Regular Lectures

Teachers using technology: aspirations and classroom activity

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ABSTRACT

This paper is about teachers using technology in 'ordinary' conditions. It addresses the discrepancies between potentialities of technology and teachers' aspirations with regard to technology use, and between teachers ' expectations and the actual carrying out of technology based lessons in the classroom as well as by unexpected episodes of uncertainty and improvisation that teachers experience when teaching these lessons. Two models are used to make sense of teachers' position and of classroom phenomena. Ruthven and Hennessy's (2002) model helps to understand how teachers connect potentialities of a technology to their pedagogical aspirations, rather than to mathematically meaningful capabilities, and to interpret their classroom activity. Saxe's (1991) model helps to analyse the flow of unexpected circumstances challenging teachers' professional knowledge in technology based lessons and to understand how teachers react to this flow. It also draws attention on the consequences of the introduction of new artifacts in the culture of the classroom. This gives tools for researchers to work in partnership with teachers, as well as for a reorientation of teacher development in technology towards reflective approaches.

Keywords

Digital Technology Integration; Teachers' aspirations; Teachers' Classroom Activity; Emergent Goals; Practitioner Model

This article is about teachers using technology in mathematics teaching/learning. I am interested here by teachers using technology in 'ordinary conditions', that is to say not in the frame of experimental projects. My question is what they really expect of technology and how these expectations impact upon their classroom activity. This question comes from observations we did in a French research group whose name is GUPTEn (Genèses d'Usages Professionnels des Technologies par les Enseignants)[1]. This group worked observing teachers with diverse methodologies and observed a series of gaps. The first gap is between institutional demand and few actual uses by teachers. In some parts of the French curriculum there are strong institutional demands towards technology use, but, as international studies like PISA show, students rarely mention having used technology in the classroom. The second gap is between the potentialities brought by technology and the actual uses by teachers. In some parts of the French curriculum technology use is compulsory. However, uses prepared and carried out by the teachers appear to be deceiving in comparison to the potentialities of technology emphasised by research studies and innovating projects. When our group looked more closely to classroom uses, a third gap appeared. The carrying-out of the lesson in the classroom was often different from what teachers had expected. Very frequently, we observed episodes where teachers seemed to be quite uncertain of how to carry out the lesson and had to improvise.

Some authors explain these gaps by teachers' conservativeness, saying that teachers are reluctant to change especially because using technology would also oblige them to modify their teaching habits and style. In my meaning, these discourses underestimate the constraints that teachers face. Cuban (1989) emphasized the crucial role of these constraints : "*Teachers teach the way they do simply to survive the impossibilities inherent in the workplace*". For me it means that opportunities exist for changing teaching practices, but, constraints make them not many, and one has to consider them carefully, which is a goal of this paper. Another explanation of the gaps would be insufficient teacher education. Certainly teacher education does not contribute efficiently enough to technology integration, but in my view, and from the observation that in France many efforts have been devoted to teacher education in this field, I take for granted that the gap is more qualitative than quantitative.

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[1] The web site http://gupten.free.fr presents this research group and its main activities

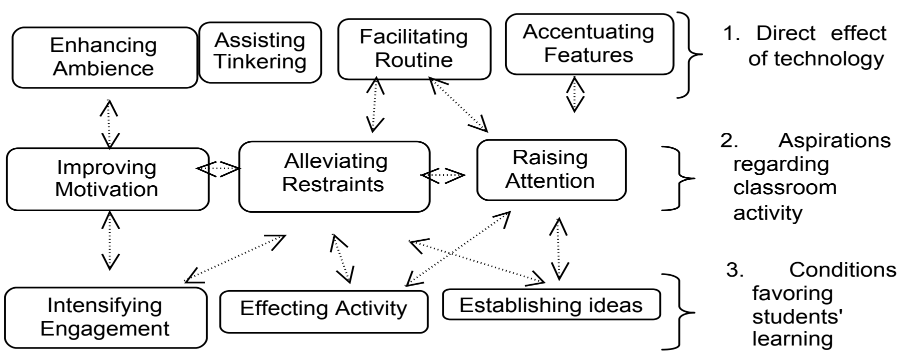
1. HYPOTHESES, FRAMEWORKS, CASE STUDIES AND QUESTIONS

In my view these gaps originate from a poor conceptualisation of teachers' position towards technology and we need to understand better this position. That is what this article wants to contribute to. Studying teachers' expectations relatively to technology and how they impact upon their classroom activity is a means for that. This study is based upon two hypotheses. The first one is of a discrepancy between *potentialities* emphasised by researchers that come from a didactical analysis of software uses and teachers' *expectations* towards supposed effects of technology. Potentialities derive from a cognitive didactical analysis of software uses by researchers, while expectations are marked by teachers' aspirations regarding students' activity.

The second hypothesis is that analysing classroom episodes where teachers meet uncertainty and have to improvise could help to reveal hidden constraints and obstacles. I will use two different frameworks for these two hypotheses. Conceptualising teachers' expectations with regard to technology will be done using Ruthven and Hennessy's (2002) "practitioner model of the use of technology to support mathematics teaching and learning". The framework that I will use to address the complexity and uncertainty of teachers' activity involving classroom use of technology is Saxe's cultural perspective (the four parameters model) that Monaghan (2005) introduced to analyse technology based lessons.

1.1 A practitioner model

Figure 1: Ruthven and Hennessy' practitioner model

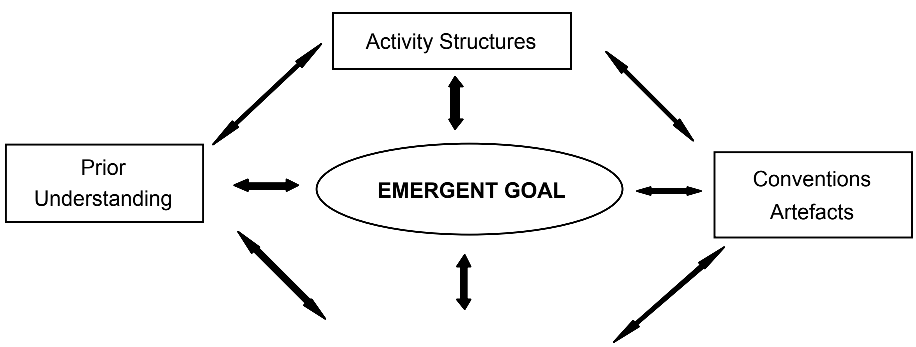


Ruthven and Hennessy built the model outlined in figure 1 from interviews conducted with mathematics teachers in the UK. These were not individual interviews but collective discussions in mathematics departments leaded by the researchers about what is for these teachers a successful use of technology. From the script of the interviews, Ruthven and Hennessy determined ten themes and organized these themes in three levels. The first level is where we find the reasons for success directly following technology use. There is a better ambience because students generally like working in pairs on a computer, it makes experimental approaches possible, routine tasks are facilitated, etc. At the second level we find aspirations regarding classroom activity is the consequences of these effects of technology to the. At the third level, we find teachers' general views regarding conditions favoring students' learning. Teachers consider that students learn better when they engage more, in a more effective activity, and when the ideas are better established.

The links between themes were established by a statistical procedure. Links are between the themes that appear together more often in the same teacher's statements.

1.2 Saxe's cultural perspective (the four parameters model)

Figure 2 : Saxe's four parameters model



Monaghan (2005) proposed to use Saxe's cultural perspective to address the complexity of classroom practises with technology as a whole. My assumption is that Saxe's idea of emergent goal might help to understand what I called above episodes of uncertainty and improvisation: «*Goals are emergent phenomena,* *shifting and taking new forms as individuals use their knowledge and skills alone and in interaction with others to organize their immediate contexts*" (Saxe, 1991).

Monaghan (2004) described the four parameters in the context of technology use (figure 2).

Activity structure:

Monaghan considers here the way teachers organize their classes and prepare students' tasks, and the decisions they take relatively to their role and activity as well as those of the students: he observed that the tasks and cycles of these lessons varied considerably across teachers and, in most cases, varied over time, technology tasks being 'unsafe' as compared to usual tasks.

Conventions – artefacts:

While recognizing that Mathematics teaching involves many cultural artefacts including systems of convention and notations, Monaghan privileges software and written resources in the study of teacher activity in technology-based lessons, considering that the way a software transforms mathematics is an important concern for a teacher and also that the shift towards technology use brings him/her to widely re-evaluate the content of his/her written material as well as the way they use it.

Social interactions:

Monaghan observed a variety of ways in which technology affected social interactions in observed classrooms. Although technology lessons were notable for their diversity, most changes appeared in relationship with specific constraints and did not denote a clear developmental path towards adopting new roles.

Prior understandings: For Monaghan, mathematics teachers' 'prior understandings' of learning and teaching incorporate a range of beliefs and professional knowledge. Beliefs are globally independent of whether or not the lesson uses technology and were not reconsidered. In contrast, teachers' knowledge of their teaching, generally tacit in ordinary lessons, had to be rethought deeply in order to incorporate technology use.

1.3 Case studies and Questions

The two case studies analyzed in this paper come from doctoral theses. The first one is Caliksan-Dedeoglu (2006). It concerns Dynamic geometry (DG) at middle school level. In this context there is a strong institutional demand and many resources should help the teacher. It is also a context where teachers feel often not easy to teach because of difficulties to maintain student's attention and motivation. The questions are about the discrepancies between potentialities of technology and teachers' expectations, and between teachers' expectations and the actual carrying out of the lessons in the classroom.

The second case study comes from Ozdemir-Erdogan (2006) and deals with spreadsheet use at upper secondary level by non scientific students. The context is a new curriculum where spreadsheet use is compulsory. The questions and hypotheses are about teachers' understanding of this curriculum and the goals emerging in teachers' classroom activity.

2 POTENTIALITIES OF TECHNOLOGY AND TEACHER EXPECTATIONS

In the first case study Caliksan-Dedeoglu studied successive gaps between the potential of Dynamic Geometry as seen by research, by the curriculum, by textbooks and by teachers. She also observed the classroom activity of a panel of teachers.

2.1 Successive gaps between views about Dynamic Geometry

Dynamic geometry in research studies

Caliksan-Dedeoglu looked for the potentialities that didactic research studies attribute to Dynamic Geometry, but also for the conditions that researchers find important in order that DG really contribute to learning.

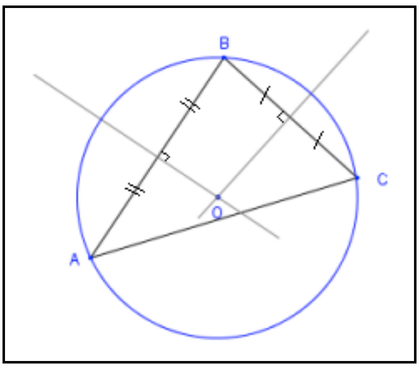
She found that the construction and creation tools can help students to draw quickly accurate figures and that the dragging tool is appropriate to distinguish between a drawing and a figure (Laborde 1994) and to explore invariant properties.

She also noticed that these features are considered in interaction. A good example of interaction is the case of robust and soft constructions (Healey 2000).

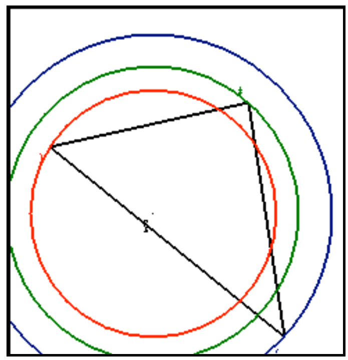
A robust construction (figure 3, left) is the dynamic equivalent of a mathematical geometrical construction. For instance, constructing the circumcircle of a given triangle *ABC*, the students classically construct O at the intersection of two perpendicular bisectors, for instance of *A* and B and of B and C. It is a robust construction because it resist to a change in the position of the given points. More generally, in a robust construction activity, dragging gives evidence of a valid construction.

In a soft construction, (figure 3, right) students drag free points to obtain a given configuration. For instance, to obtain the circumcircle of a given triangle *ABC*, students create a free point O and three circles centred in O, passing respectively by *A*, B and C. They move O in order that the first two circles overlap. Then they move O carefully in order that the two circles continue to overlap, until the three circle overlap. This is a soft construction because it does not resist to a change in the position of the given points. When a student or the teacher moves a vertex of the triangle, the three circles disjoin. In a soft construction dragging helps to explore constructions and can play a role in the proof process: "*dragging tools enable students to examine their constructions, both to identify relationships which remain invariant and to impose further relationships visually*" (Healey 2000).

Figure 3: Constructing the circumcircle of a triangle



A robust construction



A soft construction

Another important idea that we found in research studies is how classroom activities should be organized. For instance Falcade R., Laborde C., and Mariotti (2007) used Dynamic Geometry to develop students' understanding of functions and they insist on three types of uses. One is in the computer lab, students working in pairs on a task, the second is individual writing, possibly using the computer to check some functioning and the third is classroom discussion where a video projector can be used. In their view, it is important that these three types of uses are coordinated, the individual writing fostering a reflection on computer activity, and the classroom discussion helping to give a mathematical meaning to the observations.

2.2 *Dynamic geometry in Textbooks*

Caliksan-Dedeoglu made a comprehensive study of textbooks for the middle school with regard to Dynamic geometry and she found first a discrepancy with the curriculum. While the curriculum says that DG can be used for most tasks as well as paper/pencil, only 5% of the geometric tasks tasks involved Dynamic Geometry. And also there were some subjects like 3D geometry that the curriculum specially emphasised as interesting for the use of Dynamic geometry software and for which textbooks did not mention its use. This is for us an indication that integrating DG, as recommended by the curriculum is not so easy because textbooks authors find difficulties to propose tasks that the teachers could really put into operation.

These were also clear discrepancies between Dynamic Geometry in Textbooks and in research. First, tasks in textbooks separated construction and dragging. Task for creating constructions were typically reproducing a paper pencil figure. Tasks involving dragging objects typically aimed to recognize invariant properties. And also there was emphasize on measure in invariant properties, for instance invariance of ratios in the theorems of the parallels.

Then textbooks separated two types of uses. Textbooks considered mainly two types of work, one in a computer room, students working alone or in pair on a computer and following a worksheet, and the other in an ordinary classroom with a computer hooked to a video projector and activated by the teacher. As a difference with research, textbooks did not insist on coordination between these types of uses.

Because textbooks influence teachers' practices, but also are influenced by what is possible in teachers' practice, it was an indication that there could be a distance between these practices and what research studies found necessary for a successful use of DG. Actually, the computer room is seen as an environment for student autonomous work, while the computer hooked to a video projector and activated by the teacher is seen as a tool for the teacher to illustrate a lesson without necessary connection between those uses.

2.3 *Classroom observation*

As said above, Caliksan-Dedeoglu also observed teachers. She had difficulties to find teachers using dynamic geometry and accepting that a phD student observe their classroom. She found nevertheless a panel of five teachers in three schools that were not ordinary teachers in the sense of randomly chosen teachers, but rather experienced teachers with some contact with groups of research and action. Two teachers in this panel developed uses in computers labs, what we called above GD Environment for student autonomous work and three used systematically a computer hooked to a video projector that they operated themselves.

DG= Environment for student autonomous work

Anne was one of the teachers that developed uses in computers labs. She taught at 7th grade and she was positive towards technology. Especially, she saw advantage in GD use like speed and accuracy of drawings by students, avoiding mistakes by not confusing words for instance perpendicular bisector ('médiatrice' in French) and median in a triangle. To her, dragging was a means for students to experience an invariant property.

Figure 4: Anne's instructions for students

Create a triangle A B C
Create a free point O
Create three circles centred in O passing by A, B, C
What can you say of these circles?
Create the perpendicular bisector of segment AB.
Put O on this line
What do you observe?
...
Write instructions to construct the circumcircle of a triangle

Figure 4 presents a task that Anne proposed in her 7th grade class. The objective was to introduce students to the topic of the circumcircle of a triangle and more precisely that they understand the position of the circumcenter and find a paper pencil construction..

Students had to create a triangle *A B C*, a free point *O* and three circles centred in *O* passing by A, *B*, *C*, they had then to observe that the circles are different, because point *O* is randomly positioned. Then Anne did not ask her students to find experimentally some position where two or three circles could be the same, like in a soft construction strategy. In contrast she directly asked to draw the perpendicular bisector of segment *AB*, to drag *O* on this line and to observe that the two circles passing by *A* and B are the same. At the end, she did not ask for a hard construction and to verify by dragging, but to write a program of construction from the guided tasks they did before.

In our understanding, Anne transformed a problem of soft construction into a series of construction-observation tasks that should lead students to a paper pencil construction. She separated creating and dragging objects, thus loosing the challenging aspects of the problem.

The observation showed that students actually had much difficulty when operating the software, not understanding for instance that the three circles had to be named and how. Anne tried to help them individually, but there was much disturbance: the task was prepared for just half of a 50mn session, and few students could actually make sense of the observation.

DG= a tool for the teacher to illustrate a lesson

I now contrast Anne with another teacher, Bruno who had systematically developed the use of DG as a tool to illustrate a lesson. He was also positive towards technology and particularly stressed that technology helps to visualize dynamically and allows easy construction and easy measurements. He said that he preferred to use a computer hooked to a video projector because he tried before to use GD in a computer room and found that it was much work for him and little gain for the students. He said he would have needed worksheets to strictly guide students. He taught nearly all geometry lessons with DG and video projector**,** operating the computer and showing various configurations. His students had to answer questions and to copy the configurations on paper.

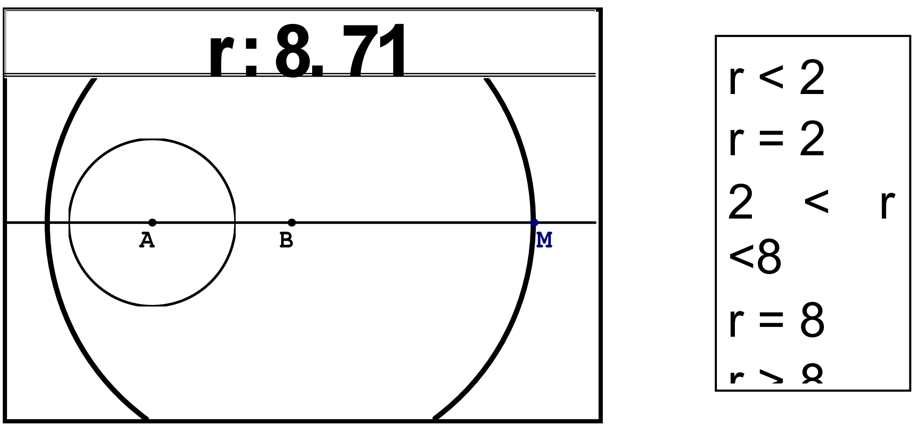
A typical situation was a lesson about the triangle inequality for an 8th grade class. The triangle inequality is the theorem stating that for any triangle, the length of a given side must be less than or equal to the sum of the other two sides but greater than or equal to the difference between the two sides. Note that only the first inequality is mentioned by the curriculum at this level.

Starting the lesson, Bruno presented the goal with the following question:

Consider a triangle ABC, with AB= 5cm, AC= 3cm. What can be the length of BC?

In a first phase, students had to try do draw a triangle on their paper. Bruno's intention was that students guess an adequate measure for the distance BC and use the standard compass procedure for drawing a triangle of given sides. Then five phases followed. Theses phases were supported by a GD figure animated by the teacher. In this figure, *A* and B were two fixed points with a distance of five. M is a free point on the line (*AB*). A fixed circle centered in *A* and of radius 3, and a circle centered in *A*, passing by M were also drawn. The teacher animated the second circle by dragging M and the radius was displayed in the form r =.... (figure 5). Bruno's idea was that the students would easily connect this figure with the standard compass procedure activated in the first phase.

Figure 5: Bruno's screen and blackboard

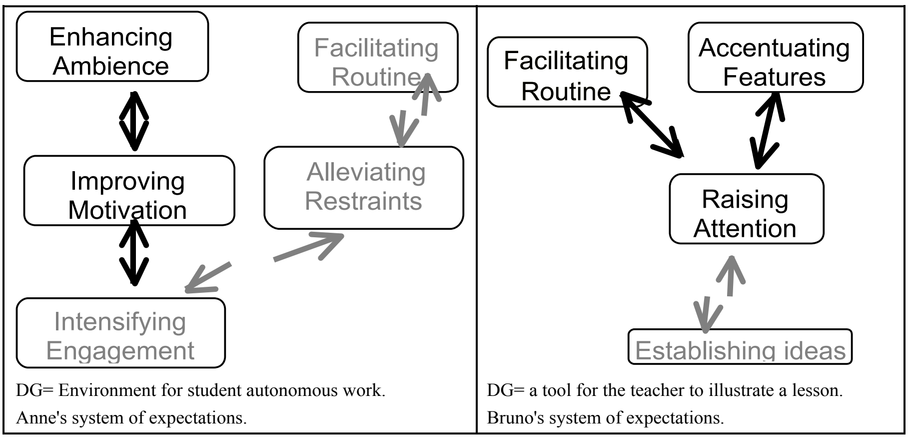


In each of the five phases, the teacher considered a specific configuration: the circles were successively separated externally, tangent internally, secant in two points, tangent internally and separated internally. After discussing each configuration with the students, the teacher wrote a condition relatively to r on the blackboard. In a last concluding phase, he tried to make students pass from the condition $2<r<8$ to a statement involving the measures of the sides, that is to say the triangle inequality. He did this by raising the students' attention on the numbers 2 and 8 (respectively the difference and the sum of 5 and 3) rather than by considering the relationship between the radius of the circles and the distance of the center in a generic configuration.

Caliksan-Dedeoglu's observation showed that the visualisation did not really make sense for the students. They did not understand the relationship between the problem and the GD figure. In the first phase they had used the compass, but they did not see the relationship with the circle drawn and animated by the teacher. They did not understand the role of the free point M and of the variable r. They saw that the different cases correspond to r<2, r=2, 2<r<8, r=8, r>8 but they did not put these facts in relationship with properties of the lengths of the triangle. In an interview after the classroom session, Bruno said that he was disappointed: students should have found themselves the triangle inequality after observing the configurations.

2.4 *Teachers' expectations versus actual carrying out*

Figure 6: a summary of the analysis



The figure 6 summaries the above analysis of the two different types of use that we pointed out in the textbooks' analysis and that we observed respectively in Anne's and Bruno's classrooms. For each teacher, from their declarations, we retain in Ruthven and Hennessy's model the themes that he (she) privileges, and the links between them. In Anne's case (left), she insisted that students would work better in the computer lab, and then that their motivation, which was one of her great concern, would improve. Easy construction would also help students to go faster in the task. The two lines of themes would converge towards intensified engagement. Really for Anne, student's engagement in the task is condition for learning. For Bruno (right), DG helps to facilitate constructions and to put the focus on relevant features of the situation. It should help students to be attentive and to make clear the relevant ideas of the situation. As we can see, the two systems realize nearly a partition of Ruthven and Hennessy's model. I take this as an evidence of two distinct systems of expectations underlying the two types of uses. This is clearly different as compared to research where different types of uses are articulated.

The two systems connect the anticipation by the teachers of effects of technology uses to deep personal aspirations regarding students' access to knowledge. For Anne, students' engagement in tasks is a condition for conceptualising, whereas Bruno privileges a good visualization of math properties in order to retain the main ideas. These personal aspirations are clearly related to the context: as said above, at middle school level in France, teaching conditions can be difficult, a situation that teachers often explain by students lacking in motivation and concentration. Teachers' use of technology seems to be driven by these context related aspirations rather than by the didactical potentialities evidenced by research on cognitive aspects of technology use.

In the figure 6, the broken arrows pointing to themes written in grey reflect phenomena observed in the classroom: in some cases expectations of the teachers were not fulfilled. Anne's students experienced difficulties with the software. Routine was not really facilitated. Then for Anne there was a danger that students did not fully engage in the task. That is why she devoted most of her time and energy to help individually students in their constructions. In Bruno's case, the link between "Raising attention" and "Establishing ideas" did not work. That is why, like so many teachers in the same circumstances, Bruno finally obtained the statements he needed by what Brousseau (1997, p.25) named "a Topaze effect". This means that the systems of expectations that I outlined can explain both how teachers prepare a lesson with technology, but also how they carry out the lesson, including the way they face unexpected circumstances. Using the second model outlined in section one, the next section will consider in more depth how a teacher in another context encounters and manages these unexpected situations.

3 EMERGING GOALS IN TEACHERS' CLASSROOM ACTIVITY

The context for the second case study is the French curriculum for upper secondary non-scientific classes existing since the year 2000. It is intended for students more attracted by literature and arts than science, and who generally experienced difficulties in mathematics. It aims to strengthen mathematical basic knowledge by favouring modelling, interpreting and criticizing varied information. It recommends involving "*mathematics use visible in society*" that is to say graphs, tables, percentages... It "*systematically proposes to put all the items into operation on a spreadsheet*". It does not recommend the study of the spreadsheet for itself, but as means for exploring and solving problems.

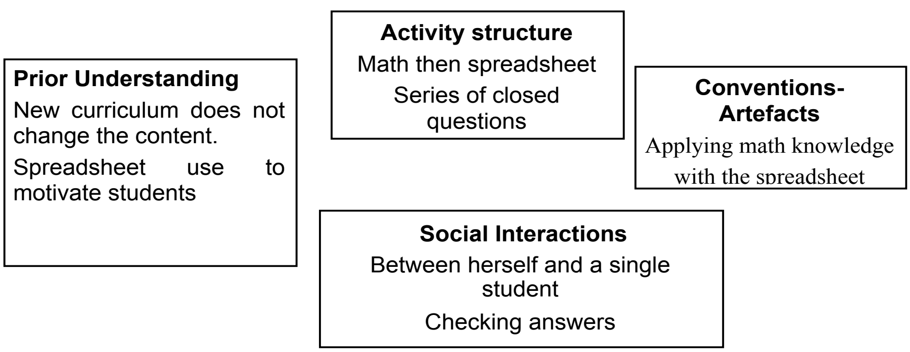
Ozdemir-Erdogan focused on lessons about linear and exponential sequences. In the curriculum, this topic derives from the focus on "visible mathematics". Students are supposed to study "types of progression" from examples of situations without the mathematical apparatus of sequences, but with the help of the spreadsheet. As a difference with other courses where most teachers ignore curricular recommendation about technology, spreadsheet's use cannot be avoided because of the national evaluation –the baccalaureate- whose texts are written in order that candidates could not succeed without spreadsheet knowledge.

3.1 *The case of Charlotte*

Among the teachers that Ozdemir-Erdogan observed in these classes we selected one - Charlotte . She was a very experienced teacher, she taught these classes for 30 years, and she had to adapt her teaching to the new curriculum although she thought that technology would not give great help. That ensured that difficulties would not come from poor classroom preparation and management. Actually Charlotte is representative of a majority of teachers who teach these classes because of the curriculum's demands rather than because they like technology. This section presents Charlotte's profile using Saxe's four parameters and then reports on fragments of a classroom session identifying emergent goals.

3.2 *Charlotte's profile*

Figure 7: Charlotte's four parameters



This is how we can see Charlotte with the help of the four parameters model

Activity Structures:

Charlotte devoted three weeks to sequences which is not much with regard to the curriculum's demands. Charlotte's structure was simple: the notion of sequence was presented to the students the first week, then arithmetical sequences the second and geometrical sequences the third. This structure is not consistent with the curriculum, since the study of situations and the notion of progression should be privileged.

The course was two hours per week, one in whole class and one in half class. Teachers had to decide how to use them. Charlotte taught the whole class in an ordinary classroom and the half class in a computer room. In both classes, whole class sessions were devoted to the presentation of the mathematical content and half class sessions to "applications" with the spreadsheet. Charlotte's students worked individually following a worksheet.

Conventions – artefacts:

We consider here the spreadsheet whose use is compulsory in this course and the written material that teachers prepared for the students. In the whole class hour Charlotte's students had to work on paper-pencil. In half-class it was clear that they had to work on the spreadsheet: Charlotte's worksheets were really specific about this use, referring to cells and formulas.

Social Interactions:

Charlotte's interactions with students were similar in the computer and in the ordinary room. These interactions were very frequent and generally between herself and a single student.

Prior Understandings:

In Charlotte's view, technology was introduced in this course in order that students learn about spreadsheets. For her, beside the use of technology, the mathematical content was not different from the previous curriculum. She thought that technology does not bring a very concrete contribution, but has a positive effect on the behaviour of her students that she considered weak and not interested in mathematics.

3.3 *Fragments of a lesson*

Figure 8: an example of a task for the first lesson about sequences

Sabine has just been born. Her grandmother opens a credit account for her, makes a first 100 € deposit and decides to make each year a new deposit of the same amount plus the double of Sabine's age.

a. Starting with u0 = 100, compute by hand the money that Sabine's grandmother will deposit on the account at year 1 : u1 =… at year2 : u2 =… at year 3: u3 =… at year 4: u4 =…

	A	B	C
1	ans	versement	total
2	0	100	
3	1	=	
4	2		
5	3		

b. Which of the following formulas should we write in cell B3 and fill down to obtain the values of the sequence?
(1) = B2 + 2*A3 (2) = B2 +2 (3) = B2+ 2*A3

The figure 8 presents an example of a task proposed by Charlotte for the first lesson about sequences. Question a. gave priority to the mathematical notion of sequence and notation, and privileged by hand calculation in the understanding of the situation. In question b., a table and three formulas were provided, and students had only to "press buttons" to complete the electronic sheet. The first formula, although congruent to the definition, does not work, because of the filling down functioning. The second formula is recursive, and then not congruent to the definition, and it works. The third formula corrects the first by way of an absolute reference (B2) to the initial deposit.

This is how the students carried out the task: they launched directly the spreadsheet, entered the values in each cell; one hundred and two in cell B2, one hundred and four in cell B3, etc.. Going to question b. they tried formula (1) that did not work. Without reflecting more they tried formula (2). Then they passed to the next question without trying formula (3). Looking at the students' screens, Charlotte was first satisfied to see the right numbers in the cells, and then she became aware that students did not enter a formula and she seemed quite surprised. She then devoted a number of individual interactions with the students to prompt them to enter formulas and fill down the sheet.

This was the first emergent goal: *making the students use the spreadsheet as a calculation tool*.

She was again surprised that students did not try the third formula and she also prompted them individually for that. This was the second emergent goal: *making the students try formula* (3).

Figure 9: Managing the first emergent goal. Two interactions

Interaction 1

Student 1: Am I right ?

Charlotte: Yes…

…And.. how do you proceed?

Student 1: I calculate

Charlotte: no, you must not calculate, the spreadsheet must calculate!

Student 1: but it is quicker than with the computer

Charlotte: but go until 200 years like that?

Student 1: but this poor girl will never be 200 years old!

Interaction 2

Charlotte: what happens to you…? No, no, do not make like that.

Student 2: me?

Charlotte: one should not type each time the calculation.

Student 2: but why not?

Charlotte: it is necessary that…

I want to be able... Take your formula and fill down.

The figure 9 displays two examples of interactions showing how Charlotte encountered and managed the first goal. The first interaction shows that teacher and student have different views of the task and of the spreadsheet. For Charlotte it was important to use the spreadsheet as a calculation tool, because, although she did not quite believe in the contribution of technology, she was aware of teaching a mathematics course and not just a course "about arranging data in columns". In the students' view the spreadsheet was not fundamentally a calculation tool. It was difficult for them to enter formulas and easy to calculate mentally the values. Some of them devoted a lot of time to neatly format the data and the columns using colours.

As can be seen in the second interaction, after trying with student 1 to give reasons for using the spreadsheet, Charlotte gave up and exercised her authority on student 2.

Figure 10: Managing the second emergent goal.

Interaction 3
Charlotte: Did you choose between the three formulas?
Student: Yes this one (She shows the second formula on her screen).
Charlotte: Did you try the third one? (with absolute reference)
Student: No, I did not.
Charlotte: Then please try.
Student: But, after that I will have to do it again!

Then Charlotte shows the student how to use a third column for the third formula.

Figure 10 shows an example of interaction relatively to the use of the third formula (second emerging goal). It helps to understand why students did not try this third formula. Because the second formula gave the data they expected, they were satisfied. They feared that the third formula would not work and then that their previous work would be destroyed. This is very consistent with common social behaviour with regard to technological tools. If one finds a way to reach a goal for instance with his (her) mobile phone, he (she) will generally not continue to seek for another way. He (she) will prefer to keep strictly to the way he (she) found. In contrast the task proposed in the text was about comparing different ways to reach the same goal.

3.4 *Parameters and emergent goals*
This is how the emergence and the management of goals can be seen under the influence of parameters (figure 11).

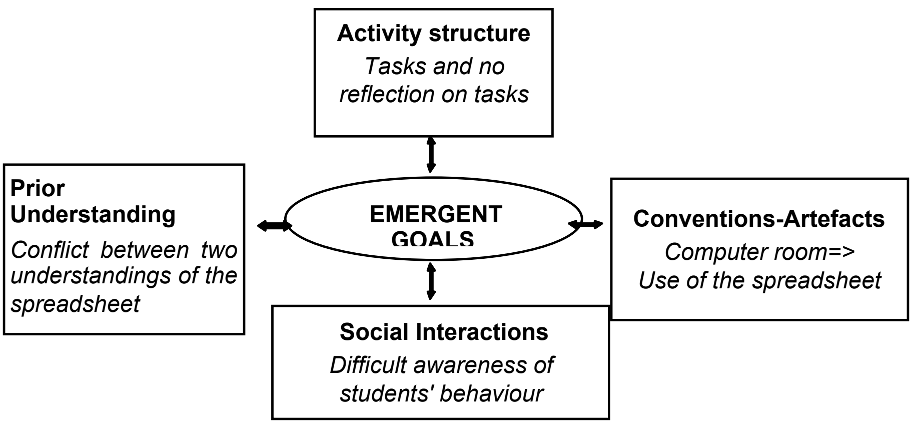
Activity Structures:
Charlotte's students worked individually following a worksheet. This did not favour reflection that would be necessary to compare the three formulas and make sense of these.

Conventions – artefacts:
The students understood that in the computer lab they had to use the computer, that is why they directly launched the spreadsheet not understanding that the text asked first a paper pencil task and then a comparison with the spreadsheet.

Social Interactions:
Charlotte's scheme of interactions with a single student made difficult for her to become aware of their behaviour. She was first satisfied to see the right values in the cells, before understanding that they had been entered one by one in the spreadsheet.

Prior Understandings:
There is a conflict between two understandings of the spreadsheet; students' understanding is related to a social view of technological tools whereas the teacher understands the spreadsheet as a calculation tool. Charlotte seemed to be not aware of social representations of tools possibly conflicting with mathematical uses.

Figure 11: Charlotte's parameters and emergent goals



3.5 *Discussion*
Other observations confirmed that Charlotte's tendency to act on an exposition/application activity format and a teacher/student individual interaction scheme, already existing before the curriculum change, had been reinforced by the spreadsheet and consequently application was replaced by narrow spreadsheet tasks. Maybe we could say that the observation of goals comes from the fact that Charlotte is particularly weak, falling into every pitfall of technology use. It is not the case, since Ozdemir-Erdogan observed another teacher, very experienced in the classroom use of technology, volunteer to teach this course

after the curriculum change, with a much better understanding of the curriculum and an innovative classroom management. The observation of a similar session showed that this teacher encountered similar emerging goals and that, although she reacted better, this reaction was not always consistent (Lagrange, Erdogan 2009).

This idea of emergent goals was means to give account of teachers' uncertainty in classroom use of technology. It brings further to think about what these teachers have in common with the New Guinea Oksapmin from which Saxe built the model. This should be that both had to deal with a new artefact involving deep cultural representations. In the Vygotskian perspective, Saxe was interested by the impact of culture upon cognition and he chose the Oksapmin people because in their case there was a conflict of cultures: these people have a traditional way of counting, using parts of the body as representation of numbers; some of them trade in the modern way, but their traditional way does not permit them the calculations that this trade requires. This comparison brings to consider cultural systems involved in classroom use of technology. Students saw the spreadsheet as a means to neatly display data. It is consistent with the social representations of technological tools. People are generally not aware of the real power of the computer, which is the possibility of doing controlled automatic calculation on a data set, even when they used spreadsheet features based on this capability. In contrast, Charlotte saw the spreadsheet as a mathematical tool. She was disconcerted because she was not conscious of the existence of other representations.

4 CONCLUSION AND PERSPECTIVES

Combined with classroom observations, the two models helped to make sense of teachers' position and of classroom phenomena. Ruthven and Hennessy's model helped to understand that teachers connect potentialities of a technology to their pedagogical aspirations, rather than to mathematically meaningful capabilities. The observation of two teachers using dynamic geometry showed what happens when the connection does not work: the teacher tries to re-establish the connection. In the first observation the teacher Anne expected that technology would help students to work alone on a task, but it did not work because of insufficient instrumental genesis. Then she tried to re-establish the connection by becoming a technical assistant. This explain why teachers teaching in a computer room devote much time to technical scaffolding when they expected that technology would help their students to work alone and that they could act as a catalyst for mathematical thinking. In the second observation the teacher Bruno expected that after the students had their attention raised thanks to technology, it would be easy for them to make sense of the situation mathematically. Again here it did not work because students did not understand the teacher's action on the computer and the teacher had to rely on a "Topaze effect".

I also noted that these teachers had different expectations directing them towards different uses of technology, one in a computer room, the other on a computer he operated himself, and that in both cases, their expectations were connected to their teaching context.

Saxe's model was chosen to analyse classroom episodes where teachers meet uncertainty and have to improvise. The notion of emergent goals was central to analyse the flow of unexpected circumstances challenging teachers' professional knowledge. The four parameters helped to understand how teachers react to this flow. Saxe's model also drew our attention to how cultural representations of a technological tool can differ between the teacher and the student, making it difficult for teachers to anticipate and understand what students do with the tool.

The upshot of the two analyses is that teachers' views of technology are influenced by general expectations and necessarily diverge from cognitive views privileged by research. It is then important that research broaden its range of concerns to include teachers' expectations and context. It is also important that research take into account the impact of cultural views associated to computer artefacts upon classroom phenomena, which is another way for broadening the range of concerns to consider the diversity of social representations relative to technological tools.

This is consistent with Fuglestad, Healy, Kynigos & Monaghan 's (2009) idea that the complexity of technology use is linked to the fact that tools are a constituent part of culture; hence the introduction of new artifacts necessarily involves the establishment of new cultural practices. The need to involve teachers as partners in research studies about technology is then increasingly evident, the focus of the partnership being on the design of learning activities and/or on the design of the digital tools themselves both resources playing the role of "boundary objects". As a tool to understand teachers' position towards technology, the two models we used are valuable for researchers working in these partnerships: learning activities and digital tools could be appreciated for their didactical relevance as well as for their adaptation to a context and for the way they can be incorporated into teaching practices.

Reciprocally, ways are to be found in order that teachers come to re-think their expectations, considering the actual support that technology is able to bring. Whilst strategies based on the transmission of "good practices" taking little care of the complexity of the integration fail to engage teachers, reflecting upon actual more or less unsatisfactory classroom activities might help them to identify possible evolutions (Emprin 2007). As a tool for teachers to clarify their beliefs, knowledge and decisions, as well as to learn to deal with shifting goals in the classroom, the two models we used here could facilitate a strategic shift towards reflectivity in teacher professional development about technology.

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A Higher Standpoint

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ABSTRACT

In 1908, Felix Klein not only became the founding president of the *Commission internationale de l'enseignement mathématique* (CIEM, anglicized as the International Commission on the Teaching of Mathematics) but also published the first volume of his groundbreaking *Elementarmathematik vom höheren Standpunkte aus* (Elementary Mathematics from a Higher Standpoint). In the introduction, Klein identifies a central problem in preparing teachers to teach mathematics: a *double discontinuity* that the prospective teacher encounters in going from school to university and then back to school to teach. School mathematics and university mathematics typically seem to have no connection. Klein's course assumes that the prospective teachers are familiar with the main branches of mathematics, and he attempts to show how problems in those branches are connected and how they are related to the problems of school mathematics. Throughout his career, Klein saw school mathematics as demanding more dynamic teaching and consequently university mathematics as needing to help prospective teachers "stand above" their subject.

In print for a century, the volumes of Klein's textbook have been used in countless courses for prospective and practicing teachers. They provide excellent early examples of what today is termed *mathematical knowledge for teaching*. Klein's courses for teachers were part of his reform efforts to improve secondary mathematics by improving the preparation of teachers. Despite the many setbacks he encountered, no mathematician has had a more profound influence on mathematics education as a field of scholarship and practice. Two later mathematicians whose contributions to mathematics education resemble those of Klein are George Pólya and Hans Freudenthal. After discussing their contributions, I suggest why *higher* is a better translation of *höheren* than *advanced* is and end by noting some problems posed when considering *mathematics education* from a higher standpoint.

Keywords
Klein, Pólya, Freudenthal, discontinuity, intuition, mathematical knowledge for teaching

In 1908, Felix Klein not only became the founding president of the Commission internationale de l'enseignement mathématique (CIEM, anglicized as the International Commission on the Teaching of Mathematics) but also published the first volume of his groundbreaking *Elementarmathematik vom höheren Standpunkte aus* (Elementary Mathematics from a Higher Standpoint). The third volume, on applications of calculus to geometry, had originally been published in 1902 but was revised and put at the end of the series because, as Klein (1924/1932) noted in his introduction to the third edition of the first volume, it had been "designed to bridge the gap between the needs of applied mathematics and the more recent investigations of pure mathematics" (p. v.), a somewhat different purpose than that of the first two volumes, which were designed "to bring to the attention of secondary school teachers of mathematics and science the significance for their professional work of their academic studies, especially their studies in pure mathematics" (p. v). The third volume (Klein, 1928) has never been translated from the original German, whereas the first two have also appeared in English and Spanish.

All three volumes in the series began as lithographed copies of handwritten lecture notes prepared by Klein's assistants Ernst Hellinger and Conrad H. Müller that were later edited for printed editions by Fritz Seyfarth and others. For some years, Klein had offered courses addressed to secondary school teachers, and in this series, he concentrated on the content of the secondary mathematics syllabus. The first volume was based on notes from a course given at Göttingen in the winter semester of 1907–1908, and the second, from a course given the following summer semester, in 1908.

ELEMENTARY MATHEMATICS FROM A HIGHER STANDPOINT

Arithmetic, Algebra, Analysis

Introduction

In the introduction to the first volume, Klein (1908, 1924, 1924/1932, 1933) identifies a central problem in preparing teachers to teach mathematics: a *double discontinuity* that the prospective teacher encounters in going from school to university and then back to school to teach. School mathematics and university mathematics appear to have no connection. Klein (1924/1932) identifies efforts to eliminate that discontinuity by updating the school curriculum, on the one hand, and by attempting "to take into account, in university instruction, the needs of the school teacher" (p. 1). His course, he says, will assume that the prospective teachers are familiar with the main fields within mathematics. His task will be to show

> the *mutual connection between problems in the various fields,* a thing which is not brought out sufficiently in the usual lecture course, and more especially to emphasize the relations of these problems to those of school mathematics. In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge there put before you a living stimulus for your teaching. (pp. 1–2)

In this quotation, one hears echoes of Klein's early views of mathematics education expressed in his inaugural address (*Antrittsrede*) of 1872 when he became professor at Erlangen at the age of 23. The problem of the secondary school curriculum was, for Klein, neither insufficient time nor inadequate content:

> What is required is more interest in mathematics, livelier instruction, and a more spirited treatment of the material! . . .
>
> At stake [for university teachers of mathematics] is the task . . . of raising the standards of mathematical education for later teaching candidates to a level that has not been seen for many years. If we educate better teachers, then mathematics instruction will improve by itself, as the old consigned form will be filled with a new, revitalized content! . . .
>
> [Therefore,] we, as university teachers, require not only that our students, on completion of their studies, know what must be taught in the schools. We want the future teacher to stand *above* his subject, that he have a conception of the present state of knowledge in his field, and that he generally be capable of following its further development. (Klein, in Rowe, 1985, p. 139)

Throughout his career, Klein saw school mathematics as demanding more dynamic teaching and consequently university mathematics as needing to help prospective teachers "stand above" their subject.

To conclude the introduction to the volume, Klein cites several recent discussions of mathematics instruction that supplement the topics he will be treating. He points out, however, that some treatments of elementary mathematics build it up "systematically and logically in the mature language of the advanced student, [whereas] the presentation in the schools . . . should be psychological and not systematic. . . . A more abstract presentation will be possible only in the upper classes" (Klein, 1924/1932, pp. 3–4). He also points out that he adopts a "progressive" stance:

> We, who are called the reformers, would put the function concept at the very center of instruction, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into instruction as early as possible with constant use of the graphical method, the representation of functional relations in the *x y* system, which is used today as a matter of course in every practical application of mathematics. . . . Strong development of space perception, above all, will always be a prime consideration. In its upper reaches, however, instruction should press far enough into the elements of infinitesimal calculus for the natural scientist or insurance specialist to get at school the tools which will be indispensable to him. (p. 4)

Klein is anticipating the emphasis that he puts in the subsequent text on applications, geometric illustrations, space perception, and the historical development of the field. The book is divided into three parts—arithmetic, algebra, analysis—together with supplementary sections on transcendental numbers and set theory.

Arithmetic

The main topics in the first part are the natural numbers; the extension to negative numbers, fractions, and irrationals; number theory; and complex numbers. An example of Klein's emphasis on practical applications is his extended treatment of the mechanism for calculating machines (see Figure 1, which shows how multiplication is performed). Later in the book, when discussing logarithmic tables, Klein (1924/1932) mentions that such a machine "makes logarithmic tables superfluous. At present, however, this machine is so expensive that only large offices can afford it. When it has become considerably cheaper, a new phase of numerical calculation will be inaugurated" (p. 174)—truly prophetic words.

Figure I

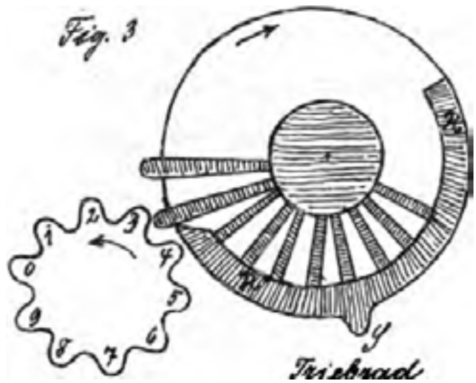


Figure 1. Driving wheel and cogwheel in a calculating machine (Klein, 1908, p. 48).

Klein ends the discussion of arithmetic with a brief survey of the modern development of mathematics. Reviewing the first edition, John Wesley Young (1910) said, "It is a mere sketch, but it is a masterpiece" (p. 258). In the survey, Klein distinguishes two processes by which mathematics has grown, each of which leads to a different plan for instruction. In Plan A, the plan more commonly followed in school and in elementary textbooks, each branch of mathematics is developed separately for its own sake and with its own methods. The major branches—algebraic analysis and geometry—make occasional contact but are not unified. In Plan B, in contrast, "the controlling *thought is that of analytic geometry,* which seeks a *fusion of the perception of number with that of space*" (Klein, 1924/1932, p. 77). Mathematics is to be seen as a connected whole, with pure and applied mathematics unified. Not surprisingly, Klein argues that Plan B is more likely than Plan A to engage those pupils "not endowed with a specific abstract mathematical gift" (p. 78). Both plans have their place, and neither should be neglected. But secondary school instruction

> *has long been under the one-sided control of the Plan A.* Any movement toward reform of mathematical teaching must, therefore, press for *more emphasis upon direction* B. [Klein is] thinking, above all, of an impregnation with the *genetic method of teaching,* of a stronger emphasis upon *space perception,* as such, and, particularly, of giving *prominence to the notion of function,* under *fusion of space perception and number perception!*" (p. 85)

Klein then argues that his aim in these books is to follow Plan B, thereby balancing existing books on elementary mathematics that almost invariably follow Plan A.

Algebra

The main topics of the second part of the book concern the use of graphical and geometric methods in the theory of equations. Klein begins by citing textbooks on algebra and pointing out that the "one-sided" approach he will take is designed to emphasize material neglected elsewhere that can nevertheless illuminate instruction. His approach to solving real equations uses the duality of point and line coordinates, and he draws on the theory of functions of a complex variable to show how to represent, using conformal mapping, the solution of equations with a complex parameter.

Analysis

The third part of the book concerns elementary transcendental functions and the calculus. It begins with a discussion of the logarithm, which provides a good illustration of Klein's approach. He first considers how the logarithm is introduced in school—by performing the operation inverse to that of raising to a power—and draws attention to various difficulties and possible confusions that accompany such an approach, including the absence of any justification for using the number e as the base for what are, for the pupil, inexplicably called the "natural" logarithms. After discussing the historical development of the concept, emphasizing the pioneering work of Napier and Bürgi, Klein proposes an introduction that would define the logarithm of a as the area between the hyperbola $xy = 1$, the x-axis, the ordinate $x = 1$, and the ordinate $x = a$, first approximating the area as a sum of rectangles and then taking the integral. The section on the logarithm ends by considering a complex-theoretic view of the function, which Klein argues that teachers should know even though it would not be an appropriate topic in school. In Young's (1910) review of the book, he points at Klein's treatment of the logarithm as the only one of his proposed reforms that would not be practical in the United States (and perhaps not even in Germany) since pupils need to use logarithms before they encounter hyperbolas, not to mention integrals.

The trigonometric functions and hyperbolic functions are also treated from the point of view of the theory of functions of a complex variable, and the part ends with an introduction to the infinitesimal calculus that relies heavily on Taylor's theorem and that includes historical and pedagogical considerations. The supplement at the end of the volume contains a proof of the transcendence of e and ϖ and a brief, lucid introduction to set theory.

GEOMETRY

In the second volume, Klein (1909, 1925, 1925/1939) takes a different approach than in the first. Arguing that there are no unified textbook treatments of geometry, as there are for algebra and analysis, he proposes to give a comprehensive overview of geometry, leaving all discussion of instruction in geometry for a final chapter (unfortunately not included in the English translation). Two supplements to the third edition that were prepared by Seyfarth in consultation with Klein "concern literature of a scientific and pedagogic character which was not considered in the original text" (Klein, 1925/1939, p. vi; the supplements were not translated into English either).

The volume, like the first, has three parts. The first concerns the simplest geometric forms; the second, geometric transformations; and the third, a systematic discussion of geometry and its foundations. Not surprisingly, Klein's innovative characterization of geometries as the invariants of their symmetry groups, from his famous Erlangen program (see, e.g., Bass, 2005; Schubring, n.d.), forms the basis of his discussion of the organization of geometry. In the discussion of foundations, Klein (1925/1939) emphasizes the importance of non-Euclidean geometry "as a very convenient means for making clear visually relations that are arithmetically complicated" (p. 184):

> *Every teacher certainly should know something of non-euclidean geometry*. . . . On the other hand, I should like to advise emphatically against bringing non-euclidean geometry into *regular school instruction* (i.e., beyond occasional suggestions, upon inquiry by interested pupils), as enthusiasts are always recommending. Let us be satisfied if the preceding advice is followed and if the pupils learn really to understand *euclidean* geometry. After all, it is in order for the teacher to know a little more than the average pupil. (p. 185)

The third part ends with a discussion of Euclid's *Elements* in its historical context.

In the final chapter, Klein surveys efforts to reform the teaching of elementary geometry in England, France, Italy, and Germany. The supplement contains some additional observations on questions of elementary geometry and updated material on reform in the four countries, particularly reports prepared for the CIEM surveys of teaching practices and curricula that had been initiated during Klein's presidency.

In print for a century, the volumes of *Elementary Mathematics from a Higher Standpoint* have been used in countless courses for prospective and practicing teachers. Although both of the first two volumes provide much useful material and are excellent early examples of what today is termed *mathematical knowledge for teaching* (Ball & Bass, 2000; Bass, 2005), the organization of the first volume, with pedagogical issues and difficulties facing the teacher taken up after each topic rather than relegated to a final chapter, seems much superior to that of the second. The organization of the first volume allows Klein to make specific suggestions for instruction and references to textbooks and historical treatments of topics, whereas the comments in the second volume tend to be more general.

KLEIN AND MATHEMATICS EDUCATION

Like many mathematicians, Felix Klein spent much of his time working on issues of mathematics education once he was no longer doing research in mathematics. Unlike most of them, however, he had pursued such issues throughout his career. As noted above, Klein's Erlangen inaugural address of 1872 dealt with mathematics education (Rowe, 1983, 1985). In it, he deplored the lack of mathematical knowledge among educated people. He saw that lack as symptomatic of a growing division between humanistic and scientific education, a division in which mathematics is uniquely positioned: "Mathematics and those fields connected with it are hereby relegated to the natural sciences and rightly so considering the indispensability of mathematics for these. On the other hand, its conceptual content belongs to neither of the two categories" (Rowe, 1985, p. 135). Observing that like all sciences, mathematics is undertaken for its own sake, Klein goes on to argue that "it also exists in order to serve the other sciences as well as for the *formal educational value* that its study provides" (p. 137).

By "formal educational value," Klein did not mean the attention to form over content that dominated German mathematics education at the time: "Instead of developing a proper feeling for mathematical operations, or promoting a lively, intuitive graphs of geometry, the class time is spent learning mindless formalities or practicing trivial tricks that exhibit no underlying principle" (Rowe, 1985, p. 139). Instead, Klein saw mathematics as a formal educational tool for training the mind. He was not especially concerned with pupils' mastery of formal procedures; he wanted them to understand the procedures they were using. He also wanted those pupils who would become gymnasium teachers to have, if possible, some experience in doing an original research study in mathematics, which was at the time a requirement in Prussia to become certified as a mathematics teacher. Klein was not concerned with which mathematical topics they studied as long as they learned to work independently.

In the inaugural address at Erlangen, Klein expressed a neohumanistic view of how mathematics ought to appear in school and university instruction, a view he was later to modify in light of his experience. After teaching at the technical institute in Munich from 1875 to 1880, for example, he adopted a more expansive outlook on the mutual roles of mathematics, science, and technology in modern education. When he became professor of geometry at Leipzig in 1880, he began to promote the teaching of applied mathematics in universities as well as in technical institutes. Klein's ultimate goal was to make mathematics a foundational discipline in higher education, and to achieve that goal, he initiated a reform of secondary mathematics education so that it would include the calculus. In Erlangen, however, he had said that livelier teacher rather than new subject matter was what the secondary schools needed: In autobiographical notes he made in 1913 (Rowe, 1985, p. 125), he summarized what he had said in that address: "An den Gymnasien auszubauen: Interesse. Leben und Geist. Kein neuer Stoff [To develop in the high schools: Interest. Life and spirit. No new material]." He then added a marginal remark reflecting his revised opinion that the secondary curriculum did need new material: "Da bin ich nun anderen Sinnes geworden [I have changed my mind about that]." After 40 years of teaching, Klein also reversed his view that prospective teachers should conduct an independent study on any topic whatsoever. In private notes made available to his colleague Wilhelm Lorey (1916, quoted in Rowe), he wrote:

> I would now suggest that teaching candidates of average talent should confine themselves to such studies as will be of fundamental importance in the later exercise of their profession, while everything beyond this should be reserved for those with unusual talent or favorable circumstances. (p. 128)

A final comment in Klein's (1913, quoted in Rowe) autobiographical notes suggests the toll his battles for reform had taken: "When one is young, one works much more hastily and unsteadily, one also believes the ideals will soon be attained" (p. 126).

Nonetheless, Klein was successful in reforming the secondary school curriculum as well as in creating university courses for teachers. His goal had long been to raise the level of mathematics instruction in both the technical institutes and the universities, and he came to realize that the key to achieving that goal would be to raise the level of secondary mathematics instruction to include the calculus, thereby raising the level of tertiary instruction (Schubring, 1989). To push for reform in secondary and tertiary curricula, Klein forged an alliance among teachers, scientists, and engineers, and he also helped the international commission become an agent for curricular change. His courses for teachers were part of reform efforts to improve secondary mathematics by improving the preparation of teachers. Despite the setbacks he encountered and the resulting changes in approach he made, no mathematician has had a more profound influence on mathematics education as a field of scholarship and practice.

PÓLYA AND FREUDENTHAL

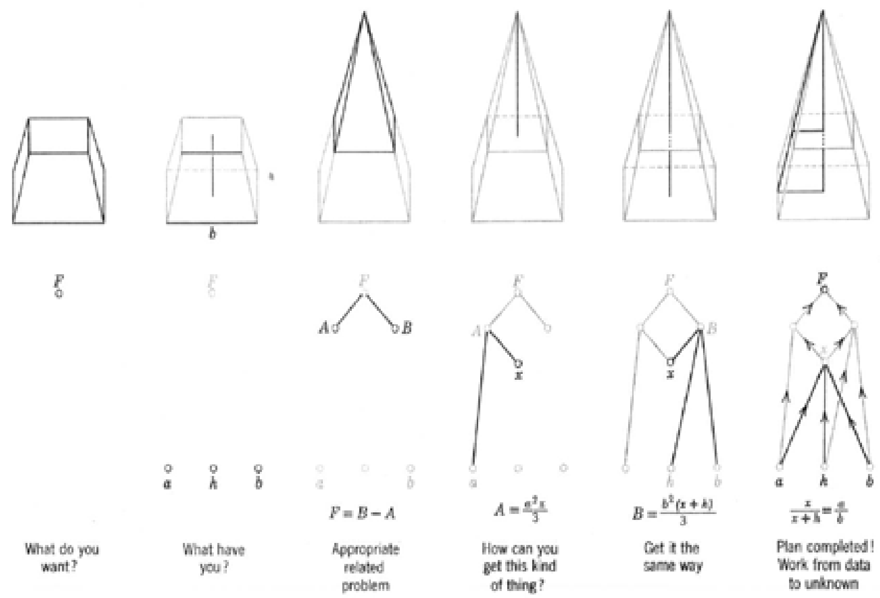
Two mathematicians whose contributions to mathematics education resemble those of Klein are George Pólya and Hans Freudenthal. Like Klein, Pólya was interested in number theory, theory of functions in the real and complex domain, mathematical physics, applied mathematics, and the art of teaching mathematics. Both were also strong proponents of the role of intuition in doing and learning mathematics. In 1912, Pólya went to Göttingen for postdoctoral studies, where he met Klein although did not take any courses from him. Talking about the connection between polyhedra and groups, Pólya later said, "I learned it from the master—Felix Klein" (quoted in Alexanderson, 2000, p. 27).

Pólya's interest in pedagogical questions began at an early age. While doing postgraduate studies at the University of Vienna in the academic year of 1910–1911, he had taken a tutoring job. At the beginning of the second volume of *Mathematical Discovery,* he recounts that experience:

It happened about fifty years ago when I was a student; I had to explain an elementary problem of solid geometry to a boy whom I was preparing for an examination, but I lost the thread and got stuck. I could have kicked myself that I failed in such a simple task, and sat down the next evening to work through the solution so thoroughly that I shall never again forget it. Trying to see intuitively the natural progress of the solution and the concatenation of the essential skills involved, I arrived eventually at a geometric representation of the problem-solving process. This was my first discovery, and the beginning of my lifelong interest, in problem solving. (Pólya, 1965, p. 1)

Pólya then shows graphically, using a problem on the volume of the frustum of a right pyramid, how the solution can be visualized as a sequence of connections, building a bridge between what is given and what is unknown (Figure 2). Pólya's (1919) first publication on problem solving and heuristics made use of this means of expressing how a solution might progress. Two years earlier, when he was only 30, Pólya had delivered a speech on teaching at the city hall in Zürich (Alexanderson, 1987, p. 18), and his publication repeated the argument he had given in the speech (Pólya, 1938, p. 119).

