ON SOME ASPECTS IN THE TEACHING OF MATHEMATICS AT SECONDARY SCHOOLS IN SWITZERLAND

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1 Introduction

In this paper I would like to discuss one aspect of teaching which seems to reflect a change in the way Mathematics is taught at the Swiss Gymnasia: *Individualization.*

"New Math" didn't have as significant an influence in Switzerland as elsewhere. This holds true in particular for the non-French speaking part of the country. But even without this disappointing experience it was noticed eventually that teacher-driven instruction with its high frequency pingpong of quick questions and short answers was not truly successful in forming mathematically educated students.

A decade ago Karl Frey became Professor of Didactics and Pedagogy at ETH. Frey, among many other things, pointed out the weaknesses of the traditional teaching style, e. g. that on the average only the top 30% of a class participate in the game. In his lectures Frey proposes a variety of alternative teaching formats, which were not widely known to Gymnasium teachers in Switzerland before. They all have in common that the student's activities are considerably enhanced and they foster his responsibility for his learning processes. We will discuss some of these schemes in the next three sections of this paper. In Section 4 we introduce some remarkable and somewhat related ideas by P. Gallin and U. Ruf, two Swiss Gymnasium teachers. The last section contains a few remarks on the impact of technology.

2 Puzzles

Frey's alternative teaching schemes have meaningful German names. In some cases it is somewhat difficult to provide reasonable

English translations. I will use the German terms and try to explain what they mean. Here are five of these methods

Puzzles
Werkstattunterricht
Lernaufgaben
Fallstudien
Leitprogramme

I will not discuss Werkstattunterricht nor Lernaufgaben, but briefly describe puzzles. The goal of the *puzzle technique* is that pupils make an attempt to explain a piece of Mathematics to their fellow students. A suitable topic is divided into four parts: A, B, C, D. One quarter of the class first works on part A, another quarter on part B, etc. Once the members of each group have made themselves experts in their parts, they are regrouped in such a way that each new group consists of four experts representing the four parts of the topic. The task then is that each expert teaches his fellow students.

I do not want to discuss this technique in any detail. However I would like to mention that a number of our teacher students designed various puzzles for the use at the Gymnasium. I used the method myself with a small group of upper undergraduate University students. The topic was the famous Poincaré-Birkhoff-Smale theorem on chaos. One group of students worked on stable and unstable manifolds, another on the so-called shadowing lemma and the third one on properties of the Bernoulli shift. This took place at a Math Camp. Two and a half days were needed to prepare the students, the presentations covered a bit more than a day. The students enjoyed this experience, as we know from spontaneous reactions as well as from a questionnaire.

3 Case Studies (Fallstudien)

Case studies, or Fallstudien as we call them in German, have a long history. They were introduced early in our century at the Harvard Business School. *The principal goal is that students learn to deal with highly complex and open situations.*

It appears difficult to construct case studies in Mathematics. Nontheless Albert Gächter, who teaches Mathematics at a Gymnasium in

St. Gallen, succeeded in designing six case studies in Mathematics for Gymnasium students of grade 11 and 12. Gächter's work was part of a larger endeavour conducted by K. Frey and supported by ETH's Vice President for Research R. Hütter.

The Titles of Gaechter's Fallstudien are

- What degree of precision is appropriate?
- QED!
- Recursion
- Shape and Number
- Algorithmic Geometry
- Computer Games

Here are some features of these Fallstudien with special emphasis on the first example. A case consists of a number of *documents and a few key questions.* The Fallstudie on precision contains among other material an article by H. Freudenthal entitled "Wie genau ist die Mathematik", a paper by G. Schierscher on computer arithmetic, a section from the book "Descartes' Dream" by P. Davis and R. Hersh, and excerpts fom A. Wittenbergs "Bildung und Mathematik". None of these articles was written for eleven grade students. The material is in fact fairly demanding. Nonetheless, working in small groups the students can gain some insight by studying these papers and by addressing the following questions:

a) What factors determine an appropriate precision of a result?
b) Precision - a virtue?
c) Fallacies with Computer numbers?
d) Why Irrationals?

Working on a Fallstudie is apparently quite different from other forms of instruction. The material is *not* divided into small and easily digestable pieces. There is no unique answer to a given question. There is room for discussions. Participants may come to different conclusions.

A Fallstudie covers 5 - 10 lessons. The Fallstudien experts recommend one or two Fallstudien per year.

4 Leitprogramme

I will discuss Leitprogramme in some detail, because this was a major project recently, again conducted by K. Frey.

Leitprogramme are based on what is called *Kellerplan techniques and on the Bloom Mastery Learning Principle.* A Leitprogramm is dedicated to an important subject of some field. It is a booklet, some 50 to 100 pages long, say. It is handed out to each student at the beginning of the working period with the Leitprogramm. It covers the material for a month's work or so. All relevant information is contained in the Leitprogramm.

A Leitprogramm consists of a basic part, compulsory for all students, which we call the *Fundamentum*, and an optional part for fast learners. This part is called the Leitprogramm's *Additum*. At the beginning of the Leitprogramm and at the beginning of each Chapter a brief and nontechnical overview of the subsequent material is offered, and the goals to be achieved are stated. One of the key parts of a Leitprogramm is a very thoughtfully-designed, well-written exposition of the scientific contents. Another important ingredient of a Leitprogramm is the following. The text is interspersed with a reasonable number of *problems,* some routine, some challenging. Their solutions are included.

At the end of each Chapter there are *two sets of problems.* If the reader is successful with the first set she is ready for the Chapter test. Otherwise the Leitprogramm offers her some advice on what to do next and she will return to the second set before taking the Chapter test. The Chapter test may be a brief discussion between the student and the teacher or some tutor, or it may be a written test. *The key point is: The student must not go on to the next Chapter before there is evidence that he or she has mastered the previous Chapter.* Mastery learning in the sense of Bloom means that 80 to 90% of a class master 80 to 90% of the material.

A Leitprogram seems to have a number of advantages compared to the conventional teaching style in the classroom. One of the most important aspects is that *the student controls the speed of his learning process himself:*

- Slow students are not pushed too much. They may read additional explanations provided by the Leitprogramm if they wish to do so.
- As to the solutions of the problems, the author of a Leitprogramm may offer various options: results, hints and comprehensive descriptions.
- If a student gets confused in the traditional classroom dialogue he is lost and gets frustrated. A group tends to exert some pressure on its members in an interaction process. All this is eliminated.

- Fast learners are not bored because the Additum offers complementary material, possibly a variety of different and challenging options.
- It is the student's responsibility to decide if and when he is ready for the Chapter test. Some students will show up soon, after half an hour say, others may take the test the next day only, etc.

A basic goal of the Leitprogramm technique is that students get started reading scientific texts. Reading combined with problem solving fosters the student's attention and enhances his own activity.

The Leitprogramm technique bears some similarities with Programmed Instruction and Computer Assisted Learning. Indeed the following are features common to both methods

- The material is carefully prepared
- The goals towards which the student is working are stated expli-i citly and the material is organized accordingly
- The progress of the learning process is checked repeatedly

On the other hand there are major differences as well:

- A Leitprogramm is much less compartmentalized; the steps are considerably bigger; the reader is guided to make some discoveries on his own
- A Leitprogramm is more entertaining because of working at times with a partner; because of small experiments that may be part of a task, etc.
- At least some of the tests are personalized.

There is statistical evidence that Mastery Learning improves success at learning. Kulik et al. [1] in a paper published in 1990 state that the improvement due to Mastery Learning in Mathematics is 0.5 standard deviations, if compared to classical instruction. Frey concludes that the Leitprogramm method which combines the Mastery Learning technique with supplementary devices is even more promising.

Between 1991 and 1995 a group of didacticians from various fields at ETH made an attempt to implement the Leitprogramm technique. The fields involved were Biology, Chemistry, Geography, Mathematics and Computer Science and Physics. The goal was to produce Leitprogramme for the use at the Gymnasium, a few in each discipline.

I will focus on the Leitprogramme in Mathematics and Computer Science now. During the 4-year period quite a number of Leitprogramme on a variety of topics were written, first by students of Mathematics' teaching. The more promising ones were discussed with experienced Gymnasium Mathematics teachers, revised, tested in classrooms, thrown away, replaced by completely new versions, again tested, etc.

At the 1995 Swiss Annual Meeting on Mathematics Education organized by ETH and the Swiss Mathematical Society, four Leitprogramme, three in Mathematics, one in Computer Science were presented. The titles of these Leitprogramme are:

Solving Quadratic Equations
If you need a Loan you better know your Mathematics!*
Solving Systems of Linear Equations
Recursive Programming*

The main Authors were: M. Adelmeyer, M. Bettinaglio, J.P. David, W. Hartmann and the Author. I will describe some features of the third of these Leitprogramme. (Versions of the second and fourth one, are available on the Web. The address is http:/educeth.ethz.ch/mathematik/)

The Leitprogramm on Linear Systems is suitable for the ninth grade students; an abbreviated version can be covered in 12 hours, the full version requires 20 hours approximately.

Of course we try to convey that studying systems of linear equations is a great topic! In the first Chapter we indicate that this subject has a very long tradition, by quoting some famous problems from Babylonian, Egyptian and Chinese sources. At the same time we demonstrate that linear systems are a subject of current interest as well by referring to various technical applications. In particular we hint at Computerized Tomography, a subject introduced in more detail later on in the Leitprogramm.

In the second Chapter we use a sequence of carefully chosen 2-dimensional systems to let the students discover the so-called substitution method. And of course we ask them to generalize the algorithm to higher dimensions. As I pointed out before, the material was tested a number of times before it was presented at the 1995 meeting. The students were asked to give their comments. As to the way we guide our readers to discover the substitution method, one student wrote: *"Das Vorgehen auf p. 19 ist super. Man kann versuchen ein System von drei Gleichungen von* *drei Unbekannten zu lösen. Wenn einem dies nicht gelingt, dann gibt es Aufgaben bei denen man einen 'Startschubs' bekommt. Nachher kann man noch einmal die andere schwierigere Aufgabe versuchen."*

The first two chapters, together with the final chapter on Computerized Tomography, make up the short version of this Leitprogramm.

Chapters 3 and 4 are devoted to theoretical and algorithmic aspects of linear systems. In Chapter 3 we discuss linear systems with no or infinitely many solutions.

Chapter 4 starts with a brief introduction to the history of computation and the use of computing machines. In particular we mention Eduard Stiefel (1909-1978), a Swiss Mathematician. He was a famous topologist when he moved to Computer Science and Applied Mathematics after World War II. He rented Konrad Zuses' famous Z4 machine and at the same time he guided a group of young scientists in constructing the first electronic computer in Switzerland. The main mathematical theme of the chapter is Gaussian elimination. In this connection we also discuss output provided by Texas Instruments' TI 85, a graphical calculator by now widely used at Swiss Gymnasia. Of course we count the number of operations of the Gaussian procedure and relate it to the speed of computation of various computer facilities.

As mentioned before, the final chapter is devoted to an introduction to what is called Computerized Tomography. We have two modest goals here. First, we want our students to understand the basic difference beween classical X-ray photographs and image-sections obtained by Computerized Tomography. Second, we want the readers to see how Computerized Tomography relates to linear systems. To this end we use a very simple mathematical model.

The tests with this Leitprogramme revealed that Computerized Tomography is considered to be a fascinating subject by most pupils. Once, however, a girl showed very strong emotional reactions: somebody in her family had died of cancer.

This concludes the remarks on the Leitprogramm on linear systems.

The four Leitprogramme were offered to the schools at a low price. More than 100 Gymnasia have purchased a set. If a School owns a set, it is entitled to produce as many copies as needed to use the Leitprogramme in class.

At this point, unfortunately, we do not have an overview as to what extend our Leitprogramme are used in practice and how successful they are.

For more information on the Leitprogramme in Mathematics and Computer Science, see [2].

4 Some ideas of P. Gallin and U. Ruf

There is no institutionalized basic research in Didactics of Mathematics in Switzerland. Yet there are remarkably many highly imaginative Gymnasium Mathematics teachers who make substantial contributions. I will make an attempt to briefly introduce some ideas of Peter Gallin and Urs Ruf. Gallin teaches Mathematics, while Ruf teaches German Language and Literature at the Gymnasium in Wetzikon, near Zurich. This latter fact is probably a surprise. It turns out that the combination of these two disciplines is particularly significant. I will not elaborate this point more fully. What is relevant here, is that Gallin and Ruf developed an approach which puts forward the idea of individualization more than any other scheme I know of.

My remarks are based on personal discussions with Ruf and Gallin and on their publications, in particular on an article entitled *Sprache und Mathematik in der Schule, ein Bericht aus der Praxis* [3] .

What makes you go? Under what conditions do you work like mad? One possible answer is: If you are driven by an idea. This idea may be vague at the beginning. But if you are fascinated by your idea, it makes you go. Such an idea is called *Kernidee* by Gallin and Ruf. Kern means Kernel, Germ. Key idea is a suitable translation.

Kernideen are very personal. An idea may be a Kernidea for one person, while it is not for someone else. Everybody has plenty of Kernideas. A pupil in particular has a microcosmos of personal Kernideas. The critical question according to Gallin and Ruf is: can we stimulate them to generate Kernideas in Mathematics, in a certain framework, on a particular subject?

What makes Mathematics teaching so difficult? Why is the success so limited? According to Gallin and Ruf there are two worlds involved, so to speak: the learner's world, his conceptions, his private views, his feelings. And the mathematical cosmos with its network of notions, ideas and procedures. Gallin and Ruf use the word *singular* to characterize the learner's individual world, while they speak of the *regular* world of Mathematics. In their understanding it is decisive that an individual traces his way from his singular world into the regular world of Mathematics. A gentle transition is needed. In one of their papers they use a schematic representation with a triangle. The vertices represent the *task*, its *solution* and the *person*. The edges indicate activities: *to solve, to explore, to comprehend.* Gallin and Ruf advocate the detour via the "Ich" (I), which in their view is not at all a detour but a conditio sine qua non for successful learning and comprehension.

Kernideas permit a global view of a problem from the very beginning, even if it is vague. This contrasts with what happens in traditional teacher-driven classroom instruction. Gallin and Ruf again use a metaphoric representation with a geometric object to illustrate the difference. To describe the traditional scheme they use a sequence of growing segments of a cone. In such an approach the clue is revealed only at the very end. Only when the very last piece is added the cone emerges. The approach via Kernideas on the other hand is represented by a sequence of gradually growing cones. At the start there is a very tiny cone only, yet it is a cone! As it grows, a more and more complete picture develops.

It is obvious that the second approach reduces pressure and stress, for both pupils and instructor. At whatever moment a pupil stops working, he or she has gained a certain amount of insight into the subject, while in compartmentalized learning the effort pays off only if all segments fit together.

To implement their ideas Gallin and Ruf propose what one can call the method of *Reisetagebuecher.* A Reisetagebuch is a kind of mathematical diary. Reise means travel, journey. Tagebuch means diary. A Reisetagebuch in the sense of Gallin and Ruf is a note-book. The pupils write down their thoughts using their own words. They express themselves in their singular language, so to speak. They comment on their ideas. They may express their feelings. The process of writing slows down the process of thinking. This is intended. Writing is like walking in the world of thoughts. Very slowly the landscape of thoughts becomes familiar. An idea may emerge. It may gradually grow. It may 'Gestalt annehmen', as we say in German. To explore, to test an idea, a pupil has to work hard. He or she may have to make an awful lot of computations. Because she is driven by her Kernidea she is willing to carry out all these tedious computations. She is not bored because she is experimenting rather than practicing. Or as Heinrich Winter would put it: "Es wird entdeckend geübt und übend entdeckt."

The instructor reads the students' Reisetagebuecher regularly. He comments on the students' texts. By a thought-provoking remark he helps the pupil to clarify his ideas or to change his direction of thoughts somewhat, and he encourages him to continue his work. The advantages of this scheme are obvious. The pupil gets a highly individualized response because his line of thought is available in written form and the instructor is not urged to react instantanously as in a classroom dialogue. It is obvious as well that the scheme allows very easily the instructor to value and evaluate the pupil's individual learning process.

In their publications, the authors offer a number of excerpts from various Reisetagebücher of pupils, together with descriptions of the underlying situations. A particularly illuminating example is the one on the ordering of fractions, a topic in grade 7. The Kernidea here is to study the hierarchy of gears of a bike. Mathematically the key step is the determinaion of the smallest common multiple of the denominators. Two excerpts are presented. Literally speaking Astrid, does not succeed in solving the problem. From her report it is quite obvious, however, that she understands the clue, and although she cannot deliver the answer, on her way she demonstrates a surprising maturity in dealing with a problem that is hard for her. Ueli on the other hand handles the mathematics without difficulties and very clearly realizes that the obvious ordering of the gears, which presents itself naturally in the mechanism, so to speak, is not the one of interest here. Both pupils, each within his or her capability, have done a very good job.

Ideas similar to the Reisetagebuch technique have apparently been developed elsewhere as well. A study by A. McCrindle and C. Christensen entitled "The impact of learning journals on metacognitive and cognitive processes and learning performance" - in connection with a first-year University biology course - reveals, among other results, an effect size of 0.8.

5 Concluding remarks: Impact of Technology

I conclude with a few remarks on the impact of technology. As mentioned earlier, graphing calculators, in particular TI's 85 are widely used in Swiss Gymnasia. A booklet [5] entitled "Mathematik sehen, Graphikrechner im Unterricht" by M. Bettinaglio, W. Hartmann and H.R. Schneebeli , all three teaching Mathematics at the Gymnasium in Baden near Zurich, offers a collection of very nice applications and convincingly demonstrates how graphic tools can influence the teaching of Mathematics.

Computer Algebra Systems are now moving into the classroom as well. First experiments with the TI 92 are under way. Yet the TI 92 is considered to be a transitional product with more powerful follow up machines to be available soon.

As to Computer Software: Cabri Geometre is widely used. Moreover, there are two products worth mentioning: the program 3D-*Geometer* by H. Klemenz, suitable for Macintosh, and *Geometry* by E. Holzherr and R. Renner for IBM-type machines. Both programs support the teaching of three-dimensional geometry, they permit to represent basic geometric objects and to perform constructions. Klemenz teaches at the Gymnasium in Wetzikon, while Holzherr is at a Gymnasium in Lucerne, Renato Renner was his pupil when their program was first developed. For further information contact the Author.

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SOME CONSIDERATIONS ON PROBLEMS AND PERSPECTIVES OF INSERVICE MATHEMATICS TEACHER EDUCATION

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1 Inservice teacher education as a complex field

Inservice teacher education, *which focusses on teachers' personal and professional development,* is seen as a major intervention to improve the *quality of education* on different (but closely interconnected) levels: the quality of students' learning, the quality of teachers' work, the quality of schools, the quality of an education system, or the quality of interaction between the education system and the society as a whole. However, due to different contexts, different people (students, teachers, parents, mathematics educators, etc.), institutions (school boards, political parties, unions, universities, ministries, etc.) and countries have different understandings on how to improve the quality of education. Therefore it is not surprising that *conflicting expectations* on inservice teacher education are expressed by different sides (see e.g. Krainer, 1994b; Cooney & Krainer, 1996).

Inservice teacher education is therefore a complex field dealing with *enormous diversity* characterised for example by elements such as *regional circumstances, participants, designs and philosophies, topics and organizing institutions.* Typical examples of *regional circumstances* are different general conditions for education and different needs of society, schools and teachers, which lead to big differences in students' completion of secondary education (Nebres, 1988) or in class size (Howson, 1994). There is also diversity with regard to *participants:* courses can be confined to special groups of teachers (e.g. 8th grade mathematics teachers or all mathematics and science teachers of a school), but there are also programs such as Family Math (see De la Cruz & Thompson, 1992) in which parents and children work together in cooperative settings to solve problems and engage in mathematical explorations. Further projects exist, like MINERVA in Portugal, which generated a nationwide community of teachers, trainers and researchers that took as their task the "formation of teacher teams and the assertion of a project culture in schools" (Ponte 1994, 161). Inservice courses also have different *designs and* *philosophies:* A more traditional approach is for experts to come in from outside and tell teachers about new research results; by contrast, there are courses where teachers are seen as co-designers of inservice education in which they are increasingly motivated to take their further education into their own hands, e.g. organizing working groups at the end of the course, where the teacher educators are the participants (Krainer, 1994a). Such courses strive for joint learning of people coming from different institutions, an approach which seems to have become more prominent under the notion of co-reform (Frasier, 1993). Inservice education for mathematics teachers also demonstrates a broad diversity of *topics*, from dealing with mathematics content knowledge or with cross-curricular connections to considering assessment, new teaching methods or reflecting critically on new technology. Diversity of *organizing institutions* is shown when inservice education is organized by the school authority, or by institutions which had also been responsible for the pre-service education of the participants, or by institutions where the connection to the participants is less strong but where other interests have to be negotiated, such as research interests or funding, or by self-organizing groups of teachers such as the MUED in Germany (see e.g. Keitel, 1992).

An additional issue which makes *inservice education* such a complex field is the fact that it *relates our research practice to our teaching practice* and therefore challenges us to apply the theoretical conceptions and philosophies we preach. Thus, it is also our beliefs - and not only those of the teachers - that have to be considered critically. Teacher education can be seen as *our big experiment* and as our continual struggle at the heart of our discipline.

2 Fundamental shifts in mathematics teacher education

Although the recent situation in mathematics teacher education and its related research is far from being a field with well-developed standards, both for theory and practice, the last thirty years seem to have brought considerable *progress.*

First, some brief general remarks on *literature* and *conference programs*: Research on teacher education developed from being "virtually nonexistent in the 1960s and early 1970s" (Cooney, 1994a, 618) to a field with increasing literature, e.g. with the first Handbook of Research on Teacher Education published in 1990 or with the section "Social Conditions and Perspectives on Professional Development" in the International Handbook on Mathematics Education (1996). Conference programs reveal a similar picture. For example, at the International Congresses on Mathematical Education in Quebec (1992) and Sevilla (1996) there were a number of lectures, working groups, and topic groups that focused explicitly on teachers, their work and teacher education. Similar trends can be observed at PME-Conferences (Hoyles 1992, 283) or at recent conferences in mathematics education held in German speaking countries (see e.g. Krainer, 1994b).

Second, two concrete examples of developments from different parts of the world:

a) In the *United States* a lot of efforts towards mathematics teacher education have been made in the last decades. For example, Cooney (1994a) describes the change in teacher education in the last thirty years as a change of paradigm from *analytic perspectives* towards *humanist perspectives* (Mitroff & Kilmann, 1978), from discovering reality to trying to understand the contexts that shape a person's perception of his or her reality (Brown, Cooney & Jones, 1990), having constructivism as an epistemological foundation for mathematics education. Early, teacher education dealt primarily with updating teachers' knowledge of mathematics. Research mostly focused on studying connections between student achievement and teachers' characteristics, behaviours and decisions, mostly on a quantitative basis, placing an emphasis on objectivity. Then the focus moved extensively towards interpretative studies describing teachers' cognitions (beliefs, meaning-making processes, etc.) and the contexts that influence cognition. Cooney sees that progress has been made in *discarding false dichotomies* that pervaded teacher education, stating that we are now more aware of the necessity of blurring the distinction between theory and practice, content and pedagogy, researchers and teachers, conceptionalizing the latter as cognizing and reflective agents. He points out that teachers and teacher education have become focal points for research in mathematics education but that we need constructs that can meaningfully guide our research efforts.

b) Another area to examine in looking for progress in mathematics teacher education is the *activities and future plans of institutes in developing countries*. As an example, the Institute for Educational Development (IED) of the Aga Khan University in Karachi (Pakistan) describes its approach to teacher education as follows: "The IED was envisaged neither as a traditional 'school of education' nor 'teacher training college' - models of higher education that seem increasingly out of step with the real needs of teachers and schools, in both the industrial countries as well as in developing nations. The training that would be provided at IED will be guided by some crucial concerns. First, it will be field based, i.e. the training will take place within classrooms. The assumption behind this practice is that effective teaching skills are best acquired 'on the job'. A second distinguishing feature of the training will be its reflective nature, i.e. the aim would be to make the IED students 'reflective practitioners', engaged in continual self-enquiry as practicing teachers. A third major feature will be training in classroom based research." (AKU/IED 1996). The IED establishes Professional Development Centres which focus on the improvement of teaching and learning in schools and classrooms in the region. The research policy of the IED promotes research projects which are realized in collaboration with partner academic institutions from all over the world.

Third, the progress in the field of mathematics teacher education might also be seen as a process of *growing awareness of the complexity of mathematics teaching* (Krainer, 1993b). In a first shift, recognizing that teaching contains more than presenting a pre-fabricated body of knowledge grounded in formalistic theories, research and development activities aimed at yielding a *broader sense of mathematical knowledge*. This included efforts to link mathematics with real life, to place an emphasis on the historical development of concepts and theories, to foster problem solving and to reflect on heuristic strategies, and to question contents with regard to specific and general educational objectives. The increased integration of pedagogical, psychological, social, historical and epistemological aspects into the didactic discussion put the dominance of the subject matter into perspective. This shift might also be seen as the start of mathematics education's struggle towards becoming a scientific discipline in its own right, i.e. a kind of emancipation from its most closely related science, namely mathematics. The teacher's task was seen more and more as creatively engaging students in important mathematical activities like proving, problem solving and modelling. However, very often a strong belief in the "manageability" of teaching through narrowly structured and covert guidance by the teacher remained. A second shift was caused by further research in mathematics education, e.g. on students' thinking and on interaction in classrooms more and more integrating methods and results of related fields; the research showed that teaching cannot be seen as a simple transmission process resulting in pre-determinable learning by the students. This fundamentally *questioned the transferability of knowledge* and partially brought a shift of focus from teaching to learning, placing an em-phasis on students' understanding.

The students are seen less as consumers but more and more as producers and even as researchers. However, this increased awareness of the complexity of learning and teaching was also to have consequences in teacher education. The next shift, therefore, concerns again questioning the transferability of knowledge, this time from us as teacher educators to our prospective and practicing teachers. It marks a step towards meeting demands which teachers formulate in the following sorts of questions: Why do mathematics educators propagate the active and investigative learner, although we the teachers have not been educated in that way, neither in pre-service nor in inservice education (with a few exceptions)? How and from whom do we get support in that direction? It is our task to find ways to take further steps in this direction, both theoretically and practically, and partially in collaboration with teachers.

3 Dimensions of teachers' professional practice: action, reflection, autonomy and networking

The recent discussion in mathematics education shows an increasing interest in teachers' roles, beliefs, knowledge, etc., in many cases emphasising the complexity of teachers' work: Doyle (1986) e.g. characterises the demands of teaching with descriptors such as "multidimensionality", "simultaneity", "immediacy", "unpredictability", "publicness" and "history" (the accumulation of joint experiences). In parallel to this quantitative shift we can also observe a *qualitative shift*: recently more and more publications and conferences deal with topics like "teachers as experts" (Bromme, 1992), as "reflective practitioners" (Schön, 1983, 1987), as "researchers" (e.g. Elliott, 1991) and as "professionals" (Stenhouse, 1975). Several approaches have characterized *basic elements of teachers' knowledge*. Shulman (1986) e.g. proposed seven domains: knowledge of subject matter, pedagogical content knowledge, knowledge of other content, knowledge of the curriculum, knowledge of learners, knowledge of educational aims, and general pedagogical knowledge. Bromme (1992) created a topology of teachers' professional knowledge that attends to the nature of mathematics, pointing out that teaching is primarily a matter of taking "situation-appropriate" decisions based on available knowledge rather than generating additional perspectives for solving newly presented problems. Therefore the focus of teachers' work in the classroom primarily calls for a holistic and integrated view of knowledge rather than the existence of separate solutions to discrete problems. This perspective is supported by Berliner et al. (1988) who found that expert teachers are able to process a greater array of information about students and classroom situations than novice teachers and can therefore demonstrate a greater range of techniques for dealing with individual students. The conception of teachers' professional knowledge cannot be adequately described using the singular category of "knowledge", for their knowledge is a product of many types of knowledge created in quite diverse settings and often rooted in "local theories" (Brown & Cooney, 1991) specific to their classroom situation.

All the approaches sketched above for describing teachers' professional practice mainly focus on teachers' work in the classroom. Given the considerations with which this paper began, it seems a natural consequence not to reduce the quality of education simply to the quality of teaching, but to see the teachers' contribution to the quality of education in a broader context. It seems to be crucial to find dimensions which are general enough to be used in different situations and where both the competence and the attitudes of teachers are given equal consideration. On that basis, the following four dimensions aim at describing teachers' professional practice:

Action: The attitude towards, and competence in, experimental, constructive and goal-directed work;

Reflection: The attitude towards, and competence in, (self-)critical and one's own actions systematically reflecting work;

Autonomy: The attitude towards, and competence in, self-initiating, self-organized and self-determined work;

Networking: The attitude towards, and competence in, communicative and cooperative work with increasing public relevance.

The creation of these four dimensions originally arose from the question of how mathematical tasks relating to the teachers' and learners' roles should be designed in order to promote effective teaching and learning ("powerful tasks", Krainer, 1993a). The main idea is that tasks should initiate active learning processes, closely linked with reflection on action and that tasks both should be well interconnected with other tasks in order to aim at specific goals, and should promote learners to generate further interesting questions for themselves (autonomy). This approach was then broadened for describing mathematics teachers' activities within an inservice course (Krainer, 1994a) and finally for describing teachers' professional practice in general (Altrichter & Krainer, 1996).

Each of the pairs, "action and reflection" and "autonomy and networking", expresses both *contrast and unity,* and can be seen as complimentary dimensions which have to be kept in a certain balance, depending on the context. The importance of the interplay between these dimensions is supported by various theoretical and practical considerations. Here I will confine myself to indicating Schön's (1983) account of "reflective practice" formulating "tacit knowing-in-action", "reflection-in-action" and "reflection-on-action" as different relationships between professional knowledge and professional action, and by citing Stenhouse (1975, 144) who described teachers' professionality as follows: "A capacity for autonomous professional self-development through systematic self-study, through the study of the work of other teachers and through the testing of ideas by classroom research procedures."

There are at least four levels in which these dimensions can be used for reflecting on mathematics teachers' work:

a) With regard to their own further development taking into consideration the standards in their profession, at their school, in their education system, etc. The present paper will mainly fous on this level.

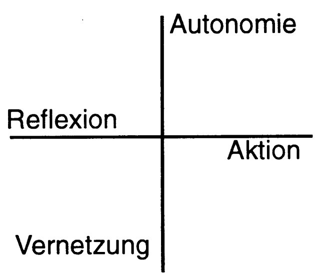
b) With regard to their students' further development, e.g. raising the following questions: In what way does teaching promote students' action and reflection? Which kind of opportunities do they have to work autonomously and also to share their conceptions with other students? How can a teacher's knowledge be effectively linked with the different meanings students have constructed?

c) With regard to their school's further development, e.g. dealing with questions such as: Is there efficient communication among the mathematics teachers in the school? Is there a fruitful collaboration between mathematics teachers and teachers of other subjects? Is mathematics seen as an important learning field at their school? Does the working climate promote innovations in classrooms?

d) With regard to the further development of their profession, their education system and its interaction with the society as a whole, for example by asking: Which role can mathematics, sciences and technology play in our society and which consequences does this role have for further developing mathematics teaching? What kind of influence do teachers have on regulations (curriculum, assessment, etc.), on standards, or on the status of their profession? Is teachers' reflection on their profession seen as a relevant contribution to the education system? Is professional communication and collaboration among teachers promoted? Is it promoted by us? Where do the rewarding effects of closer collaboration between theoreticians and practitioners (universities and schools) lie?

Of course, there are links between these levels. Here is an example which intentionally exaggerates the situation to a certain extent: teachers, who work in an educational system with narrow regulations on curriculum and assessment, who have had no influence on those regulations in the past and who will not be having any in the future, who were educated at universities where lecturing was the dominant teaching method (which leaves the audience to reflect the learned content), currently teaching at a school with a low level of communication among the teachers, and who are now confronted with inservice courses oriented towards their weaknesses, need a very strong motivation not to regard students as "received knowers", in the way themselves have been socialized. On the other hand, too often teachers complain about restrictive regulations that tend to hinder their innovations in classrooms, underestimating thereby the freedom of action they have or could establish. However, in just the same way as research on students' mathematical understanding shows that we systematically underestimate students' creative ways of thinking (when our focus is not restricted to hearing only things we want to hear), we seem to systematically underestimate teachers' creative attempts to improve their teaching. Many experiences in our work with teachers prove that fact.

Given the four dimensions: action, reflection, autonomy and networking, how would we assess mathematics teachers' position in respect of their professional development? Let us consider the following qualitative diagram in which the point in the middle means a balance with regard to each of the two pairs, whereas deviations from the middle can be interpreted as preferences for one or two dimensions (i.e. marking the point more right/higher means more emphasis on action/autonomy).



Based on various experiences, the place of most mathematics teachers (and schools) in this diagram is in the first quadrant. One could say: *there is a lot of action and autonomy but less reflection and networking,* in the sense of critical dialogue about one's teaching with colleagues, mathematics educators, the school authority, the public, etc.

To avoid misunderstandings: this assessment is in no way made with the intention of apportioning blame on teachers or on other people and institutions involved; it should simply identify particular and general problems and possible perspectives which might improve the situation; nor should make the diagram us believe that there would be an ideal position, but rather it should lead us to reflect on how teachers could react flexibly in order to meet the challenges of different contexts. To give an example: Like a medicine working on a patient, a teacher usually cannot take a time-out to reflect on the ongoing process in order to share an opinion with colleagues or to read research literature, he has to act or react immediately. However, this does not mean at all that reflecting and networking are unimportant; by contrast, given situations where time for reflecting and sharing experiences with others are limited, it is decisive to have a rich body of knowledge in which autonomous actions are soundly based on reflections and on the standards of the profession, whereby this knowledge itself is based on having the attitude of a life-long learner.

The necessity for teachers' reflection and networking is underlined by various practical and theoretical considerations. For example, Clark & Yinger (1987) and Peter (1996) stress that combining action and reflection is an essential activity of teachers. Noddings (1992, 206) states that "teachers still labor in isolation, lacking the collegiality necessary for rich professional life". In order to highlight important changes and progress in mathematics teacher education, Grouws & Schultz (1996) describe a wide range of successful projects, studies, systemic initiatives and collaboratives, e.g. stressing the importance of "reflective teaching partnerships" (447), the creation of "collaborative communities of learners - teachers learning new ways to teach and students learning new ways to do mathematics" (449), teachers' reflections "on themselves as learners", the existence of "teacher and school collaboratives, professional development schools" where "the focus of university-school linkages include practice-sensitive researchers at the university and researcher-sensitive teachers in schools ..." (452) and the facilitation of "collaboration among pre-service and in-service teachers" (453) in order to assist teachers in examining their beliefs and actions.

But what are the reasons for teachers' lack of reflection and networking? In the following three functional reasons are sketched which might give some explanation:

a) A first reason is the *culture* in which teachers have to work. In science communities we speak about "publish or perish"; with regard to teaching we might use the slogan "act or perish" (see Krainer, 1994b, 221). Researchers' and teachers' work conditions and agendas differ as researchers' culture emphasizes reflecting, analyzing, writing, making one's ideas public and discussing them with colleagues, whereas the teachers' culture usually emphasizes quickly perceiving and acting with only few contacts with colleagues. Researchers organize their own conferences in which reflection and networking are important dimensions; by contrast, meetings of teachers are mostly not designed as professional exchanges of experiences but as teacher inservice courses led by "external experts" and often organized by the school authority. This makes a difference but also shows a particular way of thinking about progress in teachers' professional development. With respect to cooperation between teachers and researchers the danger of this difference lies in possible one-sided distributions of interests, in forcing one culture on the other. However, the difference of cultures is less a counter-argument against cooperation and more a pro-argument for it because of the opportunities to learn from each other and to build a bridge between the two cultures of theory and practice. More research on the relationship between teachers and researchers is needed (see e.g. Brown, Cooney & Jones 1990; Bishop, 1992; Krainer 1994c).

b) Most teachers' *pre-service* (and often also inservice) *education* did not (and still does not) place any emphasis on reflecting or networking. In many cases teachers feel socialized as "lonely fighters" for their subject matters. Within the process of some of our inservice courses teachers expressed the experience that they firstly had to find a certain distance to mathematics in order to be able to build up a new and constructive relationship to it, allowing them to find a way to appreciate students' thinking. There are more and more international reports (see e.g. Krainer, 1994b; Grouws & Schultz, 1996; Giménez, Llinares & Sánchez, 1996) about involving (practicing or prospective) mathematics teachers into research projects and integrating research components in teacher education courses where reflection and networking are important dimensions.

c) Systematic *reflection* on one's own teaching and *sharing* it with colleagues is *unusual* and costs a lot of time and effort. However, teachers who meet this challenge in general report about positive effects on their teaching and further professional development. The following section highlights some considerations on this issue.

4 Inservice mathematics teacher education as a means of promoting action, reflection, autonomy and networking

One mathematics teacher's *self-critical investigation into his own teaching* while participating in a teacher inservice course will be examined in order to show how the activities are related to the four dimensions mentioned above.

First, some remarks on the *context*. The two-year university course "Pedagogy and Subject-Specific Methodology for Teachers - Mathematics" (PFL being an abbreviation for the German "Pädagogik und Fachdidaktik für Lehrerinnen") offers special inservice education for mathematics teachers in Austria. The PFL-*mathematics* course aims at helping teachers to improve their teaching and at making their innovative work accessible to others, thus promoting professional communication and cooperation among teachers. To meet the interests of the participants the starting point of work within the course in most cases is their practical experience. Because of the complexity of the teacher's task the interconnectedness of pedagogical and subject-specific aspects is a crucial facet. *Action research* (see e.g. Altrichter, Posch & Somekh, 1993), understood as *the systematic reflection of practitioners on action* (i.e. on their professional activities in order to improve them), is used as a framework for achieving a broader situative understanding and for improving the quality of teaching and other professional activities at their school. Within PFL courses the participants are required to do research and to write two case studies in which they have professional developmental interests. The *case study* of the participating teacher will now be discussed. For a detailed description of the philosophy and of activities within PFL-mathematics see Fischer et al. (1985), Krainer (1994a) or Krainer & Posch (1996).

