambrosio, *Etnomatemática: raízes socio-culturais da arte ou técnica de explicar e conhecer*, UNICAMP, Campinas, 1987, p.80

In other words, recognition and incorporation of ethnomathematics into the curriculum". [13]

Short overview of research on culture and mathematics (education) in Africa

A rather isolated forerunner of reflections on culture and mathematics education in Africa, was O.Raum, whose book "Arithmetic in Africa" appeared in 1938.[14] Two already classical studies, published in the USA, dealt with mathematics (learning) and African culture: "The New Mathematics and an Old Culture – A Study of Learning among the Kpelle" (1967)[15] and Zaslavsky's "Africa Counts" (1973).[16]

One of the first Africans to organize research in the field of mathematics education and culture, is S. Touré – former Secretary-General of the African Mathematical Union, and today's Minister of Higher Education and Research of Côte d'Ivoire – who introduced in 1980 a research-seminar on "Mathematics in the African socio-cultural environment" at the Mathematical Research Institute of Abidjan.[17] The seminar is now directed by S. Doumbia. One of the interesting themes analyzed by her and her colleagues is the mathematics of traditional West-African games and its use in education.[18] In the series of mathematics textbooks for secondary

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13 U. D'Ambrosio, *Socio-cultural bases for mathematics education,* UNICAMP, Campinas, 1985, p. 71. Cf. also U. D'Ambrosio contribution to ICME8 (RL5) in which he analyzes various dimensions of ethnomathematics: mathematical, socio-political, epistemological, holistic, ethical, and pedagogical: Ethnomathematics: where does it come from and where does it go?, *Proceedings ICME8,* Seville (in press)

14 O. Raum, *Arithmetic in Africa,* Evans Brothers, London, 1938, 94 pp.

15 J. Gay & M. Cole, *The New Mathematics and an Old Culture – A Study of Learning among the Kpelle,* Holt, Rinehart & Winston, New York, 1967, 100 pp.

16 C. Zaslavsky, *Africa Counts: Number and Pattern in African Culture,* Prindle, Weber and Schmidt, Boston, 328 pp. (paperback edition: Lawrence Hill, Brooklyn, NY, 1979); Hungarian translation: *Africa Szamol,* Gondalet, Budapest, 1984; French translation: *L'Afrique compte! Nombres, formes et démarches dans la culture africaine,* Éditions du Choix, Argenteuil, 1995.

17 Cf. S. Touré (Ed.), *Mathématiques dans l'environnement socio-culturel africain,* IRMA, 1984

18 See e.g. : S. Doumbia & J. Pil, *Les jeux de cauris,* IRMA, Abidjan, 1992, and: S. Doumbia, 'L'expérience en Côte d'Ivoire de l'étude des jeux traditionnels africain et de leur mathématisation', in *First European Summer University Proceedings: History and Epistemology in Mathematics Education,* IREM, Montpellier, 1995, 549-556

and high schools in francophone African countries, coordinated by S.Touré, it is tried to incorporate elements from diverse African socio-cultural environments.

Since the beginning the 1980's, the Ahmadu-Bello-University (Zaria/Bauchi, Nigeria) has been very active in doing ethnomathematical research, e.g. on the mathematics used by unschooled children and adults in daily life, and the possibilities to embed this knowledge in mathematics education. S. Ale does research on the mathematical heritage of the Fulbe (Fulani) and the possibilities to construct a curriculum that builds upon this heritage and fits the needs of the Fulbe people.

Among a whole series of research projects, all over the continent, on spoken and written numeration systems, and their role in education, A. Kane's – today Senegal's Minister of Culture – profound study on "The spoken numeration systems of West-Atlantic groups and of the Mandé", may be singled out.

Southern Africa

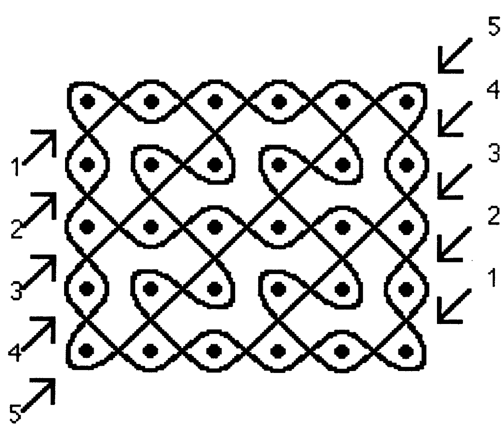
From the surviving San hunters in Botswana, H. Lea and her students at the University of Botswana have collected information. Her papers describe counting, measurement, time reckoning, classification, tracking and some mathematical ideas in San technology and craft. The San developed very good visual discrimination and memory as needed for survival in the harsh environment of the Kalahari desert. In "Common threads in Botswana" suggestions are included about the use of baskets, hair braiding, and weaving designs in mathematics education. K. Garegae-Garekwe concluded a masters thesis on "Cultural games and mathematics teaching in Botswana", and continues her research with an analysis of patterns on floors, walls, pots, basket weaving, knitting, etc. among the Tswana and possibilities to use them in mathematics education.

D. Mtetwa (University of Zimbabwe) started a research project on "Mathematical thought in aspects of Shona culture" and is also interested in the ethnography of children and its implications for mathematics education. B. Seka (Tanzanian Institute for Curriculum Development) experiments with the tradition of story telling as a didactical means in mathe-matics teaching.

The Association for Mathematics Education in South Africa (AMESA) organized at its first national congress in 1994 a round-table on ethnomathematics and education. In the same year AMESA formed a study group on ethnomathematics coordinated by D. Mosimege. Mosimege (University of the North) is preparing a Ph.D. dissertation on the exploration of string figures and traditional games from the north of South Africa in mathematics education. W. Millroy conducted an ethnographic study as an apprentice carpenter in Cape Town, to document the mathematical ideas that are embedded in everyday woodworking activities of a group of carpenters.

Mozambique

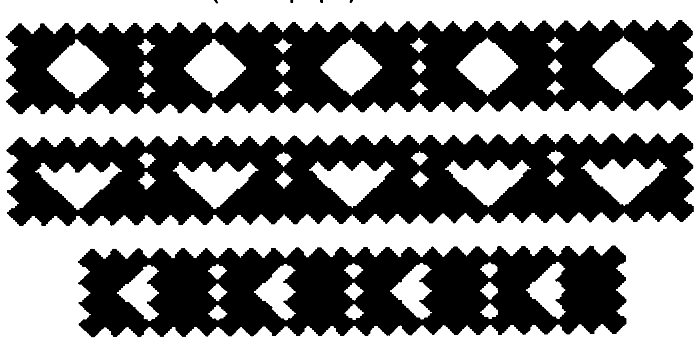
Ethnomathematical research started in Mozambique in the late 1970's. Several books have been published by the author in the context of Mozambique's Ethnomathematics Research Project. Among them are: "Culture and the awakening of geometrical thinking"; "Ethnogeometry"; "Living mathematics: Drawings of Africa" (see Figure 3); "Lusona: Geometrical recreations of Africa"; "Ethnomathematics: Culture, Mathematics, Education"; "African Pythagoras. A Study in Culture and Mathematics Education"; "Sona Geometry: Reflections on the Tradition of Sand Drawings in Africa South of the Equator" (Vol.1: Analysis and Reconstruction; Vol.2: Educational and Mathematical Exploration; Vol.3: Comparative Analysis); "Sipatsi: Technology, Art and Geometry in Inhambane" (see Figure 4); "Ethnomathematics and Education in Africa"; "Women and Geometry in Southern Africa: Suggestions for further research"; "Lunda Geometry — Designs, Polyominoes, Patterns, Symmetries".



Exploring a traditional sand drawing to reflect about arithmetical sums
Figure 3

Since the end of the 80's more lecturers and in particular young lecturers became interested in and started ethnomathematical research. So far, two collective works have been published. "Numeration in Mozambique" presents a reflection on culture, language and mathematics education. It includes studies on African systems of numeration (P. Gerdes & M. Cherinda), written and oral on spoken numeration systems in Mozambique, popular counting techniques (A. Ismael & D. Soares), comparative tables and maps on numeration (A. Mapapá & E. Uaila), and on spoken numeration and the learning of arithmetic (J. Draisma). In "Explorations in Ethnomathematics and Ethnoscience" is presented a collection of papers written by various lecturers at the 'Universidade Pedagógica', both based in Maputo situated in the South of Mozambique and in Beira in the central Sofala Province. The ethnomathematical papers reflect on some mathematical ideas involved in basket and mat making in the North of the country (A. Ismael), on languages and mental calculation (J. Draisma), on popular counting practices all over Mozambique (D. Soares & A. Ismael), symmetries on gratings in Maputo city (A. Mapapá) and on decorations of spoons in Sofala (D. Soares) and the southeastern Inhambane Province (M. Cherinda). Soon to be published are two more collective works: "Further explorations in Ethnomathematics and Ethnoscience in Mozambique" and "Geometry and Craft in Niassa and Nampula".

At this moment several Ph.D. theses in the field of culture and mathematics education are in progress: "Exploring basket and mat weaving in the mathematics classroom" (M. Cherinda); "Ntchuva and other Mozambican games in the mathematics classroom" (A. Ismael); "Gender and mathematics education in the cultural context of Mozambique" (S. Fagilde); "Traditional house building technologies and the teaching of Geometry" (D. Soares); "Language, culture and the teaching of Arithmetic in Mozambique" (J. Draisma); and "Children's games and toys in mathematics education" (A. Mapapá).



Example of woven band strips on 'sipatsi' handbags
Figure 4

Conclusion

Notwithstanding the research already realized, ethnomathematical-educational experimentation and, generally, the study of possible educational implications of ethnomathematical research, both internationally and in Africa, are still relatively in the beginning. In order to experiment a basic and radical assumption, namely that – in the words of A.Bishop – "all formal mathematics education is a process of cultural interaction, and that every child (and teacher, pg) experiences some degree of cultural conflict in that process", has to be taken into account. Established theoretical constructs of mathematics education are not based on this assumption. Generally ethnomathematical research findings oblige to reflect about fundamental mathematical-educational questions: Why teach mathematics?, What and whose mathematics should be taught, by whom and for whom?; Who participates in curriculum development?; How to organize the school practices in order to minimize the effects of the possible disruptive relationships between home and school culture and mathematics?

In the case of Mozambique, and maybe more generally in Africa, a fundamental objective of ethnomathematical research consists in looking for possibilities to improve the teaching of mathematics by embedding it into the cultural context of pupils and teachers. A type of mathematics education is intended that succeeds in dignifying and valuing the scientific knowledge inherent in the culture by using this knowledge to lay the foundations to provide quicker and better access to the scientific heritage of the whole of humanity.

A NEW ROLE FOR CURRICULUM DOCUMENTS - FROM GUIDELINES TO PRODUCTION PLANS?

Gunnar Gjone

RECENT EDUCATIONAL REFORMS

During the 1980s most industrialized countries have carried through extensive reforms in education. There are two characteristics in these reforms that should be stressed.

> One dominating element is that the reforms are directed towards greater decentralization, that is a shift in decisionmaking from central level to the periphery. There are exceptions to this picture, but these exceptions are countries with an already strong decentralized system. The other element visible in the reforms are the stressing of management by objectives.
> (Utbildningsdepartementet (The Swedish Ministry of Education), 1992, p.117. Translated by the author)

It is the second element, mentioned by the Swedish Ministry of Education that is our concern: Will a strong focus on management by objectives fundamentally change mathematics education? How is this management model related to the present situation of mathematics education?

To get knowledge on the state of education in a country, it has been a long tradition to compare with other countries. Projects in comparative education have also been used to investigate broad trends in international education. In this article we will consider the recent developments in some countries. Our first concern, however, is that the development in many countries are subject to comparisons with other countries, and this also influence the development in education in the countries studied.

THE CHANGING ASPECTS OF COMPARATIVE EDUCATION AND THE SEARCH FOR EFFICIENCY

From visits to statistics, or from qualitative to quantitative approaches

Comparative education is an active field among mathematics educators today. As mathematics educators have come to realize the international character of problems in their field, we have seen several projects where comparisons are made between programs and countries.

Comparative education is nothing new. The dominant form in earlier times was educators visiting other countries, making observations - or perhaps working there for some time, and then return to their home country with new ideas and knowledge. Such visits go far back, even to antiquity. A more systematic comparison is of a later date. Sjöstedt and Sjöstrand (1952) observed that increased activity in comparative education as a rule followed major conflicts.