Figure 2. Simultaneous progress on four levels
(Pólya, 1965, Fig. 7.8, p. 9).



Pólya (1984) saw the same discontinuity between high school and college mathematics that Klein did:

> [The prospective teacher] takes a course offered by the mathematics department about some relatively more advanced subject. He has great trouble to keep up with, and to pass, the course, because his knowledge of high school mathematics is inadequate. He cannot connect the course at all with his high school mathematics. Or he takes a course offered by the school of education about teaching methods. It is offered in accordance with the principle that the school of education teaches only methods, not subject-matter. Our prospective teacher may receive the impression, which was scarcely intended, that teaching methods are essentially connected with inadequate knowledge, or ignorance, of the subject-matter. At any rate, his knowledge of high school mathematics remains marginal. (pp. 531–532)

Pólya approached the courses he taught for teachers in much the same spirit as Klein did. He too wanted teachers to have opportunities to carry out independent projects in mathematics, and in his course assignments, he asked teachers "to discuss how the topic might be treated in school, what points students might have difficulty with, and what connections might be made to other problems or topics" (Kilpatrick, 1987, p. 92). Pólya promoted a reflective practice in which teachers looked back and critiqued their teaching, just as he did his own (p. 96).

Like Klein and Pólya, Hans Freudenthal turned to mathematics education early in life. As he said, "All my life I have been a poor teacher, and in order to make the best of it I started thinking about education at an early age" (quoted in Goffree, 1993, p. 22). Appointed a *privaat-docent* in 1930 at the University of Amsterdam at the age of 25, one of the courses Freudenthal taught was entitled Elementary Mathematics from an Advanced Standpoint (Van Est, 1993, p. 61). Early in the Second World War, while giving lessons in arithmetic to his two sons, he started studying the literature in didactics of arithmetic and making notes for a "didactics of arithmetic" book that unfortunately exists only in fragmented, manuscript form (Goffree, p. 24). Before and during the war, Freudenthal participated in the Dutch Mathematics Study Group, which discussed issues in mathematics education, attempted to develop curricula, and provided Freudenthal with what he called his "college of mathematics education" (quoted in Goffree, p. 26). In 1963, Freudenthal became a member of the reconstituted International Committee on Mathematical Instruction (ICMI) and served as ICMI President from 1967 to 1970.

Freudenthal, like Klein, was interested in applications of mathematics, emphasizing the utility of mathematics and what he termed the mathematizing process. In 1967, Freudenthal organized a colloquium in Utrecht entitled "How to Teach Mathematics So As to Be Useful," and in the introductory address laid out why he thought mathematics should be taught so as to be more useful. That address (Freudenthal, 1968) and the other colloquium papers were later published in the first issue of the journal that Freudenthal founded, *Educational Studies in Mathematics*. Freudenthal's appointment to the chair in geometry at Utrecht in 1946 had piqued his interest in geometry as a research field, and another affiliation with Klein arose when Freudenthal began to explore the connection between geometries and their symmetry groups (Van Est, 1993, p. 62). Freudenthal (1978, p. 131) credits Klein with introducing the term *model* to refer to a mathematical object that embodies a set of axioms or other conditions.

When it came to characterizing mathematical learning process, Freudenthal (1978) made the important observation that the process proceeds by moving from one "level" to a higher one: "Mathematics *exercised* on a lower level becomes mathematics *observed* on the higher level" (p. 61). Through a process of reflection, mathematical activity at one level becomes mathematical subject matter at the next level. Freudenthal criticized Klein's *Elementarmathematik* series for failing to address explicitly the need to move to a new level: "The 'high' in higher mathematics means raising the level, or at least should mean it, and if something should be made conscious in the learning process at university, it is this raising of level" (p. 71).

"ADVANCED" OR "HIGHER"?

When it came time for the American translators of Klein's *Elementarmathematik* to render the title in English, they chose to translate *vom höheren Standpunkte aus* as *from an advanced standpoint*. The term *higher* is not only a more literal translation of *höheren* than *advanced* is, but it also captures better the image Klein had for his work. *Advanced* can mean *higher*, but its connotation is more like "more developed" or "further along in space or time." Klein wanted to emphasize that his courses would give prospective teachers a better, more panoramic view of the landscape of mathematics. As noted above, he wanted those teachers to "stand above" their subject.

Discussing the mathematics a teacher needs to know, Klein (1924/1932) wrote: "The teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them" (p. 162). The metaphor here is that of guide, someone who knows the mathematical terrain well and can conduct his or her pupils through it without them getting lost or injured. Klein went on to discuss how the novice teacher needs to be equipped to counteract common misperceptions of mathematical ideas:

> If you lack orientation, if you are not well informed concerning the intuitive elements of mathematics as well as the vital relations with neighboring fields, if, above all, you do not know the historical development, your footing will be very insecure. You will then either withdraw to the ground of the most modern pure mathematics, and fail to be understood in the school, or you will succumb to the assault, give up what you learned at the university and even in your teaching allow yourself to be buried in the traditional routine. (p. 236)

Klein, Pólya, and Freudenthal all saw the value of helping teachers develop mathematical knowledge that went beyond the content they would teach and was more synoptic than the typical university mathematics course. They all saw that teachers need to know more than how to do the mathematics they are teaching; teachers need the specialized mathematical knowledge and skill that will give them a broad perspective on the field and equip them to work with learners. It is no accident that all three of these eminent figures in our field were first-rate mathematicians and also master educators.

MATHEMATICS EDUCATION FROM A HIGHER STANDPOINT

What would it mean to view mathematics education from a higher standpoint? Mathematics education as an academic field is not a school subject, and as a university subject, it belongs, at best, among the social sciences. In his inaugural address in Erlangen, Klein noted a critical difference between mathematics and other fields: "Each mathematical generation builds on the accomplishments of its predecessors, whereas in other fields it often happens that the old buildings are torn down before the new construction can proceed" (Rowe, 1985, p. 136). Consequently, the question of what is elementary and how one might adopt a higher stance to regard that elementary work becomes problematic when one moves outside of mathematics and certainly when one moves into mathematics education. What is elementary in mathematics education? Do people agree?

Where is the higher standpoint from which that elementary mathematics education can be surveyed? Does anyone know?

Mathematics educators have begun to consider the history of their field, and through the lenses of international comparative studies, they have begun to consider its geography. So we have the beginnings of efforts to get some "higher" vantage points across time and space. As mathematics education continues to develop during the next century of the international commission, the higher standpoints that Felix Klein, George Pólya, and Hans Freudenthal took with respect to mathematics may inspire mathematics educators to find similar standpoints for examining their field.

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Amongst Mathematicians and conversations on the teaching and learning of mathematics at university level: the case of visualisation

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How do students in the beginning of their undergraduate studies in mathematics cope with the requirement for rigour? Why do they so often resort to the familiarity of number? Why do they have problems with constructing examples and with identifying and accepting counterexamples? How do they manage to express in symbols their thoughts about the convergence of a sequence? Why is reference to the domain of a function so conspicuously absent in their writing? How do their teachers at university help them acquire the 'genre speech' of university mathematics and the mathematician's 'toolbox' of useful images, theorems and techniques? Do these teachers pursue the help of mathematics education researchers in these complex tasks? If at all, how? If not, why not?...

The above questions provide a flavour of the issues that the research I am reporting in this paper aimed to explore. To this purpose I am drawing on the data and analyses presented in Amongst Mathematicians: Teaching and Learning Mathematics at University Level, a 2008 Springer monograph (Nardi, 2008) that was based on this research. The study offers a perspective on how mathematicians: perceive student learning; describe and reflect on their own teaching practices; and, perceive their relationship with mathematics educators. Its evidence base is a series of focused group interviews with mathematicians from across the UK. Its analyses were presented in the format of a dialogue between two fictional, yet entirely data-grounded characters, M and RME, mathematician and researcher in mathematics education. (See the Appendix for a typical page from *Amongst Mathematicians*: each piece of dialogue between M and RME sets out from a discussion of a sample of student work, typically a piece of writing. The samples of student work exemplify topical learning and teaching issues – as highlighted in the literature and in previous research conducted by myself and colleagues at the universities of East Anglia and Oxford. Examples of relevant bibliography are cited in the footnotes accompanying the dialogue between M and RME.)

In what follows I first outline the study's background, aims and methods. I then discuss three samples of findings focusing, respectively, on: an issue of student learning (*Sample I*, the role of visualisation in mathematical reasoning and argumentation, highlighted in the literature as key to the students' early experiences of university mathematics); related pedagogical issues (*Sample II*); and, in closing, issues regarding the relationship between the respective communities of M and RME (*Sample III*).

TALUM, A NEW AND RAPIDLY DEVELOPING FIELD OF RESEARCH

TALUM, the *Teaching and Learning of Undergraduate Mathematics*, is a relatively new and rapidly developing field of mathematics education research Holton, 2001). As, particularly in the 1990s, mathematics departments started to respond to the decline in the number of students who opt for mathematical studies at university level (Hillel, 2001), the realisation that, beyond syllabus change, there is also the need to reflect upon tertiary pedagogical practice began to grow (McCallum, 2003). The research programme I am reporting here was conceived and carried out with the aim to address this need in a systematic and original way.

The study I am focusing on in this paper is underlain by a rationale for a certain type of TALUM research. The study draws on several traditions of educational research reflected in the five, essential characteristics listed below: it is collaborative, context-specific and data-grounded and, through being non-prescriptive and non-deficit, it aims to address the often difficult relationship between the communities of mathematics and mathematics education. A fundamental underlying belief of this work is that development in the practice of university-level mathematics teaching is manageable, and sustainable, if driven and owned by the mathematicians who are expected to implement it.

This rationale for collaborative, practitioner-engaging and context-specific research draws heavily on Barbara Jaworski's *Co-Learning Partnerships* (2003) and John Mason's *Inner Research* (1998). In these types of research practitioners of mathematics teaching engage with research and they, along with the researchers, become co-producers of knowledge about learning and teaching; they become educational co-researchers (Wagner, 1997). In this sense the study is a first step towards engaging with Developmental Research (van den Akker, 1999), a much needed type of research in undergraduate mathematics education. Furthermore the study has aimed to steer clear of a tendency (that sometimes studies of teaching suffer from) towards a 'deficit' and 'prescriptive' discourse on pedagogy, where the emphasis is on the identification of what it is thought teachers ought to be doing and are not doing, and on appropriate remedial action (Dawson, 1999). The work I am reporting here is located explicitly within a non-deficit and non-prescriptive discourse.

The study (Nardi, 2008) is the latest in a series of studies aiming to:

- explore students' learning in the first, and sometimes, second year of their undergraduate studies – mostly in Analysis, Linear Algebra and Group Theory and mostly through observing them in tutorials (Nardi, 1996; Nardi, 2000) and analysing their written work (e.g.: Nardi & Iannone 2001); and,
- engage their lecturers in reflection upon learning issues and pedagogical practice – mostly in individual (Nardi, Jaworski and Hegedus, 2005) and group interviews (Iannone and Nardi, 2005).

The studies were conducted at the Universities of Oxford and East Anglia in the UK between 1992 and 2004. Further studies, that aim to refine the themes emerging from the earlier studies, as well as take steps towards collaborative implementation of innovative pedagogical practice, are currently in progress.

STUDENT DATA, THE DATA THAT 'BECAME' M AND THE *RE-STORYING* APPROACH

The dialogues between M and RME that I exemplify in the following pages originate in eleven lengthy (approximately four-hour / half-day) focused group interviews with 20 mathematicians of varying experience and backgrounds from across the UK. In the interviews discussion was triggered by Datasets consisting of students' written work, interview transcripts and observation protocols collected during (overall typical in the UK) Year 1 introductory courses in Analysis / Calculus, Linear Algebra and Group Theory – see background studies listed in the previous page. Datasets had been distributed to the interviewees at least a week prior to the interview and were about a dozen pages long, split in four to six sections. A typical section of the Dataset typically consisted of:

- a mathematical problem (including its formulation as well as the suggested solution distributed to the students once they had submitted their written responses to their tutor)
- two typical student responses, often reflecting learning issues highlighted in relevant mathematics education literature

The interviews were conducted according to the principles of Focused Group Interviews (Madriz, 2001). Below I explain the narrative approach of *re-storying*

(Clandinin and Connelly, 2000) adopted in this work and the composition process through which the dialogues between M and RME came to be. In short the process of re-storying involves reading the raw transcripts, identifying and highlighting experiences to be told across this raw material and then constructing a new story that reflects these experiences. In this sense, while fictional, the new story is entirely data-grounded. In addition to the work of narrative researchers such as Clandinin and Connelly cited above a particularly helpful way of seeing the brand of re-storying I have used is Jerome Bruner's account of how the mind constructs a sense of reality through 'cultural products, like language and other symbolic systems' (1991, p3). The dialogues between M and RME in (Nardi, 2008) were constructed entirely out of the raw transcripts of the interviews with the mathematicians and then thematically arranged in *Episodes*. (For an example of the construction process see p27-28 in (Nardi, 2008)).

(Subsequently in (Nardi, 2008) chapters were constructed as series of *Episodes*, sometimes also broken in *Scenes*. Each *Episode* starts with a mathematical problem and usually two student responses. A dialogue between M and RME on issues exemplified by the student responses follows. Other examples of relevant student work are interspersed in the dialogue and links with relevant mathematics education research literature are made in the footnotes. *Special Episodes* are episodes that supplement the discussion in the main *Episodes* and *Out-Takes* are slightly peculiar or too specific incidents that stand alone and outside the more 'paradigmatic' material of the main *Episodes* but somehow address the wider theme of a chapter.)

Below I outline briefly a rationale for the dialogic format employed in the study that goes a little beyond a conventional methodological account. It may look like a digression but the brief text that follows is deeply ingrained into the study's, and the book's, *raison-d'-être*.

A BRIEF DIGRESSION REGARDING THE DIALOGIC FORMAT

> '…all you can do, if you really want to be truthful, is to tell a story'
>
> Paul Feyerabend (1991), quoted in Mason (1998, p367)

The idea for the character of M of course is not new – neither is the idea of a conversation between a researcher in mathematics education and a mathematician (Sfard, 1998a). Sfard's Typical Mathematician (1998b, p495) and Davis &

Hersh's Ideal Mathematician (1981) pre-date this study's M. Dialogue as a form for communicating and debating ideas is a format most quintessentially used by philosophers such as Plato, Galileo, Berkeley, Feyerabend and, crucially for mathematics educators, Lakatos in Proof and Refutations (1978). In theatre as well, authors such as Tom Stoppard (Arcadia) and Michael Frayn (Copenhagen, Democracy) have deployed the dialogic format in admirable attempts to help the subtle meet the artful effectively. In this sense the ultimate aim for using the dialogic format as a way of representing processed data is to employ storytelling as a different kind of science:

> Vanbrugh: [...] The plot already exists... in real life. The play and all its scenes.
> Cibber: A drama documenting facts? [...] Will you allow yourself the same liberties as Shakespeare? Taking liberties with facts converts facts into plays.
>
> Vanbrugh: No liberties... just facts in this play.
>
> *Calculus*, Scene I (Djerassi & Pinner, 2003)

SIX THEMES ON THE TEACHING AND LEARNING OF UNIVERSITY MATHEMATICS

As mentioned earlier, the dialogues between M and RME were thematically arranged in *Episodes*. Then clusters of Episodes around each one of the following six themes constituted the six chapters of data analyses presented in (Nardi, 2008):

- students' mathematical reasoning; in particular their conceptualisation of the necessity for proof and their enactment of various proving techniques;
- students' mathematical expression and their attempts to mediate mathematical meaning through words, symbols and diagrams;
- students' encounter with fundamental concepts of advanced mathematics –
- Functions (across the domains of Analysis, Linear Algebra and Group Theory) and Limits;
- pedagogical practices at university level; and,
- the often fragile relationship between M and RME as well as the necessary and sufficient conditions for their collaboration.

In the rest of this paper I collate samples of data and findings from across the above themes. Sample I reports manifestations of student perceptions of the role of visualisation as evident in their mathematical writing. Sample II reports their lecturers' reactions, mathematical and pedagogical, to these manifestations and outlines a pedagogical role for the mathematician in fostering a fluent interplay between rigour and visual insight. Finally, Sample III collates elements of the discussion between M and RME which focuses on the benefits for pedagogical practice ensuing from engagement with educational research.

Notes
All quotations that follow, except otherwise noted, are utterances of the character M – page numbers indicate pages in (Nardi, 2008). Also: the data and analyses reported in these samples have appeared partly also in (Nardi , 2009a and b).

Sample I: Students' perceptions on the role of visualisation

Students often have a turbulent relationship with visual means of mathematical expression. When they find difficulty in connecting different representations (for instance: formal definitions and visual representations), they often abandon visual representations - which tend to be personal and idiosyncratic - for ones they perceive as mathematically acceptable (Presmeg, 2006). Here we take a look at M's perspective on students' attitudes towards visualisation and on the ways in which these attitudes – and ensuing behaviour – can be influenced by teaching. The discussion eventually becomes about the importance of building bridges between the formal and the informal in constant negotiation with the students.

First and foremost M describes pictures as efficient carriers of meaning – in the case of $| \, |$ as distance, for example:

> 'What the students really need to be thinking about is what $| \, |$ means on the number line and as a distance. But they so often get stuck to the algorithmic habit of solving this without knowing what it means. And that stubbornness can be a nightmare.
>
> What I mean by *what it means* is, for example, seeing, what an equality or inequality involving $|x-1|$ means pictorially on the real line. Once you have seen it on the line, the answer to your question is obvious. That is why I am

> a huge fan of them using all sorts of visual representation: because the ones who do, almost invariably are the ones who end up writing down proper proofs.', *p238*

Instead students often feel ambivalence towards 'picture', even wondering 'are pictures mathematics?!'

> 'Students often mistrust pictures as *not mathematics* – they see mathematics as being about writing down long sequences of symbols, not drawing pictures – and they also seem to have developed limited geometric intuition perhaps since their school years. I assume that, because intuition is very difficult to examine in a written paper, in a way it is written out of the teaching experience, sadly. And, by implication, out of the students' experience. It is stupefying sometimes to see their numb response to requests such as imagining facts about lines in space or what certain equations in Complex Analysis mean as loci on the plane.', *p139*

This ambivalence can lead to a narrow, inflexible, even mutually exclusive adherence to informal or formal modes of thinking:

> '... students somehow end up believing that they need to belong exclusively to one of the two camps, the informal or the formal, and they do not understand that they need to learn how to move comfortably between them', *p140*

Now let's delve into the above general statements about student tendencies in the context of a specific mathematical problem and see how they pan out.

The premise for the discussion is the following mathematical problem (typically given to Year 1 mathematics undergraduates in a Semester 1 Calculus / Analysis course):

Write down a careful proof of the following useful lemma sketched in the lectures. If $\{b_n\}$ is a positive sequence (for each n, $b_n > 0$) that converges to a number $s > 0$, then the sequence is bounded away from 0: there exists a number $r > 0$ such that $b_n > r$ for all n. (Hint on how to start: Since $s > 0$, you might take $\frac{1}{2}s = \epsilon > 0$ in the definition of convergence.)

One acceptable approach to this is described in the notes below (written by the lecturer of the course the problem originally comes from):

Let $\epsilon = \frac{s}{2}$ in the definition of convergence. Then there is an N such that $n > N \Rightarrow |b_n - s| < \frac{s}{2} \Rightarrow b_n \geq \frac{s}{2}$. Then, for any n, $b_n \geq r = \min\{b_1, \ldots, b_N, \frac{s}{2}\}$ which is the minimum of finitely many positive quantities, hence is positive.

Questions / issues touched in the discussion included: what responses would you expect from the students to this problem; what difficulties may they face; if you were to discuss this problem with a student how would you do so?

One of the issues that emerged in the course of discussing this problem concerns the fact that, in the second line of the lecturer's notes, it is a perfectly acceptable part of the argument to 'leave out' of the inequality the terms b1, b2, …bN. Why this is helpful can also be visible in a simple picture that portrays the 'boundedness away from zero' of the significant majority of the sequence's terms. Students treated the contingency of such a picture variably. See Table 1 which shows three typical Year 1 student responses and the comments made on them – with regard to the presence, absence and quality of such a picture in the students' scripts – by M.

Overall M's insights into students', and M's own, perceptions regarding the role of visualisation revolved around the following four axes:

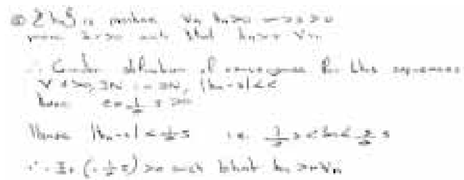
- Usefulness of visual representations: firm and unequivocal ('Graphs are good ways to communicate mathematical thought', p. 143);
- Usefulness of educational technology, e.g. graphic calculators: caution and concern ('Calculators are nothing more than a useful source of quick illustrations', p. 143);
- Students' varying degrees of reliance on graphs (both in terms of frequency and quality); and,
- The potentially creative fuzziness of the 'didactical contract' at university level with regard to the role of visualization.

We will now focus on the last two. In a nutshell, M's views are largely put forward in the light of how mathematicians employ visualisation in their own mathematical practice. The emergent perspective is of the need for a clarified didactical contract (Brousseau, 1997), in which students are encouraged to emulate the flexible ways in which mathematicians to-and-fro between analytical rigour and often visually-based intuition.

Table 1. Three ways of relating to 'pictures'.

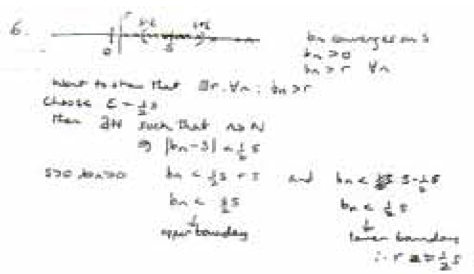
Student N, no picture

Student N has not left out of his argument a small but significant number of terms in the sequence he is working on. 'Had the student drawn a picture, he would have seen he had left them out'.



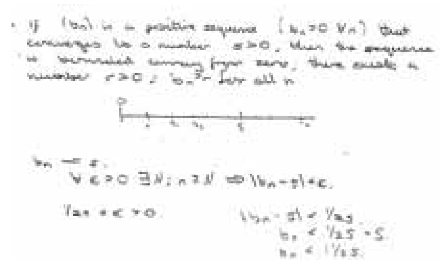
Student H, unhelpful picture

Student H, emulates 'the type of picture drawing seen in lectures'. She however 'needed a more helpful picture'. It is encouraging though that both Students N and H pinned down an understanding of | | as a 'distance between things'.



Student E, not benefiting from picture

Student E has not 'used this diagram as a source of inspiration for answering the question'. Instead 'she drew this, on cue from recommendations that are probably on frequent offer during the lectures, and then returned to the symbol mode unaffected'. So 'there is no real connection between the picture and the writing'.



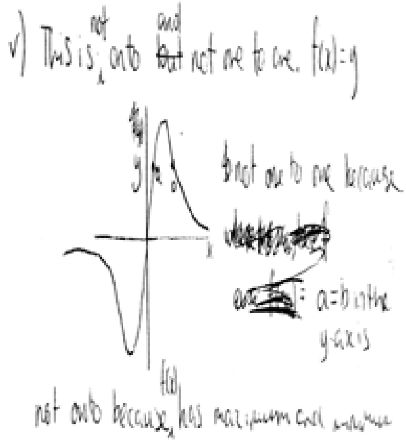
Examples and M utterances from p. 140 and pp. 195–199 in (Nardi, 2008)

The premise of the discussion is a question in which students were invited to explore whether certain functions from R to R were one-to-one and onto. In the two examples of student responses below M identifies two distinct ways in which students typically appear to rely on graphical evidence – see Table 2.

Table 2. Two ways of relying on graphs.

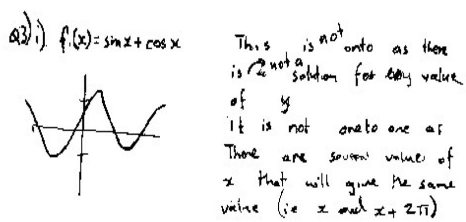
Student WD, absence of transition from picture to wording

'I am concerned about the answer being provided before the graph is produced but I also observe that the answer has been modified on the way – which may mean the graph did play some part after all in the student's decision making. If the student had drawn a line through points a and b, I would be a bit more convinced that the student is actually building the argument from what they see in the graph. I am also disappointed by the absence of a transition from the picture to some appropriate words and with the use of a=b to denote that points a and b on the curve have same y. What a use of the equals sign! In this sense…'



Student LW, no construction evidence

'… I am more sympathetic to Student LW …who may need the Intermediate Value Theorem to complete the argument in part (i) – the IVT is true after all –, the picture is almost perfect, all the shifting etc. is there, but this is still an incomplete answer. Still there is no construction evidence.'



Examples and M utterances from p. 144 in (Nardi, 2008)

M is particularly keen to stress what he calls the 'irony in using the graph to produce evidence that a function is one to one or onto' (p144) as the ability to construct this graph would in itself require this knowledge:

> '…I find this evidence compelling but still this is not a complete answer. This picture is potent and I see a certain danger in its sophistication: the fact, for example, that, if a function has a maximum, it cannot be onto is immediately graspable from the graph. However some unpacking is still necessary in order to provide a full justification of the claim.
>
> I am a proponent of starting with a diagram but I do not wish to see this placing value on starting with a diagram giving the students a false sense of obligation to do so, another hurdle to get over. I want them to think of doing so as a totally natural procedure to follow but also do it correctly.', p144

From this quotation and the one below begin to emerge some of the terms of the renewed, clarified didactical contract mentioned earlier:

> 'I would be far less frustrated if I could find evidence in the students' writing that the diagram is used almost as a third type of language, where the other two are words and symbols, as an extension of their power to understand: just drawing a diagram bigger, or, for example in the first picture, putting in a horizontal line that goes through the points *a* and *b*. I am afraid students do not use pictures to their full potential. Of course I see that relying on their power therein lies a danger but I would like to see students make a sophisticated use of this power and be alert to their potential to be misleading too.', p145

First of all students need to be alerted to what I term here 'the creative fuzziness of the 'didactical contract'. 'Fuzziness' is used here to denote the necessary acknowledgement that a clear-cut distinction between their obligation to engage with mathematics formally *or* informally, in a mutually exclusive way, is too simplistic. It is also ineffective; in fact, hence the use of 'creative', it is exactly this to-ing and fro-ing between the formal and the informal modes of engaging with mathematics that will ultimately turn out to be the most effective. M outlines two significant phases in imparting this new type of didactical contract:

Allow the use of visual insight, acknowledging that the students, by the nature of introductory university courses, are already using unproven facts:

> 'Students should be allowed at this stage to use the graphs for something more than simply identifying the answer because after all they allowed to use all sorts of other facts – the uniqueness of cubic roots is one of those facts – that have not been formally established yet.
>
> So if the Intermediate Value Theorem is implicit in their finding the answer by looking at the graph, then let that be! Of course one needs to check: an actual value of a and b there would be very reassuring. At this stage I feel sympathy for them and want to let them say this function is onto because of the uniqueness of the cubic root. Because at this stage, well, I don't want to tell you what the cube root of two is … I want to tell you the cube root of eight is. I am not sure I even know how to exhibit the cube root of two without resorting to some quite sophisticated ideas.', p145

Eventually prove, conveying that ultimately mathematics is mainly about establishing facts via proof:

> 'I am happy with using the ingredients for proving a claim and then, at some later stage, spending some time on establishing those ingredients formally. So prove that e^x is injective via the IVT and then later on prove the IVT. This to me is fine as long as I know that all along I have been leaving some business-to-be-finished on the side. That kind of rigour is fine with me.', p146

Below M concludes with two pertinent observations on this matter.

SAMPLE II: PEDAGOGICAL PRACTICE WITH REGARD TO VISUALISATION

M describes three key elements to a teacher's response to the student perceptions outlined in Sample I: acknowledgement of the innately human need for visual insight, raising students' awareness and celebration of this typically very personal need, assist them in pursuing the construction of such insights:

> '…they need to learn how to move comfortably between [the formal and the informal]. Because in fact this is how mathematicians work! I still remember acutely my own teachers' explanations of some Group Theory concepts via

> their very own, very personal pictures. I am a total believer in the Aristotelian *no soul thinks without mental images.* In our teaching we ought to communicate this aspect of our thinking and inculcate it in the students. Bring these pictures, these informal toolboxes to the overt conscious, make students aware of them and help them build their own.
>
> And I cannot stress the last point strongly enough: we need to maintain that these pictures are of a strictly personal nature and that students should develop their own. All I can do is describe vividly and precisely my own pictures and, in turn, you pick and mix and accommodate them according to your own needs.', p237

At the heart of this three-step plan of support is the frank acknowledgement that this approach to visualisation reflects the ways mathematics is understood and created by mathematicians themselves. Further elaborating the 'this is how mathematicians work' statement above M adds:

> 'Lest we forget some very clever people regarded [IVT] not needing a proof either! People like Newton. [...] there is an irony in the fact that validating the truth of the statement in IVT means that all the pictures that students have been drawing are retrospectively true – like drawing the solutions of an equation. This irony in fact is nothing other than another piece of evidence of a constant tension within pure mathematics: that you want to use these methods and occasionally you need a theory to come along and make them valid. And you need these means, diagrams etc., so badly. Yes, they are not proofs but they do help students acquire first impressions, start inventing some suitable notation.', p238

M proceeds with the presentation of examples from mathematics where the above is the case. I omit these due to limitations of space but they are available in: p238 (geometric problems in the complex plane), p240 (exponentials) and p 241 (powers) in (Nardi, 2008); and (Nardi, 2009b).

In the course of the interviews M stressed repeatedly how much of the pedagogical awareness and the potentially effective pedagogical practices evidenced above became available through participation in this study.

SAMPLE III: BENEFITS FROM ENGAGEMENT WITH EDUCATIONAL RESEARCH

M often juxtaposed the accusation for 'indecipherability', futility and irrelevance of mathematics education research often mounted by the mathematics community (Ralston, 2004) to the potent experience of participating in these interviews. Often M cited improved access to understanding students as a primary benefit of this participation:

> '… it is in these discussions exactly that these sessions have proved enormously valuable already. There are things I will teach differently. There are things that I feel like I understand better of mathematics students than I did before. And I appreciate the questioning aspects of the discussion and I realise how one should be liaising with the other lecturers simultaneously lecturing the students and discussing what things we are doing that confuse them.', p260

A substantial part of this understanding consisted of realising the extent of student difficulty:

> '…these discussions are already beginning to influence the way I think about my teaching. I think discussing the examples is a very good starting point, and a well-structured one. By seeing these often terrifying pieces of writing I am faced with the harsh reality of the extent of the students' difficulties. Too often I see colleagues who are in denial and opportunities like this are poignant reality checks! […] I am therefore grateful for this opportunity to face the music, so to speak.', p261

A significant outcome of this understanding is fostering an appetite, and capacity, for change, pedagogical innovation, even reform, away from conventional views of mathematics and how it is learnt and taught:

> 'There is substance in this; it is important.
>
> Suppose you have a schoolteacher. So, here is someone who has to run classes and, for some reason or another, their view of mathematics is no other than

> an instrumental one: you apply this rule, you put this in and you get this out. Suppose that such a person one day meets *Concept Image* and all that. All of a sudden he learns that these things are all out there and that changes that person's professional view entirely. It can change the whole classroom, it can change the whole mathematical process. That is precisely what we want.
>
> A lot of the problems you have to deal with when you meet our students is that they have a very singular view of mathematics, a rather poor view of mathematics. So, I mean, that sort of debate that is happening here is on some of the building blocks around which, it seems to me, if made available at the school level for practitioners, would be hugely interesting. To get away from this sort of mathematics which is quite poor in a way.', p262

M often concluded the discussion emphasising the gaining of awareness, and a renewed appreciation of openness regarding questions of pedagogy:

> 'I think now I don't have any more answers than when I started but certainly I don't take things for granted anymore, from colleagues or from students.
>
> I think I am much more open-minded on what might be going on inside other people brains. The material that you have got here has given the evidence that sure, it is fascinating glancing in other people's heads.
>
> And I have become much more conscious about the spoken word. What I say can have an impact, saying the right thing at the right time when you get one opportunity to introduce the students for the first time to how mathematics works and not fluff the line. That I think has made a big influence on the way I lecture.', p263

Often the discussion between M and RME signalled direct parallels with the educational literature Sample 3: M and RME – benefits, change EXAMPLE of parallel with literature – M's comment below on the importance of substantial feedback to students' written work echoes Mason's recommended tactics on this matter (*Focusing on what is mathematical; Developing a language; Finding something positive to say; Selecting what to mark; Summarising your observations; and, Providing a list of common errors or a 'corrected' sample of student argument*, Mason, 2002, Chapter 5):

> [...] examining these pieces of data was something of a reminder, if not a revelation, of the devastating importance of detailed responses to written work. In some sense every not totally perfect piece of written work has an interesting important story to tell that needs to be engaged with and responded to.', p263

Therefore it will be far from a surprise to say that the entire study incarnates rather aptly the much needed synergy between mathematicians and mathematics educations often discussed by Michèle Artigue:

> '...we, mathematicians as well as didacticians [...] have to act energetically in order to create the positive synergy between our respective competences which is necessary for a real improvement of mathematical education, both at secondary and at tertiary levels. Obviously such a positive synergy is not easy to create and is strongly dependent on the quality of the relationship between mathematicians and didacticians'. (Artigue, 1998, p482/3)

CONCLUDING REMARKS: WHAT HAS NOT BEEN AND ...A FUTURE FOR M/RME?

While the study reported in this paper focused on matters of learning and teaching that could be broadly described as 'cognitive only' (the discussion rarely turned to topical socio-affective matters such as gender, affect, equity etc.) its aspirations to meet at least two objectives were nonetheless rather wide: obviously, to listen to what M, experienced learner, doer and teacher of mathematics, has to say about learning and teaching; and, less obviously, to allow a certain image of M to emerge (characterised by pedagogical awareness, perceptiveness and sensitivity) which would be in contrast to widespread pedagogical stereotypes of university mathematicians. And, to do so through its distinctive characteristics (context-specific, example-centred, mathematically-focused samples of data, discussed in a relaxed yet focused, unthreatening and mutually respectful research ambience).

In resonance with its non-prescriptive character the study refrained from direct recommendations for practice. However, soon after its completion, a brief guide with a focus on the teaching of proof was published following a request by the UK's Higher Education Academy (Nardi and Iannone, 2006). Alongside several studies that aim to refine some of this study's findings (e.g. Ioannou and Nardi, 2009), in the (hopefully near!) future we aim to continue with more directly developmental work, namely: the construction, implementation and evaluation of innovative practice. We are currently in the process of designing a series of such interventions in collaboration with colleagues from mathematics departments in the UK.

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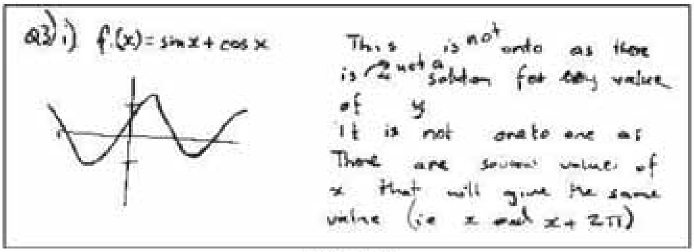
APPENDIX

A typical page from *Amongst Mathematicians* (Nardi, 2008). An example of student work at the top of the page becomes the trigger for the dialogue between M and RME in the middle. In the footnotes the reader is referred to relevant bibliography.



Student LW

M: Of course if one uses the calculator then it makes no difference: all parts of the question are equally simple or complex. But that's a big issue in itself, the use of the graphic calculator!

RME: I sense you fervently want to raise it!

M: Well, a calculator could give you a good picture in part (i), for example. The students here seem to have access to a calculator and it seems to me they have missed a terrific amount of practice in the Analysis which is how these questions ought to be tackled. Generally students lack graphing skills and awareness of what to look for when asked to produce a graph[24].

RME: But surely the technology should help them see more examples and develop such expertise.

M: Yes, they have access to more pictures, and more easily, but have no feel for what to look out for.

[24] Intertwined here are at least two issues: constructing graphs and extracting mathematical meanings from interpreting graphs – for a review on graph comprehension and definitions of what constitutes good graph sense see (Friel et al, 2001) and the reference to (Roth & Bowen, 2001) in E7.4, Scene III. The two are linked as, particularly in the absence of a graphic calculator, for example, resorting to an understanding of a function's properties is a necessary step towards the construction of its graph. Researchers have used graphing tasks to investigate graphing skills and students' developing comprehension. E.g. drawing on the theoretical construct of APOS (Action, Process, Object, Schema) as well as Piaget & Garcia's triad of levels for schema development (intra, inter, trans) Baker et al (2000) studied undergraduates' comprehension of a non-routine graphing problem in Calculus. The triad was applied for properties (condition-property schema) and for intervals (domain, interval schema). Several student difficulties were observed: with cusp point, vertical tangent, removal of the continuity condition and second derivative. Overall co-ordinating information about properties and intervals was a problem – as was the resilience of incorrect images and overly emphasis on first derivative. Generally the two-schema interplay (property, interval) was difficult for the students

Celebrating the first century of ICMI (1908-2008) Some aspects of the history of ICMI

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ABSTRACT

In this paper we report on the events in 2008 that commemorated the Centennial of the International Commission on Mathematical Instruction. This celebration offered the occasion to look back at the history of ICMI and outline the evolution of mathematics education until it achieved its present status as an academic discipline. The years after WWII up to the late 1960s were crucial in this evolution for both the settlement of some institutional aspects (mainly concerning the relationship with mathematicians) and the establishment of new trends of the activities. In this paper we outline – on the basis of unpublished documents - the role of two important figures in those years: Heinrich Behnke and Hans Freudenthal. First as secretary and later as president, Behnke faced the difficult task of reshaping the newborn ICMI after WWII and clarifying the relationship with mathematicians. His mission was completed by Freudenthal, who, as president of ICMI, definitively broke with the past and promoted important initiatives that fostered the emergence of mathematics education as an academic field.

Keywords
History, ICMI, first century, mathematicians, mathematics education

1. CELEBRATING THE FIRST CENTURY OF ICMI: A SYMPOSIUM, A BOOK AND A WEBSITE

The IV International Congress of Mathematicians, which took place in Rome from 6 to 11 April 1908, was memorable. An exceptional chronicler, the French mathematician Henri Poincaré, wrote:

> "The number of participants was the highest of any of the preceding Congresses, which is doubtless due to the attraction of the Eternal City, but this is not the only reason. [...] France was brilliantly represented [...] there were also very distinguished representatives of German science [...] No country was absent. [...] It goes without saying that Italy had the most and most brilliant representatives [...] The sessions were held at the Palazzo Corsini, home of the Accademia dei Lincei [...] a beautiful palace in Trastevere [...]" (Poincaré, 2008, pp. 19-20, our translation).

It was during this congress that an international commission on the teaching of mathematics was founded; its first president was Felix Klein, an eminent mathematician and promoter of an important reform for the teaching of mathematics in Germany. This commission may be considered the first incarnation of the International Commission on Mathematical Instruction (ICMI).[1]

To celebrate the centennial of the foundation of the ICMI, an international symposium entitled "The First Century of the International Commission on Mathematical Instruction. Reflecting and shaping the world of mathematics education" was held in Rome from 5 to 8 March 2008. Once again, as it did a hundred years ago, Palazzo Corsini, home of the Accademia Nazionale dei Lincei, provided the splendid venue for the congress, along with Palazzo Mattei di Paganica, home of the *Enciclopedia Italiana*.[2]

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[1] In the first decades of its existence the commission was mainly called *Commission Internationale de l'Enseignement Mathématique* (CIEM) or *Internationale Mathematische Unterrichtskommission* (IMUK). In the following we will use the acronym ICMI to refer to all periods.

[2] The International Programme Committee (IPC), composed of 16 members, was coordinated by Ferdinando Arzarello, while Marta Menghini represented the Organising Committee within the ICP. The permanent website http://www.unige.ch/math/EnsMath/Rome2008/ provides full documentation of the Symposium: the program, the papers presented in the Working Groups, and photos.

The Congress was attended by about 180 participants representing 43 countries. The program included ten plenary lectures, eight parallel lectures, five working groups and a panel discussion. An afternoon was reserved for the Italian teachers, with talks by scholars from Italy and abroad. The talks were broadcast via videoconferences to more than fifty schools throughout Italy; the afternoon session reached more than 1000 teachers.

The last day featured an excursion that recalled that of a century ago, and took the participants to visit the Villa d'Este and Hadrian's Villa in Tivoli, both rich in historical evocations.

The Symposium proceedings have been published by the *Enciclopedia Italiana*, as a volume of their book series *Scienze e Filosofia* (Menghini et al., 2008) and the talks of the Italian afternoon appeared in the journal *Progetto Alice* (n. 25, 2008).

Taking as a point of departure the themes connected to ICMI activities over the course of its hundred year history the symposium sought to identify future directions of research and initiatives for improving mathematics culture in the various countries. The conviction that history is a powerful means not only for giving an account of the past but also for building the future, inspired the activities of the Symposium as well as the publication. The papers in the Proceedings touch on a wide variety of themes: the origins of the ICMI; its rebirth at the end of the 1960s and the emergence of a new field of research; the dialectic between rigour and intuition; the relationships between pure and applied mathematics and the emphasis to be given to each; the interactions between research and practice; the comparison between centres and peripheries of the world; the relationships between mathematics and mathematics teaching; the training of teachers; and the relationship of mathematics education to technology, society and other disciplines. It emerges that ICMI has mirrored the development of mathematics education as a field of study and practice, and stimulated new directions of research, opening new horizons.

The comparison of the "historical" and the "didactical" papers points out the evolution of the research in mathematical education from a collateral aspect of mathematics to an autonomous scientific discipline, whose interactions with mathematics are continuously evolving. The richness of contributions, both in plenary lectures and in working groups, show how varied and deep its current landscape is. The historical papers point out the extent to which ICMI activities in past years had prepared for the growth and development of important themes in didactics of mathematics and the role played by some important figures.

In parallel with the evolution of didactics of mathematics, many papers, and particularly the panel "ICMI's challenges and future", underline a social-geographical evolution from the centre to the peripheries of the world: the rooting of mathematical education in the different cultures gives it a further dimension, which is a richness in today's globalised landscape.

On the occasion of the symposium, a permanent website was created to present the history of the ICMI (see Furinghetti & Giacardi, 2008). Its aim is to delineate the most significant events and the key figures through documents, images and interviews, and make available for the scholars all the tools which are necessary to reconstruct the complex network of relationships that featured this century of history.

The site is divided into six sections: *Timeline; Portrait gallery; Documents; Affiliated Study Groups; International Congresses on Mathematical Education; Interviews and Film clips*. The *Timeline* pinpoints the most important moments in the history of the ICMI: each fact is documented with references to the original sources, in particular to its official organ *L'Enseignement Mathématique* (hereafter *EM*)[3] with links to its website. Many images, photos and quotations by the protagonists have been inserted. The *Portrait gallery* provides biographic cameos of the ICMI officers during the first hundred years of the Commission who have since passed away, stressing their roles within the ICMI, their publications and their contributions on the problems of teaching. As far as possible the contributions to this section were written by scholars from the country of the officer concerned. Among the *Documents* we find the digitalised versions of the publications of the Central Committee, the texts of the questionnaires proposed for the inquiries and the relative reports, the successive Terms of Reference of ICMI, and the list of the documents kept in the ICMI Archives.[4] The section dedicated to *Affiliated Study Groups* (HPM, ICTMA, IOWME, PME, and WFNMC) presents the history of these groups. The section dedicated to the *International Congresses on Mathematical Education* offers general information about each of them, with bibliographical references, and the Resolutions of the Congress. Finally, *Interviews and Film Clips* provide

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3 About this journal and its link with ICMI see (Furinghetti, 2003; 209)

4 The documents referring to ICMI are in the folders 14 A-G of the IMU files stored at the central Archives of the University of Helsinki. In the following we refer to them as *ICMI Archives* (*IA*). Cf. (Giacardi, 2008b)

the testimony of some of the protagonists of the history of the ICMI – Emma Castelnuovo, Trevor Fletcher, Geoffrey Howson, Maurice Glaymann, Jean-Pierre Kahane, Heinz Kunle, André Revuz and Bryan Thwaites.

2. Glimpses of the first century of ICMI

2.1. The five periods of the history of ICMI

The celebration of the hundred years of ICMI[5] has provided an occasion of going through its history, also outlined in (Bass & Hodgson, 2004; Howson, 1984; Lehto, 1998). The book that resulted from the Rome Symposium organized to celebrate the ICMI centenary (Menghini et al., 2008) sheds light on some aspects of the life of this Commission. Donaghue (2008) and Schubring (2008a) document the inception of ICMI, the dissolution and the ephemeral rebirth between the two World Wars. Furinghetti et al. (2008) identify the conditions of ICMI's renaissance after WWII. The recent years, which saw the definitive establishing of mathematics education as an academic discipline, are illustrated in chapters by Bass (2008), Bishop (2008), and Kilpatrick (2008). The website built on the occasion of ICMI's centenary provides detailed information on people and events (Furinghetti & Giacardi, 2008).

It is possible to identify five main periods in the history of ICMI that were produced by both the external events that influenced the life of the Commission as well as by the changing centres of interest and activities of the Commission itself: the early years; the years between the two World Wars; the reconstitution after WWII; the renaissance; the recent decades.

During the early period, from its founding up to WWI, rightly called the "Klein Era", an important international network of national subcommittees was established for the preparation of reports on the state of mathematical instruction as well as on thematic issues. The original aim was to make an inquiry and publish a general report on the current trends in the secondary teaching of mathematics in various countries. However, already in the first meeting after its foundation ICMI acknowledged the need to consider all school levels. The work of the national subcommittees was really impressive. In 1920, at the moment of disbanding, in addition to the eleven Publications of the Central

5 Website of ICMI. ttp://www.mathunion.org/ICMI/

Committee there were about 300 reports of the national subcommittees of eighteen countries, for a total number of more than 13,500 pages. At the same time, eight inquiries had been launched and international congresses had been organized. The methodological tenets that underpinned Klein's conception of mathematics teaching and to some extent inspired the early Commission's work concern: bridging the gap between secondary and higher education; the early introduction of the concepts of function and transformation; the applications of mathematics across all the natural sciences; the applications of algebra to geometry and vice-versa; giving importance to the *Approximationsmathematik*, that is, "the exact mathematics of approximate relations"; fostering of intuition in teaching; the approach to topics from a historical perspective favouring a genetic teaching method; the role of elementary mathematics as seen from an advanced standpoint in teacher training.

After WWI the new conditions imposed on official international scientific relations forced international commissions or associations created before the war to either dissolve or reorganize and the shocking decision to ban the researchers of the Central Powers from most international activities was made. ICMI dissolved in 1920 and it was reconstituted only during the ICM in Bologna in 1928, when international collaboration among mathematicians was re-established, reintegrating the countries that had been excluded. However, the Commission was not able to produce new ideas and projects, and was limited to carrying out the old agenda, until WWII forced a second arrest of activities.

In 1952 during the First General Assembly of the reconstituted International Mathematical Union (hereafter IMU), which took place in Rome, the Commission was transformed into a permanent sub-commission of IMU. The new president of ICMI for the period 1952-1954 was Albert Châtelet, dean of the Faculty of Sciences in Paris; the secretary was Heinrich Behnke from the University of Münster, who two years later would become president of ICMI. In the following years, ICMI defined – not without difficulty – several basic structural issues (composition, relationship with the IMU, the organisation of regional international groups, etc.), and established collaborations both scientific and organisational with other associations. These led to a greater internationalism and to the organisation of numerous thematic congresses in various parts of the world, as well as to a broadening of the fields of interest of ICMI.

The actual renaissance and projection into the future took place in the late 1960s. It was Freudenthal who, by establishing almost with a coup de main the tradition of the ICMEs and by founding the new journal *Educational Studies in Mathematics*[6], marked a turning point in the history of the ICMI.

In the last decades there has been an evolution of the relationship between ICMI and IMU, which produced the recent changes in the governance of ICMI, giving more power to the ICMI General Assembly. The focus of the present paper is on some moments that have fostered these changes and, in particular to the two actors in these moments who played a fundamental roles.

2.2. *Problems and frictions at the rebirth of ICMI*

If the first president and enthusiastic promoter was Felix Klein, a considerable role in establishing the Commission was also played by David Eugene Smith, a professor at Teachers College of New York, who was deeply interested in education and in the history of mathematics. Thus the Commission was born of the closest collaboration between mathematicians and educators. The construction of the present status of the Commission and the birth of the discipline "Mathematics Education" are linked to the process of clarification of the relationship with the community of mathematicians and to the carrying out of initiatives independent from this community. The two main characters in the process of constructing new trends were Heinrich Behnke and Hans Freudenthal: both were able to grasp the changes in the world and in mathematics happening in their times and act consequently. Broadening our contribution to the celebration of the centennial we like to reflect on their role stressing the importance of their institutional actions.

Behnke (1898-1979) was a well-known mathematician in the field of complex analysis, professor at the University of Münster, and editor of *Mathematische Annalen* from 1938-1972. Deeply concerned with mathematical education, Behnke invested great energy in teacher training: he had founded the journal *Semesterberichte zur Pflege des Zusammenhangs von Universität und Schule aus den mathematischen Seminaren*, which he edited together with Otto Toeplitz, another mathematician strongly committed to education. The journal was aimed at encouraging the connection between school and university.

In 1955 Behnke also proposed the *Idee völlig utopisch* (Behnke, 1959, p. 148) of realizing of an international encyclopaedia of elementary mathematics, and ardently hoped for the collaboration of mathematics teachers of all levels in order

[6] About the foundation of this journal see (Furinghetti, 2008; Hanna & Sidoli, 2002)

to maintain contacts between schools and universities. On that occasion he wrote: "The work done should, if possible, be compiled in a book for the congress of Edinburg in 1958. The Italian encyclopaedia of elementary mathematics can be regarded as a model in certain ways".[7] In fact in 1958 the German subcommittee of the ICMI produced the first of the five-volume *Grundzüge der Mathematik für Lehrer an Gymnasien sowie für Mathematiker in Industrie und Wirtschaft*. In the preface to the first volume the editors Behnke and Kuno Fladt stress that the work was above all aimed at teachers: "They have always been uppermost in our thoughts. The destiny of future generations of mathematicians depends on their mastery and their love of our science" (p. V-VI). The group of collaborators – which numbered more than 100 members – included scholars not only from Germany but also from Yugoslavia, the Netherlands, Austria and Switzerland, and, significantly, comprised university and high school teachers. Each article has two authors, of which one is a university professor, the other either a high school teacher or someone coming from this career.

As Schubring (2008b) puts it, "[h]aving in so many respects become a true successor of Felix Klein, he eventually followed Klein's footsteps in organizational respect as well". He was the first secretary-general of the renewed Commission, then president of the ICMI from 1955 to 1958, vice-president from 1959 to 1962, and member of the Executive Committee from 1963 to 1970. Behnke's political action was particularly incisive in the period of his secretariat and presidency.[8] That he was completely aware of the problems he had to face in revitalising the Commission emerges from his correspondence: the difficulty of finding mathematicians active in research who were interested in teaching; the difficulty of being recognised in the world of mathematics, and thus how important it was that the work of the commission be visible at the international congresses; the difficulty of obtaining funding; and finally the relevance of the collaboration of mathematics teachers at all levels. In a confidential letter to IMU president Marshall Stone, he wrote:

> "It is a very difficult matter to engage mathematicians, well-known for their research work, into problems of instruction. Most of our colleagues refuse

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[7] *Program of Work of the International Commission for Mathematical Instruction for the period of 1955/58*, in *IA*, 14 A 1955-1957.

[8] See (Lehto, 1998) and the documents from the ICMI Archives we report below

> to be active for our commission because they regard this kind of work of little value, and they even neglect to forward circulars. [...] The work of our commission reveals its purpose and meaning only when we give lectures and exhibitions at the international mathematical congress".[9]

When Behnke became president he too underlined the need to improve the terms of reference for governing the activities of the Commission and the relationships with IMU; the importance of having regional groups so to decrease the euro-centricity in ICMI; and the need of having ICMI Congresses. In his long report of April 1955 on the activities of ICMI to the IMU president Heinz Hopf, he affirmed:

> "The national sub-commissions suffer from being ruled by university professors, for their influence is predominant through the national adhering organizations … although the number of the university professors in their countries (at least in Europe) represents but a very small part of the teachers of mathematics [...] As president of the International Commission of Mathematical Instruction it is my duty to see that the members of the Commission are not university professors only [...] the presidency of the national adhering organizations does not appreciate questions of mathematical instruction and inconsiderately uses its national power.
>
> [...] The presidency of the IMU has [...] to look upon the national sub-commissions – as was the case already before 1914 - as sub-commission of the ICMI, and not of the national adhering organizations. Otherwise the work of the ICMI is made impossible.
>
> [...] I regard it a special, honorary mission of the ICMI to establish a contact among the teachers of all levels. The teacher have to get interested in the research work, and those active in the field of research have to get interested in the work of the teachers".[10]

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[9] Behnke to Stone, Oberwolfach, August 11, 1954, in *IA*, 14A, 1952-1954.

[10] *Report of the president of the International Commission of Mathematical Instruction to the president of the International Mathematical Union*, April 20, 1955, in *IA*, 14A, 1955-1957.

In fact the beginnings of the new commission were not easy and relations with the IMU were characterised by constant friction, derived from the lack of precise terms of reference for governing the activities of the Commission.
Let us quote some passages from the correspondence in ICMI Archives. Stone, IMU president, wrote to Châtelet:

> "It is my understanding that the Commission has proposed an arrangement whereby it will seek the adherence of several nations and set up special national committees in the adhering nations to work with the Commission. I believe that activity of this kind is inappropriate for a Commission of the Union and that it would lead to intolerable confusion as to the relations between the Union, the Commission, and the nations adhering to one or the other".[11]

Stone sent a similar message to Enrico Bompiani, secretary of IMU:

> "There seems to be a great deal of confusion in connection with the ICMI. I hope we can get it cleared up. [...] The difficulties [...] make me particularly aware of the fact that we need to clarify our procedure for appointing the members of Commission [...]".[12]

As well, William Hodge, member of the Executive Committee of IMU wrote to Stone:

> "About ICMI, I agree very strongly that something must be done to curb its activities. At a recent meeting of our national committee very grave concern was expressed at the fact that so many of the Commission's activities were carried on behind our backs and that we were being let in for responsibilities we know nothing about. [...] I think it will be necessary to lay down very precise terms of reference for the Commission, and to define its powers very rigidly. It will also be necessary to select a president very carefully".[13]

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[11] Stone to Châtelet, Chicago, November 3, 1952, in *IA*, 14A, 1952-1954.
[12] Stone to Bompiani, Chicago, July 10, 1953, in *IA*, 14A, 1952-1954.
[13] Stone to Châtelet, Chicago, July 29, 1954, in *IA*, 14A, 1952-1954.

And again Stone to Châtelet:

> "In connection with the Constitution of the National Sub-Commissions, I recall our agreement that each such Sub-Commission is to be in the first place a Sub-Committee of the National Committee for Mathematics in the Country which it represents".[14]

In any case Behnke tried to make up for President Châtelet's lack of initiative, and succeeded in organising the intervention of the ICMI at the International Congress of Mathematicians in Amsterdam in 1954 notwithstanding both the difficulties in relationships between ICMI and IMU, and the resistance on the part of the organising committee of the congress. In a confidential letter to IMU president Stone, he wrote:

> "They [mathematicians] all expressed the idea that lectures on mathematical instruction might not be worthy enough for the Congress. Thus I showed them the reports of previous congresses and pointed out that after 1912 in Cambridge (England) Section VII (history and instruction) was as strongly accentuated as Section II (analysis). [...] After a report stating this fact I recommended to rebuilt Section VII. Finally I succeeded [...] I was given a highly unfavourable time for the report on the work of our commission, which would never happened in the case of my scientific lectures".[15]

The intention of Behnke was to recreate the climate of fervour and international collaboration that existed during Klein's chairmanship, and he meant "to extend the influence of Section VII (Instruction) so that it will equal the importance it had at the Congress in 1914",[16] the first congress organised by the Commission in Paris.

During the General Assembly of the IMU in The Hague (31 August - 1 September 1954) the Terms of Reference were established, the Executive Committee of the ICMI was renewed and Behnke was nominated president.

[14] Hodge to Stone, May 31,1954, *IMU Archives*, quoted in (Lehto, 1998, p. 111).

[15] Behnke to Stone, Oberwolfach, August 11, 1954, in *IA*, 14A, 1952-1954.

[16] Behnke to Bompiani, Münster, July 8, 1954, in *IA*, 14A, 1952-1954.

According to the new Terms of Reference ICMI had a relatively free hand in its internal organisation, but IMU retained control on important points: the President and the ten members-at-large of ICMI would be elected by the General Assembly of IMU on the nomination of the Union's President. Moreover, the national delegates would be named by each National Adhering Organisation of IMU.

2.3. *New trends and political issues in 1954-1967*

The period from 1954 to 1967 is characterised by important developments both of scientific and organisational kind, which will smooth the way for the ICMI renaissance.

a. New themes to investigate emerged thanks to the interaction with other organisations, such as the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM), which focused on new issues such as: the relevance of psychology in mathematics education; the attention to methodology; the key role of concrete materials; the need to take in consideration all the school levels (from primary to university); empirical research; the relation between mental and mathematical structures (Furinghetti et al., 2008). This different point of view emerges, for example, from the report presented in 1958 at ICM XIII in Edinburgh by Hans Freudenthal (1959) on the comparative study of the methods used in the initiation to geometry: he goes on to a rich and in-depth examination of the teaching subjects and the teaching methods, also considering the impact that psychological and pedagogical research may have on geometrical instruction in the initiating phase.

b. There was an effort to renovate the journal *L'Enseignement Mathématique* by starting a second series with renovated objectives that include space for psychology and methodology in mathematics education:
   "This second issue will deal with the subject of mathematics giving special care to the modern theories treating the subject in an easy form studying the methodology and organisation of the teaching, studying the psychological formation of mathematical knowledge, publishing reports on the activity and inquiries of the ICMI. Each issue will contain a bibliographical index".[17]

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[17] *Note concerning the Review "Enseignement Mathématique"*, in *IA*, 14 A, 1955-1957; see also *EM*, 1955, s. 2, 1, pp. 270-271

c . ICMI tried to gradually reduce its Eurocentric nature by attempting to extend the Commission's activities beyond Europe. In 1955 the Indian Ram Behari was nominated a member of the Executive Committee; in 1956 the ICMI was officially represented by its vice-president Stone in the Conference on Mathematical Instruction in South Asia in Bombay; in 1958 Behnke suggested forming regional groups like the European one to foster international collaboration; in 1961 Stone, president of ICMI, contributed to the organization of the First Inter-American Conference on Mathematics Education in Bogotá.

d. The effort to improve the organization of the Commission led the Executive Committee to discuss the Regulations of the ICMI during the meeting in Brussels on 3 July 1957. In 1958 Behnke proposed a new draft of the by-laws, the main points of which are the following:[18] the reduction to a single representative of each of the National Sub-Commissions; "ICMI is also authorized to accept appropriate organizations as National Sub-Commission even from countries which are not members of IMU"; the right of the National Sub-Commissions to co-opt additional members; "Each National Sub-Commission shall elect a Chairman. Generally, the Chairman shall be the representative to ICMI from his Sub-Commission [...] but he also is entitled to delegate a substitute who will have full voting power"; and finally the creation of Regional Groups".