In his first case study in the course, the teacher dealt with the topic *"On the emergence of noise"* in order to investigate the noise level in his classes and to find out means for improving the situation. In order to obtain relevant data, he administrated a questionnaire to his students and wrote a research diary in which he regularly took note of his observations on the topic "noise" in one of his classes for a period of seven months. He found out, to his surprise, that it was primarily he himself who judges his instruction as too loud, and that a considerable part of this "noise" is caused by content-related communication between students. This discovery motivated him, in a second case study, to reflect on possible changes in his approach to teaching and to test them in practice.

How can we interpret the teacher's work with regard to our four dimensions? Let us place an emphasis on the dimensions *action* and *reflection* first: As mentioned above, the course starts out from teachers' practical experiences and needs. For this teacher, "noise" was a problem which influenced his classroom activities enormously. Collecting data and reflecting on this data brought him new information and new insights he would not have got by simply referring to his original "practical theory" of the situation, which mainly said that noise has to be seen as a factor that hinders teaching. The process of reflecting led to a more complex and deeper understanding of the situation, e.g. realizing that noise may be an expression of students' need for content-related communication and that *"noise may emerge through monotonous modes of instruction"* (Kliment, 1994, 1); but this also had consequences on his actions, from an alternative way of dealing with "noise" to starting *"occasionally to design lessons in another way"*. This process shows the impact the close interplay of the teacher's actions and reflections had on his beliefs with regard to teaching and his concrete actions in the classroom. In the following considerations about the teacher's further progress we place our emphasis on the interplay between *autonomy* and *networking.*

For this second study, entitled *"Mathematics instruction for one's different"* (Kliment, 1994), the teacher read literature on teaching and learning objectives, formulated his own objectives and found consequences for his teaching, e.g. stating: *"It is clear that - with regard to the objectives formulated in the preceding section - frontal instruction now plays only a small role. But what are the alternatives? One of the most effective incentives in changing my teaching was a study by colleagues."* Here he refers to a case study written by participants and one staff member of a former course. This shows one advantage of writing down teachers' investigations and of making it accessible to others: teachers' local knowledge can be linked with the experiences of others through being published in studies available for larger community. This means that teachers' autonomous work can be networked networked and therefore used as one contribution to increasing professional communication among teachers.

The teacher increasingly turned to a child-centered, application-oriented and computer-supported form of instruction. He realized his ideas in a teaching experiment which lasted for a period of about 10 weeks (including a written examination). His new way of teaching also had some consequences with regard to assessment: *"Significantly more than before, my teaching gave me opportunities to observe students and to appreciate their involvement ... It is an enormous relief to the students if not everything depends on their performance on two days. It is also advantageous that during examinations a resort to individual or group work is possible. ... "* The evaluation of the teaching experiment was predominantly very positive. In the concluding section "Criticism and outlook" of his study the teacher writes: *"During this school-year I have had numerous experiences. It is no longer my intention to teach year in, year out in the same manner. Now it is time to sift out those methods which have been successful and can be retained. Apart from some details I am content and can with assurance build on the experiences of this school year. It would be arrogant to claim that I succeeded in realizing all my chosen objectives; Moreover, this catalogue of objectives is not definitive. Some changes were not easy for me, old habits had first to be thrown overboard. I did not succeed in encouraging students to participate more actively in the classroom experience. (...) I must further reflect on this question!"*

All in all this shows how motivating and helping teachers to write case studies can improve professional development. More reflection on action improves practitioners' activities which in turn lead to new questions and reflections, etc. But this *interplay between action and reflection* is not only confined to the learning of individual teachers, it can also be used as a starting point for professional exchange among teachers: the learning process is directed towards autonomy as well as towards networking - it is the interplay of both which leads to progress.

Of course, not all teachers of such courses change their teaching in this fundamental way. It was the intention to show here what is possible. Indeed, this is one of the main ideas of the PFL-program: to show teachers (and other interested people) that things can be done, within the existing general conditions or after successfully fighting for their change; it is worth making the good work of teachers visible and available for discussion. More than 100 studies have been written by German, English, mathematics and science teachers within the framework of PFL-courses, which have since been bought by teachers, researchers, schools, etc. and their feedback demonstrate its value.

The teachers' systematic reflections on their own practice can not only improve their own teaching but can also have consequences for the further development of teacher education (see e.g. Krainer & Posch, 1996), for mathematics education (see e.g. Fischer & Malle, 1985) or for the personal and professional development of team members (see e.g. a mathematics educator's reflective paper on his activities within the course; Peschek, 1996).

The fact that writing case studies causes some problems for both the teachers and the teacher educators supporting them, should not be withheld. In writing case studies, teachers have to do at least three things which are rather unusual in their normal practice: They have to gather data and to reflect on them systematically (and not only take action), they have to write down their findings (and not just communicate them orally), and they have to formulate these results for other people (and not just practice something within their own classrooms).

That this is more difficult for teachers than for us - living in a "culture of publishing" - should be taken into consideration. Nevertheless, it seems to be worth promoting teachers' investigations for at least four reasons: Systematic reflection on their own work creates new knowledge which in turn positively influences their teaching; writing down is an additional opportunity to learn; writing a study (to be read by others) increases the opportunities for communicating and cooperating with interested people (teachers, theoreticians, administrators); and finally, it gives us an additional opportunity to learn from them.

However, such an approach is also problematic on quite another level. Inservice teacher education based on voluntary participation is usually confronted with problems of realization and dissemination: Given a good seminar or course, the motivation to change one's mathematics teaching might be high and many participants might try out new things and might apply learned methods and ideas; but it might also happen that they rarely find colleagues who really want to join in their efforts, that their motivation and the perseverance to realize changes are (in the long run) not high enough, that innovative things at schools are often regarded rather critically and cause (open or hidden) resistance or opposition, that the participants on seminars are "always the same" and those who really would need some improvement do not come, and that links between different subjects are used too rarely.

Experience shows that participants engaged in long-term teacher inservice courses which place an emphasis on professional communication among teachers through promoting the discussion of their case studies, more efficiently support teachers' efforts to bring about change, at least during the course. That such a course gives birth to self-organized groups which remain together for a longer period (see e.g. Krainer, 1994a) is more an exception to the rule. However, a lot of participants act as "agents of change" in their region, are engaged in inservice courses or teacher pre-service education, and actively participate in conferences in which innovative work of teachers is presented.

An alternative approach to inservice education which recently has become more popular is school-based inservice. Examples with regard to mathematics education are rare. To sketch the perspective of this approach, a pilot project in this direction will be considered briefly.

5 Continuous work with groups of mathematics teachers at schools

In contrast to working with mathematics teachers from different schools, working continuously with a group of colleagues from one school might yield some advantages which should not be underestimated: the "culture" of this school (as a decisive general condition of what is possible or not) can be taken into consideration, and maybe steps towards changes could be considered; the collaboration among individuals might develop towards the establishment of a group; the teachers now could have the encouragement of others or even colleagues who were ready to join their efforts to improve their mathematics teaching; innovations would be more likely to become a relevant component of mathematics teaching (of the whole school); mathematics teaching could be more visible and could play a greater role at this school.

The necessity of considering the whole entity, in this case the school, seems to be growing in importance, the more so as general conditions change. For example, the growing tendency to more autonomy for schools (including e.g. the question of whether the number of mathematics lessons should be decreased or increased, or mathematics should be combined with other subjects) or due to budget problems a lot of inservice education will be organized at schools to save on travel expenses.

Another argument for placing *greater emphasis on the level of individual schools* is the fact that it allows the context in which the teachers live and work to be taken really seriously, and it provides opportunities where individual and social demands on mathematics teaching meet and can thus be analyzed, discussed and negotiated. This argument is supported by considerations in section one of this paper stating that the quality of education cannot be reduced to the quality of teaching and also by the belief expressed in the preceeding sections that more reflection and networking among teachers are needed.

In the following, we briefly look at the design and the outcome of a recent *pilot inservice seminar* at an urban school in Austria. As agreed at a preliminary meeting, the two-and-a-half-day seminar with the eight participants was based on a “teacher as researcher” philosophy and covered three major issues: 1) Interviews with pupils in order to understand better how they see mathematics, mathematics teaching, etc.; 2) A little experiment towards more “open” mathematics teaching in order to experience new approaches, methods, etc.; 3) Investigating connections between mathematics and other subjects in order to experience the potential of bringing the real world into mathematics teaching. Some of the results were: all teachers intended changes in their teaching, eight weeks after the seminar most teachers reported on concrete realizations of ideas, some indicated preparations for small teaching experiments in the next school-year; a mathematics education journal from which different volumes were analyzed by different teachers during the third part of the seminar was subscribed to after the principal was persuaded that this would contribute to the further development of mathematics at this school; with one participant there was an exchange of articles with regard to connections between geometry and chemistry; most colleagues were interested in commenting on the teacher educators' analyses of the seminar; the group expressed interest in further collaboration with the universitary institute.

6 Summary and outlook

The *promotion of teachers' reflection and networking* seem to be an important challenge for inservice mathematics teacher education in the future. In particular, designers of inservice courses should ask themselves questions like: To what extent do we succeed in motivating the participants to reflect (self-)critically on their own activities and the collaborative work on the seminar and on using it as an opportunity for corresponding learning processes? To what extent do we succeed in promoting deeper communication and collaboration among the participants and in linking individual and social learning experiences meaningfully?

Questions like these express an understanding of inservice education which sees teachers not as receivers of pre-fabricated knowledge and complete solutions but as reflective practitioners who develop their own knowledge and solutions, and fit them into the context in which they work. To avoid misunderstandings: building on teachers' self-critical investigations into their own work does not decrease the importance of research in mathematics education. By contrast, more research is needed, in particular on inservice teacher education. But we should find *new ways* of mediating between theory and practice, of collaborating with teachers on different levels, of taking into consideration the culture in which they live and work, of (re)constructing our beliefs on teacher education while at the same time questioning them thoroughly, and of (re)defining teacher change and inservice education as an inevitable part of professional practice.

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MATHEMATICS AND GENDER: A QUESTION OF SOCIAL SHAPING?[1]

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> *It's true that Hypatia, the ancient Greek mathematician, was the first eminent woman mathematician ... that anyone seems to know about.... She had a huge following, and distinguished students came from Europe, Asia and Africa to hear her.... Cyril was fearful of her popularity and her religion, and incited a mob of fanatics, who dragged her to a church, murdered her with shells, and then burned her. This happened at the height of her fame, when she was 45. All her writings have been lost. (Woolfe, 1996, p. 7)*

Introduction

There is no need, at a meeting of the International Congress on Mathematical Education, to dwell on the intrinsic worth of mathematics as an area for study, nor the important role played by mathematics as a critical barrier to further educational, career, and life opportunities. Nor is it necessary to repeat that mathematics and related occupational fields have been identified, internationally, as areas in which males predomin and females are believed to be disadvantaged. Regrettably, in many countries females' disadvantages go well beyond these areas, however.

> Most of the roughly 100 million homeless people in the world are women and children.... (O)f the estimated 1.3 billion people living in poverty, 70 percent are women and girls.... Some 50,000 people - mostly women and children - die daily because of poor shelter, polluted water and bad sanitation... ("End girls poverty", 1996, p. A18)

Educational opportunities are affected by class and ethnicity, as well as by gender. Females from economically and socially advantaged backgrounds are more likely to complete secondary education than males from

1 At the time of writing this paper, I was engaged in a collaborative project with Helen Forgasz (La Trobe University) and Claudie Solar (Université de Montréal). I'd like to acknowledge, with thanks, their influence on this work!

families in poverty. Decisions about continuing with the study of mathematics beyond the compulsory years are also influenced by many, often interacting, factors - including and beyond gender. In other words, research directed at gender issues in mathematics education is contextually situated, whether or not this is acknowledged or recognised by those engaged in these investigations. In the remainder of this paper I want to trace some of the important directions and developments of work conveniently grouped under the label "gender and mathematics education". The focus is necessarily on the broader issues and research dimensions. More detailed and comprehensive reviews of relevant research can be found, for example, in Fennema, 1995; Fennema and Hart, 1994; Fennema and Leder, 1993; Forgasz, 1994; Joffe and Foxman, 1988; Leder, 1992; Leder, Forgasz and Solar, 1996; Linn and Hyde, 1989; Solar et al., 1992; Willis, 1989.

Where should this review begin? It is tempting to start with the life and work of Hypatia "the first eminent woman mathematician ... that anyone seems to know about" (Woolfe, 1996, p. 7). Following the path taken by (Kimball, 1995) also has its appeal. Her more general exploration of the traditions of gender similarities and differences took as its starting point the very different lives, trials and triumphs of Leta Hollingworth and Karen Horney, both born, on different continents, well before the beginning of this century (in 1886 and 1885 respectively). Since current research and debate rely heavily on the psychological and psychoanalytic perspectives within which they worked, such an approach has much to offer. Pragmatically, given the constraints of time and space, it is, however, more appropriate to focus on activities of the past two decades or so.

IDENTIFYING A "PROBLEM"

During the 1970s, much research effort was directed at documenting gender differences in participation in mathematics courses and in performance on mathematical tasks and tests. A then timely "state of the art" summary read as follows:

> Are there sex differences in mathematics achievement? ... No significant differences between boys' and girls' mathematics achievement were found before boys and girls entered elementary school or during early elementary years. In upper elementary and early high school years significant differences were not always apparent. However, when significant differences did appear they were more apt to be in the boys' favor when higher-level cognitive tasks were being measured and in the girls' favor when lower-level

> cognitive tasks were being measured.... Is there "sexism" in mathematics education? If mathematics educators believe that there is a sex difference in learning mathematics (as was evidenced in the reviews cited) and have not attempted to help girls achieve at a similar level to boys, then this question must be answered in the affirmative. (Fennema, 1974, p. 137)

SOME EXPLANATIONS

Explanations consistently proposed by researchers for the subtle but persistent performance and participation differences identified in mathematics have included:

- the almost exclusive depiction in mathematics textbooks and related materials of males and activities that appeal to them in contrast to the virtual invisibility of females and their interests
- the setting in which learning takes place
- other organisational structures within the school
- differential (mathematics) course taking and unequal engagement in mathematical activities during leisure times
- liking of mathematics, perceptions of mathematics as useful, worthwhile, and relevant
- expectations of attaining success in mathematics and related fields
- being encouraged by significant others to continue with mathematics
- and less frequently, inherent, genetic differences rather than the environmentally related factors enumerated above.

My own master's thesis, completed in 1972, was typical of the "empirical-scientific-positivist" investigations conducted at that time. It was "concerned with sex differences in mathematics achievement. In particular, the effect of changes in the contextual setting of certain problems on the mathematics performance of a group of ... boys and girls attending Victorian metropolitan high schools (was) examined" (Leder, 1972, p. iii). The study confirmed that boys preferred and performed slightly better on problems set in a "male" context, while girls preferred and performed slightly better on "female" context problems. Furthermore, boys, but not girls, performed better if they were given problems contextualized in their preferred setting. Was this, I speculated at the time, because

> the girls have become conditioned to doing mathematics problems with content that is not sex-appropriate for them? ... For them to be given a non-preferred content version is a state of affairs to which

> they have become accustomed over the years. The boys, on the other hand, have become used to doing problems with sex-appropriate content. Apparently, being given non-preferred content has affected the performance of some boys sufficiently to produce a significant difference in performance. By analogy, this finding could be said to indicate the likely effect on girls' mathematics performance of prolonged exposure to non-preferred content. (Leder, 1972, p. 178)

This line of reasoning suggests:
(i) that males and females have similar learning characteristics and motivations to achieve the same educational goals (the *assimilationist* model)
and
(ii) that the gender differences in learning mathematics, documented in a multitude of investigations, are the result of inadequate educational opportunities, social barriers, or "biased" content and that they can be minimized, if not removed, by compensatory actions and initiatives built into an educational program and typically targeted at females. Thus the removal of barriers, and if necessary the resocialization of females, were seen as paths to equity. Explanations which encompass these thrusts have been labelled a *deficit* model.

The emphasis on the rights of females to equal the achievements of males is consistent with the tenets of *liberal feminism* and of the first wave or generation of feminism. While alternative perspectives have gained in popularity in recent years, our current understandings of gender issues in mathematics owe much to the pioneering studies conceived with this assumption as its central focus.

PRACTICAL INTERVENTIONS

Curriculum initiatives and other intervention strategies aimed at achieving gender equity have attracted considerable research attention as well as significant amounts of funding. Relevant programs can be categorised in various ways. The five-stage model described by McIntosh (1983) and adapted for mathematics education by Rogers and Kaiser (1995) is particularly evocative. Key elements of the three earlier phases of this model are summarised in Table 1.

Table 1: Towards gender equity in mathematics education (1)

PHASES	BRIEF DESCRIPTION	MESSAGES CON VEYED
Womenless mathematics	Mathematics textbooks and other materials focus on males, their interests and activities.	Mathematics is unambiguously portrayed (and typically perceived) as a male domain. This view persists in many contemporary socieites.
Women in mathematics	Portrayals of males and male contexts and themes are supplemented with references to a very small number of exceptional women, who, throughout history have been successful in mathematics. Examples commonly cited include Hypatia, Sonya Kovalevskya, and Emmy Noether.	Female mathematicians are perceived as rare, exotic and exceptional. Valuable contributions by females are scarce. Success in the field is a threat to femininity ; failure an indication of lack of personal worth and ability.
Women as a problem in mathematics	Appropriate intervention programs will raise females to the performance and participation levels of males: e.g., exposing female-friendly settings in text books, single-sex rather than coeducational groupings, highlighting the importance of mathematics as an entry to educational and career opportunities, ...	Females can become successful learners of mathematics. Their deficiencies can be overcome. It is assumed that success and continued participation in mathematics, beyond the compulsory years, is a universal goal. The nature and delivery of mathematics are not questioned.

The presentation of these phases as sequential is simplistic and convenient rather than an accurate chronological representation. Elements of each stage are still present and continue to attract instructional and other interventions. For example, some of the recent studies which have examined gender-stereotyping in mathematics textbooks continue to report a bias towards males. Others have noted that attempts to be gender-neutral seem to have resulted in an overemphasis on numerical questions and a mathematics portrayed as lacking in human dimensions (see Leder et al., 1996 for more details). Attempts to focus on women with exceptional and rare mathematical talents are also not without their problems. Some of these portrayals, it is argued, simply confirm how difficult it is for an "ordinary" (female) student to become an "extraordinary" mathematician, what hardships need to be endured, what challenges to be overcome, what prices to be paid. Preoccupation with the mathematical "deficiencies" of females has been criticized for reinforcing and perpetuating existing stereotypes and for all too readily "blaming the victim" rather than questioning whether the implicit assumptions of using "male" standards as the accepted and most appropriate norms might not more fruitfully be challenged. These alternate perspectives are the focus of later sections.

SUBSEQUENT DEVELOPMENTS

A change in terminology has clearly occurred since the 1970s. *Sex* differences featured in the titles of the early reports and investigations began to be replaced by *gender* differences. This change was of more than linguistic significance. To a sensitive reader, it could be argued, *sex* differences seemed to emphasize innate, genetic characteristics not readily amenable to change. *Gender* differences, on the other hand, instead appeared to highlight the role played by the environment - personal and situational - in which learning occurs. For those engaged in education, it seemed more constructive to concentrate on factors that are at least potentially able to be changed. Thus the current emphasis on *gender* differences signifies the primary interest of those in education with cultural conventions and pressures as well as socialization processes.

With time, the earlier crude comparisons between groups of males and females became more refined: gender differences between- as well as within-groups began to be acknowledged. Simplistic interpretations of "equal" exposure to, and involvement in, mathematics began to be questioned. So, too, were the assumptions that females' norms and preferences were necessarily inferior to those of males and that the largely monocultural definition (white, middle-class, male) of what constitutes worthwhile mathematics are inviolate. No longer was it thought appropriate to ignore the value and diversity of different ways of knowing, nor the harm done in the past, to individuals as well as to larger groups, of denying this diversity. Perhaps, it began to be asserted, it is not females who need to change but mathematics: the way it is conceptualized, defined, taught and assessed. Thus the focus moved to calls for a reconsideration of the nature of the discipline of mathematics and a re-examination of the pedagogical methods used in mathematics. Such lines of reasoning seem to suggest:

(iii) that the goals of education are not necessarily the same for all groups. Learning characteristics may vary within and between groups. These differences should not be regarded as deficits. Rather, the educational environment should be structured to take account of these differences. In this *pluralistic* model, diversity among learners is an expected outcome of the educational process, and

(iv) even more specifically, if justice and equity are to be achieved (the *social justice* model), then differences need to be respected and catered for appropriately. Identical treatment of different groups may be necessary at certain times; different treatment and actions at others.

These models, which value and respect "difference" are congruent with convictions of radical feminist research. Or, to use a somewhat different terminology, the focus on the special attributes of females and the rejection of an uncritical assimilation of females into a male world capture the critical concerns of the second wave or generation of feminism.

THE "PROBLEM" REVISITED

Work conceived and developed in the broader research community has been influential in shifting the directions of those concerned with mathematics education. The themes fuelled by Gilligan (1982) *In a different voice* and the feminist critiques of the sciences and the Western notions of knowledge have been particularly powerful. It is convenient to trace the more recent thrusts to achieve gender equity in mathematics education through the final two stages of the modified McIntosh, 1983 model, referred to earlier. Key elements are summarised in Table 2.

Table 2: Towards gender equity in mathematics education (2)

PHASES	BRIEF DESCRIPTION	MESSAGES IMPLIED
Women as central to mathematics	They system, as well as the contents of mathematics, are changed to be less alienating	Women´s experiences and interests are perceived as central to the

	to females. Rather than expect females to aim for male norms, females' experiences and interests are used to shape the mathematics taught and methods of presentation.	development of mathematics. In this phase the "blame" is shifted away from females through attempts to change not them, but the system.
Mathematics reconstructed	"Cooperation and competitiveness are in balance and mathematics will be what people do" (Kaiser and Rogers, 1995,p.9).	Debate continues about the changes required in the conceptualisation of mathematics, its delivery and applications if the subject is to be (gender) inclusive rather than exclusive

The assumptions of the "women as central to mathematics" phase are not without danger. In particular, programs which value and nourish qualities and characteristics presumed to be exclusively female may imply, directly or indirectly, that these are innate to females and alienate those who do not possess them. This essentialism also risks perpetuating traditional gender stereotypes rather than redressing gender inequities. Nevertheless, recognition that previously unchallenged assumptions, traditions, and cultural exclusivity need to be examined and possibly redefined is overdue.

FURTHER INVESTIGATIONS AND EXPLANATIONS

Recent reviews of research on gender and mathematics education reveal that

* research in the empirical tradition continues to dominate
* quantitative, as against qualitative, studies are more prevalent, irrespective of the country in which the research is located
* affective variables continue to attract considerable research attention
* increasingly authors draw on multiple research methods to plan their studies and analyse their data. This is an important development.
* drawing on publications other than those found in the traditional,

mainstream journals yields a much broader perspective. In particular, work exploring feminist theories is more likely to be found in edited books, collections of articles, and conference proceedings than in mainstream mathematics journals. However, there is some overlap in the authorship, ideas, and studies found in collected volumes of research on gender and mathematics education.

* scholarly evaluations of intervention programs and strategies are all too rarely reported in mainstream mathematics journals
* authors tend to draw on publications written in their own language. Work recorded in English appears to have the most extensive international penetration.
* gender inequities most relevant for those engaged in post compulsory mathematics education are now also attracting research interest.

CONCLUDING COMMENTS

In brief, gender equity concerns represent a significant item on the research agenda of (mathematics) educators in many countries - in highly technological societies as well as developing nations. International comparisons, formal and informal, highlight the role of culture. For a given society, the status of mathematics in the lives of females is linked to their status in that society. Male norms, and acceptance of difference without value judgments, are more likely to be challenged in countries with active and long standing concerns about equity issues. Collectively, the body of work on gender and mathematics education reflects an increasing diversity in the inquiry methods used to examine and unpack contributing factors. More radical feminist perspectives are being adopted, females are less frequently considered as a homogeneous group, and scholarly evaluations of interventions are becoming more prevalent.

There is a continuing need for traditional empirical research which monitors females' participation and performance in mathematics and related educational and career activities. It is important to remember that the various approaches that can be used to extend our knowledge and understandings of females' mathematics learning - whether informed by more classical approaches or by feminist critiques - are complementary, valuable in different ways, and share a common goal: attaining gender equity in mathematics education.

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SCHOOL STEREOTYPE WORD PROBLEMS AND THE OPEN NATURE OF APPLICATIONS

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Introduction

Teaching word problems is an enormous effort in which most of us fail. Why do we continue teaching them? The typical attitude is that by teaching students how to solve word-problems we teach them the applicativity of mathematics; that the heart of learning mathematics is to know when and how to apply mathematics. Unfortunately, many students after struggling with many types of problems, do not consider them to be part of real life, as demonstrated by my favorite example (Nesher, 1980):

In a visit to a second grade in Israel, the teacher has asked the kids to compose their own problems for the mathematical sentence 2+7=9 (which later we will call mathematical model). Here are two examples of their creations:

Joseph: Mother had 2 irons and she bought 7 more, how many irons does mother have now? or,

Rivka: Johnny ate 2 spoons and 7 forks, how many spoons and forks did Johnny eat altogether?

Today, when the math education community is talking so much about 'Realistic Mathematics' (de Lange, 1996, Reusser, 1988, Verschaffel, De Corte et al. 1994) I find it quite absurd to believe this sort of "realistic" problems coming from the children. In fact, the kids have interiorized a habit which seems to be saying something like this: "There is no connection between real life and what the teacher is asking us to do. If we get a 'problem'. i.e., a text with numbers in it, we have to do something with the numbers and to get a numerical reply. The text is irrelevant..."

They also realize that it is not that simple to know what calculation should be done. Many of the little kids will turn to their teacher saying: " I understand the problem well, just tell me whether to add or to subtract".

I am not surprised that this is the case. There is no clear understanding even among math educators what is the purpose of this entire activity. For many decades people felt that it was important to teach word problems, yet, we still don't know how to do it. In a little piece of research that I conducted in an old library at Harvard, I found that nothing has changed concerning the teaching of word problems since the eighteenth century, except for the context. Many problems in the eighteenth century are taken from farm life, or navigation, but they all have a very similar structure, and the children then engaged in the same activities as today's children.

Efforts have been made in the past twenty years to better understand the processes involved in problem solving. Most of these were within the paradigm of cognitive science research. My main thesis in this presentation is that in order to teach problem solving we should learn what cognitive processes are involved and give the children the opportunities to cope with them. Modeling real life situations with mathematical tools means being acquainted with mathematical schemes of actions. School usually starts with the most simple ones (those which unfortunately become stereotyped) and extends to more complex ones. There is a danger that starting with simple stereotyped situations will contradict the open nature of real application. Thus, teaching word problems has two aspects: learning the mathematical tools used for modeling, but at the same time freeing the activities from being artificially restricted.

Findings from Cognitive Research

Most of the findings from cognitive research are well known and have even penetrated to the teaching and learning at the primary level, yet I would like to mention them here in order to get the full picture. It was found that cognitive processes involved in problem solving are driven by schemes. Applications that call for a certain mathematical model have certain characteristics. Research has made these characteristics explicit. Let us examine the additive structure; for example,

Here is a typical word problem text.
.......7......2.....
...........?

This is a typical structure of a simple word problem text (Nesher and Katriel, 1977), This is also how the kids understand this kind of text: namely, ignore the words (the "bla-bla") and do something with the numbers. Unfortunately, without reading the words, one cannot know what to do: to add, subtract, multiply, divide, or else? But once we fill the words, as in the following case:

Johnny had in the morning a certain amount of dollars and he spent some of them. How many dollars does he have now?

We now know for sure that we have to subtract in order to find the answer although no number is mentioned in the text. Thus, the mathematical model is defined by the situation and not by the quantities mentioned in the text. Moreover, a text that has full textual and numerical information such as the following:

In the group there are 4 girls and 8 boys.

will not suffice, and does not have enough information for deciding about the correct mathematical model. Finding the mathematical model will depend on the missing string, which is the question. For example, we can end the above text with one of the following questions:

a) How many boys and girls are there altogether?
b) How many more boys are there than girls?
c) How many different couples can one arrange in this group?
d) What is the proportion of girls to boys?

and so on.

Each of the above questions will lead to a different mathematical model. Thus the full information that determines which mathematical model to employ, resides in the full text that on one hand, describes the situation to be modeled and on the other hand, fulfills some requirements that I would like to describe now.

The Additive and Multiplicative Schemes

A semantic analysis of additive texts demonstrates that any text in its minimal form has three strings holding some dependencies among them. Without going into a formal description, the main characteristics of such texts are that if the addition operation is to be selected for modeling the situation, the description is of three sets of objects. Two of these are disjoint sets, the third is a superset of the previously mentioned sets, and no other objects are involved. If more objects are mentioned they will be considered as superfluous information emphasizing the three argument relation necessary for a binary mathematical operation. This actually forms a scheme that is called into action whenever the real situation is like the conditions described above. It is interesting to note that the same conditions also hold for the subtraction operation, since addition and subtraction are two aspects of the same mathematical structure.

Actually, the entire variety of additive word problems should fulfill the above conditions, and these conditions can be described by a variety of linguistic means. Similarly, conditions can be specified for situations in which the multiplicative scheme is called into action and the multiplication or division mathematical operations are to be executed (Nesher, 1988; Vergnaud, 1985).

We, in Israel have, therefore, introduced into primary schools the teaching of schemes, rather than separate mathematical operations. We teach the additive structure which serves as an underlying structure for many arithmetic sentences and the same for the multiplicative structure. After clarifying constrains under which the additive model can fit, it is our responsibility to open it to variety of open situation (CET, 1980-1997).

I would like to present some illustrations of the instruction we use which is based on findings from the above research. In a primary school math program in Israel, we teach (i.e., we give the child an opportunity to be active and construct such schemes). The child is engaged with schemes before he attempts to formal mathematical sentences. Here are some examples of activities for learning the characteristics of the additive scheme.

Children engage in activities in which they have to deal with subsets and supersets that are relevant to the additive scheme; i.e, they should notice, via their activities, whether the subsets are disjoint; the superset includes the same object and nothing else enters the additive scheme. They realize that for deciding about the underlying structure of a text, the numbers are neither sufficient, nor even crucial, while the sets involved are. They realize that the same structure fits both, addition and subtraction, and that one real situation can call for several mathematical schemes (See appendix A for demonstrations).

It is now clearer to math educators which are the real world situations that call for the additive schemes. Three main contextual structures were identified (Greeno 1978a; Nesher, 1982; Carpenter, Moser et al. 1982):

a) Situations where two sets of objects are combined.
(COMBINE).

b) Situations in which one amount is increased or decreased.
(CHANGE).

c) Situations in which two sets are compared.
(COMPARE).

The same holds for multiplication and division. Contexts in which two dimensions are to be arranged or compared, enlarged etc., were identified and became the target of more directed teaching, as preparation for modeling.

More Complex Problems

It was found in the last decade that the basic schemes mentioned above are the building stones of more complex problems. Below, I elaborate on complex schemes, using the illustrations of our instructional program. After becoming acquainted with the two basic schemes, the multiplicative and the additive, it was established (by our group in Israel, and Shalin from LRDC) that when two schemes are combined, some additional constraints appear, and that complex situations have their own structures (Shalin and Bee 1985a; Hershkovitz, Nesher et al. 1990).

Though a problem that combines two full schemes (additive and multiplicative) has in its underlying structure 6 components (strings in the verbal form), in a combined 2-step problem only 3 components are explicitly mentioned and one string states the question. The latent components, which are not explicitly stated, are the main source of difficulty for children.

In analyzing the structure of the 2-step problems we also found that there are only three possible schemes which account for all 28 possible different combination of these two basic mathematical operations (taking into account the order of the operations). For a full exposition of these findings see (Nesher and Hershkovitz 1994).

The question that always bothers researchers in cognitive studies is, are our understandings about the schemes underlying our actions teachable?, Does presenting these structures to students enhance their learning , understanding and performance? I can only hint here about some of our findings, which were the thesis of my Ph.D. student who built two computerized environments to study this issue (Hershkovitz and Nesher 1996). One of the environments represented the schemes approach and the other, though also computerized, represented the more conventional approach. Her tutoring experiment demonstrated that emphasizing the schemes enhances the children's understanding and performance.

The open nature of applications

So far I have succeeded in showing how to reveal the underlying structure of variety of word problems. I also claimed that every well formed problem, where one knows which operation to apply to it, is well defined and can be reduced to a basic underlying structure. Such an analysis seems to lead to stereotyped. behavior. However, this is only one aspect that emphasizes the construction of mathematical tools for modeling. Let us now switch to the other end which is our target in teaching the above tools, i.e., the open nature of applications and how we model them mathematically.

What will be considered an open-ended problem?.

There are several interpretations for the notion of open-ended problems. such as:

1. *Problem with no single solution.*
2. *Problems that lack numerical information*
3. *Superfluous or missing information.*
4. *Descriptions of situations with undefined question.*
5. *An unusual problem*
6. *Only certain numbers can considered as solutions.*

(Verschaffel, De Corte et al. 1994; Greer 1996; Wyndhamn and Saljo 1996)

All the above are important as part of mathematical education. They all call for non-mechanical style of work, but rather for meaningful elaboration which fit a variety of student abilities, as in the following example:

There are 30 children in the second grade. There are more girls than boys. How many boys are there in the class?

Sara: There are 14 boys.
Ruth: It can also be 13 boys.
Eve: There are many possibilities.
Ofra: I know all the possibilities.

These children above kids demonstrated different levels of solutions and generalization However, all of them, when offering a reply have used the additive structure, explicitly or implicitly. Ofra has used it explicitly when she explained her statement.

Let me elaborate now on another problem:

The children in the camp are divided between those who prefer pizza anf those who prefer pita. Pizza costs 4 I.S. each, and pita costs 2 I.S. each. They spent 100 I.S. altogether. What did they buy?

Please note, that there is not sufficient numerical information in the problem. However, if one tries to answer the question, he or she will find that this problem has a well defined underlying structure We gave this problem to 2nd graders and suggested:

Ask your own questions.
Make up your numbers
What else do you want to know

The 2nd graders, usually, gave one solution, but many of them were aware that there are other solutions. Among 5th and 6th graders many could cope with the general structure of the problem, as was demonstrated from their solutions. Some wrote a table of options, the others drew diagrams that demonstrated their level of generalization.

FINAL COMMENTS

In order to help children cope with applications, we should teach the basic **mathematical** schemes and also where and when they apply. Pedagogy is the art of teaching things gradually, of assisting the construction of cognitive tools. The main dilemma is, how to avoid falling into the trap of stereotyping the learning when teaching the basic cognitive tools in a gradual manner. How to first present simple cases and yet keep problems open.

While we are teaching the basic tools of mathematics, we should teach also their constraints, when they apply and when they do not apply. We should present children with problems with missing information, as well as with superfluous information. These are necessary right from the beginning of the learning, as they strengthen the understanding of the mathematical structures. The gist of our teaching should be to have children understand that the game is of finding the mathematical model and not just the numerical solution.

We should not throw the baby with the water. The main role of schools is to enrich the child's cognitive tools, in our case with mathematical concepts, structures and procedures. Being able to cope with open situations successfully means having an enriched set of schemes.

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LOS SISTEMAS DE CÁLCULO SIMBÓLICO EN LA ENSEÑANZA DE LAS MATEMÁTICAS

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Abstract

Symbolic Computation Systems can and must play an important role in mathematics teaching. With adequate planning they can assist in bettering understanding, studying in depth numerous concepts, be a valuable educational instrument in problem solving and influence curriculum planning in terms of content, selection and order. Their use must be placed within what is known as "experimental mathematics teaching" and must not be hidden in activities aimed at learning as a set of fixed Algebra Computer Systems to resolve determined routine exercises. The software in question has been selected on a basis of characteristics accumulated from studies, from students and from other available sources. Alongside an overview of its advantages and inconveniences in relation to its educative tasks, the presentation will incorporate activities directed towards university students.

Introducción

Desde que, hace casi tres años, se fraguara esta conferencia, la evolución del "software" matemático ha sido imparable. Han aparecido sucesivas versiones, mejorando cada una a la anterior, de los distintos Sistemas de Cálculo Simbólico: DERIVE, MATHEMATICA, MAPLE, MACSYMA, etc. La experimentación del uso de estos sistemas para la enseñanza de las matemáticas se ha extendido de forma muy sensible, así como la investigación de su influencia en el proceso de aprendizaje, a las que posteriormente haré referencia.

El debate sobre los efectos del uso de los Sistemas de Cálculo Simbólico como recurso didáctico, en la enseñanza de las matemáticas, a muy diversos niveles y desde muy diversas ópticas, no ha hecho más que empezar. Los profesores de universidad discutimos sobre sus efectos positivos y negativos, sobre la viabilidad de su incorporación sistemática en la labor docente, y muchos profesores de bachillerato (16--18 años) se han incorporado también a la controversia; e incluso aquellos que apuestan por su uso, no llegan a conclusiones unánimes sobre la forma de hacerlo.

Deseo alejarme de esta polémica, circunscrita básicamente a la órbita del uso didáctico de esta herramienta, y abordar la cuestión desde una perspectiva más amplia.

Podemos afirmar que el ordenador es el elemento central del proceso de revolución tecnológica en que vivimos. Desde la primera generación de aquellos, allá por 1945, hasta nuestros días la evolución de la informática ha sido enorme; entre otros avances están: el acceso remoto a máquinas de elevadas prestaciones en multiproceso de tiempo compartido, las redes de comunicaciones informáticas (comúnmente conocidas como autopistas de la información), la extensión del mercado de ordenadores personales de elevadas prestaciones y precios asequibles y un largo etcétera. El avance en la investigación de "software" ha variado sustancialmente los métodos de trabajo a todos los niveles y, por supuesto, los de la investigación de muy diversas ciencias, a la vez que se han abierto nuevos campos de desarrollo científico. Es incuestionable que un objetivo de los sistemas educativos ha de ser capacitar a los alumnos para enfrentarse con adecuada formación a sus futuras actividades profesionales, así como favorecer la necesaria adaptación a los continuos avances de la tecnología.