They observed three periods of special activity in comparative education. The first they found after the Napoleonic wars, the second after the first world war, and the third after the second world war. In the first IEA[1] study (Husen, 1967) this fact is also presented, and the reasons discussed:

> As all who have followed events on the educational scene know, there has been an upsurge in comparative education since the end of World War II. ... The more we have recognized education as an investment in human resources and as an instrument for bringing about economic growth and social change, the stronger has been the need to investigate the roots of the educational systems of which the world around us shows such a striking diversity. (Husen, 1967, p.19)

But where the earlier forms of comparative education were qualitative studies, we have since the mid 1960s seen qualitative studies increasingly being replaced by more quantitative studies. The most wellknown such studies are the IEA studies. A strong focus of these studies has been testing of students, but there has also been a qualitative element in these studies, comparing curricula, textbooks as well as other elements of the educational system.

In recent years we have seen a large number of comparative studies. We can mention here the IAEP[2] studies. In the second study in 1990-91, a total of 20 countries participated in surveying mathematics and science performance of 13 year old students.

Another study to mention is the Exeter - Kassell study. This study not only included England and Germany, but Scotland, Hungary and Norway as well. A special feature of such comparative projects has been the media coverage in some countries: Results from the Kassell-Exeter project were presented in the Times, London, with the headlines: *Britain gets a minus in maths* (The Sunday Times, 14 May 1995).

Results from international comparison tests in reading/writing have been presented in the newspapers for some time. Comparisons and status

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1 IEA - International project for the evaluation of Educational Achievement

2 IAEP - The International Assessment of Educational Progress

reports concerning mathematics education have also been covered extensively in the newspapers. The United States is involved in several international and national projects where the goal is to "measure" the state of American education. "The Nations Report Card" NAEP[3] is an issue presented yearly in newspaper articles.

It seems that in some countries there are comparisons that get extensive media attention. In the United Kingdom there seems to be strong interests in comparing results with continental Europe (Prais, 1995). In the US there is a comparison with Japan, but also with the rest of the world. The notion "world class (mathematics) education" has been used from time to time to denote the goal for mathematics education in the United States - but it is not clear what the formulation means.

However, if we look at the situation in Norway, up to the early 1990s, we have been somewhat reluctant to compare our mathematics education with other countries, a notable exception being a more qualitative comparison with Sweden. The situation in Sweden has been somewhat different. Sweden participated in the Second International Mathematics Study (SIMS)[4] . It should be added that as a result of that study a large effort to reform mathematics education in Sweden was launched. We will most likely see a change in this situation for all the Nordic countries, since now most Nordic countries are participating in TIMSS[5] .

Another type of investigation which is qualitative, but not comparative, is an OECD evaluation of educational policies of various countries. In this type of evaluation, a group of experts travels quite extensively, visiting schools and school-authorities, as well as colleges and universities, e.g. in Norway. The report is a feedback to the Norwegian authorities on the impressions of the experts, who - in the Norwegian study - were from Ireland, Sweden and England. Important features of this evaluation were management and control at all levels of the school system, but some issues also considered the content of school subjects. These studies were clearly focused on the efficiency of the educational system in a country.

A feature of several more recent comparative studies has been the notions "effective" and "efficient". The volume examining data from the IEA Second International Mathematics Study (SIMS/SISS) had the title: "In Search of More Effective Mathematics Education". The notion "effective" is referred to in the index, but it is not given a descriptive definition in the

3 NAEP -The National Assessment of Educational Progress

4 SIMS/SISS - The Second International Mathematics/Science Study

5 TIMSS - The Third International Mathematics and Science Study

sense of Israel Sheffler (Scheffler, 1960). In a study, relating also to the second IEA study (Westbury, 1989) another similar notion is used: yield (due to Neville Postlewaite).

In the document laying the foundation for the third international study, efficiency is not mentioned explicitely, but various other similar notions are being used, conveying a more neutral connotation. The aims behind this type of comparative studies are outlined by David Robitaille and Cynthia Nicol:

> In addition, such studies provide valuable international perspectives for current national discussions and debates on development of efficient, effective, and qualitative mathematics education. (Robitaille & Nicol, 1994, p.403)

To strive for more effective and efficient mathematics education has been a goal for a long time in educational management, but it is not always clear what is meant by these words. In the report from the OECD experts on Norwegian education, effective education is linked to the use of resources and results, but again the concept results is not given a clear (descriptive) definition. We will conclude that comparative education has a double role: On the one hand it provides methods to study the development in a country compared to other countries, on the other hand it might strongly influence the developmetn within a country, with aiming towards larger efficiency.

What then should we mean by more effective and efficient education?

MAKING EDUCATION EFFICIENT

Efficient (effective) education is nothing new. It was in the forefront of discussion in the new math movement. In Bruner (1960) Bärbel Inhelder discussed over several pages the question if students can accelerate through the (Piagetian) stages of development.

Overview of trends

What is meant by more efficiency in education? Trying to look into this question, we find that some elements are often mentioned: resources, results, quality, and evaluation - to mention some. Concerning efficiency, important factors are the relationships between these elements. What results - and with what quality - can be obtained with given resources, and how can we evaluate the quality of the product?

The search for efficiency has been performed on all levels - from the central school authorities to teachers and students in the classroom. These questions have also been in the forefront concerning governement funded research. There is a clear parallell between education and research in this respect. We will come back to this relation at the end of the article.

An important element in Norway's school reforms was the relationship between money for education provided by the parliament, and the "output". "Output" is in this context taken to mean the performance of the students (on national tests).[6] The relationship between economy and education is important today, and has been given much consideration by educators and administrators. One such special case - that has been much debated in several countries is comprehensive education.

The case of comprehensive education

Comprehensive education is a term usually linked to secondary education. In the United States the following formulation has been taken as a definition:

> It is called comprehensive because it offers, under one administration and under one roof (or series of roofs), secondary education for almost all the high school age children of one town or neighborhood. ... It is responsible for educating the bright and the not so bright children with different vocational and professional ambitions and with various motivations. (John W. Gardner in the foreword to (Conant, 1959)).

The term comprehensive education can also be used in a more general sense. In Scandinavia the corresponding term "enhetsskole", is used to denote a school system where all children in a district are taught together for a large part of their education. Comprehensive education is moreover not an absolute entity, there are degrees of comprehensiveness.

In March 1996, a matter concerning education was on the front page of several English newspapers. The reason was statements that the labour shadow government had made concerning comprehensive education. From a tradition of fully supporting an extensive form of comprehensive education, newspaper articles stated that some changes were underway in the labour party - not favoring comprehensive education to the same extent anymore. In the debate and articles that followed the initial article we find:

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6 In Norway, in contrast to the United States, little is stated in official documents on what is meant by output and results.

> Mr. Blubkett called for more setting by ability more specialization and a greater emphasis on vocational alternatives to a narrowly academic. curriculum (The Daily Telegraph, February 28, 1996)
>
> The Office for Standards in Education said many schools were not even aware of the targets, which were launched five years ago in an effort to increase skill levels throughout the workforce and strengthen international competetiveness. ... Heads were concerned that the targets were nearly always promoted in relation to increasing Britain's industrial and commercial competitiveness. (The Daily Telegraph, March 5, 1996)

It was thought provoking to read this short and intense newspaper sequence, before other news again would dominate the front pages. In all the Nordic countries we have a well established comprehensive education. In Norway, with the present school reforms, we moreover have extended compulsory education from 9 to 10 years, and in addition all students will have the right to 3 years of further education after the first 10 years.

The issue of comprehensive education has an obvious link to efficient (mathematics) education. The discussion on comprehensive education has come up in several countries from time to time, e.g. also the German discussion on Gesamtschule, in the 1970s and 80s. An argument in such discussions has often been that comprehensive education might not be very efficient and competitive in the international marketplace.

In some countries this has lead to a search for models on how to manage education. In some countries models have been found in the private sector (models for production and control). This has - in most cases - not been explicitly stated. One exception, however, is the Swedish government. Following the quote in the introduction to this article it is stated that:

> The kind of management by objectives that has developed has elements and knowledge from management of other types of systems, especially from private sector. ... As time goes, management by objectives in education has found it's own form. (Utbildningsministeriet, 1992, p.117)

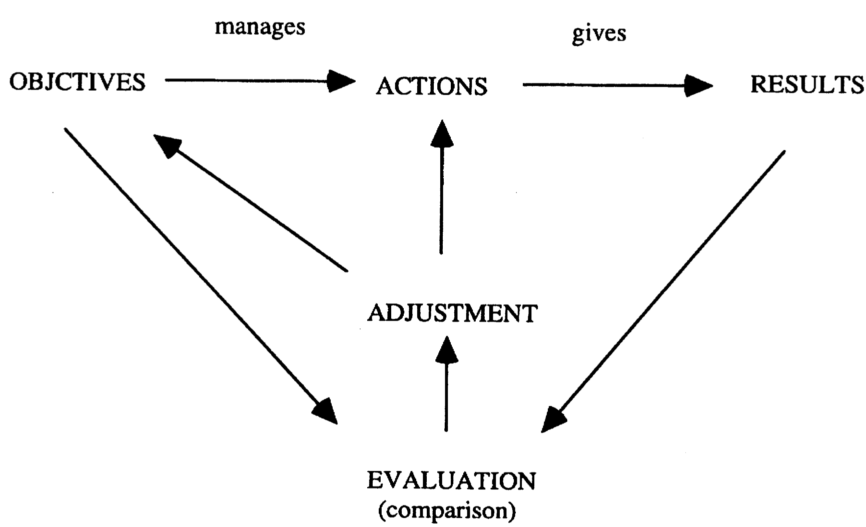
It is interesting to note that the Swedish ministry is stating this relationship explicitly. However, it is not clearly explained what this "own form" is. To find out what is usually meant by the concept *management by objectives* we will look to management literature on organizational theory.

MANAGEMENT BY OBJECTIVES (MBO)

We will present the introduction to a chapter on management by objectives from a Norwegian textbook in organizational theory. This passage is not written for school systems, but could easily be applied to such systems as well:

> Management by objectives from the leadership is that an upper instance manages by giving a message about a wished result and sets frames of resources. The subordinate unit is free to choose how and by which means the objectives should be reached. In addition to stating the objectives the reporting of results is the central element in management by objectives.
>
> Objectives, have in this context, two different functions on two different levels. On the one side it gives subordinates a goal (objective) to reach for. On the other side management by objectives is a tool for an upper instance to evaluate their subordinates. (Flaa, et al., 1995, p.118. Translated by the author)

The mechanisms of MBO are also presented in a diagram and discussed :



(Figure 1: From Flaa et.al.,1995, p.121. Translated by the author)

In the book we also find a discussion of MBO in public sector which is especially relevant for education. A thorough discussion of prerequisites for MBO is presented, as well as parts of the Norwegian discussion.

Relating this to education, we find that management by objectives can be introduced at several levels: It can be introduced in the classroom, but also at school level, hence the present interest in "school evaluation". We also find "program evaluation", as well as projects to "evaluate" several aspects of the school organization.

The Norwegian philosopher Hans Skjervheim, has introduced the two notions:

(1) MBO as a *system*, and (2) MBO as a method. (Flaa et al, 1995, p.122)

MBO as a *system* means that this way of management is introduced at all levels. In a sense reality is adjusted to the model. Reality is simplified and delimited. One concentrates on concrete and restricted objectives (aims). MBO as a *method* is a more pragmatic use of the management model - the model is adjusted to reality. Taking into account the wealth of evaluation procedures now being used in education, it seems that it is MBO as a system we see developing in education. An important element of *education* (German: *Bildung*) is in danger of being lost using management by objectives as a system. *Education* in this sense has more elements than can be regulated by an MBO model.

In Flaa etal. (1995) there is also a discussion of the use of MBO in the public sector. Two reasons are given for this use: (1) Use for political control; (2) Use to increase efficiency, to get more and better results

The authors also outline possible conflicts that may arise between the different levels of management if the objectives are very precisely stated. They argue that conflicts may seemingly be avoided by stating vague objectives, but that this will introduce new conflicts at different levels. In education, the formulation of broad and general goals and objectives on a high national level, might introduce conflicts when one seeks to make these concrete, say for a subject like mathematics. As a consequence, to avoid this situation, we find, that very precise objectives (targets) are being formulated at a national level in several countries.

Student assessment is also given broad attention in the process. Students' performance on tests can be seen as the outcome of education. An international study like TIMSS focuses on this dimension. We will come back to student assessment later in this article.

What are the consequences of this "philosophy" of education found in the curriculum materials that are prepared for school systems? If we look at curriculum material, constructed in the present reform wave, can we find reflections of the elements in the diagram presented above (Figure 1)?

Curriculum documents

Curriculum documents are reflecting the educational ideas of a country, "carries a message about a system" (Howson, 1991). Authorities have, for a long time, shown faith in the role of formal curriculum documents. (This faith, however, has not been shared by the (mathematics) education community in general, e.g. the popularity of the notion of the so-called hidden curriculum.) MBO has lead to some important changes in curriculum documents. There is a marked change in the formulation of aims for education in some countries. Formulations of the targets are moreover constructed to address an objective that can be easily assessed. As an illustration let us look at an example from Norway (upper secondary mathematics education, grade 10):

TARGET 7: Functions

Pupils should become familiar with the function concept and learn to draw graphs with and without IT aids. They should be given a first introduction to the derivation concept and be made familiar with it by means of simple examples.