This proposal was criticized by Stone, the new president of the ICMI who wrote to Beno Eckmann, Secretary of the IMU:

> "[...] the way is cleared for the elimination of any real influence in ICMI from the side of the mathematicians who are acquainted with the higher levels of their subject and who are interested in research as well as in teaching and preparation for research".[19]

[18] *(Draft) International Commission for Mathematical Instruction (ICMI) New Terms of Reference,* 1958, in *IA*, 14 A, 1958-1960; also EM, 1958, s. 2, 4, pp. 216-217.

[19] Stone to Eckmann, Chicago, January 5, 1959, in *IA*, 14 A, 1958-1960.

The New Terms of reference for ICMI would be adopted in 1960: Behnke's proposals were scaled back, and a control on the part of the IMU was sought. Behnke was obviously opposed.[20]

e. Collaborations with other Institutions (OEEC, UNESCO) were sought both for economic reasons and to widen the range of ICMI's influence. During his term as ICMI president Stone (1959-1962) promoted important activities and symposia in collaboration with local organisations, and the Organisation for European Economic Co-operation (OEEC). In 1959, he chaired the influential conference of mathematicians and educators at Royaumont devoted to the new thinking in mathematics and in mathematical education. Other important seminars and symposia were held, from which the guidelines on how to introduce "modern mathematics" into secondary schools emerged.[21] The next president of ICMI, André Lichnérowicz (1963-1966) especially promoted collaboration with UNESCO: UNESCO representatives were officially invited to the congresses organised by the ICMI; ICMI members were consulted as experts by UNESCO; and they were often sent on missions in various countries (Lichnérowicz, 1966). Thanks to the collaboration with UNESCO and other institutions, important international colloquia were organised,[22] and contracts stipulated directly between the ICMI and UNESCO led to the publication of the books in the series *New trends in mathematics teaching*.

2.4. *The renaissance in the late 1960s and the projection into the future: the role of Freudenthal*

In 1967 the presidency of the ICMI passed to Hans Freudenthal (1905-1990). He was a charismatic personality whose broad mathematical knowledge was

[20] Stone to Rolf Nevanlinna, April 5, 1960, in *IA*, 14 A, 1958-1960.

[21] We mention the following meetings: Royaumont, France (23 November- 4 December 1959); Aarhus, Denmark (30 May-2 June 1960); Zagabria – Dubrovnik, Yugoslavia (21 August-19 September 1960); Belgrade, Yugoslavia (19-24 September 1960); Lausanne, Switzerland (26-29 June 1961); Bologna, Italy (4-8 October 1961). See (Furinghetti et al., 2008) and (Giacardi, 2008, *1955-1959, 1960-1966*).

[22] We mention the following meetings: Frascati, Italy (8-10 October 1964); Utrecht, Holland (19-23 December 1964); Dakar, Senegal (14-22 January 1965); Echternach, Luxemburg, 30 May - 4 June 1965). See (Giacardi, 2008, *1960-1966*).

joined by a profound interest in culture, and who possessed a talent for organisation and the independent spirit necessary to mark a turning point in ICMI activities. In 1963, he entered the Executive Committee of ICMI, as a member until 1966, but from 1967 to 1970 as its president (and thereafter, from 1971 to 1978, as ex-officio member). The story of ICMI was deeply influenced by Freudenthal, who laid the foundations for its renaissance. The inspiring principles of his action were put forward in January 1967 at the UNESCO Colloquium in Lausanne on "Coordination of Instruction of Mathematics and Physics", in which he participated together with other important protagonists of mathematics education of those years, such as Anna Zofia Krigowska and Willy Servais. These principles were published as "Propositions on the teaching of Mathematics" in the first issue of the journal *Educational Studies in mathematics*, which Freudenthal founded in 1968. The main points were:

> "Mathematics constitutes a unique and characteristic activity of human mind. All children have a right to be educated through mathematics".
>
> "[Mathematics education] must provoke and develop in the first place the capacity of intellectual action instead of merely piling up knowledge".
>
> "Mathematics develops more and more towards a general science of structures. These structures charge it with a remarkable power of application, information and unification. The knowledge and the mastership [sic] of these structures, its utilization in the grasp of reality are the real objectives of mathematics teaching".
>
> "The reformation of mathematics teaching has to be considered a permanent process. This implies a continuous retraining of the teachers which is based on regular pedagogical research."[23]

As Howson (2008, p. 15) observes, "When he assumed the ICMI presidency (January 1967) he was faced with two alternatives: he could carry on as usual

[23] *Propositions sur l'enseignement mathématique*, in *IA*, 14B 1967-1974. Propositions on the teaching of mathematics. *Educational Studies in Mathematics*, 1, 1968, 244.

or he could try to break what was by then becoming the ICMI mould. He chose the latter". In fact, from the very beginning Freudenthal intended to open all future ICMI activities to discussion. His request for a permanent ICMI secretariat was rejected by the IMU, which saw neither an urgency nor a purpose for such a move; thus Freudenthal had the greater part of the work of secretary done by his office.[24] He sought and obtained funding beyond that provided by the IMU. He continued the collaboration with UNESCO already well established by his predecessors and stipulated a contract directly with UNESCO for the second volume of *New Trends in Mathematics Teaching* (1970).

The turning point took place in 1967 at the meeting of the Executive Committee of the ICMI held in Utrecht (August 26, 1967), following the colloquium "How to teach mathematics so as to be useful". There, the ideas for the ICME congresses and a new journal were launched. Freudenthal expressed his disappointment about the modality of ICMI participation at the quadrennial Congresses of Mathematicians and supported the idea of a congress of ICMI to be held a year before the ICM. He noted that, in general, the national reports were not useful, so he suggested that future congresses not include the topics of programs and scholastic organisation. He also listed the new subjects for discussion: mathematisation; motivation; how to teach mathematics without a schoolteacher; comparative evaluation of the contents of mathematics courses; criteria of success; evaluation of the results of research in mathematics education; and finally research methodology.

The assembly accepted the project of a Congress of ICMI to be held in 1969. The French delegate Maurice Glaymann proposed holding the congress in France. André Revuz asked for a new journal closer to secondary teachers, because *L'Enseignement Mathématique* was at too high a level. Freudenthal also proposed increasing the number of ICMI members-at-large in order to enliven the Commission, and returned to the question of a permanent secretariat.[25]

On December 2, 1967 IMU secretary Otto Frostman wrote to Freudenthal in an attempt to dissuade him from both initiatives:

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[24] Frostman to the International Commission on Mathematical Instruction, Djursholm, June 29, 1967, in *IA*, 14B 1967-1974.

[25] *EM*, 1967, s. 2, 13, pp. 243-246. *Compte-rendu de la séance de la CIEM tenue à Utrecht, le 26 août 1967*, in *IA*, 14B 1967-1974.

> "I must admit that I am not too happy about the new pedagogical journal. Do you really think that there is a market for two international journals of that kind (I do not)? If you are not satisfied with *L'Enseignement*, ICMI's official journal, perhaps it would be better to try to reform it. And I am afraid too that in a new journal the "modernizers" of the extreme sort would try to be very busy. At least I ask you to be cautious. I can agree with very much of your criticism of the meetings of ICMI at the International Congresses, but I am not sure that ICMI should isolate itself from those who have, primarily, a scientific interest but who have, nevertheless, very often taken part in the discussions of ICMI. And a special ICMI congress in France in 1969 will cost a lot of money".[26]

On December 20 Freudenthal replied:

> "I would like to reassure you about the new pedagogical journal. The provisional list of editors does not include any "radical". In spite of its name, *Enseignement* has never been a pedagogical journal. Its contributions on education were not pedagogical but organisatory [sic] and administrative. I do not believe it is possible to reorganize a journal so fundamentally".[27]

Frostman wrote to Freudenthal once again on January 2, 1968:

> "I am still a bit afraid that the market will be hard for two publications, even if the new journal will mainly stress other points than *L'Enseignement*."[28]

In March 1969 Frostman complained about not having received any report on ICMI activities[29]. ICMI secretary André Delessert answered him, announcing: the title of the new journal, *Educational Studies in Mathematics*, with Freudenthal as editor and saying that two issues had come out, the first in May 1968, and the second in January 1969; the date and place of the first ICME (Lyon, 24-30 August 1969); the preparation of the journal *Zentralblatt für Didaktik der Mathematik*,

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[26] Frostman to Freudenthal, December 2, 1967, in *IA*, 14B 1967-1974.

[27] Freudenthal to Frostman, Utrecht, December 20, 1967, in *IA*, 14B 1967-1974.

[28] Frostman to Freudenthal, January 2, 1968, in *IA*, 14B 1967-1974.

[29] Frostman to Delessert, March 16, 1969, in *IA*, 14B 1967-1974.

published as a collaboration between the ICMI and the *Zentrum für Didaktik der Mathematik* of the University of Karlsruhe. He excuses the delay in providing information by saying that the greater part of the work of secretary is performed by Freudenthal's secretary's office.[30] Thus the IMU was faced with decisions already made. In August 1969 the *First International Congress on Mathematical Education* was held in Lyon. The Congress, attended by 655 active participants from 42 countries, was a big success.[31] The main resolutions concerned: the modernisation of the teaching of mathematics, both in content and method; the collaboration between teachers of mathematics and those of other disciplines; international cooperation; the permanent training of the teachers; the place of "the theory of mathematical education" in universities or research institutes.[32] In the course of the ICMI meeting that took place during the first ICME, Freudenthal explained the reasons for the changes made: although the small congresses dedicated to well-defined topics can be useful, "today we need to go beyond the circle of specialists and reach the teachers, thus large congresses are necessary".[33] (our translation) He further underlined the fact that it is necessary to make it so that all the national sub-committees work and collaborate, and for this it is indispensable that people who are genuinely interested in teaching take part. IMU president Henri Cartan underlined that in any case there had to be retained a section of the ICM dedicated to education and that the members of the ICMI sub-committee had to be designated by the IMU sub-committee.[34]

On August 1970, during the General Assembly of the IMU in Menton, the IMU President Cartan noted the important work accomplished by outgoing ICMI President Freudenthal, and expressed his desire that the measures that he had begun will come to fruition in the future.

However, he did not even acknowledge the first ICME congress held in Lyon the previous year. During the Assembly James Lighthill was elected ICMI President for the coming four-year term. Shortly before the meeting, Cartan had written to Lighthill suggesting that Freudenthal be kept in the Commission

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[30] Delessert to Frostman, Riex, March 22, 1969, in *IA*, 14B 1967-1974.

[31] ICMI Bulletin, 5, 1975, 20-24 and http://www.icmihistory.unito.it/icme1.php.

[32] Cf. ICMI Bulletin, 5, 1975, 20-24 and http://www.icmihistory.unito.it/icme1.php.

[33] *Compte-rendu de la séance de la CIEM tenue à Lyon, le 23 août, à 14 heures, à l'occasion du premier Congrès International de l'Enseignement Mathématique*, in *IA*, 14B 1967-1974.

[34] *Ibidem*.

as past president of the ICMI and that a new secretary be chosen who would not be reduced, as Delessert had been, to a "mail box"[35].

In the ICMI session held in Nice on the occasion of ICM XVI (1-10 September 1970), the outgoing president Freudenthal presented the decisions of the IMU regarding the composition of the ICMI Executive Committee for the period 1971-1974. From the discussion that followed, two objections emerged: first, all the members at-large of the Commission were appointed by people who were not particularly competent in secondary education (Georges Papy, Freudenthal, Behnke, Đuro Kurepa), second, the members appointed did not represent the various trends in the teaching of elementary mathematics (Papy)[36]. Therefore two important recommendations were formulated: that the regulations which establish the ways that ICMI members are designated had to be modified, and that ICMI members had always to be chosen from among those who are effectively involved with mathematics teaching. Later discussion concerned the organisation of ICME 2: the Congress would take place in Exeter (UK) and would be structured differently from the preceding one, the number of plenary lectures on themes of general interest would be limited, and working groups would be constituted for addressing more specialised topics.

Even at the end of his term, Freudenthal made decisions that irritated the IMU. In fact, even while having to step down as president, he tried to insure that the directions he had opened would be followed with the same aims and guidelines. In October 1970 he sent a letter to the ICMI Executive Committee with a proposal for the Program Committee for ICME 2, which did not include Lighthill, the future president of the ICMI[37].

Three days later Cartan wrote back, letting it be known that it would be the new ICMI who would decide about the organisation of ICME 2; he also requested the rectification of the sentence in the minutes of the session on Sept. 5, 1970, concerning the constitution of the new Executive Committee of the ICMI, underlining that the regulations state that the new Executive Committee had to be designated by the entire new ICMI, that is, after every national sub-

[35] Cartan to Lighthill, Die, August 20, 1970, in *IA*, 14B 1967-1974.

[36] *EM*, 1970, s. 2, 16, p. 198.

[37] Freudenthal to the Executive Committee of ICMI, October 11, 1970, in *IA*, 14B 1967-1974.

committee had designated its representative, and that this, above all, concerned the new president Lighthill[38].

Freudenthal replied that everything was done in agreement with Lighthill. As to the sentence, in the minutes affirming that ICMI members were often elected by persons who were not effectively competent in mathematics teaching, Freudenthal wrote to Frostman:

"I admit it looks strong. This, however, reflects the actual discussion in which much stronger terms have been used. The disapproval of the way in which the new members at large were appointed was unanimous. As an attendant to this elections I could only say that the procedure was in complete agreement with the formal regulations. I would suggest that this is taken up as a serious problem by the new Executives of IMU and ICMI".[39]

During his term as president Freudenthal was completely independent with regards to financial matters as well. On November 1970 Frostman wrote to Cartan:

> "I have not paid anything to the ICMI secretariat during the last years [...] in fact I don't have exact information about ICMI's affairs".[40]

In a letter to Lighthill, Cartan wrote that the IMU had provided no funding for ICME in Lyon for the simple reason that nothing was ever requested, and that he had not asked for any funding from UNESCO because Freudenthal had gone to UNESCO directly. He also underlined that the decision to hold ICMI congresses independent of the ICMs was made by Freudenthal without him having ever consulted the IMU, and hoped that Lighthill would establish closer and more confidential relations with the IMU[41].

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[38] Cartan to Freudenthal, Paris, October 15, 1970, in *IA*, 14B 1967-1974. Writing to Frostman, (Paris, October 15, 1970) Cartan states: "Freudenthal once again worries me [...] he overdoes it somewhat by putting the new commission in front of decisions already made" [our translation].

[39] Freudenthal to Cartan, October 19, 1970, and Freudenthal to Frostman, Utrecht, October 23, 1970, in *IA*, 14B 1967-1974.

[40] Frostman to Cartan, November 15, 1970, in *IA*, 14B 1967-1974.

[41] Cartan to Lighthill, Paris, November 30, 1970, in *IA*, 14B 1967-1974.

3. AN EPILOGUE

The two important events, the inauguration of the tradition of International Congresses on Mathematical Education (ICMEs), and the launch of journals related to research in Mathematics Education, were made possible thanks to the talent for organisation and the independent spirit of Freudenthal. He realised stable landmarks for its successive development, and the story of ICMI was deeply influenced by him. Continuing the work started by his predecessors in the 1950s he allowed mathematics education to be a discipline in its own right, and not just an appendage to the world of mathematics, so that Steiner (1997, p. 28) was able to write about the ICME-7 in Quebec that

> "for the first time didactics of mathematics showed itself in great clarity as a scientific discipline which under increasing theoretical orientation and empirical foundation is dynamically growing within an international frame of complex cultural, political and interdisciplinary interrelations."

The actions of Behnke and Freudenthal make evident the friction between ICMI and IMU, as well as that between educators and mathematicians active in research, but often inattentive to education, contrasts that have been resolved by the most recent decisions: in fact, starting with the election of the 2010-2012 Executive Committee of the Commission, the election of the ICMI EC is to take place during the General Assembly of ICMI.[42]

Retracing the events that fostered the emancipation of ICMI from the community of mathematicians and the two key figures involved was a further way of celebrating the centenary and expressing our gratitude to them. Bernard of Chartres used to say that we are like dwarfs on the shoulders of giants, so that we can see more than they. So too can we, thanks to our great predecessors, who have contributed to the development of Mathematics Education into an autonomous scientific discipline.

We are very grateful to all those who have helped us with suggestions and advice, in particular to Michèle Artigue and Bernard Hodgson for their continuous help. We are also grateful to the Associazione Subalpina Mathesis of Torino and to the University of Helsinki for funding the journey to Helsinki to explore ICMI Archives.

[42] See http://www.mathunion.org/organization/ec/procedures-for-election/#ICMI

APPENDICES

1. Excerpt from Report of the president [H. Behnke] of the International Commission of Mathematical Instruction to the president of the International Mathematical Union. April 20, 1955, in IA, 14A, 1955-1957.

[…] 5. *The work of the ICMI during 1955/58*

The program of work planned for the ICMI cannot be adopted before the session of the newly constituted Executive Committee has been held. The first session of this Executive Committee will probably take place in Geneva this coming July. But, according to a discussion on the work of the ICMI for the next years at the last session of the former Executive Committee in Paris, Oct. 1954, the following program was suggested:

1. the proposition is to be made to the national sub-commissions to work on the subject of "The Scientific Basis of School mathematics" and to compose for their countries or groups of countries a book for the scientific consultation of the teachers. For this book it is of primary importance that teachers of mathematics of all levels cooperate.

   I regard it is a special, honorary mission of the ICMI to establish a contact among the teachers of all levels. The teacher have to get interested in the research work, and those active in the field of research have to get interested in the work of the teachers. I have already succeeded in being assured of the readiness of cooperation for the second volume of the German ICMI report ("Mathematical Instruction for the early Youth in the Federal Republic of Germany") from professors of the academies for education (Pädagogische Akademien) and through them from the teachers of primary level.

   At the interim meeting in 1956 (symposia for the scientific basis of school mathematics) the experiences shall be compared gathered by the different nations in projecting this book.

   In this context I may mention the suggestion of create an international encyclopedia of elementary mathematics. I do not yet see a way to realize this project because the school systems and therefore the material of instruction deviate too considerably from one another in the different countries. Yet this project will be submitted at the session of the Executive Committee in July. This way it may be possible to approach the suggestion made by M. Stone to create an international work of instruction. This plan might at first sound simply phantastic for everyone who knows the diversity of national conditions of instruction in different countries. It would, indeed involve an entirely new way of working for the ICMI since, for the first time, it would not simply have to coordinate national work, but would have to realize an important international work.

As a matter of course, considerable financial means would be necessary for the realization of such a project, because the different collaborators would have to be in constant communication during the time of accomplishing this work.

2. The inquiry "The part of Mathematics on Contemporary life" has to be examined more fully and with much more gravity than has been done up to new. The investigation of this bulk of questions is closely connected with the technical development of the different countries.

   In America, f. i., there exists a supervision of production at the instant of production. This plays a particularly important part in iron industry of small quantities, thus preventing refuse. The establishment of such a supervision and such a controlling office is, to a high degree, dependent on exact mathematical calculations. Our colleague Ulrich Graf, who died last year, was about to introduce the same establishment in Germany. If this is done on a larger scale in the region of the Ruhr, for example, large numbers of mathematicians will be required. [...] Similar questions arise for the use of large-size calculating machines in the industrial field. It is thinkable that, in the coming years, the applications of these machines might expand enormously. This involves the new vocation of the industrial mathematician. The firm Siemens-Halske in Munich has now opened a large department for the development of calculating machines and has already called from Münster four of ours of young doctors of mathematics.

   Thus questions are raised which have to be discussed on an international basis.

All pains taken by the ICMI can be summed up by this formula:

To contact people of different qualities and abilities, people of different nations and different teaching professions (as long as they are seriously interested in mathematics) in order to make them work together.

There resides the great obligation and chance for the ICMI.

I personally try to be an example for this possible, rather comprehensive kind of work

1. with my meetings at Whitsuntide aiming at the maintenance of relation between universities and schools (Pfingsttagungen zur Pflege des Zusammenhangs von Universität and Gymnasien); regular attendance of approximately 250 persons;
2. with the international interim meetings of the ICMI which will be introduced (symposia for school mathematics) and the sessions of the Section at international congresses;
3. with our series of books on mathematical instruction in the different countries;
4. with the national encyclopedias of school mathematics;
   4a. possibly with an international encyclopedia of school mathematics.

2. [Memorandum von Herrn Behnke über die Bildung von Gruppen] in IA, 14A, 1958-1960.

Suggestions on the subject of forming „Regional Groups" within ICMI

a. Since its foundation in 1908 in Rome, ICMI's aim has been to compare experiences in the teaching of mathematics in all types of educational establishments, and to discuss possible reforms in the teaching of mathematics. This program includes the following points:

   1. Reports on mathematical instruction
   2. Discussions on eliminating obsolete parts of material hitherto used
   3. Suggestions and discussions on introducing new mathematical points of view into curriculum, (for instance to introduce the concept of "structure" already at Secondary School level)
   4. To establish and cultivate contacts between various types of schools – as far as mathematical teaching is concerned – particularly where pupils progress from one school to the other.

b. In dealing with any particular problem included in this general program, we must be quite clear about the particular age group and the particular educational level of those pupils to whom this problem applies. But this is not easy because conditions vary considerably from country to country, as the national school systems are very often based on different principles. Therefore the work of the *National Sub-Commissions* must be the basis of all life in ICMI. It is then one of the main tasks of ICMI to create and cultivate the exchange of ideas and experiences between the sub-commissions of different countries. This exchange is obviously easiest between those countries where the school systems are similar. Therefore, it is rather natural that the sub-commissions of the countries sharing the old European traditions in educational matters (namely France, Germany, Great Britain, Italy, Scandinavia etc.) – which I shall briefly call the WNE-countries (*W*est and North European) – have up to now found closer contact with one another than with national sub-commissions from other parts of the world. Consequently the activity of ICMI during the 50 years of its existence was mainly concerned with these WNE-countries.

If ICMI now makes the attempt to extend its activities to all parts of the world, then it would be appropriate to form "*groups*" of national sub-commissions, so that countries whose educational systems are similar, belong to the same group. This is in accordance with the resolution passed by the General Assembly of the IMU at St. Andrews, August 1958.[43]

c. At the International Congresses of Mathematicians, ICMI plays a relatively small role, since these congresses are dominated by reports and discussions on matters of research. It might, therefore be more appropriate for ICMI to hold smaller symposia in the years between Congresses. This has, in fact, been the case in the past; especially during the periods 1909-1914 and 1953-1958, such symposia have taken place annually. For financial reasons, it is necessary to restrict each of these meetings to some countries not too far from each other. Thus the geographical aspect must also be considered in the formation of the proposed groups. Fortunately, these two aspects, the similarity of educational systems and the geographical one coincide in the most cases.

Led by these considerations, I make the tentative suggestions that to begin with, the following "Regional Groups" of ICMI must be formed:

[43] Cf. (Giacardi 2008a), 1955-1959.

1. The WNE-countries
2. The East European Countries
3. South East Asia
4. Central and South America
5. Australia and New Zealand.

One might imagine that very large countries, like USA and USSR, would have little interest in joining a regional group.

The organization of the symposia mentioned above would be such that each year a certain group arranges a meeting in which the reports and discussions deal in first place with the particular interests of the members of that group. However representatives of other groups should be invited to take part. With regard to financial arrangements, it remains to be seen whether enough money would be available from IMU or whether the group itself would have to find other resources.

d. The climax of the whole work would be a meeting of all national sub-commissions (at present numbering 23). It would be appropriate for this to take place in connection with each International Congress of Mathematicians.

3. *Proposition sur l'enseignement mathématique in IA, 14B, 1967-1974.*[44]

A la suite de l'enquête de DIALECTICA, les professeurs de mathématique* rassemblés au colloque de Lausanne[45] ont constaté qu'un accord presque général est actuellement réalisé sur les points suivants:

1. La mathématique est une activité inaliénable de l'esprit humain. Tout enfant a le droit d'y être formé.
2. Dans un monde changeant, il convient que cette formation éveille et développe plutôt des aptitudes d'action intellectuelle qu'elle ne fixe des connaissances.

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[44] These principles were published as "Propositions on the teaching of Mathematics" in the first issue of the journal *Educational Studies in mathematics* (1, 1968, p. 244).

[45] It is the UNESCO Colloquium in Lausanne on "Coordination of Instruction of Mathematics and Physics" (16-10 January 1967).

3. La mathématique évolue de plus en plus vers une science générale des structures. Celles-ci lui confèrent un pouvoir considérable d'application, d'information et d'unification. La connaissance et la maîtrise de ces structures, leur mise en ouvre dans la saisie de la réalité sont les vraies buts de l'enseignement mathématique.
4. Certaines de ces structures ont un caractère élémentaire : il y aurait intérêt à chercher à s'en servir dès l'enfance.
5. Un certain nombre de structures plus élaborées devraient être acquises au terme des études secondaires.
6. La réalisation d'un niveau valable exige une formation mathématique et pédagogique appropriée des maîtres.
7. La réforme de l'enseignement mathématique doit être considérée comme un phénomène permanent. Cela implique une formation continue des maîtres appuyée sur une recherche pédagogique suivie.
8. En ce domaine, une collaboration efficace sur le plan mondial devient indispensable. Il est urgent de fonder un organisme international des informations en matière d'enseignement mathématique.

Lausanne, le 18 janvier 1967

* A savoir : M. W. Servais (Belgique), M. R. Guy (Canada), M. J. Lichtenberg (Danemark), M. C. Pisot (France), Mme [illegible: P. Gadon ?] et M. A. Renyi (Hongrie), M. C. Cattaneo (Italie), M. H. Freudenthal et M. L. N. H. Bunt (Pays-Bas), M. Z. Krygowska et M.S.Straszewicz (Pologne), M. E. Blanc, M. A. Delessert, M. E. Emery, M. F. Gonseth, M. K. Grimm, M. J. de Siebenthal (Suisse) et M. I. Smolec (Yougoslavie).

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Collaborative learning for mathematical level raising, what does it take?

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Key words
Collaborative learning, interaction, mathematical level raising

SUMMARY

In this contribution I will give an overview of my work as researcher of collaborative mathematics learning during 20 years. I will focus on characteristics of learning materials, a helpful theoretical model, the role of the teacher, the size of small groups and new research lines.

1. INTRODUCTION

At ICME-6 in Budapest in 1988, I gave a presentation about the learning of mathematics in heterogeneous small groups. I was a PhD student and completely involved in classroom observations and the designing of good learning materials for small group learning (Dekker, 1987). Freudenthal, whose ideas about the heterogeneous learning group had influenced me, was in my audience, giving me support with his presence. Now, 20 years later, I have been involved in many research projects on collaborative learning of mathematics. We know a lot more about the process of interaction which stimulates mathematical level raising. We also know more about the characteristics of the learning materials. For level raising isolated problem solving activities are not sufficient, we need at least a series of problems, with special problems in it to provoke level differences between the students. We know more about the favorable size of small groups, the pros and cons of couples: easily accessible for research, but less rich for a critical discussion between students. And we start to know more about the role of the teacher. Which interventions stimulate the interaction and the process of level raising? Which interventions can be disturbing? Which sort of whole class discussions supports the learning in small groups? Some say that whole class discussions are crucial to establish good social and socio-mathematical norms and to consolidate level raising. Others think that they are mainly time-consuming and evoke all sorts of stereotypical behavior of the students, including off-task behavior. I will present some of our research findings over the last twenty years and I am sure we will have enough to discuss!

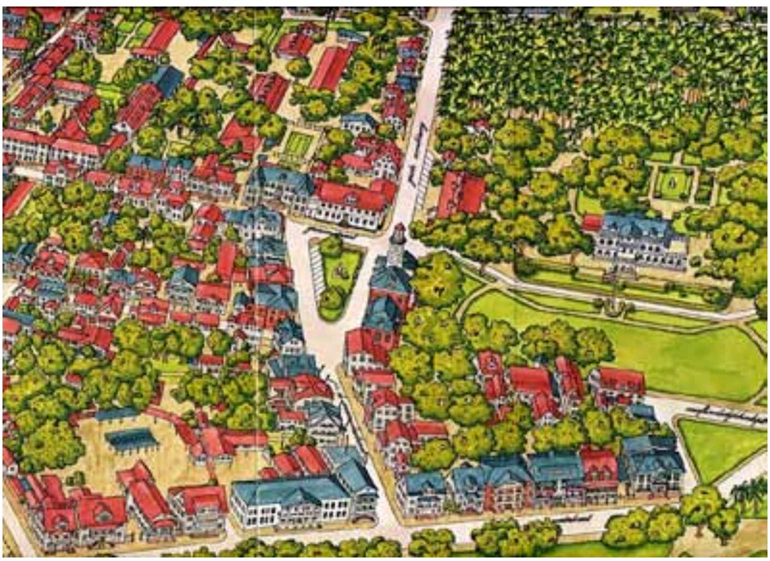
2. LEARNING MATERIALS

While finishing my PhD, one of my supervisors asked me to formulate characteristics of learning materials which evoke interaction and level differences between children, which I did in my thesis (Dekker, 1991). First, the problems are placed in a *realistic context* in order to appeal to the students and to make it possible for them to realize the situation. Second, there are problems in the learning materials which are *complex*, in order to stimulate interaction between the students. To solve these problems different abilities are needed, like finding relevant information in a text, measuring precisely, making calculations well.

They also have to take into account all sorts of different information, data from a text, a map, a table or from earlier solved problems. A third characteristic is that something has to be made, *constructed*, like a graph, a table, a model, a little story. That stimulates students to draw, write or make calculations. In that way they can see each other's work and the differences in it. An important characteristic of the learning materials is the aiming at *level raising*. At certain places in the learning materials there are problems which, when approached on a too low level, cannot be solved well. I will make the characteristics concrete by giving an example from the learning materials I have developed for my PhD research.

The learning materials for small, heterogeneous groups of students age 12, 13, consist of one map for each small group and a letter of a girl Merlien, living in Paramaribo, Surinam. Figure 1 shows a fragment of the map. Figure 2 shows a fragment of the letter.

Figure 1. Fragment of the map.



Fragment of the letter:

'It was raining too hard, so we waited for a moment. Fortunately it was cooling down a bit.
Suddenly the shower stopped, we walked on and soon the sun was burning again.
We walked slower and slower.
But when we strolled into the Palm Garden, it was pretty cool under the trees.'

The letter of Merlien is about a walk she makes with her friends from school till the Palmgarden (see the upper right corner on the map). She tells about differences in temperature because of the heath and a sudden tropical rain shower and about differences in their speed of walking, strolling by the heath, and running by the shower. In the first problems in the learning materials the small groups are asked to tell Merlien's story in graphs: a temperature/time graph about the differences in temperature during the walk, a speed/time graph about the differences in speed during the walk, and finally a distance/time graph about the growing of the walked distance during the walk.

The learning materials are clearly placed in a *realistic context*. Many children never have been in Surinam, but the map, the letter and the presence in many Dutch classes of children with parents from Surinam, make the situation very well realizable. The problems are also *complex*. In order to make the graphs, the map has to be studied, the letter as well, some measurements have to be made and all has to be combined. The graphs have to be *constructed*; decisions about the axes, about some numbers on the axes and about the global shape of the graphs have to be made. Van Hiele once explained that the making of the temperature/time graph and the speed/time graph are activities on the visual level. Changes in the temperature and in the speed are in direct contact with the changes in the graph: when the temperature or the speed goes up, the graph goes up as well and when the temperature or speed is constant, the graph is flat. Although making the graphs is not an easy thing, students don't have to know much about graphs to construct them well. However, the making of the distance/time graph is a different thing: when the speed is constant, the walked distance grows regularly, when the speed is zero, the walked distance remains constant. One really has to understand the construction of the graph, which means a jump to the descriptive level where not the objects themselves, but their properties are central (Van Hiele, 1986). So the learning materials aim at *level raising*.

Analysis of audiotapes of the small groups revealed that the construction of the distance/time graph leads to level differences in the answers of the students, which are intensively discussed. Students frequently explain their work and criticize each other's work. Level raising is already evident in some students.

3. A HELPFUL MODEL

During my PhD work I was puzzled by the question which elements in the interaction between students contribute to level raising. Freudenthal mentioned the role of explaining as a mean for reflection (Freudenthal, 1978). I thought about the role of critic. I made a model in which I described what I thought was crucial for level raising. After my PhD I became a researcher of mathematics education and I started to collaborate with Marianne Elshout-Mohr, a cognitive psychologist with whom I shared interest in learning processes. I showed her my model and we discussed it in detail. She was very interested, but also raised some sound critic. She convinced me that criticizing the work of someone else is not crucial for one's own level raising, but the justifying that it evokes, is. We reconstructed the model together and published it in Educational Studies of Mathematics (Dekker & Elshout-Mohr, 1998). The model is presented in Figure 2. For an extended explanation and theoretical justification of it, I refer to that publication. Here I will explain parts of it.

In the process model for interaction and mathematical level raising we divide *key activities*, *regulating activities* and *mental* activities. Key activities for a person A, who is working on a mathematical problem, are the main activities for A's level raising. They are:

A tells or *shows* her work
A *explains* her work
A *justifies* her work
A *reconstructs* her work

In a collaborative learning setting a person B can regulate the level raising of A by performing the regulating activities:

B *asks* A *to show* her work
B *asks* A *to explain* her work
B *criticizes* A's work

I will show her two parts of the model to give insight in the relation between the key, regulating and mental activities:

B asks A to explain her work (regulating)
A thinks about her work (mental)
A explains her work (key)
B criticizes A's work (regulating)
A thinks about B's critic (mental)
A justifies her work (key)

The main idea for level raising is that when A justifies her work and notices that her justification fails, she will criticize her own work and come to reconstruction of it. The reconstruction can reveal A's level raising.

A help for bringing the process model alive is to read only the middle column. That way one can imagine what kind of discussion between students can stimulate level raising.

Figure 2. Process model for interaction and mathematical level raising.

A and B are working on the same mathematical problem. Their work is different.		
A is working		B is working
A asks B to show his work	*What are you doing?* *What have you got?*	*B asks A to show her work*
A becomes aware of her own work		B becomes aware of his own work
A shows her own work	**I am doing this…** **I have got this…**	**B shows his own work**
A becomes aware of B's work		B becomes aware of A's work
A asks B to explain his work	*Why are you doing that?* *How did you get that?*	*B asks A to explain her work*
A thinks about her own work		B thinks about his own work
A explains her own work	**I'm doing this, because…** **I've got this, because…**	**B explains his own work**
A thinks about B's work		B thinks about A's work
A criticises B's work	*But that's wrong, because…*	*B criticises A's work*
A thinks about B's criticism		B thinks about A's criticism
A justifies her own work	**I thought it was right, because…**	**B justifies his own work**
A thinks about her justification		B thinks about his justification
A criticises her own work	Oh no, it isn't right, because…	B criticises his own work
A reconstructs her own work	**I'll better do it like this…**	**B reconstructs his own work**

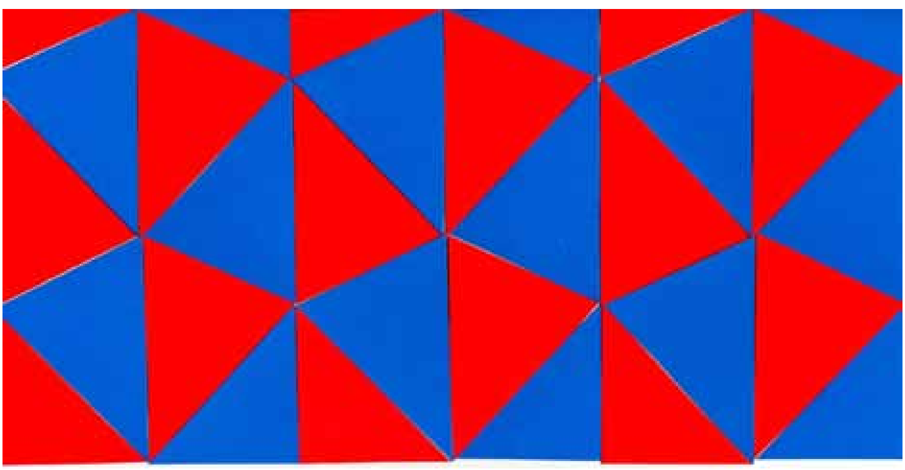
bold: key activities
standard: mental activities
italic: regulating activities

4. ROLE OF THE TEACHER

After reflecting on the findings of my PhD research and the development of the process model, Marianne Elshout-Mohr and I discussed the role of the teacher during collaborative mathematics learning. We argued that if we take our own model seriously, then a teacher who promotes the activities as described in the model is more effective in relation to level raising, than a teacher who gives 'normal' help. We assumed that in both cases help should be minimal, in order to stimulate independent learning of the small groups. To make a clear distinction of both roles, we wanted the process teacher not to give any product help and to make this clear to the students. The focus is to stimulate the students to perform key and regulating activities and the process teacher should make this clear to the students. The other teacher, we called the product teacher, as for content help to small groups the product of the small group is an important source of information for the teacher, should refrain himself from process help.

We prepared an experiment, this time with older students, age 16, 17, working in triples on learning materials about geometrical transformations (see Figure 3). Normally they follow a program on abstract mathematics.

Figure 3. Fragment of new learning materials about transformations.



The main finding of our experiment was that students with a process teacher reach more level raising than students with a product teacher. This was in line with our hypothesis, but as the quality of the help of the product teacher was very high and the help of the process teacher was almost absent, this was not what we expected during our experiment. We have described our findings, including more details about the learning materials and teacher interventions in Dekker and Elshout-Mohr (2004). Another finding from our experiment was that the discussion in triples is very intense. In the meantime my PhD student Monique Pijls also started research on the role of the teacher during collaborative mathematics learning. She developed learning materials on chances, partly on the computer. For that reason she worked with couples. Her students were younger, age 15, 16 and did a program on applied mathematics. She also worked with a process teacher and a product teacher. Her main finding was that students with a process teacher reach as much level raising as students with a product teacher. She also found that couples got stuck at level raising problems and giving process help without content help was very frustrating for the process teacher (Pijls, 2007; Pijls, Dekker & Van Hout Wolters, 2007a, 2007b).

5. SIZE OF THE SMALL GROUPS

In my PhD research I worked with groups of 4. It was very hard to listen and work out the audiotapes, but the mix of students and the level differences in their solutions led to rich discussions with a lot of showing and explaining.

In our research about teacher interventions we worked with triples. Also with triples the level differences led to rich discussions, but more than with the groups of 4 the discussions in triples were very intense. Monique Pijls worked with couples, in this case because of the computer. On the other hand, in much research on collaborative mathematics analyses of conversations between couples is dominant. Together with Terry Wood, Marianne Elshout-Mohr and I analyzed a protocol of a couple, age 8, working on a mathematical problem. We analysed the protocol from different perspectives and studied how the students regulated their own learning (Dekker, Elshout-Mohr & Wood, 2004, 2006). We felt that in a couple their can be an implicit division of roles, which can disturb the level raising process. That became more evident in the work with Konstantinos Tatsis. Tatsis analyzes collaborative mathematics learning from the perspective of the role theory of Goffman (Tatsis & Koleza, 2006). We combined our perspectives in an analysis of the protocols of couples, future primary school teachers, working on mathematical problems. We studied the influence of the different roles, students take in pairs, on the performing of the key and regulating activities. One of our findings is that a smooth collaboration can lead to shared knowledge building, but at the same time level raising is at risk, as during smooth collaboration there is less need for explaining and justifying, which are key activities for level raising. We continued our analysis on a protocol from the research of Pijls and also found a division of roles, which is in some parts counterproductive for level raising (Tatsis & Dekker, in press). It seems that working in a triple gives more chances for level raising. As a student, age 16, once said:

> "I prefer to work in a couple, because then you really have to build upon each others thoughts… …
> But in a group of three there is more knowledge."

Or is expressed in an old Chinese saying:

> 'Where three deliberate, wisdom arises.'

6. MORE RESEARCH

Monique Pijls and I reflected on our research projects and the role of the teacher. We were convinced that a process teacher gives chances for level raising, but that the role of a process teacher is not 'normal' for teachers. Teachers like to explain. That is crucial for them. So we started to think how we could persuade teachers to stimulate students to perform key and regulating activities. We were also curious if teacher maybe already do that in unexpected ways. So we observed 'normal' teaching in search of (chances for) key and regulating activities, discussed our observations with the teachers, deliberated how key and regulating activities could be stimulated more and observed more experimental lessons. It led to mixed results and feelings, as expressed very clearly by one teacher:

> "I like very much to explain. Now I had to say, ask your neighbor and then go away quickly, because otherwise they keep on asking me. I found that very hard!"

> "I saw students really working more intensely together and sometimes that worked very, very well. They really started to ask each other to explain and they have helped each other."

Monique Pijls and I described our findings in an article to be published (Pijls & Dekker, submitted). In the meantime Monique Pijls has started as a professional trainer of process help. I am trying to find new ways to implement process help in the daily practice of mathematics teachers. Sonia Palha, my new PhD student is developing switch problems to be used during the work with a chapter from a textbook that is very popular with teachers. The idea is, that during their normal teaching at certain moments, when the learning is hard, the teacher forms triples of students of mixed levels, give them switch problems to work on collaboratively in stead of working on problems in the book, and takes the role of a process teacher during the work on the switch problems. We use the word switch problems in a double meaning. The teacher switches role, from 'normal' to process teacher and the problems are to stimulate level raising, so to switch from one level to the other. The problem of making a distance time graph, presented in the beginning of my talk, is an example of such a switch problem. Sonia Palha will compare this working with the normal teaching of the chapter. Our hypothesis is that working with the chapter with switch problems, leads to more level raising than working with the chapter in the normal way. The first findings during try-outs are promising (Palha & Dekker, 2007). To be continued…

7. AN OVERVIEW

So, to sum up 20 years of research of the question 'Collaborative learning for mathematical level raising, what does it take?' We can say:

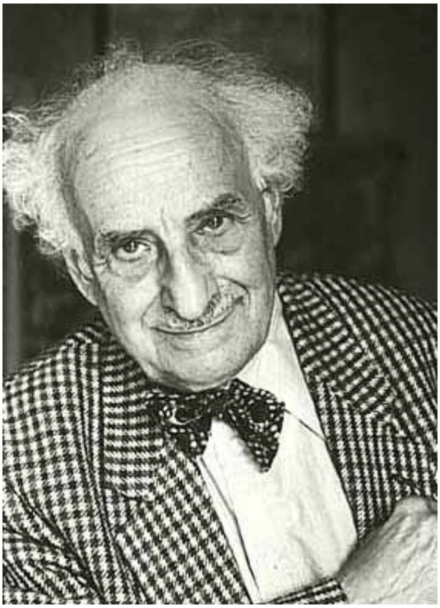
- Carefully designed learning materials with switch problems
- A teacher who stimulates students to perform key and regulating activities
- Small groups of 3
- More research!

Not a normal teacher

Ending my overview I come back to the person who once stimulated me to do research on collaborative mathematics learning. His genuine interest in my developing ideas and experiences with collaborative mathematics learning and his encouragement by saying 'go on', gave me the courage to continue my research.

And I did go on…
I still do.

Figure 4. Hans Freudenthal (1905–1990).



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Conceptualizing the Learning of Algebraic Technique: Role of Tasks and Technology

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This article is divided into four parts. The first part presents some introductory remarks on the use of Computer Algebra System (CAS) technology in relation to the long-standing dichotomy in algebra between procedures and concepts. The second part explores the technical-conceptual interface in algebraic activity and discusses what is meant by conceptual (theoretical) understanding of algebraic technique – in other words, what it means to render conceptual the technical aspects of algebra. Examples to be touched upon include seeing through symbols, becoming aware of underlying forms, and conceptualizing the equivalence of the factored and expanded forms of algebraic expressions. The ways in which students learned to draw such conceptual aspects from their work with algebraic techniques in technology environments is the focus of the third part of the article. Research studies that have been carried out by my research group[1] with a range of high school algebra students have found evidence for the kinds of theoretical thinking that can be fostered by specific types of technique-oriented tasks within CAS environments.

The fourth part of the article then shifts to the perspective of teaching practice and discusses some of the issues that, according to this research, are to be taken into account by teachers when planning for the orchestration of such task-technique-theory activity in technological environments.

Keywords

Tasks, technology, technique, theory, algebra at secondary school level, conceptual learning of algebraic technique

[1] My appreciation to the colleagues, post-doctoral fellows, and collaborators who have contributed to the research being presented in this article: André Boileau, Caroline Damboise, Paul Drijvers, José Guzmán, Fernando Hitt, Ana Isabel Sacristán, Luis Saldanha, and Denis Tanguay – as well as the teachers and students of the participating schools, and our project consultant, Michèle Artigue. I also express my gratitude to the Social Sciences and Humanities Research Council of Canada and the Québec Ministère des Relations Internationales who have funded this research.

INTRODUCTION

1.1 What is Computer Algebra System (CAS) technology?

A Computer Algebra System (CAS) is a software program that facilitates symbolic mathematics. The core functionality of a CAS is manipulation of mathematical expressions in symbolic form (Wikipedia, Sept. 5, 2007). In 1987, Hewlett-Packard introduced the first hand-held CAS calculator with the HP-28 series, and it became possible, for the first time with a calculator, to arrange algebraic expressions, to differentiate, to do limited symbolic integration and Taylor series construction, and to solve algebraic equations. The Texas Instruments company in 1995 released the TI-92 calculator with an advanced CAS, based on the software Derive. This calculator, and its successors (including TI-89, Voyage 200, and TI-Nspire), have featured a reasonably capable and relatively inexpensive hand-held Computer Algebra System with symbolic, graphical, and tabular capabilities.

1.2 CAS use in secondary school mathematics classes

Ever since the appearance of computers and calculators enabled with symbol-manipulating capabilities, educators have considered these tools to be quite appropriate for student use in college-level mathematics courses, and in calculus courses offered at some upper-level high schools (see, e.g., Heid, 1988; Shaw, Jean, & Peck, 1997; Zbiek, 2003). However, these tools have generally not been adopted for secondary school mathematics up until quite recently. Many secondary school mathematics teachers have, for several years, tended to stay away from CAS technology in their classrooms, preferring that their students first develop paper-and-pencil skills in algebra (National Council of Teachers of Mathematics, 1999).

However, these attitudes are changing – based both on research findings and on the leadership of interested teachers and mathematics educators, as well as on the greater availability of teacher resources for using this technology at the Grade 9, 10, and 11 levels of secondary school. The result is that student access to this technology is increasing in schools (Hoyles & Lagrange, 2009).

1.3 What does the research have to say?

CAS technology has been found to encourage the use of general mathematical reasoning processes and to improve student attitude, according to research reported during the five- year period from 2003 to 2008 at the annual conferences of the International Group for the Psychology of Mathematics Education (PME):

- "It allows for generating, testing, and improving conjectures"
- "It allows for developing awareness and intuition"
- "It leads students to explore their own conjectures"
- "It provides non-judgmental feedback"
- "It develops the learner's confidence."

This research has also found that CAS can help develop students' knowledge of algebraic content: their understanding of equivalence (Ball, Pierce, & Stacey, 2003), parameters and variables (Drijvers, 2003), and literal-symbolic algebraic objects in general, without "leading to the atrophy of by-hand symbolic-manipulation skills or to the slower development of these skills" (Heid, Blume, Hollebrands, & Piez, 2002, p. 586).

Since the mid-1990s, in France, when CAS technology started to make its appearance in secondary school mathematics classes, researchers (Artigue, Defouad, Duperier, Juge, & Lagrange, 1998) noticed that teachers were emphasizing the conceptual dimensions while neglecting the role of the technical work in algebra learning. However, this emphasis on conceptual work was producing neither a clear lightening of the technical aspects of the work nor a definite enhancement of students' conceptual reflection (Lagrange, 1996). From their observations, the research team of Artigue and her collaborators came to think of techniques as a link between tasks and conceptual reflection, in other words, that the learning of techniques was vital to related conceptual thinking. The implication of these findings, as Michèle Artigue stated in her plenary presentation at this ICME-11 conference (Artigue, 2008), is that the dichotomy between techniques and concepts in algebra is a false one. It is argued not only that the two are complementary, but also that, within appropriate learning environments, techniques and concepts co-emerge and mutually support each other's growth.

1.4 The Task-Technique-Theory framework

Chevallard describes four components of practice by which mathematical objects are brought into play within didactic institutions: task, technique, technology, and theory. Chevallard (1999, p. 225) states that *tasks* are normally expressed in terms of verbs, for example, "multiply the given algebraic expression." He defines *technique* as "a way of accomplishing, of carrying out tasks." In his theory, Chevallard separates *technique* from the discourse that justifies/explains/produces it, which he refers to as *technology*. But he also admits that this type of discourse is often integrated into technique, and points out that such technique can be characterized in terms of theoretical progress. According to Chevallard, *theory* takes the form of abstract speculation, a distancing from the empirical. Thus, within the anthropological approach, discourse can be viewed as bridging technique and theory.

Artigue (2002a) and her research collaborators adapted Chevallard's anthropological theory by collapsing *technology* and *theory* into the one term, *theory*. This gave the theoretical component a wider interpretation than is usual in the anthropological approach; it also reserved the use of the term *technology* for digital devices. Furthermore, Artigue (2002a, p. 248) has emphasized that *technique* also has to be given a wider meaning than is usual in educational discourse: "A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work."

Lagrange (2002, p. 163), one of Artigue's collaborators, has expressed the interrelationship of task, technique, and theory as follows:

> Within this dynamic, tasks are first of all problems. Techniques become elaborated relative to tasks, then become hierarchically differentiated. Official techniques emerge and tasks lose their problematic character: tasks become routinized, the means to perfect techniques. The theoretical environment takes techniques into account – their functioning and their limits. Then the techniques themselves become routinized to ensure the production of results useful to mathematical activity. … Thus, technique has a pragmatic role that permits the production of results; but it also plays an epistemic role (Rabardel and Samurçay, 2001) in that it constitutes understanding of objects and is the source of new questions. [my translation]

Elsewhere, Lagrange (2003, p. 271) has further extended this latter idea: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency."

Our research group was intrigued by the theoretical notion that algebra learning at the high school level might be conceptualized in terms of a dynamic among Task-Technique-Theory (T-T-T) within technological environments. And so it came to be that we began a series of studies in 2002, which continue to this day, that explored the relations among task, technique, and theory in the algebra learning (and teaching) of Grades 10, 11, and 12 students (15-18 years of age) in CAS environments. I will be elaborating on aspects of this research in a short while; nevertheless, I summarize briefly here our main findings so as to situate my underlying theme.

As reported in Kieran and Drijvers (2006), technique and theory emerged in mutual interaction. Techniques gave rise to theoretical thinking; and the other way around, theoretical reflections led students to develop and use techniques.

As reported in Kieran and Damboise (2007), a comparative study of a CAS class and non-CAS class involving the same tasks, the CAS class improved much more than the non-CAS class in both technique and theory, but especially in theory; and the sequence of lessons was one where the technical component was clearly in the forefront.

This brings us to the main question to be addressed in this paper: How does the learning of algebraic technique in a CAS environment lead to the emergence of students' theoretical/conceptual growth? In other words, how is technique rendered conceptual? What does it mean to have a conceptual understanding of algebraic technique?

2. THE INTERFACE BETWEEN TECHNIQUE AND THEORY IN ALGEBRA

Note that, within this text, I will be using the terms *conceptual* and *theoretical* interchangeably. I also wish to point out that the context of this article is related to the letter-symbolic aspects of algebra. There are two reasons for this. On the one hand, a great deal of research exists already with respect to the benefits of multi-representational approaches (e.g., graphical representations) in making algebraic objects more meaningful to students (Kieran & Yerushalmy, 2004). On the other hand, algebra involves more than representational activity; symbolic transformational activity lies at its core. However, the amount of research related to the ways in which the literal-symbolic transformational activity of algebra can be viewed as being conceptual is limited, to say the least.

2.1 *What is meant by a conceptual understanding of algebraic technique?*

I propose that *a conceptual understanding of algebraic technique* includes:

- Being able to see a certain form in algebraic expressions and equations, such as a linear or quadratic form;
- Being able to see relationships, such as the equivalence between factored and expanded expressions;
- Being able to see through algebraic transformations (the technical aspect) to the underlying changes in form of the algebraic object and being able to explain/justify these changes.

Some classic examples of conceptual understandings in algebra include: (a) the distinctions between variables and parameters, between identities and equations, between mathematical variables and programming variables, and so on; as well as (b) the knowledge of the objects to which the algebraic language refers (generally numbers and the operations on them) and the need to include certain semantic aspects of the mathematical context so as to be able to interpret the objects being treated. But these classic examples deal more with objects than with techniques.

2.2 *Some examples of a conceptual understanding of algebraic technique*

Example 1.

Seeing through symbols to the underlying forms, e.g.,

(a) seeing $x^6 - 1$ as $((x^3)^2 - 1)$ and as $((x^2)^3 - 1)$, and so being able to factor it in two ways.

(b) seeing that x^2+5x+6 and x^4+7x^2+10 are both of the form ax^2+bx+c.

Example 2.

Conceptualizing the equivalence of the factored and expanded forms of algebraic expressions, e.g., awareness that the same numerical substitution (not a restricted value) in each step of the transformation process of expanding will yield the same value:

$(x+1)(x+2)$ – factored form –
$= x(x+2) + 1(x+2)$
$= x^2 + 2x + x + 2$
$= x^2 + 3x + 2$ – expanded form – and so substituting, say 3, into all four expressions produces the same numerical result – in this case, 20 – for each expression.

Example 3.
Coordinating the "nature" of equation solution(s) with the equivalence relation between the two expressions that comprise the original equation, e.g., for the following task,

Given the three expressions: $x(x^2\text{-}9)$, $(x+3)(x^2\text{-}3x)\text{-}3x\text{-}3$, $(x^2\text{-}3x)(x+3)$,

(a) determine which of these three expressions are equivalent;
(b) construct an equation using one pair of the above expressions that are not equivalent, and find its solution;
(c) construct an equation from another pair of the above expressions that are not equivalent and, by logical reasoning only, determine its solution.

So, for the three given expressions,

Exp1: x(x2-9)
Exp2: (x+3)(x2-3x)-3x-3
Exp3: (x2-3x)(x+3)

(a) Which are equivalent?
Only Exp1 and Exp3 are equivalent.

(b) An equation using a pair of non-equivalent expressions from the three given expressions? And its solution?
One could use Exp1 and Exp2 in the equation: Exp1 = Exp2.
Its solution (with CAS or with paper and pencil): x = -1.

(c) An equation from another pair of non-equivalent expressions from the above three expressions? And its solution (by logical reasoning only)?

This time, one uses Exp3 and Exp2 in the equation: Exp3 = Exp2.
One deduces that the solution has to be the same as in (b): (x = -1). (A conceptual/theoretical understanding involving substitution of equivalent expressions and transitivity leads to this deduction.)

2.3 *The importance of fostering a conceptual understanding of algebraic technique*

Having just seen some examples of what is intended by the phrase, a conceptual understanding of algebraic technique, I now argue, briefly, for the importance of this aim for algebra instruction.

National and international mathematics assessments during the 1980s and 1990s reported that secondary school students, in order to cover their lack of understanding, resorted to memorizing rules and procedures and that students eventually came to believe that this activity represented the essence of algebra (e.g., Brown, Carpenter, Kouba, Lindquist, Silver, & Swafford, 1988).

Although some of the recent reform movements have attempted to make algebra more meaningful for students – at least during the earlier years of high school – by infusing "real-world" problem-solving activities and multiple representations of these problems into algebra curricula, these same curricula have tended to maintain the traditional dichotomy of procedures and concepts when dealing with the transformational activity of algebra in the later years of high school. When students are then faced with the literal-symbolic transformational activity of algebra, it is presented, by and large, as a primarily concept-free domain.

Although Skemp (1976) described "relational understanding" as knowing both the rules and why they work, there has never been much movement in the direction of describing what this might mean for algebra.

The point I wish to make is that this dichotomy between procedures and concepts in algebra is both unnecessary and unproductive for students, and in fact can lead to depriving them of the conceptual insights that can make their work with procedures meaningful. But before looking at how techniques can be approached so that the conceptual component might co-emerge along with the technical, we need first to consider the issue of tasks.

2.4 *The role of tasks in the T-T-T triad*

At a recent PME Research Forum on "The Significance of Task Design in Mathematics Education", Ainley and Pratt (2005) – the organizers of the Forum – argued that, "We see task design as a crucial element of the learning environment … [and contend that] the nature of the task influences the activity of students." Hoyles (2002) has emphasized that a focus on the design of task situations is at the heart of the "transformative potential of [technological] tools in activities" and that, with this focus, "knowledge and epistemology are brought back to center stage" (p. 284). Lagrange (1999) has suggested that task situations ought to be created in such a way as to "bring about a better comprehension of mathematical content" (p. 63) via the progressive acquisition of techniques in the achievement of a solution to the task. Guin and Trouche (1999) have added that tasks should aim at fostering experimental work (investigation and anticipation).

More specifically, Drijvers (2003) has pointed out that more attention needs to be paid to the role of paper-and-pencil work throughout CAS task activity. For Hitt and Kieran (2009), a main consideration in task design is the nature of the theorizing that is to be elicited by the specific tasks and techniques of a teaching sequence. Artigue (2002b) has suggested that CAS tasks can capitalize on "the surprise effect that can occur when one obtains results that do not conform to expectations and that can destabilize erroneous conceptions, as well as on the multiplicity of results that can be obtained in a short space of time when exploring and trying to understand a certain phenomenon" (p. 344, my translation).

Zehavi and Mann (2003) have described how the tasks they developed had the potential to intertwine student work, CAS performance, and student reflection. Ball and Stacey (2003) have argued that students' written task records ought to focus principally on the reasoning that has been evoked.

As is suggested by all of the above studies – research that has involved mathematical activity within technology environments – there is an undeniable importance accorded to the design of tasks, tasks whose goal is to promote conceptual reflection and development, even in technique-oriented work! Absent are task sequences whose main purpose is for students simply to provide answers to procedural questions.

2.5 To sum up

Because of the (a) recent advances in the development of theoretical frameworks, such as that of Task-Technique-Theory, (b) increasing use of technology in schools, for example, CAS at the secondary school level, and (c) attention being paid to the role that the nature of the task/situation plays in students' mathematical learning, we are well poised to make headway in reflecting upon the ways in which technique can be viewed from a conceptual angle in the teaching and learning of algebra and, in fact, how technology can enhance the conceptualizing of technique.