Desde este punto de vista es claro que la enseñanza de las matemáticas debe asumir y utilizar los recursos tecnológicos de cada momento. Si para abordar muy diversos problemas de la técnica y de la ciencia se están utilizando Sistemas de Cálculo Simbólico, parece adecuado que los alumnos, tanto los de estudios preuniversitarios (16 - 18 años) con una orientación hacia ramas científico-técnicas, como los de estudios universitarios de este mismo campo, tengan una cierta familiarización con este tipo de "software" matemático de propósito general. Por tanto, la incorporación de estos *asistentes matemáticos* a la enseñanza ha de superar su concepción como mero medio didáctico, entre otros recursos, y ha de significar una innovación sustancial que conducirá sin lugar a dudas a profundas transformaciones de los objetivos, contenidos y métodos de enseñanza en los niveles educativos señalados.

2. Matemáticas y Sistemas de Cálculo Simbólico

Si aceptamos la argumentación anterior y, por tanto, la necesidad de la familiarización de los alumnos, en los niveles señalados anteriormente, con los programas de Álgebra Computacional, es claro que éstos deberían tener presencia específica en los curriculos correspondientes, pero ¿cómo hacerlo? Tal vez podría orientar nuestra respuesta el uso que de este tipo de "software" hace el matemático en su investigación.

Creo que es comúnmente aceptado que el fin esencial de la educación matemática es formativo y que radica en el desarrollo de la capacidad de lo que podríamos denominar pensamiento matemático, es decir conseguir que los alumnos hagan matemáticas, cuestión ésta que puede realizarse a muy diversos niveles. Llegar a resolver problemas matemáticos (hacer matemáticas) es el objetivo fundamental de la enseñanza de las mismas, por cuanto en ello se condensa la capacidad de saber usarlas. Consecuentemente merecen especial atención la consecución de capacidades para:

1. Reconocer, seleccionar y saber aplicar estrategias y técnicas como la analogía, la particularización, etc.
2. Reconocer, plantear y resolver problemas a partir de situaciones dentro y fuera de las matemáticas.
3. Aplicar el proceso de formulación de modelos matemáticos a situaciones prácticas, relacionadas con los contenidos curriculares.

En esencia, la forma de hacer matemáticas del matemático y de un estudiante es la misma; varía, eso sí, los conocimientos y el grado de capacitación personal, pero sustancialmente la cualidad del proceso es la misma. Pues bien, ¿cómo los sistemas de Cálculo Simbólico están influyendo en la forma en que hoy se está investigando en matemáticas? y ¿en qué medida ha de traducirse esta situación en la enseñanza de esta disciplina?

Voy a exponer un caso concreto, parte de un problema que un colega mío[1] ha trabajado.

Él estaba interesado en el estudio de las soluciones de la conocida *ecuación de Laplace:*

$$u_{xx} + u_{yy} + u_{zz} = 0,$$

donde u es una función $u=u(x,y,z)$. El objetivo básico era determinar y "clasificar" todas las soluciones que presentan un cierto tipo de invarianza frente a grupos de transformaciones (soluciones de similaridad"), lo que entre otras virtudes le permitiría determinar y construir nuevas soluciones, conocidas otras.

Podemos interpretar una solución $u(x,y,z)$ como una superficie en $\mathbf{R}^3$ x $\mathbf{R}$ y la ecuación de Laplace como una hipersuperficie en el espacio ampliado $\mathbf{R}^3$ x $\mathbf{R}$ x $\mathbf{R}^9$, donde en $\mathbf{R}^9$ se representan las coordenadas dadas por las derivadas de primer y segundo orden: $u_x, \ldots, u_z, u_{xx}, u_{xy}, \ldots, u_{zz}$.

El procedimiento consiste en buscar grupos de transformaciones:

$$\begin{aligned} x^* &= X\,(x,y,z,u,\varepsilon) \\ y^* &= Y\,(x,y,z,u,\varepsilon) \\ z^* &= Z\,(x,y,z,u,\varepsilon) \\ u^* &= U\,(x,y,z,u,\varepsilon) \end{aligned}$$

que dejan invariantes a la *hipersuperficie* y que también preservan las condiciones de contacto, es decir de forma que los *hiperplanos* tangentes sigan siendo, mediante el grupo de transformaciones, *hiperplanos* tangentes, y donde ε es un parámetro. Obviamente a partir de soluciones tan sencillas como $u = k$, con k constante, se pueden determinar múltiples soluciones nuevas. Estas soluciones invariantes por la transformación del grupo se conocen como *soluciones de similaridad.*

Ahora bien, buscar la expresión analítica (exacta) del grupo de transformaciones requiere una cantidad de cálculos extraordinarios, que llevaría días de trabajo (si se aborda sin asistencia del ordenador) y al final no se estaría seguro de no haber cometido errores en el proceso. Los cálculos son de tipo algebraico, es una tarea semiautomática cuya complejidad radica exclusivamente en el volumen de los mismos.

Los sistemas de Cálculo Simbólico en este punto dan adecuada respuesta en pocos segundos. De hecho, a partir de 1988 hay una gran explosión de investigación en el campo de la resolución de ecuaciones en derivadas parciales, gracias al desarrollo de estos asistentes matemáticos.

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En la primera fase de la resolución, el Sistema de Cálculo Simbólico nos devuelve un sistema sobredeterminado de ecuaciones en derivadas parciales de segundo orden:

$$\rho_{xx} + \rho_{yy} + \rho_{zz} = 0,$$
$$\zeta_{xx} + \zeta_{yy} + \zeta_{zz} = 0,$$

$$2\zeta_x - \xi_{xx}^{(1)} - \xi_{yy}^{(1)} - \xi_{zz}^{(1)} = 0,$$
$$2\zeta_x - \xi_{xx}^{(2)} - \xi_{yy}^{(2)} - \xi_{zz}^{(2)} = 0,$$
$$2\zeta_x - \xi_{xx}^{(3)} - \xi_{yy}^{(3)} - \xi_{zz}^{(3)} = 0,$$

$$\xi_{x}^{(1)} - \xi_z^{(3)} = 0,$$
$$\xi_x^{(2)} - \xi_z^{(1)} = 0,$$
$$\xi_x^{(3)} - \xi_z^{(1)} = 0,$$
$$\xi_x^{(2)} - \xi_z^{(3)} = 0,$$
$$\xi_x^{(3)} - \xi_z^{(2)} = 0.$$

en el que las funciones que aparecen lo son en las variables x, y, z. Su resolución requiere destreza, intuición y cierta maestría, esta es una segunda fase del problema.

Hasta ahora el programa ha actuado como una supercalculadora. Pero ha nacido un campo nuevo de estudio, ¿cómo resolver de forma automática este tipo de sistemas?; este es un problema en el que actualmente se está investigando.

En una tercera fase, se intenta clasificar las soluciones de forma que queden identificadas. Todas las familias de soluciones se distribuyen en clases de equivalencia y se busca un representante de cada clase, entre los infinitos que se pueden elegir. Para esta tarea no hay un procedimiento automático, no existe un algoritmo; el procedimiento estará guiado por la *intuición* matemática, pero los cálculos para verificar la corrección de las sucesivas hipótesis planteadas son muy grandes. El *manipulador simbólico* permite buscar en caminos que sin su asistencia no se podría. El programa ha actuado ahora para permitir la experimentación, para conjeturar y verificar o refutar.

Este no es sino un ejemplo, de los muchos posibles, en los que los Sistemas de Cálculo Simbólico están permitiendo investigación en terrenos hasta ahora inabordables y de cómo la propia existencia de este sofisticado "software" está planteando nuevos campos de investigación matemática.

Se tiende hacia una automatización de procesos de resolución de amplias partes de la matemática. El resultado será que se dispondrá de métodos más fiables y a menor costo, más y mejores herramientas para atacar y resolver problemas y más tiempo para dedicarlo al enfoque de nuevos problemas y ahondar en los misterios de su resolución.

3. Sistemas de Cálculo Simbólico y enseñanza experimental de las Matemáticas.

Algunos profesores estiman que el uso de *Manipuladores Simbólicos* puede representar una pérdida de destrezas básicas e incluso que ésta llegue a ser un obstáculo para un adecuado desarrollo de la capacidad de abstracción y razonamiento. Quienes así piensan (aunque no siempre, ni en todos los casos, sean conscientes de ello) dedican grandes dosis de esfuerzo a la enseñanza de algoritmos y a la realización de ejercicios que tienen una respuesta más o menos inmediata en el uso de uno o varios algoritmos, digamos estándares. La situación actual podría resumirse en el siguiente esquema:

- Se dedica una enorme cantidad de esfuerzo para que los alumnos adquieran destreza en el uso de los algoritmos más usuales.

- Los que fueron alumnos olvidan los algoritmos que aprendieron.

- Muchos algoritmos, de diversos campos de la matemática, pueden construirse sobre "software" científico y particularmente con el lenguaje de programación de diversos sistemas de Cálculo Simbólico y funcionar sobre máquinas baratas.

- Muchos profesores que exponen las virtudes de aprender algoritmos rutinariamente, usan conceptos, procesos y modelos para los que no conocen ningún algoritmo relevante.

Esta situación es lógicamente inestable y presagia profundas revisiones de los sistemas educativos.

Si en el quehacer de un matemático profesional hay elementos importantes de experimentación y ésta alcanza posibilidades inimaginables hace tan sólo dos décadas con los Sistemas de Cálculo Simbólico, parece razonable utilizar este potencial también para la enseñanza. La realización de procesos inductivos, el contraste de hipótesis, la verificación o refutación, el cambio de postulados, el sometimiento de los mismos a nuevas pruebas, la formulación de conjeturas apoyadas en la construcción de modelos que responden a las exigencias del problema, son elementos distintivos de la experimentación matemática. Estos asistentes matemáticos pueden emplearse como recursos didácticos que favorezcan niveles crecientes de pensamiento formal y de adecuada conceptualización matemática. Una tal línea de actuación se inscribiría en la conocida corriente de la "enseñanza experimental" de las matemáticas, que se apoya en las tesis constructivistas y que conecta con la línea metodológica de "resolución de problemas".

3.1 Un ejemplo: La órbita de un satélite

El siguiente problema es uno de los que proponemos a nuestros alumnos de primero de universidad de estudios técnicos [2]. Las prácticas de laboratorio que venimos realizando con nuestros alumnos están centradas en la resolución de problemas, con las necesarias matizaciones y, en principio, son adaptables a distintos sistemas de Cálculo Simbólico, aunque ésta concretamente se ha desarrollado con el programa **MATHEMATICA.** La resolución está orientada y asesora puntualmente, en los momentos que se estiman pertinentes, sobre las órdenes que pueden resultar útiles en el proceso de resolución.

TRANSIT NUMBER 59 SATELLITE
fecha de observación: 15 de Octubre de 1.984

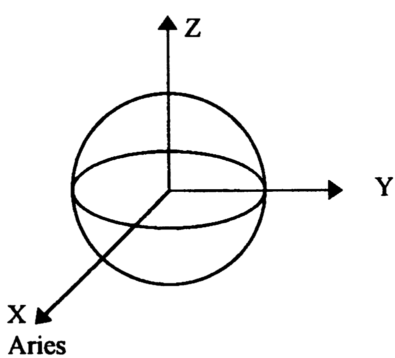
Hora	x	y	z
11h. 13m.	7347.2083	640.4444	-1083.2656
11h. 14m.	7400.8405	614.8864	-647.2850
11h. 15m.	7428.7207	586.8517	-209.0549
11h. 16m.	7430.7346	556.6627	229.9017
11h. 17m.	7406.8609	524.6504	668.0597
11h. 18m.	7357.1717	491.1516	1103.8969
11h. 19m.	7281.8320	456.5073	1535.9001
11h. 20m.	7181.0993	421.0605	1962.5701
11h. 21m.	7055.3220	385.1546	2382.4273
11h. 22m.	6904.9388	349.1307	2794.0170
11h. 23m.	6730.4703	313.3265	3195.9142
11h. 24m.	6532.5472	278.0735	3586.7290
11h. 25m.	6311.8480	243.6957	3965.1112
11h. 26m.	6069.1561	210.5077	4329.7553
11h. 27m.	5805.3269	178.8124	4679.4044
11h. 28m.	5521.2907	148.9000	5012.8552
11h. 29m.	5218.0487	121.0459	5328.9617
11h. 30m.	4896.6697	95.5092	5626.6389
11h. 31m.	4558.2862	72.5316	5904.8669
11h. 32m.	4204.0892	52.3354	6162.6936

Haremos una somera descripción del mismo, señalando los puntos más relevantes, toda vez que un relato pormenorizado requeriría más tiempo del que disponemos.

Deseamos determinar la órbita de un satélite artificial, así como el barrido del mismo sobre la superficie terrestre, determinando su posición minuto a minuto, a partir de 20 observaciones del mismo.

Nuestro punto de partida es un conjunto de 20 observaciones del satélite artificial TRANSIT 59 en coordenadas geocéntricas (en kilómetros), tomadas de [4]. En la tabla anterior aparecen las posiciones del satélite así cómo el día y hora de cada una de ellas. Las coordenadas geográficas (x,y,z), están referidas al siguiente sistema ortogonal

- Su centro (0,0,0) es el centro de la tierra.
- El eje X está situado sobre el plano del ecuador y en dirección Aries (sentido positivo hacia este punto)
- El eje Y está situado sobre el plano del ecuador ortogonal al eje X (el semieje positivo es el que se halla en el sentido contrario a las agujas del reloj desde el semieje positivo de las X).
- El eje Z es la recta norte-sur con sentido positivo hacia el norte.



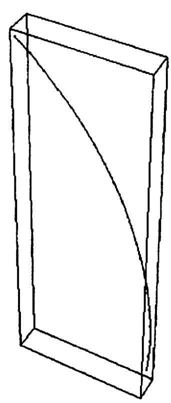
Comenzamos introduciendo la matriz **efemerides** formada por las coordenadas geocéntricas:

In[1]:
efemerides={
{7347.2083,640.4444,-1083.2656} , {7400.8405,614.8864,-647.2850} ,
{7428.7207,586.8517,-209.0549} , {7430.7346,556.6627,229.9017} ,
{7406.8609,524.6504,668.0597} , {7357.1717,491.1516,1103.8969} ,
{7281.8320,456.5073,1535.9001} , {7181.0993,421.0605,1962.5701} ,
{7055.3220,385.1546,2382.4273} , {6904.9388,349.1307,2794.0170} ,
{6730.4703,313.3265,3195.9142} , {6532.5472,278.0735,3586.7290} ,
{6311.8480,243.6957,3965.1112} , {6069.1561,210.5077,4329.7553} ,
{5805.3269,178.8124,4679.4044}, {5521.2907,148.9000,5012.8552} ,
{5218.0487,121.0459,5328.9617} , {4896.6697,95.5092,5626.6389},
{4558.2862, 72.5316,5904.8669} , {4204.0892,52.3354,6162.6936}
};

3.1.1 Arco orbital observado

Durante estos 20 minutos el satélite ha recorrido parte de su órbita que podemos representar usando la orden:

PuntosObservados=Show[Graphics3D[Join[{Line[efemerides]},
Table[Point[efemerides[[i]]],{i,20}]]]]



3.1.2. Cálculo de las coordenadas geográficas

Vamos a determinar el ángulo en el momento de la primera observación, para ello se hace preciso la determinación de:

1. El número de días julianos transcurridos y, por tanto, el número de vueltas dadas por la Tierra:

**DiaJuliano=Floor[365.25*1984]+Floor[30.6001*11]+
15+(11+13/60)/24+1720994.5+
2-Floor[1984/100]+Floor[Floor[1984/100]/4]**

2.A partir del número de días, según el calendario Juliano, se calcula el Tiempo Universal referido al año 2000 (no nos extrañemos, por tanto, del signo) con la fórmula:

TiempoUniversal=(DiaJuliano - 2451545.)/36525.

3. Ahora calculamos la Hora Sidérea, en segundos, con la fórmula:

HoraSiderea=24110.54841+8640184.812866*TiempoUniversal+

0.093104*TiempoUniversal^2-6.2*10^(-6)*TiempoUniversal^3

4. A partir de la hora sidérea, quitando las vueltas completas, determinamos el ángulo en el momento de la primera observación:

AnguloInicial=Mod[HoraSiderea*360/(24*60*60),360]

Se obtiene que el número de grados girado por la Tierra en la primera observación es de 24.1717.

Dado un punto (x,y,z) de la trayectoria del satélite podemos conocer la proyección sobre la Tierra hallando la latitud y longitud correspondiente de la siguiente forma:

latitud = ángulo del vector (x,y,z) con el plano XY
longitud = ángulo, sobre el plano XY, del vector (x,y) con la parte positiva del eje X menos el ángulo girado por la Tierra.

3.1.3 Proyección

Para determinar la sombra de los 20 puntos sobre la tierra se procede como sigue:

Latitud[{x_,y_,z_}]:= ArcSin[z/Sqrt[x^2+y^2+z^2]]*(180/pi)

**Longitud[{x_,y_,z_}]:=
2*ArcSin[Sign[y]*Sqrt[(1-x/Sqrt[x^2+y^2])/2]]/Degree**

El siguiente paso será definir una matriz con las coordenadas geográficas y en donde vamos a tener en cuenta ya el ángulo girado por la Tierra, tanto el inicial como el correspondiente a cada minuto.

```
Geograficas=Table[
{Latitud[efemerides[[i]]],Longitud[efemerides[[i]]]-AnguloInicial-360/1440(i-1)},
{i,Length[efemerides]}]//N;
```

Ello nos permite observar la proyección del satélite sobre el mapamundi.

3.1.4 Cálculo de la órbita del satélite

A continuación vamos a determinar la órbita del satélite. La elipse que describe el satélite puede obtenerse como intersección de un elipsoide y un plano, de forma que el punto de coordenadas (0,0,0) debe ser un foco de la órbita, de acuerdo con las leyes de Kepler.

1. Ecuación del elipsoide:

```
elipsoide[x_,y_,z_]:=
a x^2 + b y^2 + c z^2 + d x y + e x z + f y z + 1000;
```

2. Ecuación del plano:

```
plano[x_,y_,z_]:= p x + q y + r z + 1000
```

Se determinan los coeficientes por mínimos cuadrados. Respecto al plano del ecuador XY podemos considerar dos ramas de la órbita del satélite una superior y otra inferior. Estas ramas se calcularán haciendo la intersección del elipsoide y el plano antes calculados.

```
Z[x_,y_]:= Solve[plano[x,y,z]==0,z]
```

```
Y[x_]:= Solve[elipsoide /. z -> Z[x,y],  Y]
```

Los dos valores obtenidos corresponden con las dos ramas de la órbita. Finalmente se definen dos funciones, correspondientes a cada una de ambas ramas, de forma que para cada valor de x nos da las coordenadas geocéntricas de la posición del satélite.

```
Rama1[x_] := {x, Y[x] [[2]], Z[x, Y[x] [[2]]]}
```

```
Rama2[x_] := {x, Y[x] [[1]], Z[x, Y[x] [[1]]]}
```

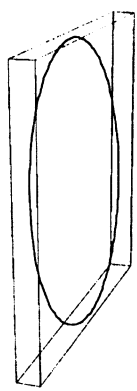
Los valores mínimos y máximos de la variable x serán necesarios para la representación grafica de la órbita. Estos valores se calculan usando el hecho de que son los puntos comunes entre las dos ramas:

{xmax,xmin}=x/. Solve[Y[x][[1]]==Y[x][[2]], x]

Se representa seguidamente la órbita hallada:

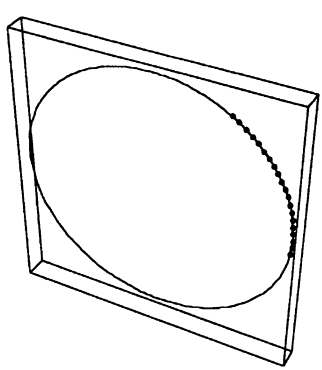
Orbita=ParametricPlot3D[{{Branch1[x][[1]],Rama1[x][[2]],Rama1[x][[3]]}, {Rama2[x][[1]],Rama2[x][[2]],Rama2[x][[3]]}}, {x,xmin,xmax},ViewPoint->{3,1,1},Ticks->None]

Y obtenemos:



La representación de los puntos observados junto a la órbita nos muestra que tal han ido nuestros cálculos:

Show [{Puntosobsevados,Orbita}]



3.1.5 Predicción orbital minuto a minuto

Una vez conocida la órbita del satélite ya estamos en condiciones de encontrar su posición en los minutos siguientes al de las observaciones.Para esto usaremos la segunda ley de Kepler: *el satélite tiene momento angular constante.* Efectuamos una simplificación aproximando el área del triángulo mixtilineo determinado por dos puntos consecutivos de la órbita y el foco por el área del triángulo formado por los tres puntos.

Definamos en primer lugar una función que nos proporcione el área de un triángulo dados dos de sus vértices (el tercero es el foco (0,0,0)):

```
Area[{u1_,u2_,u3_},{v1_,v2_,v3_}]=1/2 Sqrt[
(u2*v3-u3*v2)^2+
(u1*v3-u3*v1)^2+
(u1*v2-u2*v1)^2
];
```

Parece razonable considerar, como constante para el área de los triángulos, la media de las áreas de todos los triángulos que se forman con los pares de puntos consecutivos observados. Esto lo hacemos así:

```
AreaTriangulo=N[Sum[Area[efemerides[[i]],efemerides[[i+1]]],{i,19}]/19]
```

Posición del satélite en el minuto 21

Procedemos a determinar la posición del satélite en el minuto 21. Esa posición se encuentra en la rama superior **Rama 1** y el triángulo que forma con el punto de la posición 20 debe tener el área calculada anteriormente; así pues, resolvamos la ecuación:

```
NextPoint=x/. FindRoot[
         Area[ efemerides[[20]],Rama1[x] ]-AreaTriangulo==0,
        {x,efem[[20,1]]-5}
                    ]
```

3835.34

La posición en el minuto 21 será

Rama1[NextPoint]

{3835.34, 17.7762, 6398.92}

Añadimos este valor a la matriz de posiciones

efemerides=Join[efemerides,{Rama1[NextPoint]}];

Determinación completa de la órbita

Realizamos a continuación un procedimiento que nos halle el resto de las posiciones correspondientes a la rama superior:

```
For[i=21,efemerides[[i,1]]-xmin>15,i++,
     efemerides=Join[efem,{Rama1[x]}]/.FindRoot[Area[efemerides
[[i]],Rama1[x] ]-
     AreaTriangulo= =0, {x,efem[[i,1]]-5}                ]
   Print[ efem[[i+1]] ]
     ]
```

```
{3452.71, -5.59435, 6613.45}
{3058.04, -27.4481, 6805.07}
{2652.69, -47.7089, 6973.09}
{2238.07, -66.3063, 7116.95}
{1815.62, -83.1755, 7236.13}
{1386.83, -98.2578, 7330.23}
{953.181, -111.501, 7398.91}
{516.184, -122.858, 7441.94}
{77.3616, -132.29, 7459.16}
{-361.758, -139.764, 7450.53}
{-799.645, -145.254, 7416.05}
{-1234.78, -148.741, 7355.87}
{-1665.63, -150.212, 7270.18}
{-2090.72, -149.663, 7159.28}
{-2508.55, -147.095, 7023.56}
{-2917.67, -142.518, 6863.49}
{-3316.65, -135.946, 6679.63}
{-3704.11, -127.404, 6472.61}
{-4078.69, -116.919, 6243.16}
{-4439.1, -104.53, 5992.07}
{-4784.06, -90.2788, 5720.23}
{-5112.39, -74.2151, 5428.56}
{-5422.93, -56.3948, 5118.1}
{-5714.6, -36.88, 4789.92}
{-5986.39, -15.7386, 4445.16}
{-6237.34, 6.95602, 4085.02}
{-6466.58, 31.1247, 3710.76}
```

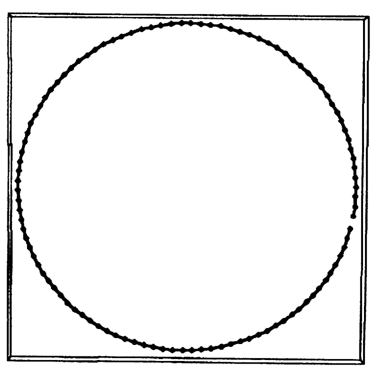
{-6673.32, 56.6834, 3323.68}
{-6856.81, 83.543, 2925.13}
{-7016.44, 111.61, 2516.5}
{-7151.64, 140.787, 2099.2}
{-7261.92, 170.972, 1674.71}
{-7346.92, 202.059, 1244.49}
{-7406.33, 233.942, 810.038}
{-7439.94, 266.507, 372.88}

Con la orden **Lenght[efemerides]** podemos saber que hemos hallado la posición del satélite durante 56 minutos.

Un procedimiento similar nos permite encontrar las posiciones sucesivas hasta completar la órbita.

La gráfica de la órbita junto a las posiciones observadas y a las calculadas nos muestra que nuestras simplificaciones producen resultados satisfactorios:

AllPoints=Show[Graphics3D[Join[{Line[efem]}, Table[Point[efem[[i]]],{i,106}]]],ViewPoint->{-0.072,-3.381,0.109}]



Para completar nuestra aventura vamos a dar vueltas a nuestro satélite suponiendo que su periodo es exactamente de 106 minutos. Damos 14 vueltas.

Turning=Table[efem[[Mod[i-1,106]+1]],{i,106*13}];

Realizamos las transformaciones necesarias tal como hicimos antes:

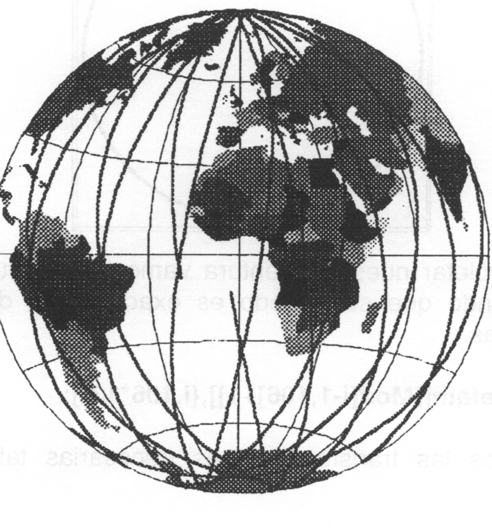
```
Geographics=Table[
{Latitude[Turning[[i]]],Longitude[Turning[[i]]]-InitialAngle-360/1440 (i-1)},
{i,Length[Turning]}]//N;

For[i=1,i<Length[Turning]+1,i++,
Geographics[[i,2]]=Mod[Geographics[[i,2]],360]
];

For[i=1,i<Length[Turning]+1,i++,
If[Geographics[[i,2]]>180,
Geographics[[i,2]]=Geographics[[i,2]]-360
]];

Geographics=60 Geographics;
```

Representamos ahora las 14 vueltas

```
EarthTrace=WorldGraphics[{RGBColor[1,0,0],
Line[
        Geographics
              ]}
                    ]
```

Y así obtenemos nuestra última gráfica

```
Show[{mapamundi,EarthTrace}]
```



Algunos comentarios

Vemos cómo a partir de datos reales los alumnos, con ayuda de un Sistema de Cálculo Simbólico, se enfrentan a un problema *real,* y cómo mediante la modelización del mismo les lleva a trabajar con variados contenidos matemáticos:

- Geometría en el espacio euclídeo.
- Coordenadas geocéntricas y geográficas.
- Funciones trigonométricas.
- Sistemas de ecuaciones sobredimensionados.
- Mínimos cuadrados.
- Ecuaciones.

En el proceso han podido aprender más acerca del programa en cuestión[2] ; así han tenido que aprender el significado y uso de órdenes como:

Graphics3D, ParametricPlot3D, Table, If, Solve

Sum, Show, For, Mod, QRDescomposition, Length

y el *Paquete WORLDPLOT,* haciendo uso de las órdenes:

WorldPlot, WorldData, WorldGraphics

Por ello, los alumnos se han enfrentado a la tarea de *resolución de problemas* (objetivo primordial de la educación matemática), que al ser de modelización aporta un elemento formativo adicional y una carga de motivación añadida. Además han profundizado en la comprensión de diversos conceptos matemáticos, al tener la necesidad de identificarlos y de aplicar sus correspondientes técnicas de cálculo (éste es el caso de mínimos cuadrados). Finalmente han aprendido mucho acerca del manejo y potencia del programa utilizado.

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[2] MATHEMATICA

4. Ventajas e inconvenientes del uso de los Sistemas de Cálculo Simbólico en la enseñanza

Aún cuando estos asistentes matemáticos sean poderosas herramientas, muchos profesores están en desacuerdo sobre su utilidad pedagógica. Las razones anteriormente esgrimidas a favor de su uso, pueden parecerles, a algunos, pertenecer más al campo de la mera disquisición que al de la realidad constatable.

Es ya abundante la literatura sobre el uso didáctico de estos programas:

1. Recientes investigaciones (cf. [9], [13], [10], [6], etc.) han puesto de manifiesto en distintos campos de las matemáticas, entre otras, las siguientes ventajas para el aprendizaje:

- Se constata que ayuda a progresar hacia niveles superiores de pensamiento formal.
- La capacidad gráfica facilita la integración de las diversas imágenes conceptuales (en el sentido de Vinner) que son un obstáculo para el aprendizaje.
- La técnica del zoom resulta extremadamente útil para la adecuada conceptualización de los procesos de paso al límite. Estos resultados enlazan con los postulados de la enseñanza dinámica de Gattegno, en la medida que el proceso de visualización se revela como primordial para la abstracción.
- Amplía el abanico de manipulaciones posibles y el de visualización.
- Mejora la actitud de los alumnos frente a las matemáticas.
- Favorece la interiorización de los conceptos y procedimientos, de forma que estos permanecen a más largo plazo.
- Desarrolla nuevas estrategias de razonamiento.
- Es de una gran ductilidad para crear situaciones de trabajo.
- ...

2. Las experimentaciones que en muy diversos lugares y con muy diversas condiciones se vienen realizando (cf. [1] , [4] , [7], etc.) ponen también de manifiesto que:

- Al tratarse de poderosas herramientas de cálculo y representación gráfica, permiten variar el enfoque de los problemas, la cantidad y la cualidad de los mismos, así como abordar problemas de modelización, dando menos tiempo para la adquisición de destreza en el uso de algoritmos y disponiendo de más tiempo para la conceptualización.

- Propician la investigación y el descubrimiento

- Pueden servir para provocar la reflexión y el razonamiento matemático.

- Permiten el trabajo autónomo del estudiante.

- Facilitan el desbloqueo del estudiante en la resolución de un problema, en la medida que le permite experimentar con rapidez y seguridad.

- Su carácter interactivo provoca una retroalimentación inmediata.

- ...

Sin embargo, no hemos de ocultar un cierto peligro tecnocentrista. La conveniencia y necesidad de adaptarse a la realidad tecnológica, el uso de los últimos avances en este campo para la docencia y la aceptación de que ello conlleva cambios curriculares, no significa que ésta tecnología sea el elemento sobre el que se deban vertebrar los cambios en la enseñanza. Un aula con ordenadores *Pentium* con un Sistema de Cálculo Simbólico instalado, en cada uno de ellos, no conlleva por sí sólo una calidad superior en el proceso de enseñanza—aprendizaje. *Los manipuladores simbólicos* no van a solucionar los problemas de aprendizaje que a diario se nos presentan, no es la panacea, pero sí pueden contribuir (integrados dentro de una metodología que contemple propuestas mucho más amplias) a mejorar la calidad de la enseñanza, desde una doble vertiente: como medio para mejorar el aprendizaje y como herramienta con la que los alumnos deben familiarizarse para una adecuada formación para el futuro.

Algunos investigadores y profesores han advertido sobre peligros no desdeñables de un uso inadecuado de estos *sistemas informáticos,* entre ellos:

- Riesgo de que una actitud tecnocentrista convierta al programa en sujeto, en lugar de las matemáticas Las clases podrían llegar a ser clases de un determinado programa.

- Se confíe a la mera interacción entre alumno y ordenador el proceso de aprendizaje.

- Pérdida de destrezas básicas.

- Confianza ciega en la máquina. Se acepta cualquier solución. Creo que todos los que tenemos alguna experiencia en este terreno hemos tenido oportunidad de constatar sobradamente este riesgo.

- Incapacidad para valorar las dificultades de los problemas

- Las dificultades de aprendizaje de un programa determinado lleguen a ser un obstáculo para el aprendizaje de las matemáticas.

- Excesiva dependencia del asistente matemático

5. Dificultades para el uso de los Sistemas de Cálculo simbólico en la enseñanza. Algunas posibles soluciones

No menos desdeñables son las dificultades que algunos señalan para su uso de manera sistemática, entre ellas:

1. Las propuestas de trabajo para los estudiantes no pueden ser exclusivamente, ni principalmente, la realización de algoritmos usuales. Ello representa una dificultad adicional para los profesores: la de diseñar tareas de aprendizaje que no sean triviales ante el medio de que se dispone. Buena parte de los problemas tradicionales, del tipo *hágase tal límite, resuélvase tal integral, calcúlese la inversa de la matriz, resuelva el sistema de ecuaciones, represente tal función, etc.}* son simplemente inadecuados. ¿Qué hacer entonces? Esta situación provoca ansiedad y rechazo por parte de muchos profesores.

2. ¿Cuándo es adecuado el uso de estos asistentes matemáticos?

3. ¿Cómo se evalúa?

4. La formación de los profesores en ejercicio no siempre es la más adecuada para afrontar este nuevo medio.

5. Dificultades logísticas:
   (a) Elevado número de alumnos.

   (b) Escasez de equipos informáticos.

   (c) Las aulas de ordenadores han de atender no sólo los requerimientos de las asignaturas de Matemáticas.

   (d) Organización de la actividad docente. El profesor suele cambiar de clase y resulta complicado trasladar a cada aula el ordenador portátil y la pantalla de cristal líquido (cuando los hay). En los centros de bachillerato la situación suele ser más dura: se carece, en general, de ordenador portátil, de retroproyector de gran intesidad y de pantalla de cristal líquido, con lo que los alumnos sólo podrán acceder al uso de los ordenadores desplazándose al aula correspondiente, con un número de ellos inferior al de alumnos, presentándose además problemas de desdoblamiento de grupos. Por otra parte esta situación provoca que la posible utilización de estos asistentes matemáticos sea puntual, con lo que desaparece la posibilidad real de incorporarlos como recurso didáctico al proceso de aprendizaje.

   La realidad es que en la mayoría de los centros de bachillerato el aula de informática no se usa en matemáticas (tampoco en otras disciplinas), limitándose su uso para la docencia de informática.

   Incluso en la universidad, la realización de clases prácticas con estos sistemas, separadas de las clases ordinarias, resultan con frecuencia inadecuadas desde una perspectiva estrictamente didáctica. Estos programas, en la mayoría de los casos, no se utilizan sistemáticamente para introducir conceptos, quedando reducidos a su uso como potentes calculadoras.

A mi juicio muchas de las dificultades y peligros señalados son subsanables:

- Algunos, con sólo tener presente el sentido común: este tipo de "software", siendo muy importante, es un elemento más, pero las explicaciones del profesor, con y sin ordenador, los ejercicios con lápiz y papel, el uso de libros adecuados y otros muchos aspectos de una metodolgía ya tradicional son también necesarios.

- Los aspectos logísticos tienen diversas soluciones:

1. Una de ellas, la especialización de aulas por materias, de forma que en éstas dispongamos de los medios necesarios, al menos de un ordenador, un potente retroproyector y una pantalla de cristal líquido.

2. La adecuada integración, dentro del desarrollo de la asignatura, en el uso de los *manipuladores simbólicos* por parte de los alumnos es un tema que la propia evolución tecnológica está resolviendo: ya existen calculadoras que incorporan Sistemas de Cálculo Simbólico, por lo que no tardará mucho en que esta sea tan usual entre los estudiantes como lo es ahora la calculadora científica.

Sin embargo otros obstáculos serán más difíciles de superar. Mientras que los profesores se sientan abrumados por la tecnología que les desborda, ésta estará proscrita. La única solución está en el aumento de bibliografía específicamente diseñada para utilizar en el aula, haciendo uso de estos asistentes matemáticos y unos adecuados programas de formación en ejercicio para profesores.

6. A modo de epílogo

En el número del pasado mes de Junio de la revista *Suma* el profesor Claudi Alsina, presidente del Comité Internacional de Programas de este congreso, publicaba un artículo titulado *Unas reflexiones sobre el ICME--8,* en el que formula algunos interrogantes educativos, entre ellos:

- *¿Cómo podría mejorarse la comunicación en clase, favorecer la motivación y aumentar las actitudes positivas?*
- *¿Qué debe cambiarse de los curricula de matemáticas, cómo hacerlo y por qué hacerlo?*
- *¿Qué tratamiento debe darse a la diversidad?, ... , ¿qué deberíamos ofrecer a los jóvenes con talento?, ¿qué ayuda podemos dar a las personas con dificultades de aprendizaje?*

- *¿Cómo deberían influir las posibilidades tecnológicas en nuestra labor?*

- ***¿Qué nuevas posibilidades ofrece la enseñanza del Cálculo a la luz de las nuevas tecnologías?***

- *¿Deberíamos preocuparnos más por la modelización y matematización de la realidad?*

- *¿El laboratorio de matemáticas es imprescindible?*

- *¿Qué ventajas introducen calculadoras, ordenadores y todas las nuevas tecnologías?*

No cometeré la petulancia de pretender que he contestado a estas ni a otras preguntas sustanciales para la enseñanza de las matemáticas en los años que vivimos, pero unas veces abiertamente y otras de forma implícita hemos tocado, aunque sólo sea tangencialmente, algunos caminos que podrían contribuir, con otros muchos, a dar respuesta a aquellas.

Creo que la penetración de los Sistemas de Cálculo Simbólico es imparable, tenerlos presente en el proceso educativo simplemente una obligación y sacarle el mayor provecho didáctico posible un acto de inteligencia.

Finalmente espero que si al menos no les he contagiado de mi fe en el futuro, al menos no les haya aburrido.

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PROGRAMAS DE DOCTORADO E INVESTIGACIÓN ACADÉMICA: EDUCACIÓN MATEMÁTICA EN LA UNIVERSIDAD ESPAÑOLA

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Introducción

Dentro de las Ciencias de la Educación, la Didáctica de la Matemática ha experimentado en los últimos 50 años un desarrollo sostenido tanto en extensión como en profundidad. Este desarrollo es efecto y causa, simultáneamente, de las importantes funciones que desempeña el conocimiento matemático en la sociedad y cultura contemporáneas. La importancia de las matemáticas en los currícula de la enseñanza obligatoria ha provocado, recientemente, una amplia reflexión teórica y un gran esfuerzo de implementación práctica, sostenidos por un cuerpo de investigación en Educación Matemática amplio y sistemático (Romberg, 1992; Popkewitz, 1994).