Main points:
Pupils should
...
7c be able to find the points of intersection of curves both by calculation and by means of graphs.
(Norwegian Ministry of Education, Research and Church Affairs, 1994)

It is interesting to compare today's language of education found in curriculum documents, with the language used some years back.

The language of education - revisited

In his well known book *The Language of Education* (Sheffler, 1960) Israel Scheffler lists some words and formulations related to education: "knowing", "learning", "thinking", "understanding" and "explaining" (p. 8). He adds a few more "mental discipline", "achievement", "curriculum", "character development", and "maturity" (p. 9). If we were to write down such a list from some curriculum documents written today, would we find the same words and formulations for inclusion on a list? Would it not contain word such as "assessment", "evaluation", "quality", "standards", "(attainment) targets", "aims" - perhaps "outcomes" and "results", would be listed as well. We could ask if this set of words, also to be found in other areas of society like production and management, signifies a new view and role for education?

As seen from form the diagram on MBO above (Figure 1), *evaluation* played a crucial role in the process. Let us try to relate evaluation/assessment in (mathematics) education, to the MBO model.

A ROLE FOR ASSESSMENT

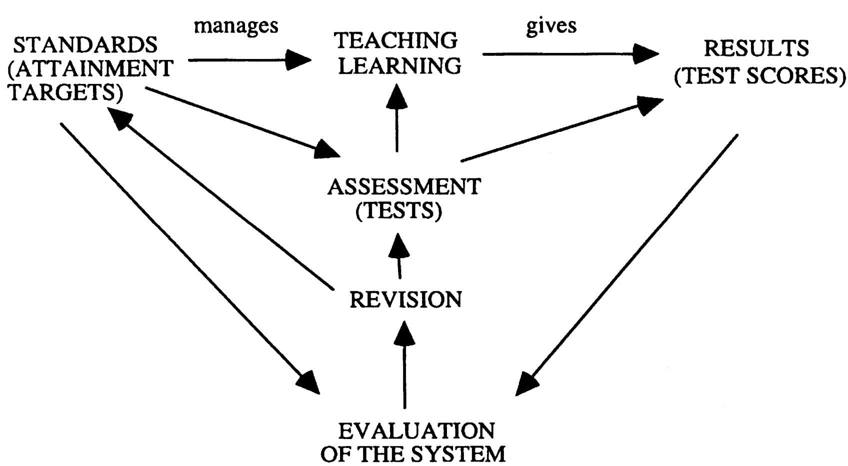
Assessment of students has traditionally had several aspects:

• the information aspect, giving information to the student, the parents, the teacher, the school system

• the motivational aspect: to motivate students (and teachers)

• the selection aspect: to select students for further advance through the educational system

However, using the diagram above (Figure 1) we might see student assessment in the MBO model in a somewhat different perspective from what is usually the case. Below we will present a revised diagram focusing on assessment of students in education in an MBO model. The categories have been given slightly different names, to be closer to "the language of education" being used today:



(Figure 2)

The diagram focuses on the importance of assessment (tests) in MBO. The testing system in some countries like Norway and the United Kingdom fits very well into this model. It has been a well known "fact" in education, that assessment (tests) has a strong influence on the teaching-learning process. In this diagram we try to show that in the MBO model it will have a central position in managing education (the teaching-learning process).

In this discussion focus has been on centralized systems, like we find in the Scandinavian countries. However, the elements would be found in many decentralized systems as well. It might be, like in the United States, that there are various instances being responsible for the various parts, such as curriculum and (some form) of testing within the system. I will argue that the model is valid also for countries like the United States.

A first conclusion

Will using this model for education bring changes to the system? The answer will be depending on the educational tradition.

The central element in the model is assessment and testing. Linked to this is the formulation of goals, standards or targets. The important relation to realize is that in such a model, one seeks to formulate quite specific targets for student performance at some level, in mathematics.

In some countries this will mean a major change, since testing/assessment within the system in a comprehensive type of education, will have a different function from the one presented in the diagram (Figure 2). We will use this diagram to show how different countries have different traditions in education, and that they are moving in a direction of a more complete MBO model.

In the United States there is a strong tradition of testing, and looking at the result of tests (the right side of the diagram). Recently we have seen advances in the process of establishing more precise standards (or benchmarks) at national and hence at state level. (NCTM, 1989; AAAS,1993)[7] In England, with *the National Curriculum* they seem to try to adapt to all elements of this model in one step, starting from a very decentralized system.

We will also note that the model we have presented for MBO in education is a variation of the general MBO model (Figure 1). We will therefore challenge the opinion in the Swedish document cited above, that

[7] NCTM - National Council of Teachers of Mathematics
AAAS - American Association for the Advancement of Science

there is an "own form" of MBO for education. We argue that it at most will be minor adjustment to the general model.

We will also argue that the change of the "language of education" (in the spirit of Sheffler), signifies a deeper change in the same direction, in the state of educational philosophy.

The model of management by objectives depends on goal formulations to be quite specific. The question if it is reasonable to give such specific formulations of goals in mathematics education, is an important question. On the one hand mathematics is often presented and thought of as a strong hierarchical discipline, which would be ideal for a structure of attainment targets. However, this view is not the only possible of mathematics for education.

MANAGEMENT BY OBJECTIVES IN MATHEMATICS EDUCATION

Mathematics in school is closely related to some other areas. We will here consider some of these areas: mathematics, development of mathematical knowledge, and the research discipline of mathematics education (didactics of mathematics).

The nature of mathematics

The structure of mathematics as a deductive system is - in some sense - well suited for MBO. It is possible to formulate a hierarchy of attainment targets, e.g. the National Curriculum in England.

Such a hierarchical view of mathematics, however, is being challenged by many mathematics educators. Activities like investigations, explorations, and problem solving do not fit the structure as well as facts and skills do. The following quote expresses what many mathematics educators see as important:

> As a matter of fact, we want students to understand that mathematics is, essentially, a human activity, that mathematics is invented by human beings. The process of creating mathematics implies moments of illumination, hesitation, acceptance, and refutation; very often centuries of endeavors, successive corrections and refinements. We want them to learn not only the formal. deductive sequence of statements leading to a theorem, but also become able to produce, by themselves, mathematical statements, to build respective proofs, to evaluate not only formally, but also intuitively the validity of mathematical statements. (Fischbein, 1994)

With this view of mathematics - and hence of school mathematics - it is problematic to construct a structure of precise attainment targets. Moreover, a step by step process implicit in such a structure of targets is not the only possible process of building competence in mathematics. This view of building competence has been realized in other subjects as well as mathematics. Some interesting models in teaching reading, writing and mathematics, of what is called cognitive apprenticeship are presented and discussed in (Collins, et al., 1989) [8]

An important type of consideration in education is how knowledge is developing in the individual. Constructivism in one form or another has a strong influence in the mathematics education community.

The nature of learning (knowledge development)

There are various forms of constructivism, but they all have some common elements. One element present in most forms of constructivism is the following: *Knowledge is constructed actively by the individual, it is not passively received.* (e.g. Kilpatrick, 1987). How does a constructivist view of the teaching-learning process correspond to the MBO philosophy?

With a constructivist view of the teaching-learning process, there must be a freedom of interpretation and choice for the teachers:

> Using their own mathematical knowledge, mathematics teachers must interpret the language and actions of their students and then make decisions about possible mathematical knowledge their students might learn. (Steffe, 1990)

There will be a problem with teachers' freedom to choose if the targets are very precise, and issued at classroom level. One might argue that this is not the case in most school systems today, but the combination of extensive use of tests linked to targets will clearly have consequences for the teacher.

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8 I would here like to add a personal observation. There are many similarities between mathematics and music, and between learning mathematics and learning to perform music. To learn to play a piano, was considered a step by step process, building up from simple exercises to more and more complicated tasks. A lot of "meaningless" exercises were necessary to advance through the levels of playing the piano. This was my experience when learning to play. When my daughter started piano lessons, she was going almost directly to playing melodies. Music was "in context". The"meaningless" exercises were introduced late when/if needed. I notice today that my daughter has more confidence in playing piano than I have.

Research in mathematics education

Education and research are two areas that in many countries are managed and funded by the government. What we have seen, e.g. in Norway in later years, are attempts by the authorities to manage research in the same way as outlined above - management by objectives.

The restructuring of government funded research has been an issue in Norwegian research administration. The establishement of new councils and units for administrating research has gone on for some time. The driving force has been efficiency of research - more results for the resources. These attempts have been argued against by researchers doing basic or fundamental research.

A Norwegian philosopher and scientist, Nils Roll Hansen, has argued that basic research does not contribute to solve the problems in society, but rather it helps formulate the problems (Roll Hansen, 1995). Basic research is in this respect not very efficient, not very economical to use this term.

Would this not be the case for much research in mathematics education? Mathematics education research has a tradition of close ties with practice - the teaching learning situation in classrooms. However, there is also a sizeable amount of what could be termed as basic research, seeking to formulate fundamental questions. Much of what is developed concerning knowledge development (e.g. constructivism) is also not directly applicable in the classroom. There would be a very different situation for mathematics education research if the model MBO should be used to manage research in the field.

CONCLUDING REMARKS

If we consider the reasons for MBO stated in (Flaa et.al., 1995) there would be several different functions for use of MBO in education:

(1) Objectives are formulated for the teachers and students to reach for,

(2) MBO will be a tool for school authorities to evaluate teachers and students,

(3) To increase efficiency.

There is a long tradition of setting aims or objectives in education. As Diane Ravitch argues (Ravitch, 1995), "standards" were used in the

United States in the 19th century. Also, the strict criteria of minimum competency that were used in some countries, could also be considered as part of the same philosophy. Few would argue that there are need for aims in education. However, what is new in the more recent development, is the structured approach to MBO in education. Large efforts in many countries now go into formulating very precise targets/standards for education, and systems of testing are being developed to go along with the targets. The consequences are that school subjects are presented as a set of "targets" or "standards". Students' performance on "targets" will be the "outcome" or "results" of education. For school authorities to evaluate the system (teachers and students), they would look to results.

Again, few would argue that good results are "good", but there are several problems: Is it possible to state precisely the type of objectives wanted in mathematics? There will be a danger of having skill related objectives - that are easy to assess. With the responsibility on how the objectives are attained in the end, solely on teachers and students, drill and practice activities might dominate classroom instruction.

There is also an obvious link to efficiency in education, which we have tried to show is a problematic concept. In MBO economic priorities will have to be decided in the "lower levels" of the system. Hence using MBO is also a way for school authorities to limit resources, but insisting on the same objectives being attained. It is a question if school systems can - or should - function like corporations in such cases.

Is it possible for us - with our market economy thoughts penetrating all layers of society - to think of alternatives to the MBO model? An essential factor of an alternative would be to give more freedom to teachers, and at the same time having the school authorities take a more direct responsibility in the teaching learning process.

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WHAT RESPONSABILITY DO RESEARCHERS HAVE TO MATHEMATICS TEACHERS AND CHILDREN?

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Research in mathematics education is mainly carried out by students working towards higher degrees. They are essentially apprentices. Research by professionals, designed to answer wider more significant questions must be for longer periods and needs external funding so that it can be pursued full-time.

There are different viewpoints on what constitutes research, as witnessed at the ICMI Conference "What Is Mathematics Education Research" in Maryland, 1994, the report of which is expected soon. In 1989 in Paris a discussion group of the Psychology of Mathematics Education Conference agreed that research in mathematics education should contain the following:

1. There is a problem.
2. There is evidence/data.
3. The work can be replicated.
4. The work is reported.
5. There is a theory.

My position is that research should at least contain all these.

Research and Evidence

For whom is research carried out? My point of view is that researchers in mathematics education should be trying to answer classroom questions concerned with teaching and learning mathematics. We know very little about how humans, especially children learn mathematics, how to teach effectively, what is the most advantageous order in which to present topics, etc, etc. These are questions of importance to millions of children and their teachers.

The responsibility for informing teachers carries with it the need to provide evidence that is sufficiently full and convincing for teachers to accept. Teachers in turn have the responsibility of looking at evidence carefully and then making decisions on its acceptance not letting prejudice and beliefs cloud their judgement.