3. HOW 10TH GRADE STUDENTS IN OUR PROJECT DREW CONCEPTUAL ASPECTS FROM THEIR WORK WITH ALGEBRAIC TECHNIQUES IN A CAS ENVIRONMENT

Two preliminary remarks are in order, the first concerning the tasks, the second concerning the technologies. With respect to the tasks: The tasks went beyond merely asking technique-oriented questions; the tasks also called upon general mathematical processes that included observing/focusing, predicting, reflecting, verifying, explaining, conjecturing, justifying. With respect to the technologies: Both CAS and paper-and-pencil were used, often with requests to coordinate the two; in general, the CAS provided the data upon which students formulated conjectures and arrived at provisional conclusions.

3.2 Conceptualizing that emerged while learning new techniques with the aid of CAS technology

The examples in this section are drawn from Kieran and Drijvers (2006) and Hitt and Kieran (2009). The two-lesson task-sequence was related to factoring (adapted from Mounier & Aldon, 1996). It involved the family of expressions, $x^n - 1$. The aim of the task sequence was to arrive at a general form of factorization for $x^n - 1$ (for integer values of $n \geq 2$) and then to relate this to the complete factorization of particular cases for integer values of n from 2 to 13. Proving one of these cases was part of the sequence, but is not included in this article (for details on the proving component and its unfolding in class, see Kieran & Guzmán, 2010).

One of the initial tasks of the sequence involved the following questions, which have been compressed for this article into Figure 1.

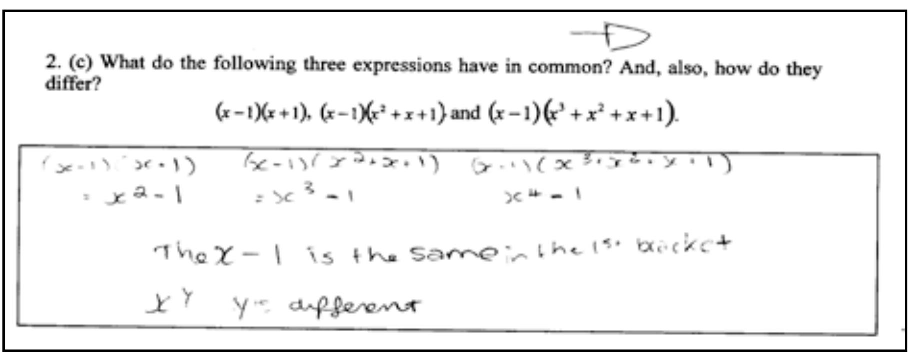
Figure 1. Some of the initial task questions of the x^n - 1 sequence.

1. Perform the indicated operations: $(x-1)(x+1)$; $(x-1)(x^2+x+1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product $(x-1)\left(x^3+x^2+x+1\right)=$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ? $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)\left(x^3+x^2+x+1\right)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. Is your explanation valid for the following equality: $(x-1)(x^{134}+x^{133}+x^{132}+\ldots+x^2+x+1)=x^{135}-1$? Explain.

After students had worked on these questions, either in groups or individually, the teacher opened up a whole-class discussion and asked students to state their responses to one particular question (Question #4 of Figure 1). Different students noticed different things in the pattern of expressions. The teacher's aim in having the whole-class discussion was to encourage students to learn from what some of their peers had noticed. Figures 2 and 3 provide some samples of their responses to the given question. (As an aside: the issue of what students notice when doing exploratory mathematical work with technology is one that has received little research attention.)

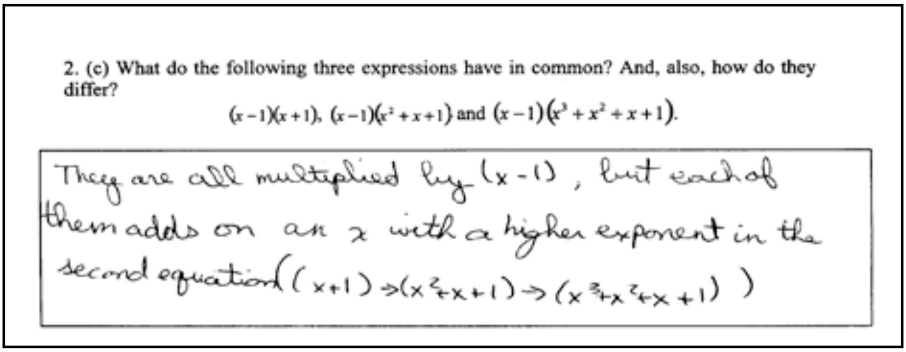
The particular student whose work is shown in Figure 2 focused on the $(x-1)$ in the factored form and on the exponent in the expanded form.

Figure 2. For this question, this student focused on the $(x-1)$ and the exponents.



The student whose work is displayed in Figure 3 helped others to "refine their noticing" when she described during the whole-class discussion what she had focused on. She noticed more than did some other students and was also able to express herself with a certain clarity – even if she misused terminology. Linguistic imprecisions such as this one, where *equation* was used for *factor*, were a common occurrence among the students in the classes we observed.

Figure 3. This student helped others in the class to "refine their noticing".


2. (c) What do the following three expressions have in common? And, also, how do they differ?

$(x-1)(x+1)$, $(x-1)(x^2+x+1)$ and $(x-1)(x^3+x^2+x+1)$.

They are all multiplied by $(x-1)$, but each of them adds on an x with a higher exponent in the second equation $((x+1) \to (x^2+x+1) \to (x^3+x^2+x+1))$

The class then moved on to a general form of factorization for x^n-1 based on the above prior examples: $x^n - 1 = (x-1)(x^{n-1}+x^{n-2} + \dots x+1)$ (see Sacristán & Kieran, 2006, for student work related to this component of the task sequence). After arriving at this general form, the students worked on the Factorization Task where they were confronted with the completely factored forms produced by the CAS and where they were requested to reconcile their paper-and-pencil (p/p) factorizations with those produced by the CAS. One of the ways in which students attempted to reconcile their expected factorization of, for example, x^4-1 with the CAS factorization is suggested by the work displayed in Figure 4. Here the student multiplied the 2nd and 3rd CAS factors to yield the same second factor that she had obtained with paper and pencil. Other students reconciled their p/p and CAS productions either by factoring more completely their 2nd p/p factor or by asking the CAS to multiply its 2nd and 3rd factors so as to see whether that produced the same polynomial as their 2nd p/p factor.

Figure 4. Reconciling paper-and-pencil and CAS factorizations for x^4-1.

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	N/A
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	N/A
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x+1)(x^2+1)$ $(x-1)(x^3+x^2+x+1)$

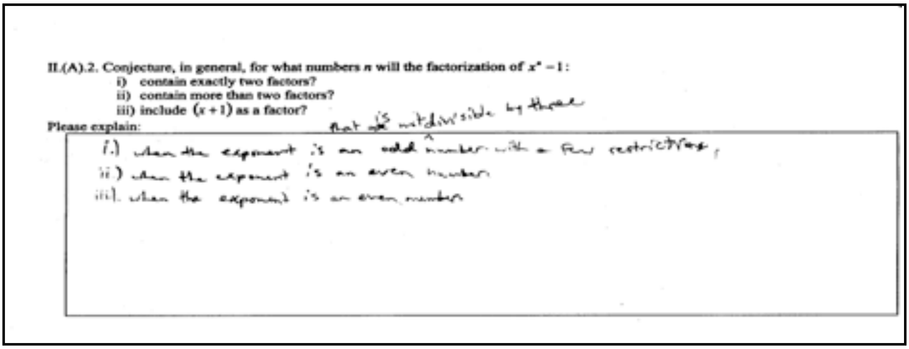
After completing the Factorization Task for n = 2 to 6 in x^n - 1, students were presented with the Conjecture Task: "Conjecture, in general, for what numbers n will the factorization of x^n-1: (i) contain exactly two factors? (ii) contain more than two factors? (iii) include (x+1) as a factor? Explain." The following pair of students, Chris and Peter, incorrectly conjectured that, for all odd ns, the complete factorization of x^n-1 would contain exactly two factors (see Figure 5). The last line of the transcript extract indicates the moment of surprise when their initial conjecture proved false (this extract is drawn from Hitt & Kieran, 2009).

Figure 5. The role played by the CAS in disproving t he initial false conjecture.

Chris	'Two factors' means two separate sets of brackets, right?
Peter	Yeah.
Chris	The only time it contains two factors is when it is odd, I think, which means it can be, [pause] like, our pattern can't be broken down anymore. 'Cause it always ends up being all positive. And uh, then, because, it's sort of hard to explain.
Peter	When the exponent is [pause], when the exponent is an even number you'll have more than two factors, but when the exponent is not an even number, you'll have exactly two factors all the time.
Chris	Yeah. [Types Factor (x^7-1) into the CAS] Yeah, because any time you plug in an odd number as the exponent power, it's uh, the calculator always stays at the most simplified [pause] and [Types in Factor (x^9-1); the CAS displays: $(x-1)(x^2+x+1)(x^6+x^3+1)$] And, no!!! [a look of utter surprise on Chris's face]

The two students then began to wonder: If it is not the case that all odd ns produce exactly two factors when x^n-1 is completely factored, then which ns will produce only two factors? The CAS allowed them to test a variety of values for n, including the extreme case of $n = 99$, which led to a first revision of their initial conjecture (see Figure 6).

Figure 6. A first revision of their odd-number conjecture: *Exclude multiples of 3.*


II.(A).2. Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x+1)$ as a factor?

Please explain:

i.) when the exponent is an odd number that is not divisible by three with a few restrictions,

ii.) when the exponent is an even number

iii). when the exponent is an even number

But they had not quite finished with their conjecturing, and testing of conjectures, with the CAS. In addition to eliminating multiples of 3 as possible values for n, they soon were able to eliminate multiples of 5 and 7 as well. Then one of them suggested trying x^{60} - 1 because, as he said, "I think it has to do with how many numbers can go into it." This led to the "eureka" moment: that n had to be a prime number in order for the complete factorization of x^n-1 to contain exactly two factors.

From these samples drawn from Chris and Peter's activity, we have had a glimpse at the role that CAS technology, within a thought-provoking task sequence, can play in supporting algebraic conjecture-making and conjecture-refining – allowing these two students to focus their trials on certain multiples of the exponent, to try out extreme cases, … in short, to arrive at a new conceptualization of the factors for expressions from this family of polynomials – all this within an activity related to technical work on factoring.

3.2 Further evidence for the emergence of theoretical/conceptual ideas arising from work with CAS techniques

The second set of examples to be presented is pulled from a comparison study that we carried out with two classes of weak Grade 10 algebra students (Kieran

& Damboise, 2007). Some of the characteristics of the task and test design were as follows:

- A set of tasks was developed on the topic of factoring and expanding.
- Tasks were identical for the two classes except that, where one class was to use p/p only, the other class was to use CAS or a combination of CAS and p/p (see Figures 7 and 8 for an example of the parallel task-sets for each class).
- Some tasks were technique-oriented; others were theory-oriented.
- A pretest and posttest were also created with some questions being technical and others theoretical.

Note that, in both task-sets of Figures 7 and 8, the technical is the focus of the first question; the theoretical is the focus of the second question with its four subparts. Note as well that, in the CAS version of Question 1, students are asked to enter onto their worksheet the output produced by the CAS, while in the non-CAS version they are to record their paper-and-pencil factorizations and expansions. (N.B.: The "dissected" form of the first column was one with which both classes were quite familiar by the time that they encountered this Activity.)

Figure 7. One of the task-sets for the CAS class.

Activity 3 (CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Use the calculator in completing the table below.

Given trinomial (in "dissected" form)	Factored form using FACTOR	Expanded form using EXPAND
(a) $x^2 + (3+4)x + 3 \bullet 4$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why did the calculator not factor the trinomial expressions of 1(d) and 1(e) above?
2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?
2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.
2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

Figure 8. The parallel task-set for the non-CAS class.

Activity 3 (non-CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Complete the table below by following the example at the beginning of the table.

Given trinomial (in "dissected" form)	Factored form	Expanded form
Example: $x^2 + (3+4)x + 3 \bullet 4$	$x^2 + (3+4)x + 3 \bullet 4$ $= x^2 + 3x + 4x + 3 \bullet 4$ $= x(x+3) + 4(x+3)$ $= (x+3)(x+4)$	$x^2 + 7x + 12$
(a) $x^2 + (5+6)x + 5 \bullet 6$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why could you not factor the trinomial expressions in 1(d) and 1(e) above?

2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

In this study, the technology was found to play several roles in the CAS class:

- It provoked discussion;
- It generated exact answers that could be scrutinized for structure and form;
- It helped students to verify their conjectures, as well as their paper-and-pencil responses;
- It motivated the checking of answers; and
- It created a sense of confidence and thus led to increased interest in the algebraic activity.

Of all the roles that the CAS played in this study, the fact that CAS generated exact answers that could be scrutinized for structure and form was found to be crucial to the success of these weak algebra students. It proved to be the main mechanism underlying the evolution in the CAS students' algebraic thinking. Ironically, the importance of this role was first made apparent to us by the voicing of frustration on the part of one of the students in the non-CAS class. This student from the non-CAS class, when faced with Questions 2(c) and 2(d) of the task shown in Figure 8, remarked:

> *"How can we describe the relation between the factored form and the expanded form of these trinomials? – we don't even know if our paper-and-pencil factorizations and expansions from Question 1 are right."*

Students in the non-CAS class were at a loss to answer these explanation-oriented questions. They stated emphatically that they were not sure of their paper-and-pencil answers to Question 1, and could hardly use these as a basis for answering, say, Question 2d. In contrast, the students in the CAS class had at their disposal a set of factored and expanded expressions that had been generated by the calculator. They thus had confidence in these responses and could begin to examine them for elements related to structure and form.

This study analyzed the improvements of two classes of weak algebra students in both *technique* (being able to do) and *theory* (i.e., being able to explain why and to note some structural aspects), in the context of tasks that invited technical and theoretical development. At the outset, both the CAS class and the non-CAS class scored at the same levels in a pretest that included technical and theoretical components. However, the CAS class improved more than the non-CAS class on both components, but especially on the theoretical component.

We see this finding as being of some interest. Being able to generate exact answers with the CAS allowed students to examine their CAS work and to see patterns among answers that they were sure were correct. This kind of assurance, which led the CAS students to theorize, was found to be lacking in the uniquely paper-and-pencil environment where students made few theoretical observations. The theoretical observations made by CAS students worked hand-in-hand with improving their technical ability. In other words, *their technique had become theorized*, which in turn led to further improvement in technique.

4. THE ROLE OF THE TEACHER

Are good tasks and CAS technology all that are needed to render technique conceptual, that is, to develop a conceptual understanding of algebraic technique? It would seem not!

Another deciding factor is the nature of the teacher's orchestration of classroom activity that gives rise to the conceptualizing of technique in technology environments. It is the teacher who is pivotal in encouraging the students to struggle with the task, who asks them key questions at appropriate times, who helps them to see the overarching themes within the tasks, who makes the instrumental genesis converge to a common set of techniques and insights, and who leads the classroom discussions that provoke this convergence through discourse. However, not all of the teachers in our research study proved to be equally successful in orchestrating the co-emergence of technique and theory within their students.

Currently our research group is analyzing teaching practice with the aim of identifying some of the key characteristics of teachers' orchestrations of classroom activity with CAS technology that relate to drawing out the conceptual aspects of technical work in algebra. Some of the characteristics we have begun to identify include the following: (a) importance accorded to the mathematical aspects of the task – both technical and conceptual; (b) emphasis on the mathematical-technological similarities/differences; (c) interest in inquiring into the students' thinking regarding the mathematics of the task at hand, by asking for their conjectures, their observations, their elaborations, and their justifications; and (d) awareness of the many possible roles that the technology can play. These possible roles encompass, for example, creating surprising results, generating results for the purpose of exploration, verifying other results or conjectures, and serving as a computational assistant. However, teachers also need to be able to capitalize on these roles in such a way as to encourage student learning.

Other characteristics of teachers' orchestrations of classroom activity with CAS technology that we have been observing include having a repertoire of tasks that engage a variety of learning approaches and evoke different processes, such as, provoking cognitive conflict and seeking to resolve the conflict; looking for patterns; generalizing; activating general mathematical processes, such as observing, comparing, extrapolating, conjecturing, and predicting; and having considered, before the lesson begins, possible student responses and how to encourage further evolution of their thinking within the ensuing lesson. Promising teacher orchestrations also consider the ways in which to incorporate additional artifacts (e.g., worksheets, paper and pencil, the blackboard (or the equivalent), electronic projection devices, etc.) and the roles they might play, namely guiding the work of pupils and structuring their explorations (worksheets), focusing their attention (blackboard), and leading to a convergence of ideas (blackboard).

In sum, effective teaching practice with CAS would appear to embody planning that takes into account at the very least the following:

1. Starting with a key mathematical idea.
2. Thinking about both the technical and theoretical aspects of the key idea.
3. Trying out, when planning the task, some technical examples on the CAS to see how best to take advantage of the technology (does it produce any surprises that could be integrated into an interesting sequence?)
4. Deciding what role the technological artifact should play in the task (generate examples, create surprises, serve as calculation assistant, ...)
5. Deciding on the epistemological processes to be engaged by the task (pattern matching and generalization, conjecturing, seeking connections between representations, resolving cognitive conflict, predicting, ...)
6. Reflecting on how to draw out effectively within class discussions the mathematical-technological links.

Last, but not least, our research observations so far suggest that the one aspect of teacher's practice in CAS environments that seems to be most crucial to students' becoming aware of the conceptual aspects of their technical work in algebra is the following: Orchestrating classroom discussion in such a way as to draw out students' thinking regarding the mathematics of the task at hand, by asking for their conjectures, their observations, their elaborations, and their justifications. When such orchestration is accompanied by tasks that (a) go beyond merely asking technique-oriented questions and which (b) call upon mathematical processes that include: observing/focusing, predicting, reflecting, verifying, explaining, conjecturing, justifying, and which (c) require at times that students coordinate CAS techniques with paper-and-pencil techniques, as well as (d) seek consistency between surprising CAS outputs and existing theoretical notions, then algebraic techniques will have a greater likelihood of being rendered conceptual.

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Conceptions for Relating the Evolution of Mathematical concepts to Mathematics Learning - Epistemology, History, and Semiotics Interacting

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ABSTRACT

Despite all the intense and international efforts of research into the teaching-learning processes of mathematics, Euclid's famous dictum is still valid according to which there is no royal way to mathematics. A growing number of approaches has as its focus the nature of mathematics and investigates whether, by taking into account this nature, the teaching-learning processes might be improved. A common pattern of these approaches can be called to be a "genetic" one, i.e., to establish a relation between the historical evolution of mathematics and the learning of mathematics.

The paper ten discusses how interactions between epistemology and history of mathematics can contribute to better qualify teachers to cope with the conceptual problems inherent to the nature of mathematics. An outlook to the importance of semiotics within the history of mathematics is given for reflection within mathematics education.

INTRODUCTION

The integration of historical elements is a longstanding issue in mathematics education. The ICMI Study of 2000, History in Mathematics Education, represents its most elaborate state of the art (Fauvel&Maanen 2000). Yet, the mainstream of approaches and proposals for the use of mathematics history in teaching mathematics takes history of mathematics as a ready-made collection of facts, easily transposable to the aims of teaching.

In fact, the main justification usually given for the direct use resides in practical methods of classroom teaching: historical elements are claimed to increase the motivation of the pupils, by showing them that the seemingly abstract mathematical system is a living system, that it was developed by human beings and that it is related to the cultural history of mankind - or of a particular nation. Even if not explicitly reflected, the underlying epistemological assumption about the nature of mathematics is that of a continuistic growth.

I should like to refer here to a still not sufficiently known but seminal paper by Antonio Miguel of 1997 where besides the positive effects the problematic issues of the use of history in classrooms are reflected (Miguel 1997). He was only followed by Man-Keung Siu in 2004 with his 16 thought-provocative arguments for not using history in classrooms (Siu 2004).

Actually, all the approaches concerning a use of history are based on certain epistemological views about the nature of mathematics, but in general they remain implicit, and use underlying assumptions. And in order to make the approaches productive, these views should not only be made explicit, but also be reflected within the frames of theoretical discussions in historiography and sociology of science as well as in mathematics education.

What I am interested in, is, whether there exists - beyond the merely accidental contribution of the motivational function - a productive function of the history of mathematics for the actual mathematical practice and for research in the learning process. If one wants to tackle such a question one has to challenge a view of mathematics which is deeply grounded in the common-day philosophy of many mathematicians: I do mean the view of an essentially cumulative nature of the development and growth of mathematical knowledge. According to this common-day philosophy (or epistemology), modern mathematics contains already all fruitful achievements of earlier periods, in an abridged and rationalized manner - so that one could say that contemporaneous mathematics presents in a condensed form the "logic" of history. Consequently, there would be no inherent reason for analyzing the processes of development of mathematical concepts. Likewise, no intrinsic moment would exist for a use of history in teaching – other than to constitute an exhibition of remarkable facts and dates. There would be left just one dimension for historical research: the dimension of factual data like those of priority - who invented first Lemma X, who invented first theorem Y? - and those on ordering and connection of the propositions and of regional/geographical distribution of mathematical knowledge. I confess that such a restricted view or epistemology is too unsatisfactory for me.

Gladly enough, there are recent conceptual developments in historiography of mathematics and in didactics of mathematics, which allow to question the traditional cumulative view and which allow new insights in the relations between history, teaching and learning. The common feature in these developments resides in new approaches to consider the subjectivity in the development of knowledge - as regards the learning person as well as the researching person.

THE GENETIC PRINCIPLE: KEY APPROACHES

Let us begin to look at some prominent genetic approaches and how they conceptualized the role of mathematics history.

In fact, it was an outstanding mathematician and a mathematician who probably was the one who did the utmost for a productive relation between mathematics and mathematics education and who decisively promoted the genetic principle: this person was Felix Klein, at the turn from the 19th to the 20th centuries. Klein was deeply convinced of the pedagogical superiority of the genetic principle – yet he never gave concrete suggestions for practising it. Nevertheless, from his assertions, one can deduce some of the intended characteristic features.

Firstly, he expressed, in 1907, the conviction that this didactical principle had won the dominance within mathematics education:

> "While a systematic manner of exposing mathematics instruction dominated earlier on, which overemphasized the formal aspects of knowledge, this did change more and more over the last years. Today, in German schools, this methodology is overcome. You can remark this victory of the genetic methodology, in the most impressive way, by the establishment of the already mentioned propedeutic geometry teaching" (Klein 1907, 24; my transl.).

A first concrete hint is, hence, that Klein understood a "genetic ordering of the teaching subjects" as opposed to the traditional "systematic" teaching. A next hint is that he recommended the so called biogenetic law as the basis for establishing a good syllabus:

> "This basic law should apply mathematics instruction, too, like any instruction, at least in general: teaching should, by tieing to the natural disposition of the youth, lead them slowly to higher things and eventually even to abstract formulations, by following that same path on which the entire mankind struggled to climb from its naïve primitive state upwards to more developed insight. […] A decisive obstacle for a dissemination of such a natural and truly scientific teaching methodology seems to be the lack of historical knowledge, which becomes so often evident" (Klein 1911, 590 f.; my transl.).

Klein mentioned here a factual restriction regarding a general application of this teaching method, which he had characterized as being simultaneously natural and truly scientific: the lack of sufficient historical knowledge – apparently he meant the teachers of mathematics. Another hint how Klein conceived of the genetic curriculum is that he postulated mathematics instruction should begin with the continuous, i.e. with geometry, like mathematics itself he claimed, and only after that proceed to the discontinuous, i.e. to the number concept and to algebra (Klein 1899, 136).

It is highly revealing that the genetic principle became prominent again in almost the same wording in the 1960s, as a reaction against the so-called modern mathematics, against a one-sided orientation of school mathematics at the structure of mathematical science. It was in the famous memorandum of 65 mathematicians from Canada and the USA – among them Birkhoff, Courant, Kline, Polya, André Weil, and Wittenberg, published in 1962, which argued for the genetic principle:

> "in order to explain an idea (one should) refer to its genesis and retrace the historical formation of the idea. This may suggest a general principle: The best way to guide the mental development of the individual is to let him retrace the mental development of the race – retrace its great lines, of course, and not the thousand errors of detail. […] On the whole, we may expect greater success by following suggestions from the genetic principle than from the purely formal approach to mathematics" (Memorandum 1962).

As you will note, both in Felix Klein's view as in that of these North-American mathematicians, the biogenetic law featured prominently. I will discuss this issue soon. But first let me discuss some works, which have been esteemed as realizations of the genetic principle sensu Felix Klein.

Alexis-Claude Clairaut (1713-1765) was an important French mathematician and physicist. He wrote tow textbooks, one on geometry in 1741 and the other one on algebra in 1746. They have often been claimed to be realizations of the genetic principle. This characterization is misleading, however: it is better to attest them a problem-oriented or heuristic approach (see Schubring 1983a, Glaeser 1983, Schubring 2003, 54ff.).

The geometry textbook intends to develop geometry step by step, always motivated by practical questions like measuring quantities in fields, in the landscape, in farming, and generally in land surveying. At a first glance, the geometry textbook realizes Felix Klein's demands to develop the geometrical notions – beginning from natural, "primitive" questions.

A closer analysis shows, however, that Clairaut did not succeed in a "natural" evolution of the conceptual field, according to an unfolding of "original" problems and of their consequences. Rather, he imposes what should be the next, seemingly practical question to be solved. Moreover, Clairaut's approach does not realize the claim to lead from simple notions to abstract knowledge. Rather, he refrains from all abstraction and theorization. And his claim to follow the historical evolution of geometry is not realized, neither: Clairaut postulates, in fact, *how it might have been*, how the "inventors" did proceed – his historical-genetic claim can hence at best be appreciated as a "rational reconstruction" – in the sense of Lakatos.

The lack of abstraction was consciously intended: The book was produced for a mundane public, not for use in schools and systematic teaching. Actually, it was written for a marquise who desired to be instructed in some leisure mathematics. This explains Clairaut's main methodological concern: ne pas rebuter les commençants – not to scare off the beginners. For the algebra textbook, the problem-oriented approach was even more difficult to realize.
In the famous *Encyclopédie* by Diderot and d'Alembert, in the key entry about textbooks, Clairaut's textbooks were sharply criticized for omitting essential proofs and hence for lack of rigor. Moreover, they were criticized for providing nothing but a sample of propositions instead of a methodically constructed architecture (d'Alembert 1755, 497 r).

A much more elaborated and theoretically reflected conception has been presented by Otto Toeplitz (1881-1940) - a German mathematician whose main book is translated at least into English and who was quite active for improving the teaching of mathematics in schools and in universities between the two world wars. Toeplitz pleaded for using history as a pivotal didactical means - he called this the "genetic methodology" and introduced the distinction between a "direct" genetic methodology and an "indirect" genetic one.

In a key paper of 1927, Toeplitz proposed to return to the "roots" of the concepts and to present them thus as living beings. As Toeplitz said, one could pursue two different ways to realize this goal in the teaching practice:

> "One can either present the discoveries to the students with all its dramatic circumstances and let thus grow for them the questions, concepts and facts - I would call this the direct genetic methodology - or one can learn oneself from such an historical analysis what is the real meaning, the true essence of each concept, and one can draw conclusions from such an analysis for the teaching of this concept which are no longer tied to the historical development - I am calling this second approach the indirect genetic methodology" (Toeplitz 1927, 92f.).

While the direct genetic methodology corresponds to the already discussed direct use of history in teaching, the second, indirect approach is interesting since it takes into consideration the role of the teacher and understands the teacher as actively reflecting the historical processes and as transmitting their essence by his teaching. Toeplitz's indirect approach looks not so much on knowledge, but on meta-knowledge and his main focus is on how to provide teacher-students in their training with such a meta-knowledge about mathematics.

Toeplitz has used this methodology in his own courses at the university, in particular on the infinitesimal calculus. This course has been published as a book: "The development of the infinitesimal calculus, exposed according to the genetic methodology" (Toeplitz 1972/1963). Unfortunately, despite its promising approach, this book cannot really serve as a model for the proposed methodological use of history, since Toeplitz's program to reveal the decisive turning points and ruptures in the historical processes is hardly realized: Toeplitz discerned mainly three fundamental concepts, which determined, by their development, the emergence of the infinitesimal calculus. For two of them, the "infinite process" and the number concept, Toeplitz tries to show that the ancient Greeks did already achieve all essential steps and that later developments were but an unfolding and a change of exterior form of these first achievements. For instance, in the famous dispute between Dedekind and Lipschitz, whether Dedekind's concept of real numbers was new or identical with the notions of the Greek Eudoxos, Toeplitz took the part of Lipschitz in claiming that Eudoxos already operated with the concept of real numbers while Dedekind had insisted that the notion of completeness was missing entirely in Greek mathematics and could not be derived, not even implicitly from geometrical ideas. Toeplitz admitted for the function concept only that it emerged as a new concept in modern times, but even here he tried to show that Ptolemy was already aware in Hellenist time of this concept (see Schubring 1978).

We can see therefore that Toeplitz remained attached to the traditional view of a continuous, cumulative development in the history of mathematics so that his own notion of an indirect approach could not become fruitful. His underlying conception seems, too, to be effected by that notion, which is commonly called the "biogenetic law": Toeplitz claimed that the development of mathematical concepts uses in general to follow "the easy ascent from the more simple to the more complex" and that this historical ascent might be used didactically (Toeplitz 1927, 95).

The example of Toeplitz's conception therefore again shows that the main problem for a revealing use of history resides in an adequate conception of historical development. While most of the other scientific disciplines are discussing - since Thomas Kuhn's famous book on scientific revolutions - revolutions in their field and ruptures in the conceptual development, mathematics seems to close its mind to realize an analogous epistemological change. The traditional epistemology stressing the uniform, continuous and cumulative character of this "queen of the sciences" is, apparently, too strong. A telling example for this exceptional position of mathematics has been formulated by the French philosopher Gaston Bachelard who has convincingly analyzed epistemological ruptures in the exact sciences, but who has consciously excepted mathematics from these analyses:

> "The history of mathematics is a miracle of regularity. There are periods of standstill, but it knows no periods of errors" (Bachelard 1975, 25; my transl.).

Actually, the notion of error will provide a key to challenge this epistemological view.

RELATION BETWEEN RESEARCH AND TEACHING

In order to tackle this question let me present you some of the mentioned new approaches in historiography of mathematics. Their main feature is constituted by studying the interrelationship between the system of production of new mathematical knowledge and the systems encompassing and supporting mathematics. These new types of historical research, which have evolved over the last decades, focus in particular on one specifically related social sub-system: on the education system, since the dissemination of mathematical knowledge is essentially bound to the education system and since teaching positions were for a long period the only relevant professional careers for mathematicians. The analysis of the relationship between mathematics seen as a social system and its surrounding systems has progressed much beyond the fruitless dichotomy of internal versus external determination of mathematical ideas and has particularly contributed to better understanding the circumstances of mathematical production.

A primordial element in these analyses is given by a re-evaluation of the relation between teaching and research. The traditional view of this relation has been that the scientific part exclusively plays the active, productive role and that the didactical side always is the passively receiving partner, which transposes the received into the instruction system (a view, still perpetuated by Chevallard's concept of transposition didactique). The relation between scientific knowledge and school knowledge was therefore understood as operating only in one direction. This one-directional view has been denounced in 1978 by Willem Kuyk – the author of "Complementarity in mathematics" (1977) - by comparing it with the relation between stalactites and stalagmites (Kuyk 1978, 5):

"Mathematics is not a stalactite hanging over a stalagmite", thus denying the view that mathematics education grows but by receiving some drops from above, from the supreme instance. The instructional system cannot be understood in the simplistic way of a stalagmite, which receives some drops from the stalactite while it is growing. My intention is to show that the re-evaluation of the relation between research and teaching allows at arriving at another understanding of historical development.

An important publication on this way has been the article by Judith Grabiner of 1974: "Is mathematical truth time dependent?" At the same time, Hans Wußing had remarked that the new system of teaching higher mathematics, emerging in France since the end of the 18th century, contributed decisively to establishing new standards of rigor, to promote research on the foundations of mathematics and to falsify propositions, which had been thought to be true (Wußing 1974, XVIII).

My own research on the development of mathematics in Prussia (a leading state in Northern Germany) in the 19th century done in the early 1980s, has shown that the profession of mathematics teachers at secondary schools constituted the social basis which enabled the establishment of mathematics as an autonomous discipline within the university system. Moreover, the type of interest of these teachers in mathematics decisively moulded the production of pure mathematics for which Prussian and later German mathematics has become so well known: Actually, the interest of these teachers - themselves regarded as "scholars" - in rigor and in a consequent architecture of mathematics yielded important achievements in foundational questions and in clarifying basic notions (see Schubring 1983b, 158 ff.).

Resuming these briefly outlined researches and results on the history of mathematics in its context, one can say:

- firstly, the teaching of mathematics has influenced the development of mathematical research. The dimension of instruction and teaching has therefore to be considered for an adequate notion of historical understanding of mathematics (see Schubring 2001);
- secondly, ruptures and emergence of novel directions in history of mathematics are largely due to epistemological changes, which are connected to changes in the systems related to the system of scientific activities;
- and thirdly, didactical research on learning processes can reveal means and categories which are usefully applicable to analyze also processes of scientific development.

The last two propositions aim at including the subjectivity of the student and of the scientist into the theoretical framework. In order to explain and to apply these propositions I want to discuss two aspects on which much didactical research has been done over the last decades in order to study the subjective element in the learning process. These two aspects are the errors and the obstacles.

Errors

The investigation of pupils' errors in the learning process constitutes a major field of didactical research since several decades – actually, as one of the main features of the emergence of mathematics education as a scientific discipline. Didactics of mathematics has increasingly established more refined experimental instruments to analyze pupils' errors and discusses theoretical models for interpreting errors.

As major results of these researches I need here to mention only briefly: errors are not merely expressions of an individual's "defects", of missing attention, or the consequence of missing knowledge or due to an accidental specific situation. Errors can therefore not be simply remedied by increasing discipline, attention and diligence of the pupils.

Empirical research has shown that errors are rather causally determined and often of a systematic nature. Errors can be analyzed and described as resulting from patterns and notions, which can be internally consistent but which do not coincide with the notions and operations as intended by the teacher. A first consequence of these researches has been to identify as causes of the errors either difficulties of the pupils in grasping the new information in teaching or problems in the interaction of the variables influencing mathematics instruction (teacher, curriculum, pupil, context of the school). But even in this research, errors of students were understood as indicators for individual difficulties (Radatz 1979). Further research has, however, increasingly questioned that these specific patterns are signs for merely individual difficulties.

A radical research program developed in this field is that of social interactionism, initiated and developed by the Bielefeld group: Bauersfeld, Krummheuer, and Voigt, since the 1980s:[1] in this program the status of errors is challenged. The basis for this program is the philosophy of constructivism as developed in particular by Glasersfeld: There exists no objective meaning of notions and concepts. Each individual constructs his own meanings given his experience and background. It is only by the social interaction between the individuals that communication takes place and that the individual constructions can gain a certain convergence. It is by the process of social interaction, that a specific construction becomes acknowledged as common knowledge, as "objective" (Bauersfeld 1983).

[1] Paul Cobb, in his speech at ICME 11, after having received the ICMI Freudenthal medal, remembered the formative significance of his cooperation with this group.

Mathematics teaching is particularly suited for studying the processes of establishing a common knowledge shared by the participants of the communication in a class since there is no direct exterior reality, which would allow testing the validity of the individual constructions. The teaching process can be described as a negotiation between teacher and students and where the teacher tries to establish working procedures, which may be more or less stable. The original Bielefeld group has used particular experimental instrumentations like video-recording of the teacher-student interactions and developed methods for transcribing the interactions in order to make them analyzable and reproducible to other researchers. This research program has yielded very remarkable results and shown that what is usually seen, by the teacher, as errors are in fact misunderstandings: the students use to "see" other notions in the material presented by the teacher than the teacher had in mind.

For instance, in the teaching materials used for introducing the notions of the first natural numbers several objects from the real world are shown. The student should "abstract" from the real world features and just retain the cardinal number. The analysis of the interaction process shows, however, that the students direct their attention to other elements in the pictures and effect therefore other "abstractions". It takes a long time until the students can divine what the teacher wants to hear and that conventions become routinized upon signals given by the teacher. This learning "success" can be a merely superficial one and the working procedure can break down when the teacher uses a different symbolization (see Voigt 1985).

This concept of social interactionism need not remain restricted to school teaching and didactics. It can equally well be applied to research in mathematics and therefore to history, too. How does it happen that a new theory is adopted in mathematics, that a concept is regarded as rigorous or rejected as not rigorous, that a proposition is regarded as false? This is neither by the decision of an individual nor by the universal insight of an eternal truth, rather we find, here too, negotiating processes in the mathematical community, interactions in this social community, which determine about acknowledgement or refutation. Before I discuss consequences of this view for the growth of mathematical knowledge, there is to mention yet another dimension relevant for history in the didactical research on students' errors.

In fact, in the didactical research on errors one does not locate all problems in the modes of interaction and in the communication process, but one also emphasizes possible causes in the mathematical content of the communication: One analyzes the teaching material or the exposition of the teacher if they might not be correct from the mathematical point of view or if they contain missing links which could have caused that a student did not grasp a mathematical notion and its operations. This is surely a legitimate approach in didactics of mathematics but it is not a sufficient one: In almost all didactical theories, the mathematical knowledge is taken as objective or absolute precondition for learning which will not be questioned. This starting point of didactics is, however, insofar not sufficient, as there exists no *a priori* evidence that the mathematical knowledge used for teaching really is complete, organized consequently and coherently and without missing links. Didactical research should be aware of inherent problems in mathematics itself: unsolved or even undetected problems in the logic or in the epistemology of mathematics, ambiguous or even misleading notations. The teacher who has been initiated to the language of mathematics and its peculiar operating procedures will not be able to remark such inconsistencies, but the student as naive, as non-initiated, might be hindered by such problems inherent to mathematics - what the teacher marks as error can be an indication for deficiencies within the mathematical knowledge.

It is particularly this dimension of unsolved internal or epistemological problems in mathematics by which the teaching process can effect an impetus for progress in mathematics or can even effect ruptures within the established system of mathematics.

Since school mathematics represents to a greater degree the condensed essence of the historical development than the actual research knowledge we did arrive at a first productive use of mathematics history for didactical research, namely by supplying the means for analyzing those conceptual, notational or epistemological problems of mathematics which are due to certain stages of the historical development and which effect errors or misunderstandings by the side of the students.

Obstacles

We can deepen the discussion of the use of history for teaching by the means of didactical categories if we regard the specific contributions by French didacticians. The emphasis on the knowledge itself - what one can call the epistemic dimension -, which is largely missing in German and North-American didactical research on errors, constitutes in French research one of the main issues. One uses in France a category for didactics of mathematics, which has originally been established for studies in history of science. I mean the category of *obstacles épistémologiques*, of epistemological obstacles, put forward in 1936 by the already mentioned French philosopher Gaston Bachelard. It gained particular influence after a re-edition of his works in 1975. Bachelard's conceptions have been transposed by Guy Brousseau to didactics of mathematics, who has developed a didactical theory of obstacles. Its main aim is to overcome to attribute errors only to subjective causes in the students. Brousseau discerns in particular the following types:

- didactical or didacto-genetic obstacles: by this he means learning difficulties or bareers which originate from the conception or structure of the curriculum, from the particular teaching concept, from didactical concepts,
- and, secondly, epistemological obstacles. According to Brousseau, these obstacles to learning are rooted in the nature of mathematical knowledge and can therefore not be avoided. They are constitutive for the respective knowledge, they become visible in some stage of the historical development and can be identified by historical analysis.

According to Brousseau's theory, where a model of stages is applied, there are inherent contradictions within the types of knowledge tied to the lower stages: the knowledge shows itself effective as long as applied within these restricted areas, but reveals to be an obstacle when it becomes applied to situations of a higher stage. Some knowledge can therefore, due to inherent reasons, function as an obstacle against progress on the next stage (Brousseau 1997, 84).

One can therefore understand his theory as a "transposition" of Bachelardian ideas to didactics. Both theories on whom Brousseau relies, by Bachelard and by Piaget, imply a teleological vision: the certainty to be able to achieve the most "mature", the most elevated level of science, of human thinking.

A number of studies has been carried through on the basis of this research program, for instance on the difficulties of students with the limit concept in calculus, with the notion of infinite, and on students' notions of basic geometric concepts. A particularly profound study of the limit concept, both for the historical and for the didactical side is the collective work published in 2005 by a group of Italian researchers: *Oltre ogni limite* – Beyond any limit.

The quality of such research relies to an important part on the reliability of the historical analysis: otherwise, the empirical findings on students' difficulties are interpreted according to prejudices or to a common-day understanding about the nature of breaks, ruptures and problems in the historical development. The demand for detailed and qualified historical research is the more imperative as historiography of mathematics traditionally tended to restrict itself to the ideas of the "great men", the "heroes" - an emphasis by which the real difficulties experienced by the larger contemporaneous mathematical community can hardly be taken into account.

We arrive thus at a second, "indirect", use of history for didactical research: In order to fill the enormous gaps of knowledge about the mathematical thinking and practice in the larger group of mathematical practitioners, it constitutes a challenging task for the historiography of mathematics to study debates and controversies about the status and nature of relevant mathematical concepts.

This second use is not thought of in the way of deriving recipes for teaching, but as elements for the didactical research on epistemological obstacles and to enrich the meta-knowledge of teacher students and of teachers.

Starting from such a conception, I have done extensive research on the history of negative numbers. The results were significant contributions for history and for didactics, namely on the role of errors for mathematicians and for the teaching process and, likewise, on the notion of epistemological obstacles in history and in teaching (see Schubring 2005a; 2005 b; 2007).

The function of history in this French conception

Understanding Brousseau's theory is facilitated by comparing its two versions, of 1976 and of 1983, which is easy, since many of his publications were translated in the volume Theory of Didactical Situations (1997). He uses to emphasize that obstacles are unavoidable, but also that one should not reinforce them explicitly:

Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative rôle in the knowledge being sought. (Brousseau 1997, 87).

In 1983, after his controversy with Georges Glaeser about the meaning of the term "obstacle" and basing himself now on the study by Duroux –

the first concrete investigation in French didactique to identify epistemological obstacles, this time regarding the notion of absolute value – Brousseau relied much more on history of mathematics and attributed it a decisive function:

But it can prove itself to be fruitful for teaching insofar as:

- the obstacles in question are truly identified in the history of mathematics;
- they have been traced in students' spontaneous models;
- the pedagogical conditions of their "defeat" or their rejection are studied with precision in such a way that a precise didactical project can be proposed to teachers,
- the assessment of such a project can be considered positive. (Brousseau 1997, p. 93-94)

This strengthened function of history reveals, however, a weakness of the conception: history has to serve as source for errors committed by mathematicians. Thus, history has no productive function; it serves as an element of a recipe for research:

From the outset, therefore, researchers should

a. find recurrent errors, and show that they are grouped around conceptions;
b. find obstacles in the history of mathematics;
c. compare historical obstacles with obstacles to learning and establish their epistemological character. (Brousseau 1997, p. 99)

Here, one finds no active role for history. This seems to be related to the fact that Brousseau did no integrate a key element of Bachelard's conception: the notion of a rupture between empirical knowledge and scientific knowledge, which is of enormous importance expressly for didactical research. Furthermore, in the 1983 conception, there is not the supposed symmetry between the side of history and the side of the learner: Since obstacles were declared to be insurmountable - "incontournables" and "insurmontables" (Brousseau 1989, quoted in Brousseau 1998, 154) -, while students' errors should be surmountable, there is a drastic asymmetry. And scientific progress would be impossible when obstacles could not be overcome. Glaeser's understanding of obstacles as "difficulties", hence without a normative character, lent itself better for historical investigations.

The general weakness of the conception of epistemological obstacle resides in the problem that the history of mathematics is regarded as a fixed collection no longer open to questions and research. Traditional historiography, onto which this didactical conception would base itself, is not adapted for answering to these new questions, for a use in didactics and learning: they did not look for the "normal" mathematicians who would better reveal the obstacles sought for than the traditional heroes.

Actually, it had never been investigated whether one of the normative pillars of the concept of epistemological obstacles is really justified, namely whether an historical obstacle necessarily shows up as a learning obstacle. I have therefore undertaken a case study to test this issue: it concerned the multiplication of quantities, which proved to constitute over various centuries a genuine conceptual obstacle in arithmetic and which characteristically no longer constitutes an obstacle in learning – essentially due to an epistemological switch which had happened in the meantime: quantities no longer constituting the conceptual fundament of mathematics (Schubring 2005 b).

Critique of the biogenetic law

In almost all the genetic approaches, which I have presented to you so far, almost inevitably the so-called biogenetic law showed up – either implicitly or explicitly. This is true since Felix Klein's first pleas. Even in Brousseau's conception it shows up implicitly. He uses to relate to "spontaneous" reactions of students, i.e. to answers before teaching the respective concept (Brousseau 1997, 93). According to him, these spontaneous answers reveal epistemological obstacles and correspond at the same time to the naïve hypotheses of the first scientists. This implies not only the implicit acceptation of the biogenetic law, but negates at the same time the profound social and cultural changes, which effect that children of today start at decisively different conditions than earlier generations.

The recapitulation hypothesis originated from a transfer of biologism to cognitive development. It was in particular Haeckel's famous law for biological development of the. species which was grafted to psychology. The graft from biology on psychology and education was effected, among others, by the philosopher Herbert Spencer

> "the education of the child must accord, both in mode and arrangement, with the education of mankind, considered historically. In other words, the genesis of knowledge in the individual must follow the same course as the genesis of knowledge in the race." (quoted from Branford 1908, 326).

This grafted biogenetical principle, or principle of parallelism, had become a largely shared topic in education by the end of the 19th and the early 20th centuries and, remarkably enough, in particular in mathematics education. In fact, it would seem that mathematics was, and still is, the only school discipline where this principle has become so prominent. I cannot remember anybody to have claimed it being applicable, say, to physics or to chemistry. Strangely enough, the biogenetic law, no longer prominent in the first half of the 20th century, made a more or less explicit return to mathematics education in its second half, and in particular in approaches for using mathematics history in teaching (see Schubring 2004).

An instructive and concise introduction to the entire problematic of parallelism and of the biogenetic law is the excellent paper of 2002 by Luis Radford and Fulvia Furinghetti. They elaborate not only Piaget's and Garcia's deficits in conceiving of cultural and social impacts on cognitive formation, but they also present L. Vygotski's alternative approach as that of one of the few psychologists to have profoundly investigated socio-cultural influences on cognitive processes. As they put it, "the merging of the natural and the socio-cultural lines of development in the intellectual development of the child definitely precludes any recapitulation" (Radford/Furinghetti 2002, pp. 634 – 642; here: 637).

The major flaw in all the approaches based on parallelism is that they presuppose history of mathematics as a definitely established corpus of knowledge, which is beyond controversy. This is, however, far from being true. The historiography of mathematics has hitherto concentrated on the "peaks", on the "heroes" of mathematics, and it has practiced a resultatist view, searching for forerunners of the results of present mathematics, and thus ever and again reproducing the continuist view of development we always find in how didacticians assess the history of mathematics.

For uses in education, another type of historiography and of research has to be attained, however, a view which unravels the contributions of scientific communities at large, identifying and assessing conceptual ruptures, and in this way documenting conceptual developments in different relations of subsystems to their encompassing systems (cf. Schubring 2002). This will make it possible to better establish the social and cultural contexts and their impact on scientific development – an approach hitherto only postulated, but never really elaborated.

Resuming our discussion of the conceptions of epistemological obstacles and of the biogenetic law (or parallelism) we have to state that both are not adapted for a productive use of the history of mathematics. Both are normative approaches and do thus hamper experimental research in both domains, in history and in mathematics education – they are prejudicial for open-ended research.

Furthermore, al the discussed genetic approaches and these last two in particular presuppose a universally homogeneous conceptual development over time. However, there does not exist a "Gesamt-Intellektueller", an all-comprising intellectual. Conceptual developments occur within determinate and specific groups, the so-called scientific communities which have as primary references for their conceptual frames the values and norms of their particular cultural environment, their directly surrounding systems – which one may shortly call "context". Therefore, there does likewise not exist an absolute simultaneousness or parallelism of conceptual developments in different cultures.

Errors in mathematics

I can now come back to my proposed approach to start form the subjectivity of the person and its group: I spoke already of this approach for the learner, within the conception of social constructivism. I should now turn to the other side, which is relevant here, to the scientist – and now not limited by *a priori* assumptions about a *Naiveté* of early scientists etc., but based on a productive role of interaction between research and learning. In such a sense, one is able to investigate more freely possible errors of scientists, and in particular of mathematicians.

In present day convictions it seems to be unthinkable to acknowledge the possibility of serious errors in the history of mathematics, as exemplified by Bachelard's exclusion of errors in mathematics. Earlier generations seem to have had less problems with such a possibility. A telling example is provided by Martin Gebhardt, the author of the first ICMI Study on the role of mathematics history for mathematics instruction in 1912. He assured:

> "With the proof by history that error and controversy play their role and are important in mathematics, too, the abysm, which separates it from other sciences, in particular also from the natural sciences, will disappear to a considerable degree" (Gebhardt 1912, 83).

And by errors he meant, as he emphasized, not those which can happen to each mathematician, but those which are characteristic for an entire epoch – like the conviction of convergence of the series 1-1+1-1+1- … having as limit ½, defended by Grandi, Leibniz, and Euler, among others. And in 1904, E. Maillet, a French mathematician had called to collect remarkable errors of mathematicians, as an instance of self-reflection. The resulting collection was published in 1935 by Maurice Lecat, a specialist in variational calculus. It is not well known, neither in historiography nor in mathematics education. The collection documents about 500 errors, attributed to 330 mathematicians – among them many minor figures, but also famous mathematicians. Lecat stated that there was only one famous mathematician who never committed an error: Evariste Galois. Thus, Lecat dedicated to him an honorary page, i.e. an empty page (Lecat 1935, 39).

Given this dimension and extension of committing errors in mathematical research, on the one hand, and the acceptance of "errors" as good mathematics over extended periods, I am now able to formulate my main hypothesis/research guideline/proposition:

It is a consequence of the program of social constructivism resp. social interactionism that so-called students' errors can no longer be called "errors" if they follow a definite strategy, jointly shared by that entire social group. Analogously, this applies to communities of scientists, too, and in particular to mathematicians. Regarding chemistry, I should like to recall the phlogiston theory, which was accepted by chemists over centuries (see Kuhn 1962).

This specific claim of such a constructivism has to face the objection: where remains the objectivity of mathematics, which has always been maintained to be the major characteristic of this science?

In fact, the consequence of my conception is that there exists no objectivity, at least no overall objectivity. Not only in learning, meanings of concepts are subject to negotiation processes, so that differences in meanings established by various groups will disappear as result of interactions when these groups get into communication and achieve shared meanings, but also in science a common understanding will at first be restricted to social communities, which are tied together by certain conditions to form a basic unit of communication, say by sharing a common culture and language. Let me call this basic unit a scientific community of first order. In general, one can assume that they will share, too, the epistemological view of their subject. While there might co-exist different epistemological and conceptual views of mathematics in separated mathematical communities, there should begin processes of interaction at the moment when such separated communities come in contact with each other. Consequently, either the values and conceptions remain mutually alien so that – if there are no other pressures for establishing shared conceptions – the communities will continue to be separated, or a negotiating about the differences will begin with the effect of certain compromises or dominations.

This hypothesis about a relative objectivity as result of negotiation processes between originally separated mathematical communities can be tested by investigating – not a "clash" of cultures – but the effect when two cultures with different conceptions of knowledge are colliding.

A first such test is presented by the transmission of number signs and of decimal fractions from India to the Arab civilization, studied by Mahdi Abdeljaouad. As is well known our so-called Arab number signs are in reality Indian signs, as well as the establishment of zero and of the decadal number system. The Arabs used, like the Greeks, the Phoenician manner of designing numbers by letters of the alphabet. And for fractions, they either used Babylonian sexagesimal fractions or Egyptian unit fractions. In the main period of Islamic culture, from the 8th to the tenth centuries, the Indian numbers and the decimal fractions had not found acceptance. Al-Uqlidisi who had tried to introduce them, by a significant textbook in 952, had no success and his book was forgotten, until a re-edition by Saidan in 1966. The resistance against the Indian way of mathematics is clearly documented by a polemic appreciation uttered by Al-Biruni in the 11th century, in his introduction to the book "History of India":

> "The Indians to not dispose of philosophers like the Greeks who have exposed their subjects in their texts entirely scientifically. They have produced almost no book, which is not a downright collection of rubbish and where get mixed all varieties of popular beliefs. The spirit of authority dominates in them. As far as I am concerned, I can assure that their books of arithmetic and mathematics are comparable with nothing else than stone cairns containing some fragments of ceramics or with pearls hidden in the dung/manure of camels." (quoted from Abdeljaouad 1978, 14; my transl.)

I have published a more recent example of mutually exclusive visions of mathematics last year: the case of Edmund Külp, the teacher of Georg Cantor who in his youth was educated according to the values of French mathematics – and that meant of physico-mathématique: a vision of applied and applicable mathematics. Becoming transferred to Germany, Külp had to suffer a purely formal, inapplicable mathematics – the mathematics of permutations and transpositions of the German combinatorial school. Due to the incompatible meanings of that French and that German mathematics, Külp failed with his project to pursue an academic career at a German university and had to serve for decades in primary teacher education to make his living – until he managed to become a teacher at a trade school where some French mathematics was admissible (Schubring 2007).

Role of semiotics: the development of signs

A particularly illuminating quotation by Destutt de Tracy, a French philosopher, of 1801 underlines the productive role of teaching for research, for obtaining new knowledge, which I am emphasizing in my approach to the use of history of mathematics. This quotation presents an evaluation of the historically first experience to disseminate scientific knowledge, to elementarize science and making it accessible to a general public. Reflecting the ambitious projects of the French Revolution to produce such truly elementary textbooks, Destutt de Tracy resumed:

> "When one is about to expose a scientific fact, one often remarks that it necessitates to undertake before new observations, and – better investigated – it presents itself by a quite different point of view. At other occasions, it proves that it is the principles of science itself, which need to be revised, or one has to fill numerous gaps to connect them mutually. Briefly, the matter is not to disseminate the truth, rather one has to detect it" (vol. 1, p. 4 f., of his *Projet d'Éléments d'Idéologie*; quoted from Schubring 1982, 114 (my translation).

A particularly important dimension of the challenge to research by teaching as explained by Destutt de Tracy is presented by the representation of mathematical objects, by the sign function of concepts where essential elements use to be hidden and where it is in particular the effort to teach them which effects an explication of implicit and hidden assumptions and conceptual moments.

The last part of the lecture will be devoted to briefly expose the role of semiotics for such a new approach to the relation between history and didactics. In fact, there is a forerunner for the present approaches to introduce semiotics into mathematics education: It is Karl Menger (1902-1985), the important philosopher of the Vienna circle, logician, mathematician and economist. He had to flee the Nazis and emigrated to the United States. Since the late 1940s, Menger has published several papers and even a seminal book, which give excellent descriptions and analyses of inconsistencies in mathematics and notational ambiguities, most of them remain even today to be solved and pupils and students are left with the obstacles to get through the misleading paths.

His publications are not only well instructed in history, in semiotics, and in teaching, but they are written with such a deep humour that it is a real pleasure to read his profound analyses.

It is highly remarkable that a review of his seminal book refounding the teaching of the calculus emphasizes the same points as Destutt de Tracy:

> "It becomes clear after reading the book that the invention of the new notation was an essential step toward the clarification of the basic ideas and their applications and is thus amply justified" (Review of: Karl Menger, *Calculus, A Modern Approach*, by H. E. Bray, in: *American Mathematical Monthly*, vol. 61, Sept. 1954, 483-492, on p. 483).

A key starting point for Menger are notational ambiguities in mathematics, which use to be ever again transmitted to the next generation as time honoured and therefore not questionable. A particularly striking example are the twelve different meanings of the seemingly so innocent letters x and y. In fact, the meanings range from numerical variables, over indicators for the identity function, indeterminates, specific fluents, function variables, to "dummies" (Menger 1956a).

Menger has sharply criticized the negation of notational and conceptual problems arising from the weight of unchallenged history. Summarizing the mainstream thinking of mathematicians at least of his time, he lets them say:

> "Since for the past two hundred years and to this day, all mathematicians and scientists have achieved complete mastery of mathematics with its time honored procedures and in its traditional presentation, and since furthermore,

> the difficulties here discussed do not disturb any accomplished mathematician in the least, youngsters who cannot cope with them must be mathematically utterly incompetent. To revise procedures or symbols for their sake is not worth anyone's while since their study of mathematics cannot, under any circumstances, be profitable either to those mathematical morons themselves or to anyone else" (Menger 1956b, 584).

In a perfectly satirical manner, Menger has denounced the sticking to historical traditions in his series of papers on Gulliver, in particular in the first one entitled: "Gulliver in the land without one, two, three".

His starting point is the juxtaposition – historically to be often found – of the first numbers being treated as quantities (or "named numbers") and the greater ones as numbers. And he ridicules a didactical retrogression by which all numbers are treated as quantities or named numbers. Here, the mainstream mathematicians defending this anti-didactic transformation are called the IMMORTALS, abbreviation of: The Island's Major Mathematicians of Real Talent and Learning (Menger 1959).

This invented example of retrogression, of a use of history where one needs to get liberated from historical dust, serves as an introduction for what is a key element in Menger's theories: the establishment of an algebra of functions. For this, he first criticizes a notational ambiguity, which causes many learning problems: the often missing distinction between a function and a value of this function - both being usually designated by f(x) (or cumbersome formulations like: "the function which is expressed by f(x)"). Rather, one has to designate a function by its name; one is thus able to distinguish the function from its value at a certain point.

More generally, however, his conclusion is that one does not need variables in calculus, that they constitute but dummies, and that one has rather to reflect on naming functions to be able to operate with functions. In this sense, he calls variables "dummies" and shows that these are elements of historical tradition, from which teaching has to be liberated.

On the other hand, he develops his algebra of functions by the introduction of a notational innovation: the basic element of this algebra is the identity, the neutral element. He calls it the function I, namely:

$$I : x \to x, \quad I^2 : x \to x^2$$

And this otherwise neglected neutral element enables him to introduce an operational calculus with functions. Therefore, this new approach of semiotics implies a double role of history of mathematics for teaching: on the one hand, it reveals outdated mathematical practices, which need to be deleted to improve the teaching-learning process. On the other hand, it reveals forgotten or marginalized conceptions which had been established in some mathematical community in an earlier period and which need to be valourised and updated for present day teaching purposes. In fact, Menger's operational calculus is a direct continuation of the Derivation Calculus established by Arbogast in the wake of the French Revolution – exactly as a realization of the *méthode analytique* of the Enlightenment, which should contribute to disseminating the scientific knowledge. It is not by accident that in these analytic approaches the role of symbols is decisive for clarifying the meaning of the concepts and for enhancing their teaching and learning.