Los especialistas coinciden en que las características distintivas más apreciables del conocimiento matemático son su carácter formativo y su utilidad práctica, tanto al considerar la dimensión individual como la social (Romberg, 1991; Niss, 1996). La enseñanza de las matemáticas afecta a millones de jóvenes y adolescentes; como campo de actuación profesional son cientos de miles los profesores y educadores que trabajan sobre la enseñanza y el aprendizaje de las matemáticas (OCDE, 1995).

El carácter eminentemente social y cultural de la enseñanza de las matemáticas, junto con la complejidad y dificultades detectadas en el aprendizaje de estas disciplinas, han contribuido a que la preocupación por el estudio de los procesos de comunicación, transmisión y comprensión de las matemáticas hayan interesado a una amplia comunidad científica, que viene realizando desde hace más de un siglo investigación cualificada en este campo (Kilpatrick, 1992).

La investigación, junto con la innovación curricular y la formación del profesorado, concentran la mayor parte de los esfuerzos de la comunidad de educadores matemáticos (Rico y Sierra, 1991).

El despegue de la investigación en Educación Matemática en los últimos años se ha sostenido sobre unas determinadas claves; entre ellas destacan su incorporación a la universidad (Long, Meltzer y Hilton, 1970), el control y la validación académica de sus resultados, el sostenimiento de revistas especializadas con alto nivel de rigor y exigencia científica, la celebración de encuentros y debates periódicos entre especialistas, la delimitación y puesta en práctica de agendas de investigación y el estímulo a los grupos y líneas de investigación en Educación Matemática llevado a cabo por organismos y agencias de promoción de la investigación (Kilpatrick, Rico y Sierra, 1994).

Dentro de este marco general la investigación española en Educación Matemática ha tenido su propio desarrollo en los últimos 25 años.

Antecedentes: Investigación española en los 70

Durante la década de los 70 en España se implanta la Ley General de Educación. Hay dos datos principales, de interés para la Investigación en Educación Matemática:

* una nueva organización del sistema educativo, que desarrolla un currículo basado en las Matemáticas Modernas para todos los niveles de la educación obligatoria (6-14 años) y post-obligatoria (15-18);

* la creación en cada una de las universidades de los Institutos de Ciencias de la Educación (I.C.E.), mediante los que se incorporan la investigación educativa y la formación del profesorado a las competencias universitarias.

Como consecuencia derivada de estas reformas se plantea la necesidad de dar respuesta fundada a los problemas de enseñanza detectados en el Sistema Educativo y justificar la adecuación de los nuevos programas a las necesidades formativas de los escolares. Son varias las iniciativas que surgen en estos años para abordar estos problemas entre las que, por su orientación investigadora, destacamos dos:

* el Equipo de Investigación Granada-Mats, que se constituye en 1971, y

* los grupos de innovación Grupo Cero y Grup Zero, cuyos trabajos se inician en 1975.

Una descripción detallada de la actividad investigadora de estos grupos puede verse en Rico y Sierra (1997).

En el marco de las reformas a que da lugar la Ley General de Educación aparece la disciplina *Didáctica de la Matemática* por primera vez en la Universidad española. Esto ocurre en los nuevos planes de estudio para la formación inicial de los Profesores de Educación General Básica (1971), en primer lugar, y, posteriormente, en la Licenciatura de Matemáticas de algunas universidades (1975). En este contexto hay grupos específicos de investigadores en algunas universidades que comienzan a desarrollar trabajos de investigación en Didáctica de la Matemática; también se logra la valoración académica de algunos de los estudios realizados en este campo. Esto sucede en la Universidad de Granada donde en el año 75, con carácter pionero, se presentan dos tesinas de licenciatura en Didáctica de la Matemática.

Sin embargo, las oportunidades y condiciones institucionales en esta época son restringidas y limitadas, dificultando un desarrollo adecuado de la investigación en Educación Matemática.

La Ley de Reforma Universitaria

En 1984 se promulga la Ley de Reforma Universitaria (L.R.U.). A partir de la nueva estructura universitaria derivada de esta Ley se diversificaron las disciplinas tradicionales en un nuevo catálogo de Areas de Conocimiento, adaptado a un desarrollo actualizado de las ciencias.

La Ley estableció las Areas de conocimiento "como aquellos campos del saber caracterizados por la homogeneidad de su objeto de conocimiento, una común tradición histórica y la existencia de comunidades investigadoras nacionales o internacionales". En este marco surge el *Area de Conocimiento Didáctica de la Matemática* como uno de los campos del conocimiento mediante los que se estructura la Universidad, reconociendo el esfuerzo realizado por la comunidad de educadores matemáticos de nuestro país en años anteriores.

La constitución de Departamentos universitarios en los que está integrada el Area de Didáctica de la Matemática y, en especial, los Departamentos de Didáctica de la Matemática han supuesto un paso importante para la Educación Matemática en España, disponiéndose de nuevos medios personales y materiales y potenciándose la docencia e investigación en el Area.

Pero el punto de partida no resultó fácil; las necesidades y carencias desbordaban ampliamente los medios disponibles y el reto que se asumía parecía estar fuera de cualquier posibilidad de realización. Al comenzar su andadura institucional el área de Didáctica de la Matemática tiene ante sí grandes retos; uno de los más destacados es la investigación académica y la validación de sus resultados mediante tesis doctorales.

Las prioridades de la investigación en Educación Matemática se centran durante estos años, a nivel internacional, en la delimitación, explicitación y enunciado de los principales problemas sobre los que debe trabajar esta investigación (Shumway, 1980; Freudenthal,1981; Wheeler, 1984) y en la conexión necesaria de los resultados de las investigaciones con la práctica escolar (Bell, Low & Kilpatrick 1985). Además, cada comunidad científica nacional tiene que adaptar este programa a las propias condiciones locales, definir sus prioridades, adaptar los medios a los fines e iniciar una propuesta rigurosa de investigación académica. En este contexto se dan los primeros pasos para poner en marcha los programa de Doctorado en Didáctica de la Matemática.

Programas de Doctorado

La nueva ley universitaria establecía que "corresponde a los Departamentos la articulación y coordinación de las enseñanzas y de las actividades investigadoras de las Universidades." Uno de los logros mas importantes derivados de la nueva regulación universitaria ha sido la organización y desarrollo de Programas de Doctorado específicos en Didáctica de la Matemática, como ha ocurrido en la Universidad Autónoma de Barcelona, Universidad de Valencia y Universidad de Granada; posteriormente se han incorporado las universidades de Sevilla y Extremadura.

La importancia de los Programas de Doctorado se resalta en el Real Decreto que regula el Tercer Ciclo de Estudios Universitarios, donde leemos:

"El Tercer Ciclo, como demuestra la experiencia comparada, constituye condición esencial para el progreso científico y, por ello, para el progreso social y económico de una comunidad por cuanto de la profundidad de sus contenidos y la seriedad en su planteamiento depende la formación de los investigadores."

A estos efectos la Ley de Reforma Universitaria se plantea cuatro grandes objetivos en el campo de los estudios de postgrado:

* Disponer de un marco adecuado para la consecución y transmisión de los avances científicos.

* Formar a los nuevos investigadores y preparar equipos de investigación que puedan afrontar con éxito el reto que suponen las nuevas ciencias, técnicas y metodologías.

* Impulsar la formación de nuevo profesorado.

* Perfeccionar el desarrollo profesional, científico y artístico de los titulados superiores.

Queda claro que el desarrollo de un Area de Conocimiento pasa por el mantenimiento continuado de un Programa de Tercer Ciclo mediante el que se realicen y logren los anteriores objetivos. En este contexto, las Universidades de Granada, Valencia y Autónoma de Barcelona iniciaron durante el curso 88-89, el Programa de Doctorado en Didáctica de la Matemática, que se ha continuado a lo largo de estos años.

Programa de Doctorado en la Universidad de Granada

El Departamento de Didáctica de la Matemática de la Universidad de Granada viene ofreciendo un Programa de Doctorado bianual para la formación de investigadores. Hasta el momento se han desarrollado 4 bienios y va a comenzar un nuevo programa en el próximo bienio 96-98. Según la normativa establecida los Programas de Doctorado deben comprender, con carácter general:

a) Cursos o Seminarios relacionados con la metodología y formación en técnicas de investigación.

b) Cursos o Seminarios sobre los contenidos fundamentales de los campos científico, técnico o artístico a los que esté dedicado el Programa de Doctorado correspondiente.

c) Cursos o Seminarios relacionados con campos afines al del Programa y que sean de interés para el proyecto de tésis doctoral del doctorando.

Siguiendo estas directrices generales, el Programa de Doctorado de Didáctica de la Matemática presenta la siguiente organización de materias para el bienio 96-98:

Curso 96-97

Investigación en Educación Matemática: Avances Metodológicos.
Teoría de la Educación Matemática.
Diseño, Desarrollo y Evaluación del Currículo de Matemáticas.
Seminario de Didáctica de la Matemática I.
Epistemología de la Probabilidad y la Combinatoria.
Análisis de Datos I.
Pensamiento Numérico.

Modelos para Investigación en Etnomatemática, Formación de Profesores y Curricular I.

Curso 97-98:
Seminario de Didáctica de la Matemática II.
Modelos para Investigación Etnomatemática, Formación de Profesores y Curricular II.
Introducción al Análisis de Datos Multivariantes.
Diseño de Investigaciones Educativas.
Investigación en Resolución de Problemas.
Pensamiento Numérico Avanzado.
Semiometría y Ecología de los Objetos Matemáticos.
Creencias y Concepciones de los Profesores, Investigación en Formación de Profesores.
Evaluación en el Aula de Matemáticas.
Epistemología y Didáctica de la Inferencia Estadística.

Estos 18 cursos totalizan una oferta de 52 créditos de formación, y están impartidos por 11 profesores.

Regulación del Programa de Doctorado

El alumno inscrito en los estudios de doctorado deberá cursar y aprobar en el plazo de dos años, prorrogables a tres, un total de 32 créditos (320 horas) mediante los cursos y seminarios incluidos en el programa, así como con créditos obtenidos por la realización de un trabajo de investigación obligatorio, hasta un máximo de 9 créditos. Se exige un mínimo de 16 créditos en materias del área de conocimiento o fundamentales; el resto puede cursarse con asignaturas afines.

El trabajo de investigación consiste en una primera aproximación a la Tesis. El Departamento entiende que la mejor forma de aprender a investigar consiste en realizar un trabajo de investigación, para cuyo fin se estimula a los alumnos a que presenten una Memoria de Tercer Ciclo con los resultados obtenidos en este trabajo. Los alumnos de doctorado han de presentar en el Departamento, antes de terminar el Programa, un proyecto de tesis doctoral avalado por el que vaya a ser su director o directores. La tesis deberá terminarse en el plazo de cinco años desde la fecha de inicio de los estudios, ampliables en otros dos años a juicio de la Comisión de Doctorado(Rico, Batanero y Diaz, 1994). Hasta el momento son 16 las tesis doctorales, 2 tesinas y 12 trabajos de investigación los leidos en el programa de Doctorado de Didáctica de la Matemática de la Universidad de Granada.

Desarrollo de la Investigación

A los cuatro objetivos generales señalados por la ley relativos a la investigación, antes mencionados, el Area de Didáctica de la Matemática ha añadido los siguientes:

* Establecer y mantener un espacio de crítica, debate y comunicación sobre el estado actual y desarrollo reciente de la investigación en el Area de Didáctica de la Matemática, así como de sus avances teóricos y metodológicos.

* Impulsar la delimitación de problemas relevantes en la enseñanza y aprendizaje de las matemáticas para su estudio sistemático, que permita obtener información significativa para su diagnóstico y tratamiento y dé lugar a materiales y recursos adecuados para el aula de matemáticas.

* Constituir grupos de investigación estables, que trabajen metódica y continuadamente sobre líneas específicas de investigación en Didáctica de la Matemática, que sirvan de referencia para los especialistas y estén conectados con la comunidad investigadora internacional.

* Producir investigación propia cualificada, que suponga una aportación específica y original a las cuestiones de indagación prioritarias en el Area de Conocimiento, y presentar regularmente los resultados obtenidos en los foros y medios de comunicación de la comunidad de investigadores de Didáctica de la Matemática.

El logro principal de los Programas de Doctorado en Didáctica de la Matemática no se limita a las tesis doctorales sino que avanza hacia la constitución de grupos estables de investigación y la consolidación de una comunidad de investigadores en Educación Matemática en la Universidad Española. Varios son los datos que avalan esta consideración.

Los profesores del Area de Didáctica de la Matemática se han presentado a las evaluaciones de la actividad investigadora realizadas por el Ministerio de Educación; hay un grupo significativo de estos profesores con tramos evaluados positivamente. En el Plan Nacional de Formación de Personal Investigador se han concedido becas para trabajar en Didáctica de la Matemática.

En las convocatorias anuales para la Promoción General del Conocimiento, convocados por la Dirección General de Investigación Científica y Técnica (DGICYT) se han presentado y aprobado proyectos de grupos de investigadores de Didáctica de la Matemática; igualmente en el Plan Nacional de Investigación Educativa de la Dirección General de Renovación Pedagógica, así como en las convocatorias del Plan Andaluz de Investigación de la Consejería de Educación y Ciencia de la Junta de Andalucia y de otras Comunidades Autónomas.

En los departamentos de Didáctica de la Matemática hay una organización mediante líneas de investigación. Así ocurre en la Universidad de Granada donde el Departamento está estructurado según cinco líneas, con el fin de organizar su actividad investigadora. Estas líneas son:

Didáctica de la Matemática: Pensamiento Numérico
Didáctica de la Probabilidad y de la Estadística
Diseño, Desarrollo y Evaluación del Currículo de Matemáticas
Formación del Profesorado de Matemáticas
Teoría y Métodos de Investigación en Educación Matemática

Cada uno de estos grupos está teniendo un desarrollo considerable, que permite ubicar las investigaciones de los proyectos y tesis doctorales en un marco más general y coordinado, que da continuidad y profundidad a estos estudios especializados.

Las relaciones internacionales son también un rango distintivo de la situación actual. A las invitaciones individuales para impartir cursos y conferencias, hay que añadir que los Departamentos de Didáctica de la Matemática de las Universidades Autónoma de Barcelona, Valencia y Granada tienen firmados convenios entre sí y con Centros de Investigación en Educación Matemática y Departamentos Universitarios de otros países, principalmente de la Comunidad Europea y de Latinoamerica. Estos departamentos han participado en proyectos de la Unión Europea, tales como el Erasmus, Tempus, Sócrates y Alfa, coordinando y formando parte de redes de investigadores en Educación Matemática. La contribución a la formación inicial de investigadores latinoamericanos en Educación Matemática de estos tres centros es sistemática y productiva.

También merece mención la incorporación de investigadores españoles a grupos internacionales (PME, CIEAEM, ICOTS, ICMI-Studies, etc.) y la participación internacional destacada en comités de evaluación, comités científicos y editoriales, paneles de expertos, conferencias, ponencias invitadas, grupos de investigación, redacción de libros, informes de investigación y artículos.

Sociedad Española de Investigación en Educación Matemática

En el marco descrito, las relaciones entre los investigadores españoles en Educación Matemática han ido creciendo y centrándose en los trabajos y tareas de investigación. Todos los debates y actividades realizados han impulsado el sentimiento y han desarrollado la percepción de formar parte de una misma comunidad de investigadores.

Se delimita así un grupo profesionalizado en investigación sobre Educación Matemática, reconocible por sus trabajos académicos, por la pertenencia a grupos internacionales, y por su producción sistemática de trabajos de investigación en este campo, sometidos a la crítica y control de la comunidad.

Este grupo, no muy extenso, necesita su propio espacio de encuentro, debate y reflexión. Surge de este modo la necesidad de una sociedad formalmente establecida, en la que se incardinen y organicen los investigadores profesionales en educación matemática.

La Sociedad Española de Investigación en Educación Matemática se constituye en marzo de 1996. Entre sus principales objetivos están:

* Promover el impulso a la Educación Matemática en los organismos e instituciones relacionados con la investigación. Promover la participación en las convocatorias de ayudas a la investigación, institucionales y privadas.
* Contribuir y participar en el desarrollo, evaluación y aplicación de investigaciones en Didáctica de la Matemática.
* Contribuir a la presentación de resultados de investigación en los foros, encuentros y revistas de Educación Matemática.
* Mantener contactos y promover la colaboración con grupos de investigación en Educación Matemática.
* Favorecer activamente la cooperación e intercambio entre investigación y docencia en todos los niveles educativos.
* Transmitir y divulgar institucionalmente la actividad de la Sociedad.

La Sociedad ha iniciado el debate sobre los campos de investigación prioritarios en Educación Matemática para facilitar la constitución de grupos de trabajo estables en la comunidad.

Tras una revisión y análisis de los tópicos y campos de investigación en los que los investigadores españoles han venido desarrollando sus trabajos, se optó por constituir los siguientes grupos:

1 Didáctica del Análisis. Pensamiento Matemático Avanzado.
2 Aprendizaje de la Geometría y Nuevas Tecnologías
3 Didáctica de la Estadística, Probabilidad y Combinatoria.
4 Pensamiento Numérico y Algebraico.
5 Formación de Profesores de Matemáticas.
6 Metodología de Investigación en Didáctica de la Matemática.

Perspectivas de futuro

Hemos hecho una descripción de los cambios e innovaciones ocurridos en España en el campo de la Investigación en Educación Matemática durante los últimos 25 años, destacando el desarrollo acelerado de los 10 últimos años. Ninguno de estos cambios hubiera tenido lugar con el vigor y profundidad con que se han presentado si no se hubiese producido una evolución en la sociedad española y, en particular, en su sistema educativo que han dado lugar a unas condiciones idóneas para el desarrollo de la investigación.

Entre tales cambios hay que destacar, principalmente, el gran desarrollo cultural y social ocurrido en España y destacar la renovación económica, cultural, política y educativa realizada en estos últimos 25 años.

Para la estabilidad de la comunidad de investigadores en Educación Matemática ha sido determinante que, previamente, se haya consolidado una fuerte comunidad de investigadores en las diferentes disciplinas matemáticas. El hecho de que se investigue en Matemáticas en la Universidad española da sentido a los problemas de su enseñanza y aprendizaje, debido a las dificultades de comunicación y transmisión que se plantean. También es determinante la consolidación de diferentes comunidades de investigadores en las diversas disciplinas que denominamos Ciencias de la Educación, ya que proporcionan marcos de referencia teóricos y metodológicos adecuados y sirven de crítica y contraste a las investigaciones realizadas en Educación Matemática.

Igualmente tiene un efecto determinante para esta investigación la constitución de las Sociedades de Educadores o Profesores de Matemáticas. Estas sociedades han realizado una actividad vigorosa con aportaciones al diseño y desarrollo del curriculo de las matemáticas escolares asi como a la formación inicial y permanente del profesorado. La contribución de las Sociedades de Profesores al desarrollo de la Educación Matemática y, en especial, a la reflexión sobre las conexiones entre teoría y práctica ha sido destacable en España; gran parte del trabajo realizado en investigación se ha planteado y discutido en los encuentros y jornadas organizados por las Sociedades y se ha difundido mediante las actas, revistas y otras publicaciones editadas por estas Sociedades.

La especificidad de los problemas de la Investigación en Educación Matemática desde una perspectiva profesional y académica no deben hacernos olvidar el compromiso y la ineludible conexión con la práctica profesional del educador matemático. Los profesores de matemáticas tampoco pueden contemplar al investigador como ajeno a su trabajo; antes bien, deben exigirle rigor en sus planteamientos, claridad en sus realizaciones y practicidad en sus resultados.

Los investigadores en educación matemática forman parte, por derecho propio, de la comunidad de educadores matemáticos, pero tienen su campo profesional específico al que deben atender con prioridad. Esta situación obliga a reflexionar sobre los problemas comunes, que deben abordarse conjuntamente. La coordinación sistemática entre estos dos colectivos permitirá alcanzar unas señas de identidad bien fundadas y consolidar ambas comunidades.

Todos los educadores matemáticos españoles comparten la misma finalidad: mejorar la calidad de la enseñanza y aprendizaje de las Matemáticas en España y, cada uno, debe asumir este objetivo en el ámbito de sus competencias profesionales. Los investigadores tienen un campo profesional bien delimitado que no pueden eludir. Por razones éticas, cívicas y profesionales han de llevar a cabo el inaplazable desarrollo de la investigación española en Educación Matemática. Se trata de un deber prioritario cuya realización les compete y del que no pueden sustraerse; el éxito o fracaso en esta tarea dará en el futuro la dimensión auténtica de su contribución a la Educación Matemática en España.

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SEMANTIC STRUCTURES OF WORD PROBLEMS - MEDIATORS BETWEEN MATHEMATICAL STRUCTURES AND COGNITIVE STRUCTURES?

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- Shortened Version -

(1) Introduction

Referring to a considerable body of research on solving word problems in addition and subtraction as well as in multiplication and division by elementary school students (cf.,e.g., Carpenter, Moser & Romberg (1982); Hiebert & Behr (1988)), taking into account epistemological reflections (Ernest (1994), v. Glasersfeld (1995)) and results of neural science (Roth (1995), Kandel et al. (1995)) and - finally - considering the language game perspective of the late Wittgenstein (Wittgenstein (1953)) we discuss this question: What ought to be the role of semantic structures of word problems in mathematics education?

Discussing this question mathematics education is considered in a threefold meaning: as a scientific discipline, as an environment for teacher training, as the background for the activities of the teachers in the mathematics classroom of the elementary school.

(2) Semantic Structures of Multiplicative Word Problems in the elementary School: A Survey of Different Theoretical Frameworks

Concerning semantic structures of simple word problems involving multiplication - and division - in the elementary school different proposals have been made for classifications. Already a global survey of the several classifications of semantic structures of multiplicative word problems reveals: The underlying categorical accentuations are different and there do not exist one-to-one correspondences.

- Vergnaud (1983, (1988)): isomorphism of measures - product of measures - multiple proportion.

- Bell et al. (1989): Multiple groups (repeated addition - sets) - repeated measure (repeated addition - measures) - rate - change of size (same units [enlargement], different units) - mixture (same units, different units).

- Schwartz (1988) (I: intensive quantities; E: extensive quantities): multiplication of I x E = E' - multiplication of E x E' = E" - multiplication of I x I' = I".

- Nesher (1988): mapping rule - multiplicative comparison - Cartesian product.

- Mulligan (1992): repeated addition (types (a),(b),(c)) - rate - factor - array - Cartesian product.

- Schmidt & Weiser (1995): forming the n-th multiple: part-whole structure, iteration structure, multiplicative change, multiplicative comparison, proportion - combinatorial multiplication - composition of operators - multiplication by formula: product structure, quotient structure.

Nesher (1988) presents this example as a paradigmatic one for a "problem describing a 'mapping rule' ":

> There are 5 shelves of books in Dan's room. Dan puts 8 books on each shelf.
> How many books are there in his room?

According to Schwartz (1988) this example has to classified to I x E = E x I = E': 5 shelves and 40 books interpreted as the extensive quantities E and E', respectively, and 8 books/shelf as an intensive quantity.

Referring to the system of Vergnaud (1983) two interpretations of the category "isomorphism of measures" are possible: using the multiplier 5 as a scalar operator or using it as a function operator.

According to Bell et al (1989) "multiple group" is the appropriate category; Mulligan would classify it to "repeated addition, type (a).

According to Schmidt & Weiser (1995) two interpretations of the category "forming the n-th multiple" are possible depending whether the situation is interpreted as a static one (part-whole structure) or as a dynamic one (iteration structure).

Comparing this example just considered and the following one from Schmidt & Weiser (1995):

> Mr. Brown fills the tank of his car with 17 liters of gas. Mr. Miller fills his with 3 times as much. How much gas does Mr. Miller take?

both have to be classified to the same category according to the systems of Vergnaud (1983), namely "isomorphism of measures", and of Schwartz (1988), namely "I x E = E' " [I = 3 l/l, E = 17 l, E' = 3 l/l x 17 l = 3 × 17 l], but to different categories according to the systems of Nesher (1988), namely "mapping rule" or "compare problems", respectively, Bell et al. (1989), namely "multiple groups" or "mixture (same units)", respectively, Mulligan (1992), namely "repeated addition", perhaps "array" or "factor", respectively, Schmidt & Weiser (1995), namely "part-whole structure" or "multplicative comparison", respectively.

Already this short consideration of examples illustrates this: Sometimes one system appears to differentiate more refined, sometimes another one does so.

Evidently the systems of semantic structures under discussion differ from one another: They "tell", in fact, "different stories". And considering empirical results we get to know that not any system of semantic structures reveals the "whole story". How shall we deal with this situation?

Referring to the threefold meaning of "mathematics education" mentioned above we get three questions:

- Shall this situation of theorizing be considered as a deficit situation which has to be overcome towards a unified theory? [*mathematics education question*]
- What appears to be reasonable or recommendable for us when teaching student teachers or when discussing with them? [*teacher question*]
- Which help can a teacher get from research and different theoretical frameworks on semantic structures (of simple word problems)? [*classroom question*]

(3) Semantic Structures - Results of Constructions or Mappings? Or: The Convergence of Epistemological Reflections and Research in Neural Science (on Perceptions and Knowledge)

Nesher (1988) states, that

> "it is generally agreed that the main source of difficulty for the learner lies in the transition from the problem given in natural language into its presentation in mathematical language" (Nesher, 1988, p. 19).

For our purpose it is appropriate to transform this statement into this diagram.

natural language:	*mathematical language:*
word problem - e.g.:	arithmetical problem - e.g.:
There are 5 shelves of books in Dan's room.	[according to Bell et al.,1989]
Dan puts 8 books on each shelf.	8 × 5 =
How many books are there in his room?	[books] [books]

↓ student /learner ↑
or her/his
↓ cognitive structure ↑
→ *transition* →

Mr. Brown fills the tank of his car with 17 liters of gas.	[according to Schwartz, 1988]
Mr. Miller fills his with 3 times as much.	3 × 17 =
How much gas does Mr. Miller take?	[l/l] [l] [l]

↓ student /learner ↑
or her/his
↓ cognitive structure ↑
→ *transition* →

One may be inclined to interpret this situation under discussion like this:

- There is something outside the cognitive structure of the student, namely the word problem considered as a set of syntactically correct sentences and - mostly - a concluding question in written form or orally uttered.

And this word problem in natural language is bearing a certain structure - a semantic structure.

- The student has to *extract* the meaning of the word problem, i.e., she/he has to *extraxt* the semantic structure .
- This semantic structure shall help her/him to find and to utter a presentation of this structure in mathematical terms - thus finding a transition from natural language into mathematical language.

Where are the structures?

- Are the semantic structures - 'really'- outside the cognitive structures of the students and, thus, have to be transferred into the cognitive structures of them?

- And what about the arithmetical structures - where have they to be located?
- Which component can be considered as a mediator between the others - or:
- Must we - finally - take an epistemologically different position?

In the history of epistemology the following questions have been discussed since the time of pre-Socratic philosophy of ancient Greece:

- Is it possible to get objective empirical knowledge?
- Considering this to be the case: How can we get such objective empirical knowledge?

For reasons of brevity we confine to a simple 'epistemological coordinate system' making explicitly only the difference *between* realism and *idealism* that dominated epistemology since the 17th century. Distinguishing between ontological and epistemological realism or idealism, respectively, we already get a certain differentiation.

Thesis of *ontological realism:*
There is a reality existing - and structured - distinctly from human beings experiencing, interpreting or recognizing it.

Thesis of *epistemological realism*:
It is possible to get objective knowledge - i.e., to get knowledge on the independently existing reality. There may be difficulties or constraints to get such knowledge but - at least - in parts we can get knowledge from reality how it actually is.

Thesis of *ontological idealism:*
Reality only consists of human subjects and their ideas:
esse est percipi vel percipere (to be is to be perceived or to perceive).

Thesis of *epistemological idealism*:
Only "phainomena" are objects of our experience.
Phainomena are ideas (Descartes, Locke, Berkeley), sensations, or phenomenons (Kant); phainomena are mental inner objects of human beings - not 'external objects'.

The epistemological realism presupposes an ontological realism. An ontological idealism presupposes an epistemological idealism. Thus an epistemological realism is not compatible with an ontological idealism. But an ontological realism does not imply an epistemological realism: I. Kant (1724-1804) represents a famous combination of ontological realism and epistemological idealism: The "noumena" ("Dinge an sich") cannot be recognized, only their "phainomena" ("Erscheinungen") can be recognized and are processed by the fundamental forms of intuition ("reine Anschauungsformen") of space and time and the categories of reason ("Kategorien", "reine Verstandesbegriffe") - quantity, quality, relationship, modality.

		Epistemological	
		realism	idealism
Ontological	realism	+	+ (!)
	idealism	-	+

Mapping theories of epistemology consider mental objects and their attributes as models of 'real' - external - objects and their 'real' attributes. What would such a 'mapping perspective' mean for our situation under discussion? The interpretation sketched above can be regarded as determined by the mapping perspective: The student has to extract 'the' meaning from the word problem by mapping the semantic structure into her / his own cognitive structure. Then she / he has to map it from her / his cognitive structure into a presentation in mathematical language. Combining an ontological realism with an epistemological idealism must not necessarily lead to a mapping theory - as, e.g., the "Critique of Pure Reason" (1781, 1787) of I. Kant shows. Without discussing the solutions of this combination of ontological realism with an epistemological idealism in the area of philosophy we argue that a convergence is emerging between this philosophical orientation, on the one side, and research in neural science, on the other one.

Regarding the capacity of stimulating and the capacity of interpreting within the human visual system and the neutrality of the neural codes I want to make this plausible.

Considering the different stages of processing of our human visual perception - starting with the photoreceptors and the retina and going on to the primary, the secondary, and the tertiary visual areas of the cerebral cortex there are growing numbers of cells and neurons processing the signals received from rather a small number of ganglion cells in the retina: For one ganglion cell in the retina our human visual system has got 100

000 central neurons. This rate is even more impressive for the auditory system: One hair cell - in the inner ear - corresponds to 16 000 000 central neurons. Not all 200 billion neurons of the brain are active simultaneously: About 100 000 to some millions are active simultaneously, but the patterns of active neurons are perpetually changing. Results of brain research show that objects are not represented by a sole detector cell - the famous "grandma neuron" does not exist nor exists - let us say - a single neuron for detecting the digit "1": There is no single neuron or a set of neurons that can represent a concrete object in all its details and all its different meanings. (For details of this part of scientific discussion cf. Roth (1995, chap. 7,8) or Kandel et al. (1995, Chap. 18,21-25).)

What is important, too, is the neutrality of the neural codes. The different physical and chemical stimulations of our sensory apparatus have to get changed into the 'language' of the brain, i.e., into neuroelectric and neurochemical signals. And those signals are transmitted to the ganglion cells and the neurons in the cerebral cortex and processed there. Considering such neuroelectric or neurochemical signals without any further informations it is not possible to determine whether they have been caused visually or acoustically, chemically or mecanically: Only when knowing in which area of the brain the active neurons are located we can conclude, e.g., that the visual system is activated.

The brain as a neural system gets only its own stimulations in the mode of neuroelectric and neurochemical signals, and it is its task to detect from which sensory system the stimulations come and - being more important - which meaning can be assigned to them. The brain is forced to interpret the received content free signals, i.e., it is forced to give them meaning. All in all the different areas in the brain have to do a lot of work in order to assign meanings to the relatively few informations of the eyes or the ears or the other sensory systems; and there does not exist a unique correspondence between the stimuli of the environment and the processes within the brain. Insofar we get this result:

- Even perceiving is already an active constructing.

And without discussing in detail we state another result:

- The most important part for the assignments of meaning is played by the memory.

These shortly sketched results of brain research can be condensed to this thesis:

The world cannot be mapped by the brain, the world - the knowledge of the world - must be constructed by our neural system on the occasions of sensory stimuli.

Epistemologically speaking this thesis can be interpreted as a support of the combination of ontological realism and epistomological idealism based on results of neural science; and it is quite compatible with reflections of v.Glasersfeld (1995). Using this background we can modify our analysis from the beginning of this section:

> A word problem as a set of written or orally uttered syntactically correct sentences is peripherically coded by our visual or auditory system and then centrally coded into neurochemical or neuroelectrical signals, and after this being processed in the cerebral cortex.
>
> Referring to our memory a specific interpretation of this "exterior" word problem is established - and it is only this interpretation the human subject can deal with, all further processing is based on this interpretation: There is no direct access to the "exterior" word problem.
>
> And where are the semantic structures or where can they be? In the process of interpreting the incoming signals and while using the own memory semantic structures may shape the interpretation which is the basis for all further processing in the cortex.

Insofar we get:

> It is always the reader, the listener who constructs meaning: There is "no direct reading either of a text or a diagram, chart or picture" (H. Bauersfeld, 1995, p. 275) nor a direct hearing of a saying. Objects are only objects for a certain - human - subject.

Hence neither a mathematics educator nor a mathematics teacher in the classroom can directly transfer meanings or intentions to the students: Meaning does not travel.

Coming back to the *'mathematics education question'* we can state:

On the epistemological basis of a combination of ontological realism and epistemological idealism it is quite normal and not at all astonishing that there exist different systems of semantic structures of, e.g., simple multiplicative word problems depending on those aspects considered to be relevant for the ordering of a piece of our experiental world. It is quite normal that the systems of semantic structures proposed tell "different stories" and not any tells the "whole story" - the latter because we do not know - and we cannot know - what the whole story is.

What we can do and, of course, what we ought to do is this:
- to make explicitly the foundations and arguments when establishing a certain system of semantic structures, and
- to show what we can achieve when using it.

But, nevertheless, the idea of a *"unified theory"* in the end is - very likely - only a belief that attracts more epistemological doubts than supports!

What can be the consequences for the *teacher question and the classroom question?*

The teachers - as reflective practitioners - must learn as well as the mathematics educators that the diversity of systems of semantic structures is an epistemologically quite normal situation: A word problem ought better to be considered as a set of constraints for the problem solver that allows some - "viable" - interpretations - for instance, some fitting a certain system of semantic structures.

Sticking too rigorously to certain semantic structures by the teacher can cause obstacles for the development of the intuitive meanings of, e.g., multiplication of the students.

A system of semantic structures is a set of perspectives and using it we - the mathematics educators or the teachers or the students - can reduce the vast diversity of one-step multiplicative word problems to a finite system of types:

> The teacher can analyse students' proposals and ascriptions of meanings - but in an open minded manner; she / he can control whether the set of problems used in the mathematics classroom is of sufficient diversity.
> For the students a prototype like usage of semantic structures may serve as an aid in order to become conscious of their own interpretations of different situations and to get to know other interpretations, for example, of their classmates.

Semantic structures - grounded in whatsoever categories - ought to be used open mindedly as descriptive notions; the interpretation of them as normative notions appears to be too narrow and not enough productive.

(4) Knowing and Social Practices - the "Language Game"Perspective of the Late Wittgenstein and Semantic Structures as Suggestions for Practices

In the preceeding section (3) our discussion was focused on the human being as an individual: The cognitive acts of constructing meanings were considered to be personal and to be located within the individual. Now I will turn to the social aspect using as a background the "language game" perspective of the late Wittgenstein. (The "late Wittgenstein" is the author of the "Philosophical Investigations" (PI), 1953 and the "Remarks on the Foundations of Mathematics" (RFM), 1956.)

Let us not enter here into the primacy debate about the individual versus the social; in particular, let aside the discussion of the question: Who is right - Piaget or Vygotsky? As a foundation a position shall be adopted here proposed by the sociologist Elias (1969/1990):

> The single human being ought to be considered as a being that is continuously in an "open process within indissoluble interdependencies with other single human beings" (Elias, 1990, p. LI). As an "open personality" a human being has got only a "relative autonomy" in comparison with the other fellow-beings. Neither are human beings the 'really existing things' beyond society nor is a society the 'really existing thing' beyond the individual human beings: A human being is not 'outside' a society, a society is not 'outside' the individual human being.

In the philosophy of the late Wittgenstein knowledge is not an object any longer for which language is only a box of neutral tools. Language is not considered as an objectively existing mediating factor between the given human subject and the - given - object as it is done in the Aristotelian view. Language is a universal medium - thus it is impossible to describe one's own language from outside: We are always and inevitably within our own language (cf. Hintikka & Hintikka, 1986/96). Knowledge appears as knowing, and knowing is always performed in language games. Language as languaging or playing a language game is equal to constituting meanings and, thus, constituting objects. There are no objects without meaning, and meaning is constituted by a specific use of language within a respective language game:

> "For a *large* class of cases - though not for *all* - in which we employ the word 'meaning' it can be explained thus: the meaning of a word is its use in the language" (Wittgenstein, PI, 43).

"Every sign *by itself* seems dead. *What* gives it life? - In use it is *alive*" (Wittgenstein, PI, 432).

"I shall call the whole, consisting of language and the actions into which it is woven, the 'language game' " (Wittgenstein, PI, 7).

"There are countless kinds: countless different kinds of use of what we call 'symbols', 'words', 'sentences'. And this multiplicity is not something fixed, given once for all; but new types of language, new language games, as we may say, come into existence, and others become obsolete and get forgetten. (...)
Here the term 'language game' is meant to bring into prominence the fact that speaking of language is part of an activity, or of a form of life" (Wittgenstein, PI, 23).

Language games are "systems of communication" between human beings in which the connections between

- language (as languaging),
- (non-verbal) actions, and the
- environment of the utterances count.

Language games are always parts of a social practice, and non-linguistic components become a necessary condition for the understanding of a language - even gestures and pictures or patterns can be important components of a language game.

Accepting the 'language game perspective' as a productive and challenging perspective for analyzing teaching-learning processes in mathematics as well as for the teacher's activities in the classroom

- also mathematical knowledge ought to be considered as a part of a - specific - social practice;
- and it is by communicating - by playing a language game - that we give witness for the existence of a particular use - be it use of a formula, of a definition, of a theorem, or something else. Thus it is by communicating - in a language game - that we give witness to a certain meaning.