The National Curriculum of England

In the United Kingdom we have had the same political party in power for 16 years. There is no child in British schools who has known or been educated under any other government. We have had at least 16 'Education Reforms' and each has been far-reaching and enforced immediately by law. Since 1988 there has been a National Curriculum instituted in England (a different one is implemented in Scotland). Britain never had such a curriculum before, although there always was considerable unity because of the united advice from an independent inspectorate, national school leaving examinations and a choice made by the majority of schools to use the same textbooks. Since 1988 Mathematics has been one of the three 'Core' subjects in the school curriculum to be studied in state schools during the ages of compulsory schooling (5-16 years). Private schools do not need to follow the contents of the National Curriculum. The contents of the mathematics curriculum are listed in 'levels' (there were ten but there are now eight). The set of levels provides a progression of mathematical topics which are supposed to match the development of children. The first set of level descriptions was very detailed, drawn-up by a committee of mathematics educators. These persons placed considerable emphasis on using and applying mathematics, thus showing what they valued. The curriculum has been changed three times by others since then and is now composed of four topic strands rather than 14. It still contains a strand called "Using and Applying Mathematics".

Testing

Tied to the content of the National Curriculum is the legal requirement for the schools to test their pupils with statutory externally written (and marked) tests at the age of 7, 11 and 14 years. There are national school-leaving tests at age 16 years, set by private examination boards. Additional to the statutory tests teachers are required to give their own assessments. Parents are important recipients of this information and every home in the country has received a copy of the "Parents Charter" which says the following:

> Pupils will be tested in the core subjects in English, maths and science at about the ages of 7, 11 and 14. The tests are designed to be easier for teachers to manage than they were in the past. But they will still be challenging. They will show what pupils are able to do at the key stages of their time at school. They will help you to know how your child is doing. And assessing pupils against national standards will be made easier for teachers.
>
> (Parents Charter 1994)

In the absence of multiple items to exemplify the rather general descriptions of the national curriculum we now have, the tests are used to interpret the meanings and requirements. They have as well become instruments which record a) the achievement of children, b) the effectiveness of the teaching in delivering the contents of the curriculum and c) keep teachers and schools 'in line'. Notice that schools are to improve through no extra training, provision of materials or RESEARCH but simply through fear of what test results might bring.

The inevitable 'teaching to pass the tests' has of course become apparent and the Review of Assessment and Testing published by the School Curriculum and Assessment Authority (1995) admits that schools are narrowing the range of questions children are given.

Expectations

In the first version of the Mathematics Curriculum (1989) it was stated that during the key stages 5-7 years, 7-11 years, 11-14 years and 14-16 years the content of school mathematics would encompass the following levels:

		Mathematics Levels
(5 - 7 yr)	KS1	1 - 3
(7 - 11 yr)	KS2	2 - 6
(11 - 14 yr)	KS3	3 - 8
(14 - 16 yr)	KS4	4 - 10

It is expected that the attainments of the great majority of pupils at the end of key stages will fall within the specified ranges of levels.

(Mathematics in the National Curriculum 1989)

More important was the statement of expectations held concerning pupils' performance.

> In formulating the statements of attainment which define the levels, the working assumption, which will need to be tested in practice, has been that pupils should typically be capable of achieving around levels 2, 4, 5/6 or 6/7 respectively at or near the reporting ages of 7, 11, 14 and 16. The levels at the outer limits of the ranges will be achieved by a minority of pupils only.
>
> (Mathematics in the National Curriculum 1989)

Since 1989 the content of the levels has changed, for example the subject 'Vectors' no longer appears BUT the expectations are the same. This has given rise to the continued harassment of English mathematics teachers by the media with headlines such as:

England's maths wins yet another brickbat

Primary maths in trouble

The 'expected' levels with no research to show whether the expectations are all reasonable have become the standards by which schools succeed or fail. At age 11 years the expected performance is success at level 4 or higher. The highest scoring English geographic region had 58 per cent of its eleven year olds reaching this score. Perhaps the expectations were wrong. Our politicians do not consider this; they declare English children have failed. Nobody declares we need to redefine the expectations because that would be tantamount to saying 'standards' should be lowered. This word 'standards' is usually based on myth e.g. (1) "In the past everybody could calculate their shopping bill"; (2) "Everybody in my class could do long division". Not true! You only have to read reports of school mathematics 50 years ago to realise how little has changed. This is not a cry for complacency but a request for a thorough investigation of what it is we are trying to preserve. There are people who think a good standard is preserved when 60 per cent of the candidates in an examination fail. This is surely a terrible waste of human resources - failure is not motivating for any learner.

Myths and Folklore

Over the last 30 years mathematics educators have espoused many causes but done little to show whether these 'good ideas' work with an ordinary class. We have said "using bricks are good things in the primary school". We had no research evidence of the circumstances in which the manipulatives might aid learning and when children still did not add fractions successfully we said 'the bricks have not been used enough'. Teachers knew better. Manipulatives are not the miracle products we pretend. Perhaps the bricks are made of coloured wood. Why should two pieces of pink wood and one of green lead the child to solve $2x - 5 = 17$? There is a very large gap between these two statements, one in wood and the other in symbols. As I have quoted before of a child I interviewed, who showed great sense when I asked whether the concrete aids used by her class in learning subtraction with decomposition and the symbolic subtraction she was doing in her book were connected. "Sums is sums and bricks is bricks".

We have told teachers that part of their work is to write the materials for their class. We have not thought how difficult (and different) it is to write mathematics text and questions. Why should teachers be good authors? It should be enough that they be good teachers. Two of my students at the Shell Centre, Beth Moren and Dora Santos have been researching teachers' use of textbooks. Let us not suppose that teachers have stopped using mathematics textbooks (as is shown by the survey of Johnson and Millett, 1996)

Table 1 Proportion of pupils' mathematics work done from a commercial scheme (question 15) (%)

	n	0	1 - 5	6 - 20	21 - 50	51 - 80	>80
KS1	215	11	15	19	23	27	6
KS2	282	4	6	7	24	49	10
KS3	237	4	4	5	8	41	38

KS1 - 7 yr
KS2 - 11 yr
KS3 - 14 yr
(Implementing the National Curriculum (Johnson & Millett))

Teacher-trainers and advisers have often been heard to say 'you should dip into a textbook, do not follow it page by page'. "Slavishly" is sometimes added, gratuitously. This really appears to be strange advice when one considers that the author of the book has presumably sequenced the content so that it made sense and in order to address prerequisites. "Dipping-in" destroys this sequence. If a teacher was seeking examples to illustrate or extend material already available, then the textbook may be being used for a purpose not intended by the author but better served by lists of examples/word problems.

We have started some research on styles of using printed material in mathematics classrooms. Initially with one student's PhD work in which secondary school mathematics teachers are observed in class. The teachers are also interviewed and asked how the material has been chosen. Is there a set of textbooks chosen by the department? What part was played in the choice by this individual? The observations are for a number of reasons for example:

a) to ascertain whether a new topic is introduced by the teacher or by the child reading the textbook.
b) How close to the book is the teacher's introduction?
c) Do the examples that follow match it?
d) How do the children use the material?
e) How close to the teacher's objectives for a lesson (or series of lessons) is the class performance? In other words "How effective is the teaching?"

The simple recognition of different styles of textbook use should allow teacher-trainers the opportunity to discuss what is likely to be the principle teaching aid with future teachers. It should be possible to suggest a variety of styles amongst which the young teacher could choose and which might be adapted for individual needs.

What is in textbooks is often decided by an editorial assistant working in the publishing house and not one who necessarily knows anything about teaching or mathematics. The author might have considerable control over the words and exercises in the textbook but little influence on the illustrations. Modern mathematics textbooks are full of illustrations. These are not diagrams or graphs which might be considered to be part of the content of mathematics but pictures showing "real-life contexts" and meant to be informative and motivating. The quality of the drawing is often poor and the scene depicted is often irrelevant or more dangerously it is inconsistent with the information given in the text. Children make the best effort they can to cope with the printed page (we teachers do not explain how to use a textbook) and sometimes the child-rule is to ignore all pictures. Textbooks are very important in children's mathematical lives but we know very little about how they might be produced to achieve greater effectiveness.

Conclusion

This paper based on a lecture given in Seville is a cry for more research. The citizens of our countries should be demanding evidence of benefit before their children are subjected to yet one more education reform. Our teachers should be demanding research before they try out yet one more 'good idea' and we, the professionals should be making our presence felt. Decisions on what is given to children in the name of mathematics, are being taken by amateurs.

MATHEMATICS AND COMMON SENSE

Geoffrey Howson

Let me begin by explaining what prompted my current interest in this topic, and also by emphasising that here I shall be presenting very much a personal view, and not a survey of all recent writings, such as the report of a meeting held in Berlin on this subject (*Keitel et al, 1996*) or the chapters in a forthcoming volume written by members of the Basic Components of Mathematics Education for Teachers project (Hoyles, Kilpatrick and Skovsmose, to appear).

About three years ago I became involved in two pieces of work: one was linked with the Third International Mathematics and Science Study and involved the study of Grade 8 texts drawn from eight different countries (Howson, 1995); the other was part of the BACOMET group's investigation of "meaning" within mathematics education. One important aspect of the latter was the way in which we help students to construct not only "meaning" for concepts, but also to provide "meaning" for their study of mathematics itself.

With one exception, the authors of the texts I considered all sought to supply "meaning", in both senses, through the use of real world examples and the obvious utility of the subject. Moreover, it seemed mathematics was to be viewed as a specialised form of "common sense" and the various approaches appeared to be based on the principle that all we are doing in school mathematics is trying to codify common sense and to extend commonly held notions. (The one exception was a book which still showed the influence of the 1960s French approach to mathematics teaching. It laid considerable stress on algebraic structures introduced in a "semi-formal" way, although now tempered by reference to a number of "motivatory" real-life examples. That textbook, however, was in the process of being superseded.) Even more recently, I have seen various materials which have placed less emphasis on the "real world" and more on mathematical "activities", some of which were quite abstract. Yet even these texts appeared to assume that the use of common sense would lead to the desired mathematical outcomes, and, because of their lack of emphasis on definitions, facts or skills to be learned, that the aim of the teaching was to develop a kind of specific mathematical common sense which could then be applied to any mathematical problem.

Now it is important to emphasise at this point that I should not want entirely to abandon such approaches, particularly in the early years of education. However, my study of the Grade 8 texts alerted me to the tensions which will naturally arise if the shortcomings and possible dangers of such an approach are not recognised.

My aim, then, is to look at some of the interactions between mathematics and common sense - for we are not talking of an interface, the two are not disjoint; to consider the implications for the ways in which we both think about mathematics and also teach it; and to cause us to re-examine certain ideas, beliefs and teaching practices.

First let us begin with a dictionary definition of "common sense":

Common sense: average understanding; good sense or practical sagacity; the opinion of a community; the universally admitted impressions of mankind (Chambers' *Twentieth Century Dictionary*)

We see immediately, then, that "common sense" is far from being a scientific term: it is a vague, culturally dependent, but nonetheless extremely valuable concept. In essence, it provides us with a means to talk about mathematics, and with a rudimentary, one-way form of logical reasoning. Normally, "use your common sense" is a call for someone to come to a conclusion or devise a course of action based on local knowledge, past experience and simple reasoning. Common sense is distinguished by the way in which it depends upon evidence, accepted truths and conventions, and upon "innate" operating systems of perception, meaning and understanding. There is no doubt that it provides a powerful tool for survival in social life. It cannot be denied, then, that "common sense" is something which educators must try to develop in students and, conversely, something on which we must draw in our teaching.

But common sense is not just something which can aid mathematics learning, it can be seen as the foundation upon which mathematics is erected. In his "China Lectures" (1991), Freudenthal asks whether or not common sense is not the primordial certainty: the most abundant and reliable source of certainty within mathematics. He goes on to point out the extent to which number and elementary geometry (eg, ideas of similarity) are grounded in common sense. The significance of this to me is as great as the implications of common sense for mathematics teaching. Many of you will be familiar with, at least, the title of Morris Kline's book, *Mathematics: the loss of certainty,* and will know of the implications of

Gödel's work. I am certain that much will be made of this in other talks at this Congress. Nevertheless, such findings have not led to a loss of faith, changes in practice, or to an increase in neuroticism, within university mathematics faculties. The brilliance and significance of Gödel's work is readily acknowledged, but essentially mathematicians take refuge not in their axiom systems, but in the nesting of logical models which, for example, rest the consistency of hyperbolic geometry upon that of the natural numbers, and these, as Freudenthal remarks, are firmly grounded in common sense - the ultimate basis for the mathematician's confidence in what he or she is doing. I shall return to such matters, including questions of "fallibility", later in my talk.