In fact, Menger's algebra of functions confirms again the systematic relation between processes of algebraization and reflection on the use of symbols. Semiotics promises fruitful impacts on the use of history for teaching!

Menger's ideas have had some impact and influence in the great Curriculum-Projects in the USA, during the 1960s, in particular within CSMP, but with their end they remain rather neglected. His legacy constitutes treasures, which still remain to be excavated and brought to light and to use!

CONCLUSION

Although conceptually attractive, approaches to use history of mathematics for mathematics teaching show theoretical shortcomings as well as problematics in the experimental designs. As a major reason, continuistic visions of mathematical development proved to be underlying so that history of mathematics was not able to exert a productive function. The most promising conception, the indirect genetic method of Toeplitz, suffered in his realization from his peculiar teleological view of development: all the essence being already contained as a germ in Greek mathematics. But the kernel of his vision, to unravel the conceptual depth and meaning from turning points in the history, provides a precious approach at least for teacher training. Yet, historiography of mathematics still has to broaden its research areas to comply with such a vision. Semiotics provides promising contributions.

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Different profiles of 'negative attitude toward mathematics'[1]

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Attitude toward mathematics, affect, narratives, students' difficulties in mathematics

[1] This is a short version of the lecture given at ICME 11: an extended one has been published in the *Journal of Mathematics Teacher Education* (Di Martino P., Zan R.: 'Me and maths': toward a definition of attitude grounded on students' narratives), vol.13, n.1, pp. 27-48.

1. INTRODUCTION

In this communication some reflections about 'negative attitude toward mathematics' are proposed, which link theoretical issues with problems emerging from both teachers' practice and students' experiences.

After a brief theoretical introduction, in which some issues from research about attitude toward mathematics are summarized, a study from an Italian National Project about negative attitude toward mathematics is presented, aimed at investigating teachers' use of the construct in their practice. Then a second study from the Project is discussed, based on students' narratives about their own stories with mathematics. From this study a characterization of attitude emerges, that strictly links theoretical issues with students' experience. Implications for research and teachers' practice are then discussed.

2. THEORETICAL ISSUES ABOUT *ATTITUDE*

Research on attitude has a long history in mathematics education. The construct was borrowed from the field of social psychology (Allport, 1935), where attitude is viewed as the predisposition to respond to a certain object either in a positive or in a non positive way. The early studies in mathematics education aimed at investigating the relationship between attitude towards maths and achievement. This kind of studies led to often ambiguous or even contradictory results, as highlighted by the meta analysis carried out by Ma and Kishor (1997).

Research about attitude in ME more recently developed in the field of affect. In the classification of Mc Leod (1992) attitude is considered together with emotions and beliefs as one of the constructs that constitute the affective domain (De Bellis & Goldin, 1999, propose values as a fourth construct).

With the development of the field, the need also grows for a theoretical framework for affect in ME (Zan et al., 2006).

This claim for theory also involved the construct of 'attitude to mathematics', and led to identify some critical issues in research:

i) Need for theoretical clarity about the definition of attitude and of positive / negative attitude (Leder, 1985; Ruffell, Mason & Allen, 1998; Daskalogianni & Simpson, 2000; Di Martino & Zan, 2001, 2002, 2003).

Most studies about attitude do not provide a clear definition of the construct itself. Attitude tends rather to be defined implicitly and a posteriori through the instruments used to measure it (Leder, 1985; Daskalogianni & Simpson, 2000).

When a definition is explicitly given, or can be inferred, it mainly refers to one of the two following types:

- A 'simple' definition, that describes attitude as the positive or negative degree of affect associated with a certain subject (McLeod, 1992; Haladyna, Shaughnessy J. & Shaughnessy M., 1983).
- A multidimensional definition, which generally recognizes three components in attitude: emotional response, beliefs regarding the subject, behaviour related to the subject (Hart, 1989).

If in the case of the simple definition the characterization of positive / negative attitude seems *natural* (identified with positive / negative emotional disposition toward mathematics), in the case of multidimensional definitions it requires several choices that need to be made explicit (what do 'positive' or 'negative' refer to? To each dimension individually? What does 'positive' belief mean?)

ii) Need for instruments consistent with the research problem and with the chosen definition of attitude, and capable of capturing the deep interaction between affect and cognition. In particular several scholars question the possibility of 'measuring' attitude through questionnaires (Ruffell et al., 1998).

iii) Need for overcoming the limits of a normative approach. Most studies try to point out a general cause / effect relationship between attitude and behaviour, but this approach does not seem compatible with the fact that the interaction affect / cognition depends on the individual. In particular, the same belief can elicit different emotions in different individuals: for example some individuals associate the belief 'In mathematics there is always a reason for everything' with a positive emotion, others with a negative one (Di Martino & Zan, 2002).

3. AN ITALIAN NATIONAL PROJECT ABOUT ATTITUDE

The points made above about the need for a theoretical framework for affect, together with the importance given to linking theory and practice, have been fundamental issues of an Italian Project about attitude, named 'Negative attitude towards mathematics: analysis of an alarming phenomenon for culture in the new millennium'.

One of the first studies carried out within the Project was an investigation aimed to recognize how teachers actually use the construct 'negative attitude' in their practice (Polo & Zan, 2006).

With this aim a questionnaire with 6 multiple choice questions and 6 open ended questions has been designed and administered to 146 teachers from various school levels.

The study highlighted that the diagnosis "This student has a negative attitude toward mathematics" is frequently given by most teachers. Furthermore it emerges that to describe 'negative attitude' teachers do not refer to a simple negative emotional disposition toward mathematics: their answers regard students' beliefs about maths, students' beliefs about self, students' emotions, students' behaviour.

What the study mostly suggests is that the diagnosis "This student has a negative attitude toward mathematics" is not an accurate interpretation of the student's behavior, capable of steering the teacher's future action. Rather, it is a generic causal attribution of the student's failure, that the teacher perceives as global and uncontrollable and gives as the final step of a series of unsuccessfull didactical actions.

To make the 'negative attitude' construct turn into a useful instrument for both practitioners and researchers, it is necessary to clarify it from a theoretical viewpoint, while keeping in touch with the practice that motivates its use.

4. AN INVESTIGATION BASED ON STUDENTS' NARRATIVES ABOUT THEIR OWN STORIES WITH MATHEMATICS

With this aim a second study was carried out with students. We meant to get over the normative approach that characterises most research on attitude, and that we consider one of the reasons underlying both the lack of theoretical clarity and the difficulties encountered in getting significant results. Therefore we adopted an interpretive approach, aimed at studying attitude in its natural context.

The need for studying affect in its natural context is particularly stressed in the field of affect, and it is the basis for the use of non-traditional methods, such as narratives (da Ponte et al., 1998; Ruffell, Mason & Allen, 1998; Hannula, 2004).

In our study we collected and analysed students' narratives about their own story with mathematics (hence autobiographical writing, according to the classification of Connelly and Clandinin, 1990), investigating students' relationship with mathematics 'from the bottom' and trying to spot in their descriptions the dimensions involved.

In order to stimulate students' narration of their story, they were proposed the essay "Me and mathematics: my relationship with maths up to now".

In choosing autobiographical essays we are interested in what the student thinks he/she has done, the reasons underlying these actions, the type of situations he/she believed to be into and so on: it is not important whether the story told is actually «contradictory» or «likely» (Bruner, 1990).

In the end, our hypothesis is that the narrative and autobiographic data collected allow us to identify the dimensions students use to describe their relationship to mathematics and therefore may suggest a characterisation of attitude towards mathematics (in particular of negative attitude) that strictly links to practice.

We collected 1656 essays ranging from grade 1 to grade 13: 867 from primary school (grade 1-5), 369 from middle school (grade 6-8), 420 from high school (grade 9-13).

The essays were anonymous, assigned and collected in the class not by the class mathematics teacher.

As already mentioned, we adopted an interpretive approach, trying to understand how students interpret their own experiences with mathematics, rather than to explain their mathematical path in terms of cause / effect.

Final outcome of this analytical process is expected to be the construction of a set of categories, properties, relationships: what Glaser and Strauss (1967) call a *grounded theory*, i.e. a theory based on collected data, the construction of which requires a continuous back and forth between the different research phases.

In our case the essays were read in the light of both pre-existing categories (for instance liking and disliking mathematics) and in a free way, trying to identify meaningful categories a posteriori.

As regard the analysis, we refer to Lieblich et al. (1998), who, looking at different possibilities for analyzing life stories and other narrative materials, identify two main independent dimensions:

(a) Holistic versus Categorical approaches
(b) Content versus Form

Combining these dimensions results in four modes of reading a narrative:

(1) Holistic – Content mode of analysis
(2) Holistic – Form – based mode
(3) Categorical – Content mode ("content analysis")
(4) Categorical – Form mode

Each of the four modes of analysis is related to certain types of research questions. Both quality and quantity of collected data and the aims of our research led us to use all these four types in our analysis.

Here we will only present some results about:

- the 'dimensions' used by the students to describe their own relationship with mathematics
- some particularly meaningful types of stories, and precisely stories characterized by changes in the quality of the relationship with mathematics, and stories characterized by difficulties and unease.

5. THE 'DIMENSIONS' USED BY THE STUDENTS TO DESCRIBE THEIR OWN RELATIONSHIP WITH MATHEMATICS

From a repeated reading of the essays, supported by a quantitative analysis carried out through the software T-LAB (consisting of linguistic and statistical tools to analyse texts), we identified three main expressions: the most frequent is 'I like / dislike mathematics' (in the different forms: I like / I don't like / I used to like …), followed by 'I can do it / I can't do it', and then 'mathematics is...'.

Therefore we identified three core themes:

- the emotional disposition towards mathematics, concisely expressed with 'I like / dislike mathematics'
- the perception of being /not being able to succeed in mathematics, what often is called *perceived competence* (concisely expressed with: 'I can do it / I can't do it')
- the vision of mathematics, concisely expressed with 'mathematics is...'.

Sometimes an essay develops around one of these three themes. More often, it makes reference to all the themes, although it is centred on one of them, which therefore we called the 'core theme' of the essay.

The three themes are explicitly and deeply interconnected: the most frequent connection is associated with the word 'because'.

Starting from the most recurrent theme, i.e. the emotional disposition (expressed with 'I like / dislike') it is a motivation ('I like / dislike *because* …') that leads to the other two themes.

The motivation 'because' may link the *emotional disposition* to the *vision of mathematics*:

> I never liked to learn things by heart (except for some formulae) and this subject, together with Physics, gives me a chance to think and discuss. I like it, because it is a subject which needs reasoning. [3H.16[2]]

> I don't like it because there are many rules to make a tiny little operation you must divide one number by the other one, take away the number you had before and so on. Moreover, if you forget a rule you run into troubles! [1M.16]

From these essays, two different visions of mathematics emerge, that Skemp (1976) respectively calls *instrumental* and *relational*: on the one hand 'rules without reasons', leading to the need of remembering / memorizing; on the

[2] The first number refers to the class level, the letter refers to the school level (Primary / Middle / High), the last number indicates the progressive numbering of the essay within the category.

other hand 'knowing both what to do and why', thus stressing the role of reasoning.

Combining emotional disposition ('I like/I dislike maths') and instrumental/relational vision of mathematics, we have, in theory,four possible combinations: I like / relational, I don't like / instrumental (excerpts 3H.16 and 1M.16 reported above), I like / instrumental, I don't like / relational.

Interestingly enough, we did not find the combination I don't like / relational in any of the 1656 essays.

Getting back to the link between emotional disposition and the other themes pointed out by the causal conjunction 'because', we also found a strong connection between emotional disposition (expressed with 'I like') and perceived competence (expressed with 'I can / can't do it').

This connection comes out so strong from the essays, that sometimes the expressions "I like" ("I dislike") and "I can do it" ("I can't do it") are used as synonyms:

> Since primary school, I remember when the teacher asked us to number by 2, 3, 6, 9 up to 800, 900 … I used to hate it.Then I changed school and I started to hate it even more because of the expressions. Let's not talk about middle school I changed 4 teachers in the 3 school years and therefore if I didn't understand anything before, now I really understand zero. [1H.3]

One of the most interesting outcome of the reading of the essays is that 'success' in mathematics has many deeply different meanings.

In some essays 'succeeding' is identified with school success, i.e. with getting good marks, and thus it is up to the teacher to acknowledge one's success.

In some other cases, 'succeeding' is identified with 'understanding' (and therefore it is the student that acknowledges his/her own success): sometimes 'understanding' is used with an instrumental meaning and it is identified with knowing the rules and being able to apply them correctly, in other cases a relational-type 'understanding' appears, referring to one's awareness of why the rules work and how they are linked to one another.

As a consequence, the themes 'perceived competence' and 'vision of mathematics' turn out to be deeply intertwined within the beliefs the student has about success in mathematics.

6. STORIES

Although the title asked students to write their own 'story' with mathematics (also stressed by the expression 'up to now'), not all the essays are 'stories', since a story involves a sequence of events in time (a beginning state, a middle action, a final state): what is called a plot.

The development of the plot over time has been analyzed by Lieblich et al. (1998), who identified three basic formats:

- in a 'progressive narrative', the story advances steadily
- in a 'regressive narrative' there is a course of deterioration or decline
- in the 'stable narrative', the plot is steady, and the graph does not change.

These three basic formats can be combined to construct more complex plots. In our sample the most frequent plot is characterized by jumps.
Actually our stories always tell a 'change', i.e. we did not find 'stable' narratives.

The moments in which this change occurred, what Bruner (1990) calls the turning points, are described by the writer in great detail, thus giving more information about the possible causes.

We are particularly interested in those stories in which change involves an inversion in the quality of the relationship with maths, since - due to our goals - we are interested in the possibility of change from 'negative' to 'positive', and in understanding the reasons of a change from 'positive' to 'negative'.

Actually we found examples of these kinds of inversion, but most frequently we found ups and downs:

> The first time I met mathematics was in the first year of primary school, and then my hatred for mathematics started because of the times tables. (…)

And then I went to lower secondary school and there I most hated mathematics, as a matter of fact I didn't understand anything, mathematics was Arabic to me. We were not made to be together, but then, who knows why, in grade 8 there was a Boom, I was like a sort of mathematician, I was so good that equations and problems and theorems seemed to be brothers of mine, I almost appeared as a genius in mathematics. But, as it happens in dreams, good things never last for long and actually my achievement dropped in grade 9, but nothing serious: my relationship to mathematics depends on the moment. [1H.42]

As we said earlier, turning points are described in great detail, thus giving information about the factors that caused the change.

Among these factors we sometimes found specific episodes, topics / activities (mainly algebra, equations, sets ...), moves from one school to another one.

But, above all, the teacher emerges as the most important factor:

> My relationship with mathematics did not start well, because my primary school teacher only looked after the best pupils and this was not fair to me. My relationship with mathematics at lower secondary school got betterm because I had a teacher who looked after me; whereas my relationship at higher secondary school is rather good, maybe because the teacher is looking after me enough. [1H.27S]

Another interesting point related to the stories is that in most essays that tell a story of difficulty or unease, a recurrent pattern emerges. This pattern is characterized:

- by an instrumental vision of mathematics, which implies the need of memorizing many products perceived as unconnected;
- by a low perceived competence, also witnessed by causal attributions of failure (see Weiner, 1974) to causes often external and stable, but mainly perceived by the student as uncontrollable: for example the teacher, mathematics itself, some characteristics of the student himself.

7. CONCLUSIONS

The study highlights that to describe their own relationship with mathematics students use the three main following expressions, deeply interconnected in the essays mainly through the word 'because':

- I like/dislike maths
- Mathematics is…
- I can / I cannot succeed in mathematics.

This result suggests – as a first implication for research - that attitude toward mathematics may be described through the three corresponding dimensions:

- Emotional disposition
- Vision of mathematics
- Perceived competence.

In this way a multidimensional description of attitude emerges, based on emotions, beliefs about mathematics, beliefs about self.

This multidimensionality, together with the richness of the students' narratives, underlines the inadequacy of the positive / negative dichotomy, and rather suggests the opportunity of considering *profiles* of negative attitude, depending on the dimension that we can define as 'negative'.

In particular, and oversimplifying, we can reduce the complexity of each of the three dimensions to a dichotomy:

- Emotional disposition: like / dislike
- Vision of mathematics: relational / instrumental
- Perceived competence: high / low.

In this way we obtain eight different profiles, out of which, seven are 'negative' in some sense.

Interestingly, the profile 'I dislike / relational vision / high self-efficacy' did not emerge from any essay.

A second implication for research is that the essays underline the deep interaction among the three dimensions (in particular between the vision of mathematics and the emotional disposition toward maths) and the subjectivity of this interaction. More generally, the essays confirm the role of affect in learning mathematics.

This study also suggests some implications for practice.

Drawing on the study on teachers' use of the construct 'attitude' we observed that the diagnosis "This student has a negative attitude toward mathematics" is not an accurate interpretation of the student's behavior, capable of steering the teacher's future action: it rather is a generic causal attribution of the student's failure, that the teacher perceives as global and uncontrollable and gives as the final step of a series of unsuccessfull didactical actions.

The three dimensions that emerge from students' narratives about their own story with mathematics suggest that different and targeted diagnoses of negative attitude are possible, through the identification of one or more negative components. This diagnosis in turn, would suggest a teaching intervention aimed at changing those components .

Particularly interesting profiles that emerge from the study are those characterized by an instrumental vision of mathematics and by a low perceived competence.

Both these two kinds of profiles lead the student to the perception of not being capable to have *control* over mathematics, a sort of 'fatalism', that can result in giving up thinking, and therefore in a failing behaviour, such as avoiding answering, or answering randomly.

This interpretation suggests a didactical action aimed at overcoming low perceived competence and the instrumental view of mathematics: an activity centred on mathematical processes rather than on products – such as problem solving – may be a valuable strategy to either prevent or overcome both these negative profiles.

The study also emphasizes the teacher's role in the vision of mathematics constructed by pupils, in the idea of 'success' they develop, in their perceived competence.

But most of all, by highlighting teachers' role in stories of change, the study suggests that it is never too late to change one's own relationship to mathematics.

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Equity: The Case for and Against Gender

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ABSTRACT

In recent years there has been renewed interest in gender differences in education generally, and in mathematics achievement and participation in particular, not only from researchers but also from practitioners and policy makers. In this paper I provide a brief overview of historical evidence describing females' involvement in mathematics and illustrate that research on gender and mathematics education has increasingly reflected a greater diversity of inquiry methods used to examine and unpack critical factors. I examine changing perceptions over time – with boys now perceived by some as disadvantaged compared to girls, highlight insights to be gained from cross cultural perspectives, and document that our understandings of, and reactions to, gender differences in mathematics are affected by a lesser reliance on methods favored in psychology, and a greater acceptance of traditions prevalent in other disciplines. Theoretical considerations are supplemented by reference to "cases". Assessment practices, changing beliefs about the perceived advantages and disadvantages of single-sex and co-educational settings and of diverse grouping practices are among the examples explored.

Keywords

Gender, equity, social perceptions, research foci

INTRODUCTION – SOME HISTORICAL NOTES

Reviews of gender differences in mathematics learning frequently start with a discussion of the situation that prevailed in the early 1970s. Yet concern about the education of females can be traced to much earlier times. Over three centuries ago, for example, the English writer Daniel Defoe noted:

> I have often thought of it as one of the most barbarous customs in the world that we deny the advantages of learning to women … If knowledge and understanding had been useless additions to the sex, God almighty would never have given them capacity; for he made nothing needless (Defoe, 1697, pp. 283-284).

The experiences of Mary Somerville, who is often included in historical lists of successful female mathematician, provide a revealing picture of the education available to females in earlier times.

Is mathematics suitable for girls? The case of Mary Somerville

Mary Somerville was born in 1780, in Scotland. From descriptions of her early life we can glean some insight into the prevailing educational customs. A tutor was engaged to teach Mary's brothers. Appropriate books were available in the home library. Yet for Mary it was initially deemed sufficient to be taught to read. Learning to write was not considered a priority. Eventually, at the age of ten, Mary was sent to a fashionable boarding school for 12 months. From there she emerged "with a taste for reading, some notion of simple arithmetic, a smattering of grammar and French, poor hand writing and abominable spelling (Patterson, 1974, p. 270). Although she subsequently had lessons in ballroom dancing, playing the piano, horse riding, cookery, drawing and painting, Mary's year at boarding school was her only formal education. Some years later, fortuitously, she came across a problem which aroused her curiosity. In her own words:

> At the end of the magazine, I read what appeared to me to be simply an algebraic question, but on turning the page I was surprised to see strange-looking lines mixed with letters, chiefly Xs and Ys and asked 'what is that?' 'Oh', said (my) friend, 'it's a kind of arithmetic; they call it algebra; but I can tell you nothing about it.' … On going home I thought I would look if any of our books could tell me what is meant by algebra. (Tabor, 1933, p. 98)

Mary continued studying mathematics, very much against her father's wishes, but with some help from her brother's tutor. Over time, she was fortunate enough to obtain support from other sources. The early death of her first husband gave her financial independence, freedom, and an opportunity to pursue her studies. William Wallace, a professor of mathematics at Edinburgh University and the editor of The Mathematical Repository, one of several periodicals catering for popular mathematical interests, was a supportive friend and mentor. Mary's second husband accepted and encouraged her mathematical endeavors.

This brief vignette illustrates how opportunities to engage with mathematical studies can be affected by the social and economic environment – an observation still relevant today.

The "girls should study/not study mathematics" debate

The United States

The literacy and numeracy rates of males and females in the early days of Colonial America are useful for gauging differences in the educational opportunities available to the two groups. By the middle of the eighteenth century literacy rates of 80% for males and 45% for females were not uncommon. Girls were usually not taught arithmetic "because it was assumed that women had no need of it in adult life" (Cohen, 1982, p. 140). Over time, with improved schooling and levels of participation in education, this perception changed.

> In the 1820s, with the spread of the common-school system and the insertion of arithmetic into the elementary curriculum, female pupils for the first time encountered arithmetic, and educators, also for the first time, were forced to articulate the reasons why arithmetic beyond the Rule of Three was inappropriate for girls to learn. A whole corpus of books and articles asserted that it was useless or even impossible to teach girls to reason logically about mathematics…. It seems supremely ironic that at the precise moment when arithmetic was finally within the reach of the female half of the population, because it was not decently taught in local schools, the stereotype of the non-mathematical feminine mind became dogma (Cohen, 1982, p. 139).

The United Kingdom

The desirability of girls studying mathematics beyond elementary arithmetic was also questioned in the United Kingdom. The headmistress of a leading college for girls maintained

> …I do not think that the mathematical powers of women enable them generally – (their physical strength, I dare say, has a great deal to do with it) to go so far in the higher branches (of mathematics), and I think we should be straining the mind (which is the thing of all things to be most deprecated) if we were to try to force them to take up such examinations … (Evidence given by Dorothea Beale in 1868 to the Schools Inquiry (Taunton Commission, quoted in Clements, 1979, p. 317)

Yet in earlier evidence given to the Commission Beale had argued: "suppose there is a taste for mathematics (in a girl), I would like to encourage it. I do not see why we should limit it where we find a special taste, …[but] I would not insist upon it for all" (Clements, 1979, p. 316).

In their summation of the evidence presented, the Commission concluded that

> as far as higher mathematics for girls is concerned … mathematics do not appear to be much in use.… But in favourable circumstances, … girls who have an aptitude for the subject are said to make good progress, and the study of it is approved by some of the ablest mistresses" (Clements, 1979, p. 317).

Australia

Educational authorities in colonial Australia were heavily influenced by the debate in England about girls and mathematics. Examination records from the time females were first allowed to matriculate and enter university indicate "that in the 1870s and 1880s many of the girls who presented for matriculation took two, and some even took three of arithmetic, algebra, and Euclid" (Clements, 1979, p. 318). Some of these girls performed well. For others the hurdle of being taught by "persons with minimal qualifications in the subject" (Clements, 1979, p. 319) was reflected in the moderate results obtained. Significantly, the first female to win the matriculation mathematics exhibition (in 1890) attended a "Ladies' College" with sufficient financial resources to employ a highly qualified, specialist mathematics teacher.

In brief

With appropriate support, personal and institutional, females were able to cope well with the mathematics curriculum deemed suitable for males. More frequently, however, girls wishing to study mathematics had to manage with teachers whose own knowledge of mathematics was limited, and with social ambivalence, if not disapproval, about the wisdom of doing so. These obstacles inevitably influenced their performance in mathematics and reinforced the beliefs of those who argued that girls could not cope with more advanced mathematics and should not be encouraged to do so.

TOWARDS THE PRESENT

In the 1970s, gender differences in mathematics performance and participation in post compulsory mathematics courses began to attract considerable research attention. A careful reading of the literature consistently revealed a substantial overlap in the performance of males and females. When found, gender differences in performance – typically in favor of males – were small and influenced by many factors - including the students' grade level and the format, scope, content, and setting of the test. Gender differences in favor of boys were also more likely to be found when the sample consisted of high achieving students.

Over the years, means to achieve gender equity have been introduced in many countries. These have included putting in place legislation to address discriminatory practices in fields such as education and employment, media campaigns to encourage females to continue with mathematics and enter traditional male fields which rely on strong mathematical background, and welfare grants to schools to initiate special intervention programs. What have these intervention programs achieved?

CURRENT EVIDENCE: GENDER DIFFERENCES IN MATHEMATICS PERFORMANCE – DATA FROM SELECTED LARGE SCALE TESTS

International Examples

The Programme for International Student Achievement [PISA]

More than 400 000 (15-year-old) students from 57 countries participated in PISA 2006. Overall, relatively few changes in performance were found when data from successive testings were compared. "For most countries, performance in mathematics remained broadly unchanged between PISA 2003 and PISA 2006….The performance advantage of males (also) remained unchanged … at some 11 score points" (OECD, 2007, p. 320). More specifically, boys performed significantly better in mathematics than girls in 35 of the participating countries. No significant differences were found in 21 countries. Girls outperformed males in only one country, Qatar.

Trends in International Mathematics and Science Study [TIMSS]

In many countries no statistically significant gender differences in mathematics performance were found in the TIMSS 2003 testing and when such differences were found they varied by country. The United States was among those in which males performed statistically significantly better than females at both the eighth and fourth grades level; Australia and Japan among those in which males performed somewhat but not significantly better than females at both these levels; and Singapore among those in which females performed significantly better than males at both the grade four and grade eight levels[1].

Gender differences by content area (in TIMSS 2003) also showed considerable between-country variations. For students in grade eight, the most striking gender differences were found on the algebra items, with females significantly outperforming males in 22 of the participating countries. Fewer differences were found for the number, measurement, and geometry items with males outperforming females in 12, 13, and 11 countries respectively. At the grade four level males outperformed females on the measurement items in well over half the participating countries.

National Examples

National Assessment of Educational Progress [NAEP]

The NAEP program provides a nationally representative and sustained overview of the performance of America's students in grades 4, 8, and 12 in various

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[1] Subtle changes to these findings were reported in the TIMSS 2007 data which were released after the ICME 11 conference was held. Males again outperformed females in Australia and the USA. In the former the difference was statistically significant at grade 8 but not at grade 4; in the latter the difference was statistically significant a grade 4 but not at grade 8. No difference was found in the performance of males and females in grade 4 in Japan, but females performed non-significantly better at grade 8. Females again scored significantly higher than males in Singapore, at both grades 4 and 8.

subject areas, including mathematics. The tests are administered in selected American schools each year. Results are reported at various levels: overall and by specific group (e.g., by grade level, gender, race/ethnicity, region, and state). McGraw, Lubienski, and Struchens (2006) examined NAEP data from 1990 to 2003 and concluded that

> Gender gaps favoring males (1) were generally small but had not diminished across reporting years, (2) were largest in the areas of measurement, number and operations (in Grades 8 and 12) and geometry (in Grade 12), (3) tended to be concentrated at the upper end of the score distributions, and (4) were most consistent for White, high-SES students and non-existent for Black students. (p. 129)

Australian data

The Australian Mathematics Competition [AMC] and the Victorian Certificate of Education [VCE]

Leder, Forgasz, and Taylor (2006) compared the performance of grade 12 students in two large scale testings: the AMC and the VCE. The former is a highly respected voluntary national competition; the latter is a high stake State-wide examination, compulsory for students enrolled in grade 12, the final year of high school for students across Australia who wish to proceed to university as VCE results are converted into a score used for tertiary entrance. The authors concluded:

> retention rates in the final year of secondary schooling are higher for females than for males Australia-wide. Yet more grade 12 males than females engaged in formal (VCE) and informal (AMC) mathematical endeavours. At the highest levels of achievement, males outperformed females in both of the tests monitored, whether comparisons were made with or without adjustment for the differences in cohort sizes. Male dominance was more marked and more consistent for the voluntary AMC than for Mathematical Methods, the important VCE gate keeping subject. (p. 39)

In brief

Gender differences in performance, most often in favor of males, continue to be reported, particularly on selected mathematical tasks assessed through standardized or large scale testings, for students in advanced post compulsory mathematics courses, and when above average performance is considered.

The emphasis in this section of the paper on continuing gender differences must not be allowed to obscure the large overlap in the performance of males and females. As pointed out by Hyde (2005),

> It is time to consider the costs of over inflated claims of gender differences. Arguably, they cause harm in numerous realms, including women's opportunities in the workplace, couple conflict and communication, and analyses of self esteem problems among adolescents. Most important, these claims are not consistent with the scientific data. (p. 590)

BEYOND LARGE SCALE TESTING: THEORETICAL CONSIDERATIONS

Elsewhere (Leder, 2004) I have sketched the changing lenses through which gender and mathematics learning have been viewed as follows:

> Gender differences in achievement in areas such as mathematics were typically assumed to be the result of inadequate educational opportunities, social barriers, or biased instructional methods and materials…. It was generally assumed that the removal of school and curriculum barriers, and if necessary the resocialization of females, would prove to be fruitful paths for achieving gender equity. Male (white and Western) norms of performance, standards, participation levels, and approach to work were generally accepted uncritically as optimum. Females were to be encouraged and helped to *assimilate*. This notion, helping females attain achievements equal to those of males, was consistent with the tenets of *liberal feminism*…. Undoubtedly influenced by work developed in the wider research community, those working within the mathematics/science area also began to frame research questions guided by a different set of assumptions. The themes fueled by Gilligan's (1982) *In a different voice*, and the feminist critiques of the sciences and of the Western notions of knowledge proved particularly powerful. New questions began to be asked …. Rather than expect them to aim for male norms, attempts were made to use females' experiences and interests to shape curriculum content and methods of instruction. The assumptions of *liberal feminism* that discrimination and

> inequalities faced by females were the result of social practices and outdated laws were no longer deemed sufficient or necessary explanations. Instead, emphasis began to be placed on the pervasive power structures imposed by males for males. …Some researchers …wished to settle for nothing less than making fundamental changes to society. Advocates of this approach, often classed as *radical feminists,* considered that the long-term impact of traditional power relations between men and women could only be redressed through such means. (pp. 106-107)

Others have used different theoretical perspectives and nomenclature to chart the developments in research on gender and education. In the comprehensive two tomes of Gender and education (Bank, 2007) gendered theories of education are discussed under a number of headings, listed in Table 1 below.

Table 1: Gendered theories of education – selected perspectives

Academic Capitalism
"in times of financial stress or uncertainty, individuals and organizations often adopt market like strategies to strengthen or bolster their relative position in the economy (Metcalfe & Slaughter, 2007, p. 7)

Black Feminism, Womanism, and Standpoint Theories
"Black feminist perspectives stress how various forms of gender, race, and class oppression work together to form a matrix of domination. These perspectives are deeply interwoven into social structures …" (Wheeler, 2007, p. 22)

Cultural Capital Theories
"… insightfully draws attention to the power dimensions of cultural practices, dispositions, and resources in market societies …. Cultural capital theories have rarely been utilized to explain inequalities of gender or race…" (Reay, 2007, p. 23)

Feminist Reproduction Theory
"… arguably the form of educational feminism aligned most closely with Marxist and neo-Marxist feminist thought…. (Its proponents argue) that education and other social forces in the cultural field (e.g., media) play a very substantial part in reproducing … gender, race, and class divisions in the state"(Dillabough, 2007,p. 31)

Liberal and Radical Feminisms

"Liberal feminism has argued that women are as rational as men and that gender should not affect the forms that education takes… radical feminism criticized existing educational provisions as part of a patriarchal order … and argued for education for women that would enable them to resist and transform the patriarchal order"(Weedon, 2007, p. 38)

Multicultural and Global Feminisms

"… are two related modes of feminist thinking that emphasize women's differences, disagreements, and situated identities, even as they strive to identify both commonalities in women's experiences and opportunities for women to work together to achieve shared goals"(Tong, 2007, p. 47)

Postmodern and Poststructural Theories

"Poststructuralism is a branch of postmodernism that places particular emphasis on the ways in which socially and culturally produced patterns of language … construct people and the power relationships among them … (it) has also challenged feminism, particularly its tendency to categorize people by gender and its claims to being a movement that will emancipate women" (Francis, 2007, p. 55)

Queer Theory

"Informed by lesbian and gay studies, as well as feminist and poststructural theorizing, queer theory is less a systematic method or framework than a collection of approaches to questioning normative assumptions about sex, gender, and sexuality" (Talburt, 2007, p. 64)

Relational-Cultural Theory (RCT)

"In reframing relationships as the context in which we experience optimal psychological development and emotional well-being throughout our lives, RCT articulates as a means by which we can create and nourish mutually empathic growth-fostering relationships in therapy and life" (Comstock, 2007, p. 78)

Sex Role Socialization

"Sex role socialization … involves developing beliefs about gender roles, the expectations associated with each sex group, and … gender identity, an understanding of what it means to be a male or female" (Stockard, 2007, p. 79)

Social Capital Theories

"… social capital can be seen as an investment of a resource with an expectation that there will be a return on this investment. Theorists' definitions of the concept have varied" (Horvat, 2007, pp. 87-88)

Social Constructionism
"… social constructionism occupies an important position in questioning the so-called positivist research paradigm in which the world can be understood only through the ways in which it is mediated by culture and through ways in which people understand and interpret their experiences" (Gordon, 2007, p. 93)

In brief

The theoretical stances summarized above are at times overlapping, sometimes complementary, and sometimes contradictory. The different perspectives encapsulate a variety of personal values and beliefs. They are based on different assumptions which can directly or indirectly shape the research undertaken, the selection of research methods and design employed, and the conclusions ultimately drawn. Collectively they capture the ingenuity with which subtle and elusive gender differences continue to be explored.

CASES – THE INCONSISTENCY OF GENDER DIFFERENCES

Beliefs "they are a-changing"

The Fennema-Sherman [F-S] Mathematics Attitudes Scales [MAS] (Fennema & Sherman, 1976) were published in 1976 and have been widely used since then to examine gender differences in mathematics learning. An extensively modified version of one of the subscales scales, the Mathematics as a male domain subscale [MD] was administered several years ago to a sample of approximately 860 students in coeducational high schools in Victoria, Australia. The questionnaire was used to tap students' perceptions about the learning of mathematics and possible gender-linked differences in those perceptions (see Forgasz, Leder, & Kloosterman, (2004). For each of 30 statements students were asked to indicate whether they believed (1) the statement to be definitely more likely to be true for boys than girls, (2) probably more likely to be true for boys than girls, (3) there was no difference between boys and girls, (4) probably more likely to be true for girls than boys, or (5) definitely more likely to be true for girls than boys. In Table 2, the data obtained from the administration of that questionnaire were compared with findings previously reported in the relevant research literature.

Table 2. Research findings (in italics) and predictions based on previous research

ITEM	*Pred*	*Find*	*ITEM*	*Pred*	*Find*
1 Mathematics is their favourite subject	M	*F*	16 Distract others from mathematics work	M	M
2 Think it is important to understand the work	F	*F*	17 Get wrong answers in mathematics	F	M
3 Are asked more questions by the mathematics teacher	M	M	18 Find mathematics easy	M	*F*
4 Give up when they find a mathematics problem too difficult	F	M	19 Parents think it is important for them to study mathematics	M	*nd*
5 Have to work hard to do well	F	M	20 Need more help in mathematics	F	M
6 Enjoy mathematics	M	*F*	21 Tease boys if they are good at mathematics	M	M
7 Care about doing well	M/F	*F*	22 Worry if they don't do well in mathematics	M/F	*F*
8 Think they did not work hard enough if don't do well	M	*F*	23 Are not good at mathematics	F	M
9 Parents would be disappointed if they don't do well	M	*F*	24 Like using computers to solve mathematics problems	M	M
10 Need mathematics to maximise employ opportunities	M	M	25 Teachers spend more time with them	M	*nd*
11 Like challenging mathematics problems	M	*nd*	26 Consider mathematics boring	F	M
12 Are encouraged to do well by the mathematics teacher	M	*nd*	27 Find mathematics difficult	F	M
13 Mathematics teacher thinks they will do well	M	*F*	28 Get on with their work in class	F	*F*
14 Think mathematics will be important in their adult life	M	*F*	29 Think mathematics is interesting	M	*F*
15 Expect to do well in mathematics	M	*F*	30 Tease girls if they are good at mathematics	M	M

There were only eight items, it can be seen from Table 2, for which the responses were consistent with previous findings consistently reported in the research literature. These items were largely related to the learning environment and to peers. For example, boys were still believed more likely to distract others from their work (Item 16) and to like using computers to solve problems (Item 24). Girls, it was still indicated, were more likely to get on with their work in class (item 28). In the past, boys were generally believed to have more natural ability for mathematics than girls, were considered to enjoy mathematics more, and to find it more interesting than did girls. Yet the more recent data revealed that, on average, students considered boys more likely than girls to give up when they find a problem too challenging (Item 4), to find mathematics difficult (Items 27 & 18), and to need additional help (Item 20). Girls were considered more likely than boys to enjoy mathematics (Item 6) and find mathematics interesting (Item 29). Responses on so many items inconsistent with previous findings surely implies that changes have occurred over time in gendered perceptions related to mathematics education, that, in other words, the energy expended on documenting gender inequities in Australia and attempting to redress them have left their mark.

Administration of this instrument in other countries has yielded similar results, i.e., with some changes over time in perceptions of gender differences in mathematics learning[2].

Assessment practices – do they matter?

In Victoria, Australia, the final examination program at the end of secondary school contains three different grade 12 mathematics subjects. These are Further Mathematics (the least difficult option), Mathematical Methods (the most popular mathematics subject and a prerequisite for a large number of university courses) and Specialist Mathematics (the most challenging mathematics subject and a prerequisite for tertiary courses with a strong mathematics component). Some years ago, the format of the examination for these subjects was changed. Three Common Assessment Tasks, or CATs, were introduced. These were set by a central body for all three subjects. The first, CAT 1 consisted of an investigative project or challenging problem, to be solved during school time and at home. Initial solution attempts were expected to be redrafted after

[2] Relevant publications include Leder and Forgasz (2000) – Australian students; Barkatsas, Forgasz, and Leder (2001) – Greek students; Forgasz, Leder and Kaur (2001) – Singaporian students; Forgasz, Leder and Kloosterman (2004) – American students; and Brandell, Leder, and Nyström, (2007) – Swedish students.

some teacher input. CATs 2 and 3 were traditional timed examinations, to be completed under supervision. CAT 2 contained multiple-choice questions and questions requiring a short answer. Questions in CAT 3 typically required more extended written answers. All students in a given year needed to complete each of the three CATs.

Clearly, the new assessment procedures offered a unique opportunity to explore the affect on student performance of different types of assessments – for under the new examination structure, the same group of students was required to sit for three distinct examination tasks during the school year. The format of CAT1 was less traditional: time constraints were less rigid. Solutions had a strong language component as considerable explanations were required of the methods used and solution steps taken. The other two components, CATs 2 and 3, were traditional timed examination papers. The students' performance on the different test components are shown in Table 3, for Mathematical Methods, the most popular grade 12 mathematics subject.

Table 3: Mean scores (converted to percentages) obtained in Mathematical Methods, by gender, for 6 successive years

Year	*CAT 1 (%)*	*CAT 2 (%)*	*CAT 3 (%)*
	M F	M F	M F
1994	72.0 75.5	64.4 63.6	53.3 49.3
1995	64.1 67.6	56.2 54.8	36.5 32.6
1996	64.0 66.0	50.9 48.9	42.0 38.9
1997	68.0 70.4	55.8 54.3	44.5 40.7
1998	65.0 67.0	47.6 45.9	41.7 40.0
1999	69.3 72.2	55.8 55.1	38.1 36.6

Consistently, it can be seen, girls outperformed boys on the more innovative examination task CAT 1 while boys outperformed girls on CAT 2 and CAT 3, the more traditional examination formats. Clearly, the format of the examination task influenced students' performance and hence their perceived mathematical ability. Who is apparently good at mathematics can be affected by the nature of the assessment task.

Single-sex v co-educational settings

Australia has a long history of single-sex schooling. Concerns about educational outcomes for girls initially fuelled research on the advantages and disadvantages of single-sex and co-education. More recently perceived disadvantages in boys' educational outcomes have often been the driving forces behind such work. Investigations – often with inconclusive findings – typically involve comparisons of single-sex schooling; single-sex classes in co-educational settings; other single-sex models; and sex-segregation differences by subject area.

Exhibit 1

In a series of articles my colleague Helen Forgasz and I (see Leder & Forgasz, 1994; 1997a; 1997b, Forgasz & Leder, 1995) reported on an evaluation of the implementation of single-sex mathematics classes in one public co-educational high school in Australia. We were invited to evaluate the program not long after it had first been implemented at the grade 10 level, and were then invited back three years later to re-evaluate it. The single-sex classes at that time were at the grade 9 level. Data were gathered from students, teachers, and parents through questionnaires and interviews. The first evaluation did not:

> provide unequivocal evidence that single-sex mathematics classes per se address well-documented gender differences in mathematics learning outcomes. The program evaluated did not appear to have been damaging to the majority of Grade 10 students in the school investigated, and may well have benefited many. Although the school's aims for the program, and the students' and parents' beliefs were that females would benefit most from single-sex classes, there were signs that males derived equal, if not more, benefit from the program than the females (Forgasz & Leder, 1995, p. 44).

Three years later, it was found (see Leder & Forgasz, 1997) that relatively fewer males and females had enjoyed their single-sex classes, and relatively fewer females wanted the single-sex classes to continue into the next year. The teachers were also found to have adopted different teaching approaches in the boys' and the girls' single-sex classes. Over the two evaluations, parents' support for the program overall had waned. The parents of daughters were much less supportive than initially, but the mothers of sons were more supportive. They seemed to believe that their sons' education at the school was in need of special attention. Thus at the time of the second evaluation, parents' and students' perceptions had changed: boys rather than girls were deemed to be disadvantaged educationally.

Exhibit 2

After a recent survey of research comparing students' performance in single-sex and co-educational settings (for an unpublished study) Helen Forgasz and I summarized the findings with respect to mathematics education as follows:

Collectively, the findings reported from mathematics classes mirrored those drawn from the broader classroom setting. When differences were found, girls typically liked the single-sex setting and performed somewhat better academically than in coeducational classes. In a number of the studies surveyed, boys were more ambivalent than girls about the single-sex setting with some indicating a firm preference for coeducational classes. These differences, however, could often be attributed to differences in student background factors rather than the sex-segregated setting per se. In the majority of studies, the focus was on the shorter term effect of single-sex / coeducational grouping. In the few studies in which longer term effects were examined, earlier advantages attributed to the single-sex grouping appeared to dissipate and those students who originally favored single-sex groupings seemed less enthusiastic. Two explanations for the equivocal findings emerged: certain groups of students (e.g., those being harassed in a coeducational setting) benefited from a single-sex environment while for other groups it made no difference. Teacher strategies and the prevailing school climate, rather than the gender grouping in the mathematics class, seemed critical to students' success and perceptions of the class environment.

In brief

Gender differences in mathematics, if found, do not occur in a vacuum but are susceptible to societal expectations and environmental and contextual influences.

THE FUTURE

Research on gender differences in learning mathematics, per se or interactively with other factors, continues – as is evidenced, for example, by the continuing stream of papers on the topic published in peer reviewed journals, presented at national and international research conferences, and reported in the popular media. From the different theoretical perspectives summarized in Table 1 above, there are clearly many different lenses through which research can be planned, gender issues can be explored, and data gathered can be interpreted.

Contradictory findings continue to emerge. At times, females are considered to be the educationally disadvantaged group; at others it is males for whom it is considered that more support is needed.

The media undoubtedly capture and reinforce current expectations and beliefs about gender issues and have a more than negligible input into shaping future directions. A focus on some recent print media articles, then, concludes this paper.

Seemingly contradictory reports appear, as can be seen from two articles printed in the New York Times on the same day, December 4, 2007. From the one it might be concluded that females are doing well.

> This year, more than 1,600 students nationwide entered the Siemens competition [a prestigious math/science/technology award]. After several rounds of judging, 20 finalists were chosen to present their projects at N.Y.U. and to vie for scholarships ranging from $10,000 to $100,000. Eleven of the finalists were girls. It was the first year that girls outnumbered boys in the final round. Most of the finalists attend public school. … Three-quarters of the finalists have a parent who is a scientist. (New York times, December 4, 2007)

From the other, it appears that gender stereotyping is persistently robust:

> Dr. Hopkins helped start a national discussion about girls and science two years ago when she walked out of a talk by Harvard University's president, Lawrence H. Summers, after he suggested that innate differences between men and women might be one reason that fewer women than men succeed in math and science careers. Dr. Summers apologized during the ensuing furor; he announced his resignation as Harvard's president 13 months later. (New York times, December 4, 2007)

Articles such as these should not be allowed to disguise a broader problem identified in many countries: the drift away from the mathematical sciences and related careers. How best to counteract this trend is a topic of intense interest, and indeed some interventionist activity, in many countries.

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Ethnomathematics at the Margin of Europe – A Pagan Calendar

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ABSTRACT

In 930, at the close of the settlement period in Iceland, a week-based calendar was adopted. Observations of the solar cycle soon revealed errors of the calendar, which were cleverly amended. In the 12th century, the week-based calendar, called misseri calendar was adjusted to the Roman Calendar used by the Christian Church. It remained in common use for secular purposes until the 19th century, and detailed guides to it were written. Special occasions related to it are still celebrated.

Keywords

Ethnomathematics, week-based calendar, Roman calendar, Julian Calendar, Gregorian Calendar, *misseri*, finger-rhyme counting method.

INTRODUCTION

> The construction of calendars, i.e. the counting and recording of time, is an excellent example of ethnomathematics (D'Ambrosio, 2001, 12).

What is ethnomathematics? "The term was coined by Ubiratan D'Ambrosio to describe the mathematical practices of identifiable cultural groups ... in its broadest sense the "ethno" prefix can refer to any group – national societies, labor communities, religious traditions, professional classes, and so on. Mathematical practices include ... measurement in time and space ... and other cognitive and material activities which can be translated to formal mathematical representation." (ISGEm, website).

In this article, practices of an ethnic group, a subgroup of the Viking culture, and their translatation into mathematical representation will be explained. The Vikings established a society in Iceland, an island in the Mid-North Atlantic Ocean, around 900 AD. The settlers came from different parts of Norway, the British Isles and Ireland. The calendar they had in common included a seven-day week and an empirical lunar calendar (Richards, 1998, p. 204). They constructed a new system of recording time, a calendar later called *misseri* calendar. On the basis of the above quotations, the *misseri* calendar will be considered as an example of ethnomathematics:

A parliament, *Althingi*, for the inhabitants of the relatively large country, 100,000 square kilometres, was established in 930. Its meetings were held for two weeks every summer. The short summer in Iceland and its vulnerable nature demanded that the meetings take place after certain farming duties were done, and before others arose. A fairly accurate calendar was therefore needed for the gathering. The seven-day-week calendar was extended to measure the length of the year, as will be explained in this article.

The parliament agreed to accept the Christian faith in about 1000 AD. The Christian Church as an institution was established around 1100, and due to its influence the Icelanders became literate in the first quarter of the twelfth century. Once literate, the Icelanders began to write voluminously, initially to document the laws of the newly-founded Commonwealth (Kristjansson, 1980, p. 29). A thirteenth-century manuscript of the law code *Grágás* contains a concise description of a week-based calendar, created in

Iceland in the tenth century (*Grágás*, 2001; 1980–2000). The Church introduced the Roman system of Julian calendar in the twelfth century, with one extra day added to the 365 days every fourth year: the leap year.

During the Commonwealth period, Icelanders had a close connection to Norway and in 1262 they submitted to the Norwegian King. By the establishment of the Kalmar Union in 1397, Iceland followed Norway into the Danish realm, to stay there until 1944 when the Republic of Iceland was established (Thorsteinsson and Jonsson, 1991).

THE VIKING CALENDAR

The common calendar of the settlers included a seven-day week, the days being named after the Norse gods (Bjornsson, 1990, pp. 71–74; 1993, pp. 18–19, 665–660):

Sunnudagur, Sunday, the day of the sun.
Manadagur, Monday, the day of the moon.
Tysdagur, Tuesday for Tyr, the god of war.
Odinsdagur, Wednesday for Woden, the cunning god.
Thorsdagur, Thursday for Thor, the thunder god.
Frjadagur, Friday for Freyja/Frigg, the goddesses of love/marriage.
Laugardagur, Saturday, the day of bathing.

The pagan names have survived in English and other Nordic languages than Icelandic, where they were abandoned by the Icelandic Church in the twelfth century for *thridjudagur* (Third Day) for Tuesday, *miðvikudagur* (Mid-week Day) for Wednesday, *fimmtudagur* (Fifth Day) for Thursday and *föstudagur* (Fast Day) for Friday. *Sunnudagur*, *manadagur* (later *manudagur*) and *laugardagur* have remained intact to this day.

Probably some of the settlers counted the time according to the cycle of the moon, which is 29.52 days. In Iceland the nights are light from April until late August, so the moon is barely seen. Counting the lunar months in summer was therefore abandoned and counting the summer weeks was taken up instead. Moreover, difficult weather conditions may mean that the moon cannot be seen regularly in wintertime and in time winter months were standardized at 30 days each (Richards, 1998, p. 204).

When a yearly parliamentary gathering was agreed upon in AD 930, some way to count the time had to be accepted. An agreement was reached that the next meeting would take place after 52 weeks or twelve 30-day months plus four extra nights. The year was divided into two terms, *misseri*, and accordingly the calendar was called *misseri* calendar. The winter *misseri* was to last six months, the summer *misseri* six months, and the four extra nights were added at mid-summer, after the 13th week of the summer *misseri*. The parliamentary meeting was to be in the tenth week of summer (Benediktsson, 1968, pp. 9–11, 15; *Almanak fyrir Ísland* 2008).

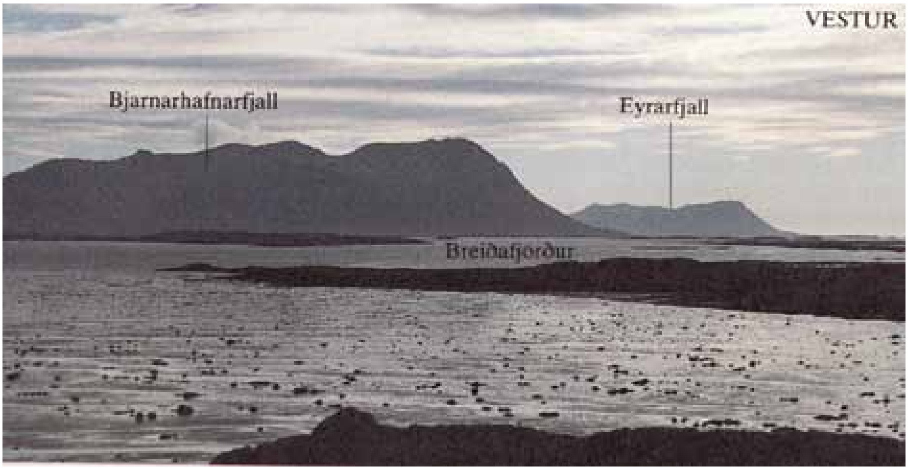
This system quickly revealed the need for a more reliable system of time-computing. By the 950s it had become clear that the summer 'moved back towards the spring', i.e. the summer according to this calendar began earlier and earlier vis-à-vis the natural summer. This was inconvenient, as the parliamentary gathering had to assemble after the completion of certain necessary farming tasks, and before others were due to begin. This is recorded in a brief history of Iceland, *Íslendingabók* (*The Book of Icelanders*, *Libellum Islandorum*), written by Ari the Learned in the period 1122–1133 that exists in manuscripts from seventeenth century (Benediktsson, 1968, pp. xvii–xvii, xliv–xlvii).

> This was when the wisest men of the country had counted in two *misseris* 364 days – that is 52 weeks, but twelve thirty-night months and four extra days – then they observed from the motion of the sun that the summer moved back towards the spring; but nobody could tell them that there is one day more in two semesters than can be measured by whole weeks, and that was the reason. But there was a man called Thorsteinn Surtur … when they came to the Althing then he sought the remedy … that every seventh summer a week should be added and try how that would work …[1] (Benediktsson, 1968, pp. 9–11).

Figure 1 below shows the view from Thorsteinn Surtur's farm where he may have studied the motion of the sun. Only at the summer solstice does the sun set on the right of Mt. Eyrarfjall (Vilhjalmsson, 1990, p. 21).

[1] All Icelandic texts have been translated by the author, KB.

Figure 1: The view from Thorsteinn Surtur's farm. Photographer: Gretar Eiriksson.



Thorsteinn Surtur thus realized the error around 955 AD by an observation of the location of the sunset, which in northern areas moves rapidly clockwise along the horizon before the summer solstice, and subsequently anti-clockwise. The extra week that Thorsteinn Surtur recommended every seventh year to be inserted at mid-summer is called *Sumarauki / Summer's Extra Week,* making the average year 365 days.

By 1000 AD parliament was meeting a week later than before, which indicates that the eleven missing leap years had also caused the start of 'summer' to move progressively earlier in the year, as explained above. In the *Book of Icelanders* it says: "Then it was spoken the previous summer by law, that men should arrive at *Althingi* when ten weeks of summer had passed, but until then it had been a week earlier." (Benediktsson, 1968, p. 15).

The reason why summer solstice may so easily be recognized in Iceland is that the track of the sun is flatter at northern latitudes than closer to the equator. Recalling that the declination of the axis of the earth is 23.5°, figure 2 is a simplified graph of the path of the sun, which shows how the sun moves rapidly along the horizon near the solstice.

The Mediterranean or 'Mid-Earth Sea' is known by that same name in Old Norse and in modern Icelandic: *Midjardarhaf*. The great city, Rome, was regarded as the middle of the earth. Rome is at 42°N. At the equinoxes the altitude of the sun there at noon is 90° – 42° = 48°. At the summer solstice the sun is 48° – 23.5° = 24.5° below the horizon at midnight and the night is completely dark. Thorsteinn Surtur lived at Thorsnes, near the modern town of Stykkishólmur, at 65°N (*Almanak fyrir Ísland*, 2008, p. 59). The altitude of the sun there at noon is 90° – 65° = 25° at the equinoxes. These computations are in agreement with the true altitude of the sun in Reykjavík at 64°N (*Almanak Háskóla Íslands*, website).

At summer solstice the sun is therefore only 25° – 23.5° = 1.5° below the horizon at its lowest position. Since the sun is so close to the horizon at that time, the night is bright enough for reading a book. The official calendar for Iceland does not record darkness in Reykjavík at 64° N from May 19 until July 23 (*Almanak fyrir Ísland*, 2008, pp. 22, 30).

As the path of the sun may be approximated fairly well by a graph of the cosine function, which becomes increasingly flatter when approximating its minimum value, one may understand the fast displacement of the sunset's position along the horizon in the period around the summer solstice (and the winter solstice as well). In figure 2 the altitude of the sun at 65°N and 42°N is approximated by

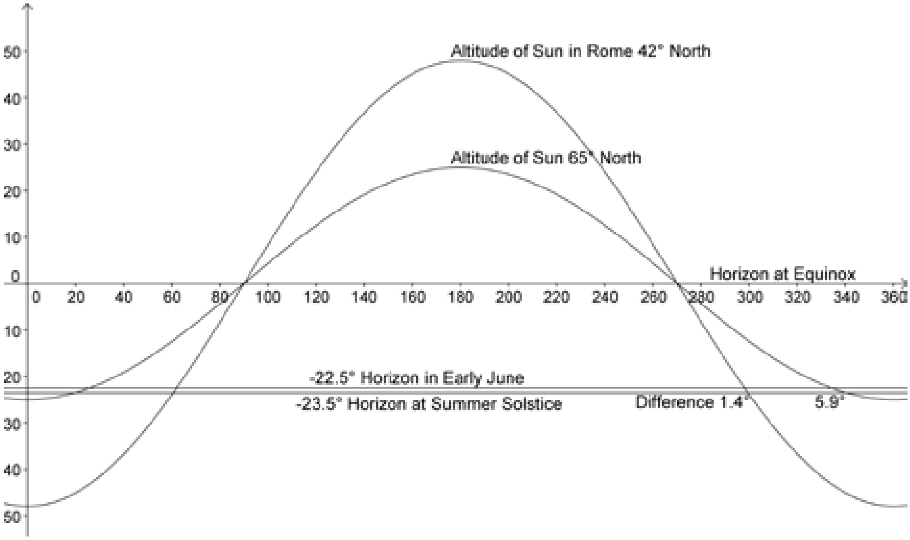
$f(x) = -(90-65)\cos(2\pi(x)/360)$ and $g(x) = -(90-42)\cos(2\pi(x)/365.22)$ respectively.

The scale on the horizontal axis, 0 – 360°, denotes the direction of the sun at the various times of the 24 hours' day, while the scale on the vertical axis denotes the altitude in degrees.

Figure 2: The altitude of the sun with respect to directions on the horizon.

The horizontal lines at -22.5 and -23.5 denote the horizon in early June and at summer solstice respectively. Their intersections to the two graphs denote the directions of sunrise and sunsets and their differences in Mid-Iceland and Rome at the indicated time of the year.

The time between the 22.5° and 23.5° lowering of the horizon is about 17 days. During that period the sunset moves about 1.4° at 42°N in Rome, while at 65°N in Thorsnes, it moves 5.9° along the horizon in the same number of days, or more than four times as far.



Figures 3 and 4 below show that this fact can also be easily realized in Reykjavík at 64°N. The pictures were taken at eight days' interval in mid-June 2008.

Figure 3: Sunset at 64°N on June 11, 2008 at 23:55
Figure 4: Sunset at 64° on June 19, 2008 at 24:04

According to the time difference of the two sunsets, 9 minutes, the sunset has moved 9/(60*24)*360 = 2¼° clockwise along the horizon in 8 days at 64°N.

ADVENT OF THE CHRISTIAN CHURCH AND ROMAN CALENDAR

After the establishment of the Christian Church as an institution in the twelfth century, the Roman Julian calendar was introduced as the calendar of the Church, with one extra day added to the 365 days every fourth year, in the leap year.