How do students - in the elementary school - use simple word problems? When pupils are asking - for example - "Is it to add, Miss?" or "Is it to multiply, Miss?" or when children consider the verbal cue "times" as indicating multiplication or the cue "more" as indicating addition we can interpret such - and similar - questions and such 'cue hunting' as indicators for a certain language game in the classroom, for a certain social practice

- namely: Word problems are only hidden arithmetical problems, and how to use them is equal to reveal the hidden arithmetical problem whereby the teacher is the institution to evaluate the proposed solutions and where you have preferably to look for certain cues. This social practice is a result of the practice in the classroom and is - rather often - supported by the textbook used.

If we want to alter a social practice, if we want to initiate another language game - thus assigning a new meaning to word problems, e.g. - this can be done by initiating another set of social practices:

> "Social practices can be readily criticized: by appeal to another set of social practices. The possibility of criticism resides in diversity" (Bloor, 1974-75, p. 185).

We have to bear in mind: New language games are not initiated by learning new rules but - vice versa - it is by mastering a language game that we learn new rules. Insofar it does not make sense when teaching student teachers or elementary school students, respectively, to make them learn descriptions of semantic structures first in order to start a new language game. Moreover, we - the teachers - first ought to use the semantic structures as an implicit background in order to practice an 'anti cue hunting' language game when dealing with simple word problems.

The wit of a 'cue hunting' language game with word problems is to produce an arithmetic sentence in symbolic form the teacher is satisfied with. The wit of a 'modelling oriented' language game with word problems is to answer questions like these:

- How can I / we interpret a word problem situation in a contextual and in an arithmetical (mathematical) meaningful manner?
- Which interpretation
  - fits certain non-mathematical contextual aspects (criteria),
  - fits certain arithmetical (mathematical) aspects (standards, criteria),
  - is intended (by the author(s) of the word problem presented)?

  (These references may be in conflict with another!)
- Which procedure(s) or strategy(ies) to solve a word problem - or certain interpretations of it - appear to be appropriate?
- Does a comparison of several procedures or strategies help me / us to construct more powerful procedures or strategies?

Working within such framing can result in making explicitly certain semantic structures thus making them explicit 'rules' for this language game, and having them available - on the other hand - can enable the learner to work more effectively by using these 'rules' consciously.

Concluding this section we hold: Playing such 'modelling oriented' language games - taking this as a 'didactical demand' for teachers in the classroom as well as for the mathematics educators - is our proposal for the threefold guiding question as a whole.

(5) Concluding Remarks

Concluding this paper I want to focus on two points aiming at the *mathematics education question* and at the *teacher and the classroom question* all at once.

The following statements of Bernstein (1983) and of Kuhn (1970) are adopted as appropriate descriptions concerning the situation of the choice of models or theories:

> "Theory-choice is a jugdemental activity requiring imagination, interpretation, the weighing of alternatives, and the application of criteria that are essentially open" (Bernstein, 1983, p. 56).

> "There is no neutral algorithm for theory-choice, no systematic decision procedure which, properly applied, must lead each individual or group to the same decision" (Kuhn, 1970, p. 200).

In educational environments in Germany this Latin saying is well known: Docendo discimus - it is by teaching that we are learning. As an important complement we can add: Communicando discimus - it is by communicating that we are learning.

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ON ACQUISITION METAPHOR AND PARTICIPATION METAPHOR FOR MATHEMATICS LEARNING

Anna Sfard

1. Introduction: Theories as metaphors

Theories as metaphors. In the moving novel *Ardiente Paciencia* (turned into an unforgettable movie *Il Postino -- The Postman*), the author, Antonio Skarmata, tells the story of the Chilean poet Pablo Neruda who explains the concept of metaphor to his young admirer Mario, the postman. To Mario's question : "[Metaphor], what kind of thing is this?", the poet replies: "In order for you to have some sense of it, let's say that this is presenting something by help of something else". Quite a classic treatment, so far. It is the uneducated postman rather than the sophisticated poet who, after a little sampling and additional explanations, draws a conclusion similar to the one which in this talk is going to be grounded: "The entire world is like a metaphor of something else". The immediacy of Mario's insight indicates that one does not need more than scrutinizing look around to realize the ubiquity of metaphors and their power to create for us the world in which we live.

Although the indispensability of the metaphors may render them practically transparent, philosophers of science agreed quite a long time ago that no kind of research would be possible without them (see e.g. Ortony, 1993). As Scheffler (1991) put it, "The line, even in science, between serious theory and metaphor is a thin one -- if it can be drawn at all.... there is no obvious point at which we may say, 'Here the metaphors stop and the theories begin'" (p. 45). Indeed, there are no clear boundaries which would separate the metaphorical from the literal; there is no background of genuinely non-figurative expressions against which the metaphorical nature of such terms as "cognitive strain", "closed set" or "constructing meaning" would stand in full relief. The fact that concealing the metaphorical origins of ideas is a mandatory part of the scientific game makes the figurative roots of scientific theories fairly difficult to reconstruct. As an aside let me notice how the basic distinction between "literal" and "metaphorical" loses its ground when it comes to concepts that grew out of metaphors.

Conceptual metaphors. Quite often, when we choose a concept, say *teaching*, and then look carefully at the language in which we use to talk about it, we are able to notice a striking phenomenon: while there may be a great variety of common expressions concerned with this concept, a sizable subset of these expressions takes us in a systematic way to a certain well-defined domain which does not seem to be a "natural setting" of the concept at hand. Thus, for example, whether we talk about 'conveying ideas', 'delivering [getting] a message' or 'putting thoughts into words', we make it clear that our image of *communicating* is borrowed from the domain of *transferring) material goods*. This observation was first made in late seventies by Michael Reddy (1978) in his seminal paper entitled *Conduit metaphor*. Since then, systematic conceptual mappings came to be known as *conceptual metaphors* and became an object of a vigorous inquiry (Sacks, 1978; Lakoff & Johnson, 1980; Lakoff, 1987, 1993; Johnson, 1987). What traditionally has been regarded as a mere tool for better understanding and more effective memorizing, was now recognized as the primary source of our conceptual systems.

The strikingly systematic character of conceptual mappings such as the ones presented above, and the fact that such mappings can only arise and be dissipated through language, point out to the social, supra-individual character of conceptual metaphors. Being by-products of interpersonal communication rather than of a solitary effort of a lone thinker, they enjoy the status of public possessions. No wonder, then, that deeply rooted metaphors such as the one that ties human communication to transferring goods are often thought of as externally determined, natural, and mind-independent. As such, they also tend to be "dead" metaphors, their metaphorical nature being hardly recognizable behind their apparent self-evidence. Another noteworthy aspect is the cultural embeddedness of metaphors -- their being a product of associations that are specific to the culture within which they arise. One may say, therefore, that metaphorical projection is a mechanism through which the given culture perpetuates and reproduces itself in a steadily growing system of concepts.

Elicitation of the metaphors which guide us in our work as mathematics teachers and as mathematics education researchers is the aim of the present paper. Before I proceed, however, let me remark that the things I am going to say (as well as those I have said already) are, in themselves, metaphorical. For those who accept the claim about the constitutive role of metaphor, this fact should be easily understandable: if we create our conceptual systems with the help of metaphors, then the mechanism of metaphor is essentially recursive -- self-referential. Or, as Ricoeur (1977, p. 66) has observed, "The paradox is that we can't talk about metaphor except by using a conceptual framework which itself is engendered out of metaphor".

2. Learning mathematics: The *Acquisition Metaphor* vs. *Participation Metaphor*

In the quest after metaphors that guide our work as mathematics teachers and as researchers I decided to make a search of professional literature, looking for characteristic expressions and keywords. It did not take much effort to notice that there seem to be two leading motifs in what we do and what we say. In fact, mathematics education research seems to be caught in between two metaphors, which I decided to call *Acquisition Metaphor* and *Participation Metaphor.* Both these metaphors are simultaneously present in most recent texts, but in any given paper one of them is usually more dominant than the other. In my search I quickly noticed that the Acquisition Metaphor is likely to be more prominent in older texts while the Participation Metaphor took the lead mainly in the more recent studies. It is also quite obvious that at present, some researchers are making a strenuous effort to free themselves from the former metaphor for the sake of the latter.

Acquisition Metaphor. Ever since the dawn of civilization, human learning is conceived as an *acquisition* of something. Indeed, *The Collins English Dictionary* defines learning as "the act of gaining knowledge". Since the works of Piaget and Vygotsky, the growth of knowledge in the process of learning has been analyzed in terms of *concept development.* Concepts are to be understood as basic units of knowledge which can be accumulated, gradually refined, and combined together to form ever richer and ever more complex cognitive structures. The picture is not much different when we talk about the learner as a person who *constructs meaning.* This approach, which today seems self-evident and natural, brings to mind the activity of enriching oneself with material goods. The language of 'knowledge acquisition' and 'concept development' makes us think about human mind as a container to be filled with certain materials, and about the learner as becoming an owner of these materials.

Once we realize the fact that it is the metaphor of acquisition that underlies our thinking about learning mathematics, we become immediately aware of its being present in almost every common utterance on learning. Let us have a look at a number of titles taken from publications that appeared over the last two decades: *Acquisition of mathematical concepts and processes, Building up mathematics, Rachel's schemes for constructing fraction knowledge, The development of ... ratio concept, Children's construction of number, Extending the meaning of multiplication and division, On having and using geometric knowledge, The development of the concept of space in the child, Conceptual difficulties ... in the acquisition of the concept of function.* The idea that learning means acquisition and accumulation of some goods is evident in all these titles. They may point to a gradual *reception* or to an acquirement by *development* or by *construction*, but all of them seem to imply gaining ownership over some kind of self-sustained entity.

There are many different types of entities that may be acquired in the process of learning. One finds a great variety of relevant terms among the keywords of the frameworks generated by the Acquisition Metaphor: *knowledge, concept, conception, idea, notion, misconception, meaning, sense, referent, schema, fact, representation, material, contents, mathematical process, mathematical object.* There are equally many terms which denote the action of making such entity one's own: *reception, acquisition, construction, internalization, transmission, attainment, development, accumulation, grasp.* The teacher may help the student to attain her goal by *delivering, conveying, facilitating, mediating,* etc.

This impressively rich terminological assortment was necessary to mark differences, sometimes substantial and sometimes quite subtle, between different schools of thought. Over the last decades, many different suggestions have been made as to the nature of the mechanism through which mathematical concepts may be turned into the learner's private property; however, in spite of the many differences on the issue of "how", there was no controversy about the essence: the idea of learning as gaining possession over some commodity persisted in the wide spectrum of frameworks, from moderate to radical constructivist, and then to interactionist and socio-cultural theories. The researchers have offered a range of greatly differing mechanisms of concept development. First, they were simply talking about passive 'reception' of knowledge (thus of concepts), then about its being actively constructed by the learner; later, they analyzed the ways in which concepts are transferred from a social to individual plane and interiorized by the learner; eventually, they envisioned learning as a never-ending, self-regulating process of emergence in a continuing interaction with peers, teachers and texts. As long, however, as they investigated learning by focusing on the 'development of concepts' and on 'acquisition of knowledge', they implicitly agreed that this process can be conceptualized in terms of the Acquisition Metaphor.

Participation Metaphor. The learning-as-Acquisition Metaphor is so deeply entrenched in our minds that we would probably never became aware of its existence if another, alternative metaphor did not start to develop.

Indeed, when we search through recent issues of professional journals (e.g. *For the learning of mathematics, Learning and Instruction*) and newly published books, the emergence of a new metaphor becomes immediately apparent. Among the harbingers of the change are such titles as *Reflection, communication, and learning mathematics; Democratic competence and reflective knowing in mathematics; Developing written communication in mathematics; Reflective discourse and collective reflection; Discourse, mathematical thinking and classroom practice; Mathematics as being in the world. New researcher talks about learning as a legitimate peripheral participation* (Lave and Wenger, 1989) *or as apprenticeship in thinking* (Rogoff, 1990).

A far-reaching change is signaled by the fact that although all these titles and expressions refer to learning, none of them mentions either "concept" or "knowledge". The terms which imply the existence of some permanent entities have been replaced with the noun "knowing" that indicates action. This seemingly minor linguistic modification marks a remarkable ontological and epistemological shift in the research on learning (compare Smith, 1995; Cobb, 1995). The talks about the *states* have been replaced with attention to *activities.* In the image of learning that emerges from this linguistic turn, the permanence of *having gives way to a constant flux of doing.* While the concept of acquisition implies that there is a clear end-point to the process of learning, the new terminology leaves no room for immutable states and halting signals. Moreover, the ongoing mathematical activities are never considered separately from the context within which they are taking place. The context, in its turn, is rich and multifarious, and its importance is pronounced by talks about *situatedness, contextuality, cultural embeddedness, and social mediation.* The set of new keywords which, along with the noun *practice,* prominently features the terms *discourse* and *communication*, signals that the learner should be viewed as a person interested in *participation* in a certain kind of activities rather than in accumulating private possessions. To put it differently, learning mathematics is now conceived as a process of *becoming a member of a mathematical community.* This entails, above all, the ability to *communicate* in the language of this community and acting according to its particular *norms*. The norms themselves are to be *negotiated* in the process of consolidating the community. While learners are the newcomers and potential reformers of the practice, the teachers are the preservers of its continuity. From a lone entrepreneur the learner turns into an integral part of a team.

For obvious reasons, this new view of learning can be called *Participation Metaphor*. The decision to view learning an integration with a community in action rather than as an attempt to enhance an individual possession gave raise to quite a number of different approaches, the *theory of situated learning* (Brown et al, 1989; Lave and Wenger, 1991), the *discursive paradigm* (Foucault, 1972), and the *theory of distributes cognition* (Salomon, 1993) being the best known among them. As I will soon explain in a more detailed way, all these are theories of a new kind, differing from the old doctrines not only in their vision of learning but also, and perhaps most importantly, in their basic epistemological beliefs and the underlying assumption on the mission of the research on learning. The profoundness of the change and its revolutionary quality is sensed by many, but its exact nature has yet to be understood and made explicit. It is by no means restricted to research in mathematics education. For example, one relevant attempt at capturing the revolutionary character of the "discursive turn", which is the direct derivative of the change of the metaphor, has recently been made by Harre and Gillett (1995) in the book entitled *The Discursive Mind.* While presenting the latest developments in the study of human thinking as an emergence of *discursive psychology*, the authors name this event "a second cognitive revolution, the final apotheosis of the New Paradigm". According to their account, this second cognitive revolution" aims to accomplish what the first one failed to achieve, namely to "push the transformation of psychology right through" freeing it from the shortcomings of the behaviorist approach as well as of those inherent in the computer metaphor mind.

Twilit zone -- mathematics education in between metaphors. It is now worthwhile to pause for a moment in order to reflect on what is happening to us, mathematics teachers and educational researchers, in the twilight zone in between the two metaphors.

Perhaps the most salient indication of the switching allegiances is the change of the professional language. Since such change can only be gradual, initially one can hardly avoid linguistic hybrids. This period is also marked by the appearance of an interim language, where inverted commas around an old word are used to signal its demoted status. It is obvious that this word will only be left in the discourse for as long as proves necessary to find an eligible replacement. This is how we use today the words "fact", "knowledge", "real" world, etc.

Another way to preserve the existing terminology is to provide the old words with new definitions. Thus, Lave and Wenger (1991) propose to re-define the old terms *learning and knowing* as "relations between people in activity in, with, and arising from the socially and culturally structured world" (p. 51). With Foucault (1972) we can re-describe the term *concept* in discursive terms and say that it is a virtual entity "constituted by all that was said in all the statements that named it, divided it up, described it, explained it, traced its developments, indicated its various correlations, judged it..."

Such a "face-lifting" job on the old terminology may be not acceptable, however, in the eyes of the most devoted adherents of the new metaphor. They would claim that the switch to a new metaphor cannot be regarded as complete until the professional discourse is thoroughly purged from words that bring to mind the old metaphor. Thus, for example, they would object to preserving the words "knowledge" and "concept" as the central elements of the language of Acquisition Metaphor (see Bauersfeld, 1995; Smith, 1995). Harbingers of revolutions tend to believe that the old and the new are mutually exclusive. Are they really? Let me leave this question open, at the moment. For now, I will only remark that it is only natural that the profound change like the one we are witnessing nowadays is marked by a doze of single-mindedness and zealousness. One must declare his or her full allegiance to the new metaphor if the other metaphor -- the one by which we lived for centuries -- is to be ever elicited and questioned.

A schematic comparison between the Acquisition and Participation Metaphors is presented in Figure 1.

Fig.1: The metaphorical mappings

individual enrichment	**goal of learning**	community buding
acquisition of something	**learning**	becoming a participant
recipient (consumer), (re-)constructor	**student**	peripheral participant apprentice
provider, facilitator, mediator	**teacher**	expert participant preserver of practice/discourse
property, possession, commodity (individual, public)	**knowledge concept**	aspect of practice/discourse/ activity
having, possessing	**knowing**	belonging, participating, communicating

3. What does the Participation Metaphor change?

The Acquisition Metaphor is the one which underlies probably all the theories of cognitive development. Up to now, this metaphor has been promoting research molded in the image of natural sciences (after all, natural science is the place the metaphors of acquisition and development come from). Such research considers human cognition in its "pure" form and does not leave a room for any "noises". This means, among others, that in the acquisition-based theories almost no space is left for the role of the genuine interests of those who learn, those who teach, and those who decide what should be taught. It is therefore quite obvious that if one expects these other issues to be considered as well, quite a different kind of theoretical endeavor must be undertaken. Since the Acquisition Metaphor can hardly be expected to remain sufficient when this other kind of purpose is being pursued, the need of re-consideration, and then of another metaphor for learning becomes evident.

The shift from the Acquisition Metaphor to the Participation Metaphors makes an essential difference in almost every possible aspect of both theory and practice: it means a new epistemology, a different type of theory, a reformed visions of mathematics, of its learning and teaching, and a novel research paradigm. Let me say a few words on some of these shifts.

For one thing, our thinking about learning has always been plagued by epistemological and ontological quandaries which would not yield to the finest of philosophical minds. Further, the teaching of mathematics that followed the lead of the Acquisition Metaphor has invariably been producing to disappointing results while continuously deepening our sense of helplessness. Moreover, for some time now it has been becoming increasingly clear that in the pragmatically-minded post-modern world, the idea of a solitary activity aimed at accumulation of some esoteric goods that can hardly be shared with others is rapidly losing its allure. Let me now give a closer look to each one of these problems.

Foundational change. Nowadays it is quite obvious that the critical reconsideration of the Acquisition Metaphor can no longer be put off. First, there is a foundational dilemma that was first signaled by Plato in his dialogue *Meno* and came later to be known as *the learning paradox* (Berieter, 1985; Cobb et al., 1992). Although appearing in many different disguises throughout history, the quandary is always the same and its gist is embarrassingly simple: How can we want to acquire a knowledge of something which is not yet known to us? Indeed, if this something does not yet belong to the repertoire of the things we know, then being completely unaware of its existence we cannot possibly want it or inquire about it. Or, to put it differently, if we can only become aware of something by recognizing it on the basis of the knowledge we already posses, then nothing that does not yet belong to the assortment of the things we know can ever become one of them. Conclusion: learning new things is inherently impossible.

Thinking about the epistemological and ontological foundations of our conception of learning intensified a few decades ago, when the doctrine of radical constructivism entangled psychologists into a new dilemma. Without questioning the thrust of the Acquisition Metaphor, the constructivists offered a new conception of the mechanism which turns knowledge into a private possession of a person. In their hands, passive recipients of knowledge turned into builders of their own conceptual schemes. This image of the learner was forcefully promoted by many contemporary thinkers, notably by Piaget and by Vygotsky who, although divided on the questions of the role of social interaction and of the primary sources of learners' inspiration, were nevertheless in full agreement as to the constructive nature of learning. It is this central idea of the individuals as constructing or re-constructing their private conceptions from external materials which, at a closer look, turns problematic. Whatever version of constructivism is concerned -- the moderate, the radical or the social, the same dilemma must eventually pop up: how do we account for the fact that the learners are able to build for themselves concepts which are fully congruent with those of other people? Or, to put it differently, how do people bridge between individual and public possession?

The Participation Metaphor liberates us from these paradoxes by *disobjectivation of knowledge,* namely by providing an alternative to the talks about learning as making an acquisition. In doing so, this new metaphor does not *solve* the old quandaries by rather pulls the ground from under the vexing questions and renders them meaningless. Within its boundaries, there is simply no room for the dichotomy between internal and external (concepts, knowledge), which is the basis of the objectification. The new metaphor replaces the old selective outlook with the attention to the whole, and with the view of the learner as being a part of a community in a most essential way. Consequently, science or mathematics cannot be considered as self-contained entities anymore; rather they have to be regarded as aspects of ongoing social activities. The researchers must no longer insist on isolating knowledge from the totality of social interactions.

The change in the vision of learning mathematics. For the sake of later comparison, I will begin with drafting *the picture of learning as conveyed by the Acquisition Metaphor.*

There is hardly a more forcible expression of the vision of mathematics as an accumulable commodity and there is no better source for insights about the metaphor's entailments than the classical pamphlet "A Mathematician's Apology" by the Cambridge mathematician G.H. Hardy. For Hardy, mathematical knowledge is a means for a personal advancement and success. Many times in his brief essay he speaks about the *superiority and seriousness* of mathematics, thus stressing the superiority and seriousness of people who have an access to this special commodity. Like material goods, mathematics has the permanent quality, which makes the special merits and the privileged position of their owner equally permanent: Thus, learning mathematics means insuring one's future with the help of one's past. In fact, according to Hardy it means not less than immortality: "Immortality may be a silly word, but probably a mathematician has the best chance of whatever it may mean" (p. 81).

Within the acquisition paradigm, not only the mathematical knowledge, but also the *means* for gaining it count as a private possession of the learner. "Man's choice of a career will almost always be dictated by the limitations of his natural abilities" (p. 69) says Hardy, implying that one has to have a special mathematical *talent* to become a successful learner or creator of mathematics. This characteristic is believed to be given, not acquired. It is a person's permanent 'quality mark'. Student's achievements may depend on environmental factors, but the teachers feel they can tell students' *real* (permanent) potential from their actual performance.

Let me now try to show how the Participation Metaphor changes the overall picture. According to Rorty (1991, p. 21), there are two principal ways in which people can give sense to their lives: they can do it by describing themselves "as standing in immediate relation to a nonhuman reality" or by "telling the story of their contribution to community". Clearly, Hardy has chosen the first of these ways. Adherents of the Participation Metaphor opt for the other. They seem to be saying, together with Rorty, that "whatever good the ideas of 'objectivity' and 'transcendence' have done for our culture can be attained equally well by the idea of a community which strives after both intersubjective agreement and novelty" (*ibid*, p.13).

While the Acquisition Metaphor puts forward human personal ambitions as a principal drive for learning, within the participation framework the most important prerequisite for learning is student's wish to be a part of a certain community. Further, while Acquisition Metaphor presents *cognitive skills* as a most valued characteristic of a learner, the Participation Metaphor stresses qualities which till now, have been regarded as social rather than intellectual, and as such have not been an integral part of research on learning: being able to negotiate norms of behavior and then observe them, being able to develop a good communication with other members of the group, having a good influence on others and, preferably, leadership qualities.

Another important change induced by the Participation Metaphor is the fact that there are no more talks about permanence -- permanence of human possessions or of human traits. The new metaphor promotes an interest in people in action rather than in people "as such", and views the reality as being in a constant flux. The awareness of the constant change means refraining from any permanent labeling. It is action that can be clever or unsuccessful, not the actor. For the learner, all the options remain open in spite of failures of the past. To sum up, the Participation Metaphor brings a much more optimistic message for the learner. Since nothing is viewed as permanent anymore, and there are no more talks about factors that determine the learner's fate once and for all, the new metaphor's main message seems to be that of an everlasting hope: today you act one way, tomorrow you may act in quite differently.

It the light of all this, it is quite obvious that the Participation Metaphor has a potential to lead to a new, more democratic practice of learning and teaching mathematics. It is significant, however, that I said "has a potential to lead" rather than just "leads to". It is extremely important to understand that the outcomes of the use of a metaphor are not inscribed in the metaphor itself but rather are a function of the intentions and skills of those who harness the metaphors to work. All this is obviously true also about the NCTM's *New Standards* for teaching and learning mathematics, which seem to favor the Participation Metaphor, but which cannot bring the desired change by their mere existence. In the final account, it is up to those who translate ideas into practice rather than to the legislators, whether the introduction of the new metaphor will, indeed, lead to a democratization of learning and to the improvement of learner's condition.

4. Concluding question: Is this either-or choice?

In this talk I have elaborated on the drawbacks of the Acquisition Metaphor and on the advantages of the Participation Metaphor. It would be a mistake, however, to let you leave this room with the impression that I have preached a clear-cut preference for the latter while suggesting the abandonment of the former. Nothing could be farther from what I really intended to say. If I did not put any effort in showing the advantages of the Acquisition Metaphor, it is only because this metaphor, being still the default option for the majority of researchers, did not seem to me in a need of defense; and if I tried to show the bright sides of the Participation Metaphor, it is because of its being a relatively new idea, and as such --- in a need of explanation and justification. But now, it is time to remind ourselves that the Acquisition Metaphor does have much to offer, while the Participation Metaphor has shortcomings which, if not controlled, may lead to undesirable consequences (see e.g. Sierpinska, 1995; Thomas, 1996). Besides, even if we don't like the objectifying quality of the Acquisition Metaphor, we can hardly escape it. The perceptual, bodily roots of all our thinking compel us to talk in terms of objects and processes that can be applied to these objects even when we reach the regions of pure abstraction. I committed the "objectification crime" in this very papers when presenting its central notion -- the metaphor -- as a "conceptual transplant".

It is my deep belief that most powerful theories are those that stand on more than one metaphorical leg (compare Sfard, 1996). Metaphorical pluralism seems to me an absolute necessity. An adequate combination of metaphors would allow for bringing to the fore the advantages of each one of them while keeping their respective drawbacks under control. I fully agree with Freudenthal (1978) who said that "education is a vast field and even that part which displays a scientific attitude is too vast to be watched with one pair of eyes" (p. 78). The Acquisition and Participation Metaphor, when combined together, run a good chance of gratifying all our needs without perpetuating the drawbacks of each one of them.

Considering the fact that the two metaphors, while offering competing outlooks and conflicting ontological claims about the same phenomena, seem to be mutually exclusive, one may wonder how the suggested metaphorical crossbreeding could be possible at all. The problem, however, is definitely not new, and it is not restricted to the research on learning. We can turn to contemporary science for many more examples of similar dilemmas, as well as for ways in which the difficulty can be overcome (think, for example, about the Niels Bohr's complementarity principle which settled the ontological debate in physics without resolving the wave-particle controversy; or of chemistry and physics which deal with the same natural phenomena , but they do it in completely different ways).

Whichever of the possible solutions is adopted, one thing transpires from the dilemma itself and from the assortment of ways in which it may be tackled: one can only arrive at a peace of mind if one accepts the thought of reality constructed from a variety of metaphors. The metaphors we use while theorizing are good enough to fit small areas, but none of them can suffice to cover the entire field. We have to satisfy ourselves with only <u>local</u> sense making. Realistic thinker knows she has no choice but to give up the hope that the little patches of coherence will eventually combine into a consistent <u>global</u> theory. It seems that the sooner we accept the thought that our work is bound to produce a patchwork of metaphors rather than a unified, homogeneous theory of learning, the better for us and for those whose lives are likely to be affected by our work.

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CRITICAL MATHEMATICS EDUCATION - SOME PHILOSOPHICAL REMARKS

Ole Skovsmose

Introduction

If mathematics education can be organised in a way so that it will challenge undemocratic features of society, it can be called critical mathematics education. This education does not provide any recipe for teaching. Nor does it provide a recipe for researching mathematics education.

Critical mathematics education refers to educational concerns. The notions of students' interest and of reflective knowing are important in clarifying those concerns. Students' interest cannot be described in terms of students' background only, but must be discussed with reference to students' foreground as well. Reflective knowing is introduced as a constituent of the notion of mathemacy. Reflecting knowing refers to a broad range of considerations having to do with already developed understandings and misconceptions.

A *thought experiment*: Imagine that I am a dictator. I run this country. I decide everything. Imagine that you are the, rather naive, people of the country. You are naive because you want to do what I ask you to do. You are very kind and try to please me in every respect. Nevertheless, it is not an attractive job for me to be a dictator, at least not in this country with these naive people. I write down what I want my people to do, but unfortunately my people are not able to read my orders. Everybody ask me to explain what they have to do. So I decide to teach my people how to read and write. It turns out to be a successful educational programme, and now I do not find difficulties in getting my orders carried out.

However, the dictator has run a risk. When the people was taught how to read and write, they also acquired a competency which can be used for a different purpose. The competency of literacy can be used to interpret the situation in which the learner is engaged. Seen with the eyes of the dictator, it is 'risky' to develop literacy as a general competency. The people may use this competency and reinterpret the power relationships of society. Literacy can be used for critical purposes, and the nature of a dictatorship may be put on the agenda. Literacy is a double-edged-sword competency.

Nevertheless, the story is favourable to the dictator. He is not overthrown. Even though literacy may have a double-edged-sword quality, it need not be applied. The dictator can live peacefully together with his people, who continue to follow his written orders.

Industrialisation and the need for modernisation also reach this peaceful dictatorship. Representatives of the new industry show nice things which can be produced if the country is industrialised. The dictator decides to develop his country in accordance with these new suggestions. All sorts of glittering machinery are installed, but the people look at the machinery and at the dictator. What are they supposed to do?

The dictator starts to teach the people how to handle this technology, and this of course presupposes that he teaches them mathematics. The dictator introduces mathematics at all levels of the curriculum, and the entire work-force acquires the competency needed in order to meet the demands of industrialisation. Mathematical knowledge and literacy become the two pillars of the educational system of the dictatorship.

Does the dictator run any risks introducing mathematics into the curriculum? Will mathematical knowledge turn into a double-edged-sword competency? The dictator includes 'everything' in the mathematical curriculum: set theory, functions, graphs, algebra, group theory, calculus, etc. The people really learn mathematics. But is this learning a threat to the dictatorship?

This question concerns the possibility of establishing a critical mathematics education. If mathematics education can be organised in such a way that it challenges undemocratic features in society, we can call it *critical mathematics education.*

1 The notion of critical mathematics education

Critical mathematics education does not provide methods for teaching and researching. Critical mathematics education refers to educational *concerns.*

In the chapter, 'Critical Mathematics Education', from the *International Handbook of Mathematics Education,* Lene Nielsen and I outline such concerns. They have to do with: (1) preparing students for citizenship; (2) establishing mathematics as a tool for analysing critical features of social relevance; (3) considering the students' interest; (4) considering cultural conflicts in which the schooling takes place; (5) reflecting upon mathematics which as such might be a problematic tool; (6) communication in the classroom, as personal interrelationships provides a basis for democratic life.[1]

The emphasis on these concerns can be explained with reference to the technological paradox: Technology can be interpreted as a response to human needs, but the very attempt to solve problems is itself an instrument for creating problems. In the middle of this paradox of technology we find mathematics.[2]

In 'Cultural Framing of Mathematics Teaching and Learning', Ubiratan D'Ambrosio states the paradox in the following way: "In the last 100 years, we have seen enormous advances on our knowledge of nature and in the development of new technologies. ... And yet, this same century has shown us a despicable human behaviour. Unprecedented means of mass destruction, of insecurity, new terrible diseases, unjustified famine ... are matched only by an irreversible destruction of the environment. Much of this paradox has to do with an absence of reflections and considerations of values in academics, particularly in the scientific disciplines, both in research and in education. Most of the means to achieve these wonders and also these horrors of science and technology have to do with advances in mathematics."[3]

Could mathematics as such be considered problematic? On the other hand: Could anybody imagine a more critical thinking than the mathematical? However, the notion of critique cannot be restricted to certain forms of logical reasoning, nor to a problem-solving competency. Critical thinking cannot be reduced to a form of strict reasoning. The interpretation of critical thinking as logical reasoning has its roots in rationalism. René Descartes has emphasised that all truths can be grasped by reasoning and by reasoning alone. But the paradox of technology indicates that a more fundamental interpretation of critical thinking is necessary.

Human reason has not grasped the nature of its own creation.[4] To the rational eye, the social implications of technology are hidden below the

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1 See Skovsmose and Nielsen (1996).

2 For a discussion of the relationship between mathematics and technology see also Keitel (1989, 1993), Keitel, Kotzmann and Skovsmose (1993) and Skovsmose (1994, Chapter 3).

3 See D'Ambrosio, 1993, p. 443.

4 The Vico-paradox refers to this phenomenon, see Skovsmose (1994, Chapter 3).

horizon. Therefore, critical reasoning must be developed in a much broader way. This will be an educational task, if mathematical understanding is to be developed as a double-edged-sword competency.

Mathematics education as a global concern assigns educators a particular responsibility. Mathematics education provides, on the one hand, an introduction to participation in the technological development but, on the other hand, this education might provide a basis for criticising this particular development. This challenge calls for a critical mathematics education.

I do not think of critical mathematics education as a special branch of mathematics education. It is a global concern which reminds us that we cannot develop mathematics education on a blind assumption that mathematical knowledge, as such, ensures critical thinking.

2 The extended family of critical mathematics education

Critical mathematics education has different roots. One is found in Europe. As a reaction to the Second World War, the idea developed that education must invite students into a democratic life.[5] Education must prevent the upbringing of followers: Why did so many accept the Nazi way of thinking? How was it possible to turn technology into a production of mass-destruction? It was suggested that education should be part of a democratic life. Education should mean education for citizenship as well. This leads to the notion of critical education.

According to Paulo Freire, education is also a way of grasping the political and social constrains in which the learners are situated. In particular, these constrains are leftovers from colonialism. In this way education can be seen as a reaction to imposed social structures[6]. Ethnomathematics can be interpreted along similar lines of thought.[7] A particular development of critical mathematics education, with reference to the work of Paulo Freire, has been carried out by Marilyn Frankenstein.[8]

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5 See Adorno (1971) and in particular the chapter 'Erziehung nach Auschwitz' ('Education after Auschwitz') which was published in 1966.

6 See Freire (1972, 1974).

7 See for instance D'Ambrosio 1985), Borba (1990) and Knijnik (1993).

8 See Frankenstein (1983, 1989).

I am involved in a project in South Africa, which has to do with establishing frameworks for research in mathematics education.[9] From a South African perspective, it becomes obvious that it does not make sense to import a European variant of critical mathematics education. Nor does it make sense to import an ethnomathematical perspective. The prefix 'ethno' has an awkward connotation in South Africa.[10] Critical mathematics must be rethought anew.

This is the present challenge of critical mathematics education: To rethink its conceptions and concerns in terms of new challenges to education. Are we able to, in this new situation, to identify a competency, mathemacy, which can support critical thinking?

Can mathemacy, similarly to literacy, be developed as a double-edged-sword competency? Is it possible to relate both literacy and mathemacy to sociological imagination which means to imagine that a given situation could be formed differently. [11]

3 Two warnings

Let me give two warnings concerning critical mathematics education:

(1) Critical mathematics education cannot be imposed on students nor on teachers. The only possibility seems to be to make an invitation to being critical. Critique cannot be itemised and incorporated bit by bit into a curriculum.

(2) It is not possible to provide a specific description of a certain amount of knowledge which contains the essence of being critical. Mathemacy, as part of critical mathematics education, must be searched for as a complex network of understanding.

In what follows I shall concentrate on two issues: the notion of students' interest, keeping in mind that critical competency cannot be imposed on students; and the notion of *reflective knowing* in order to emphasise the complexity as well as the fragility of knowing.

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9 The persons involved in the project are Jill Adler, Mathume Bopape, Jonathan Jansen, Herbert Khuzwayo, Mzwandile Kibi, Cassius Lubisi, Manikam Moodley, Anandhavelli Naidoo, Nomsa Sibisi, Renuka Vithal and John Volmink.

10 For a further discussion of the notion of ethnomathematics, see Vithal and Skovsmose (in press).

11 See Mills (1959).

4 'Energy'

Before going into this discussion, let me shortly mention an example of classroom practice. The example (together with other examples) is described in my book *Towards a Philosophy of Critical Mathematics Education*. The examples in my book are not to be thought of as examples of critical mathematics education. However, I certainly conceive them as interesting, and it is possible for me to explain some of the idea of critical mathematics education by referring to them.

Here I have chosen the example 'Energy', which was planned and carried out by the teacher, Henning Bødtkjer. As it is described in details elsewhere, I shall make only a brief summary.[12]

In the project 'Energy', students discussed the input-output figures for the 'use of energy'.[13] The first part of the project work concerned the students' own breakfast. What energy supply is contained in an ordinary breakfast? This energy-supply was calculated using statistics about the 'energy-content' of bread, butter, cheese, etc. 'Use of energy' consisted in a ride on a bike. By means of certain formulas, including the parameters, velocity, time and 'front area of the cyclist', it was possible to calculate the use of energy during the ride.[14]

The project then turned to input-output figures for farming. First: What energy input is needed to grow barley in a specific field? The input includes, for instance, the use of petrol for ploughing. The students then calculated the energy supply contained in the harvested barley. The result of these calculations showed that the energy-output in the barley produced is six times the energy-input. The farmer used the harvested barley as pig food, and the input-output figures for pig-breeding were finally calculated. The result was that the energy output was five times less than the energy input. Therefore, according to the students' calculations, pork-production is very expensive, seen from the perspective of energy supply. Finally, these results were interpreted in a global perspective.

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12 See Skovsmose (1994), Chapter 7. In Skovsmose (1996) the project 'Energy' is discussed with reference to 'meaning in mathematics education'.

13 Physics states that energy does not disappear but changes from one form to another. Naturally, it is this phenomenon of changing which is referred to by the everyday expression 'use of energy'

14 Three different formulas could be used to determine the 'bike resistance' r which depends on the type of bike, the velocity v, and the 'frontal area' a:

Normal bike: $r = 1.1av^2 + 7$

Sports bike: $r = 1.0av^2 + 6$

Racer: $r = 0.7av^2 + 5$

5 Students' interest

With reference to research in ethnomathematics it has been suggested that mathematics education must consider the cultural background of the students as a source for developing mathematical activities. The students cannot be 'objects' of an educational process. They must be seen as participants, and *students'* interest therefore plays a crucial role.

The concern for students' interest is important. Let me, however, add a few comments on the notion of background. There is a difference between paying attention to the students' background and respecting the students' interest. Students' interest cannot be described in terms of students' background only. We do not do students a favour by relating the content of the curriculum solely to their background.