In reading the Grade 8 texts, I became aware of how these, in many countries, represent a "fault line" so far as arguments based on common sense are concerned, for it is at this level that many countries introduce the multiplication of negative integers. Here, probably for the first time at a school level, the link between common sense and mathematics breaks down. Negative numbers have caused difficulties to more than Grade 8 students. For example, in the late 1700s and early 1800s, many in England still looked upon such numbers with considerable misgiving. Indeed, in 1796, Frend, a Cambridge mathematician, produced an algebra text in which he avoided their use. He argued that "multiplying a negative number into a negative number and thus producing a positive number" finds most supporters "amongst those who love to take things upon trust and hate the labour of serious thought", for "when a person cannot explain the principles of science without reference to metaphor, the probability is that he has never thought accurately upon the subject"[1] . Frend's son-in-law, the better-known mathematician, De Morgan, was to write in his *On the Study and Difficulties of Mathematics* (1831) that "the imaginary expression $\sqrt{(-a)}$ and the negative expression -b ... are equally imaginary *as far as real meaning is concerned* (my italics). [One] is as inconceivable as [the other].[2] "

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1 Frend was removed as a tutor from Cambridge, not because he lacked belief in negative numbers, but because of his unorthodox religious beliefs: as a Unitarian, he did not believe in the Holy Trinity. He was later banished from the university when he publicly denounced the war against France. Cambridge was prepared to tolerate mathematical unorthodoxy, but not religious or political dissent.

2 In the early 19th century negative numbers were referred to by many English authors as "imaginary numbers". Nowadays, see below, negative integers have become part of "local knowledge", thus adding weight to Paul Langevin's claim that "the concrete is the abstract made familiar by usage".

Frend's objections to negative numbers were essentially based on common sense: how could multiplication of two of these mysterious objects yield something with which he was familiar - something which was part of common sense? Many other examples illustrate what happens when the links between mathematics and common sense break down. When Gauss suppressed his findings on hyperbolic geometry because, as he wrote, he feared the shrieks which they would elicit from the Boeotians (a Greek tribe traditionally considered dull-witted), he surely meant that many of his fellow mathematicians would reject, as an affront to their common sense, the assumption that given a point and a line in the plane then there could possibly be more than one parallel to the line through that point. Du Bois-Reymond objected to Cantor's set theory because "it appears repugnant to common sense" (1882, quoted in Kline, 1972, p. 998). Clearly, the theory of infinite sets **does** contradict common sense. Let us take a simple example. Suppose there are footballers on a field, some in red kit and some in indigo. If I ask them to form a line according to the rule that between every two in red there must be one in indigo, and between every two in indigo one in red, and then note that the line begins and ends with a person in red, common-sense tells me that there is one more person in red than in indigo. But what happens on the number line in the closed interval 0 to 1? The interval begins and ends with a rational. Between every two rationals there is an irrational and between every two irrationals a rational. But there are not more rationals than irrationals: indeed the number of rationals is insignificant compared with that of the irrationals. What has happened to common sense? Other such examples exist outside of set theory, but there is not time to give them here.

Mathematics should not be confused with, or constrained by, common sense. The latter, if it is to become genuine mathematics, must be systematised, organised and, if necessary, formalised. Arguments must be based on more than that elementary logic and inferential reasoning which underpin common sense. These seemingly obvious remarks can create problems for the textbook author or curriculum developer. Let us take one simple example based on exercises to be found in one of the series I studied. Students were provided with photographs and asked, on the basis of the plans provided, to identify the church photographed, or, using a town plan, to say from which point a particular photograph was taken. These I see as useful activities to develop "spatial awareness" (whatever that means!), which mathematicians will wish to draw upon. Yet in these sections the textbook had no "kernels" to offer, that is, no mathematical definitions, results or procedures which students might identify, learn or follow, or which might help them to crystallise what they had learned into a usable form.

Opportunities were provided for students to use their "common sense", presumably in order to develop this further. It was hoped that in so doing they would also develop desirable, idiosyncratic traits. However, the logicality and systematics which lay behind these were never discussed or assessed. Is this the best which we as mathematics educators can offer? Strangely enough, 1960s textbooks carried within them a language of systematisation that might be applied to this problem. Looking at a town plan we could consider the set of all points from which, say, the spire of St Peter's church would appear to be on the left of that of St. John's. Consideration of the intersection of various such sets would lead to the desired answer. In the 1960s such context-based questions were rarely, if ever, set. In the 1990s there is a danger that context and a dependence upon common sense are driving out mathematics. There is a nice balance to be observed. We have to build upon common sense, but we need to demonstrate that, unlike common sense, mathematics relies upon the structuring, organisation and sharing of knowledge, experiences and techniques.

Already, then, we begin to see certain problems arising in connection with "mathematics and common sense".

(a) Although founded upon common sense - ie, the "universally admitted impressions of mankind" - mathematics is more than common sense. Indeed, the latter can be a constraining force on the learning, comprehension and development of mathematics.

(b) In our mathematics teaching there will frequently be a need for us to develop our students' "common sense" (ie, bring a student up to "average understanding" as in the example just cited on spatial awareness) in order for him or her to make progress in mathematics. This may be a legitimate part of mathematics education, although perhaps scarcely qualifying as "mathematics". Moreover, children from different social communities will bring with them different types of "common sense", for this is not a universal constant: indeed, it has been described elsewhere as "local knowledge". Discussions of "ethnomathematics" might well gain, then, by focussing more on the issue of the common sense peculiar to a particular society.

There is a great need to recognise and build upon such "local knowledge" and to be aware that such knowledge is never static - changing social contexts ensure that there are new mathematical entries and omissions. Textbook writers have, to some extent, recognised this and, for example, most of the books I studied made use of the fact that weather forecasts on television have now made most children from non-tropical countries very familiar with negative numbers. These are now part of "local knowledge", although the operations on them are not. But many opportunities would still seem to be lost. One of the most interesting national reports to come out of the Second International Mathematics Study described how in one developed country teachers had great difficulty in forecasting which items their students could answer successfully. All too often they expected failure simply because they had not yet taught the topics: in fact students were able to answer the questions from what had become "local knowledge". To take an example from another country: when the English National Curriculum was devised in 1988, scientific notation was thought appropriate to be learned by only about half the students by age 16. Yet, again, SIMS had shown that in 1981 almost 40% of English 13-year-old students answered an item on that topic correctly. Whether they had learned it in science lessons, in mathematics, or simply by playing around with a calculator or computer, I do not know - but for them it had become "local knowledge". I look forward to the publication of the results of the Third International Mathematics and Science Study in order to see how 13 year-olds in those countries which have not taught probability or, in some cases, devoted much time to the graphical representation of data, perform on such items. My suspicion is that the elementary aspects of these topics are fast becoming "local knowledge" amongst young adolescents.

We all know that it is wrong for teachers to assume that "what has been taught has been learned", but it is equally dangerous to assume also that "what has not been taught is not known". An immediate consequence is, of course, the influence of the student's social background - for this automatically defines the "local knowledge" associated with the student.

(c) Mathematicians as "a community" have their own brand of "common sense". A major aim of mathematics education is to develop this type of common sense in students - to add to what they consider to be normal mathematical behaviour, to develop that knowledge and those methods of thinking which are often ascribed to "(mathematical) common sense". Does, for example, "It is obvious that ...", really mean: " given my knowledge of mathematics, it is (mathematical) common sense that ..."? (Perhaps one should add a note here concerning the differences between "the mathematician's common sense" and those other two useful, but ill-defined terms, "intuition" and "insight". I remember being embroiled in arguments about the meanings of the last two terms over twenty years ago - and the meanings attached to them are still largely idiosyncratic. However, I believe "intuition" is like "common sense" in that it is based on local (although frequently highly professional) knowledge and experience - but I do not see its results being derived even by the simplistic logic utilised in "common sense". On the other hand, "insight" goes far beyond "common sense" in demanding an appreciation of the structure of the problem or of the mathematical topics involved.)

(d) If the Greeks (in the case of irrational numbers), and mathematicians such as Frend and Lazare Carnot (negative numbers), and Du Bois-Reymond (set theory) had difficulty in accepting new ideas which contradicted their "common sense" views, then it should not surprise us if our students have problems.

All these points would repay much more detailed study. In particular there would seem to be close links between "mathematics" and "common sense" in what Skovsmose, following Bourdieu, has described as different "spheres of practice", whether these refer to the arithmetical practices of Brazilian street children or to the activities of professional research mathematicians. The aim of developing specific mathematically-oriented common sense within a particular sphere of practice, as in the case of the example just given on spatial awareness, is, of course, present in the writings of, for example, Greeno (1991) on number sense, Eisenberg (1992) on function sense, and Arcavi (1994) on symbol sense. We note, however, the difficulty of expressing specific objectives, ie, the desirable "senses" to be developed, in those simplistic, knowledge-based terms which fit happily in national curricula for mathematics.

All of us present will no doubt be able to call on personal experiences in which the difficulties mentioned in (c) and (d) arose. Certainly, they were clearly present in the case of a university student observed on video by the BACOMET group. When asked, in a course on linear spaces, to prove that there can be only one zero vector, his reaction was that this was "common sense". This inference appeared to be based on the study of a particular geometrical representation of a vector space and he saw no reason to apply reasoning other than that associated with common sense. The student's difficulties seemed also to be reinforced by the request that he should "prove" something. For does the concept of proof have any standing within "common sense" or is it lodged firmly within mathematics? This latter problem arose also in relation to the Grade 8 texts I studied, for two countries attempted to introduce the notion of formal geometrical proof at this level. Although both texts had many good features, nevertheless there seemed to be a marked and largely unexplained and unmotivated jump from the "common sense" approach which had characterised the books up to that point and the more formal notions of the proof sections.

Another video seen at the same time showed a similar problem, hinging on a limited representation, but this time arising in the primary school. This concerned fractions discussed in the context of the ubiquitous pizza. Did 4/4 equal 5/5? One child supplied the required response, but another actually *used* her common sense and argued correctly that a pizza divided into four equal parts was not the same as one divided into five. The former would be useless should five children wish to share it! Of course such subversive reasoning had to be corrected! But the child was right. It has to be an odd equivalence relation which treats many real-life, or common sense, situations as equivalent. Someone tying a parcel will need a lot of convincing that 100 pieces of string each 1 cm long are equivalent to one piece of one metre in length. Abstract mathematical equivalence relations do not always translate sensibly into real life terms.

There is no easy way to teach fractions, but that which leaned too heavily on pizzas had probably resulted in at least one girl believing that mathematics teachers could be very selective when deciding what comprised "common sense". I am not here offering a solution to this difficult problem, but unless we recognise its existence, we are unlikely to make significant headway towards its solution.

There comes a time in mathematics when the constraints of common sense must yield to the demands of structure - whether this accompanies the introduction of new types of number and the operations upon them, or of algebraic and topological structures per se. In some ways this parallels developments in the foundations of mathematics, from Frege, who attempted to build mathematics upon the foundations of a "common sense" logic, to those of Hilbert and the development of formal systems. Thus, for example, in 1919, Hilbert stressed that the concepts of mathematics were built up "systematically for reasons that are both internal and external" (quoted in Rowe, 1994). Common sense, supplies "external" motivation, but we must look to mathematics for the internal reasons.

It will never be easy to make the change from a view based on common sense, to one which takes into account the internal structure and demands of mathematics. However, attempts to disguise the need for this transition will almost invariably lead to confusion in the minds of students and teachers.

Certainly, I felt that none of the textbooks I studied provided a satisfactory introduction to the multiplication of integers. As I have already mentioned, one approached the subject from the "internal" standpoint of algebraic structure, but in such a way as to make little sense either to an eighth grade student or a mathematician. Other texts tried to make the result "convincing" through a variety of approaches: for example, by an appeal to the authority of a calculator; by a weird fable which ignored the laws of physics and, like several other approaches, essentially fudged the mathematical issue by taking the product of two different representations of the integers; or by assuming that the result followed by applying common sense to the extension of numerical patterns (see Howson, to appear, for further details). It was interesting to compare these approaches with two from the 1930s. The first, told by the poet, W.H. Auden, was by being made to memorise:

Minus times minus equals plus.
The reason for this we need not discuss.

which had at least the virtues of honesty and being easy to remember. The second, more serious example, concerns the advice given by the schoolteacher, C.V. Durell, in his 1931 book, *The teaching of elementary algebra*. What was most frightening was that I could not detect in any of the modern texts the pedagogical and mathematical understanding shown by Durell [3]. It would seem to be a serious criticism of mathematics education

3 Durell stressed history as strong evidence of the difficulty of the concepts involved and remarked that the introduction of negative numbers had, in recent years, tended to be delayed as a result of the underlying theory and difficulties being better appreciated. He, himself, strongly believed that "it is inexcusably wrong to teach pupils to use symbols to which they attach no meaning". One consequence of this was his wish to distinguish between a signed negative number and the operation of subtraction on unsigned numbers. (Only one of the texts I studied did this.) The approach he suggested for the introduction of directed numbers and for addition and subtraction are those most favoured today: temperatures, gain and loss, etc. Here, however, he wished emphasis to be placed on the fact that "the 'rules of signs' are definitions which have been framed to establish correspondences between similar processes in different number systems. The question of proof does not arise here, though it does so in connection with the consequences of the rules, but [such work] is for specialists". On multiplication, he suggested beginning with consideration of repeated addition, eg (-5) x 3.