But the Julian calendar also contained errors. By adding a day to 365 days every fourth year, the average length of the year became 365.25 days, while in reality it is approximately 365.2422 days. The Julian calendar assumed the summer solstice to be on June 21, decided upon in Nicea AD 325, while in the twelfth century it fell on June 15, six days earlier, due to the addition of six too many leap-year days, which would have be skipped at years 500, 600, 700, 900, 1000 and 1100 according to the correction of the Gregorian calendar.

In the first half of the twelfth century Oddi Helgason, called Star-Oddi, a farm labourer, made observations of the annual motion of the sun, of which an account is found in the ancient treatise *Odda-tala/Oddi's Tale* (Beckman and Kålund, 1914–1916, pp. 48–53).

The treatise *Oddi's Tale* is preserved in several ancient manuscripts. In some of them it is a part of a chronological treatise, *Rím I* ('*rím*'/rhyme meaning calendar), while in the oldest manuscript, GKS 1812, 4to, written around 1192 (*A dictionary of old Norse prose*, 1989, p. 471), it is a separate treatise. *Oddi's Tale* comprises three sections, treating different aspects of the sun's motion. Firstly, Star-Oddi observed the summer solstice and the winter solstice to be a week earlier than the official date, i.e. on June 15 and December 15 instead of June 21 and December 21. Secondly, he explained the curve of the height of the sun during the year by counting the weekly increase in the first half of the year and decrease in the second half. As a measuring scale, he used the diameter of the sun, the sun rising a total of 91 diameters. The third part of *Oddi's Tale* concerns the time of dawn (Vilhjálmsson, 1991, pp. 27–34).

The Icelandic chronological treatise, *Rím II*, written in the late thirteenth century, says:

> Solstice in summer is four nights before the mass of John the Baptist ... It is so in the middle of the world. Some men say that it is close to a week earlier in Iceland (Beckman, and Kålund, 1914–1916, p. 121).

The error of the Julian calendar had thus been discovered in Iceland in the twelfth century. Better estimates of the year than that entailed by the Julian calendar had been made earlier, as listed in Table 1.

Table 1: Examples of early estimates of the length of the year (Richards, 1998: p. 33).

Researcher	*Location*	*Year*	*Length of the year*
?	Babylon	c. 700 BC	365.24579 days
Hippachus	Egypt	150 BC	365.2466 -
?	Mexico[2] (Mayan)	700 AD	365.2420 -
Da Yen	China	724 AD	365.2441 -
Al-Battani	Arabia	900 AD	365.24056 -
Al-Zarqali	Arabia	1270 AD	365.24225 -

The Icelandic week-based *misseri*-calendar was adjusted to the Julian calendar in the early twelfth century, in Oddi's time. By this adjustment the *Summer's Extra Week* was to be inserted every sixth year, or every fifth year if there were two leap years in between. The computations thus depended on the Julian calendar.

THE *MISSERI* CALENDAR

The pagan *misseri* calendar is adjusted to the Julian calendar but there is a basic difference between the calendars in determining dates. The year in the *misseri* calendar is counted in weeks. The years therefore have two different durations: 52 weeks with 364 days or 53 weeks with 371 days.

The *First Day of Summer* marks the beginning of the secular year. It was to fall on Thursday in the week April 9 to 15. In the late middle ages, April 9 was the beginning of the light-night period in Northern Iceland.

Thus the summer-*misseri* begins on a Thursday, and lasts 26 weeks and 2 days plus *Summer's Extra Week*. The winter-*misseri* begins on a Saturday in late October and is 25 weeks and 5 days. Dates are expressed in terms of days of a specified week of summer or winter. The following examples are taken from the 1920 national census of Iceland:

[2] So in the source.

Sigurður Jónsson born on Sunday in twelfth week of winter 1859.

Guðlaug Einarsdóttir born on the sixteenth Saturday of summer in 1850 (National Archives of Iceland, Statistics Iceland).

The Gregorian calendar was a reform to correct the discrepancies of the Julian calendar. By 1700, when the Gregorian calendar was adopted in the Danish Realm, eleven days were omitted, November 17–27 (Saemundsson, 1972, p. 131). The *First Day of Summer* was transferred to Thursday in the week April 19–25, and other dates, mid-summer, and beginning of winter were adjusted accordingly (Bjornsson, 1993: p. 16).

The *Misseri* calendar is basically a week-based calendar. However, there are also twelve thirty-day months and the four extra nights as quoted in the *Book of Icelanders*. The three last winter months have definite names that have remained unchanged through the centuries: *Thorri*, *Goa* and *Einmanudur*. *Thorri* and *Goa* were also names of pagan gods. *Einmanudur* means One-Month or Lone-Month. The name is believed to derive from the fact that when the month commences there is one month left until summer begins (Thorkelsson, 1928). The beginning of *Thorri* marks mid-winter and has been an occasion for mid-winter festivities.

Thorri. (masculine) begins on Friday in the 13th week of winter (in late January); this was Husbands' Day.

Goa. (feminine) begins on Sunday in the 18th week of winter (in late February); this was Wives' or Women's Day.

Einmanudur. (masculine) begins on Tuesday in the 22nd week of winter (late March); this was the Young Men's Day.

Harpa. (feminine), the first month of summer, begins on Thursday in April 19–25, First Day of Summer; this was the Young Girls' Day. (Björnsson, 1993, pp. 766–783).

The *First Day of Summer* has been a public holiday in Iceland for centuries. Youth and child-care organisations organize festivities in cooperation with local authorities. Furthermore, international Mother's and Father's Days are not much celebrated in Iceland: rather the first days of *Thorri* and *Goa* (Bjornsson, 1993, p. 31, 44–45, 766, 778, 780).

DACTYLISMUS ECCLESIASTICUS OR FINGER-RHYME

Great many calendars were preserved in manuscripts from the twelfth to the eighteenth centuries. The two first printed calendars in Icelandic were published at Hólar, one of the two episcopal sees, the latter *Calendarium: íslenzkt rím* (1597). It was a perpetual Roman calendar, but also explaining the *misseri* calendar. The information contained in the calendars was often partly built into verses and rhymes (Björnsson, 1990, pp. 68–69, 91–98).

Bishop Jon Arnason published in 1739 a detailed guide, *Dactylismus Ecclesiasticus or Finger-Rhyme (eður Fingra-Rím)*, to computing the calendar according to the new Gregorian calendar style, both by mathematical formulas and by counting on fingers. The title, *Dactylismus*, is drawn from the Latin word *dactylus* which again is drawn from the Greek word *dactylos*, meaning finger.

The *Dactylismus* was reprinted in 1838. On its front page it says that it is completely similar to the 1739 edition. A photographic facsimile of the 1838 printing was published in 1946 as a rare and appreciated book of earlier age. The facsimile has been used as a source to this article.

In his foreword, Bishop Arnason wrote:

> It is distressing to know that the art of finger-rhyme is mostly extinct in this country, which however was in my young days properly applied and used; many unlearned men and women could in a moment compute on their fingers both the dates of new moons and festivals ... (Arnason, 1739, 1838, p. 11).

The introduction of the Gregorian calendar in 1700 may have adversely affected the knowledge of the common people in this respect, but Bishop Arnason was hopeful that the lack of knowledge and skills concerning the calendar could be remedied by his work. The art had previously been practised with Latin rigmaroles so Arnason hoped that his *Dactylismus* in the vernacular would be a great support. His foreword concluded with a statement that he had composed the *Dactylismus* for the common people in the country. In this he differed from foreign authors he mentioned, who dedicated their works to the nobility, dukes and counts. The fact that there was no such class in Iceland meant that literature had to be aimed at least at the clergy and landowners and every common person, who could afford to own books.

The main bulk of the *Dactylismus Ecclesiasticus* is a guide to computing ecclesiastical moveable festivals, such as Easter, while a calendar of the 'farming-year', the *misseri* calendar, was attached as a second section. For both calendars, the Gregorian and the *misseri* calendar, so-called Sunday letters, or dominical letters, are important.

Each day of the year is assigned a letter, called *calendar letter*, A, B, C, D, E, F or G. Thus January 1 is assigned the calendar letter A, January 2 has B, January 3 has C, and so on. February 29 and March 1 have the same calendar letter. Each year is then assigned a letter, *dominical letter*, according to the calendar letter of the Sundays that year. As an example, the dominical letter of year 2010 is C as January 3 is the first Sunday, and so all Sundays in 2010 have calendar letter C. A regular 365-day year begins and ends on the same weekday, which entails that the dominical letters of succeeding years are displaced back one place (G for the next year after A). The rule is broken on leap years. As the leap-year day has the same calendar letter as the following day, the leap years need two dominical letters, one for January and February and another for the rest of the year.

Every fourth year is a leap year and the week counts 7 days. The lowest common multiple of 4 and 7 is 28, so that the sequence of dominical letters, called the Solar Cycle, was repeated every 28 years in the Julian calendar. Thus each year is assigned a number in the interval 1 to 28 in the Solar Cycle, beginning with 1 in the year 1600. Accordingly, years 1628, 1656, etc. were allocated the number 1 (Arnason, 1739, 1838, pp. 200–217).

The relation between the position of a year in the Solar Cycle and its dominical letter is found in Table 2. The years of *Summer's Extra Week* are marked by an asterisk by their dominical letter (Arnason, 1838, pp. 200–217).

Table 2: The Solar Cycle and the corresponding dominical letters. Years of *Summer's Extra Week* are marked by an asterisk.

Year of the cycle	*Dominical letter*	*Year of the cycle*	*Dominical letter*
1	B, A	15	E
2	G*	16	D
3	F	17	C, B
4	E	18	A
5	D, C	19	G*
6	B	20	F
7	A	21	E, D
8	G*	22	C
9	F, E	23	B
10	D	24	A*
11	C	25	G, F
12	B	26	E
13	A, G*	27	D
14	F	28	C

In the Gregorian calendar the leap years were skipped in years 1700, 1800 and 1900, so the Solar Cycles including these years were lengthened to 40 years. This is done by subtracting 12 from the number in the sequence at the turn of the century. For example the year 1699 is number 16 in the Solar Cycle, while year 1700 is number 5, year 1799 is number 20, but year 1800 number 9, and year 1899 is number 24 but year 1900 number 13. Year 1999 was number 28 and year 2000 number 1. The turn-of-the-century years, not divisible by 400, have only one dominical letter as they are not leap years, the latter letter of the two assigned to their number in their sequence (Arnason, 1739, 1838, pp. 200–217).

The *Summer's Extra Week* is inserted at mid-summer, beginning on Sunday after 13 weeks of summer. The years of *Summer's Extra Week* are those which begin on Monday, that is, when the dominical letter of the year is G, and those which begin on a Sunday, having dominical letter A, the year before a leap year. In that case, next year will begin with dominical letter G, only applying to the first two months of the year, and the remaining months of the year have dominical letter F.

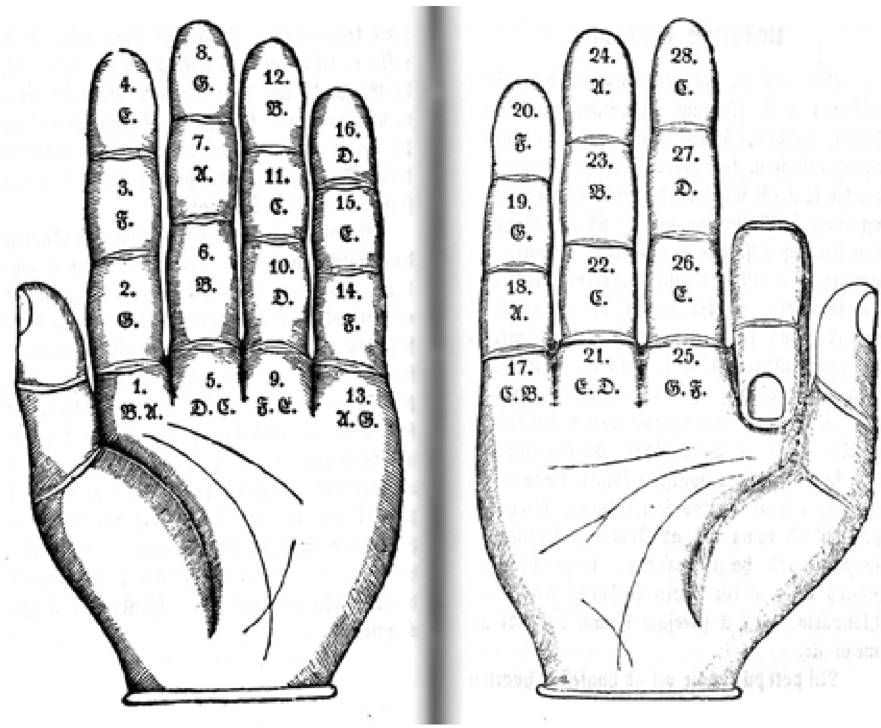
Dominical letter G brings July 22 on Sunday. This means that *Summer's Extra Week* always begins on Sunday July 22, except in the case of a leap year beginning with dominical letter G. In that case, *Summer's Extra Week* is inserted the year before, beginning on Sunday July 23 in a year with a dominical letter A.

Those *Summer's Extra Weeks* next before a leap year, beginning on a Sunday with dominical letter A, are called *Rímspillir* / Rhyme Spoilers in the *misseri* calendar. The Rhyme Spoiler moves all dates forward one day from *Summer's Extra Week* until leap-year day. This happens once in the 28-year Solar Cycle, in year 24, see table 2 above. In the Solar Cycles that contain the years 1700, 1800, 1900, 2100, and so on, when leap years are skipped, the Rhyme Spoiler year is number 36 of the cycle.

In his *Dactylismus*, Bishop Arnason explained how the dominical letters were remembered by their position on the fingers. Figure 4 shows how years number 1 to 28 in the Solar Cycle were assigned dominical letters in reverse alphabetical order.

The year 1600 was, as mentioned before, the first year in the Solar Cycle and had, as a leap year, two dominical letters, B and A. Thus for example, the year 1614 was number 15 in the Solar Cycle and had dominical letter E. The year 1623 was number 24 and had dominical letter A. It was a Rhyme-Spoiler year, as 1624 was a leap year.

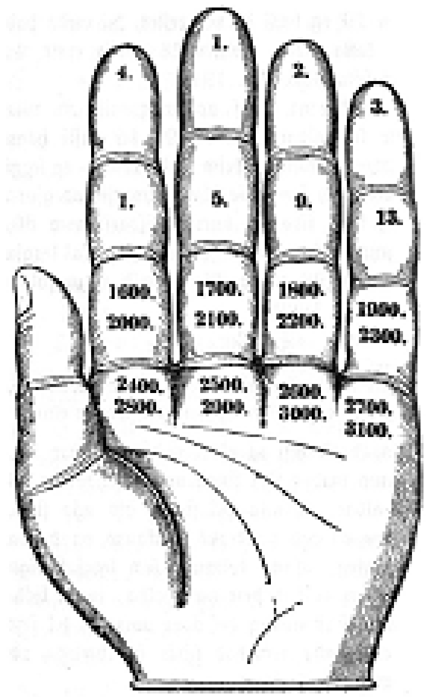
**Figure 4: The numbers in the Solar Cycle
and their dominical letters placed on palmes and fingers.**



When finding a dominical letter for a year such as 1674, one could say that it is number 75 in the Solar Cycle, but it is far too high. The remainder when 75 is divided by 28 is 19 which becomes the number in the Solar Cycle so the dominical letter of year 1674 is G, according to Table 2 and the palms in Figure 4. The last year of the century, 1699, is number 100, with remainder 16 when divided by 28, so the dominical letter is D.

Due to dropping leap years at the turns of centuries, except when the year is divisible by 400, the larger Solar Cycle is 400 years. The *Dactylismus – Finger-Rhyme* helps to find the number of each year in the Solar Cycle, explained above. To correct the cycles due to missing leap years, the years at the turn of the centuries are assigned new numbers to be remembered on the left-hand palm and mid-bones of the fingers as shown in figure 5 below.

Figure 5: A memory scheme to find the number in the Solar Cycle at turns of centuries.



Years 1600, 2000, ... # 1
Years 1700, 2100, ... # 5
Years 1800, 2200, ... # 9
Years 1900, 2300, ... # 13

As an example, the years 1700, 2100, etc., instead of being assigned # 17, they go back to being #5, marked on the mid-bone of the middle finger. The length of the Solar Cycle is thus increased by 12 years to become 40 years. The years in-between the turns of the centuries are counted onwards in 28 year cycles explained earlier. The numbers 9 and 13 similarly correct the Solar Cycles due to missing leap years at the turns of centuries 1800, 2200, etc., and 1900, 2300, etc., respectively.

The numbers 4, 1, 2, 3 at the top bones of the four fingers denote the classes of the centuries within the 400-year cycle (Arnason, 1739, 1838, pp. 102–103).

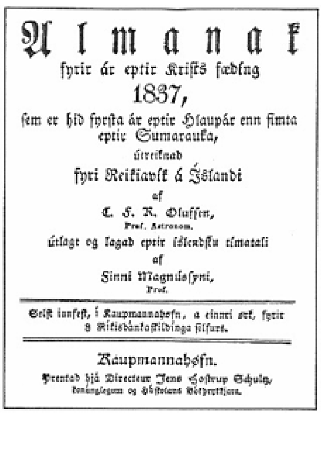
THE *MISSERI*-CALENDAR AND ALMANACS

Before Bishop Arnason's *Dactylismus*, Danish calendars were in use for a few centuries, but these did not meet the needs of Icelanders, most of whom were more familiar with the *misseri* calendar. The *Dactylismus* therefore must have been the main handbook for Icelanders during the 18th century.

Icelandic calendars of the years 1800 to 1836 exist in manuscripts, adjusted to the environment in Northern Iceland, but they were not continuous. In response to these, which were deemed to violate the University of Copenhagen's monopoly on publication of calendars, a calendar in Icelandic, the *Iceland Almanac*, was first published in 1837 by the University of Copenhagen. The calendars were computed by professors at the University of Copenhagen until 1923, and translated into Icelandic by prominent Icelandic scholars. They added the *misseri* calendar with all its features to the regular *Almanac*, which otherwise contained the ecclesiastical calendar of the Evangelical Lutheran Church in addition to local geographical information, such as time of sunrise and sunset in Reykjavík, the capital. (Sigurgeirsdóttir, 1969).

Figure 6 below shows the cover of the first issue of the *Iceland Almanac*. In translation it says:

> Almanac for year after Christ's birth 1837, which is the first year after leap year but the fifth after Summer´s Extra Week, calculated for Reykjavík on Iceland, by C. F. R. Olofsen, Prof. Astronom, translated and adjusted to the Icelandic calendar by Finnur Magnússon Prof. (Sigurgeirsdóttir, 1969).



Almanak
fyrir ár eptir Krists fæding
1837,
sem er hid fyrsta ár eptir Hlaupár enn fimta
eptir Sumarauka,
útreiknad
fyri Reikiavík á Íslandi
af
C. F. R. Olufsen,
Prof. Astronom.
útlagt og lagad eptir íslendsku tímatali
af
Finni Magnússyni,
Prof.

Selst innfest, í Kaupmannahøfn, a einni ark, fyrir 8 Rikisbankaskildinga silfurs.

Kaupmannahøfn.
Prentad hjá Directeur Jens Hostrup Schultz,
Konunglegum og Háskólans Bókprentara.

Figure 6: The cover page of the first issue of the *Iceland Almanac*.

The publication of the *Almanac* was transferred to the University of Iceland in 1917, and from 1923 the computations have been made by Icelandic mathematicians or astronomers (Sigurgeirsdóttir, 1969).

WHY DID THE *MISSERI* CALENDAR SURVIVE IN A SOCIETY OF CHRISTIAN CULTURE?

The *misseri* calendar had been in use for two centuries before the introduction of the Roman calendar used by the Church. It was maintained as a secular calendar by parliament, which gathered in summer during the period 930-1800, and the calendar was registered in the law (*Grágás*, 2001; 1980–2000). It is rooted in the medieval literary heritage that was preserved and studied in Iceland through the centuries. Bishop Arnason respected it in his 1739 *Dactylismus*, as did the nineteenth century scholars at the establishment of the *Iceland Almanac* and its later calculators did so too.

Registrations of births and deaths, carried out by the Church, were only prescribed in Iceland from 1746. For this reason, the official Roman calendar was not in common use among the general public until after the 1739 publication of Bishop Arnason's *Dactylismus*. An inspection of the official census in 1920 reveals that a number of people, born before or around the 1860s, recorded their birthdates according to the *misseri* calendar, at a certain weekday in a certain week of summer or winter.

In northern latitudes like Iceland, the difference between darkness in winter and light in summer is extreme. Celebrating mid-winter *Thorri* and *First Day of Summer* and counting the weeks and months in between is a tribute to the light, and is intimately related to life in northern nature.

Figure 7: Celebration of First Day of Summer in 2008.

Photographer: Heida Helgadóttir.

CONCLUSION – THE *MISSERI* CALENDAR AS ETHNOMATHEMATICS

Iceland was settled while the northern people in Europe were still pagan. The settlers formed a new society on their own terms. A new system had to be made from scratch. The country lies on the margin of the North Pole area with continuous light in summer time and conversely long-lasting darkness during winter-time. The first Icelanders observed a new perspective on the heavenly bodies, the sun and the moon, the universal facts on which recording the time is based. Their intellectual instruments for recording time were deeply influenced by this environment. Their farming duties during the short summer also demanded more accurate dating than they had brought from their earlier domicile. They managed to establish a cleverly made dating system, adjusting a primitive week-based calendar to observations of nature phenomena, less visible in more southerly regions. The effort to adjust the length of the calendar year to observations of nature are examples of empirical adjustments of a mathematical model.

D'Ambrosio (2001) has explained the term *ethnomathematics* in the following way:

In the same culture, individuals provide the same explanations and use the same material and intellectual instruments in their everyday activities. The set of these instruments is manifested in the manners, modes, abilities, arts, techniques – in the *tics* of dealing with the environment, of understanding and explaining facts and phenomena, of teaching and sharing all this, which is the *mathema* of the group, of the community, of the *ethno*. That is, it is their *ethnomathematics* (D'Ambrosio, 2001, p. 24)

The Icelanders developed new instruments for dealing with their environment and explaining phenomena, different *tics*, from those living closer to the 'middle of the earth', the Mediterranian area, from where the Roman system of Julian and Gregorian calendars originated, adjusted to different modes of environment and different perspectives. The new environment and instruments created the settlers' own ways of sharing agreements on time-reckoning, manifested by the law, their own *mathema*.

Later adoptions of the Roman style did not overtake the domestic system of the week-based *misseri* calendar but served to refine it, to refine the *tics* and the *mathema* of the Icelandic community, the *ethno*. The Icelandic week-based *misseri* calendar may indeed be considered as an excellent example of the ethnomathematical concept.

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Exploring and investigating in mathematics teaching and learning

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ABSTRACT

This paper assumes that investigating, exploring and solving problems are central elements of the mathematical activity. It presents examples of students investigating mathematics that illustrate important aspects of an exploratory approach to mathematics teaching and its consequences to mathematics learning. This approach depends on the nature of tasks and on the roles of teachers and students in the classroom. It requires an overall organization of content and processes in meaningful mathematics teaching units. This kind of teaching is rather demanding and teachers' professional competence in carrying it may be developed by collaborating, researching our own practice and getting involved in the professional community. The paper analyses the relationships of investigating, teaching, and learning, arguing that, as students explore and investigate mathematics, teachers profit in investigating their own practice in professional collaborative settings.

Teaching mathematics as a finished product has always been problematic. For many students, this subject is meaningless and it not worthwhile to make an effort to learn it. Others, striving to survive, develop partial meanings that often conceal deep misconceptions. For a long time, mathematicians and mathematics teachers have tried to find alternative ways of presenting mathematics to students. One of the most promising of such ways is to regard mathematics as an activity (Freudenthal, 1973) and emphasizing exploring and investigating mathematics situations.

1. INVESTIGATING AS A KEY FEATURE OF THE MATHEMATICS ACTIVITY

There are many perspectives about mathematics. Most dictionaries present this subject as the "science of number and form" (Davis & Hersh, 1980). For many mathematicians, it is the "science of proof". This is the notion that Bertrand Russell had in mind when he said: "mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true" (Kline, 1974, p. 462). Jean Dieudonné put the same idea in a shorter way: "qui dit mathématiques, dit demonstration". The structuralist movement of the first half of the twentieth century encouraged the view of mathematics as the "science of structures", and that framed the Bourbaki program and influenced a deep educational reform in the 1960s known as "modern mathematics". Still another view claims that mathematics is best described as the "science of patterns", aiming to describe, classify and explain patterns in number, data, forms, organizations, and relations (Steen, 1990).

When we think about mathematics we may focus on the mathematical concepts or on the body of knowledge encapsulated in articles and books. We form an image of a complex building or of a tree with many branches – in any case, a finished product. Alternatively, we may focus on the activity of people doing mathematics. Regarded in this way, mathematics is indeed a dynamic science. That is captured by George Pólya (1945), who says "mathematics has two faces; it is the rigorous science of Euclid, but it also something else [...] Mathematics in the making appears as an experimental, inductive science" (p. vii). That is also sustained by Irme Lakatos (1978) who states that mathematics "does not develop through monotonous growing of the number of theorems unquestionably established but through the increasing improvement of conjectures by speculation and critique, by the logic of proofs and refutations" (p. 18).

Mathematics can be an interesting and involving activity not only for the mathematician but also for the teacher and the student. Singh (1998) refers that Andrew Wiles, now famous for his proof of a long standing theorem, recalls the role of his teacher in getting him involved in mathematical explorations:

> Since I found for the first time Fermat's Last Theorem, when I was a child, this has been a major passion... I had a high school teacher who did research in mathematics and gave me a book on number theory and provided some hints on how to attack it. To begin with, I started from the hypothesis that Fermat did not know much more mathematics than me... (p. 93)

Another mathematician, Jacques Hadamard (1945) states that there is no major difference in the mathematical activity of a student and a mathematician when they are working on challenging mathematical situations:

> Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention [of a mathematician], one can say that there is only a difference in degree, a difference of level, both works being of a similar nature (p. 104).

Investigating in mathematics is finding out about some issue for which we do not know the answer. It includes the formulation of questions, often of many related questions that evolve as the work proceeds. It also involves the production, testing and refinement of conjectures about those questions. And finally, it involves proving and communicating results. In mathematics, the starting point for an investigation may be a mathematical or a non-mathematical situation from other sciences, technology, social organization, or daily life. As we try to get a better perception of the situation, we are "exploring" it. Later, when our question is clearly formulated and drives all our work, we may say that we have a "problem". Carrying out a mathematical investigation involves conscious and unconscious processes, aesthetic sensibility, and connections and analogies with mathematical and non-mathematical situations. It is undertaken in different ways by people with different cognitive styles – analytic, visual, conceptual (Burton, 2001; Davis & Hersh, 1980). But it is for all of them an involving and rewarding activity.

2. STUDENTS INVESTIGATING MATHEMATICS IN THE CLASSROOM

Let us consider some examples of students working as mathematics researchers.

Example 1. Working with numbers

The first example comes from a class led by Irene Segurado, a grade 5 teacher working with 10 year old students (see Ponte, Oliveira, Cunha & Segurado, 1998). The task is the following:

1. Write in column the 20 first multiples of 5.
2. Look at the digits of the units and tens. Do you find any patterns?
3. Now investigate what happens with the multiples of 4 and 6.
4. Investigate with other multiples.

This task was presented at the beginning of a 50-minute class. The teacher had planned for group work, but she found the students very agitated at the beginning of the class and decided to work instead with the class as a whole. She asked the students for the multiples of 5 and wrote them on the board. The students began looking for patterns:

Tatiana, raising her arm, answered quickly: *The units' digit is always 0 or 5*, and that was accepted by her colleagues, echoing around the room: *it is always 0, 5, 0, 5...*

Teacher: *What else?*
Octávio, with a happy face: *The tens digit repeats itself: 0-0, 1-1, 2-2, 3-3...*
Carlos agitated: *I discovered something else... May I explain at the blackboard? (...)*
At the blackboard, he continued: *0 with 5 is 5, 0 with 0 is 0, 1 with 5 is 6, 1 with 0 is 1, 2 with 5 is 7, 2 with 0 is 2, 3 with 5 is 8, are you getting it? There's a sequence. It's 5, it jumps one, it's 6, jumps one, it's 7... Or it's 0, jumps one, it's 1, jumps one, it's 2...* (Ponte et al., 1998, pp. 68-69)

We see that the students were able to identify different kinds of patterns. They noticed simple repetition patterns (such as 0 5 0 5 ...) and more complex patterns combining linear growth and repetition (such as 1 1 2 2 3 3 ...). They also identified linear patterns as subsequences of rather complex patterns (0 5 1 6 2 7 3 8 ...).

The class also analyzed patterns in the multiples of 4. Then, they turned to the multiples of 6 that were put in a column alongside with the multiples of 5 and 4.

0	0	0
5	4	6
10	8	12
15	12	18
20	16	24
25	20	30
30	24	36
35	28	42
40	32	48
45	36	54
50	40	60
55	44	66
60	48	72
65	52	78
70	56	84
75	60	90
80	64	96
85	68	102
90	72	108

Students' discoveries were coming in bunches. They were rather excited, thus creating some difficulties to the teacher in recording and systematizing their contributions:

The units' digit is always 0, 6, 2, 8 and 4.
The units' digit is always even.
The tens' digit does not repeat from 5 in 5.

The teacher tried to handle this enthusiasm: *Take it easy! Let us verify if what your colleague said is true. Attention! Look! Look how interesting what your colleague discovered!* Suddenly, Sónia said: *There are the same digits that for the multiples of 4.* Even before this statement made any sense to the teacher, Vânia continued: *But they are in a different order.* The teacher figured out that the students were comparing the multiples of 4 and 6, and she indicated that to the class. Other students went on:

It also begins with 0.
The other digits are in a different order.
There are multiples of 4 that are also multiples of 6.
The multiples of 6, beginning at 12, are alternately also multiples of 4.

The students expressed their generalizations in natural language. They could find again complex repetition patterns (such as 8 2 6 0 4 8 2 6 0 4 8…) and, more interesting, they were able to compare features of different patterns. In this activity they developed their number sense, they got a better grasp of the behaviour of multiples, and they did a lot of mental computation.

In her reflection, Irene Segurado indicates that the students surpassed all her expectations. She says: "I had not foreseen the hypothesis of comparing the multiples of the different numbers, because I had never put them side by side. Therefore, I experienced their discoveries with great enthusiasm" (p. 71). She also reflects on the implications of working as a whole class, as compared to small groups: "The contribution of a student was 'picked' by all his colleagues, yielding a greater number of discoveries" (p. 72). It would seem that in curriculum topics such as multiplication facts, multiples, and divisors, at the elementary school level, one can just do routine exercises. This example shows that, on the contrary, these topics allow for much exploratory and investigative work.

Example 2. How is the typical student in my class?

A second example comes from a class of Olívia Sousa, a grade 6 teacher working with students aged 11 (see Sousa, 2002). The task was organized as a statistical investigation: "Imagine you want to communicate to another student in a distant country, or, who knows, to an ET, how students in your class are?…" This was meant to have students taking all kinds of measurements about their bodies and collecting data about their families, which usually raises high levels of students' enthusiasm.

Six 90-minute blocks were used to carry out this task, with students working in small groups. The teacher divided the whole task in four main steps: (i) preparation of the investigation questions; (ii) data collection; (iii) data analysis; and (iv) reporting the results. In each step some written instructions were provided to the students. For example, the directions for step 2, were:

With your colleagues:

- Write as a question each one of the characteristics that you are going to investigate.
- What answers do you expect to obtain for your questions?
- How (through observing, measuring or a questionnaire) can you get the answers to your questions?
- Prepare data sheets to collect the data.

The statistics measures (mean, median, mode) had not been taught to this class yet. A major decision in this experiment was to have the students working with their previous knowledge of these notions, instead of teaching them formally and after propose application exercises to practice. Therefore, the students were asked to find the mode (that is, "the most frequent value"), the median (the "middle" value), and the mean (assuming that they knew about it). In fact, they had no trouble in finding the most frequent value. To find the median took more time, but when they realized that they could order the values, it became easier. There were a few problems as some students forgot to count repeated values or took the median as the average of the extremes but the class discussion was a good setting to sort these things out. And, finally, the students had already a strong intuitive notion of mean as something halfway between two values:

Inês: *Then we put 1 and 35.*
Alexandre: *1 and 40.*
Prof. *How did you get 1 and 35? (...)*
Inês and Estelle: *It was an estimation!*
Inês: *It is not as Mauro (1,20 m) nor as myself (1,50 m),. It is in the middle.*
Estelle: *It is between.*
Inês: *It is between the two.*
Estelle: *Mauro and Inês.*

To find the mean of more than two numbers, with the help of the teacher, they were able to generalize the intuitive notion of adding two numbers and dividing by two.

In her reflection, Olívia Sousa considered that carrying out this task was a significant learning experience, in which the students worked mathematics notions of two domains, statistics and numbers and computation, in an integrated way. Decimal fractions obtained from measuring quantities associated to the body, were no longer abstract entities but something with meaning. Working with these numbers – comparing, sorting, and operating – in a significant context contributed towards students' better understanding of them. She considered that, regarding statistics topics, the contact with different kinds of variables and different ways of collecting, organizing, and representing meaningful information, promoted students' understanding of the statistics language, concepts and methods that went much beyond simple memorization. This example shows that an investigation based on the students' reality can be the starting point to develop investigation competences, to learn new mathematics concepts (in this case, statistics notions), and to practice and consolidate previous mathematics knowledge.

Example 3. How to amplify?

The next example concerns an experience carried out by João Almiro (2005), a grade 8 teacher:

The Visual Education teacher wants to amplify the picture below but she put the following condition: the area of the amplified picture must be 400 times larger than this. The teacher is going to do a overhead transparency with the picture and project it in the wall. But she has a big problem: At what distance she must put the overhead projector from the wall? How can we help her? Write a report that includes the description of your investigations, the computations that you made, your conjectures and possible solutions.

(M. C. Escher, 1965)

The students had to design their own strategies. João Almiro prepared the room with four overhead projectors (each one to be used by two groups of students) and gave a metric strip and a ruler to each group. The room was a little small for the projectors but, anyway, it was possible to work. The teacher did not provide any further instructions.

The reactions from the groups were very different. Some were lost, not knowing what to do. As one student wrote in a final questionnaire: "I felt some difficulties with the overhead projectors since in the beginning we did not know where to start". Others, immediately started trying to find ways of doing the task. The teacher was pleased to notice that all the groups understood that the projected rectangle would need to have length and width 20 times larger than the initial picture, so that the area was 400 times larger. The students had solved problems involving enlargements before and were able to mobilize this previous knowledge.

The big difficulty of the students was finding the distance that they should put the overhead projector from the wall so that the length and the width amplify 20 times. All the groups constructed a rectangle with the dimensions of the picture. They projected, measured what they found, and then figured out how many times the length and width were now larger. They quickly understood that they did not have space in the room to enlarge the projected dimensions 20 times and, therefore, they had to use some strategy to know what distance the overhead projector had to be from the wall.

In one of the groups, the students understood that there was a direct proportion between the distance of the overhead projector to the wall and the number of times that the dimensions were amplified and quickly solved the problem. Four other groups, however, had much more difficulty. Helping each other, they went on measuring and arguing and when a group arrived to a conclusion, they shared it with the others. Sometimes they made conjectures that the other groups refuted and showed that were not correct. Finally, they arrived to solutions that the teacher considered acceptable. This is the final part of the solution of one of the groups that used the notion of unit rate and cross products:

Measuring the picture, they found that it was a rectangle with 11,2cm by 7,9cm. Enlarging the length 20 times yields 224cm. As they found that with the projector 1m from the wall transformed this length in a 44,5cm segment, they found the required distance using the cross product. For three other groups this was a very difficult task, and they were not able to do it, even with the help from the teacher.

Some students (about 1/5) reported a negative view of this work. One of them wrote: "I didn't like these classes (…) I think that I learn more in classes doing exercises and asking questions". However, other students were happy and recognized that they had significant learning. As one of them said:

> The problems are a bit more complicated that those from other classes, at least the overhead one, in which we had to think a lot, develop, we had to think different methods, to achieve the ideal method to get the correct result. We had to begin by finding out what was to do. In textbooks, the questions are direct, they tell us immediately what we have to do.

These responses from students show that not all of them get very excited when the teacher presents challenging tasks. It is not because of "motivation" that these tasks have an important role in mathematics teaching. It is because they may promote significant learning. This problem required the students to draw on their previous knowledge of similarity, area, and direct proportion. They also had to design a strategy to collect data to figure out the relationship of the distance of the overhead projector to the wall and the size of the image.

Example 4. Numerical equations.

This example is drawn from an algebra teaching experiment carried out by Ana Matos (2007) in her grade 8 class. This teaching unit included the study of numerical sequences, functions, and 1st degree equations. The class had a high number of students that were recent immigrants from countries such as Angola, Brazil, Cap Verde, Guinea, S. Tomé and Prince, and Romania. The unit was carried out in 12,5 classes (90 minutes each). It provided several kinds of learning experiences. The first part of the unit included exploratory and investigative tasks as a mean to foster the construction of new concepts. In the tasks about numerical sequences, the students had to explore numerical patterns with different levels of difficulty (some of which presented pictorially).

These tasks created opportunities for identifying generalizations, which could be expressed in natural language at first but should progressively be expressed using algebraic language. In this part of the unit, letters were mainly used as generalized numbers and as unknowns in simple 1st degree equations. This is the overall plan of the unit:

Classes / Tasks	*Topics*	*Objectives*	*Aspects to develop*
3,5 (Tasks 1, 2, 3)	Number sequences.	- To discover relationships among numbers; - To continue sequences of numbers: divisors; multiples; squares; cubes and powers of a number.	- Searching patterns and establishing generalizations; - Representing numerical relationships in natural language, by other means and symbols; - To construct tables of values, graphics and verbal rules that represent functional relationships; - To understand the use of functions as mathematical models of real world situations; - To particularize relationships among variables and formulae and solving simple equations; - To solve problems represented by equations and to carry out simple algebraic procedures; - To translate information from a representation to another.
3 Tasks 4, 5, 6 and textbook exercises and problems	Functions - Tables; - Graphics; - Functions defined by an algebraic expression. Direct proportion as a function $x \to kx$ $x \to kx$. - Graphics of the functions $x \to kx$ $x \to kx$ and $x \to kx + b$ $x \to kx + b$.	- Read, interpret and construct tables and graphics for functions such as $x \to kx$ $x \to kx$. $x \to kx + b$ $x \to kx + b$ or other simple ones; - Relate in intuitive way the slope of a line with the rate in a function such as $x \to kx$ $x \to kx$.	
6 Tasks 7, 8 and textbook exercises and problems	1st degree equations - Equations with denominators and parenthesis; - Literal equations	- Interpret the statement of a problem; - Translate a problem by an equation; - To search solutions of an equation; - To solve 1st degree equations with an unknown; - To solve literal equations, notably formulas used in other disciplines, for one of the unknowns.	

In the second part of the unit, the study of functions was introduced by two tasks involving relationships between variables. Although the letter is used both as a generalized number and as an unknown, here the focus was on its use as a variable and on the notion of joint variation. In the third part, tasks 7 and 8 continued the study of equations that the students begun at grade 7 and revisited in previous topics, solving new kinds of problems and equations with denominators. In this phase, letters were mostly used as unknowns and as generalized numbers. All tasks allowed the students to use different strategies exploring them on their own way. This approach stimulates students' active participation, providing them multiple entry points, adequate to their ability levels.

Working with sequences and functions became an opportunity to use the algebraic language as a tool for generalizing and sharing meanings. The study of these topics required solving simple equations, which was important to create a common understanding among students, allowing them to continue learning more complex algebraic ideas. For example, in the first general discussion, the sequence with general term $3n + 5$ was considered and the following dialogue took place:

Teacher: *So, which was the order in which 300 was placed?*
Erica: *Teacher, 3 x 100…*
Teacher: *OK, but does that give 300?*
Erica: *No, that is just with 3n.*
Teacher: *Oh, but I can't change the rule like that because we would be working with another sequence, different from this one. We just need to know which is the n that makes this expression yield 300.*
Sofia: 300 – 5? I don't know. [Students talk with each other.]
Erica: So, we make 3n = 300 – 5.

Some students did not follow Erica's suggestion, and went on thinking on their own strategies. For example, Pedro claimed with enthusiasm: "3 x 98 + 5 = 299; 3 x 99 + 5 = 302. It will not pass on 300!" This discussion continued with the contributions of Isabel, who solved the equation at the board, using her previous knowledge. The discussion provided a contrast between Erica's idea, the formal approach of Isabel and the intuitive process used by Pedro to see if 300 was a term of the sequence and supported a discussion abut the advantages of each process.

This example shows how students may be encouraged to design their own strategies and how these may be discussed and contrasted in the classroom.

Such discussion helps them to realize more connections and relationships and to become more resourceful to deal with new problems in the future.

An important feature of this teaching unit is the interconnection of sequences, functions and equations. The work with sequences leads itself to formulating generalizations and using the algebraic language to express them. In turn, this language may be used in functions and equations. And equations may be used again to solve problems concerning functions and sequences.

3. DIRECT TEACHING AND EXPLORATORY LEARNING

The examples of the previous section illustrate some key ideas about mathematics teaching and learning that I now address in more general terms.

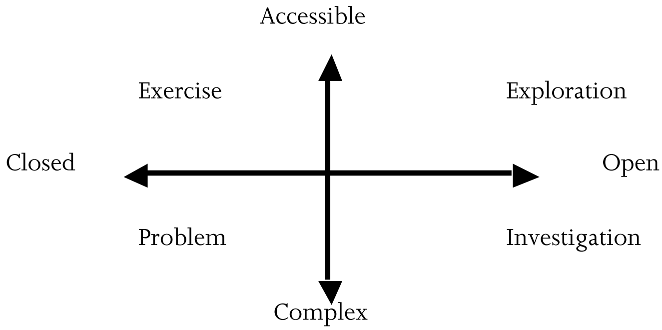
Tasks

At the core of the former situations there were investigations, explorations and problems. It is important to note how these tasks differ from usual exercises. If a student knows about equivalent fractions and use of parenthesis, an *exercise* may be the demand to simplify a fraction such as $\frac{6}{12}$ or an expression such as $\frac{3\times(10-7)}{17-2}$. That is, in an *exercise*, applying a computational procedure or doing a straightforward reasoning provides the answer. Furthermore, the question is clear as well as the given conditions. On the other hand, a *problem* may be a task such as: "What is the smallest integer number that, divided by 5, 6 and 7 all yield 3 as remainder?" A problem clearly also states what is given and what is asked, but there is no straightforward way to find the solution. And this is an example of what we may call an *investigation*:

1. Write the table for 9s, from 1 to 12. Observe the digits in the different columns. Do you notice any pattern?
2. See if you find patterns in the tables of other numbers.

Here the question is somehow open as the reader does not know what kind of "pattern" can be found. Whereas a problem states a well formulated question, in an investigation, deciding exactly what our question is, is the first thing we need to do.

We can differentiate tasks according to two main dimensions: (i) structure, ranging from closed to open, and (ii) complexity, ranging from accessible to complex as in the figure:



Explorations and investigations are both open tasks but with different complexity. Explorations are most suitable to assist the development of new concepts and representations. Investigations provide the opportunity to students to go through a real mathematical experience of formulating questions, posing and testing conjectures, and arguing and proving statements. Problems are necessary to challenge students with non-trivial mathematics questions. And exercises are important to consolidate students' knowledge of basic facts and procedures. In consequence, the teacher cannot do his/her job properly using just one kind of task – the issue is to select an appropriate mix, taking into account the students' needs and interests (Ponte, 2005).

Of course, tasks differ in other dimensions, such as the time needed to do them. For example, investigations that take a long time to complete are usually called "projects". Another dimension of tasks is pure/applied. In our examples, some tasks were framed in "real-life" contexts (Sousa; Almiro) and others in "pure mathematics" contexts (Segurado; Matos).

Classroom roles

Usually, a class in which students work on explorations or investigations has three main segments (Christiansen & Walther, 1986): (i) introduction; (ii) development of the work, and (ii) final discussion and reflection about what was done, its meaning, and new questions to study. In the introduction, the task is negotiated between teacher and students; during the development of the work the students work by themselves; and the final discussion is a key moment of sharing ideas and institutionalising new mathematical knowledge. The roles of teacher and students change during these three segments. However, at each segment, rather than a one way flow of information, centred on the authority of the teacher, we may have a classroom marked by multiple and complex interactions.

In the former examples, tasks were proposed to the students who had to discover strategies to solve them. They also had the responsibility of using logical arguments to convince the others of the correctness of their solutions. Therefore, the student had a voice, not only to ask clarification questions, but also to defend his/her claims as an intellectual authority. This is a quite different setting from the case in which students receive "explanations" from the teacher, who shows "examples" and indicates "how to do things", where the teachers and the textbook remain as the sole authorities in the classroom.

Controlling the class when the students are more agitated, as in the case of Irene Segurado, or leaving them to work with large autonomy, as João Almiro did, that is a decision that the teacher needs to take according to the particular situation. However, in all cases presented, the students are assigned a significant role in their mathematical work as a classroom community.

Classroom communication

In a standard mathematics classroom the teacher dominates the discourse, either providing explanations and examples or posing questions and providing immediate feedback. The operating IRF sequence is well known – the teacher *initiates* with a question, a student *responds* and the teacher *feedback* closes down the issue, confirming or rejecting this response. We must note, however, that not all the questions fall in this pattern. In fact, there are many kinds of questions (e.g., focus, confirmatory and inquiry questions) and appropriate questioning is one of the main resources that teachers have to lead classroom discourse (Pólya, 1945).

In our examples, the students are encouraged to share ideas with their colleagues, often working in groups or in pairs. At the end of significant work, there are discussions with all the class. These are very important moments in which there is negotiation of meanings (Bishop & Goffree, 1986). Different representations may be contrasted and the conventional representations may be analysed in detail. The proper use of mathematical language is fixed. This is also the moment when the main ideas related to the task are stressed, formalized, and institutionalized as accepted knowledge in the classroom community.

During group work, communication among students may vary a lot. Sometimes, there is a real exchange of ideas and arguments. In other cases, only one or two students conduct all the work and the others remain silent. The way the teacher interacts with the students of a group is also of great importance.

If the teacher does not respond to the students' questions, these may lose their motivation in the task. If the teacher provides all the answers, the possible benefit of the task for the students may be lost. This means that the teacher has to deal permanently with many dilemmas in conducting the classroom communication.

Teaching units

Just by itself, a very powerful task does not much. If the students are to experience some significant mathematics learning, they have to work on a field of problems for some extended period of time (at least for a couple of classes), where they have the opportunity to grasp the non-trivial aspects of the new knowledge, connect it to previous knowledge, and develop new representations and working strategies.

Teachers have to work through teaching units that, on the one hand, provide a journey that supports students' learning trajectory (Simon, 1999) on a given theme and, on the other hand, support the development of students' transversal aims for mathematics learning, including their representing, reasoning, connecting, problem solving, and communicating capacities. As Witmann (1984) indicates, designing these teaching units, according to careful criteria, is a major task for mathematics education researchers and classroom teachers.

Summing up

This analysis of different kinds of tasks, roles and communication patterns provides a characterization of two main styles of mathematics teaching that, in different grade levels, we find today in classrooms all over the world. We may call them *direct teaching* and *exploratory learning*:

Direct teaching	*Exploratory learning*
Tasks - tandard task: Exercise, - The situations are artificial, - For each problem there is a strategy and a correct answer. *Roles* - Students receive "explanations", - The teachers and the textbook are the single authorities in the classroom, - The teacher shows "examples" so that they learn "how to do things".	*Tasks* - Variety: Explorations, Investigations, Problems, Projects, Exercises, - The situations are realistic, - Often, there are several strategies to deal with a problem. *Roles* - Students receive tasks to discover strategies to solve them, - The teacher asks the student to explain and justify his/her reasoning, - The student is also an authority.

Direct teaching	*Exploratory learning*
Communication - The teacher poses questions and provides immediate feedback (sequence I-R-F). - The student poses "clarification" questions.	***Communication*** - Students are encouraged to discuss with colleagues (working in groups or pairs), - At the end of a significant work, there are discussions with all class, - Meanings are negotiated.

CHALLENGES TO TEACHERS

One must note that a class with exploration and investigation tasks is much more complex to manage than a class based in the exposition of contents and doing exercises, given the impossibility of predicting the proposals and questions that students may pose. In addition, the students do not know how to work on this kind of task and need that the teacher helps them doing such learning. Notwithstanding its difficulties and limitations, this work is essential in a mathematics class that aims educational objectives that go beyond those that are achieved by doing structured activities.

We need to ask what is necessary for a teacher to carry out such exploratory and investigative work in his/her classroom. An analysis of this activity and its contextual requirements leads us to two main areas. The first area concerns the personal relation with mathematical investigations and the second the use of investigations in professional practice.

Personal relation with mathematical investigations

1. To have a good notion about what a mathematical exploration/investigation is, how it is carried out, how results are validated (*What is it/How to do it?*)
2. To feel a minimum level of *confidence* and spontaneity in carrying out a mathematical exploration/investigation;
3. To have a *general view of mathematics* that is not restricted to definitions, procedures and rules, but that values this activity.

Use of investigations in professional practice

1. To know how to *select and adapt* exploratory and investigative *tasks* adjusted to the needs of his/her classes;
2. To know how to direct students carrying out investigative work, in the phases of *introduction, development of the work* and *final discussion*;

3. To have confidence in his/her capacity to manage the classroom *atmosphere* and the *relations* with students to carry out this work;
4. To develop a perspective about his/her role in *curriculum management*, so that mathematical exploration/investigations, in combination with other tasks, have an adequate role according to the needs of the students.

These are not competencies that teachers develop from one day to another. The teachers involved in the projects that I mentioned developed professionally for an extended period of time. As important as their projects, was the work in communicating their experiences, writing papers and presenting conferences at professional meetings. This enabled a deeper look at the experiences that become an important resource for mathematics education, showing the path that curriculum development and change of professional practice may take. The development of this competence stands on three main elements: collaborating, researching on our own practice, and getting involved with the professional community, beginning at the school level.

Collaborating

Joining together the efforts of several people is a powerful strategy to cope with the problems of professional practice. Several people working together have more ideas, more energy and more strength to overcome obstacles than an individual working alone, and they may build on the diversity of competencies. To do that, of course, they need to adjust to each other, creating an efficient system of collective work. When one of the members of the group is going through a difficult time, he/she receives the support from the others. When a member is really inspired, he/she energizes all group.

Researching professional practice

Teachers' culture has been essentially that of "knowledge transmission", bridging the gap between scholarly knowledge and students. Today, this appears as a very limited view of the professional identity. Teachers, although experts in their subject matter field, are professionals that face complex problems and need to research them. This means that they need to be able to identify problems, gather information, consider different sides of the issues, test solutions, analyse data and interpret results. They have to present their studies to the other members of the profession. This does not depend so much in learning "research methods" but, mostly, in keeping an inquiry stance (Cochran-Smith & Lytle, 1999), in knowing about defining issues and problems, and learning about theoretical notions that help in interpreting data. Investigating is a new element of the teachers' professional culture that requires an integrative view of theory and practice as two sides of a single coin since, establishing a dialogue between both is a major step towards understanding and solving problems.

Involvement with the professional community

Valuing a culture of research among teachers does not depend only on an obstinate individual agency. On the contrary, it requires a fundamental role of the collective stances where teachers carry out their professional activity, especially the schools, pedagogical movements and associative groups. In Portugal, there is an important tradition of innovative projects carried out by collaborative groups and sharing experiences in associative settings. What is still missing is reflective and transformative activity at the school level. Teachers who want to bring about change need to carry out their own projects within the schools, showing the results to other teachers, stimulating reflection, creating the need to know more, to experiment, and, hopefully to get other teachers involved in common initiatives.

CONCLUSION

Mathematical explorations and investigations can be a significant part of the mathematics curriculum. This is because of a number of reasons:

- They constitute an essential part of the mathematician's work,
- They favour the involvement of the student in work carried out in the mathematics class, indispensable for a significant learning,
- They provide multiple entry points for students at different levels of mathematical competence,
- They stimulate holistic thinking,
- They can be integrated naturally in every part of the curriculum,
- They promote complex thinking, but reinforce learning elementary concepts.

With greater or lesser emphasis, either mathematical investigations or key elements of investigating such as conjecturing, testing, and proving are recommended in the official curricula in many countries around the world (Ponte, Brocardo, & Oliveira, 2003). Investigating, teaching, and learning can be seen as an interconnected. The researcher who teaches benefits from the contact with students, as he or she listens to their questions. The teacher who investigates can use current examples and open problems, making teaching a stimulating activity. And through investigations, the student may become involved in genuine knowledge construction.

Mathematics teachers and teacher educators have interest to investigate their own professional practice, seeking to understand students' and student teachers' difficulties, the factors from the social and school contexts that influence them, and the power of teaching strategies to promote qualitative changes in students' learning. As students may explore and investigate mathematics, teachers and teacher educators may investigate students' mathematics learning and the conditions that enable it (Ponte, 2001).

In mathematics education there are at present two separate worlds. One is the world of research, as an intellectual elaboration with high rigour but with problematic practical relevance. The other is the world of practice, where problems are felt in a cogent way, but where there is often little capacity to theorize and to introduce and sustain innovative solutions. We now have an emerging reality, the world of researching practice. One may expect that it will deal with questions with strong practical relevance, with proper rigor and intellectual elaboration. Working towards such an agenda is a joint task of teachers and teacher educators.

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Generating examples: an intriguing problem-solving activity

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ABSTRACT

Generating examples of mathematical objects can be very difficult for students and it can be considered a problem solving activity. In literature, some potentialities of such activity are suggested, from different points of view and for different reasons. Our investigation aims to better identify the characteristics and the potentialities of the processes of constructing examples. The analysis, carried out by observing students' processes, reveals a high complexity of examples generation tasks. In particular, giving an example requires continuous integrations between semiotic activities on mathematical objects and argumentation, between concept image and concept definition, between cognitive and meta-cognitive resources. The study on these processes highlights the potentialities of generating examples activity as a tool for researchers in investigating many aspects of students' thinking and for teachers in promoting students' understanding and conceptualization.

Keywords
Examples generation, problem-solving, argumentation and proof

INTRODUCTION

The importance of examples in Mathematics is well recognised by mathematicians, mathematics educators and philosophers. Lakatos (1976) has considered the production of examples as one of the basic activities in the development process of this science. Mathematicians are aware of the relevant contribution of examples both in problem solving (see Polya, 1945) and in education, and they have provided to collect examples and counterexamples in Analysis (Gelbaum & Olmsted, 1964), Probability and Statistics (Romano & Siegel, 1986; Stoyanov, 1987), Topology (Steen & Seebach, 1978; Khaleelulla, 1982), Graph Theory (Capobianco & Molluzzo, 1978), and in general in Mathematics (Gelbaum & Olmsted, 1990).

In the last years, there has been an increasing interest in the examples also in mathematics education, as we can see by the high number of journal publications and sessions dedicated to this topic at the conferences. It is worth reminding, for example, the Special Issue (vol. 69, n. 2, 2008) "*The Role and Use of Examples in Mathematics Education*" of the Journal *Educational Studies in Mathematics* and the Research Forum "*Exemplification: the use of examples in teaching and learning mathematics*" at the Conference of the International Group for the Psychology in Mathematics Education in Praha in 2006 (see Bills et al., 2006).

Nowadays, we can read studies on examples in mathematics education carried out by different approaches. In this paper, I refer to examples of mathematical objects and I consider in particular the examples generation task. This is an activity with many potentialities in education (see Watson & Mason, 2005), which has been studied in different situations from cognitive and epistemological points of view, as in defining (Dahlberg & Housman, 1997), in generation of conjecture, argumentation (Boero et al., 1999; Antonini, 2003; Alcock, 2004) and proof (Balacheff, 1987; Harel & Sowder, 1998). The act of generating an example offers also to teachers and researchers a diagnostic tool "*that provides a 'window' into a learner's mind*", because the examples produced by students "*mirror their conceptions of mathematical objects involved in an example generation task*" (Zazkis & Leikin, 2007, p. 15).

One of the important approaches in studying examples production is the analysis of cognitive processes involved in it, a study that could answer to one of the research questions proposed in (Bills et al., 2006, p. 125): "*What is entailed and revealed by the process of constructing examples and how does construction of examples* *promote mathematical understanding?*" In this article, I aim to show the complexity of processes involved in examples generation, and at the same time, to present a tool to analyse these processes.

THEORETICAL FRAMEWORK AND METHODOLOGY

Giving an example is often an open problem, without an algorithm to solve it, and with a not unique solution (in general and if there exists): "*the state of generating examples can be seen as a problem solving situation, for which different people employ different strategies*" (Zaslavsky and Peled, 1996, p. 76).
In this article, according to Zaslavsky and Peled (1996), I consider the construction of examples as a problem solving activity. This point of view makes the study of strategies for producing examples and of the underlying cognitive processes meaningful. The processes are analysed with particular attention to both strategies and subjects' control over the efficacy of the strategies, according to the role of these aspects emphasized in the studies about mathematical problem solving (see, for instance, Schoenfeld, 1992).

Moreover, the analysis of processes takes into account those aspects that are specific in the construction and treatment of mathematical objects: in particular, I consider the semiotic representations of objects and the cognitive part of concepts. I respectively will refer to the notion of *semiotic register of representation* (Duval, 1995), and to the classic distinction between *concept image* and *concept definition* (Tall & Vinner, 1981), together with the notion of cognitive category, prototype and metaphors, (Rosch, 1977, Presmeg, 1992, Lakoff, 1987).

Collection of data of these studies was carried out through interviews, in which students were asked to produce mathematical objects. The subjects were students at university level (see Antonini et al., 2007; Antonini et al., 2008) and PHD students in Mathematics (see Antonini, 2006). The analysis of processes carried out by experts is interesting as a form of mathematical thinking, and in particular it is common in problem solving research for the richness, complexity and efficiency of their reasoning.

We present here only the problems that will be analysed in this article. All the tasks have an open form ("*Give an example, if possible*"), so that the students must explore the situation to solve the problem. When the example does not exist (problem 5), an argumentation or a proof of this impossibility is required. In order to stimulate experts' exploration processes, I propose them the problem 1 and 2 which are particularly difficult. These two problems, in general, were not proposed to university students. The following is the list of the problems (in brackets we put the label identifying the problem within the paper):

1. Give an example, if possible, of a real function of a real variable, non constant, periodic and not having a minimum period (*the periodic function*)
2. Give an example, if possible, of a function f:[a,b]∩Q→Q (a,b∈Q) continuous and not bounded (*the function on* Q)
3. Give an example, if possible, of a binary operation that is commutative but not associative (*the operation*, modified from a problem discussed in Zaslavsky & Peled, 1996)
4. Give an example, if possible, of an injective function f:[-1,1]→R, such that f(0)= -1 and $\lim_{x\to 1} f(x) = \lim_{x\to -1} f(x) = 2$ (*the injective function*)
5. Give an example, if possible, of a twice differentiable function f:[a,b]→ R, such that f is zero in three different points and its second derivate is positive in the domain (*the convex function*)

Some students' solutions of these problems will be presented in the following sections.