My father was a tailor, and I can assure you that much mathematics, especially geometry, is involved in tailoring. Mathematicians have faced a tremendous task in projecting the three-dimensional globe into the two-dimensional map. But tailors are involved in the complicated task: projecting the two-dimensional cloth unto the three-dimensional body. Let us imagine that some ethnomathematicians had come to my place. They studied my background and the mathematical content of my father's work. They could tell him that in fact he was doing very advanced mathematics. He would not understand a single bit of what they were saying. (Maybe he would have asked the ethnomathematicians if they wanted him to make them a new dress.) Assume that the ethnomathematicians had suggested to my teacher that he was to develop some special mathematics for me, based on (the so fascinating and advanced) tailor's mathematics. I would feel insulted. I am sure that I would not want to be treated differently in this respect.

The point of this story is not to say that students' interest is unimportant. It is essential to consider the students' interest. But the interest cannot be examined simply in terms of the background of the students. Equally important is the *foreground* of the students.

By foreground I refer to the students' interpretation of opportunities which society reveals as opportunities for the students. The foreground contains hopes and aspirations. Instead of tailor's mathematics it might have been just as rewarding to introduce me to pilots' mathematics (although I had no idea that I would ever become a pilot). The students' interest is not to be reduced to 'background'.

The frontal area was estimated by each student individually by using a video print of themselves riding towards the camera. To get the right scale each student had attached a little piece of cardboard on which were drawn a few squares, one dm^2 each. The top of the cardboard was fixed to the student's sweat-shirt by two safety pins. This meant that the cardboard took a vertical position independent of whether the student was sitting in an upright position or bent forward when riding the bike towards the camera. The whole video print was divided by drawing a lattice of squares over it, and that made it possible to count the number of squares needed to cover the whole picture of the person. The front area a has to be measured in m^2 and the velocity v in m/s. The formulas then gives the bike resistance r as measured in N (Newton).

The project 'Energy' was developed with aim of relating it to the students' interest. Is it possible for the teacher (and other planners of the educational process) to design a project and then claim, so to say, that they have considered the students' interest? This is a difficult question. I do not think a simple answer exists, which again means that I think it is possible for the teacher to plan a project considering the students' interest.

For me it is essential that the project 'Energy' is presented within a framework making sense to the students. It should be possible for the students to negotiate the purpose of doing different things. For me the question is not simple how student' interest can be used as a source of motivation. I do not think in terms of using students' interest as a device for planning and managing the curriculum. It is essential that students' interest is respected, which again implies establishing projects the relevance of which can be challenged by the students.

The project 'Energy' was planned by the teacher who presented the topic to the students. However, the topic was presented in a way that provided opportunities for the students to question the points of the different tasks.

We can separate two different notions of understanding. The students might come to understand *how* a certain algorithm works. They might also come to understand *why* certain algorithms are used and exercised. Respecting students' interest means ensuring an understanding not only directed towards mathematical concepts and algorithms but also towards the nature of the educational tasks. What was learned in the project 'Energy' was not linked to the students' background nor immediate interests. While the introduction of the topic, breakfast-biking may be seen as part of a motivational device, the kernel of the project, coming to grasp the essential elements of input-output figures of farming, refers to a different interpretation of students' interest.

This interpretation of interest has to do more with students as citizens. Such long-term interests are also real interests, and the students will also be aware of these, if they are involved in projects facilitating discussions of the meaning of educational tasks.

6 Reflective knowing

Mathematics educators are very concerned about students' development of mathematical knowledge and, in specific, about students' understanding (or lack of understanding) of mathematical concepts. However, much more is on stake in mathematics education than mathematical knowledge.

This is the reason why I have introduced the notion of *reflective knowing* as an important constituent of mathemacy.[15] Mathematical knowledge in itself does not provide a double-edged-sword competency.

The students can reflect upon many things. Let us relate to the example of 'Energy'. Some formulas were used for the calculation of bike resistance. They contained different parameters: What do the different parameters mean? Is the result reasonable? How are the results justified? How are the formulas justified?

The students may reflect on how they did the calculations: How do we calculate the energy supply in the field? They may consider their own modelling activity: Have we considered the relevant features? Are some aspects forgotten?

The students may reflect upon the actual results of the modelling process: What can the input-output figures for farming tell us? Are our results similar to other results? Are other 'official' results reliable?

The students may consider their results in a global perspective. Their reflections may take the form of exemplary thought: What does this mean for the food production of the earth?

The students may also consider their own activity: Maybe the project was not that interesting but what did we actually learn? The reflection may consider the whole school situation.

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15 The notion of reflective knowing is developed in Skovsmose (1994). See also Skovsmose (1990).

The activity of reflecting is essential in the development of knowing. I do not see reflecting as similar to the notion of reflective abstraction which plays a critical role in Jean Piaget's genetic epistemology. Piaget concentrates on the description of the development of mathematics knowledge, and he studies operations as the basis for mathematical knowledge. However, I am not only interested in the development of mathematical knowledge. My concern is the development of the much broader competency of mathemacy. In this competency, mathematical knowledge is only one of the elements. Reflecting knowing refers to a broad range of considerations having to do with already developed understandings and misconceptions.

The concern of mathematics education cannot be simply to produce mathematical knowledge. To develop a reflective knowing is a much wider concern which calls for the development of a mathemacy.

7 Conclusion

As already emphasised, there is no recipe for critical mathematics education. We must be open to an experimental practice, and such practice might give ways to identifying ideas for the further development of critical competencies.

Why are the competencies important? This has to do with the paradox of technology which indicates that conditions for democracy may be hampered by the actual technological development for which mathematics education serves as a preparation. This is the challenge of a critical mathematics education.

A *thought experiment*: Imagine we are joining a Middle Age conference concerning religious education. Sitting in the dim light in the conference centre, we are listening to many and interesting lectures. One lecturer makes an exegesis of the teaching of the Holy Trinity. He explains that he has investigated also the sources of the holy text and come to new conclusions concerning the basic structures of the Trinity. He suggests that the curriculum in religious education, including the teaching of how to pray, is changed in accordance with his findings. Instead of relating to a patch-work of information about the Trinity, the prayers can simply be structured, and a few basic prayers can serve as 'mother-structures' in the development of all sorts of advanced praying. To change the curriculum in accordance with this insight would make sure that the children were better equipped for further education (which also means further praying).

Another lecturer has found it possible to interpret the basic concepts expressing the structure of the Trinity in a way comprehensible to every child, independent of the child's intellectual development. This lecturer has obviously already been listening to the first one. Other lectures announce that they have produced textbooks in accordance with these new ideas about how to teach the Holy Trinity.

A critique, however, is passed about. Why not listen to the way children already produce simple rhymes? Children are already preoccupied with praying. The only thing needed is to push the children smoothly in the direction of praying for the real Trinity. In this way the teaching of the Holy Trinity can be based on already established rhymes of the children.

This suggestion is supported by scholars, obviously not living in the city of the conference. They explain that in their country they have observed many sorts of old and well-established prayers. Already long before the praying for the Holy Trinity was institutionalised, people have been praying, and in many respects these old prayers anticipate the prayers for the Holy Trinity. In fact, they are to be conceived as genuine prayers. The foreign scholars suggest that the religious curriculum take into consideration these good old prayers, and on the basis on these the teaching can move in direction of paradigmatic prayers for the Holy Trinity. But why, in fact, call some prayers 'right' ones? Why stick to the old paradigm? All sorts of prayers seem equal.

During the happy hour of the conference somebody raised the questions: Why this concern for religious education? Why teach children to memorise players? What are the social and political functions of teaching everybody to pray? What is the purpose? Why not discuss the actual function of religious education? This might be the voice of critical religious education.

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MATHEMATICS FOR WORK - A DIDACTICAL PERSPECTIVE

Rudolf Straesser,

"... mind is an extension of the hands and tools that you use and of the jobs to which you apply them."
Jerome Bruner 1996

0 Slogans on mathematics for work and its learning

Here are some slogans which are frequently offered to comment on the role of mathematics at work and the way of learning to prepare for the workplace:

- The world of work is full of Mathematics.
- Abstract Mathematics is the most powerful mathematics for work.
- Computer use implies sophisticated mathematics at work.
- The average employee / worker must learn (no) mathematics for her/ his work.
- The best way to learn mathematics for work is training on the job.

After briefly describing the most important concepts - namely "work" and "vocational education", the lecture reviews some research findings on mathematics for / at work and identifies consequences, problems and potentials of vocational mathematics education.

1 Work and Learning for Work: some basic "definitions"

The "Advanced Learner's Dictionary of Current English" offers the following two first (of seven) definitions of "**work**":

- "bodily or mental effort directed towards doing or making something; the expenditure of energy (by man, machinery, forces such as steam, electricity, etc. or by forces of nature)" and
- "occupation; employment what a person does in order to earn money"

(p. 1492 in Hornby et al. 1960, 11th impr.).

From this definition, it is obvious that "work" is one of the central human activities - if not *the* central one as in certain philosophies. The lecture comments on the role of mathematics in this human activity and on learning mathematics related to these activities. Following the definition, I will concentrate on work with an identifiable purpose and within a certain social system to secure one's own life. Consequently, I will not go into details on "informal", everyday-activities related to mathematics or topics like "out of school mathematics" or "Ethno-Mathematics" which nevertheless are somehow related to the subject of my lecture.

In this lecture, learning mathematics for work will be referred to as "learning mathematics in technical and vocational education" - following the definition given more than a decade ago. According to the terminology of UNESCO, technical and vocational education is

> "the educational process ... (which) involves, in addition to general education, the study of technologies and related sciences and the acquisition of practical skills and knowledge relating occupations in various sectors of economic and social life"
> (UNESCO 1978, p. 17).

On purpose, the lecture will start from aims, problems and potentialities inherent in mathematics at work and vocational mathematics education. As a consequence, only minor attention is paid to general psychological competencies fostered or destroyed by vocational mathematics education. Nevertheless, by analysing mathematics at work and vocational mathematics education, general mathematics education could be informed on fundamental issues of using and learning mathematics. In my view - and somehow in contrast to the UNESCO-definition, vocational education is not an addition or a field of application of general education. To the contrary: general education tends to present too narrow a perspective on education, a perspective hampered by the concentration on schools, classrooms and mathematics watered down to school mathematics. Vocational mathematics broadens this narrow perspective by opening a window on mathematics at work, deeply intertwined with its applications and social life.

2 Mathematics for Work: findings on workplace practice

In this paragraph I present three cases in mathematics for work and its learning in order to show the variety of relevant situations as well as some features which I think are most pertinent to learning mathematics for work. I start with "street mathematics" at work, then present a study on geometry in technical drawing and finish this paragraph with research on mathematics in the banking sector. "Street mathematics" is a reminder of the omnipresence of mathematics at work, while the other two cases analyse standard professional use of mathematics in economic enterprises. These activities are firmly institutionalised in a workplace hierarchy and executed to earn one' s living. Both cases serve to look more deeply in major features of mathematics at work. Technical drawing in some sense presents the "usual" potentials and problems of mathematics at work while mathematics in the banking sector throws some light on the role of modern technology, especially computers, with mathematics at work.

2.1"Street Mathematics" at work

"Street mathematics" is usually not analysed under the heading "work" but under the heading "out-of-school" or "informal" mathematics. If you recall the definition given above, it nevertheless is quite obvious that it is a case of "mathematics for work".

What I have in mind is the series of studies undertaken by Carraher, Nunes and Schliemann (and a whole lot of co-workers) recently published as a book entitled "Street mathematics and school mathematics" (cf. Nunes et al. 1993). I do not want to rephrase all the findings and careful analyses of the book. From this material it is obvious that mathematics can and - sometimes - must be learned at the workplace, that mathematics learned at the workplace seems to be somehow different from school mathematics and that mathematics learned in school sometimes even seems to hinder workplace activities. As an illustration I want to read out only one quote from the study on "Mathematical Knowledge Developed at Work: The Contribution of Practice Versus the Contribution of Schooling" where Schliemann&Acioly (1989) analyse the use and understanding of mathematical knowledge among Brazilian bookies working in a special lottery game:

> "Procedures to solve problems for which bookies do not have a ready answer are usually oral procedures not taught in schools. ... experience in solving problems at work can be a source of mathematical knowledge" (op.cit., p. 217).

As a consequence, we start from the assumption that participation in a workplace community of practice seems to foster learning for work - even learning mathematics for work. Nunes et al. even go further, concluding a study on the concept of proportionality and its transfer by stating:

> "... this series of study demonstrates that the concept of proportionality does not have to be taught. It can develop on the basis of everyday experience. The resulting conceptual schema ...

> models relationships in everyday situations but clearly surpasses the procedures used in everyday practice. It is not unidirectional, as everyday practices tend to be, and it can be applied to new situations"
> (Nunes et al. 1993, p. 126).

Is this true for every type of mathematics at work ? Does this also hold for the qualified technician ? Are we entitled to dissolve all technical and vocational schools and colleges around the world in favour of learning at the workplace ? The next case will shed some favourable light on this wrong assumption.

2.2 Mathematics for the qualified technician: Geometry and technical drawing

To learn more about the actual use of mathematics, especially geometry, at work, we "climb up" the qualification ladder and look into a study with technical drawers in metalwork analysing the role of geometry in technical drawing (for a detailed report cf. Bromme, Rambow & Straesser 1996). As a consequence of difficulties in identifying mathematics at work, the study in technical drawing approached the problem of finding mathematics at work by individually interviewing thirty draughtspersons near their workplace and during their usual working time. All but one interviewee had passed an examination as a qualified draughtsperson. They had 2 to 32 years of professional draughting experience (median: twelve years). At the time of the interviews, thirteen of them did not use CAD-techniques at their workplace.

A prepared set of sixteen technical drawings was offered to the draughtspersons who were asked to classify them in a way they would classify if they had to draw these. In order to relate the drawings to the work of the interviewees, the drawings were selected to represent three dimensions: measurement versus no measurement in the drawing, symmetry or non-symmetry of the pieces drawn, and type of drawing (projections versus orthogonal views).

With two possibilities in each dimension, 16 drawings can have two drawings for every combination of possibilities (cf. Fig. 1 showing two sample drawings). When the interviewees had formed groups with the drawings, they were asked to comment on their classification by giving a short description, at least a catchword to every group they had formed.

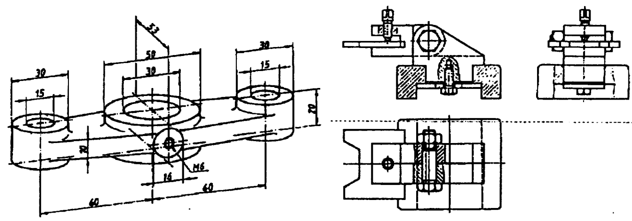

Fig. 1: Two sample drawings

The groupings and catchwords of the draughtspersons can be summed up as follows:

• Symmetry does not play a role in the mental representation of the technical draughtspersons, while type of representation of the drawings shown is respected by the interviewees and measurement seems highly important to them.

• On average, the draughtspersons give a classification which can be represented by the following two-dimensional drawing with the "standard" catchwords given.

This classification rather well duplicates the two ex-ante dimensions of measurement and type of drawing. One could even say: the draughtspersons rather well respected two major mathematical classifications "hidden" in the drawings.

• Nevertheless the "standard"-descriptions illustrate that the interviewees more or less classify according to similar uses of the drawings (such as production, fitting, or presentation of tools) and only once in four "standard" descriptions relay on categories from descriptive geometry - a subdiscipline from applied geometry, if classified in disciplinary mathematical terms. What is most important to the draughtspersons is the professional, the production aspect of the drawings.

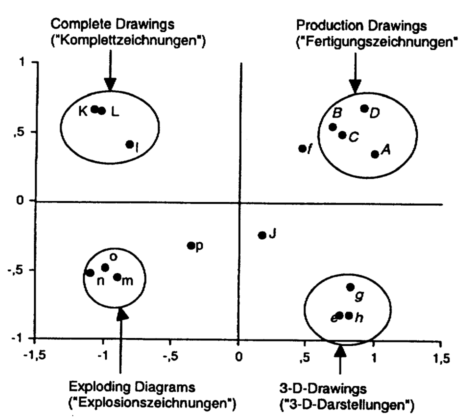

Fig. 2: Classification of drawings in a 2D-space
(Drawings with measurement are represented by italics,
orthogonal views are represented by capital letters)

In order to know more about the actual workplace situation, the draughtspersons were also asked to evaluate the respective importance of vocational training against workplace-experience and aspects of technical drawing which they came to learn at the workplace. We asked the interviewees to linearly rank qualifications according to their importance for an experienced draughtspersons. With nine qualifications offered, the draughtspersons came up with three qualifications rather clearly ranked most important. "Comprehension of the purpose and functioning of the object to be drawn" was clearly ranked in the first place (mean: 1, mode: 1), while "comprehension of geometrical relations of the drawing" (mean: 3, mode: 1) and "comprehension of a sketch or an order" (mean: 3, mode: 2) were ranked behind. The result is some sort of confirmation of the utmost importance of production-related aspects of a technical drawing - while mathematics, esp. geometry seems to be embedded in workplace matters. According to the interviewees, this "vocational mathematics" is better learned at the workplace than in colleges.

Interpreted more globally, the study clearly shows that mathematics and vocational knowledge are intimately interwoven at work. Workplace practices do not distinguish mathematical knowledge from other knowledge helpful to cope with the professional problem. It is by means of a "problem-oriented integration of concepts" from various sources (cf. Bromme et al. 1996, p. 166) that the draughtspersons cognitively organise their work. A separation and piecewise analysis and piecewise learning seem to be inappropriate.

2.3 Information technology and mathematics for banking

To come to the topic of vocational use of mathematics when the professional activity is deeply characterised by use of modern information technology, I turn to a recent study by Noss & Hoyles (1996) analysing the mathematics of banking. They were asked to help with a problem in a London bank. The senior management thought that

> "many employees did not have a robust grasp of the mathematics underpinning their work — they had little feel for the mathematics which would enable them to appreciate the models on which financial instruments were based and to recognise their limitations. There was apparently a widespread reluctance to think mathematically about transactions, and a recipe-book mentality which relied on technology without understanding what it could and could not do" (loc.cit., p. 6).

To illustrate the situation, Noss & Hoyles (loc. cit., p. 8) quote from a meeting with an employee:

> In short, ... 'I press the button and see what it says'.
> What then?
> 'I look at the answer. If it seems to indicate what I think we should do, I use the number to justify my decision. If not, I ignore it, or put in figures which will support my hunch'.

Noss & Hoyles accepted to work on the problems and took a twofold approach:

> "First, we set out to understand more clearly what was the essence of the problem. ... What did employees do that was (or was not) mathematical? ... What would be a useful and valid way to simplify and mathematise the banking situation for learnability? ...
> Second, we agreed to implement an educational programme which could begin to tackle the problem." (loc.cit, p. 7)

The most difficult part of the study was to identify mathematical relations in the practice of the bankers, to handle the dialectics of the specific procedures used by the bankers and the mathematical commonalities underpinning their transactions. Noss & Hoyles took the concept of function as a bridge between banking procedures and the mathematical models. They decided to place modelling by programming at the core of their course in order to foster the mathematical culture of the employees. The conceptual focus was on percentages and graphs which where (re)presented by simple program procedures which could be changed and linked in co-operation between the employees and the mathematics educators.

I will not go on with Noss & Hoyles' description of the course they designed to make the bankers better comprehend their banking mathematics. More important here are two aspects of the use of modern technology:

(1) If nothing is prepared to counterbalance the 'natural' development, application of modern (computer) technology seems to imply an additional step to a total invisibility of mathematics at the workplace. Even in the number driven world of banking, numbers and commercial arithmetic disappear from the consciousness of the average employee. Mathematics hide in computer algorithms which are applied without paying attention to the underlying mathematical model of the banking process. Even somewhat complicated procedures (like calculating the present value of a treasury bill by discounting from face value in

dependence of the day of maturity) go unrecognised by the average employee who relies on the programs designed by an unknown specialist in an unknown software house or department.

> "... these models were almost entirely hidden from view. Understanding and reshaping them was the preserve of the rocket scientists; the separation between use and understanding was absolute and the models' structures were obscured by the data-driven view encouraged by the computer screens"
> (cf. Noss & Hoyles 1996, p. 17).

The use of modern (computer) technology implies the use of sophisticated mathematical models - but this normally goes without recognition by the average employee.

(2) Nevertheless this practice of using sophisticated mathematics can be brought to the foreground and consciousness of the user by appropriate courses designed to open up the black boxes of the programs and partially degreying these boxes. And it is modern computer technology and appropriate software again who can be successfully used in this process to really explore and understand the underlying banking mathematics.

To put it differently: Modern computer technology itself has a dual role in the process of using mathematics at the workplace: It can be used as a way to hide mathematics in sophisticated software. Mathematics as a tool disappears in workplace routines - and modern technology speeds up this disappearance. On the other hand, the very same technology can be used to foster understanding of the professional use of mathematics by explicitly modelling the hidden mathematical relations and offering software tools to explore and better understand the underlying mathematical models.

3 Vocational Mathematics Education: Learning for work

The last remark on technology's potential to further understanding brings me to my second important topic: What about learning mathematics for work ? What about vocational mathematics education ?

If we look for the organisational patterns of technical and vocational education around the world, we find a whole variety of models from full-time technical colleges organised by the government (as in France) over part-time colleges partly run by governmental agencies (e.g.: some areas of vocational education in Germany) to isolated activities in colleges and/or private enterprises (as for instance in England or the USA). Sometimes, a mix of all these ingredients is offered (see e.g. Australia, showing the typical feature of a country which only recently acknowledges the potential of technical and vocational education). The so-called developing countries often have no technical and vocational education at all.

3.1 Two Pedagogies: Modelling versus Legitimate Peripheral Participation

Taken as an indicator of the underlying pedagogy, the organisational features show two extremes of learning principles and the standard oscillation and insecurity of political decisions on this matter.

Classroom type of vocational mathematics education tends to present mathematics as a separate body of knowledge, sometimes even structured along a disciplinary system from mathematics. In this case, mathematics has to be linked to work and workplace practice by building mathematical models and applying mathematics by the well-known modelling cycle of "situation - (mathematical) model - interpretation of the situation". The situation is to come from the workplace, the mathematical model rests upon mathematical structures and algorithms known before or taught on the spot and the solution of the model hopefully can be interpreted in a way to cope with the given professional situation (for a summary of this approach see Blum 1988, related Theme or Topic Groups at various ICMEs and the series of conferences under the "ICTMA" heading). In this pedagogy, mathematics can come first and can be taught / learned along its own, disciplinary structure while applying it to work via modelling may come second, sometimes never or inappropriately. As can be seen from this description, the modelling approach clearly distinguishes two types of knowledge - namely professional and mathematical knowledge, which have to be brought together by the individual to cope with the professional problem. In most cases, modelling vocational problems by applying mathematics is a major difficulty for the future worker - especially the extraction of the mathematical model from a professional situation at hand.

The other extreme and contrasting pedagogy is training on the job, where learning takes place at the workplace whenever it is needed by the workplace practice and its problems. The focus is on coping with the situation at hand - and mathematics may come in or not when solving a workplace problem. Apprenticeship may offer a chance to gradually develop from a beginner to an expert at the workplace. With this approach, learning may be identified with taking part in a "community of practice" and gradually developing from a beginner to a full practitioner by means of situated learning (for thorough discussion of the underlying concept of "legitimate peripheral participation" see Lave & Wenger 1991). This pedagogy starts from a uniform concept of knowledge present in a community of practice (not in individual workers),

> "knowing is inherent in the growth and transformation of identities and it is located in relations among practitioners, their practice, the artefacts of that practice and the social organization and political economy of communities of practice"
> (Lave&Wenger 1991, p. 122).

As a consequence, mathematics can continue to go invisibly, embedded in the workplace practice and serving as a tool used to cope with professional problems if needed. A problem-oriented integration of concepts tends to hide mathematical relations under the uniform workplace practices. Following this approach, studies on "street mathematics" (like Nunes et al.) had to detect and bring back to light the mathematical procedures in workplace activities, to describe them and to show the competence of the practitioners in using mathematics.

3.2 Transfer - a Focus of the Didactical Debate

The starting point of the research on "street mathematics" was a twofold observation: (a) With little or no schooling, the children working in the streets were able to solve their "mathematical" problems at work. (b) Even if the children had attended school, children successful at work could not or worse solve "isomorphic" college type word problems. How to understand this obvious lack of transfer from classroom to work ?

After more than a decade of research (cf. the monographs Lave 1988, Lave & Wenger 1991, Nunes et al. 1993, Saxe 1991), the protagonists of situated learning in a community of practice can easily understand the dilemma described above: Mathematics used in the street is learned there, is efficient in solving the street problems and fundamentally different from the one learned in school or researched in a mathematics department at university. To rephrase it in a more general way: Mathematics learned in a specific context is part of a subjective domain of experience (cf. Bauersfeld 1983) and cannot easily be isolated, taken away, transferred and applied in a different situation.

In contrast to that, the "mathematical modeller" starts from the assumption that a piece of mathematics once learned will come to mind whenever it models (adequately) a given situation, that - after appropriately modelling the situation - it can be applied easily and will offer a decent way to cope with the problem at hand. In fact, reality seems to be less convenient: The learner usually has difficulties to mobilise her/his knowledge in so-called isomorphic situations, s/he has problems to transfer a procedure, a solution from one situation to a different, maybe unknown, one. The widespread and well documented lack of easy transfer definitely contradicts the plea for modelling and easy application.

3.3 Training on the Job versus Learning in Vocational Colleges

As a consequence of this preference for the situated learning and community of practice approach, why not dissolve any classroom type of training at least in vocational mathematics and totally rely on training on the job for vocational mathematics? I want to draw your attention to a finding which might be forgotten when closing vocational / technical colleges: In the study on Brazilian bookies, the protagonists of street mathematics state :

> "... the influence of schooling is not limited to topics explicitly taught in classrooms but ... school experience provides a different way of analyzing and understanding everyday activities. ... Schooled bookies ... seem to have a different attitude toward procedures for solving problems as a result of their schooling. ... school experience has an effect on how people deal with more academic questions, such as explaining their everyday procedures or making explicit the mathematical structures implicit in their everyday activities. School experience is also related to better performance on solving problems that differ from those usually encountered at work"
> (Schliemann&Acioly 1989, p. 216 ff).

Obviously, classroom type of activities can offer an opportunity to broaden the perspective of the future worker, to empower her/him with solving problems not common to workplace practice and to foster understanding of the workplace procedure. Classroom type of activities can offer an understanding which goes beyond the narrow confines of the actual situation, which transcends the situation and the problem where and when knowledge is developed. Classroom type activities in schools or colleges can show mathematics as a way to transcend the context with more general problem solving strategies and structures. But how to cope with the transfer dilemma described in part 3.2 ?

As far as I can see there is a "way out". Modelling with the help of mathematics should not be taken as a means to get rid of the dirty specialities of the concrete workplace to solve the abstracted problem by means of pure mathematics. It is by exploiting the interplay of the professional, concrete situation and the structural, mathematical model that one can cope with the given professional problem. In doing so, one can develop a mathematical structure maybe adaptable to a variety of different problems linked to the initial professional situation. Noss & Hoyles in their paper call this to set up a "domain of abstraction" where the "dialectic between concrete and abstract" closely ties together mathematical ideas and practical knowledge of the professional domain (Noss & Hoyles 1996., p. 27). In doing so, mathematics is not reduced to the general type of activity of theorising, analysing language and seeing structures implicitly devaluing situated learning as learning no mathematics (for a recent claim of this reduced point of view cf. Sierpinska 1995, p. 5).

If mathematics is taught as a bridge between the concrete, maybe vocational situation and the abstract, maybe systematic structure, even classroom vocational education can show mathematics as a "general" tool which is of larger an importance than just coping with the narrow tasks of the everyday work practice or the inculcation of algorithms. If college type education aims at presenting (vocational) mathematics in this way, one condition for success seems to be that mathematics is taught in a way it is "meaningful to the individual" who is learning. Technical and vocational colleges then have to strive for problems from the workplace which are as realistic as possible. And the problems should be taught in a way as close as possible to the actual concerns of the students (for an elaboration of this cf. Boaler 1993).

An additional case for learning mathematics not in a too narrow workplace context is expressed in a reminder I would like to place at the end of this section:

> "Mathematics in vocational education is serving more as a background knowledge for explaining and avoiding mistakes, recognising safety risks, judicious measurement and various forms of estimation. ... Not practice at the workplace but deepening of the professional knowledge, education to a responsible use of tools and machines and the understanding of and coping with everyday mathematical problems legitimise mathematics in vocational education"
> (Appelrath 1985, p. 133/139; translation R.S.).

4 The slogans revisited

To end the lecture, I will comment on the slogans of the beginning of my talk.

"The world of work is full of Mathematics."

Indeed, the world of work is full of mathematics, but vocational mathematics is different from disciplinary mathematics - insofar as it is interested in solving the workplace problems, not disciplinary mathematical problems.

Vocational mathematics is also different from "school mathematics" in general education - insofar as it is always specific to the workplace in question, hardly interested in links to other mathematics and sometimes far more complicated than school mathematics.

"Abstract Mathematics is the most powerful mathematics for work."

Mathematics at work is not primarily interested in structures and logico-mathematical statements per se. It is not used the way mathematics is developed within its discipline. At work, it is used as algorithms, black boxes, prepared forms like worksheets etc.

"Computer use implies sophisticated mathematics at work."

Computer based mathematics is developed by few specialised people in a sophisticated way - to give it to the majority of workers encapsulated in algorithms and black boxes. Computers and appropriate software normally work to hide mathematical structures, but can be used to enhance understanding the practice of the workplace.

"The average worker must learn (no) mathematics for her/his work."

The mathematics of the average worker depends on the vision one has of a worker. To train the qualified and autonomous worker, you have to give the opportunity of learning and understanding ("legitimate peripheral participation" together with understanding algorithms and "degreying" black boxes). This is not only the case in the presence of modern (computer) technology, but also holds in traditional workplaces and societies.

"The best way to learn mathematics for work is training on the job."

Training on the job is a very good way to learn vocational mathematics. Nevertheless, learning deeply related to, but within a certain distance from actual workplace practice offers the opportunity to develop a broader understanding of the task to be mastered. There may be even technological and/or social conditions of work where learning is only feasible and affordable in a certain distance from productive work, e.g. in simulation scenarios. If "peripheral" is realised in an empowering way - not necessarily in classrooms, legitimate peripheral participation may be the best way to learn mathematics for work.

As a reminder of the more general aspiration of my lecture, I would like to offer you a slight change of my title's wording. Why not change from "Mathematics for Work - a Didactical Perspective" to "Vocational Mathematics Education - a New Perspective for Mathematics Instruction" ?

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UNA TEORÍA DE PROCESOS Y SISTEMAS GENÉRICOS EN LAS MATEMÁTICAS Y EN LA EDUCACIÓN MATEMÁTICA

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1. Introducción

La enunciación de categorías ontológicas explícitas pareció quedar desacreditada después de Aristóteles por los desarrollos neo-platónicos; pero reapareció con toda su fuerza en la filosofía escolástica. Esas categorías fueron rechazadas nuevamente por los filósofos y científicos de la Ilustración, pero reaparecieron en el ingenioso intento de diseñar nuevas categorías que instauró Immanuel Kant. Los neo-positivistas y los empirstas lógicos desacreditaron todo tipo de ontología como metafísica sin sentido. Pero esa breve historia puede significar también que ya está maduro el tiempo para intentar una nueva organización de las categorías del discurso contemporáneo.

Un primer intento de este estilo fue la Teoría General de Sistemas de Ludwig von Berthalanffy, que derivó del lenguaje de la biología en los años treinta, y tuvo gran florecimiento en los sesenta y setenta. Pero fue criticada duramente por los existencialistas y por los neo-marxistas, hasta el punto de que en los años ochenta y noventa se experimenta ya como pasada de moda.

Un esfuerzo por repensar todas esas objeciones a la Teoría General de Sistemas y por recuperar muchos aspectos válidos de ella me llevó a proponer una teoría general de procesos, en la cual los sistemas juegan un papel secundario, pues sirven sólo como modelos para esos procesos que parecen escaparse de nuestra comprensión. La he llamado indistintamente 'Teoría General de Procesos y Sistemas' o 'Teoría de Procesos y Sistemas Genéricos', según si el énfasis está en la generalidad de la teoría o en la genericidad de los procesos y los sistemas de que ella trata. Para esta conferencia voy a utilizar la segunda manera de nombrarla, haciendo énfasis en los procesos y sistemas genéricos y por ello, para abreviar, me referiré a ella como 'la teoría PSG'.

La inspiración para esta teoría la obtuve de la lógica matemática, que estudia todas las ramas de las matemáticas por medio de la teoría de modelos, utilizando un dispositivo muy sencillo, que los matemáticos suelen llamar 'una estructura', y que consiste de un conjunto básico de elementos o universo de la estructura, un conjunto de operaciones y un conjunto de relaciones. El nombre de 'estructura' es cuestionable, pero la idea es lo suficientemente sencilla y poderosa para intentar ensayarla en disciplinas diferentes de las matemáticas, desde la biología y la ecología hasta la sociología y la ciencia política. También resultó muy adecuada para describir la actividad de los matemáticos y para diseñar proyectos de investigación y materiales curriculares en la educación matemática.

Esta conferencia intenta hacer una breve descripción de esta teoría, omitiendo los aspectos puramente filosóficos y concentrándonos en las descripciones y esquemas que se vuelven posibles desde la teoría PSG cuando tratamos de capturar las maneras como tanto los matemáticos como los investigadores en educación matemática conducen sus investigaciones, y en las implicaciones que esto tiene para el trabajo curricular en matemáticas en todos los niveles, desde el Jardín Infantil hasta la Escuela de Postgrado.

Para decirlo de una vez, no estoy hablando de meras aplicaciones futuras de esta teoría PSG. Ya fue utilizada para desarrollar todo el currículo de matemáticas para los grados primero a noveno de la reforma de la educación colombiana que fue adoptada en 1984 para los dos primeros grados, y extendida luego grado por grado hasta alcanzar el noveno en 1993.

Un informe sobre este tratamiento del currículo de matemáticas se publicó en inglés en un libro sobre experiencias transculturales de educación apoyado por el Proyecto de Potencial Humano de la Universidad de Harvard y la Fundación Bernard van Leer, que salió en 1985 y fue traducido al español en 1990. Dirigí un seminario sobre ese tema en la Universidad de Harvard en 1986, y he publicado una serie de trabajos en español sobre esta teoría PSG, tanto respecto a las matemáticas como a la educación matemática, así como a otras disciplinas.

En esta conferencia voy a exponer la última versión de la teoría PSG que acaba de ser publicada en febrero pasado en el segundo volumen de una serie de siete tomos que contienen las contribuciones de la Misión de Ciencia, Educación y Desarrollo de la Presidencia de la República de Colombia, de la cual Misión tuve el honor de haber sido miembro y comisionado coordinador.

2. La teoría PSG

Para las matemáticas, la idea básica de la teoría PSG es que un matemático selecciona mentalmente un subproceso específico de entre los muchos y muy complejos procesos que vive, siendo él mismo o ella misma un subproceso más que se entrelaza con el subproceso seleccionado. La investigación original en matemáticas involucra ante todo la creación de modelos mentales de esos subprocesos, con el fin de reproducir, a través de la manipulación mental de esos modelos y a través de la manipulación de símbolos externos para ellos, los esquemas y patrones que se observaron en esos subprocesos. La tarea de construcción de modelos de esos subprocesos fue bautizada por Hans Freudenthal como 'matematización', palabra que ya es usual entre los educadores matemáticos. Pero con la ayuda de la teoría PSG podemos decir mucho más que el nombre de esa actividad matematizadora: podemos describir precisamente los productos de ese esfuerzo. El producto de un esfuerzo de matematización es un sistema conceptual con uno o más universos básicos, que llamo el substrato o el aspecto material del sistema; con ninguna, una, dos o más operaciones, posiblemente de ariedades diferentes (o sea no sólo binarias, sino tal vez unarias, ternarias, o hablando en general, n-arias), que llamo la dinámica o el aspecto activo del sistema, y con una, dos o más relaciones, que también pueden ser unarias, binarias, ternarias, o en general n-arias, que llamo la estructura o aspecto formal del sistema.

Utilizo con frecuencia la analogía de los juegos de salón para explicar lo que son los sistemas matemáticos: el substrato o aspecto material de un juego está compuesto por las fichas y el tablero; la dinámica o aspecto activo del sistema está compuesto por las jugadas válidas del juego, y la estructura o aspecto formal está compuesto por la red de relaciones definida en las reglas del juego.

Por lo tanto, insisto en distinguir el sistema de su substrato, distinción que ya es común entre los matemáticos, quienes saben cómo construir sistemas matemáticos diferentes utilizando el mismo conjunto básico de fichas; por ejemplo, con un conjunto de sólo cuatro fichas básicas se pueden construir dos grupos no isomorfos, 48 grupos superficialmente diferentes, y más de cuatro mil millones de grupoides. Pero también insisto en distinguir claramente la estructura del sistema de lo que es el sistema mismo, con el argumento de que es necesario mantener la distinción de los aspectos material y formal, o sea del substrato con respecto a la estructura o red de relaciones. Así pues, la palabra 'estructura' se refiere a esa red de relaciones que hace que un juego dado sea diferente de otros que utilizan las mismas fichas. Un argumento fuerte para apoyar esta distinción está tomado del lenguaje matemático mismo: se puede decir correctamente que un sistema tiene una estructura particular, pero no tiene mucho sentido decir que una estructura tiene un sistema particular. En el ejemplo de los grupos de cuatro elementos, hay sólo dos estructuras de grupo diferentes, cada una de ellas compartida por 24 grupos superficialmente diferentes pero isomorfos. Tiene pues mucho sentido decir que los 24 miembros diferentes de una clase de isomorfismo tienen la misma estructura, pero no tiene sentido decir que tienen el mismo sistema.