"This does not prove that (-5) x 3 = -15; it would merely make matters very awkward if it were not so, and suggests what kind of definition is most useful." Further justification for a definition is sought through contextualised examples. Here we want to substitute negative numbers into physical formulae which involve products. Durell suggested simple equations arising in kinematics and, for example, temperature rising at a steady rate in a boiler. He then suggested that students be asked to supply interpretations of these formulae when none, one or two of the variables have negative values. And so his advice proceeded.

and of mathematics educators that in over 60 years we seem to have made no progress in our thinking on how a key topic to be found in the curriculum of every country, namely the multiplication of negative numbers, should be taught.

We have already seen some of the problems which arise when we do not distinguish clearly enough between the world, and our common-sense conceptions within that, and the abstract model of it which we form within mathematics. In his *Pathway to Knowledge* (1551), Recorde tackled this problem head on:

> A point [the spelling has been modernised throughout this extract] or a prick is named of geometricians that small and insensible shape which hath in it no parts, that is to say, neither length, breadth or depth. But as the exactness of this definition is [more suited] for only theoretic speculation, than for practice and outward work (considering that my intent is to apply all these principles to work) I think it [more suitable] to call a point or prick that small print of pen, pencil or other instrument which is not moved nor drawn from the first touch. ...

Similarly when defining a line he was led to observe how geometers "in their theories (which are only mind works) do precisely understand these definitions". Whether Recorde shared our appreciation of the distinction which he drew between the abstract nature of the geometer's system and the artisan's real world of which it is a model is, of course, doubtful. Nevertheless, it demonstrates the care with which he approached his task. Perhaps at the other extreme to Recorde, in so far as it confuses abstract mathematics with the real world, is an example I saw recently which sought to bring sense to irrational numbers through contextualisation. It was a proposed test item which described a clock having a minute hand of $\sqrt{5}$ cm and an hour hand of length 2 cm. The students were asked to state whether the distances between the tips of the two hands at 12, 6 and 9 o'clock were rational or irrational. The question demonstrates a degree of bizarre ingenuity. The outcome, however, is more irrational than the numbers involved. Irrational numbers are not measuring numbers in the real world - indeed it is difficult to imagine what an irrational measurement could mean in physical terms. What degree of accuracy is reasonable in a measurement? What clock has its minute and hour hands moving in the same plane? This example, then, provides us with a misguided attempt to supply meaning - driven not by the demands of mathematics, but by a belief that meaning and motivation must be provided through "real-world(!)" contextualisations. We note also not only a reluctance to move into what Recorde calls the geometer's system, in which the diagonal of a square of side 1 has length root 2, but an utter confusion between the real world, and measurements within it, and that of formal geometry. Let me give another example. In England it is now common to "motivate" and add "reality" to mathematics by always placing geometrical problems in the "real world" of measures. So a question such as "is a triangle with sides 12, 12 and 17 right-angled?" would be replaced by "is a triangle with sides 12 cm, 12 cm and 17 cm right-angled?" However, there are enormous differences between these two questions. The answer to the first question is a straightforward "No". The second question is much more difficult - what, in the real world do we mean by "12 cm long" or "right angled"? The obvious answer this time is "more or less"; but it is not simple to calculate the probability and this will depend upon what assumptions are made en route.

Such confusion is not only to be found in school texts. I see it, for example, in some of the writings about the "fallibility" of mathematics. Many of you will be familiar with the problem of adding together two numbers, A and B, say, each some 60 or so digits in length and constructed using only the digits 1 and 7. One writer asks the question: "[Since] we have no certain answer by which to check or verify [this addition]... what are we left with, as far as certainty in mathematics goes?" (Lerman, 1994, p. 203) Again, this to me would seem to confuse the problems caused by doing mathematics in the real world and those inherent to mathematics itself. If A + B is not uniquely defined, then we can all go home and abandon the teaching of mathematics. Physical limitations may prevent my adding the two numbers together correctly, and might cause me to have doubts about the validity of a proof which hinged on that calculation being correctly done (although I find it hard to envisage how such a proof might arise). However, I am certain that A + B is even and that AB is odd - and of other similar results. To dismiss "certainty" because of our human limitations would seem to me to be only one step on from Lewis Carroll's:

> "What's one and one and one and one and one and one and one and one and one and one?", [asked the White Queen].
> "I don't know", said Alice, "I lost count".
> "She can't do Addition", the Red Queen interrupted.
>
> (*Alice through the Looking Glass,* 1871)[4]

4 Lewis Carroll's allusions normally had a serious point. A colleague suggested that here he might have been influenced by thoughts on Peano's "successor" notion, but since Peano did not publish his axioms until 1892, I think this unlikely. Could it be that, in the section from which this excerpt was taken, Carroll was poking fun at the simplistic way in which student attainment and teacher effectiveness were being measured following the establishment of the "payment by results" scheme in 1862 (see, eg, Howson, 1982)? If so, one can only wonder what Carroll would have made of educational developments in England since 1988!

However, this is in some ways taking me away from my main theme. Let me finish, then, by summing up some of the points I wish to make.

First, our mathematics teaching, like mathematics itself, must be founded upon common sense.

However, as we have observed, common sense is a vague and unstructured notion: we must ensure that students see mathematics not merely as an extension of common sense, but as a structured, organised discipline that builds upon accepted, shared results and techniques.

Sooner or later, the "shared" results and techniques of mathematics transcend common sense. It is necessary to be aware of this and to ensure that students are properly prepared for this shift and that the transition is achieved honestly - that they are not misled by half-truths.

Common sense is rooted in everyday experiences: when we make use of common sense in our teaching we must appreciate that there are significant differences between this world and that which mathematicians have created. Care must be taken to ensure that students do not become confused by this.

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THE CHANGING FACE OF SCHOOL ALGEBRA

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Traditionally, school algebra has been associated with literal symbols and the operations that are carried out on these symbols. But for the past decade or so, this vision of school algebra has gradually been widening to encompass activities and perspectives that were not previously considered part of algebra. The broader term, algebraic thinking, is being employed more and more often as a vehicle for describing the kinds of encounters students are having with algebra. This paper examines a couple of these newer perspectives in the light of a distinction between algebra and algebraic thinking, discusses some recent research that shows what we might expect from these approaches, and offers a suggestion as to the direction in which we ought to be heading.

1. ALGEBRA VERSUS ALGEBRAIC THINKING

1.1. ALGEBRA

In order to distinguish algebra from algebraic thinking, let's begin with what I believe to be a fairly widespread definition of what has traditionally been meant by algebra, at least elementary algebra: Algebra is a tool whereby we not only represent numbers and quantities with literal symbols but also calculate with these symbols. Thus, in North America at least, school algebra has been viewed as the course in which students are introduced to the principal ways in which letters are used to represent numbers and numerical relationships--in equations as unknowns and in expressions of generality as variables--and to the corresponding activities involved with these uses of letters--mainly equation solving and expression simplification. In this paper, school algebra is taken to be synonymous with elementary algebra.

Implicit in this definition of school algebra is both the "acting upon" (or "actions") and the "objets" themselves. We might ask ourselves at this point whether there are specific algebraic objects. For example, are the objects of algebra different from the objects of, say, calculus? Some would say "yes": "In the calculus, differentiation and integration are perhaps most readily thought of as operations on functions as entities" (Schwartz &

Yerushalmy, 1992, p. 266). In other words, the functional expressions that are the objects of the operation of differentiation are viewed as entities rather than as sequences of operations (see also Sfard, 1991). Others, possibly a majority, might say "no," that the expressions that are the objects in calculus have the same form as the expressions that are the objects of school algebra, even though they might be interpreted differently and have other referents. In line with this perspective, there are no distinct objects in algebra, thus making algebra an all-purpose tool, handy in more than one mathematical domain. But then are the activities of algebra different from the activities of, once again, calculus? Many would be of the opinion that the basic activities of algebra--simplification and equation-solving--can be distinguished from the basic activities of calculus--differentiation and integration--even though the former activities are clearly engaged in when doing calculus.

Let us at this moment look a little more closely at these tool-based activities of algebra. We can view them as being of different types. First, there are what we might call the *generational* activities. These involve the generating of expressions and equations that are the objects of algebra, for example, equations containing an unknown that represent quantitative problem situations (see, e.g., Bell, 1995), expressions of generality from geometric patterns or numerical sequences (see, e.g., Mason, 1996), and expressions of the rules governing numerical relationships (see, e.g., Lee & Wheeler, 1987). Another way of looking at these generational activities is from the perspective of translating, that is, translating a situation into an algebraic representation--however, the use of the term translation suggests a fidelity with respect to maintaining all the information of a situation, which is clearly not the case in algebra. Then, there are the *transformational*, rule-based activities of algebra, for example, collecting like terms, factoring, expanding, substituting, solving equations, simplifying expressions, and so on. Much of this activity is concerned with equivalence and the preservation of essence despite the apparent transformation of form. And lastly there are the *global, meta-level, mathematical* activities for which algebra is used as a tool but which are not exclusive to algebra and which could be engaged in without using any algebra at all, such as, problem solving, modeling, finding structure, justifying, proving, and predicting (Kieran, 1.997). In these latter activities, the algebraic representations that have previously been generated or manipulated are interpreted or closely examined so as to provide insights into, for example, the underlying mathematical structure of a situation or to yield answers to specific or conjectural questions. The case could be made that, since these latter activities are not restricted to the algebraic domain, they are, in fact, not part of algebra. But attempting to divorce these meta-level activities from algebra removes any context or need that one might have for using algebra.

This, in summary, we have these three types of algebraical activities- the generational, the transformational, and the global, meta-level -all of which I will be referring to again and again in this paper.

Another lens with which to view the activities of algebra is that of *mathematical doing* versus *mathematical thinking*. We are clearly "doing" mathematics when we are engaged in any of the above activities. But the question of whether we are "thinking" mathematically is more problematic. If we look first at the generational and transformational activities of algebra, Love (1986) has described the kind of thinking that is entailed as follows:

> Algebra is now not merely "giving meaning to the symbols," but another level beyond that; concerning itself with those modes of thought that are essentially algebraic--for example, handling the as yet unknown, inverting and reversing operations, seeing the general in the particular. Becoming aware of these processes, and in control of them, is what it means to think algebraically. (p. 49)

For experts, much of the transformational activity can become quite automated. Once one makes the transformation rules one's own, the algorithms of algebra can be executed, in a sense, without thinking. Wheeler (1989) has pointed out that this is, indeed, one of the main advantages of working in the symbolic system of algebra. This is the power of algebra! One need not be thinking of, for example, the operations one is carrying out or the referents of the expressions. In fact, once one has generated the expressions or equations (or has been provided with them) and knows what one's goal is, they can be treated in an almost mindless fashion. But not quite! We have all seen students who in trying to solve equations or simplify expressions knew the techniques, but kept going in circles (Wenger, 1987), not having a sense of where they were going or when they should stop. For these latter students, even the doing was not under control. Boero (1993) has discussed this issue in terms of anticipation, that every algebraic manipulation contains an anticipatory element, a sense of the direction in which you want to be going and of what the desired expression will look like once you get there. Pimm (1995) has remarked that the development of this sense of anticipation can provide an alternative to the "blind" manipulation that is so often found in beginning algebra students. Thus, even the doing of algebra, for it to be successful, must involve a particular kind of thinking.

But what about the global, meta-level activities of algebra, or more precisely those special activities in which algebra can be used as a tool (e.g., problem solving, proving, etc.)? These are usually considered the arenas of higher-plane mathematical thinking. Examples of the kinds of thinking engaged in with respect to some of these activities is one of the topics of this paper. But, before going further in a discussion of these meta-level activities and their role in the changing face of school algebra, I wish to shift gears slightly and address a broader question, one that involves a different interpretation of the term algebraic thinking.

1.2. ALGEBRAIC THINKING

In all of the activities described above, the view of algebra has been limited to a very narrow and quite traditional perspective, that involving letter-symbolic representations and an implicit set of numerical references--sometimes referred to as "generalized arithmetic" (see also Kieran, 1989). In this regard, Bell (1996) reminds us that there are several other algebras, each involving its own objects and actions, such as, "geometric algebras in which the elements are geometric transformations and the (single) operation is function composition" (p. 173) and "Boolean algebra, where the letters may denote propositions, and the operations are *and* and *or*" (p. 174). However, these have rarely been the domain of school algebra. My aim is not to move in this direction, but rather to enlarge our view of what might be considered algebra by including a broader range of algebra-related tools, with the purpose of ultimately arriving at a much wider vision of what we deem to be the content of school algebra.