THREE PROCESSES

From the analysis of the transcripts, I identified three processes (see Antonini, 2006) that can be the basic components of more complex processes of generating examples.

1.Trial and error:

The example is sought among some recalled objects; for each example the subject only observes whether it has the requested properties or not.

Excerpt: Franco (last year of the degree in Physics, the operation example)

> "Which operations do I know? Sum, multiplication,... but they are no good.... The product of matrices!... No, no, it is associative ... and it is not commutative at all. Let's see... division is not associative. No, it is no good, it is not

> commutative. ...The exponential! No, it is not a binary operation. ...Well, if I take a^b it is binary... but it does not commutate, so... Which other operations are there? [...]"

As we can see, if an example does not satisfy the required properties, another example is considered: after any unsuccessful attempt, the process starts from the beginning. It is interesting to compare this excerpt with Sandro's solution of the same problem (see the next session). Sandro as well considers the division and, differently from Franco, when he realizes that this operation is not commutative, he does not consider another operation, but he modifies the division transforming it into an operation which is a solution of the problem.

I underline that in trial and error process the subject does not necessarily recall the objects by chance. For example, Filippa (PHD in Mathematics) considers the binary operations in set with one element, then in set with two elements, and so on, testing the required properties for every operation. Her process is carried out by trials and errors but the examples are generated with a precise and planned order.

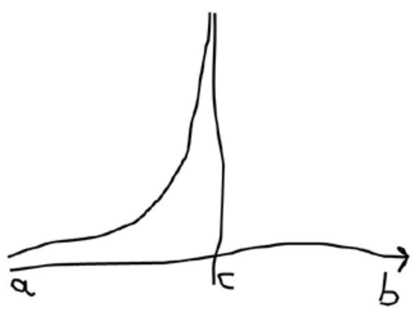
2. Transformation:

An object that fulfils part of the requested properties is modified through one or more successive transformations until it is turned into a new object with all the requested characteristics.

Excerpt: Stefano (PHD student, the function on Q example)

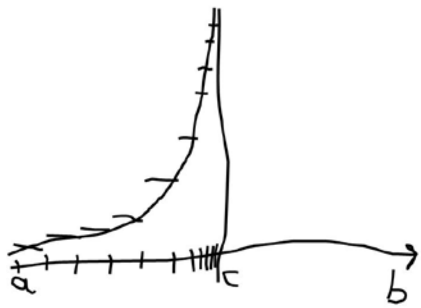
> "Now… [sketching a graph, figure 1]… where c will be an irrational. Of course this one does not have [all] values in Q. Let's make it have values in Q."

Figure 1



"I might take a sequence [in the rest of the interview it will be clear that the subject means a sequence of irrational numbers], so... [drawing, see figure 2]... and there, in each little interval, taking a sort of maximum or minimum. Well, right, any rational number between the maximum and the minimum value. Is it continuous? [...] Then on the other side [meaning in the interval between c and b], the same."

Figure 2



Stefano considers a not bounded function and then he modifies it in such a way that it assumes rational values.

In general, transformations and adjustments are physically carried out on one of the objects' representations, which works as provider of the raw material to be shaped in order to obtain the final object. In fact, Stefano really acts on the graph, drawing and transforming signs. In this sense, the transformation processes is similar to a process of construction and modification of physical objects in real situations.

We can see another solution of this problem. The process is the same, but the register of semiotic representation is different. Sandro, a PHD in Mathematics, generates his example transforming the analytical representation of the function:

"[...] example $f(x)=\frac{1}{x-\sqrt{2}}$ with $f:[0,2]\cap Q \rightarrow R$. It is continuous in any points, not bounded. Let us look for $f:[a,b]\cap Q \rightarrow Q$ with such properties.

I make it go into Q, but how? ...if I take the first three decimal digits?

...well, let us see before by integer part. [...]
$f(x)=\left[\frac{1}{x-\sqrt{2}}\right]$ [the square brackets denote the integer part]"

As in Stefano's solution, Sandro transforms the first function in such a way that it is a solution. It is not surprising that Stefano and Sandro start from the same function, a familiar object that seems to be a prototype of not bounded and continuous function. The only difference between these processes is the choice of the function representation and consequently, of the transformations that force the function to have values in rational numbers.

At this point, I think it is clear that by *transformations* I refer here to a very wide class including transformations on graphs of functions, movements of parts of geometrical figures, transformations of an algebraic formula into another (not necessarily equivalent) and so on, that is any transformation of the signs representing mathematical objects.

If the transformational process requires an intensive semiotic activity, the following process is performed by a sequence of inferences.

3. *Analysis:*

Assuming that the object is been constructed, and possibly assuming that it satisfies other properties added in order to simplify or restrict the search ground, further properties are deduced up to consequences that may evoke either a known object or a procedure to construct the requested one, that is a solution.

Excerpt: Sandro (PHD student, periodic function example)

> "It seems to me that if it is continuous it is no good ...or maybe I should make it on Q. Well, let's not complicate things... ... The examples I know are continuous enough periodic functions... and even if I adjust them I cannot get out of there … no, I must construct it from scratch. ... Example, a function that every 1/n is the same.
>
> f(1/n)=f(2/n)......
>
> Ah, so f(p/q) gets the same value! Now it will be enough to put another value for non rational numbers, for instance $f(x)=0$, if $x \in Q$ and $f(x)=1$, if $x \notin Q$."

I named this strategy *analysis* for the analogy with the equally named method used by ancient Greeks for both geometrical constructions and search for proofs:

> "in both cases, analysis apparently consists in assuming what was being sought for, in inquiring where it comes from, and in proceeding further till one reaches something already known" (Hintikka & Remes, 1974, p.1).

ANTICIPATION AND TRANSFORMATION

The empirical data show that often all the three processes are involved in generating examples, even if the transformational process seems the most common, both in experts and in students' solution.

In mathematics education we can read many articles in which processes involving a transformation are analysed. Even if these studies are carried out with different points of view and are based on different theoretical assumptions, it is often underlined that one of the most important ingredients of transformation is *anticipation* (see, for example, Simon, 1996; Harel e Sowder, 1998; Boero, 2001): to perform an efficient transformation, one has to foresee some aspect of the final shape of what is transformed.

Also in example generation processes, we can observe the role of anticipation in leading the transformations, as we can see in the following excerpt (Sandro, PHD student, operation problem):

> "[...] So, a non-associative operation is division: a*b=a/b. Well, I should take out 0, I will adjust the definition set later. Now, the problem is that it is not commutative. Can I use it anyway? ... Ah! I can make it commutative by making it symmetrical! a*b=a/b+b/a ...[...]"

Sandro deals with a non-associative and non-commutative operation. Transformation of the considered operation into a new operation is performed within the algebraic register and seems to be caused by the fact that the subject translates the commutative property in this register into symmetry between representation's symbols and non-commutative property into non-symmetry. This translation seems to allow the subject to anticipate the possibility of constructing a new operation having the commutative property, by means of a *treatment*[1] within the

[1] Duval (1995) describes two types of transformations of *semiotic representations*: *treatments* and *conversions*. The former ones are transformations of representations within one single register, the latter ones are transformations of representations consisting of a change of register without changing the denoted object.

algebraic register that aims at "*symmetrising*" the symbolic writing so that the operation may become commutative ("*I can make it commutative by making it symmetrical!*").

From the experimental data, it seems that experts choose the register of representation in such a way to perform efficient transformations foreseeing some aspect of the final form of the modified object. The lack of anticipation makes a transformation a blind attempts and the sequence of transformations could become a trial and error process. Some other examples can be found in (Antonini et al., 2008).

METACOGNITIVE PROCESSES

Metacognitive processes have the function of planning and monitoring and have a fundamental role in problem solving (Schoenfeld, 1992). The following excerpts show these processes in the particular case of examples generation.

Excerpt: Marco (PHD student, the function on Q example)

> "It is like… [he sketches a graph of a function with a vertical asymptote in x=c].
>
> […] This is of the type $\frac{1}{x-c}$ but it is not in Q. How can I map it into Q? I don't really know how I could handle this one [in Italian: "non so proprio come potrei aggeggiare"]. […] Well, the typical one like this is $\frac{1}{x-\sqrt{2}}$. But how can I map it into Q?...... Well, let's write what the problem asks …"

Marco sketches a graph and writes the analytical expression of a non bounded function. Therefore he has two representations of a starting object on which he can work and he asks himself how to do. It is interesting the use of the metaphor "to handle": I have translate in this way the unusual Italian verb "*aggeggiare*", that recalls a manual activity related to the explorative use of a device. Marco realizes that the problem is forcing the function to have rational values and he makes explicit that he does not know how to do. I underline that Marco does not state that there are no transformations but that he, in this situation, does not manage to identify transformations that could respond to his goals. This awareness leads him to change the strategy initially based on transformation and to activate the analysis process:

> "[...] Then, let's write what the problem asks… $f:[a,b]\cap Q \rightarrow Q$ continuous: that is $\forall a,b \in Q$, $f^{-1}([a,b]\cap Q)$ is open and not bounded: $\forall n\ \exists x\ |f(x)|\geq n$, well, actually, the absolute value is not so important, if I find it negative I will find also positive.
>
> Maybe it is sufficient the integer part, because I see there $|f(x)| \geq n$ then it is sufficient $f(x)=n$. Then $f(x)=\left[\frac{1}{x-\sqrt{2}}\right]$."

Here Marco studies some properties of the required function until one of the properties evokes the integer part and the solution is constructed modifying the initial function. Therefore, the analysis process, activated by a metacognitive control, has allowed to identify one efficient transformation.

Now I propose an analysis of an excerpt already considered in a previous section (Sandro, PHD student, periodic function), to highlight the cultural origin (see Morselli, 2007, p. 125) of a metacognitive process.

> Sandro: "It seems to me that if it is continuous it is no good ...or maybe I should make it on Q. Well, let's not complicate things... ... The examples I know are continuous enough periodic functions... "

Sandro conjectures that the continuous functions cannot fulfil the required properties. In fact, it is possible to prove that a periodic continuous function, is either constant or has a minimum period. Sandro is also aware that the periodic function that he knows are continuous or "continuous enough", where with this expression he probably refers to piecewise continuous functions. In any case, they are functions that make valid his conjecture on the existence of a minimum period.

> Sandro: "and even if I adjust them I cannot get out of there …"

Sandro anticipates that there are no transformations to modify these functions in such a way they become neither non continuous nor non "continuous enough". We can observe here the use of two metaphors that seem to characterize two different points of view in seeing the idea of transformation: the verb "*adjust*" which evokes an action on objects, and the expression "*I cannot get out of there*" which refers to a transformation as a process from a set into another set.

> Sandro: "no, I must construct it from scratch. ..."

Without transformations, Sandro changes his strategy and activates the analysis process. Finally, as seen above, he concludes successfully with the Dirichlet function.

Therefore, while Marco analyses his own cognitive resources, available in one situation, the Sandro's process is based on an anticipation with strong cultural roots: a conjecture on periodic functions and a consideration on the possibility to activate an efficient transformational process.

PROTOTYPES, CONCEPT IMAGE AND CONCEPT DEFINITION

The examples generation activity can be an efficient tool to observe some effects and processes that can be described as prototypes effect (Rosch, 1977, Presmeg, 1992), or by the notions of *concept image* and *concept definition* (Tall & Vinner, 1981). In a previous article (Antonini et al., 2008), we have shown as referring to a prototype and to some aspect of *concept image* can efficiently support the examples production but can also generate conflicts and make difficult to solve the task.

Here, we have already seen how prototypes play a significant role in these processes (see protocols of Stefano, Sandro and Marco). I add just a brief description of the case of Marisa (PHD in Mathematics, the periodic function problem) to show that also for an expert these aspects can be significant in failing the task.

Marisa is astonished because, for her, a periodic function is "*periodic if it repeats itself in the same way [...]... something that repeats itself...*". She concludes that if a function is periodic, then it has a minimum period, and she tries to prove it. The process is based on a concept image of periodic function that makes impossible to solve the problem. We observe how the activity of examples generation, in this case, has allowed to make observable this aspect of concept image, strong enough to darken the mathematical definition and its use also for a subject with a high mathematical culture.

CONCEPTUALIZATION AND MATHEMATICAL DEFINITION

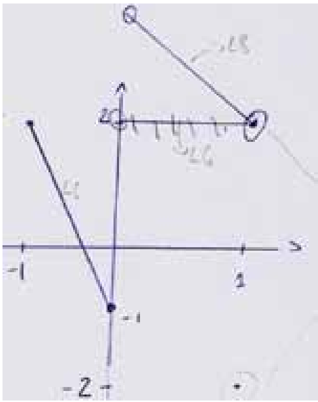
The examples generation activities reveal didactical potentialities that requires further studies. I report here a transcript in which the process of generating an example has given an important contribution to make sense of one aspect of the mathematical definition of limit (for a more detailed analysis see Antonini et al., 2007).

Letizia (forth year of the degree in Mathematics, the injective function problem), after having sketched and modified a graph, focuses on the values of the function at the end points of the interval. The problem is that, in her opinion, the limit of the function should be equal to the value it assumes.

Letizia: "I was thinking… Can I define my function in x=1, by giving any value? No, because if I define f(1)=3, then the limit for x tending to 1 of my function is 3 [see figure 3].

[…] Maybe, I want the function to be continuous in the intervals where I'm defining it, but it could even be not continuous. If I define f(1)= - 2, so that it is injective, my problem now is to see what is the value of the limit for x tending to 1 of this function. I don't know what is the value, I mean, looking at the graph I would say that the limit is –2 and not 2."

Figure 3



Interviewer: "Try to think of the definition of limit."

Letizia: "Ah, but there is a neighbourhood with a hole! I mean, I write you the definition of limit [she writes down the definition]. I must exclude the point to which the x is tending, then it is ok, the function that I drew is ok, it tends to 2 for x tending to 1. What a nice exercise! Eventually I understand why in the definition of limit it is necessary to exclude the value of the point, I understand the meaning for neighborhood with a hole!"

The suggestion of the interviewer has been essential and has the role of external metacognitive control. With the last comment ("*What a nice exercise! Eventually I understand*…") Letizia (a student who have already had some experience in Mathematics!) makes explicit that this activity has given her the possibility to refine her understanding of the meaning of the mathematical definition of limit.

EXAMPLES GENERATION, ARGUMENTATION AND PROOF

It is well known that for some students giving some examples is enough to prove a statement (see, for example Balacheff, 1987; Chazan, 1993; Harel & Sowder, 1998). On the other side, generating examples could be relevant also for experts in conjecturing, argumenting and proving (see, for example, Alcock, 2004). In a study on explorative processes, Boero et al. (1999) identify four models of production of a statement, highlighting different roles of the examples generation. In Antonini (2003), I analyse some aspects of examples that can affect the argumentative processes and the structure of argumentation.

By now, the relationships between examples and argumentation has mainly seen from the point of view of argumentation. In this article I take the opposite point of view, focusing on argumentation processes in examples generation tasks. In these activities, it is common to observe argumentation, and sometimes mathematical proof, supporting some properties that an object should have - as in the analysis process - or the impossibility of generating an object. Here, I would like to spend some words about argumentation produced to show that an object does not exist.

In general, we can observe three situations:

1. The research of examples fails, the subject is convinced that the example does not exist but the only argument is his/her failure. In this case, there are not useful arguments to construct a mathematical proof.

2. In the analysis process, a contradiction is deduced. In fact, through the analysis it is sometimes possible to deduce a property that may evoke the required object, but in other cases it might happen to deduce a contradiction.

For example, Cristiano (PHD in Mathematics, the periodic function problem) is aware of this double possibilities of the analysis and says: "I *don't know whether it exists, but I suppose it does, so either I find it or else I prove it does not exist*". The subject is not convinced that the requested object exists and believes that analysis may allow him to either find the function or prove that it does not exist.

In this case, the analysis process offers elements for constructing a proof by contradiction: there are no examples having the requested properties, in fact assuming the existence of such an example implies a contradiction. In cases like this, we can observe *cognitive unity* (in terms of Garuti et al., 1996) between exploration and proof construction processes, and *structural continuity* (in terms of Pedemonte, 2007) between argumentation and proof. On the other side, if the student plans to product a direct proof, as it could happens because direct proof is closer to his/her conception of proof (see Antonini & Mariotti, 2008), many difficulties could appear because new arguments are needed to construct the proof.

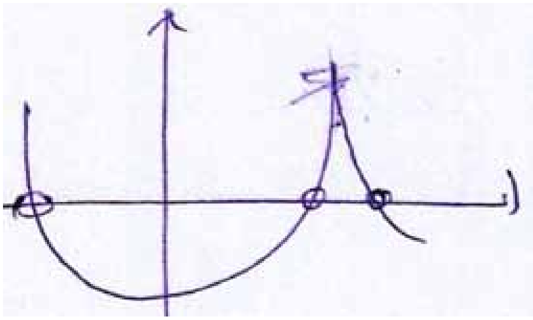
3. The transformations modify the objects in something that does not fulfil the required properties and the impossibility of generating the example is based on the reasons of the failure of the transformation process.

   In this case, the proving process could be very problematic, in particular when the subject tries to produce a proof that is close to some of the arguments related to transformations. In this case, it is the search of *cognitive unity* between argumentation and proof that causes the main obstacles. In other words, the process of generating the conjecture could interfere with the proving process, causing significant difficulties, as we can see in the following excerpt.

Federica (fifth year of the Mathematics Degree, the convex function problem) tries to construct a convex function with three zeros joining two convex functions and she realizes that the problem is in the joining point:

> "We should manage to join two functions […] in a smooth way so that the result is differentiable. [...] I give you an example [see figure 4], this function is zero in at least three points but it doesn't work because there is a point where it is not differentiable."

Figure 4

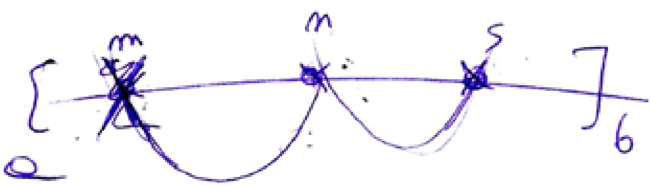


> "The problem is that, I ask myself if in order to have the derivability in points like this, I have necessarily to consider a piece of function that is concave; or, if not concave, constant, that is not good because the second derivate is zero."

I omit a part of the interview in which Federica tries to construct the function by defining the analytic expressions in two adjacent segments and in the point that separates the segments. After this work she realizes that the problem is again in joining the expressions so that the requirements are fulfilled and she produces a conjecture and an argumentation:

> "I suspect that it is absurd. Because with functions like that I wrote, when I define [the value of the function] in one point I lose the second derivate everywhere positive. However, if I define it by piecewise it is not easy to joint them [the pieces] so that it [the function] is twice differentiable. Then I ask myself if it is absurd. Let's see as this means. I write down the hypotheses. Now, if I assume that there exists a function fulfilling the hypotheses I want to arrive at an absurdity. I draw my hypotheses [see figure 5]."

Figure 5



"I ask myself what the hypotheses mean. If the function were … […] Let's see what happens in n [she is assuming that the function is composed by two convex functions joined in a point named n] […] I would like to show that the function in n either isn't continue or isn't differentiable, in order to arrive at an absurdity."

Subsequently, Federica is involved in the production of a proof that the function does not exist but she has many difficulties. The main obstacle seems to be the interference of the process of generating the conjecture in the process of the proof production, as we can see in her decision of treating the problem of the joining point also in the proof.

We can observe here a continuity (in the sense of Garuti et al., 1996 and Pedemonte, 2007) between the structure of argumentation and that of the planned proof. In fact, Federica plans to prove her conjecture by contradiction, and it seems that she does not assume only that the function exists, but, in continuity with the precedent stage, she assumes that the function is composed by two convex functions joined in the point n. In addition, she wants to look for a contradiction related to the point n, in particular she wants to prove that the function in n is not differentiable or not continue.

Only when Federica, after some suggestions of the interviewer, leaves the idea of the joining point, she manages to conclude her proof.

CONCLUSIONS

In this article, I have presented an analysis of processes involved in examples generation, showing their richness, complexity and potentialities. Constructing an example is a rich problem solving activity, efficient for didactical and diagnostic goals, for what it can reveal on conceptualization of students and with big potential from the point of view of education.

The transformation process is very common both in experts and students' protocols. Even if further investigations are needed to explore its potentialities, transformation on objects seem to have a significant role in conceptualization, as described by Piaget:

"To know an object is to act on it. To know it is to modify, to transform the object and to understand the process of this transformation and, as a consequence, to understand the way the object is constructed "(Piaget, 1964, p. 176)

One of the role of the teacher is leading students to the awareness and familiarity with transformations of mathematical objects in different registers, and promoting processes of anticipation.

The analysis process is sophisticated and not common in students' solution. On the other side, it seems a particularly significant process from cultural point of view, for the role that it assumes in scientific and, in general, speculative activities.

Further studies are necessary in different directions. It is necessary to investigate the identification of other processes, and the relationships with conceptualization, argumentation and proof. One open question is the educability of the processes, even if I think that suitable didactical activities can favour their development. Finally, a crucial research question regards the cultural and cognitive relevance of the processes of generating examples in Mathematics, in Sciences, and, in general, in reasoning.

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How Mathematics Education can help in shaping a better World?

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ABSTRACT

As educators we influence the new generations that, in two decades, will be in charge of World affairs. I address our responsibility, as Mathematics Educators, in preparing them to shape a new civilization, in which social justice and Peace with dignity for all prevail. This needs an universal ethics synthesized as 1. respect for the other/the different; 2. solidarity with the other/the different; 3. cooperation with the other/the different. History tell us that Mathematics is the dorsal spine of Modern Civilization, hence Mathematics and Mathematics Education have everything to do with the State of the World. In an era of increasing globalization in all sectors of society, the ethics of respect, solidarity and cooperation is absolutely necessary. In this talk I will discuss why and how the universal ethics of respect, solidarity and cooperation, synthesized above, is intrinsic to the Program Ethnomathematics. Through Ethnomathematics we may be effectively contributing to achieve social justice and Peace with dignity for all.

Keywords

Peace, Social Justice, Ethics, Globalization, Ethnomathematics.

THE STATE OF THE WORLD

The main issues affecting society nowadays can be synthesized in

- national security; personal security;
- government/politics;
- economics: social and environmental impact;
- relations among nations;
- relations among social classes;
- people's welfare;
- the preservation of natural and cultural resources.

Mathematics, mathematicians and mathematics educators are deeply involved with all these issues. History tells us that the technological, industrial, military, economic and political complexes have developed thanks to mathematical instruments, and that mathematics has been relying on these complexes for the material bases of its continuing progress.

It is also widely recognized that mathematics is the most universal mode of thought and that survival with dignity is the most universal problem facing mankind.

It is expected that scientists, in particular mathematicians and math educators, who have much familiarity with the most universal mode of thought, be concerned with the most universal problem, that is, survival with dignity. It is absolutely natural to expect that they, mathematicians and math educators, look into the relations between these two universals, that is, into the role of mathematicians and math educators in the pursuit of a civilization with dignity for all, in which inequity, arrogance and bigotry have no place. This means, to achieve a world in peace (D'Ambrosio 2001).

My current concerns about research and practice in math education fit into my broad interest in the human condition as understood in the history of natural evolution (from the Cosmos to the future of the human species) and to the history of ideas.

For over two decades, I have been formally involved with the Pugwash Movement and the pursuit of Peace (in all four dimensions: individual, social, environmental and military). This movement originated from the Russel-Einstein Manifesto of 1955 (Pugwash 1955). Paradoxically, the amazing progress of Western Civilization did not bring progress in the four dimensions of Peace. On the contrary, it provided more powerful material and intellectual instruments for the violations of Peace, in all these four dimensions.

My research program is to establish the responsibility of mathematicians and mathematics educators in offering venues for Peace. The Program Ethnomathematics, which will be discussed below, is a response to this.

Let me begin with a few basic questions, which guide my research program on mathematics, history, education and on the curriculum.

We need a reflection on the nature of mathematical behavior. How is mathematics created? How different is mathematical creativity from other forms of creativity?

To face these questions, there is need of a complete and structured view of the role of mathematics in building up our civilization, hence tp look into the history and geography of human behavior.

I emphasize that History is not only a chronological narrative of events, focused in the narrow geographic limits of a few civilizations which have been successful in a short span of time. The course of the history of mankind can not be separated from the natural history of the planet. History of civilization has developed in close and increasing interdependence with the natural history of the planet.

WHY TEACH MATHEMATICS?

The title of this section is the same as Session in the Third International Congress on Mathematical Education/ICME 3, held in Karlsruhe, Germany, 1976, when I was responsible for the session on "Objectives and Goals of Mathematics Education: Why Study Mathematics?". The main focus of the session was a critical perception of the objectives of education through history, in different civilizations (D'Ambrosio 1979).

We find, in every civilization and in all the times, some form of education. From initiation practices through complex education systems, the major goals are always:

- to promote creativity, helping people to fulfill their potential and rise to the highest of their capability, but being careful not to promote docile citizens. We do not want our students to become citizens who obey and accept rules and codes which violate human dignity.

- to promote citizenship, transmitting values, and showing rights and responsibilities in society, but being careful not to promote irresponsible creativity. We do not want our student to become bright scientists creating new weaponry and instruments of oppression and inequity.

The big challenge we face in education is the encounter of the old and the new. The old is present in the societal values, which were established in the past and are essential for life in a community. Since the modern state, this is intrinsic to the concept of citizenship. And the new is intrinsic to the promotion of creativity, which points to the future.

I answer the question "why teach mathematics?" simply saying that the utmost goal of Mathematics Education is to cooperate in building up a civilization in peace, which is free of inequity, arrogance and bigotry and gives the opportunity to every individual to reach the full realization of its capabilities.

The strategy of education systems to pursue goals is the curriculum. Curriculum is usually organized in three strands: objectives, contents, and methods. Every educational moment can be identified with the objective (why), contents (what) and method (how). This traditional approach must accept that three components are in solidarity, just like a point in space, This is, indeed, a cartesian model of the curriculum. This model implies accepting the social aims of education systems, then identifying contents that may help to reach the goals and developing methods to transmit those contents. Traditionally, contents are dictated by the inner structure of mathematics and give origin to methods and subordinate vague social aims to achievement of the contents.

Let us more closely look into what is going on in teaching mathematics. We immediately recognize the assumption of a form of universality, since what we observe happens in all countries at all levels. This universality is justified by arguments that I will discuss below.

The character of universality of Mathematics dominates contemporary reflections about the curriculum. Indeed, rationality is universal and mathematics is an expression of rationality. Rationality is the support for the development of strategies to deal with space and time, and ways, modes and styles of comparing, classifying and ordering, evaluating and measuring, inferring and concluding. These strategies have been developed, in very specific form, in the various natural and cultural contexts of the World. The same as religion, language, art, dressing, cuisine, medicine, they all are strategies to deal with daily life problems and situations and to explain observed facts and phenomena.

In the natural and cultural context of the Mediterranean Basin, specific strategies were developed in Sumer, in Egypt, in Israel and in Babylon and, through the dynamics of cultural encounters, were absorbed and incorporated by the Greeks to their own strategies. This gave rise to a very specific strategy, generating a concept relying on a specific concept of proof. This is the essence of what we call Greek Mathematics, which is characterized by practical achievements, for example, constructing war machines and architecture, and by theoretical aspects, that is, relying on proofs as criterion for truth. But Greek Mathematics strongly favored theory. This is clear when we analyze the works of Archytas of Tarentum (Ruffman 2007). Romans absorbed, again thanks to dynamics of cultural encounters, Greek Mathematics; Although we see significant mathematics theoretical achievements in the Roman Empire, for example Diophantus and Claudius Ptolemaeue, in Alexandria, the Romans instead privileged practice. This is clearly seen in the classical book on Roman science De Architecture, by Vitruvius, written in the 1st century BCE (Loeb 1931). With the emergence of Christianity, in the 4th century, Greek Mathematics and Philosophy were ignored. After the Crusades, 12th and 13th centuries, Greek Mathematics, which was preserved, commented and expanded by Muslim scholars, particularly by al-Kwarizmi and its Algebra, was incorporated to the Roman intellectual centers and to the quotidian. Indeed, Greek Mathematics, thanks to Arabic contributions, became a new mathematics which flourished in the European Lower Middle Ages.

A relevant feature of this new mathematics was allying numerical reasoning to the qualitative reasoning typical of theoretical Greek Mathematics. This new mathematics made its way into Education and became the central component of the curriculum throughout Europe. It was responsible for the extraordinary development of European Commerce and Economics, Science and Technology. This still prevails. Since the 17th century, this new mathematics, which is result of the dynamics of cultural encounters in the Mediterranean Basin since Antiquity became the mathematics of every European country. We clearly recognize this as the European Mathematics. Extant local mathematics can be noticed (for example, among the Euskaldunak, in Spain, the Gypsies, all over Europe, and other minorities), but they have no importance in the general European scenario.

Conquest and Colonialism, since the 15th century, imposed European education to the entire World, and with it European Mathematics. A tacit assumption of the universality of European Mathematics prevailed in the teaching of mathematics. Although it is accepted that no religion is universal, no language is universal, no cuisine or medicine are universal, European Mathematics is regarded as universal. This is clearly challenged by the eminent Japanese algebraist Yasuo Akizuki (1960, p.289), what was went unnoticed by mathematics educators, when he says that

> "Oriental philosophies and religions are of a very different kind from those of the West. I can therefore imagine that there might also exist different modes of thinking even in mathematics. Thus I think we should not limit ourselves to applying directly the methods which are currently considered in Europe and America to be the best, but should study mathematical instruction in Asia properly. Such a study might prove to be of interest and value for the West as well as for the East."

The acceptance of the universality of European Mathematics displaces all other ways of quantifying, of measuring, of ordering, of inferring. Although it is undeniable that European Mathematics is the imprint and support of the entire technological, industrial, military, economic and political behavior of the entire World, to exclude other modes of thinking, using the wording of Akizuki, may be detrimental. This was soon recognized by the pharmaceutical industry. Regrettably, the general public believes that Homo rationalis, as an evolved species of Homo sapiens sapiens, is characterized by proficiency in European Mathematics. This is intrinsic to the mounting social phenomenon of exclusion.

I have been using the concept of filters in education, particularly when referring to the prevailing evaluation and degrees system in schools and society as a whole. Important discussions on these matters are due to Alexander Grothendieck: La Nouvelle Eglise Universelle and Pierre Samuel: Mathématiques, Latin et selection des elites, in Jaulin 1974, respectively pp.11-25 and pp.147-171.

These views are supported by the results of the conference on "Comparative Studies of Mathematics Curricula – Change and Stability 1960-1980", chaired by Hans-Georg Steiner, which took place in Osnabruck, 1980, sponsored by the Institute for the Didactics of Mathematics (IDM) and the International Mathematics Committee of the Second International Mathematics Study of the International Association for the Evaluation of Educational Achievement (IEA) (Steiner, 1980).

The universality, which was the reason for calling the Osnabruck conference, can be challenged with a proper interpretation of a phrase of Ian Westbury, 1980, p.23:

> "One task of the curricular system in mathematics education is to ensure that the stock of resources for an appropriate general education contained within the culture of mathematics, as this culture is conceived and practiced within industry, higher education and learned societies, is searched and made available to our students. It also implies that one result in this search should be represented in the curricula in mathematics that appropriate numbers of our students experience."

The same challenge goes for the stability in time. In his remarkable conference in the International Congress of Mathematicians, in Paris in 1900, David Hilbert challenges the permanence of curriculum. The quote below (see Hilbert 1902, p.437) clearly states that much of current curricula should be discarded:

> "History teaches the continuity of the development of science. We know that every age has its new problems, which the following age either solves or casts aside as profitless and replaces by new ones."

This affects much of our mathematics curriculum, particularly contents. It is time to recognize that much of what we teach to our students is, in the words of Hilbert, profitless and should be replaced by new contents.

Frequently, some topics of the curriculum are justified with the argument that we have to teach subject A to be able to understand subject B, which is needed to follow subject C, and so on. This propaedeutic concept of a linear organization of the programs is one of the many myths in Mathematics Education, which are based on obsolete learning theories.

A QUALITATIVE SHIFT IN MATHEMATICS EDUCATION.

Beginning with the social critique that intensified at the end of the last century, the social dimension of mathematics education became the object of intense study. International congresses, conferences, and commissions, all affirming the universality of the discipline, have provided forums for these reflections.

In the Third International Congress on Mathematical Education/ICME 3, already mentioned above, the discussion on "Why teach mathematics?" focused on the objectives of mathematics education from a socio-cultural and political perspective. Contrary to ICME 1 (Lyon, 1968) and ICME 2 (Exeter, 1972), when there was no input from the then called Third World countries, ICME 3, in Karlsruhe, had an important presence of participants from all over the World. This created an ambience favorable to question, more profoundly, the position of mathematics in education systems. Central in the discussions was the negative effects that can result from a mathematics education that is poorly adapted to distinct socio-cultural conditions. This was a major qualitative shift proposed since ICME 3.

The qualitative shifts were discussed in two major conferences, held in 1978, sponsored by UNESCO, on "The Development of Mathematics in Third World Countries" organized by Mohamed El-Tom, in Khartoum, Sudan (El-Tom 1979); and a conference on "Mathematics and the Real World" organized by M. Niss and B. Booss at the University of Roskilde, Denmark, in 1978. This latter was held immediately preceding the International Congress of Mathematicians in Helsinki, Finland (Booss and Niss, 1978), and gave origin to a satellite session of the congress on "Mathematics and Society." I believe this was the first time an international congress of mathematicians created space to question mathematics itself, and its epistemological character. This questioning was also present in the Fourth International Congress on Mathematical Education/ICME 4, held in Berkeley in 1980 (Steen and Albers, 1981).

The Fifth International Congress on Mathematical Education/ICME 5, in Adelaide, Australia, in August 1984, showed a definitive tendency toward socio-cultural interests in mathematics education. Questions about "Mathematics and Society", "Mathematics for All", the increasing emphasis on the "History of Mathematics and its Pedagogy", and discussions of the goals of mathematics education subordinated to the general goals of education, were in the program. Surely, ICME 5 marked a qualitative shift in the tendencies of mathematics education. Besides the participation of anthropologists and sociologists in the reflections about mathematics education, a concern with the political dimensions of mathematics education and with the state of the World became part of a new concern in mathematics education.

It is impossible to ignore that the repercussions of the student movement of 1968, which was impregnated by a kind of academic cultural mystique, were felt during the 70s. Much of this mystique had a noticeable influence in the developments of research in mathematics education in the seventies and eighties. But this has not yet been properly studied.

Since the end of World War II, the major goal of education for the masses has been an equal education for all, independent of social and economic class. This should be provided by all governments. This goal dominated the political ideals and aspirations of countries. Thirty years later the illusory, and at times negative, effects of such aspirations are felt in many countries. Such disillusion also contributed to a climate of doubt, which interferes with the necessary qualitative shift.

This is an issue not only affecting less developed countries, but also in countries with advanced industrial development. Now, the increasing population of immigrants in the more developed countries, calls for priority to face socio-cultural issues and to question the universality of accepted canons of mathematics education.

THE POLITICAL AND ETHICAL DIMENSIONS OF MATHEMATICS EDUCATION

As it is generally accepted, the curriculum is organized in three strands: objectives, contents, and methods. It is the classical "Why-What-How". The political dimension of education is sometimes immersed in the discussion of objectives of mathematics education, but very rarely has mathematics content and methodology been examined with respect to this dimension. Indeed, some educators and mathematicians claim that content and methods in mathematics have nothing to do with the political dimension of education.

Since mathematics conveys the imprint of Western thought, it is not an absurd to consider a possible role of mathematics in framing a state of mind that tolerates war. This is similar to the debate about the eeffects of violent video games on aggressive behavior. For more on this, see Anderson (2001). Our responsibility, as mathematicians and mathematics educators, is to offer venues of peace (D'Ambrosio 1998). The possibility that we are conveying to our children the acceptance of the inevitability, and even normality, of a World convulsed by wars is disturbing. There is an expectation about our role, as mathematicians and mathematics educators, in the pursuit of peace. I discussed this role in a recent study commissioned by the Center for Global Nonkilling, in Honolulu (D'Ambrosio 2009).

It is undeniable that mathematics provides an important instrument for social analyses. Western civilization entirely relies on data control and management. "The world of the twenty-first century is a world awash in numbers" (Steen 2001, 1). Social critics will find it difficult to argue without understanding and analyzing data. Obviously, to make good use of these instruments, which are provided as contents, we must master them, but it is equally important to have a critical view of their potentialities and of the risk involved in misusing them. The critical view is not incorporated in contents and methods. Practically all attention is given to skill and drilling, which is supported by inadequate testing systems.

This concept of curriculum won't do anymore for our times. I propose a new concept of curriculum, based in three strands, literacy, matheracy, and technoracy, to bring the qualitative change. This is discussed in (D'Ambrosio 1999b).

To be effective in building up a civilization that rejects inequity, arrogance, and bigotry, education must give special attention to the redemption of peoples that have been for a long time subordinated and must give priority to the empowerment of the excluded sectors of societies.

The Program Ethnomathematics contributes to restoring cultural dignity and offers the intellectual tools for the exercise of citizenship which erases arrogance, inequity and bigotry in society. Ethnomathematics enhances creativity, reinforces cultural self-respect, and offers a broad view of mankind. In everyday life, it is a system of knowledge that offers the possibility of a more favorable and harmonious relation between humans and between humans and nature (D'Ambrosio 1999a).

It has, intrinsic to it, the Ethics of Diversity:

- respect for the other (the different);
- solidarity with the other;
- cooperation with the other.

A frequently asked question is: Is Ethnomathematics research or practice?

Ethnomathematics is fundamentally research in History and Philosophy of mathematics, and this is the reason for calling it the Program Ethnomathematics. But it has obvious pedagogical implications, particularly for curriculum innovation and development, for teaching and teacher education and for policy making.

The Program Ethnomathematics has, intrinsic to it, new historiographical approaches to the history of ideas. Basically, the Program Ethnomathematics goes deeper into non-Western civilizations and into comparative studies of civilizations. It is important the research on established forms of knowledge (communications, languages, religions, arts, techniques, sciences, mathematics) in different cultural environments. Indeed, the Program Ethnomathematics draws from a broad theory of knowledge, which I call the "cycle of knowledge" and from the dynamics of cultural encounters, based on what I call the "basin metaphor". All this links to the historical and epistemological dimensions of the Program Ethnomathematics, which brings new light into our understanding of how mathematical ideas are generated and how they have evolved through the history of mankind. For an explanation of this historiographical approach see (D'Ambrosio 2000).

It is fundamental to recognize the contributions of other cultures and the importance of the dynamics of cultural encounters. Culture is understood in its widest form, and includes art, history, languages, literature, medicine, music, philosophy, religion and science. Research in ethnomathematics is necessarily transcultural and transdisciplinary. The encounters are examined in its widest form, to permit exploration of more indirect interactions and influences, and to permit examination of subjects on a comparative basis. Although academic mathematics developed in the Mediterranean Basin, expanded to Northern Europe and later to other parts of the World, it is difficult to deny that the codes and techniques to express and communicate the reflections on space, time, classifying, comparing, which are proper to the human species, are contextual. Among these codes are measuring, quantifying, inferring and the emergence of abstract thinking.

Basically, research in the Program Ethnomathematics starts with three basic questions:

- How are ad hoc practices and solution of problems developed into methods?
- How are methods developed into theories?
- How are theories developed into scientific invention?

At this moment, it is important to clarify that my view of ethnomathematics should not be confused with ethnic-mathematics, as it is mistakenly understood by many. This is one of the reasons why I insist in referring to the Program Ethnomathematics. The ethnic component of Ethnomathematics is the ethno-graphic study of mathematics of a certain social group and culture and it is based on gathering empirical data on the form of mathematics practiced in the social groups and culture. Data collection is often done through participant observation, interviews and questionnaires. The Program Ethnomathematics goes beyond the ethnographical approach. It tries to explain how is this mathematics generated, socialized, organized and transmitted in these social groups and cultures. It tries to understand and explain mathematics as well as religion, art, cuisine, dressing, football and several other abstract and practical manifestations of the members of the respective social groups and the peoples of the cultures.

Of course, the Program Ethnomathematics was initially inspired by recognizing ideas and ways of doing that reminds us of Western mathematics. What we call mathematics in academia is a Western construct. Although dealing with space, time, classifying, comparing, which are proper to the human species, the codes and techniques to express and communicate the reflections on space, time, classifying, comparing, are undeniably contextual. I gained an insight into this general approach while visiting other cultural environments, during my work in Africa, in practically all the countries of continental America and the Caribbean, and in some European environments. Later, I tried to understand the situation in Asia and Oceania, although with no field work. Cultural Anthropology is a strong support for the research.

As I said above, it is important to insist that the Program Ethnomathematics is not ethnic mathematics, as some commentators interpret it. Of course, one has to work with different cultural environments and, as an ethnographer, try to describe mathematical ideas and practices of other cultures. This is a style of doing ethnomathematics, which is absolutely necessary. These cultural environments include not only indigenous populations, but labour and artisan groups, communities in urban environments and, in the periphery, farms, professional groups. These groups develop their own practices, have specific jargons and theorize on their ideas. This is an important element for the development of the Program Ethnomathematics, as important as the cycle of knowledge and the recognition of the cultural encounters.

It is important to recognize the special role of technology in the human species and the implications of this for science and mathematics. Thus, the need of History of Science and Technology (and, of course, of Mathematics) to understand the role of technology as a consequence of science, but also as an essential element for furthering scientific ideas and theories (D'Ambrosio 2004).

Once the role of technology in the development of mathematics is recognized, reflections about the future of mathematics propose important questions about the role of technology in mathematics education. Besides these more immediate concerns, there are long term concerns. Of course, they are all related to social behavior and to Ethics. It is important to recognizing that the universal ethics of respect, solidarity and cooperation is intrinsic to Ethnomathematics, Hence, Ethnomathematics favors the pursuit of Peace in its four dimensions (inner peace, social peace, environmental peace and military peace), which depends on the universal ethics.

THE FUTURE OF MATHEMATICS EDUCATION.

The increasing presence of technology in modern civilization leads, naturally, to question about the future of our species. Thus, the importance of the emergent fields of Primatology and Artificial Intelligence, Cybernetics and Human Consciousness, This is synthesized in the concept of fyborgs (which are a kind of "new" species, i.e., humans with dependence on implanted technological devices, such as an electronic pacemaker). Most of our children will be fyborgs when, around 2025, they become decision makers and take charge of all societal affairs. Educating these future fyborgs calls, necessarily, for much broader concepts of learning and teaching. The role of mathematics in the future is undeniable. But what kind of mathematics?

Understanding how, historically, societies absorb innovation, is greatly aided by looking into fiction literature (from iconography to written fiction, music and cinema). It is important to understand the way material and intellectual innovation permeates the thinking and the myths, and the ways of knowing and doing of non-initiated people. In a sense, how new ideas are diffused making abstruse theories and artifacts easier to understand to a non-specialized public. In this respect, fiction is in vantage as compared to other forms of narrative. To convey mathematical ideas through fiction, as well as metaphors, may be a good strategy for education.

How communities deal with space and time is mainly to understand how space and time became sacred in their history. The resources for the sacralization of chronology and topology are, essentially, of mathematical nature.

We have to look into the cultural dynamics of the encounter of generations (parents and teachers and youth). This encounter is dominated by mistrust and cooptation, relying on testing and evaluation practices, which dominate our civilization. In mathematics education, this is particularly disastrous. Mathematics is, usually, seen by youth as uninteresting, obsolete and useless. And they are right. Much of the traditional curriculum is uninteresting, obsolete and useless.

Standardized Testing is the main support of traditional contents. There is more concern with attaining pre-established goals of proficiency than to enhance creativity. Enormous effort and resources are aimed at rising the scores of Standardized Testing. In the opinion of Anthony Ralston (Ralston 2002)

> "rising scores on standardized tests are not only not a sign of significant learning (in mathematics and other subjects) but, as well, they hide continuing serious deficiencies in the mathematical learning of children. Still worse, they give politicians and, it must be said, some educationists something to crow about when nothing good is happening. Worst of all, they give parents a false sense that the learning of their children is improving when it is not."

The arguments to justify Standardized Testing are based on claims of the importance of current math contents are fragile. Myths surround these claims. Tests penalize creative and critical education, which leads to intimidation of the new and to the reproduction of this model of society. This favors the promotion of docile citizenship and irresponsible creativity. This is the goal of great financial corporations. This is discussed in my paper (D"Ambrosio 2009).

It is important to understand children and youth behavior and their expectations. History gives us hints on how periods of great changes affect the relations between generations. Regrettably, education, in general, is dominated by a kind of "corporate" attitude, in the sense that there is more concern with the continuity of a model of society than in giving space for the new, which needs the creativity of the youth. Traditional subjects are an instrument to achieve societal sameness. This is particularly true with Mathematics Education.

Bertrand Russell and Albert Einstein, in the most critical period of the Cold War, said "We have to learn to think in a new way." (Pugwash 1955). Paraphrasing them, we need a new thinking in Mathematics Education, bringing to our practice the interests, the dynamics and the new mathematics of the contemporary quotidian and stimulating creativity of the new generations. If we do so, there is much space for the growth of Mathematics in the curriculum. Otherwise, there lies before us the risk of Mathematics not having a place in the curriculum of the future.

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Mathematics education in Finnish comprehensive school: characteristics contributing to student success

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Keywords

Mathematics education, comprehensive school, equity, curriculum, special education, teacher training, assessment, Finland

1. INTRODUCTION

The Finnish comprehensive school system and its outcomes have received wide international attention at the beginning of the new millennium (the so-called "PISA effect"). Our education system has become an attractive and internationally examined example of a well-performing system that successfully combines high quality with widespread equity and social cohesion through reasonable public financing (Sahlberg 2006). Since 2001, hundreds of foreign delegates have visited Finland in order to learn the secrets of the high performing system. Through these visits, we Finns have benefited at least as much as our visitors. Questions and doubts presented by the visitors have helped us see what is valuable in our system and, most importantly, understand that explaining the high level of our school system is not a simple and straightforward task. Consequently, we have also started to think seriously about the special characteristics and strengths of our mathematics education. What explains the high level of mathematics performance in the studies like PISA? What kinds of policies and improvement strategies have been implemented since the 1970s in raising student achievement in mathematics?

This paper will address some main characteristics of mathematics education in the Finnish comprehensive school (Grades 1–9), starting with a brief review of Finnish comprehensive school education in general. Drawing on recent articles and reports (e.g. Aho et al. 2006, Kupari 2004, Kupari & Välijärvi 2005, Kupari et al. 2007, Linnakylä 2004, Linnakylä & Välijärvi 2005, Pehkonen et al. 2007, Sahlberg 2006, Simola 2005, Välijärvi et al. 2002, Välijärvi et al. 2007), the main part of my presentation concentrates on describing essential features in our mathematics education such as the curriculum, teaching practices, assessment policies, and teacher training. Finally, some future prospects of mathematics education in Finland will be discussed.

2. FINNISH COMPREHENSIVE SCHOOL EDUCATION

2.1 *General features of the Finnish comprehensive school system*

Finland has nine years of compulsory schooling and children generally start school at the age of seven (see Appendix). Usually, for the first six years of comprehensive school, the children are taught by a class teacher, who generally teaches all or at least most subjects. Then, during the last three years, the different subjects are taught by specialised subject teachers. Almost all of the age group (99.7%) completes compulsory schooling. (Välijärvi et al. 2007)

The school network covers the whole country and schools are primarily run by local authorities, with the exception of a small number of private schools. For children, the teaching and educational equipment are free of charge since education in Finland is publicly financed from pre-school to higher education. In addition, the pupils get a free warm meal at school every day. Transportation is also arranged by the education provider for distance of 5 km and over. Presently, the smallest schools have fewer than ten pupils, and the largest ones about 900. There are some 3200 comprehensive schools in Finland. The amount of schools has dramatically declined because the number of pupils has decreased and municipalities have cut budgets.

At present, the National Core Curriculum for Basic Education prepared by the Finnish National Board of Education (FNBE) determines the core subjects which all pupils study, and the Finnish government determines the national goals for education and the number of classroom hours allocated to each subject. Besides this, learning usually takes place in heterogeneous groups. This means that all pupils study the same core subjects with similar instructional contents. However, about 20 per cent of all classroom hours are reserved for optional subjects freely chosen by the pupil and his or her parents. Furthermore, the schools can develop individual profiles by focusing on some area, such as languages, mathematics, sciences, sports, music or arts.

There is no actual graduation certificate or qualification to be gained upon completing the comprehensive school, but once a student's compulsory education is over, it opens the way to all secondary education options, i.e. different types of vocational training or upper secondary school.

2.2 *Strengths of the Finnish comprehensive education*

The Finnish comprehensive education system is not only a system. It is also a matter of pedagogical philosophy and practice. The comprehensive school is for child and, hence, has to adjust to the needs of each child. Instruction and pedagogy have been developed to adapt to heterogeneous student groups; no student can be excluded or sent to another school. Students' own interests and choices are likewise taken into account at schools when selecting contents, textbooks, learning strategies, methods and assessment devices. Of course, for heterogeneous groups to be successful class size must be relatively small. In fact, PISA 2003 data revealed that mathematics class sizes were among the smallest in the OECD countries (the mean was 18 students). All in all, the comprehensive education calls for a flexible, school-based and teacher-planned curriculum along with student-centred instruction, counselling and remedial teaching.

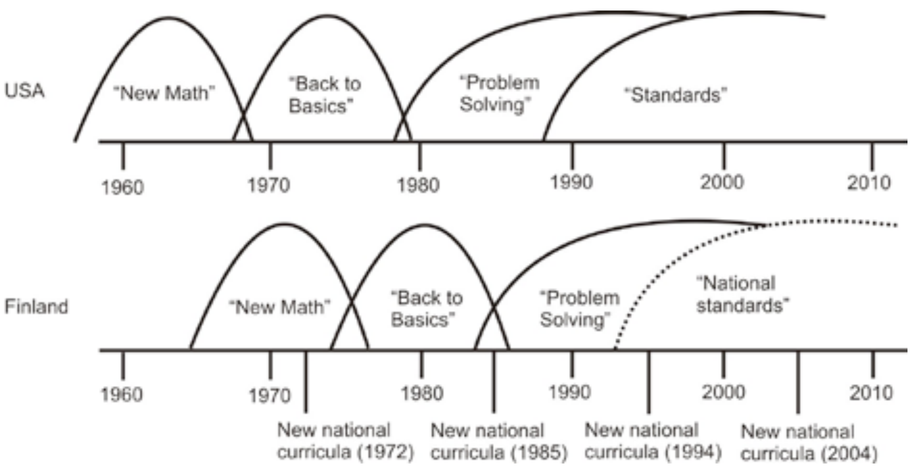
Special education has likewise played an important role in Finnish schools in catering for students who have problems following regular teaching. Special education is usually closely integrated into normal teaching and is highly inclusive by nature. Indeed, only about two per cent of students attend separate special education institutions. In practice, a student with problems for example in mathematics typically has the opportunity of studying once or twice a week in a small group of 2–5 students or even individually with a special teacher. The special teacher may, alternatively, also attend regular classes. On the primary level (grades 1 to 6), where class teachers have the main responsibility for instruction, special education is mostly focused on reading and writing skills along with mathematics skills. A student's right to special education is stipulated in the Finnish school laws.

Every student also has a right to *student counselling*. Schools are to provide students with guidance in study skills, choice of options (e.g. elective courses) and planning of post-compulsory studies. At grade levels 7 to 9, every school has a student counsellor, who provides individual guidance to those in need or desirous of it.

3. DEVELOPMENTS IN MATHEMATICS CURRICULUM

In this chapter, I will shed some light on the curricular background and development of the Finnish mathematics education. Figure 1 below describes the different phases of mathematics curriculum taken place in Finland since the introduction of the comprehensive school system in the beginning of the 1970s.

Figure 1. The developmental phases of the comprehensive school mathematics curriculum in Finland related to the curricular trends in USA



Since 1972 there has been four distinct phases in the development of mathematics curriculum in Finland (cf. Kupari 1994). The figure reveals that the curriculum changes have always tended to follow international - specifically Anglo-American – reform trends. In order to save some time, I will pass by a closer analysis of the first two phases - *New Math* and *Back to Basics* – and concentrate more on the latest curricular phases.

The agenda of NCTM at the beginning of 1980s (NCTM 1980) raised problem solving to a key position in mathematics teaching and it meant the start of the new phase in the development of mathematics curriculum in

Finland, as well (*Problem Solving –phase*). In 1985, the school legislation was reformed and simultaneously the National Board of Education (NBE) introduced the new Framework Curriculum for the Comprehensive School. The objectives of mathematics curriculum emphasised strongly both applications and problem solving and this could be seen very soon in the mathematics textbooks.

The new legislation had also impacts on the practical schoolwork especially on the upper level of the comprehensive school (grades 7-9). The number of mathematics lessons per week was reduced by one (from 10 to 9). Furthermore, the ability grouping (streaming) of students was removed and this was a very significant change for teaching and teachers. Mathematics teachers were now compelled to apply internal differentiation within heterogeneous teaching groups, but at the same time this change of the teaching environment was supported by reducing the size of teaching groups. In mathematics classes, there were about 16-19 students and it provided more opportunities for individualised teaching. During the late 1980s, both mathematics teachers and students got used little by little to work in these heterogeneous classes.

In 1994, the NBE issued again a new Framework Curriculum for the Comprehensive School. This framework curriculum started a new kind of education and curricular culture in Finland. There was a clear shift from a centralised curriculum system to a decentralised system. Instead of uniform national curricula, the NBE now issues curricular guidelines, while the Ministry of Education determines the allocation of lesson hours across school subjects, and schools then accordingly make up curricula of their own. Another important change was that learning materials no longer needed the approval of the NBE. So, schools were given more freedom and responsibility for their own curricular preparation and development (*National Standards –phase*).

Despite rather strong aspirations for reform, the 1994 mathematics curriculum included only minor changes as compared to the previous framework curriculum from 1985. The objectives of mathematics education thus continued the accepted line by emphasising problem solving and application of mathematical knowledge and skills. The main difference compared with the earlier curriculum was that now the objectives and contents of mathematics education were presented in a concise and generic form by school level (about 2 pages in total), whereas previously they had been described in great detail and by grade level.

At the beginning of 2004, the NBE introduced the National Core Curriculum for Basic Education. This latest mathematics curriculum continues the guidelines and objectives expressed in the 1994 curriculum. However, the core curriculum for grades 1-9 is again more detailed than the previous one. The overall objective is to create uniform basic education, i.e. a continuum through grades 1-9.

In summary, the mathematics curriculum has changed about once in ten years during the comprehensive school system. An important issue is that the international trends were not transferred into the Finnish practice as such. Instead, they were transformed into the solution that fitted our national situation. Thus, it was not only a question of borrowing a curricular "ideology" from some other country. A bigger change in the national curriculum system has taken place in 1994 when the directive administration was transferred from the central level to the local municipalities (Lampiselkä et al. 2007). This meant that the local authorities became responsible for the preparation and implementation of the national curriculum at school level.

Perhaps the most significant feature behind the Finnish success in PISA mathematics has been *the systematic development of comprehensive school mathematics curriculum* which has continued since the early 1980s. During the last 25 years, applications and problem solving have been important goals in the mathematics curriculum of our comprehensive school. Step by step, these goals have become more and more established in mathematics textbooks and teaching practice. As we know, the PISA approach particularly focuses on young people's capability to apply their mathematical skills and knowledge in situations that are as authentic and close to daily-life needs as possible. Thus, the Finnish mathematics curriculum has emphasized and also implemented goals and contents comparable to those assessed in PISA mathematics surveys. In this respect, our curricular decisions have been successful and produced great results.

4. HIGHLY QUALIFIED MATHEMATICS TEACHERS ARE A NECESSITY

In the following, I will describe the education of Finnish mathematics teachers. In Finland, the university-level teacher education was implemented in 1974. Today, a research-based approach is a main organising theme integrated into our teacher education programmes. From the very beginning, the objective of teacher education has been to educate pedagogically thinking teachers who are able to think reflectively over their teaching. A teacher is seen as a reflective practitioner who has a strong personal-practical theory of education. (Lavonen et al. 2007, Kansanen et al. 2000)

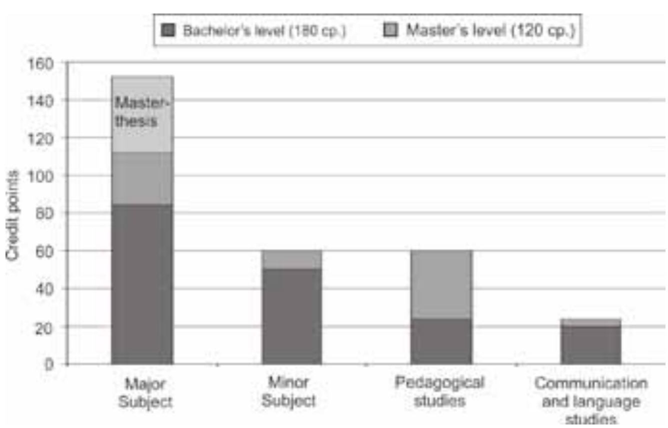
In our comprehensive school system, class teachers are teaching almost all subjects - including mathematics - in primary school at grades 1-6. Subject teachers are teaching in lower secondary school at grades 7-9. All class and subject teachers are educated in Master level programmes requiring 300 credit points (1 cp. = 27 hours work) which are offered by eight universities in Finland.

The structure of a master's degree for a class teacher and a subject teacher are rather similar. As an example, I will shortly present the content of the subject teacher programme in one Finnish university based on the article of Lavonen et al. (2007, 49-59). A typical structure of the education programme can be seen in Figure 2.

Subject teacher studies are divided into two parts: mathematics is studied at the Department of Mathematics and pedagogical studies at the Department of Teacher Education and in the Teacher Training School. In general, teacher students take a major and a minor in the subjects they intend to teach in school. Typical combinations for a mathematics teacher are: mathematics – physics, mathematics – chemistry and mathematics – computer science but the students are free to choose also other combinations of subjects (e.g. mathematics – home economics).

Mathematics in the Finnish universities is very much the same as mathematics in the western world in general. The main aim of the mathematics studies is to give university level understanding of mathematics covering those subject domains taught at Finnish schools. The utilisation of new technology in teaching and learning mathematics have recently included in the studies.

Figure 2. A typical structure of a master's degree of a subject teacher (Lavonen et al. 2007)



During the pedagogical studies, the students' mathematics knowledge, knowledge about teaching and learning mathematics and school practices are integrated into students' personal pedagogical theory. The pedagogical studies are divided into bachelor's level studies (25 cp.) and master's level studies (35 cp.). Typical contents within studies are: teaching and learning mathematics, pupils' interest and motivation in mathematics, national and local curriculum including curriculum planning, teaching methods, ICT in mathematics education and evaluation and research methodologies in mathematics education. One third of the pedagogical studies consist of teaching practice (20 cp.) placed both in the Teacher Training Schools and municipal network schools. Teaching practice has been divided into two parts: the first part takes place during the bachelor studies and the second part at the end of master studies.