También distingo claramente la estructura de la dinámica. Hay sistemas puramente relacionales, con estructura pero sin dinámica, como los conjuntos parcialmente ordenados. Pero no hay sistemas puramente operacionales que no tengan estructura. Es verdad que uno puede presentar un sistema en una forma puramente operacional, como lo voy a mostrar más tarde con el ejemplo de la teoría de categorías; pero cada operación crea estructura a través de la relación (n+1)-aria que corresponde a cada operación n-aria. Por lo menos esta estructura implícita siempre existe. Pero el hecho mismo de que haya sistemas puramente relacionales, es decir, sistemas con estructura y sin dinámica, nos obliga a admitir que la estructura es diferente de la dinámica de un sistema dado. La estructura es algo más pasivo y estático; la dinámica es algo más activo y cinemático. Y esta distinción es muy poderosa cuando se trata de construir materiales curriculares, porque los estudiantes prefieren los aspectos dinámicos a los estructurales, y los textos prefieren lo estructural a lo dinámico. La misma distinción es muy útil para revisar y planificar investigaciones sobre el aprendizaje de las funciones, pues los estudiantes las entienden principalmente como transformadores, que son dinámicos y activos; pero los libros de texto las definen como un tipo especial de relaciones, o hasta como un tipo especial de conjuntos, a saber, conjuntos de parejas ordenadas. Pero tanto las relaciones como los conjuntos son pasivos y estáticos, y no capturan las construcciones activas y dinámicas que los estudiantes producen a partir de los sistemas concretos con los cuales ya están familiarizados.

Se puede decir que la potencia de una teoría se puede comparar con el poder de resolución de un microscopio o telescopio: es tanto más poderosa, cuanto mejores y más generadoras sean las distinciones que permite esa teoría. Otra distinción importante que permite la teoría PSG viene de la observación de que los sistemas matemáticos aparecen en tres capas de profundidad diferente: la capa superficial o visible está compuesta por sistemas simbólicos externos, que tienen sus propios elementos, operaciones y relaciones: la capa central o nuclear está compuesta de sistemas conceptuales, y la capa inferior o generadora está compuesta de sistemas concretos o familiares para los sujetos activos que construyen sistemas conceptuales a partir de ellos.

La diferenciación de esos tres niveles de sistemas matemáticos se puede visualizar como un rayo de luz que pasa a través de un prisma: el espectro está compuesto de sólo tres bandas de colores: la superficial, la central y la inferior (ver figura 1).

Un rayo de luz ya refractado a través de un prisma en general no se vuelve a partir cuando pasa a través de un segundo prisma. Pero en este caso podemos visualizar los tres tipos de sistemas como rayos de luz que atraviesan un segundo prisma y que producen a la derecha otro espectro de tres bandas: el conjunto de universos básicos en donde viven sus elementos o fichas del juego, que constituyen su substrato, el conjunto de operaciones básicas que constituyen su dinámica, y el conjunto de relaciones básicas que constituyen su estructura (ver figura 2).

Este artificio sencillo, tomado de la teoría de modelos, pero con la distinción clara entre los tres niveles de sistemas, y entre los tres aspectos de cada sistema, ayuda a tratar todos los temas de las matemáticas de manera coherente desde el Jardín Infantil hasta la Escuela de Postgrado.

3. Algunos ejemplos de utilización de la teoría PSG

La teoría PSG ayuda a distinguir entre operaciones y relaciones. Esta distinción es muy enriquecedora cuando se trata de formular proyectos de investigación sobre preconcepciones de los alumnos o sobre posibles estrategias curriculares.

La reducción usual de las operaciones binarias a relaciones ternarias brilla con luz propia, lo que permite extenderla a la reducción de operaciones unarias a relaciones binarias, de las operaciones ternarias a relaciones cuaternarias, y en general, de las operaciones n-arias a relaciones (n+1)-arias, pero distinguiendo claramente la reducción de la confusión.

A su vez, se puede ver que las relaciones ternarias también pueden reducirse a operaciones n-arias externas, que producen un resultado en un clasificador, por lo general en el clasificador booleano Verdadero-Falso usual, pero también podrían utilizarse otras semánticas. Esta distinción entre relaciones y operaciones, con las dos posibles reducciones que ella habilita, me ha resultado muy fructuosa para tratar los problemas que tienen los estudiantes con las matemáticas de las calculadoras de bolsillo y los computadores, y para aclarar el carácter operatorio que tienen las conectivas lógicas, incluida la implicación material, como diferente del carácter relacional que tienen otras construcciones lógicas como la implicación semántica y la sintáctica, que no son conectivas, pues son relaciones implicativas y no operaciones, como sí lo es la implicación material.

Otro ejemplo tomado de la lógica: la distinción entre las oposiciones medievales entre proposiciones cuantificadas, que es una tripla de relaciones binarias, y las operaciones respectivas, me ayudó a descubrir un nuevo grupo piagetiano en la lógica elemental, a aclarar las misteriosas oscuridades del grupo INRC de Piaget, y a distinguirlo de otro grupo isomorfo a él, pero diferente, que actúa únicamente sobre las implicaciones.

La utilización de la teoría PSG para revisar cuidadosamente la epistemología y la psicología genéticas de Piaget me ayudó a construir una filosofía de las matemáticas, que llamo 'el constructivismo genético', con su contraparte pedagógica, basada en la actividad de construcción de modelos a partir de esquemas recurrentes que se observan en procesos concretos y familiares.

Para explicar el aspecto genético de la teoría PSG voy a utilizar otro ejemplo de lógica. En lugar de introducir las conectivas lógicas usuales por medio de sus tablas de verdad, comienzo con el sistema concreto y familiar de las discusiones entre estudiantes acerca de las distintas interpretaciones de las reglas del fútbol o de otro de sus deportes favoritos. El intento de construir sistemas consistentes que reproduzcan algunos de los esquemas recurrentes en estas discusiones produce en primer lugar una lógica de conectivas temporales, diferentes de las usuales. Por ejemplo, la 'y' temporal no es conmutativa. Trátese de conmutar la frase: 'Me quito los zapatos y me quito las medias'. Es bastante difícil hacerlo en el orden inverso. Pero se puede construir un modelito simplificado de conectivas atemporales, y si se distingue adecuadamente entre las operaciones internas de tipo sintáctico sobre proposiciones, que producen una proposición nueva, y las operaciones externas de tipo semántico sobre las proposiciones hacia el clasificador usual Verdadero-Falso, que producen un valor de verdad, los estudiantes reconstruyen sin mucha dificultad las tablas de verdad y utilizan apropiadamente las conectivas usuales, sin necesidad de obligarlos a aprenderse de memoria esas tablas de verdad.

Tomemos otros ejemplos de la geometría. Los procesos espaciales, como moverse en un salón, moldear arcilla, construir mesas y casas, dibujar decoraciones en hojas de papel, etc., son muy familiares y concretos para la mayoría de las personas. A partir de esos subprocesos se pueden construir muchos modelos diferentes, con el fin de producir los esquemas recurrentes que aparecen cuando esos subprocesos se van desarrollando en el tiempo. Piaget anotó que regresar al punto de partida de un movimiento, o llegar al mismo sitio por medio de caminos diferentes, son procesos que preparan la construcción de los conceptos de invertibilidad y de asociatividad, lo que permite la construcción del concepto de grupo sin necesidad de introducir definiciones formales o axiomas. Lo mismo se puede decir de cualquier sistema matemático relacionado con la geometría.

En realidad, ¿que es una geometría? Desde el punto de vista de la teoría PSG se puede decir que una geometría es un sistema que tiene un substrato con al menos dos universos diferentes, estructurado por relaciones de incidencia que los conectan. Se pueden pues desarrollar geometrías puramente relacionales finitas e infinitas. Al agregarles transformaciones, que en general son operaciones unarias sobre puntos o sobre conjuntos, se está marcando la transición de las geometrías clásicas a las modernas, incluida la topología. Recuérdese que la sola introducción de una operación binaria activa, llamada 'adición', guiada por una ecuación cúbica dada (que es una relación estática), transformó la teoría de las curvas elípticas.

Esta visión filosófica de la geometría, tomada de la teoría PSG me permitió introducir desde el curriculo de primer grado hasta la escuela secundaria un tipo de geometría dinámica que no es la geometría de transformaciones usual. Una breve reseña de esta geometría apareció en las pre-memorias de la Conferencia del ICMI que tuvo lugar en Catania el año pasado, y una versión más completa aparecerá en el libro sobre enseñanza de la geometría que está preparando Vagn Lundsgaard Hansen.

4. Presentaciones de los sistemas

Así como los modelos no son únicos ni unívocos, dada la creatividad de los sujetos que los construyen, así también las presentaciones de un sistema conceptual no son únicas: cada tema matemático puede presentarse por medio de distintos tipos de sistemas. (La palabra 'presentación' está tomada de la teoría de grupos presentados por medio de generadores y relaciones).

Por ejemplo, en la teoría de categorías es posible presentar cada categoría como un sistema puramente relacional, con dos universos diferentes (objetos y flechas), dos relaciones de incidencia diferentes entre flechas y objetos (ser fuente y ser meta), una relación binaria de componibilidad entre flechas, y una relación ternaria que represente la composición; pero también puede presentarse la misma categoría como un sistema puramente operacional, pensando los objetos como productos de operaciones unarias sobre flechas (la fuente de y la meta de), y postulando sólo una operación binaria, la composición de flechas. (Técnicamente hasta se podría eliminar el universo de los objetos, identificando cada objeto con su flecha de identidad).

La teoría de modelos nos ahorra mucho tiempo, al proveernos de modelos para muchos campos conocidos de las matemáticas, y darnos ideas de cómo podemos modelar los nuevos. Pero siempre es muy instructivo tratar de presentar cada sistema por medio de otros tipos no ortodoxos de sistemas.

Por ejemplo, los espacios vectoriales se pueden presentar como sistemas con un universo único y una sola operación binaria interna, bajo la cual forman un grupo abeliano, que tiene también un rico conjunto de operaciones unarias, cada una de las cuales es un endomorfismo, y que forman ellas mismas un cuerpo bajo las operaciones apropiadas; pero también puede presentarse como un sistema con dos universos, uno para un grupo abeliano y otro para un cuerpo, con sus operaciones internas respectivas, más una operación binaria externa que conecta los dos universos.

Esta diversidad de presentaciones nos ilustra uno de los postulados básicos de la teoría PSG: los procesos son lo primario; los sistemas son siempre secundarios, como estrategias subjetivas que son para representar, intervenir o predecir la evolución de los subprocesos, y por lo tanto son variables, incompletos, siempre perfectibles y desechables.

5. Las fracciones y la teoría PSG

Estudiemos ahora una aplicación de la teoría PSG a los sistemas numéricos. La distinción entre los sistemas simbólicos externos, los sistemas conceptuales y sus sistemas concretos de donde provienen, me ayudó a producir una visión más clara de lo que llamo 'el archipiélago fraccionario', basado en la teoría de Thomas Kieran sobre las fracciones como operadores y como partidores. Esta visión le quita las arrugas a esa teoría y ayuda a planear una revisión completa de la investigación sobre fracciones y sobre números racionales que existe en la literatura, pues las fracciones pertenecen a los sistemas simbólicos externos y los números racionales forman por lo menos un sistema conceptual, y probablemente muchos.

Les propongo un acertijo que muestra algo del poder de la teoría PSG en estos aspectos de la investigación sobre fracciones y números racionales. Piensen en esta paradoja: un número racional no tiene numerador ni denominador. ¿Qué les parece?

Si los tuviera, yo pregunto si el numerador de la fracción mitad es par o impar. Puede ser cualquiera de las dos cosas: 1/2 o 2/4. Por lo tanto, lo que llamamos equivocadamente 'la fracción mitad' no es una fracción, sino algo más profundo: un objeto de un sistema conceptual que no tiene numerador ni denominador, y las fracciones no son objetos de ese sistema conceptual, sino de otro sistema simbólico externo para los mismos números racionales como sistema conceptual.

Los decimales son otro tipo de sistema simbólico externo para el mismo sistema conceptual, y también los son los sistemas de rectángulos cuadriculados y los sistemas de pizzas sectorizadas que son tan frecuentes en los libros de texto de matemáticas elementales.

Por ello podemos hablar de fracciones equivalentes: porque hay varias maneras simbólicas de representar el mismo racional. Así deberíamos hablar de rectangulaciones equivalentes y tal vez de 'pizzaciones' equivalentes (ver figura 3). Pero ninguna de ellas es el número racional del sistema conceptual relevante.

La falta de una adecuada distinción entre los subprocesos familiares a los estudiantes, como partir chocolatinas en partes iguales de volumen, o partir rectángulos en partes iguales de área, o partir segmentos de recta en partes iguales de largas (que son procesos no equivalentes); los sistemas conceptuales que los maestros quieren que sus alumnos construyan (los sistemas de números racionales) y los sistemas simbólicos externos que pueden utilizarse para representarlos (el verbal, el gestual, el decimal, el fraccional, el porcentual, el rectangular y el sectorial, entre otros) es lo que hace tan difícil elaborar proyectos de investigación consistentes acerca del aprendizaje de los fraccionarios, y lo que hace más difícil todavía diseñar los currículos, elaborar los listados de contenidos, escribir los libros de texto o programar el software educativo. Manejar esta distinción fundamental no es que haga fáciles esas tareas, pero sí las hace mucho menos difíciles.

6. ¿Dónde quedó el álgebra?

La eliminación de las fracciones de la lista de sistemas conceptuales importantes en el currículo colombiano de matemáticas elementales fue una jugada radical. Pero la teoría PSG me permitió atreverme a hacer una jugada más radical todavía: la eliminación del álgebra del currículo de matemáticas de la educación secundaria en Colombia. En realidad, en los programas de matemáticas de la renovación curricular en Colombia no aparece nada que se llame 'el Algebra'. Es que para mí, desde el punto de vista de la teoría PSG no hay ningún sistema conceptual matemático que se llame 'el Algebra', así escrita en singular y con mayúscula. Habrá tantas álgebras como sistemas simbólicos operatorios podamos inventar para representar cada tipo de sistema conceptual. Por ejemplo, habrá un álgebra lineal para los espacios vectoriales; habrá una o varias álgebras lógicas para los sistemas conceptuales de la lógica; hay al menos un álgebra conjuntista para la teoría de conjuntos, etc.

Al comienzo de los cursos de álgebra de la secundaria, se puede comprobar que lo que se considera como aprender álgebra es sólo un aprendizaje de un juego particular con un sistema simbólico externo para el sistema conceptual de los números reales. Por ejemplo, la letra x se usa para un número real genérico, y es más una incógnita que una variable real. No tendría sentido preguntar cuál es la derivada de x. No tiene derivada, porque las derivadas sólo están definidas para las funciones.

O los profesores de secundaria no caen en la cuenta de que sólo están enseñando a manipular símbolos sin ningún sistema conceptual subyacente, o sí caen en la cuenta de que lo utilizan para los números reales, pero pronto se olvidan de ello, y se deslizan inconcientemente hacia los símbolos para las funciones. De todas maneras, al nivel de los cursos de cálculo, ya los profesores están utilizando lo que ellos llaman 'álgebra' para representar funciones, pero no le dicen a los estudiantes lo que están haciendo. Por ejemplo, ¿han visto Uds. a algún profesor de cálculo que explique que ahora la letra x se utiliza para representar la función idéntica sobre los reales? En realidad así se utiliza, y por ello tiene como derivada la función constante uno (no el número uno).

La teoría PSG le ayuda a uno a ver que las funciones reales, o sea de valor real, de una variable real son precisamente las operaciones unarias de los sistemas de números reales, y que ellas mismas como objetos de un nuevo universo forman el substrato de un sistema de orden superior. Esto parece un hecho obvio, pero no aparece así en ninguno de los libros de cálculo que conozco. En realidad, en ellos las funciones se reducen a relaciones binarias de cierto tipo, o aun a conjuntos de parejas ordenadas. Pero esto no está de acuerdo con el sistema conceptual subyacente que ha sido construido a partir de las operaciones unarias sobre los números reales. Se puede ver esto muy claramente al estudiar los capítulos sobre relaciones y funciones que aparecen en los libros de cálculo, si se utiliza la teoría PSG para distinguir las relaciones binarias de las operaciones unarias, que son las funciones. Este análisis de los textos usuales de cálculo (y he revisado más de veinte de ellos) muestra que, con una sola excepción, hay un error en la transición del capítulo de relaciones al capítulo de funciones, o aun dentro del mismo capítulo, de la sección sobre relaciones a la sección sobre funciones, error que sólo se notaría si se asignaran ejercicios combinados de relaciones y funciones, ejercicios que prácticamente ninguno de los libros propone.

Tomemos por ejemplo la relación de ser raíz cuadrada. Sea 'xRy' una abreviatura de 'x es raíz cuadrada de y'. La pareja ordenada (x,y) debería estar en el grafo de la relación R. Pero ensayemos con (2,4). Ya habrán adivinado que (2,4) no está en el gráfico usual de la relación raíz cuadrada (ver figura 4). Peor todavía: si analizamos esa relación R por medio de las definiciones usuales, resulta ser funcional. Tomemos un número específico x y tratemos de encontrar otros dos números diferentes, y y y', tales que se cumpla tanto xRy como xRy'. No existe tal pareja de números diferentes, y según las definiciones en el capítulo de relaciones, eso quiere decir que la relación R es funcional.

Lo que sucede es que, más tarde, en el capítulo de funciones, cuando se trata de invertir las funciones, se señala que una función que no es inyectiva, al invertirse, no produce una función sino una relación no funcional; si se trata de invertir la función cuadrado, resulta una relación no funcional. Eso es correcto dentro del capítulo de funciones, pero al compararlo con el capítulo previo sobre relaciones, hay una contradicción clara, pero que pasa desapercibida para los profesores, los estudiantes, los editores de textos, y hasta para sus críticos.

Ensayemos ahora un ejercicio combinado de relaciones y funciones. Sobrepongamos el gráfico de la relación > correspondiente a 'x es estrictamente mayor que y' y el gráfico de la función cuadrado. Tomemos la intersección de esos dos gráficos, y la proyección sobre el eje de las x. Esto debe producir el conjunto de los números reales que tienen su cuadrado estrictamente mayor que ellos mismos. Hagamos la superposición y la proyección (ver figura 5). Podremos ver que ese conjunto consiste precisamente de aquellos números reales cuyo cuadrado es estrictamente menor que el número inicial. ¿Qué sucedió?

Así pues, la famosa álgebra de secundaria no es una verdadera disciplina matemática porque no es un sistema conceptual. Es sólo un sistema simbólico externo. Hay un álgebra para los números reales, que debe estudiarse para poderlos tratar a fondo y familiarizarse con ellos. Luego se introducen las operaciones unarias sobre el cuerpo de los números reales, y se las estudia por medio de calculadoras, computadoras, tablas, gráficos, instrucciones y algoritmos, y por medio de otros dispositivos gráficos que llamo 'molinos de moler números', con el fin de familiarizar a los estudiantes con los sistemas cuyos objetos son las operaciones sobre los antiguos objetos llamados 'números racionales' y 'números reales'. Cuando los estudiantes se pueden ya olvidar de esos objetos antiguos y trabajar directamente con esos monstruos activos que se alimentan de ellos, entonces sí se puede empezar a manejar un nuevo sistema simbólico para las funciones que sea apropiado para ese nuevo sistema conceptual. Ese nuevo sistema simbólico será un álgebra de funciones, que se parece mucho al álgebra de números reales, pero no es el mismo ni tiene la misma referencia.

Toda la literatura sobre investigaciones en educación matemática acerca del álgebra debe revisarse desde el punto de vista de la teoría PSG para tratar de especificar qué es realmente álgebra para números reales y qué es álgebra para funciones y para distinguir más precisamente los sistemas conceptuales de los simbólicos. Todo el currículo de pre-álgebra, álgebra y cálculo debe reestructurarse de acuerdo con esta teoría.

7. Conclusión

Así lo hice no sólo para el álgebra, sino para todo el currículo de la escuela elemental y de la escuela secundaria en Colombia. Se identificaron siete tipos distintos de sistemas conceptuales como dignos de atención en los programas curriculares de los grados primero a noveno, y uno más para los grados sexto a noveno.

Los tres tipos de sistemas más importantes para todos los grados son los sistemas numéricos, los sistemas geométricos y los sistemas métricos (y la distinción entre sistemas geométricos y sistemas métricos es crucial para el currículo, y muy clara desde el punto de vista de la teoría PSG). Además de éstos, están los sistemas de datos, los sistemas conjuntistas, los sistemas lógicos y los sistemas generalizados de orden superior, cuyos elementos son las operaciones o las relaciones de otros sistemas precedentes. Entre estos sistemas generalizados, se selecciona como el más importante para la secundaria el sistema cuyo substrato está formado por las operaciones unarias sobre los números racionales y reales, o sea por las funciones reales de una variable real.

Se recomienda tratar los sistemas de datos desde el primer grado en adelante. Los sistemas conjuntistas, los sistemas lógicos y los sistemas generalizados de relaciones y funciones (a excepción de los sistemas analíticos), se tratan sólo como herramientas, y no como objetos, según la distinción introducida por Régine Douady, hasta los dos o tres últimos años de la escuela secundaria.

Hablando en broma, y otras veces no tan en broma, algunos profesores de matemáticas en las escuelas secundarias de Colombia han dicho que a mí me recordarán en la historia de la educación colombiana por haber tratado de eliminar del currículo las fracciones y el álgebra, que son para ellos los temas más importantes y los que saben enseñar mejor. Pero yo no las eliminé del currículo: sólo las eliminé de la lista de sistemas conceptuales relevantes. Y lo hice porque no lo son: las fracciones son sólo uno de los muchos sistemas simbólicos para el sistema conceptual de los números racionales, y el álgebra usual en secundaria es sólo uno entre otros sistemas simbólicos para los sistemas conceptuales analíticos, cuyo substrato está formado por las funciones.

Espero haberles proporcionado algunos avances enriquecedores sobre las posibilidades que nos puede proporcionar la teoría de procesos y sistemas genéricos para hacer más comprensible la maravillosa empresa que son las matemáticas, para hacer más transparente la relación de ellas con la educación matemática, para hacer más visible y clasificable el currículo de las matemáticas escolares, y para captar que los programas de investigación en educación matemática pueden desplegarse e interconectarse en forma más orgánica y, sobre todo, mucho más hermosa.

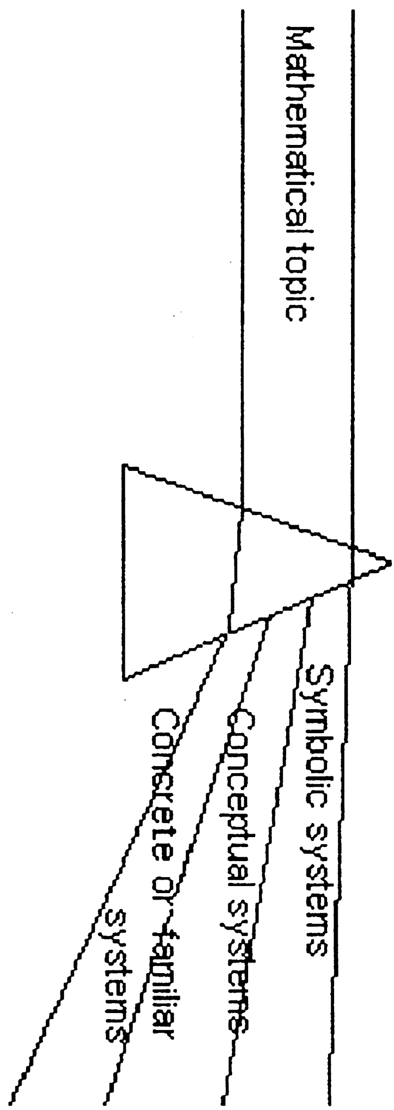

Mathematical topic
Symbolic systems
Conceptual systems
Concrete or familiar systems

Figure 1

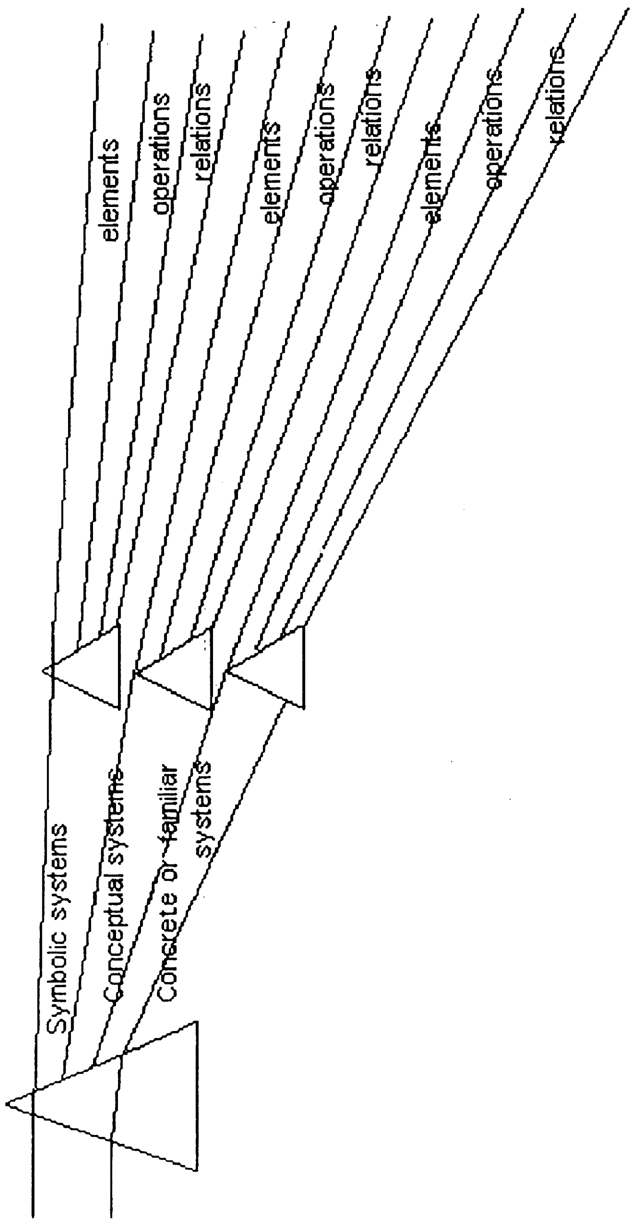

Symbolic systems
Conceptual systems
Concrete or familiar systems
elements
operations
relations
elements
operations
relations
elements
operations
relations

Figure 2

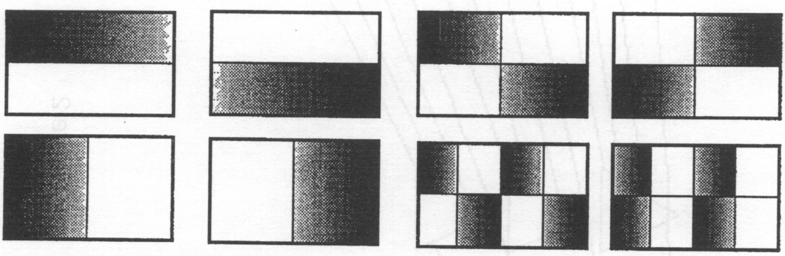
DIFFERENT DISGUISES OF THE HALVING MONSTER

DISFRACES DIFERENTES PARA EL MONSTRUO MITAD

Decimales equivalentes
Equivalent decimals   **0.5 .5 0,5 0.50 0,500**

Fracciones equivalentes
Equivalent fractions $\frac{1}{2}$ $\frac{2}{4}$ $\frac{3}{6}$ $\frac{5}{10}$ $\frac{50}{100}$ $\frac{500}{1000}$

Rectangulaciones equivalentes
Equivalent rectangulations



Pizzaciones equivalentes
Equivalent pizzations

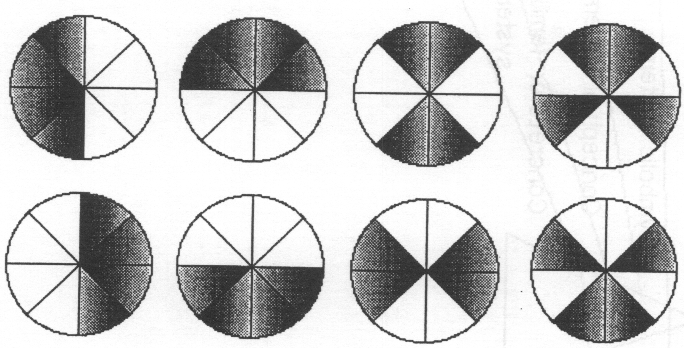


Figure 3

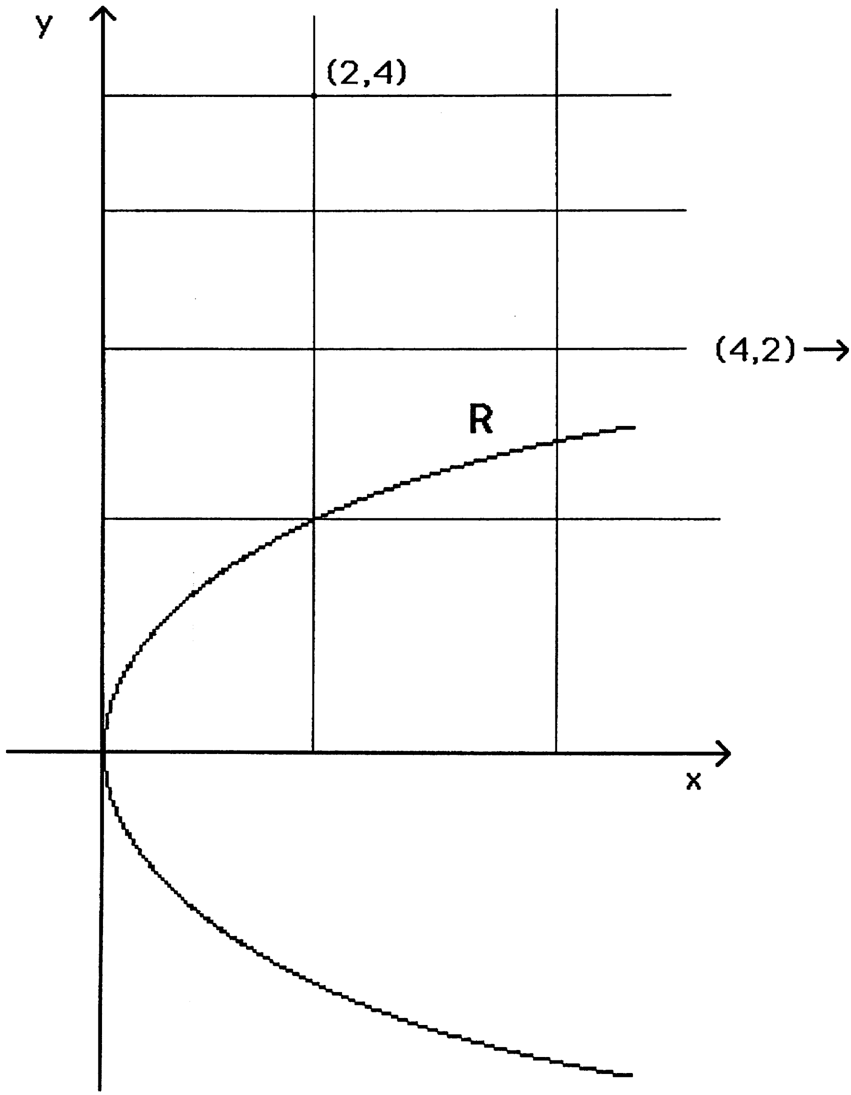


Figure 4

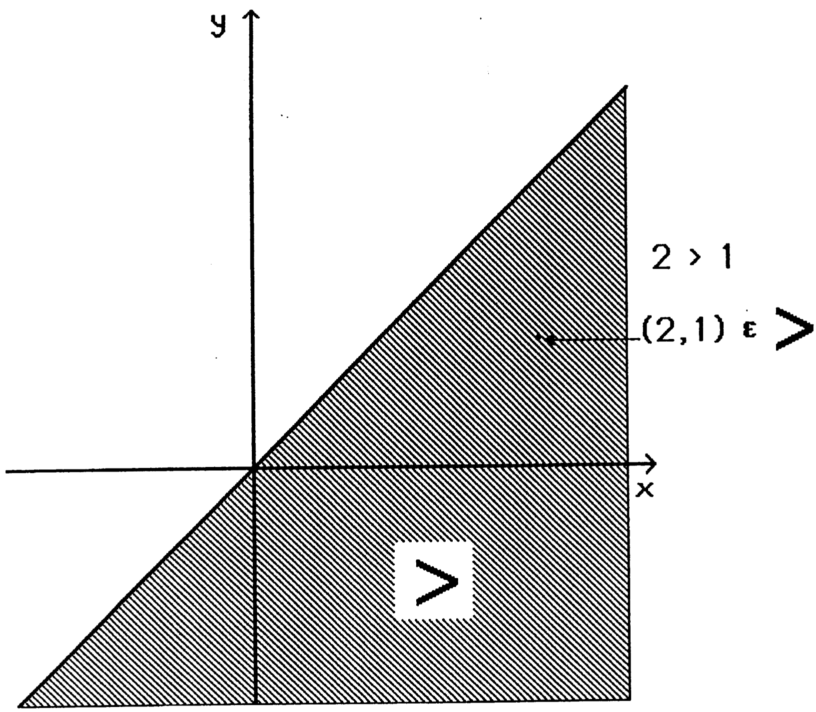


Figure 5a

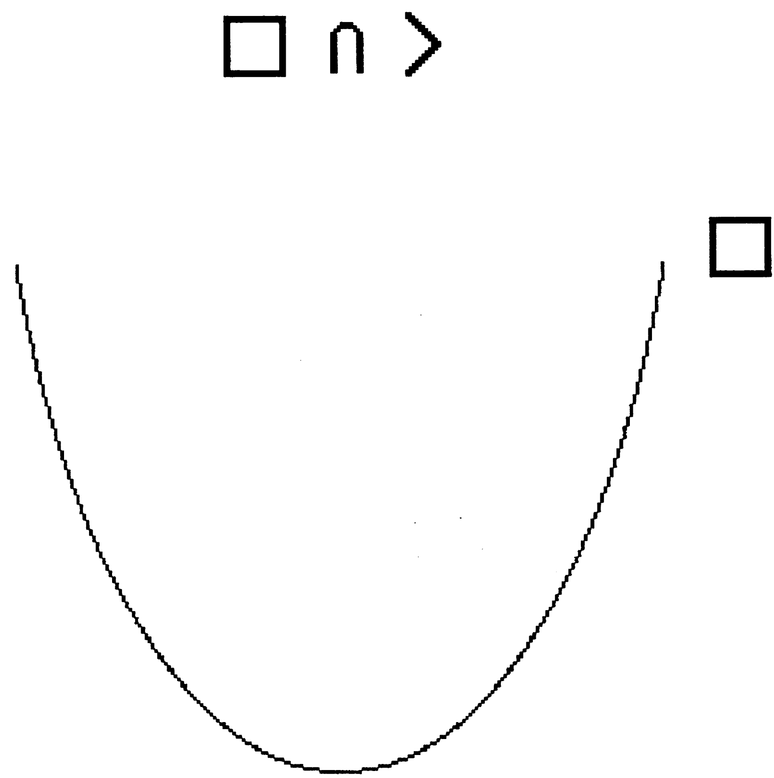


Figure 5b

PHILOSOPHY OF MATHEMATICAL EDUCATION: A PHENOMENOLOGICAL APPROACH

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I. EXPLICITING THE MEANING OF PHILOSOPHY MATHEMATICS

1.1. A little bit of history

In 1980 I begun to work with Philosophy of Education with students of a course of teachers of Mathematics. Such a experience made me wonder about the possibility of naming this school subject of *Philosophy of Mathematical Education*. I, then, suggested it to university professors who had already worked with the Teaching of Mathematics. They guaranteed me that papers having this name had never been presented either in congress or in international or national meetings. They also guaranteed me that it was very common to have papers, articles, books which focused upon Psychology of Mathematical Education, Didactic of Mathematics and specific topics of teaching Mathematics. Nevertheless, they had never seen *Philosophy of Mathematical Education* as topic of research before.

From that moment on I attempted to find out the work which had been carried out in such area of teaching and research. I realized that many themes treated by Philosophy such as Epistemology, Ontology and Axiology used to be merged in arguments and discussions in those school subjects without being highlighted and treated in accordance with the strictness and theoretical basis of Philosophy itself.

In January 1982, in the path I had followed inquiring and doing research about relevant topic to Philosophy of Mathematical Education, for the very first time, I came to know of a work which had been named Philosophy of Mathematical Education. It is a doctorate thesis of Eric Blaire[1], presented at the Institute of Education of the University of London in December 1981.

1 Blaire, Eric *Philosophy of Mathematics Education*. London. Institute of Education, University of London. 1981 (Doctorate Thesis)

It is a study which joins Philosophy of Mathematics. In its first section, while it takes on questions about mathematical objects on epistemological and ontological basis, it describes the three traditional schools in Philosophy of Mathematics, logicism, formalism and intuitionalism, aiming at constructing a fourth one which is named hypothetical in the lights of Pierce's and Lakatos ideas.

In its second section, it presents different ways of teaching Mathematics and identifies logical connections and some times contingents which the author realizes between the philosophies of Mathematics addressed in the first part and these teaching practices. It draws four perspectives, the teaching of Mathematics as language, the teaching of Mathematics as a game, the teaching of Mathematics as a member of natural sciences and the teaching of Mathematics oriented to technology. The author argues that it is possible to draw a fifth perspective as teaching of Mathematics recognized as an interdisciplinary perspective.

In its third section, it draws attention to the concept of Education, to the objectives of Education and points out what is essential to be treated in courses addressed to teachers of Mathematics.

This way, Blaire works the Philosophy of Mathematical Education, based on the Philosophy of Mathematics and on the Philosophy of Education which provide him with the support to the analysis of teaching practices of Mathematics and to introduce pedagogical proposes which focus on the formation of teachers of Mathematics.

From 1982 to 1992, I carried out my work on Philosophy of Mathematical Education. In the meantime, I came to know important pieces of research and far reaching books by authors worldwide known and recognized in the area of Education of Mathematics without having mentioned the Philosophy of Mathematical Education, however.

In order to mention some, amongst the most meaningful ones, I point out Hans Freudenthal, mainly his book on *Didatical Phenomenological of Mathematical Structures*,[2] the work *Theory of Mathematical Education* which was one of the ICME-5 topics, joining researchers such as H.G. Steiner, N. Balacheff, J. Mason. H. Steinbring, L.P. Steffe, H.Brousseau, T.J. Cooney, B. Christiansen.