Up until very recently, we have privileged the letter-symbolic in our representations of numerical relationships. But, with the advent of computer technology, we now have open to us several other means of representing such relationships and of operating on these relationships in ways that can be seen to be somewhat analogous to the generational and transformational activities of algebra described earlier. And since the relationships that are underlying these alternate representations are still numerical, it seems appropriate that these other representations and our thinking about them and with them be included in the domain of algebra, even if they look different from the representations that we are used to considering as part of algebra.

When viewed from this perspective, algebra does not require the letter-symbolic form. From now on, when I employ the term algebraic thinking, I am referring to a broader range of representations that nonetheless includes at times the letter-symbolic. This enlarged view of algebra is called algebraic thinking to distinguish it from traditional elementary algebra. Thus, algebraic thinking can be defined as the use of any of a variety of representations in order to handle quantitative situations in a relational way. Once the representation has been generated, it can be operated upon according to certain transformational rules related to the particular representation being used, within the context of global, meta-level mathematical goals, just as was the case with the activities of purely letter-symbolic algebra.

From a pedagogical angle, the non-letter-symbolic representations and their transformations can be used to make contact with or give meaning to the letter-symbolic representations that are traditionally involved in algebraic activities. Thus, algebraic thinking can be interpreted as an approach to quantitative situations that emphasizes the general relational aspects with tools that are not necessarily letter-symbolic, but which can ultimately be used as cognitive support for introducing and for sustaining the more traditional discourse of school algebra. This supporting use of non-letter-symbolic representations is not, however, meant to suggest that the primary value of these alternate representations is pre-algebraic; in the perspective being discussed here, they are considered full-fledged ongoing objects of algebraic activity. In view of my earlier remarks on mathematical thinking and doing, and how in the algebraic activities of generation and transformation, the thinking of experts often becomes routinized to an anticipatory kind of thinking, this new perspective on algebra and its being named algebraic thinking does not imply that it is at a higher level of thinking than that of traditional algebra. Perhaps it would have been simpler to redefine algebra than to create something new called algebraic thinking; but the term algebra already has so much baggage associated with it.

In the international mathematics education community, the development of algebraic thinking has recently received a great deal of research interest. Different approaches aimed at making algebra meaningful for students have been proposed which combine the use of various representations for expressing quantitative relationships and an emphasis on global, meta-level activities. Two examples of such approaches will now be presented, each of them giving priority to the meta-level activity of problem solving. In fact, of all the global activities mentioned above for which algebra has been used as a tool in schools, the one that has received the most attention in the newer approaches is that of problem solving. As is also typical of many of these new approaches, the computer plays an important role. In both examples of algebraic thinking to be described, it will be seen that the three-fold nature of the activities engaged in parallels that noted earlier for traditional algebra. More detailed information on these two approaches and others--

approaches that illustrate how the face of school algebra is indeed changing and is being widened to include algebra-related tools--can be obtained from Bednarz, Kieran, and Lee (1996).

2. TWO OF THE NEWER APPROACHES TO SCHOOL ALGEBRA

Problem solving has always been a part of school algebra; in fact, problem solving is at the heart of the historical development of algebra. But the way in which problem solving has generally been treated in traditional algebra courses is as a vehicle for applying the most recently learned symbolic form or technique. Problem solving itself has not until recently been taken very seriously in school algebra. However, during the last decade or so, several countries have shifted the emphasis in their algebra curricula from manipulation techniques to the pursuit of solving problems. The accent has moved from the typical transformation activities to the more global, meta-level activity of problem solving. A first example that illustrates some of the newer facets of this perspective on problem solving, in particular the kinds of thinking that are engaged by the use of a non-traditional algebraic representation and its related transformations, is drawn from the work of Rojano and Sutherland (1992, 1993; see also Sutherland & Rojano, 1993). Central to these new approaches to school algebra are the countless research findings related to the difficulties that students have typically encountered in the study of algebra.

2.1. A SPREADSHEET APPROACH

Rojano (1996) has argued that "trial and error, together with other strategies considered informal and which are found in students beginning the study of algebra, are indeed a real foundation upon which the methods or strategies of algebraic thought are constructed" (p. 137). This conviction, along with the large body of research evidence showing the inability of many students to achieve an integration of their problem solving and symbolic manipulation domains of knowledge, led Rojano and Sutherland to approach problem solving by the alternate avenue afforded by spreadsheet computer environments. Thus, students' own strategies are put into play in an environment that helps them symbolize their informal procedures for the problem.

In the following extract that describes the algebraic thinking activities involved in spreadsheet work, 10- and 11-year-old children had been attempting the problem: "In a rectangular piece of land the length is 4 times the width. The perimeter is 280 meters. What is the area of this piece of land?"

Throughout the spreadsheet word-problem sequence, pupils were taught a spreadsheet-algebraic approach (algebraic because it involves working from the unknown to the known; see Figure 1). The unknown is represented by a spreadsheet cell (this might be called *x* in an algebraic solution). Other mathematical relationships are then expressed in terms of this unknown. When the problem has been expressed in the spreadsheet symbolic language, pupils then vary the unknown either by copying down the rules or by changing the number in the cell representing the unknown (in a paper and pencil algebra approach, this would be the equation-solving part). In the figure below, the spreadsheet formulas are shown in order to present the pupils' solution processes. However, once the formula is entered into the spreadsheet, a number is automatically computed and a table of numerical values is produced (although the user can ask to view the formulas). (Rojano, 1996, p. 141)

	A	B	C	D
1		WIDTH (CM)	LENGTH (CM)	PERIMETER (CM)
2		25	= B2 * 4	= (B2*2) + (C2*2)
3		= B2 + 1	= B3 * 4	= (B3*2) + (C3*2)
4		= B3 + 1	= B4 * 4	= (B4*2) + (C4*2)
5				
6				
7		rule copied down	rule copied down	rule copied down

Figure 1.

One of the features that distinguishes this approach from traditional school algebra is the absence of standard algebraic symbolism; the representation used is the spreadsheet-symbolic. Nevertheless, the strategies used for solving this and other similar quantitative situations are not unlike those called into play in more typical algebra environments in which the student begins by identifying the variables and then decides on the relationship among the variables. Rojano (1996) points out the critical aspects of algebraic thinking that are favored in this approach: "The spreadsheet environment supported pupils in moving from thinking with the specific to the general, both in terms of the unknown and of the mathematical relationships expressed in the problem" (p. 144). Thus, the algebra-like tools of the spreadsheet environment are assumed to engage mental processes that are similar to those used in the more traditional algebra environment, but which are made more accessible due to the spreadsheet's unique modes of representation and transformation.

Rojano and Sutherland have also employed this same problem-solving oriented, spreadsheet method to help students who have been experiencing difficulty with the algebraic symbolism of more traditional approaches to school mathematics. The example given here was recounted by Sutherland (1993) in her story of a student named Jo. As was also the case with several of her 14- and 15-year-old peers, Jo had had some previous experience with algebra, but, as Sutherland points out, disliked mathematics and had performed very poorly on the algebra test and interview at the beginning of the study. She viewed algebraic symbols as letters of the alphabet and considered the numerical values of these letters to correspond to their position in the alphabet. During the 4-month study (one lesson per week), Jo learned how to use the spreadsheet to solve problems such as the one provided above in the Rojano excerpt. At the end of the study, Jo was given the following problem to solve (with no computer present):

> 100 chocolates were distributed to three groups of children. The second group received 4 times as many chocolates as the first group. The third group received 10 chocolates more than the second group. How many chocolates did the first, the second, and the third group receive? (p. 22).

see Fig. 2

She drew a spreadsheet on paper, and showed in her written solution "the way in which the spreadsheet code was beginning to play a role in her thinking processes" (p. 22).

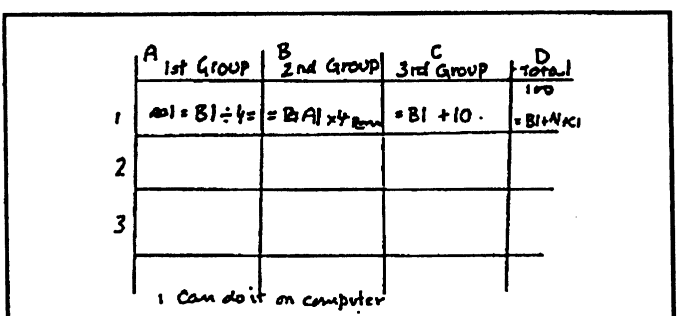

Figure 2.

When subsequently interviewed, Jo was asked, "If we call this cell X what could you write down for the number of chocolates in the other groups?" (p. 22). She wrote down the following, which shows that she was now able to represent the problem using the literal symbols of algebra:

=X	=X x 4	=X x 4 +10

In closing, Sutherland points out that "working with a spreadsheet can help pupils develop algebraic ideas which they can also use in a paper setting" and concludes that "the transfer from spreadsheet to traditional algebraic symbolism is not as difficult as many people have predicted" (p. 22).

2.2. A GRAPHICAL, FUNCTIONAL APPROACH

The previous example focused on using the spreadsheet symbolism and methods both for initiating students into the realm of algebraic thinking and for bridging with the literal-symbolic representations of traditional algebra. In the example that follows, functional graphical representations and the operations that can be carried out with these graphs are used as a foundation for approaching algebraic symbolism and its transformations. The central idea here is that students develop (a) algebraic symbols as a means for describing the functions represented by the graphs and (b) algebraic manipulations as a means for recording operations with the graphs of these functions. The meta-level activity that provides the context for these algebraic thinking activities is once again problem solving, but with two additional emphases that distinguish it from some other problem-solving approaches.

First, there are various criteria that may be used to select the types of problems that can be presented to students in a problem-solving approach, for example: (a) traditional categories such as age problems, distance-rate-time problems, mixture problems, and so on; (b) problems borrowed from the physical modeling sciences such as population growth problems; (c) problems based on situations according to their class of function. The algebraic thinking approach being described in this section focused on problems that were selected according to their function family. This functional orientation provided a means of unifying problem situations and of connecting them with their corresponding graphical, tabular, and literal symbolic representations. One of the complaints often heard with respect to traditional algebra is that students never make the links between the work they do with the literal symbolic representations and their later experiences with graphs of functions; this functional approach attempted to overcome such weaknesses and to help students see a larger picture. Significant attention was given from the outset to graphical representations and the important role played by the parameters.

Second, when solving problems in a traditional algebra environment, the letter in the equation representing the problem to be solved is viewed as an unknown. In some of the newer problem-solving approaches that do not involve equations--at least not at first--the distinction between unknown and variable is blurred. In the functional approach that we are now looking at, the initial emphasis is on representing the situation graphically before searching for the problem solution. Thus, the literal symbols that are developed in order to describe the graphically-represented function are interpreted more as variables. It is when the function is examined with a view to answering some specific question that the notion of unknown might enter. (Note that the spreadsheet-symbolic representation that was just seen also tends to blur the unknown-variable distinction.)

In this functional approach, which was developed in collaboration with my fellow-researcher Anna Sfard, the computer software *Math Connections: Algebra II* (Rosenberg, 1992) was used in conjunction with paper-and-pencil work. Over the course of 30 lessons, the 12- and 13-year-old students soon learned, for example, how to look at a linear graph, directly read from it the slope and y-intercept, and generate a story and literal-symbolic expression that would fit the observed properties. One of the classroom tasks that led to this kind of thinking was the Snowfall Problem, taken from Lesson 11, and provided in Appendix 1 certain physical phenomena associated with falling snow, such as compacting, etc., have been ignored in this introductory problem.

The students also learned to add functional expressions by adding the graphs of two linear functions. One of the problem situation that provided the initial motivation is excerpted here from Kieran (1994):

> Two brothers have savings accounts in the bank. When they opened their accounts, they each had a lump sum to deposit; this amount was to be increased by regular deposits of their allowances. The two given graphs represent how their accounts have grown each month. If they had opened only a joint account from the beginning, what would the graph look like?

After the sum-graph was generated by means of adding the corresponding vertical lengths of the two given functions (--see Figure 3), the students were asked to give the expression for the sum-function directly from its slope and y-intercept.(the expressions for the two given lines had not been provided to the students).One pair of students, after they had written down the expressions for the two lines, soon noticed that sum- they would get the same expression by merely adding together the slopes and intercepts of the two original functions. After this shortcut had spread throughout the class, the students next worked with the multiplying of a function by a constant, also done initially by means of Cartesian graphs.

They could now simplify any combination of linear functions into canonical form. Thus, equations such as $3x + 4 - 2 - 7x + 6(x + 5) = 2(3 - 4x) - 14$ could be solved graphically by first "adding both the slope and intercept terms on each side" to obtain the canonical form of each expression, and then graphing the two functions being compared by means of the slope and y-intercept terms of the two expressions, in order to find the value of x for which the two functions were equal. During the 30 lessons of the study, students were not introduced to the solving of equations by algebraic manipulation.