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2 Freudenthal, Hans. *Didactical Phenomenological of Mathematical Structures*. Dordrecht: D. Reidel Publishing Co. 1983.

I would also point out Gila Hanna, Michael Otte, Ubiratan D'Ambrósio, whose studies discuss aspects concerning the Mathematical reality, the epistemology which underlies the doing and the teaching of Mathematics. They also analyze and criticize the pedagogical practices, mathematical practices and the tendencies of teaching of Mathematics, providing explicit positions on the topics addressed.

In 1992, owing to ICME-7, I found the book under the particular name of *The Philosophy of the Mathematics Education*[3] by Paul Ernest. In this book, Ernst takes the task of explaining the title, and borrowing Higginson's[4] postulations "(he) identifies a number of subjects of foundations to Mathematical Education, including the Philosophy. A perspective of Mathematical Education, he argues, gathers a different set of problems from those regarded from any other point.[5] "

Based on this elucidation, he distinguishes, as the most relevant, four sets of problems and questions to the Philosophy of Mathematical Education, as follows:

• Philosophy of Mathematics which addresses questions such as: what is Mathematics and how can we explain its nature? Which philosophies of Mathematics were developed?

• The nature of learning, focusing on questions as such: which philosophical statements, possibly the implicit ones, underlie mathematical learning? Which epistemology and theories of learning are assumed?

• The objectives of Education, emphasizing questions as such: which are the objectives of Mathematical Education? Are its objectives valid? Who benefits and who loses?

• The nature of teaching, focusing on: which philosophical assertions, possibly the implicit ones, bear mathematical teaching? Are these assertions valid? Which means are adopted to achieve the objectives of mathematical teaching?[6]

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[3] Ernest, Paul, *The Philosophy of Mathematics Education*. London. New York. Philadelphia. The Falmer Press. 1991.

[4] Higginson, W. "On The Foundations of Mathematics Education" *For the Learning of Mathematics*. 1 (2) (1980) pp.3-7.

[5] Ernest, P. op.cit. P.XII

[6] cf. Ernest, P. op.cit. p.XII and XIII

Ernest divides his books into two parts. The first one addresses the Philosophy of Mathematics, focusing on the logicism, the formalism and the intuitionalism in accordance with the category of the Philosophy of Absolutistic Mathematics, exposing his criticism, arguing about the implicit fallacy in these philosophies and enlarging the scope of understanding with the arguments of the falibilistic vision of Mathematics. He draws out the grounds for his criticism about Lakatos' work and about Constructivism, working out a conception of one Philosophy of Mathematics which is supported by Social Constructivism.

In the second part, he exploits the Philosophy of Mathematical Education, showing that many aspects of Mathematical Education lie on philosophical assertions.

In 1994, I came to know of the book by Ole Skovsmose intitled *Towards a Philosophy of Critical Mathematics Education.*[7] The title itself indicates that the author takes the Philosophy of Mathematics Education as being critical, according to his conception. Skovsmose attributes to the criticism the meaning of an Education which is kept as social and political power in a society whose nature is critical, punctuated by crises and conflicts.

He is based on the works by contemporaneous authors such as Adorno, Habermas and Paulo Freire, without losing the sight of a historical vision of the term *criticism* shown in Kant and Hegel, interpreted by Marx. He goes farther concerning the meaning of the term, visiting the authors of the School of Frankfurt, mentioning Max Horkheimer, Herbert Marcuse.

In order to elucidate the idea of critical Education, he highlights crises, criticism and emancipation.

To Skovsmose[8] criticism has a double meaning: it means criticism of some opinion and criticism of some real situation, of some aspect of real life. Crisis "is a metaphor to a situation to which people are to react through criticism"[9] . Emancipation has various aspects. It refers to the liberation of stereotypes of thought, when it would be the result of an ideological

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7 Skovsmose, Ole. *Towards a Philosophy of Critical Mathematics Education.* Dordrecht, Klumver Academic Publishers. 1994

8 idem. chap. 1.6.

9 idem.

criticism. It also refers to the liberation of material obstacles, as it is the case of setting people free from slavery.

The author builds his concept of critical Education based on these aspects, in a way that, being critical, Education should be aware of social inequities, trying to abolish them and without lengthening the existent social relationship. He exposed the different ways through which Education could react to the critical nature of the society. He sums up his thought, stating that critical Education involves two major interests: it has to recognize the different ways through which society is reproduced, it has to attempt to compensate these reproductive forces, it has to offer an equal distribution of what school can offer, it has to provide children with competence which enables them to identify and to react to social repression. Thus, in his book, he shows how he develops topics of Mathematics, in order to provide students with competence to react to social repression, being critical.

1.2. Explaining the meaning of philosophy of mathematical education

The meaning I have given to the Philosophy of Mathematical Education has been built throughout my work with Philosophy of Education itself.

While working with Philosophy of Education I have always highlighted Education as the main focus of my attention, bringing the systematic of philosophical doing to help me to understand this phenomenon, Education, in its complexity.

What does that mean?

It means I regard Education as phenomenon and I try to see its different perspectives of taking place: in school, in family, in books, in media, in multimedia, in life. After all, I try to understand what is shown in the real life in order to understand what characterizes it and then interpret it in the lights of the ways it takes place and in the lights of the world where it occurs.

Hence, Education is not regarded through filters, that is, through theoretical conceptions which define it previously, such as those that seek for these conceptions in Philosophy, in Psychology, in Anthropology, and so forth. Education itself is regarded as the focus of the investigation which is carried out in a multi-disciplinary way.

This procedure is sustained by the systematic of work of Philosophy which has as its focus a reaching, systematic and reflexive thinking. It focuses the everydayness life of Education, providing topics to the aspects of educational doing such as the relationship between teacher and student, teaching, learning, evaluation, curriculum, school, describing the ways through which this doing takes place, analyzing them and reflecting about the meanings which are constructed. The doing of Philosophy of Education allows me to understand and to interpret what is done while education takes place. It allows me to understand pedagogical proposals and the sense that theories which study educational issues make. It is undoubtedly a meditative doing which leads to self knowledge, to self criticism, and, therefore, to the knowledge and criticism of the world.

It is important to emphasize that in the conception of Philosophy of Education which I assume, reflection only takes place if it focuses upon an action properly analyzed in its genesis. It does not take place on the top of abstract thinking, in the sphere of subjectiveness of a subject, apart from the lived reality. Accomplishing it already takes place in the process of an intervening action in the educational reality. Hence, its characteristic doing does not reject what exists, but it assumes it as being the world where action occurs and where the analyses and the reflection can be possible. Therefore, Philosophy of Education is constructing knowledge and, at the same time, it is a continuous evaluation and criticism of this construction.

This work demands the presence of the others, researchers, authors who have already written about the studied topic, subjects present to the educational studied situation. This work always takes into account the presence of the world, horizons of interpretations, field of lived experiences.

In this perspective, Philosophy of Education is characterized as a criticism of education. This criticism is understood in the philosophical sense which has reflection as its focus, and which accomplishes knowledge of the genesis, i.e. of the creation of ideas, pedagogical proposals and educational action present to the investigated everydayness life. This task is important to prevent the agents of Education from getting lost in fashionable theories, in words of command determined by political discourse, in the density of the school world, helping them to keep lucid, understanding their doing, being able to choose and defining the path to be followed.

This conception and procedure in Philosophy of Education has delimited my work in Philosophy of Mathematical Education.

I understand Philosophy of Mathematical Education as a reaching, systematic, and reflexive study of Mathematical Education, as it appears in everyday life. Mathematical Education is the main focus. Understanding it demands having it reflected on what is done. Therefore, in this perspective, Mathematical Education is regarded as a whole which appears in different ways, down the streets, in theories, in culture, in curriculum, in legislative rules, in the educational policy, in media, in multimedia.

To ask "WHAT IS MATHEMATICAL EDUCATION?" leads to the way of the investigation, in the sense of seeing what is common to the different ways through which it appears, keeping it as Mathematical Education. This investigation demands analyses and interpretation of data, a logical work to reunite what is constant in the multiple appearances and in the reflexive work to accomplish criticism, searching for the sense of what is taken in the world of Mathematical Education.

Philosophy of Mathematical Education cannot be taken as Philosophy of Mathematics, nor even as Philosophy of Education. It can be distinguished from the former for it does not have the theme reality of mathematical entities nor the construction of their knowledge as its goal. It can be distinguished from the latter because it does not deal with specific issues, nor even with the ones that are peculiar to it, such as: purposes and aims of Education, nature of teaching, of learning, of school, and of school curriculum. Nevertheless, in spite of being distinguished from both - Philosophy of Mathematics and Philosophy of Education - Philosophy of Mathematical Education is sustained by their studies, deepened in specific themes which can be detected in the interface which it has with both areas. It supports both of them with their own research and reflection, as it is supported by them, at the same time.

Philosophy of Mathematical Education deals with the issues studied by Mathematical Philosophy, regarding them from the point of view of Education. This way it focuses topics like reality of mathematical entities, on knowledge of mathematical objects, value of mathematics, characteristics of mathematical sciences by studying them through the mathematical perspective. In order to work in this perspective it is necessary to be based on studies and reflexive analyses of Philosophy of Education. By doing this it gets the necessary strength to understand how having the conceptions of reality and of knowledge of mathematical entities take place in the way the teacher of Mathematics teaches and evaluates his/her students, in the proposed curriculum, in the ways people deal with their everyday work, such as building houses, preparing and organizing the soil for plantation, trade exchanges and manipulation of technology.

From the point of view of the value of mathematical entities or of Mathematics, Philosophy of Mathematical Education approaches the matter of the position of Mathematics in the school curriculum, in the way the society values it and, how this valuation interferes in the school evaluation and in the selection of the most capable ones in a particular community. This is a study which cannot be carried out without comprehension of the ideology which is present in the way of seeing, of evaluating Mathematics, interfering, therefore, in the conceptions of the reality of mathematical entities and of the knowledge of these objects.

Hence, Philosophy of Mathematical Education imposes itself as a thinking about reaching themes to cover the field of Mathematical Education. This does not mean that it can be reduced to Philosophy of Mathematical Education. It only means that the last one reflects and thinks reflexively the Mathematical Education, trying to know and to interpret what has been and what is being accomplished. This a meditative thinking which leads to self knowledge, to self criticism, and also outlines the identity. It is in this way that Mathematical Education is strengthened and, at the same time, it discerns future perspectives and provides support to its choices.

In the field of activities of Mathematical Education, I understand that the following topics represent convergence to be taken as center of reflexive and critical analysis by the Philosophy of Mathematical Education:

- conception of Education and of the Mathematical Education;
- conception of reality and of knowledge;
- conception of reality of mathematical entities;
- attitude and pedagogical didactic aims of the teaching of Mathematics work.

2. Approaching mathematical education in a phenomeno logical perspective

I will resume the topics appointed as important to the Philosophy of Mathematical Education in order to treat them according to the phenomenological concept, demonstrating how the Phenomenology addresses them. I have decided not to establish a parallel between the conceptions and the relevant guiding to Phenomenological Attitude and those of **Natural Attitude**[10] , but to show conceptions of Phenomenology which support pedagogical practice and its analysis and reflection.

2.1. Conception of education and the mathematical education

In the phenomenological conception, Education is taken as phenomenon which is shown to consciousness which, in its intentionality, comprises it, making its sense to take place in its diverse ways of showing up.

Phenomenon means what is shown, what is manifested to consciousness. The meaning of counsciousness will be carried out in the next item 2.2

The essence of the *phenomenon Education* is understood by Phenomenology as being the care taken with the project[11] of human being taking him in his possibilities of being wordly and temporal. Project which throws man in his being, hence, in his acting.

The interpretation of this statement is constructed in the network of meanings attributed by Phenomenology to Ego, to Other, to truth, to reality of the life-world. These issues make the nucleo of the development of phenomenological thinking exposed in the works by Edmund Husserl, Martin Heidegger, Hans G. Gadamer, Paul Ricoeur and Maurice Merleau-Ponty, just to mention the most renowned founders.

The life-world[12] is the universal field of lived experiences, it is the horizon where one is always conscious of objects and of other fellows. In the school life-world there are students, teachers and cultural objects

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10 The Natural Attitude is characterized by conceiving the things of the natural world as positive contents thought as distinct of the phenomenon and its manifestations. Either the thing which becomes object to the subject, or the mind which operates the relationships of knowledge. In the Phenomenological Attitude, the thing is not taken as having objective existence in itself, thus a) it is not beyond its manifestations and, therefore, it is relative to perception and dependent of consciousness; b) consciousness is not part or region of a field larger, but it itself is a whole which does not have anything out of it. It is by being understood and comprised the world that it makes the world make sense to subject. (cf. Moura, C.A.R. *Crítica da Razão na Fenomenologia* - São Paulo - Nova Stella & EDUSP, 1989)

11 Pro-ject is the act of throwing ahead, permitting the human possibilities to be updated.

12 cf. Husserl, E. *The crises of European Sciences,* Evanston. North Western University Press. 1970.

which are already given to the consciousness of those who live in this horizon. The last ones are given as intentional objects, therefore, comprised by consciousness activity. The *others* are others of each *Self* making themselves present in their own body and in their psyche[13] . Each student, being Ego, is a pole of intentionality, a zero-origin from which the perspective of the world is drawn. This is an existential comprehension which everyone develops from himself, pole of identity, incarnated body which is modility, intentionality, desire, communication and also comprehension of *other* which makes sense by being with in the life-world, horizons of meanings of existence and the of cultural objectives.

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The school life-world is in the *School,* secular institution which is history and whose meaning has been constructed over the years. The meaning of School takes place in school everyday life experienced by its agents: teachers, students, member of the family, clerks, pedagogical and technical staff, etc. Therefore, this sense takes place in teaching, in learning, in evaluation, in wish, in want, in repudiation. It takes place where the ideal[14] objects permeate pragmatically contents from which they are transmitted by spoken language, by images and sounds, gestures and writing.

The Phenomenology works the educational project in the everydayness of school. It pursues the sense which actions of the school life-world make to their subjects and it sends the thinking of this sense in a vigorous and systematic way, putting forward to the analysis and criticism and self-criticism. This is the real work of Philosophy of Education: being aware of action or being conscious of what is done, analyzing and reflecting the done in a systematic and rigorous way, pointing out possible paths and their implications in the educational project.

The highlighted aspect in the phenomenological approach is to understand the essence of Education as human project. It means the Phenomenology does not address Education as a natural object, possible to become known by the means of the representations by language signs or signs possible of being decomposed into parts of a process programmed in sequences of aims and operations displayed in time. It means it works Education as opening, as possibilities which are accomplished in the human temporality in which actions and decisions outline paths,

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13 cf. Husserl, E. *Cartesian Meditations,* op. cit.

14 The mathematical objects are idealities, according to the exposed in Husserl, E. "The Origin of Geometry" in Husserl, E. *The crises of European Sciences,* op. cit.

making history. Possibilities, decisions and signs reflected, by searching, the Education, the consciousness of the sense which world and life make to each one and for everyone at the same time.

Therefore, the Phenomenology does not have as its starting point a concept of Education or of a particular educational proposal as the most plausible and valid one. But it searches in the school life-world itself for the sense of what one does, the sense of time and of history, and the sense of ideologies, of theories and of pedagogical practices which permeate and base accomplished action.

The Mathematical Education is also seen as phenomenon by Phenomenology. Thus, as a totality which is shown in the everyday life-world through perceptions of subject aware of it. Hence, Mathematical Education is a human being project which is projected in the possibilities of man being worldly and temporal, and understanding mathematical relationships perceived in the life-world, expanding them creatively by using in the interventive action in everydayness lived.

To assume a Phenomenological attitude when we work with Mathematical Education means to search for the sense of what is done while teaching and learning, the sense of the mathematical contents transmitted in culture, those of common sense and of the everydayness lived by subjects, those transmitted by books, specialized magazines and in the academy. The sense of ideologies permeate the network of the meanings of mathematical conceptions, of pedagogical conceptions, of educational practice. It is on attempt to understand the sense which the world makes to each participant of a particular process of teaching and learning, searching for points of intercession of the horizon of comprehension. It is to be aware of *other, co-subject* of the life-world, observer of the comprehended and nuclear presence in self-knowledge process. It is to proceed constantly and systematically to the analysis, to the reflection and to the criticism of the acceptable truth.

2.2 Conception of reality and of knowledge

In the phenomenologial perspective the real is given as an everyday temporal and historical dynamic whole, perceived in the man-world encounter, not apart from that who perceives it, who speaks of it, who interprets it, constructing a network of meanings in the inter-subjectiveness by sharing and communicating interpretation. In the preface of *Phenomenology of Perception*, Marleau-Ponty states:

> "The real is a closed woven fabric. It does not await our judgement before incorporating the most surprising phenomena, or before rejecting the most plausible figments of our imagination."[15]

The solid woven fabric which is given to the lived experience. And this is the world: lived, reflected, communicated and shared experience. "The world is not what I think, but what I live through."[16] The reading of this statement leads to the interpretation that the world is not what is postulated about it or what is stated about it or an object susceptible to be owned or representated. But, "it is natural setting of and field of all my thoughts and of all my explicit perceptions."[17]

The real is understood as reality lived in spatiality and temporality of the life-world. Therefore, in the perceived perception of time and space and its convergences which join modalities of perception of each subject and of several subjects.

From reality, the subject is integrated and constituent part who perceives together with other subjects, mates and co-subjects of this reality. Subject and reality do not separate from each other. There is a constant movement between perception and acts which generates meanings attributed to the perceived one, to the perceived thing. This movement is the noesis - *noema* process - *noesis* refers to perception of acts, of providing sense, of logical organization of those data and *noema* refers to the perceived one.

All this process is worked by Husserl under the name **transcendental reduction**[18] , that exposes how understanding of the world takes place. Being so, to clarify the meaning of reality in the phenomenological approach implies to expose meaning to knowledge.

I am going to focus on some nuclear topics of phenomenological conceptions of knowledge and of reality, considering the impossibility of treating, in this article, the transcendental reduction in its complexity and with the strictness which Husserl treats it in his several works.

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15 Merleau-Ponty, M. *Phenomenology of Perception.* London and Henley. Ronthege & Kegan Paul New Jersey. The Humanities Press. 1978 P. X

16 Idem. p.XVII

17 Idem. p. XI

18 cf. Husserl, E. *The crises of European Sciences.* op.cit.

The emphasis has to be given to consciousness and reflection, since to understand these allows me to understand the way through which Phenomenology conceives the real and the superation of momentaneous of *noétic* acts.

To Phenomenology consciousness is intentionality.[19] It is the act of being aware of..., of being drawn to... Intentionality is the essence of consciousness. It comes from the Latin verb *intento, tendi, tentum, ere* which means to be inclined to a direction, to enlarge, to be inclined to open, to make one aware of, to support, to provide intensity, to state boldly[20]. These meanings allow me to understand consciousness as expansion to the world, being opened to... In this aspect lies the difference between understood consciousness, in the natural attitude, as thing, as recipient, as modelling as part of the world and consciousness understood by Phenomenology as intentionality, as movement of being understood to...[21] This *something* to which consciousness is expanded to is not only something visually present, but it comprehends wish of acting, of effecting itself in which the existence takes place.

Hence, consciousness does not have anything out of it, for the movement of being expanded to it involves objects of its existence and because of it these objects are always intentional.

It is in this aspect that lies the core of the difference between Natural Attitude and Phenomenological Attitude. To Phenomenology, every object is an intentional object. This is a *noésis-noema* synthesis, or rather, of existential acts and of their products.

The consciousness is also expanded to it itself, to its own acts. This is a reflexive movement, through which it understands its own existence, allowing itself lucidity of its acts. Through this movement, there is the possibility of consciousness include self comprehension, self knowledge, self criticism. It is a retrospective perception focusing on manifestations of primary perceptions. This is the sense of *Erlebnis,* or of experiencing.

To reflect is an act. And as such, it is susceptible of becoming an intentional object whose acts of reflection can be focused on. It is a movement of going one step behind and regarding the lived, the done, and

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19 Husserl, E. *Ideas* op.cit. p. 198

20 Gaffiot, F. *Dictionnaire Latin / Français.* Paris. Hachette, 1934.

21 Husserl, E. *Ideas.* op. cit. p.199

the accomplished one. This involves the distance and, at the same time, the reflection on lived experience. This is the sense of *transcendence*, in Phenomenology: a retrospective perception of the invariance of the lived one. Hence, the Transcendental Phenomenology is posed as a criticism of knowledge and, also, as a criticism of the totality of human experience which has self-criticism as a founding one.[22]

This reflexive movement which leads the consciousness to self-knowledge enabled the interpretation that Phenomenology operates in an introspective way and that what it does, even if proceeding strictly to the *transcendental reduction*,[23] is to reach to a solipsist *Self* which is self-comprehended and that creates the world.

However, the possibility of a conception of solipsist ideality of the *Self* and of reality is definitely overcome when the husserlian Phenomenology considers the presence of the *other* in the life-world.

Before addressing the theme of *Other* and of intersubjectiveness, we shall focus on *perception*, for it is the key of the encounter man-world and, thus, of knowledge and of construction of worldly reality, in the phenomenological perspective.

It is important to make it clear that Phenomenology perception is not taken as sensation and it is not susceptible of being decomposed into parts and made by the sum of sensations either. In *Phenomenology of Perception*[24] and in *Primacy of Perception*[25] Merleau-Ponty carries out a very deep study of this theme, exposing philosophical consequences deriving from the attitude of admitting perception as primacy of knowledge of the world. To this author, perception does not reveal the ideal and the necessary, not even the transitory occurrence free from chains and independent of the world. He states that it occurs in scope of the whole. The subject who perceives and who takes a seen point is the incarnated-body, field of perception and of action, who makes syntheses in his aimed horizon.

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22 Zaner, Richard M. *The way of Phenomenology. Criticism as a Philosophical Discipline*. Indianopolis. The Bobbs Merril Co. Inc. 1970. p.194

23 Merleau-Ponty, M. *Phenomenology of Perception*. op. cit.

24 Idem

25 Merleau-Ponty, M. "The primacy of Perception and its philosophical consequences" in Merleau-Ponty, M. *The primacy of Perception* . Northwestern University Press - USA, 1964. p.12-43

The perception offers truth as *presences*. This means, according to Merleau-Ponty, that our relationships with the world is not that of a thinker with the object of thought or that of the unit of the perceived thing, perceived by many consciousness, that is that one which is given in proposal. It does not mean that the perceived is comparable to the real either.

Presence is to take part of the moment in which things, truths, values are constituted. It is the instant in which the sense takes place. Owing to this, Merleau-Ponty states that perception is constituted as a nascent logos.[26] The statement that *"the matter of perception with its pregnant of its form*[27] comprises the idea that every perception takes place in an horizon, in the world and in the action of the incarnated body itself. Thus, the distinction classically made between *matter* and *form* does not proceed. The matter is already pregnant with its form, for it is in the action, in the moment that the perception unifies, that the form is constituted with the matter. This unification is, according to Husserl, a synthesis of transition or of the identity which processes the unity of the perceived objects.[28] Therefore, it does not refer to an intellectual synthesis which apprehend the object as possible or necessary, but it is a synthesis in which the object is given as a series of perception in profiles, though, in its totality, it is not given in any of them. Thus, in the perception the object can be given in deformed way. It refers to the deformation deriving of the perceptible taken from the zero-origin, which is that point of the incarnated body.

However, in this point of view, although the perceived **thing**[29] is shown in multiples ways, it is not lost in the multiplicity of perceptions. According to Husserl[30] there is always an unit which permeates the multiple ways through which perception of things takes place, formed by *synthesis of transition.*

This game between multiplicity, typical of perception, which is by profile, and identity of intentional object defines transcendence of object

26 Idem p.25

27 Idem p.15

28 Husserl.

29 Thing is not something which is imposed as true to all inteligence, not something felt in privacy of individual perception, but it is what is there in its concrete aspect, in the texture of its qualities itself. cf. Merleau-Ponty, M. "An Unpublished Text" in Merleau-Ponty, M. *The Primacy of Perception.* North Western University Press. 1964, p.48.

30 cf. Husserl, E. *Cartesian Meditations,* op.cit.

related to psychological aspects. Thus, intentional object is polo of identity intrinsic to lived experiences. However, it is also transcendent to these existence by being perceived as identical or invariant in temporal flow of lived experiences.

To Husserl, the activity which joins multiplicity, so that the identical is perceived, is the *essential intuition*. It is in this act that *essence*, is intuited, enabling the essential evidence of the phenomenon. The essence defines an intentional object with characteristics, for it is given in the *essential intuition*. These are the acts which generate the ideal objects or idealities. *An empirical intuition or individual can be converted into essential intuition (ideal)*[31] *and the essential intuition is also intuition and not a representation.*[32]

Although idealities are generated in the *essential intuition*, they acquire form, they are kept and perpetuated in an objective way in language, in the relationships between subjects, in culture and in history. That is why idealities of ideal objects exceed psychological existences and perspective multiplicity, in which the phenomenon is shown to the subject. They are taken from the subjective sphere, projected to the intersubjective and to the objective ones through language and through the presence of the *other*, in the horizon of life-world. Hence, in phenomenological perspective, the idealities do not exist in the abstract level, subjective or not, and non-temporal, but they are cultural objective realities, temporal and worldly.

To move on to the sphere of intersubjectivity and of objectivity, we are withdrawing from the possibility of interpreting husserlian Phenomenology as abstract and founded in solipsism of a pure Self. And here I emphasize that the *Other* and the *Intersubjectiveness,* also central themes of conceptions of knowledge and of reality, in this approach.

For Husserl, perception is temporal, worldly and carnal.[33] He states that the act of perceiving takes place in the present. However, it always takes place in a temporal horizon where past and future are also present in a continuous flow of retention and of procrastination. Perception and perceived thing take place in horizon-world, in perspectives, when the sense gains space and the perception is processed. It is like this that inten-

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31 Husserl, E. *Ideas*, p.20

32 Husserl. E. *Cartesian Meditation*, op. cit. 116/117.

33 Husserl, E. *Cartesian Mediattions*. op. cit. p. 116 / 117.

tional objects gain existence to the consciousness having meanings attributed to them through the way they happened in the horizon-world. It in this way that the reality of the the life-world goes through subjectiveness and that certainty of the world is established.

This subjectiveness is considered by Husserl as carnal, as being the one of the incarnated body which moves, feels, desires and which perceives the movement of the physical bodies, in its concreteness with the others, present in a corporeal way, thus, intentionally.

Other, living bodies which are made present in perceptions which are communicated, making the network of intersubjectiveness, are co-subjects of experience of the world, making the horizon where the encounter of *Self* - and oneself and with the *Other* is possible.

The *other* is not made present to consciousness in a direct and primordial way, but always through his incarnated body. In this intervention the Ego is perceived in a "worldly" way. It is a process which involves analogy from body to body, lived in a existential way which enables the formation of the peer Self-Other, supposition of a strange life to *Self* which is confirmed in gestures, in expressions, in *others'* behaviors, imagination of itself and of the *other,* as a being in the place of the *other,* who is up-to-date with an experience as if the *Self* were there.[34]

The intervening notion which enables the passage of identity from a body to what is common between the *Self* and the *Other* is the perspective notion. The body itself is the zero-origin of a point of view which gives a certain orientation to its system of experience. It understands that the *other* has another perspective which orients his experience in a different way of that through which the *Self* orients his. The body of the *other* belongs concomitantly to the system of experience of the *Self* and of the *Other* and that makes possible to understand that the same object can be seen from different perspectives.

In the husserlian thinking, however, the *Other,* or the *Self* does not appear only as flesh, as incarnated body, but also as *psyche,* others that are separated and different and which are linked in the fabric woven by constructing a common world made by a community of inter-egos or of co-subjects. This is the way to the construction of all the intersubjective communities and, also, its foundation. In the common project of man, the

34 cf. Husserl. E. Fifth Meditation. in Husserl, E. *Cartesian Meditations,* op. cit

world and the time provide base to the union of men and transform the union of their bodies in an indissoluble cohesion. In this way, surrounding cultural worlds are made as if they were personalities of a higher level.

Those are *Lebenswelt's* and *Geist*'s notions, or the life-world and of the spirit.[35] The *geist* separates the boarding line between nature and culture. The latter is then constituted by the intersubjectiveness which leads to higher personalities.

The core of the life-world and of culture is in the language, in the history and in the tradition.

Language, in the Phenomenology, is not only understood as communication between subjects but also and mostly as organizer and structure of thinking. The affirmative of Mealeau-Ponty that *perception is a nascent logos*[36] makes sense when language is thought as a process which organizes the acts which generate the sense and the meaning, as expositor of these meanings generated in speech, as articulator of the perceived sense, as vehicle of senses, as maintainer of senses, as structure of the world commonalized and of the perception itself in its thinking process.

Hence, all language is founded in a discourse[37] , that is, in the articulation of the sense that the world makes to the perceiver subject. The written text vehiculates this articulation. The reader has to search for the sense which the text makes to him in the horizon-world of his comprehension, which is also that of the *others*, of culture. This is a work of hermeneutical interpretation, that is, the one of the reading which focuses on the sense and the comprehension of cultural meanings, which are historical. They are historical, for they are made in the perspective of time, in the interlacement of intentional acts in the life-world, in the network of objective meanings kept as cultural objects, altogether with all subjects in the world.

That is the network of the real worked out by knowledge. That is weft *Lebenswelt*.

In this weft, HOW CAN THE TRUTH BE UNDERSTOOD?

35 cf. Husserl. E. *The crises of European Sciences*, op.cit.

36 cf. Mealeu-Ponty. *Primacy of Perception*. op. cit. p.25

37 Discourse is understood according to Martin Heidegger, cf. Heidegger, Martin, *To be and Time.* Trad. M.S. Cavancanti. Rio de Janeiro. Ed. Vozes. 1988.

In phenomenological view truth is not understood as adequacy between the representation and the represented one, but it is understood as *aletheia,* that is, unveiling.

It is the truth given in perception, which unveils or shows without veiling the presence of the world. It is the truth exposed in the discourse which unveils the sense that the world makes to the one who interprets and communicates the interpreted. Therefore, one cannot work on Phenomenology with truth itself, a non-temporal, absolute truth which reveals sharpness. One works on convergence of unveiling got in accordance with the reality attained through perspectives.

2.3 Conception of reality of the mathematical entities

The mathematical entities, in according with phenomenological view, are ideal objects. They are constituted in the essential or eidetical intuition, thus in the psyche subjectiveness. However, its ideality exceeds this sphere and, by the means of intersubjectiveness, it is presented objectively in the life-world, and so, they are present to consciouness.

Emphasizing what has already been postulated in the previous item concerning the ideal objects, the ideality of mathematical entities is not kept in the level of abstractness apart from the lived experiences in the life-world. But it becomes wordly in the intersubjectiveness and it is embodied in the language and it is kept in *History* and in *Tradition.* It is intentional objects, but, through essential intuition exceed the psychological existences and the perspective multiplicities given by perception. The ideality of mathematical entities is kept as objective and susceptible of being perceived and developed by means of evidence, imagination, logical reasoning, practical and theoretical doings.

They are projected to the intersubjectiveness sphere and given as objectives in culture through language, which can be exposed in differentes ways: speech, exposing propositions, interconnecting judgements, enchaining reasoning. It is the writing which registers what is said in symbols through common language and in particular mathematical symbols, gesticulation which communicates through corporeal gestures of whats is understood by subjects and through pictorical language which communicate through figures; plastic language which exposes the understood through art, etc.

To guarantee the permanence of mathematical entities it is necessary to take into consideration to the life-world, horizon of civilization where the *other* is, fellow to whom we are always virtually thoughtful and to whom we are always present as own body and as psyche. It is also necessary to count with the structural characteristic of spoken language and the possibility of its structure be confirmed through writing.

In accordance with Husserl the evidence got in the essential intuition, generates the mathematical entity and it is susceptible of being communicated to other through mental structures of communicators agents, subjects co-present to the same community. Being repeated in many productions and communications, by the means of spoken language, this structure becomes an ordinary structure to the community, keeping the objectiveness of ideal objects.

Besides being objective, the mathematical entities are lasting. Its lastingness is guaranteed by linguistical documentation, throught writing.

The writing brings in itself a transformation concerning the way of being of the structure and of the sense of the ideal objects. While in oral and gesticulated communication the structure of mathematical entities can be transmitted through empathy and fellowship, reinforcing the linguistical structure which communicates the discourse, that is, the articulation of the sense that the world makes to its interlocutor subject, the written language incorporates and perpetuates that structure. Logic is inlaid in this language, understood as Theory of Sentences or of the propositions in general .[38]

To Husserl, to the reader is opened the possibility of renewing the lived in the evidences which incorporate the ideality of the ideal mathematical entities or of being reduced through language. In the former, the most general experiences should be reactivated by the sense which the idealities make to him. Going beyond Husserl, but still remaining in the phenomenological thinking, I can postulate that the reader, while searching for the sense in the written text, still has to search for the discourse which provides base to this text. In this case, I am concerned with an hermeneutic work which privileges the sense, the perception of perspectives which is made concrete in the horizon of temporality and that of the spatiality of the subject who reads, interpreting the text from the life-world and that also previleges the interpretation of cultural and social meanings of what is written. It provides dimension to it in History and in the region of mathematical and that of the sciences.

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38 cf. Husserl. E. "The Origin of Germany", op. cit.

In the case of the reader who remains seduced by the language I have a more passive reading, more restricted to repetition, to mechanical and pragmatical application of the mathematical entities, which is possible through the formula and practical doings.

2.4 Didactic pedagogical attitudes and proposals for mathematics teaching

A phenomenological didactics of Mathematics considers that the school world in its concrete worldhood and experiences lived in it form the reality in which the work of teaching is accomplished. It works with perception, exploiting the ways the mathematical entities are shown to the subjects, to each student, to the teacher and to others, also present to the teaching / learning situation. It considers the ways through which everyone feels, in accordance with his possibilities and how everyone sees the world and Mathematics, from the zero-origin given by his own body and by his culture. This regard is an incarnated one, therefore, it brings with it the action, the thought, the speech ..., after all the ways through which the subject is in his world with the motility with others. Thus, the ideality of mathematical entities is presented either in books, texts and specific articles on mathematical sciences or on teaching of mathematical sciences, as in everydayness ordinary practical doings lived by students and teachers. It is shown in perspectives.

To emphasize the perception, the pedagogical work of the teacher of Mathematics gives preference in the present moment and temporal horizon making possible that teacher and students be alert to past, to future and to their own existences which take place in the present. So, they can realize themselves feeling, reasoning, remembering, speaking about the perceived, moving around, in other words, acting. Hence, the sense of the accomplished one is presented to them, each of them considered as an individual ego and the meaning keeps on being processed.

This procedure contributes so that the life-world makes sense. The certainty of the world reinforces the knowledge of its reality, through subjectiveness, through intersubjectiveness and reaching the cultural objectiveness, without mystery.

Moving ahead towards the understanding of the reality of the ideal objects, the work of the teacher of Mathematics elects activities which enable him / her to reunite to the given multiplicity in the perception and in the individual experiences. This activity should privilege attention to similar aspects between classmates expositions and experiences individually lived by the students. It is mandatory to listen to the *Other* attentively, seeing what he does and speaks, trying to interprete it, searching for convergencies. It also is mandatory to listen to oneself trying to interprete his own feeling, doing, speaking etc.

Hence, the pedagogical doing of the teacher of Mathematics works on the *Self* and on the *Other* by the means of his own body and not in an introspective way. It privileges the perception of the *Self* and of the *Other* which are perceived as incarnated bodies making movements, wanting, acting, answering, speaking, listening, and interpreting. The perspective, the temporality are existencially understood in a primary level, and worked little by little in the sphere of cultural meanings and in the sphere of sciences.

The mathematical text - either the one produced by students or the one produced by authors - is part of these activities. To understand the meaning of writing, managing to register their own mathematical understandings of the world, sharing with learning-situation mates their evidences, the ones already elaborated and communicated through language, listening to what the *other* has to say are activities which take place in the core of mathematical doing and in the core of mathematical science. They subside the development of an active attitude of reading, participative and critical reading of texts written by other authors.

These activities generate the *trans-doing,*[39] for they re-create the data and what is already done in an endless chain of constructing the unfinished one, what is in movement, assuming that we ourselves are the horizon-world, the culture, the History after all.

Another point to be highlighted in the pedagogical practice which has Phenomenology as proposal of regarding the world is *reflection*. It involves everyday school activities which require students and teachers drawing attention to their own experiences, either the individual or the group ones in order to understand them, analysing them and criticizing them properly. In classes of Mathematics, these activities

39 Trans-doing, recreating term which does not have the same meaning of dialectics. The trans-doing refers to how the human being as an individual feels the world and from it attributes meaning. It means to go beyond, to overcome a simple doing. It is an endless re-creating, always unfinished one, for the human being is always a being of possibilities." (cf. Marins Joel (org. Vitória H.C. Espósito). *Educação como Noises*. São Paulo, 2ed.Cortez, 1992.

comprehend the exposition of operated reasoning, the analysis of the starting point, of its sequence and of anticipation of other possibilities through imagination. Or they comprehend the presentation of figures which represent the past, moving them in the depth which the regard in perspective enables one to project light and shadow and to reporting the seen thing. They comprehend the analysis of text, of programs of informatics etc.

Going backwards and returning to the accomplished thing is the turning point to learn how to regard the world phenomenologically, or rather, not taking oneself and the classmates as natural object, given objectively and susceptible of being handled, reproduced and represented according to pattern of truth. But it is by taking them as incarnated bodies to whom the world makes sense and which questions this sense itself, as well as the *self* and the world, always searching for the truth as explanation, as a clear regard or evidence of essence or essential intuition which gathers the multiplicity of ways of focusing according to perspectives.

The reflection lies in the core of strictness of phenomenological procedures which have research as goal. To work this strictness pedagogically helps the formation of a thoughtful reflexive and critical researcher. It also helps the formation of a citizen who interferes in the reality in a conscious and consequent way.

The evaluation, in accordance with the perspective assumed here, is qualitative and based on the process in which the sense and the meaning take place and which they elaborate their temporal and cultural dimension. It is always accomplished by subjects presented to the reality in which evaluated activities take place and in which they claim themselves and the others as authors and as subjects aware of their own action.

In the school world, the product of evaluation is preferably presented in a propositional language which provides values of judgement, and are objectivated in texts susceptible to be interpreted, making sense to the subjects involved.