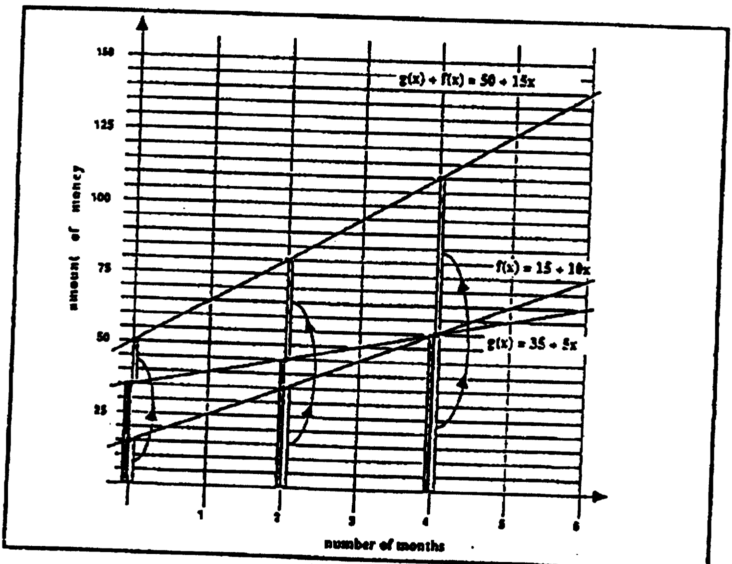

Figure 3.

One month after the above study had been carried out, several students were interviewed on an individual basis. Following is an extract from the interview with the student, Nat:

Interviewer: Can you solve 7x + 4 = 5x + 8?

Nat: Well, you could see, it would be like start at 4 and 8, this one would go up by 7, hold on, 8, 8 and 7, hold on, no, 4 and 7, 4 and 7 is 11. ... They'd be equal, like, 2 or 3 or something like that.

Interviewer: How are you getting that 2 or 3?

Nat: I'm just like graphing it in my head.

That Nat had learned to "graph in his head" by means of the parameters is quite impressive. The value of graphical representations as a foundation for the letter-symbolic was noticed by the classroom mathematics teacher, Bob, who was involved in our research project and who compared these students with others whom he had taught in the past with more traditional approaches. He said:

> They now have a visual tool to both think with and to talk with. With my past students, I often noticed that their algebra never seemed to have a handle. ... We all have different ways of understanding and of seeing things. But for many people, if you can picture something, it's much easier to understand.

3. DISCUSSION

In this brief and very selective summary of aspects from only a couple of the new approaches to algebraic thinking, we have seen two different examples of generational and transformational activities not normally considered part of algebra. Within the context of the global activity of problem solving, both made use of alternate representations (spreadsheet-symbolic; Cartesian graphic) and their related transformations (e.g., extending a spreadsheet rule down a column rather than substituting numerical values into an equation or manipulating that equation; adding graphs to arrive at a sum graph as a precursor to adding expressions) to achieve the analogs of equation solutions and expression simplifications. We have also witnessed some of the kinds of thinking that can occur while solving problems using these alternate algebra-related tools--from the informal, trial-and-error based thinking of the spreadsheet environment to the visually-oriented thinking of the graphical, functional environment. Several other illustrations of facets of algebraic thinking approaches could have been provided, such as, setting up a program of some situation in a computer language, say Logo, and running it with varying inputs, to obtain certain pieces of information in the form of output and then examining this output in order to find, for example, a problem solution or some structural pattern. In presenting these examples, I have attempted to interweave the two features--alternate representations and significant meta-level contexts--that have both tended to characterize the new face of school algebra, a subject area that is increasingly being referred to as algebraic thinking. Even though the use of computer as tool figured prominently in the given examples, in some of the new problem-solving oriented approaches to algebraic thinking it does not figure at all.

The testimony provided by Jo, Nat, and Bob suggests that these new approaches do indeed have a lot to offer. But as with any reform, finding a middle-ground is not always easy. In some national curricula, the pendulum has swung completely to the other side. In attempting to eliminate or at least greatly reduce the meaninglessness of past traditional algebra curricula, some have gone too far. The search for meaning and the consequent suppression of symbolism in, for example, the United Kingdom have led to a situation where most students now do hardly any symbol manipulation (Sutherland, 1990). In various countries, problem solving--by whatever means--has all but replaced traditional algebra. The hope was that, in focusing on understanding, the techniques would take care of themselves. But it has not happened (Artigue, cited in Lee, 1.997). Without a developed feel for the literal-symbolic representation, tasks that touch upon the class of activities related to finding structure, justifying, and proving--activities for which algebra is well suited and which have always been difficult in any case for algebra students (see, e.g., Figure 4)--are now quite beyond their reach. These students have little or no access to the power of symbol manipulation and to the role it can play whenever they want to get at the "why" of a mathematical phenomenon.

1. A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Using algebra, show that she is right. (Lee & Wheeler, 1987)
2. Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two. Now do it with another three consecutive numbers. Can you explain it with numbers? Can you use algebra to explain it? (Chevallard & Conne, 1984)

Figure 4.

I began this paper with the aim of presenting some of the new ways in which the picture of school algebra is changing and of encouraging all who have not yet begun to do so to consider viewing school algebra from this wider perspective called algebraic thinking. But I have also just shared with you what I believe to be some of the recent excesses of this movement. Thus, I would like to conclude on a cautionary note. I am suggesting that the manipulative side of algebra not be thrown out. To my mind, we are doing ourselves and our students a great disservice if, in embracing the new algebra-related tools for expressing and handling numerical relationships, we completely eliminate the original algebraic tool. The content of school algebra is changing. It had to change. It is being enlarged--and I emphasize the word enlarged--to include alternate representations and their related transformations. In encompassing what we are calling algebraic thinking, it may look at times quite different from traditional algebra. But algebraic thinking should not be a weaker resource than what we had before. For those "algebra" curricula that have all but eliminated symbolic manipulations, it is time, in my view, for the pendulum to swing back to a more middle ground--but not a return to curricula of the past that were exclusively oriented toward symbol manipulation; nor do we have to do as much of the manipulation as we used to do. In fact, there is less time for it; but, in the process of adopting these new approaches, the symbolic work has been made more meaninful than in the past.

The changing face of school algebra strikes a balance. It puts greater emphasis on a context of global meta-level activities and accommodates alternate representations with their related transformational activities, but still keeps the power of algebra. Freudenthal (1983) once remarked that "even if the formalism functions reasonably, the teacher or the one who defines the instruction should avail himself of each opportunity to return to the source of the insight [underlying the formalism]" (p. 469). This suggests not only that it is possible to combine the traditional literal-symbolic with non-traditional representations and operations, but also that it is important to do so. Without falling back into the traps of the past, we must continue to make significant room for the literal-symbolic in the content of school algebra. After all, many of us even <u>enjoy</u> working with it and appreciate its power. I am reminded here of a remark made by one of the 12-year-olds involved in the study on the Graphical Functional Approach. After solving an equation graphically during his post-study interview, he was shown for the first time how to solve an equation by algebraic manipulations. When he saw how he could arrive at a precise solution that had been only more or less correct in the graphical environment, and which had taken a fair bit of time to arrive at, he exclaimed with delight: "There is the solution--on a silver platter!

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APPENDIX 1 SNOWFALL PROBLEM

EXPLORING LINEAR FUNCTIONS
EXPRESSIONS AND GRAPHS

One day in January, a corner of the school yard had 60 cm of snow. It began to snow, and continued all day at a rate of 5 cm per hour. The height of snow in the corner of the yard is a function of time.

(1) Write the algebraic expression which tells how to calculate the height of snow at any time. Use the variable t (or x) to represent the time in hours.

(2) **Underline** the part of the expression that shows how fast the snow was falling.

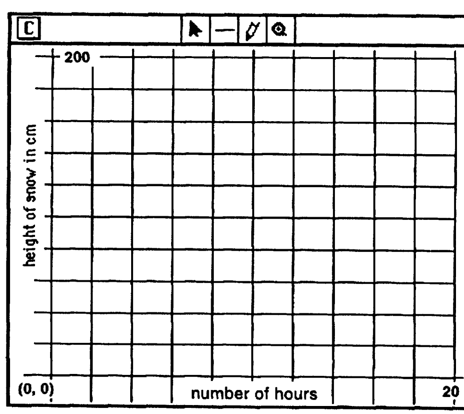
(3) **Circle** the part of the expression that shows the height of the snow at zero hours.

(4) At the computer, Copy from Floppy the file LINEAR (11.1). Enter your expression into the first expression box and click so that the computer will plot the graph.

Copy the computer graph here (use a ruler, and be accurate). Put a $\boxed{1}$ at the right end of the line.

(5) **Circle** the part of the graph which shows the level of snowfall at zero hours.

(6) **Explain** how the graph shows the rate of snowfall:



Meanwhile, across the street from the school, there was a deep snowbank of 120 cm at the beginning of the day. The snow fell at the same rate (5 cm per hour) on this side of the street as it did at the school.

(7) Write the algebraic expression for this story:

(8) Type this expression into the second expression box, and draw the line that represents this story on the graph on the previous page. Put a $\boxed{2}$ at the right end of the line.

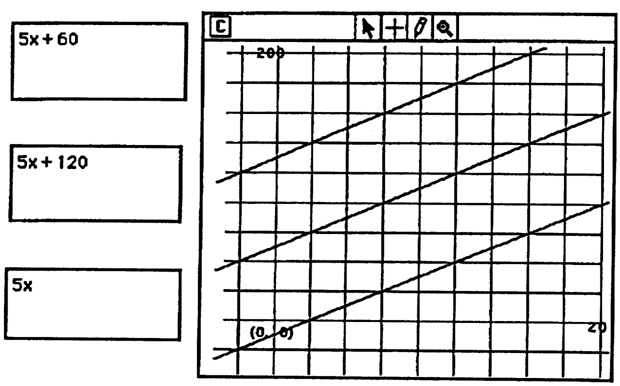
How does the graph show the height of snow at zero hours?

How does the graph show the rate of snowfall?

Someone had shovelled the sidewalk by the side door of the school early in the morning, so it was bare when the snowfall began.

(9) Write an expression for this story:

(10) Enter your expression into the third expression box in the computer, and have the computer graph it. Copy the graph onto the graph on the previous page. Put a $\boxed{3}$ at the right end of the line.



(11) Look at the three graphs which have been plotted. What do you notice about their relationship with each other?

How can you explain this?

Look at the three expressions for these lines.
How are they the same?

How are they different?

(12) Click on the first line that the computer drew. It is connected to the first expression box. You will see small black squares on the line. Put the mouse on the square which is on the vertical axis, and slide the line up and down. Release the mouse, and look at the new expression for the line (in the first expression box). Do this a couple more times.

What part of the story changes when the line slides up and down?

What part of the expression changes when the line slides up and down?

Predict what the expression would be if you slid the line to 500 on the vertical axis. Write the expression here:

Predict what the expression would be if you slid the line to -30 on the vertical axis. Write the expression here:
(13) Click on the second line that the computer drew. This time, put the mouse on the square which is at the right part of the line, and move the mouse. Don't forget to see what changed in the expression box. Do this a couple more times.

How does the line change? (also tell what stays the same)

How does the expression change each time?

What part of the story will change?

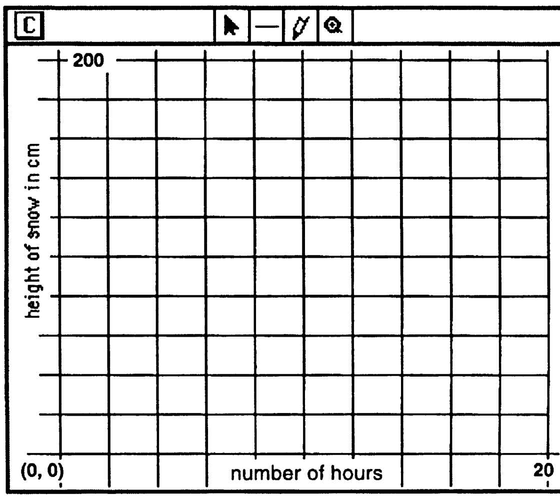
(14) Clear the graph window by clicking on C at the top left corner. In the first expression box, enter the first expression that you used (on page 47). Click to have the graph drawn.

Click on the line and move it so that it shows a rate of snowfall of 10 cm per hour.

Write the new expression:

Draw the new line on this graph, and mark it with a 4.

(15) Move the line so that it shows not a snowfall, but a melting rate of 5 cm per hour. Write the new expression:



How does the expression show that the snow was **melting**?

Draw the new line on this graph and mark it with a 5.

(16) Move the line so that it shows that the snow neither fell nor melted - the height of snow stayed the same.

Write the new expression:

How does the expression show the rate of change in height of snow?

Draw the new line on the graph and mark it with a 6.

(17) Look at the three lines that you have drawn. What do you notice about their relationship to each other?

Look at the three expressions for these lines. What is the same?

What is different?