Finally, the mathematics teacher students carry out their master thesis (40 cp.) in mathematics. Then they can choose either a pedagogical orientation or a mathematics orientation and prepare the thesis in guidance of a professor or in a research group.

Summing-up, the teaching profession has always enjoyed great public respect and appreciation in Finland, and a lot of resources have consequently been invested in teacher education. Teachers have also been trusted as true professionals of education. This basically means that the educational decision makers believe that teachers together with principals, parents, and their communities know how to provide the best possible education for their children (Aho et al. 2006). From this it has followed that Finnish teachers have considerable pedagogical independency in the classroom and that schools likewise enjoy substantial autonomy in organizing their work within the limits of the national core curriculum (Välijärvi et al. 2007). Teachers make their own decisions related to the conduct of the teaching and learning process, they are responsible and competent for developing the local curriculum, choosing teaching methods and selecting learning materials to be used. Especially, Finnish teachers are relied on when it comes to student assessment, which usually draws on students' class work, teacher-made exams, projects and portfolios. The role of teacher-based assessment is all the more important because at Finnish comprehensive schools students are not assessed by any national tests or examinations upon completing school or during the school years.

In addition, the teacher's profession, especially that of the class teacher, is greatly valued and popular among Finnish post-secondary students. This can be seen, for example, in the popularity of the class teacher's programme provided at universities. Of all the applicants for this programme, only 10-15 per cent is admitted, which implies that those accepted are highly motivated and multi-talented students with excellent academic skills. Educating class teachers at universities and the scope and depth of their study programme seem to be the factors that make Finnish teacher education stand out as special, when compared to other countries.

5. TEACHING PRACTICES IN MATHEMATICS

Efficient mathematics instruction requires an active role both from the students and the teacher. The teacher's aim is to provide opportunities for all students to have versatile and rich learning experiences. Pedagogy in mathematics teaching pays a great attention to individual needs of students. The mathematics core curriculum lays a lot of emphasis on the student's active role in studying mathematics, but still the traditional model of the mathematics lesson including certain successive stages (cf. Pehkonen & Rossi 2007) is vital.

Typical mathematics lessons in Finland include teacher's instruction and students' own working in different forms and mathematics textbooks play an important role in teaching (e.g. Törnroos 2004). Also the term "pedagogical conservatism" has been mentioned in this connection (cf. Simola 2005). The textbook dependence is stronger at the primary level (grades 1-6) than at the lower secondary level (grades 7-9). For many teachers mathematics textbooks have almost the same position in teaching as the curriculum itself (Perkkilä & Lehtelä 2007). This means that the mathematics lessons easily follow the order and contents of the mathematics textbook.

Several publishers in Finland produce mathematics textbooks for the comprehensive school and almost all students have their own textbook. In general, the mathematics textbooks are well planned and prepared. The mathematics curriculum creates the basis for the mathematics textbooks, but naturally there can be big differences between the textbooks. One additional reason for these differences can be the fact that since 1992 there is no official control of textbooks any more.

Teaching heterogeneous student body in mathematics presupposes small teaching groups and possibilities to reorganise groups if necessary. The PISA 2003 data shows that in Finland the average size of mathematics teaching groups (18 students) is among the smallest in the OECD. In addition, the time used in mathematics instruction is an essential pedagogical issue. Table 1 below presents the minimum numbers of mathematics lessons per grade in a school week. The schools have the freedom to divide these lessons between grades. For example, on the grades 3-6 schools have totally 12 lessons mathematics, and usually each grade has 3 lessons (45 minutes) mathematics in a week.

Table 1. The minimum number of mathematics lessons per grade in Finnish school week

Subject / Grade	1	2	3	4	5	6	7	8	9	Altogether
Mathematics	6		12				14			32

Assessment in mathematics is usually carried out by the teacher and it is based mainly on the summative tests but also some formative tests and the teacher's observations during instruction are utilized (cf. Lampiselkä et al. 2007). The teacher's role in assessment is very important in Finland because students are not assessed by any national tests or examinations upon completing the comprehensive school or during the school years. The final assessment takes place twice a year after the autumn term and the spring term and then pupils will have their school report including marks in all their subjects. In the Finnish school reports the marks vary from 4 to10, and 10 is the best mark.

Changes in the pupil assessment reflect changes in the curriculum. Until 1994, assessment in mathematics can be characterized rather formal in nature but since then more versatile and informal assessment methods have been applied. Teachers have started to use for example portfolio assessment and other self-assessment tools more frequently. In the 1994 Framework Curriculum, a verbal evaluation was introduced to be used on grades 1 to 4. Five years later in 1999, the NBE introduced the new guidelines for the final assessment of the basic education. These guidelines include descriptions of good performance (i.e. the mark 8) in all common subjects of the basic education. The main purpose of the guidelines is to ensure that students' final marks would be more equitable and comparable between different schools. These guidelines, however, are far from strict, allowing students' effort and activity to be taken into consideration.

National data from the 2003 PISA sample show that Finnish students view their mathematics teachers positively, and teachers are seen as strong supporters of studying and learning (Kupari & Välijärvi, 2005). The attitude measures indicate also that the Finnish school climate for learning mathematics is positive and encouraging. Stress and anxiety among pupils and teachers is not as common as it is within many other education systems.

In 2007, the Education Evaluation Council organized the evaluation of pedagogy in Finnish basic education (Atjonen et al. 2008). The evaluation focused on the key features of basic education like the teachers' pedagogic principles, the diversity of teaching methods and the effectiveness of the studying environment. The evaluated data consisted of a survey of principals (N = 410) and a survey of teachers (N = 2310) as well as 12 visits to schools. According to the evaluation results, basic education teaching can be characterized as fair and equal, encouraging and appreciative of the student. Basic education teachers aim to at doing their best for their students and prefer applying diverse methods of teaching. There were surprisingly small differences in the pedagogic characteristics expressed by the teachers of different subject (for example between mathematics and mother tongue). However, teachers seem to be strongly bound up with the structural terms of teaching environment, and therefore they are not very eager to promote changes in the existing pedagogy.

Teaching practices in mathematics are changing slowly. Moving towards active learning requires less teacher's talk during the time reserved for face-to-face teaching (Sahlberg & Berry 2003). Rather than deliver the curriculum and transfer information to students, teachers should become facilitators of the mathematics learning process and promoters of social interaction of their students. When students learn to communicate their mathematical thinking, it will also improve their attitudes towards mathematics and reinforce their self-confidence as learners of mathematics (Kupari 2007). During the last fifteen years, there has been promising signs of the positive development. For example, the development program of mathematics and science education during 1996-2002 (called LUMA) created new educational opportunities, produced active collaboration between teachers and schools and aroused new enthusiasm within mathematics education. Since then, many mathematics teachers have actively sought for alternative and more pupil-centered methods in their teaching. Mathematical modelling, activity tasks, learning games, problem solving, investigations and project work are all the more applied in mathematics lessons (Pehkonen & Rossi 2007). Explanations, argumentations and lively discussions are also more common during the Finnish mathematics lessons.

6. DISCUSSION

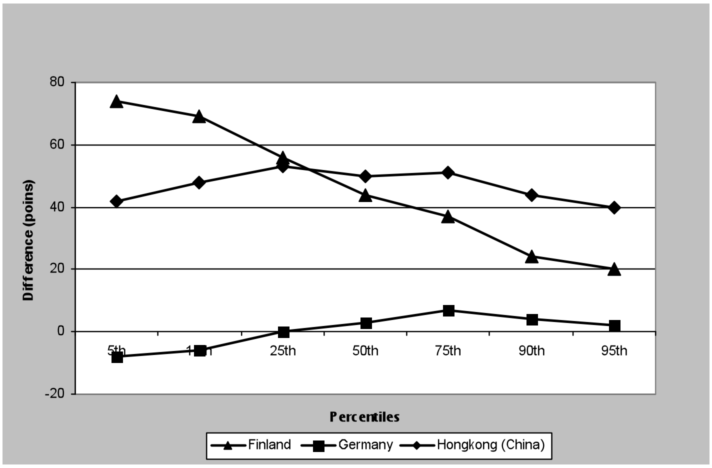
Attaining high overall performance while, at the same time, evening out disparities in performance is one of the key aims of national education policy in most OECD countries. In Finland and also in the other Nordic countries, this thinking has a long tradition. Providing all students with equal educational opportunities and removing obstacles to learning especially among the least successful students, have been leading principles in Finnish education policy for the comprehensive school system. In the light of PISA findings, Finland seems to have managed extraordinarily well in combining these two principles. (Välijärvi et al. 2007)

In this part of the paper, I briefly summarise the essential results related to equality in Finnish mathematics performance on the basis of PISA 2003 data.

6.1 Equality in mathematics achievement

The Finnish strategy for improving education is based on *the principle of equity*, and particularly on *an effort to minimise low achievement* (Linnakylä & Välijärvi 2005). One of the most important findings of PISA, therefore, has to do with the fact that in Finland the gap between high and low performers is relatively narrow. In mathematical literacy, the standard deviation for Finnish student scores was the smallest (84) in 2003 among all OECD countries. Likewise, the number of low performers – whose performance was at or below PISA proficiency level 1 – was significantly smaller in Finland (7%) than it was in the OECD countries on average (21%). Indeed, it seemed to be a characteristic of the Finnish performance profile that *the lowest scoring students performed better than their fellow students in the other OECD countries*. The difference between the top performers, on the other hand, was much less pronounced. This becomes evident when comparing the distributions of Finland and some other countries against the OECD average distribution on a percentile scale (see Figure 3).

Figure 3. Means of country percentiles compared to OECD-means (0-level) on the combined mathematical literacy scale in PISA 2003



The PISA 2003 results also revealed that in Finland, parents' socio-economic status has a relatively low impact on student performance compared to other OECD countries. The gender difference was also relatively small – 7 points in favour of boys. Furthermore, the differences found between Finnish schools were among the smallest in the OECD countries. While in 2003 these differences accounted for 34 per cent of the variation in student mathematics performance in the OECD countries on average, in Finland only 5 per cent of the total variance within the country was between schools (OECD 2004). In Finland, even the weakest performing schools achieved the OECD average in mathematics.

Small between-school variation is a characteristic of all the Nordic countries. This is largely due to the fact that these countries have non-selective education systems in which all students are provided with the same kind of comprehensive basic education from age 7 to age 16. In contrast, variation between schools tends to be more pronounced in countries where students are enrolled into different kinds of schools, streams or tracks at an early age. The results of PISA indicate that small between-school variation is one of the key factors associated with high and relatively equal performance. From this equity perspective, the PISA results are most encouraging for Finland, where the differences among schools, between the different regions, and between urban and rural areas proved small. In Finland, it matters little where a student lives or which school he or she attends. The opportunities to learn seem to be virtually the same all over the country, whether the student lives in the far North, in the remotest districts of Lapland or in the Helsinki capital area. (Linnakylä & Välijärvi 2005)

7. CONCLUSIONS

The PISA results clearly show that the Finnish comprehensive school yields high achievement in mathematical literacy. In all three PISA-studies, Finland has been within the best-performing countries in the mathematical literacy. Furthermore, the mathematics performance of our seventh-graders was clearly above the international average in the TIMSS 1999 –study. However, Finnish mathematics education has many challenges to which we need to react in future. Here, I will mention just some of them.

One major challenge to Finnish mathematics education seems to be students' attitudes towards mathematics, particularly in the case of girls. Finnish students showed surprisingly low interest in mathematics in international comparison. Especially girls' interest in mathematics, girls' confidence in their possibilities of learning mathematics and enjoyment in studying mathematics were inconsistent with their high performance in PISA 2003. The high prevalence of negative attitudes is worrying because interest in and confidence with mathematics is considered to have a strong steering influence when young people select their further studies. Increasing students' confidence and enjoyment in learning mathematics is thus a major pedagogical concern that requires a critical evaluation of the methods of learning and materials used in mathematics instruction. Students' attitudes can be improved, for example, by creating more interesting and meaningful classroom practices and by providing positive experiences during mathematics lessons. In part this is, however, a larger cultural concern as there seems to be a strong tradition of labelling mathematics as a male domain in Finland.

A serious challenge in future relates to a growing number of immigrant students in our country. Although Finland is officially a bilingual state, it has been a culturally homogeneous country. The official languages are Finnish (94 per cent of the inhabitants) and Swedish (6 per cent). Both of these language groups are equally entitled to and have equal resources for education in their own language from the pre-school up to the university. Other minorities, however, are still relatively small.

The pursuit of equal opportunities to learn has been a leading principle in the development of the Finnish educational system. Despite the relative homogeneity of Finnish population, this pursuit has been put to a severe test during the last decades due to a growing number of immigrant students and growing cultural heterogeneity. This presents a special challenge to literacy education and therefore to mathematics education as well.

During the last decade or so, many suburban schools in Finland have experienced increasing social and behavioural problems as more pupils live in broken homes, engage in drugs and alcohol at younger ages, and spend more time with computers, electronic games, and television. Schools in Finland must now compete with media and entertainment more than ever. Sustaining the genuine interest of pupils in learning is the premier goal for education development in the future. (Aho et al. 2006)

In summary, all experiences in relation to Finnish mathematics education give support to the notion that a high average performance can be achieved also in mathematics by taking equally care of learning across the whole age cohort. The high overall standard of our mathematics education in the comprehensive school is an asset that allows providing support for the low achievers while also motivating the top performers to use their potential to the full. This kind of positive thinking which is founded on our own national strengths provides a good basis for the development of mathematics education that aims at even better achievements.

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Retaining the Heritage -Preparing the Future. Fundamental Ideas of Mathematics

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INTRODUCTION

Children learn to read because they are surrounded by texts. They learn to write because they want to write messages to friends. They use computers and want to drive a car. The discussion on the green-house effect stimulates interest in environmental studies. Even history and paleontology sell well on television (think of recent stories with dinosaurs and mammoths). The need of mathematical skills for all has been challenged. One reason is the progressive division of labor which leaves mathematical issues to specialists. Another reason is the pervasive use of electronic devices. Therefore mathematical education is undergoing a substantial transformation which may be expressed in the short terms "Less doing mathematics but more learning to understand the role of mathematics in society." Bruner's concept of Fundamental Ideas could be a guideline for this process. In the first part the concept of so-called Fundamental Ideas is presented. In the second section these ideas will be illustrated by an example of expository teaching.

FUNDAMENTAL IDEAS

It should be emphasized that my notion of Fundamental Ideas refers to activities. This notion is close to the view of mathematicians like Halmos 1981 ("No doubt many mathematicians have noted that there are some *basic ideas* that keep cropping up, in widely different parts of their subject, combining and re-combining with one another in a way faintly reminiscent of how all matter is made up of elements") or Mac Lane 1992 who says that mathematics begins in the human experiences of *moving, measuring, shaping, combining, and counting*. In a similar way Bishop 1991 names six basic mathematical activities, namely *counting, locating, measuring, designing, playing and explaining*. MacLane 1986 states a similar view: "[Mathematics] is not a science of time and space, but a formulation of the *ideas* needed to understand time, space, and motion. This understanding depends on *ideas*" (MacLane 1986:414).

Such catalogues are familiar to anyone who participates in discussions on curricula or mathematical standards (e.g. the Process Standards of the NCTM 2000). A short overview of the discussion of such concepts was given in Schweiger 1992 and Schweiger 2006. During the last ten years I collected some more material on this topic for a forthcoming book (Schweiger 2010). However, every year some new ideas were born and some old concepts did not look so promising as I thought before. It is not important to have a larger or shorter list of Fundamental Ideas but it is important to consider the question behind. Can mathematics and mathematical activities be organized as a bundle of coherent ideas which are helpful to communicate mathematics and to speak about mathematics as a valuable intellectual endeavor? Such a list will reflect the personal view of mathematics and stands open for revision. My notion of Fundamental Ideas is close to activities in the interplay between form and function. If you design a house there are some ideas of its form or shape but the function as a living place will also guide your considerations. In a similar way, if you design pottery the form (a jug, a cup, a plate) will give you some intuition but also possible functions (to contain water or wine) will be important. A different approach is taken from the angle of cognitive science by Lakoff & Núñez. They emphasize the importance of ideas in mathematics and provide some detailed analysis. "The intellectual content of mathematics lies in its *ideas*, not in the symbols themselves (Lakoff & Núñez 2000:xi)". When they state "... a great many of the most fundamental mathematical ideas are inherently metaphorical in nature" (their examples are: number line, Boole's algebra of classes, symbolic logic, trigonometric functions, complex plane) I would agree. However these examples do not cover my conception of *fundamental ideas*. These examples are important tools for *doing mathematics* and their invention or discovery has to do with *fundamental ideas*.

During the last years four descriptive criteria for Fundamental Ideas turned out to be useful (Schwill 1993).

1. Fundamental Ideas recur in the historical development of mathematics. They are related to "perennial notions" (Barbin 2007). The identification of techniques and patterns which recur in history is an interesting and important task of historical investigations. Here we can add an important observation of MacLane: "Mathematical ideas arise not just from human activities or scientific questions; they also arise out of the urge to understand prior pieces of mathematics" (MacLane 1986:415).The idea of recursion and iteration may be a good candidate to illustrate this point.
2. Fundamental Ideas recur in different areas of mathematics. The art of recognizing patterns and designing patterns can be found in algebra and number theory as well as in various branches of analysis. The aim to classify mathematical objects and to recognize prototypes is also widespread. This idea encompasses all types of morphisms as well as prototypical objects like the normal distribution.
3. Fundamental Ideas recur at different levels. The idea of testing and controlling illustrates this point. After solving an equation even at an early level it is recommended to insert the found solution in the given equation. Tests by congruence (modulo 9 or modulo 11, say) are easy to apply. The search for testing primality has become en vogue recently.
4. Last but not least Fundamental Ideas are related to every day activities. To recognize and to produce patterns is essential to artistic activity. Iteration is a basic human activity in preparing tools, pottery and canoes. It is very likely that classification and recognizing prototypes is indispensable for the formation of concepts and its counterpart, language. Otherwise we would not speak of dogs, flowers, houses etc. and we could not distinguish between sitting and moving. People test food and drinks before they buy greater quantities. The quality of products must be controlled.

After having seen these four descriptive criteria we add the potential use of Fundamental Ideas in educational practice. The statement "Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise" (NCTM 2000:17) is a useful guideline for teacher education. They can be seen as a guide line for designing curricula. To some extent curricula are oriented not only on mathematical contents but also on mathematical activities. In my opinion text books could be designed to make Fundamental Ideas more explicit. This can be important in connection with a change to a certain amount of expository teaching. Fundamental Ideas should be capable to elucidate mathematical practice. The difficult aim to explain mathematics to other people should be named here. Furthermore Fundamental Ideas can be useful for building semantic networks between different areas of mathematics. The classification of conic sections (illustrated by the prototypes ellipse, hyperbola, parabola) is the same idea as the basic classification of monotone dependence (increasing vs. decreasing; illustrated at the elementary level by $y = ax + b, a > 0$ vs $a < 0$). Therefore Fundamental Ideas should help to improve memory. It seems to be common sense that concepts which are understood and included in a semantic net are better memorized or can more easily retrieved.

Fundamental Ideas could also help to communicate the beauty, joy, and excitement of mathematics. Maybe this would especially have some effects on pre-and in-service education of teachers. A short glimpse at Nardi's study (Nardi 2008) of mathematics undergraduates in the UK shows that a lot of students' problems seem to be related with a lack of understanding of the basic ideas which drive mathematics. In his very revealing essay Thurston says: "We mathematicians need to put far greater effort into communicating mathematical *ideas*. To accomplish this, we need to pay much more attention to communicating not just definitions, theorems, and proofs, but also our ways of thinking" (Thurston 2006:45).

This description immediately leads to proposals for research activities. The focus could be more "theoretical" or more "practice oriented".

1. Construction of semantic nets between different Fundamental Ideas
2. Analysis of teaching materials, curricula, and standards along the lines of Fundamental Ideas

3 Connections to other important concepts like mathematical literacy, orientation on applications, orientation on problem solving, orientation on structures, 'genetischer Unterricht'
4. Experiments with learning materials which are designed according to this guideline
5. Exploring mathematical beliefs (of students and teachers) and Fundamental Ideas
6. Validation of some aspects of the human dimension

Compared with other subjects mathematical education lacks environmental input. Clearly, we all are surrounded by numbers. We look at prices of goods and inspect our bank account. Good and more often bad news are illustrated by figures but the cost of a billion dollars is nothing more than incredibly high. In every day live almost everything is done by pocket calculators and computers. Clearly, there is much more mathematics around us e.g. geometrical figures and shapes, topology in form of the subway network, fractal images and clouds. But the mathematics behind the curtain has to be detected. Furthermore our technological civilization rests on mathematics but the increasing division of labor could suggest to leave the mathematics behind to specialists. We all use computers and television but basically we are happy if engineers provide us with these items, ready for use! The intimate connections of mathematics to various parts of our culture are demonstrated in Emmer 2004, 2005.

If a society should have some coherence it could be important that there is a basic knowledge which is shared by many. Furthermore the communication among specialists needs a common language (Fischer 1993). Therefore mathematics education has at least three goals.

1. Some basic skills should be provided (comparable with reading and writing).
2. A preparation for professions which use more mathematics is important but the extent of this preparation could vary at different levels and type of school.
3. Mathematics as a cultural activity should be taught. A path to this goal can be "expository teaching" (Lóvasz 2008).

In my opinion these aims could be enhanced by reliance on Fundamental Ideas. The classical quotations which follow are from Bruner 1960.

> "It is that the basic ideas that lie at the heart of all science and mathematics and the basic themes that give form to life and literature are as simple as they are powerful."
>
> "The early teaching of science, mathematics, social studies, and literature should be designed to teach these subjects with scrupulous intellectual honesty, but with an emphasis upon the use of these basic ideas."
>
> "The first [general claim] is that understanding fundamentals makes a subject more comprehensible."
>
> "The second point relates to human memory." "Third, an understanding of fundamental principles and ideas, as noted earlier, appears to be the main road to adequate 'transfer of training'."
>
> "The fourth claim for emphasis on structure and principles in teaching is that by constantly reexamining material taught in elementary and secondary schools for its fundamental character, one is able to narrow the gap between 'advanced' knowledge and 'elementary' knowledge."
>
> "We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development. It is a bold hypothesis and an essential one in thinking about the nature of a curriculum."

EXPOSITORY TEACHING

As an example of the use of Fundamental Ideas in expository teaching I refer to Riemann's hypothesis. We also try to emphasize the importance of Fundamental Ideas related to this example: recognizing patterns, taking a new approach, confidence in formal calculations, redefining, and estimating.

Recognizing patterns is a Fundamental Idea. This activity leads to the detection of prime numbers. Some numbers like 4, 6, 9, $\cdots$ can be laid down as proper rectangles. Others like 2, 3, 5, 7, $\cdots$ cannot. If one uses the Sieve of Eratosthenes one is confronted with the surprising irregular pattern of prime numbers. Due to Euclid we know that there are infinitely many prime numbers but the path to Riemann's hypothesis uses a different idea. This illustrates the idea of *taking a new approach*. Since every number $n \geq 1$ can be written as a product of primes in a unique way the equation

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \sum_{\alpha=0}^{\infty} \frac{1}{p^{\alpha s}} = \prod_p \frac{1}{1-\frac{1}{p^s}}$$

Is valid for $s > 1$. If the number of primes is finite this relation should be valid for $s = 1$. Since the so-called harmonic series is divergent this is not possible.

Another proof is also remarkable. We know

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_p \frac{1}{1-\frac{1}{p^2}}.$$

If the number of primes would be finite we obtain that $\zeta(2)$ is a rational number but in fact we know that $\zeta(2) = \frac{\pi^2}{6}$ and π^2 is not a rational number.

Due to the invention (or discovery?) of complex functions it was tempting to consider the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s = \sigma + it.$$

Convergence for $\sigma > 1$ is no problem since $|n^s| = n^\sigma$ but there is no necessity to prove it in an expository teaching. The idea of *confidence in formal calculations* which leads to new areas stands behind. The exponential function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ can be extended to complex numbers by $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ or even to matrices by $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. The idea of *redefining* is the key for the next step. This activity can be explained at various levels. If you look at the function $f(x) = x$ for $x > 0$ then this function could be part of $f(x) = x$ for all real x but it could also belong to $f(x) = |x|$. Therefore if you have a piece of a function several continuations are possible. The formula for geometric series shows $\sum_{z=0}^{\infty} z^n = \frac{1}{1-z}$ as long as $|z| < 1$. But the function $f(z) = \frac{1}{1-z}$ is well defined for all complex numbers $z = 1$. Now we look at the equation

$$\zeta(s) = 1 + \frac{1}{s-1} + s \int_1^{\infty} \frac{[w]-w}{w^{s+1}} dw.$$

Since $[w] = n$ for $n \leq w < n + 1$ we obtain

$$s \int_1^{\infty} \frac{[w]}{w^{s+1}} = \sum_{n=1}^{\infty} n \left(\frac{1}{n^s} - \frac{1}{n^{s+1}} \right) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The last equality is very easy to understand.

$$\left(\frac{1}{1^s}-\frac{1}{2^s}\right) + \left(\frac{2}{2^s}-\frac{2}{3^s}\right)+\left(\frac{3}{3^s}-\frac{3}{4^s}\right)+\cdots+\left(\frac{N}{N^s}-\frac{N}{(N+1)^s}\right)$$
$$= \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots+\frac{1}{N^s}-\frac{N}{(N+1)^s}.$$

Therefore this equation is true for $\sigma > 1$. But since $|\,[w]-w\,| < 1$ the integral on the right hand side converges for $\sigma > 0$. This gives a definition of the ζ-function for $\sigma > 0$ (with the important condition $\sigma \neq 1$). The conjecture of Riemann now reads as follows. If $\zeta(s) = 0$, $0 < \sigma < 1$, then $s = \frac{1}{2} + it$. This conjecture was published by Riemann in the year 1854 but up to now withstood all attempts of being proved (Riemann 1990:148). In fact $\sigma = \frac{1}{2} + i14, 13472$ ... is the first zero in the upper half plane (with $s = \frac{1}{2}$).

Now we have formulated Riemann's conjecture (later on called Riemann's hypothesis if one uses this "result" in further investigations) but the question remains: Why is this an important conjecture? Let $\pi(x)$ be the number of primes $p \leq x$ then Gauss and Legendre conjectured that $\pi(x)$ is approximately $\frac{x}{\log x}$. More precisely this means $\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1$. This result was eventually proved by Hadamard and de la Vallée-Poussin in 1896. The Riemann hypothesis deals with the difference $\pi(x) - \frac{x}{\log x}$. If the Riemann conjecture is true then we would have the best possible result for the error. I will not go into further details but just mention that the idea of estimating is a key notion. If you know that a value x is correct up to $\pm x$ this means that the value $x = 100$ lies between 0 and 200. If you know that the error is $\pm\sqrt{x}$ then the value $x = 100$ lies between 90 and 110. As is well known estimating is important in numerical calculations and in statistics.

A good account of the mathematics around the ζ-function is Edwards 1974 (only suitable for students with a strong mathematical background). An interesting presentation of several mathematicians who are connected with this problem is Du Sautoy 2007 (readable for the layman). A nice novel around Riemann's life is Naess 2006. This novel could be used as a bridge between literature and history of mathematics (and may even lead to some mathematics!). As Ziegler points out it is very important to familiarize mathematics by presenting people you can talk to or write about (Ziegler 2008:341).

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Sociocultural Perspectives on the Learning and Development of Mathematics Teachers and Teacher-Educator-Researchers

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In this report I explore what we can learn from research that takes a sociocultural perspective on conceptualising "learning to teach". The first part of the report refers to selected studies of pre-service teacher education, the transition from prospective to beginning teacher, and professional development programs to illustrate what we might learn from the various sociocultural orientations employed. The second part further develops one sociocultural approach – an application of Valsiner's (1997) zone theory, and illustrates its use in my own research involving prospective and beginning mathematics teachers. The third part of the report examines, from a sociocultural perspective, what it means to "learn" from research in teacher education, leading to a proposal that zone theory might offer a sociocultural framework for understanding the work of mathematics teacher-educator-researchers.

Keywords

Sociocultural theories; mathematics teacher education; development of mathematics teacher educators.

The ideas presented in this report have developed from many years of my own research using sociocultural theories to investigate students' mathematics learning in secondary school classrooms and, more recently, the learning and development of mathematics teachers. In this report I look to extend these ideas to help me understand the learning of mathematics teacher educators who are also mathematics education researchers.

There is growing interest in theories that view teachers' learning as a form of participation in social and cultural practices rather than as an internal mental process. Recent reviews of research in mathematics teacher education have noted increasing attention to the social, cultural and institutional dimensions of teachers' learning as well as attempts to integrate social and individual levels of analysis (da Ponte & Chapman, 2006; Lerman, 2001; Llinares & Krainer, 2006). To explain what I mean by sociocultural approaches to mathematics teaching and learning I take the words of Stephen Lerman (1996), who defined such approaches as involving "frameworks which build on the notion that *the individual's cognition originates in social interactions* (Harré & Gillett, 1994) and therefore the role of culture, motives, values, and social and discursive practices are central, not secondary" (p. 4, emphasis added).

The report considers the following questions:

1. What can we learn from sociocultural research on learning to teach mathematics?
2. How might this research provide a framework for theorising the role of mathematics teacher-educator-researchers?

In the first part of the report I briefly survey the sociocultural landscape in mathematics teacher education by referring to representative studies that use different sociocultural approaches. In the second part I elaborate on one sociocultural approach – an application of Valsiner's (1997) zone theory, and illustrate its use in my own research involving prospective and beginning mathematics teachers. The third part of the report considers what we can learn from mathematics teacher education research using Valsiner's zone theory. The final part develops a proposal that zone theory might offer a sociocultural framework for understanding the role of mathematics teacher-educator-researchers.

1. THE SOCIOCULTURAL LANDSCAPE IN MATHEMATICS TEACHER EDUCATION

Sociocultural perspectives on learning and development grew from the work of Vygotsky in the early 20th century. Vygotsky introduced the now familiar concept of the Zone of Proximal Development (ZPD) to explain how an individual's cognition originates in social interaction. He proposed that the ZPD is created when a child's interaction with an adult or more capable peer awakens mental functions that have not yet matured and thus lie in the region between actual and potential developmental levels.

Recent socioculturally oriented research on teachers' learning has drawn on two perspectives: a *discourse* perspective and a *practice* perspective (cf Forman, 2003). The discourse perspective focuses on the dynamics of mathematical communication in classrooms, an approach exemplified by research undertaken by Blanton, Westbrook and Carter (2005). Their study examined how a prospective teacher's classroom discourse changed as her perception of teacher and student roles shifted from teacher as teller to student as mathematical participant. This change was no accident; it was deliberately planned by the university practicum supervisor (Blanton) in the conversations she had with the prospective teacher about classroom interactions she had observed and what this revealed about how students learned mathematics. Blanton calls this a "pedagogy of supervision", which she claims opens up a ZPD that can challenge a prospective teacher's models of teaching in the context of actual practice.

The practice perspective links classroom and professional activity structures with learning and identity. Situative and community of practice approaches typify this perspective (e.g., see Graven, 2004; Greeno, 2003; Lave & Wenger, 1991; Wenger, 1998). Peressini, Borko, Romagnano, Knuth, and Willis (2004) adapted a situative perspective on learning to develop a conceptual framework for learning to teach secondary mathematics, focusing particularly on teacher learning within multiple contexts such as university mathematics and teacher education courses, practicum experiences, and schools of employment. They noted apparent inconsistencies between the ways teachers taught in different contexts; for example, one teacher used reform-based approaches during the practicum but more traditional approaches during her first year of full-time teaching after graduation. These are not unusual or surprising observations, but Peressini et al. concluded that the inconsistencies were responses to the different affordances and constraints of the different contexts, and hence teachers' knowledge-in-practice varies with participation in different contexts. This research is useful because it helps us understand how context makes a difference to the development of mathematics teachers and their professional identities.

Krainer has noted that teacher educators have the dual roles of "intervening and investigating … of improving and understanding" (Adler, Ball, Krainer, Lin, & Novotna, 2005, p. 371). Sociocultural studies such as those summarised above help us *understand* how teachers learn from their experiences in different contexts. But perhaps sociocultural perspectives have been used less effectively to guide research on *intervening* to improve teachers' opportunities to learn. This has left the role of the teacher educator largely untheorised. I argue that a more elaborated sociocultural theory of teaching is needed to complement existing sociocultural language and concepts used to describe learning in a community of practice or in the ZPD. My approach is based on an adaptation of Valsiner's (1997) zone theory of child development, which is outlined below.

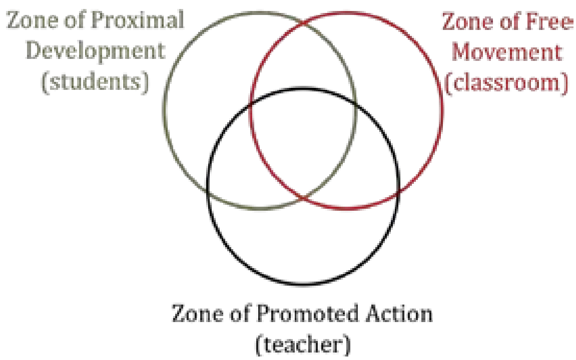
2. VALSINER'S ZONE THEORY

Valsiner (1997) sees the Zone of Proximal Development as a set of possibilities for development that come into being as individuals negotiate their relationship with the learning environment and the people in it. His theory proposes the existence of two additional zones, the Zone of Free Movement (ZFM) and the Zone of Promoted Action (ZPA). The ZFM structures an individual's access to different areas of the environment, the availability of different objects within an accessible area, and the ways the individual is permitted or enabled to act with accessible objects in accessible areas. The ZPA comprises activities, objects, or areas in the environment in respect of which the person's actions are promoted. The ZFM and ZPA are dynamic and inter-related, and are constantly being re-organised by adults in interactions with children.

2.1 *Adaptation of zone theory to mathematics education*

Mathematics education researchers have taken two contrasting approaches to applying this theory to teaching-learning interactions. The first defines the zones from the perspective of the teacher-as-teacher, with the ZPD "belonging" to the students as it is they who are learning. A teacher's instructional choices about what to promote and what to allow in the classroom establish a ZFM/ZPA complex that characterises the learning opportunities experienced by students. One possible zone configuration is represented in Figure 1; others can be imagined if overlap between zones is allowed to change. This representation implies that learning takes place at the intersection of the three zones.

Figure 1. A possible zone configuration (teacher-as-teacher)



This teacher-as-teacher version of zone theory is useful for explaining apparent contradictions between the types of learning that teachers claim to promote and the learning environment they actually allow students to experience.

My own research has taken a different approach because I have applied Valsiner's theory to teacher learning and development (Galbraith & Goos, 2003; Goos, 2005a, 2005b, 2009). Here, all zones are defined from the perspective of the teacher-as-learner. When I consider how teachers learn, I view the teacher's ZPD as a set of possibilities for their development that are influenced by their knowledge and beliefs, including their disciplinary knowledge, pedagogical content knowledge, and beliefs about their discipline and how it is best taught and learned. The ZFM can then be interpreted as constraints within the teacher's professional context such as students (e.g., behaviour, socio-economic background, motivation, perceived abilities), access to resources and teaching materials, curriculum and assessment requirements, organisational structures (e.g., timetabling, room allocation, grouping of students, subject offerings) and organisational cultures (e.g., support for collaborative planning and participation in professional development). While the ZFM suggests which teaching actions are *allowed*, the ZPA represents teaching approaches that might be specifically *promoted* by pre-service teacher education, formal professional development activities, or informal interaction with colleagues in the school setting. For learning to occur, the ZPA must engage with the individual's possibilities for development (ZPD) and must promote actions that the individual believes to be feasible within a given ZFM. It is significant that prospective teachers develop under the influence of two ZPAs, one provided by the university program and the other by the supervising teacher(s) in the practicum school, which do not necessarily coincide. A possible zone configuration for teacher-as-learner is represented in Figure 2.

Figure 2. A possible zone configuration (teacher-as-learner)

2.2 *Application of zone theory: The case of Adam*

I illustrate the application of the teacher-as-learner version of zone theory by referring to a case study of one of my own students, whom I will call "Adam" (a pseudonym). Adam was a participant in a three longitudinal study in which I followed successive cohorts of my teacher education students into their early years of teaching (Goos, 2005a, 2005b). I designed and taught the mathematics methods course so that students experienced regular and intensive use of graphics calculators, computer software, and Internet applications. Thus the course offered a teaching repertoire, or ZPA, that emphasised technology as a pedagogical resource.

I developed case studies of selected participants to capture developmental snapshots of their experience at three stages: (1) during their final practice teaching session, (2) towards the end of the first year of full-time teaching, and (3) in their second or subsequent years of teaching. I selected participants to sample practicum school settings that differed in terms of the Zone of Free

Movement (professional context) and Zone of Promoted Action (supervising teacher approaches) they offered. I visited them in their practicum schools and schools of employment for lesson observations, collection of teaching materials and audio taped interviews (see Goos, 2005a for details).

Data sourced from lesson observations, surveys, questionnaires, and interviews were categorised as representing elements of participants' ZPDs, ZFMs, and ZPAs. As the zones themselves are abstractions, this analytical process focused on the particular circumstances under which zones were "filled in" with new people, actions, places and meanings. This approach enabled me to explore how personal, contextual, and instructional factors came together to shape prospective and beginning teachers' pedagogical identities.

The school where Adam completed his practice teaching sessions had recently bought resources such as graphics calculators, data logging equipment, and software. Every mathematics classroom was equipped with computers connected to the Internet, a data projector, and a TV monitor for projecting graphics calculator screen output. A hire scheme provided calculators to all students in the final two years of secondary school, and there were also sufficient class sets of calculators for use by younger classes. Some of these changes had been made in response to new mathematics syllabuses that mandated the use of computers or graphics calculators in teaching and assessment programs. Thus the school and curriculum environment offered a Zone of Free Movement that seemed to afford the integration of technology into mathematics teaching.

Adam had previously worked as a software designer and was confident in using computers and the Internet. Although he had not used a graphics calculator before starting the teacher education course, he quickly became familiar with its capabilities and with the support of his supervising teacher began to incorporate this and other technologies into his mathematics lessons. At this stage Adam was still a little concerned that students might become dependent on the technology by "just punching things into the calculator and getting the answer straight away". However, he recognised that he may have formed this view because he had only seen other teachers use graphics calculators in class as a tool for saving time or for checking work done first by hand. In theoretical terms, then, the Zone of Promoted Action organised by the supervising teacher was consistent with the ZPA I offered in my university course and also with the ZPD that defined Adam's potential for development. Thus his zone configuration at this stage resembled that shown in Figure 2.

After graduation Adam was employed by the same school where he had completed his practicum. By this time, Adam had developed more sophisticated pedagogical knowledge about how to use technology to help students learn new concepts. For example, in a lesson about families of functions, I observed him follow the students' lead when they used their graphics calculators to explore different ways of transforming an absolute value function $y = |x|$, and he coaxed generalised findings out of the students using their own language and symbols. He described his approach to this lesson as follows:

> I had a rough plan and we kind of went all over the place because we found different things. But I think that's better anyway because they're using their calculators to help them learn.

One might expect Adam to experience a seamless transition from prospective to beginning teacher; yet I found this was not the case when I visited him near the end of his first year of teaching. By this time he had discovered that many of the other mathematics teachers were unenthusiastic about using technology and favoured teaching approaches that he claimed were based on their faulty belief that learning is linear and teacher-directed rather than richly connected and student-led. He described these beliefs and teaching approaches as follows:

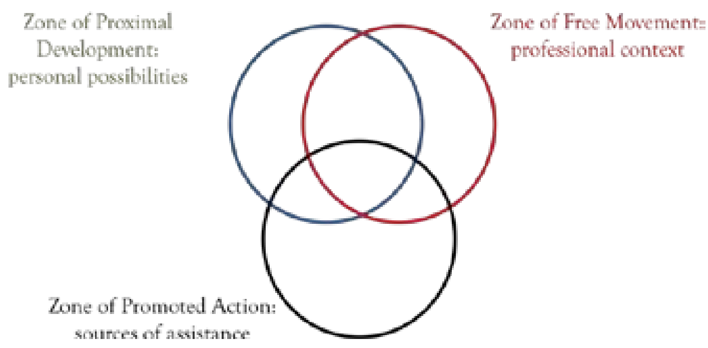
> You do an example from a textbook, start at Question 1(a) and then off you go. And if you didn't get it – it's because you're dumb, it's not because I didn't explain it in a way that reached you.

Because he disagreed with this approach, Adam deliberately ignored the worksheet provided for the families of functions lesson by the teacher who coordinated this subject. The worksheet led students through a sequence of exercises where they were to construct tables of values, plot graphs by hand, and answer questions about the effects of each constant in turn. Only then was it suggested that students might use their graphics calculators to check their work. Conflicting pedagogical beliefs were a source of friction in the staffroom, and this was often played out in arguments where the teacher in question accused Adam of not teaching in the "right" way. Compared with his earlier experience as a prospective teacher, Adam now found himself in a more complex situation that required him to defend his instructional decisions while negotiating a harmonious relationship with several colleagues who did not share his beliefs about learning. Adam explained:

> [Now I'm willing] to stand up and say "This is how I am comfortable teaching". I just walk away now because we've had it over and over and the kids are responding to the way I'm teaching them. So I'm going to keep going that way.

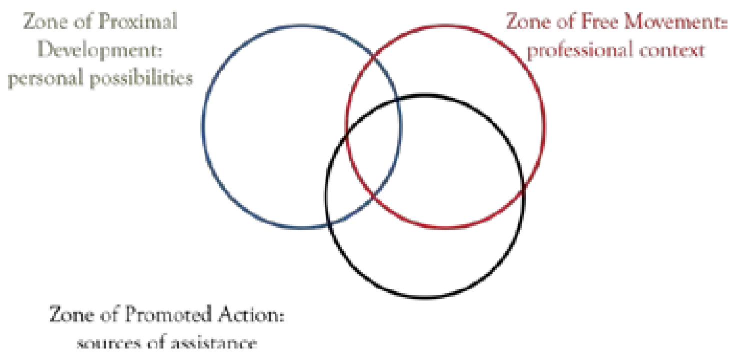
In terms of Valsiner's zone framework, Adam became aware of conflicts between his technology-rich ZFM, a ZPA that promoted, at best, fairly mundane uses of technology in his teaching, and his personal ZPD. This zone configuration is depicted in Figure 3. He responded by paying attention only to those aspects of the Mathematics Department's ZPA that were consistent with his own beliefs and goals (his ZPD) and also with the ZPA offered by the university teacher education course. This, it seemed to me, was how he was able to reconcile his pedagogical beliefs (a part of his ZPD) with the ZFM/ZPA complex within his teaching environment.

Figure 3. Adam's zone configuration for first year of teaching



The next year Adam was transferred to a different school that had fewer resources and a more difficult teaching environment. For example, there was only one class set of graphics calculators in the whole school, and most students were from low socio-economic backgrounds and could not afford to buy their own calculators. The learning environment was disruptive and poorly managed, and teachers felt frustrated at a perceived lack of support from the school's leadership team. Adam found no colleagues in the mathematics department who shared his pedagogical beliefs or enthusiasm for using technology to help students learn. This school promoted teaching approaches (Zone of Promoted Action) that were consistent with the technology-poor environment (Zone of Free Movement), but not with Adam's beliefs and aspirations as a beginning teacher (his Zone of Proximal Development). I have represented his zone configuration at this school in Figure 4.

Figure 4. Adam's zone configuration for second year of teaching



3. WHAT CAN BE LEARNED FROM SOCIOCULTURAL RESEARCH USING VALSINER'S ZONE THEORY

Earlier I wrote that teacher education research aims to *understand* how teachers learn and to *intervene* so as to improve teachers' opportunities to learn. Let me take up these themes once more to consider how using zone theory has helped me to understand and intervene in teachers' learning and development.

In my work with prospective and beginning teachers, I now have a better understanding of the scope and limitations of my role as a mathematics teacher educator. For example, for many years I addressed separately some of the key factors known to influence technology integration. I had my students carry out an annual technology audit of their practicum schools so that on their return to the university they could report on and debate the significance of access to resources and technical support and the effect of curriculum and assessment requirements on technology usage. In these post-practicum sessions I also structured small group discussion tasks in which students compared their own pedagogical beliefs about the role of technology in mathematics education with the technology-related practices demonstrated (or not) by their supervising teachers. These coursework activities have not changed in their classroom enactment. What has changed is the way I now integrate these and other elements of my course into a single zone-theoretical framework that suggests to me how and where I might intervene in the development of prospective and beginning teachers' identities as users of technology.

The question of intervention is more difficult, since I am but one of many influences on the learning and development of a beginning teacher. In Adam's case, I decided to try to change the way he viewed his context (ZFM) and the influence of other teachers (ZPA) in his second school to bring these zones into alignment with his ZPD. I encouraged him to view the single class set of graphics calculators as an opportunity he could exploit, because he was the only teacher who wanted to use them. I also supported him in increasing his involvement in the local mathematics teacher professional association where I hoped he would find a ZPA external to the school that would nurture his potential for further development. Through these quite modest interventions I aimed to help Adam change the way he interpreted his circumstances and gain a sense of agency in his own development.

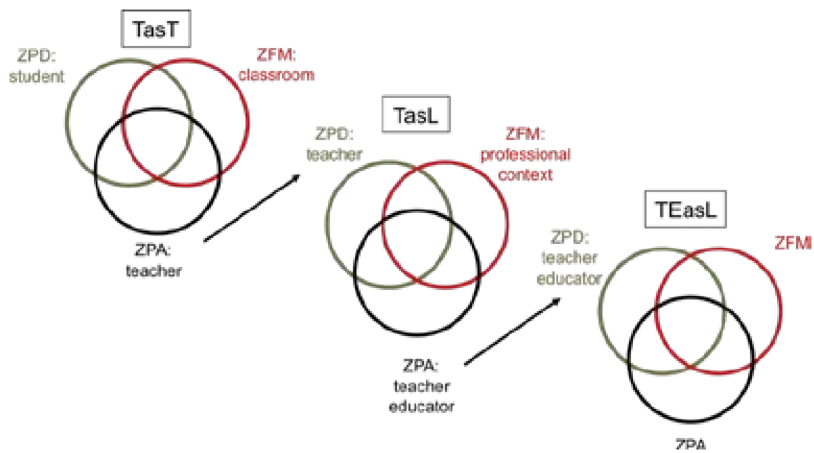
I have also used Valsiner's zone theory to better understand the issues facing experienced teachers who are unfamiliar with new teaching or assessment approaches or with new technologies. I use this theory to deliberately design professional development interventions that take into account not only teachers' knowledge and beliefs, but also with what they believe to be feasible in their professional contexts (e.g., see Goos, Dole & Makar, 2007). Again, my aim is to create a sense of agency in teachers by helping them see how they could view their circumstances differently and recognise elements of their professional context that they can change.

4. USING VALSINER'S ZONE THEORY TO UNDERSTAND THE ROLE OF MATHEMATICS TEACHER-EDUCATOR-RESEARCHERS

Zone theory is useful because it brings teaching, learning and context into the same discussion. The work outlined above shows it can be applied in two connected layers: (i) the teacher-as-teacher (TasT in Figure 5) creating classroom Zones of Free Movement and Promoted Action that structure student learning;

and (ii) the teacher-as-learner (TasL in Figure 5) negotiating the ZFM/ZPAs that structure their own professional learning. At the latter layer the teacher-educator-as-teacher comes into the picture, providing the ZPA. Now let us imagine a third layer, with teacher-educator-as-learner (TEasL in Figure 5). This theoretical extension of the zone model opens up the possibility for investigation of how mathematics teacher educators' knowledge and beliefs define a set of possibilities for their continuing development (ZPD), how their professional contexts constrain their actions (ZFM), and how they experience and benefit from different opportunities to learn (ZPA).

*Figure 5.Three lay***ers of application of zone theory: students, teachers, and teacher educators.**



Let me sketch out what such an analysis might look like by applying zone theory to my own practice in the dual roles of researcher and teacher educator. As a researcher, my Zone of Proximal Development is influenced by my growing knowledge of theories and methodologies within my discipline (mathematics education) and the sub-fields in which I work (sociocultural approaches to mathematics learning and teaching). Disciplinary epistemologies and beliefs shape my ZPD as a teacher educator in much the same way. In many respects, the knowledge needed by mathematics teacher educators is similar to that required of mathematics teachers. According to Jaworski (2008), this includes:

> … knowledge of mathematics, pedagogy related to mathematics, mathematical didactics in transforming mathematics into activity for learners in classrooms, elements of educational systems in which teachers work including curriculum and assessment, and social systems and cultural settings with respect to which education is located (p. 1).

However, mathematics teacher educators also need to know how new teaching practices are learned and the pitfalls associated with promoting this learning. This includes knowledge of how to design teacher education activities, especially activities that connect prospective teachers' learning in the university and practicum contexts (Bergsten & Grevholm, 2008).

Mathematics teacher beliefs have been extensively researched, but the beliefs of mathematics teacher educators have received little attention in studies published to date. As an element of the ZPD, mathematics teacher educator beliefs about teaching and learning are likely to be influenced by theoretical studies and research (Bergsten & Grevholm, 2008), which suggests a need to identify the theoretical and philosophical positions (e.g., constructivist, sociocultural, post-structuralist) that inform mathematics teacher educators' practice.

As a researcher, my Zone of Free Movement is constrained by academic structures and cultures within and beyond my university. These include:

guidelines for career development, identifying activities that are formally recognised and rewarded; mechanisms for managing academic workloads that seek to balance teaching and research; government programs for assessing the quality and impact of university research; competitive research grant schemes; the process of peer review of articles submitted for publication in scholarly journals.

Closely inter-related with these elements of my professional context is the Zone of Promoted Action represented by my initial research training (doctoral studies, early experiences as a research assistant), participation in research conferences and other activities of educational research associations, and formal or informal mentoring by more experienced colleagues. This ZFM/ZPA complex helps shape possibilities for my development as a researcher (ZPD) by defining what is allowed and what is promoted. The learning opportunities that arise in this way are well charted and form part of the enculturation of novice researchers into academic life.

As a mathematics teacher educator, I must negotiate a different zone configuration. Here, my practice is constrained by a Zone of Free Movement comprising the following elements: student characteristics, such as their mathematical knowledge and their beliefs about mathematics teaching and learning; curriculum and assessment requirements that are governed by external teacher registration authorities as well as university course accreditation processes; limited access to technology resources in the university; reduction of the hours allocated to teaching methods courses in the pre-service teacher education program; difficulties in finding suitable practicum placements for prospective teachers; perceptions amongst colleagues that teacher education is low status work.

My ZPA as a teacher educator is less clearly defined in that it is difficult to identify people or activities that explicitly promote my development in this role, and thus difficult to describe the ZFM/ZPA complex that shapes my teacher education practice. Llinares and Krainer (2006) point out that the growth of mathematics teacher educators as learners is a new field of study, and research in this area has so far drawn on notions of reflective practice rather than sociocultural theories that take into account the settings in which practice develops. From a sociocultural perspective, I could say that my own research in teacher education acts as a ZPA that informs my practice as a mathematics teacher educator. My research using zone theory has also influenced how I work with prospective teachers – my own teacher education students – to help them analyse tensions between the learning experiences offered by the university course and the practicum. While this approach helps give coherence to my dual roles as researcher and teacher educator, further elaboration of Valsiner's zone theory is necessary to create a conceptual framework that better explains how mathematics teacher educators learn from research into teacher education.

ACKNOWLEDGEMENT

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An Innovative Integrated Model of School-University Partnership

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ABSTRACT

This study was intended to explore an innovative integrated model for supporting future teachers learning to teach under the impact of teacher education reform of Taiwan, particularly, in the internship. It begins by introducing the change of teacher education reform issued in 1994, followed by the description of the impact of teacher education on quality control. Then, it describes an innovative approach of internship through the school-university partnership. The innovation approach is intended to enhance mentors' knowledge and skill, such that mentors have better ability in mentoring future teachers. The aspects of innovation include the course of the mentoring, the process of mentoring, an integral model of mentoring, and its evaluation of the mentoring program. The characteristics of the partnership are summarized. Several tensions and difficulties emerged under the integrated model of mentoring are described in the end.

Keywords
Mentors, integrated model, internship, school-university partnership.

1. INTRODUCTION

Teacher preparation programs across countries have made considerable efforts to improve the content and the process of the practicum (Fairbanks, Freedman, Kahn, 2000; Field & Latta, 2001; Nichols & Tobin, 2000; Nilssen, 2003; Strong, Baron, 2004; Wang & Odell, 2002). The practicum stipulated allows a future teacher (FT) to have field experience in school settings for an entire school year with the support of university faculty and school teachers (Booth, 1995). Due to fact that the responsibility for mentoring FTs in Taiwan lies with the mentor in the schools who are not subject specialists rather than with the university faculty, so that FTs have little professional learning with school teachers during practicum (National Hsinchu University of Education, 2006).

The Teacher Education Act (TEA) passed in 1994 significantly changed the way that teachers in Taiwan are trained. With the influence of economic, political and social constructs (Fwu & Wang, 2002; MOE, 1994). The TEA brought into law two important changes (MOE, 1994): 1) Teacher preparation can be offered from any institution in which has a teacher education program; 2) School-based practicum is reduced to half year from a whole year; 3) Teachers are certified by the processes of graduating from a qualified teacher certification program (4 years), completing a half-year practicum, and passing a certified teacher examination.

A great deal of teacher education researchers have paid a lot of attention on the studies of teacher preparation, but these studies are limited on the learning opportunities for FTs provided by the teacher preparation program (Huang & Chin, 2003; Lo, Hung, & Liu, 2002; NHUE, 2006). There was relatively little research on the support given to FTs until the privilege of teacher colleges or normal universities for teacher preparation was deprived. The focus of this study is on the effect of an innovative integrated model of mentoring that was designed to improve the skill and knowledge of mentors' in supporting FTs' quality of teaching during the internship of the elementary school-university partnership.

2. THE IMPACT OF TEACHER EDUCATION REFORM IN TAIWAN

Several issues regarding teacher education impacted by the teacher education reform are addressed as follows.

2.1 *Variance in Teacher Training Among Universities*

All four-year public and private universities and colleges are allowed to run teacher education programs as long as they meet the requirements of the MOE. The teacher education program in any university needs to be approved by the MOE which requires the school to meet criteria regarding the staff and faculty, curriculum, and facilities of the program(s). However, the process of instruction, training, and practicum vary with different programs (MOE, 1994). Some programs have inadequate number of faculties, while some lack of practical experience in internship (MOE, 2005). The enactment of the TEA accelerated the number of TE programs set by regular universities from 9 programs in 1994 increasing up to 88 programs in 2006 (MOE, 2005). Due to these circumstances, two teacher colleges upgraded to be a comprehensive university (MOE, 2005). The declining budget of government for higher education and the limited amount of the faculty and facilities made the transformation of universities of education or to seek for compiling into nearby universities (Cheng, 2009; Lee, 2008).

2.2 *Initiation of National Certified Teacher Examination*

Due to the decreasing birth rate (Sheau, 2006), the supply of teachers is much more than the demand. The number of teachers to be prepared is reduced by 50% from 1994 to 2009; when only 9,123 students were admitted to schools of education (MOE, 2009).

Although fewer teachers are being educated more institutions are involved; to control teacher quality, a National Certified Teacher Examination (NCTE) was initiated in 2004. The examination assesses FTs' knowledge of general pedagogy instead of subject matter pedagogy. The items of examination do not assess FTs' pedagogical content knowledge of mathematics.

2.3 *FT's Practical Knowledge Undeveloped in the Practicum*

The practicum provides FTs with an opportunity to develop the professional knowledge but it often results in FTs developing technical skills of classroom management, rather than the wisdom of professional practice (Fwu & Wang, 2002; Huang & Chin, 2003; Lo, Hung, & Liu, 2002). Within ten years, a great deal of studies on teacher preparation show that FTs complained that they are required to devote a great deal of time to administrative affairs of schools, due to the ambiguity of FT's role (Lin, 2007; Lo, et. al, 2003; National Hsinchu

University of Education, 2006). FTs in school placement were neither a student (because of their completion of courses of TE program) nor a teacher (because of no salary). FTs were required by mentors or by school administrators to devote a great deal of time to doing school administrative affairs. The FTs were afraid of rebelling school teachers' authorities because the part of their grade of internship was graded by mentors. This leads to lack of professional learning during the internship. In addition mentees complained that they were mentored by the mentors who did not have enough professional knowledge in mentoring (Lo, Hung, & Liu, 2002; Huang & Chin, 2003; Lin, 2007).

To increase the quality of mentoring, the National Science Council (NSC) associated with MOE called for research proposals. This study was developed under the situation. This study began by constructing professional standards for mentors and for FTs and followed by designing a mentoring program, developing a model of mentoring, and evaluating mentoring program.

4. AN INTEGRATED MODEL OF MENTORING FOR IMPROVING THE QUALITY OF INTERNSHIP

4.1 Courses of the Mentoring Program

The goal of the half-year mentoring program as part of the study was to enhance mentors' knowledge and skills in mentoring. The mentoring program was based on the professional standards of mentors that were conducted by the first year of study (Lin & Tsai, 2007).

The program was divided into two sections: summer workshop and half school-year mentoring practice. The course of each section covered five topics: curriculum, pedagogy, assessment, social mathematics norm, and topics about individual students. Curriculum topics refers to the objectives of instruction, the scope and sequence of the content to be learned, resources of teaching, textbook, and the plans and schedules for teaching. Pedagogical topics involve the discussions on subject matter knowledge, instructional strategies, clarity of explanation, questioning, problem-posing, and analyzing students' various solutions. Assessment analysis is for understand students' performance as well as their progress. The social mathematics norm reviewed the issues about social interaction in mathematics classroom, the norms of groups of students in a class. Individual students included discussions about the background, learners' needs, behavior, and progress of an individual student (Lin, 2007).

The courses of the mentoring program were implemented in a six-day with 36 hours summer workshop; followed by half school-year with 42 hours of instructional time. The summer workshop was to provide a learner oriented conception to mentors' and FTs' for teaching mathematics, while the half-year course was to enhance mentors' knowledge and skills in mentoring and FTs' knowledge of teaching.

4.2 Partnership of University-School

It is not possible to develop FTs' professional knowledge if the mentors' mentoring knowledge and skills have not been well developed (Cobb, & McClain, 1999). Thus, developing mentors knowledge and skill of mentoring is prerequisite before they mentor with FTs. To reduce mentors' tension and burden from their participation in the mentoring program, each mentor was only trained to specialize in one subject by a teacher educator from mathematics department of the university. For instance, the mentors A, B, C, and D were trained to be an expert in mathematics teaching assisted by the teacher educator of mathematics education, while mentors P, Q, R, and S were trained to be an expert in Chinese teaching assisted by the teacher educator from Chinese department.

Four groups involving in the partnership were: mathematics mentors group (MMG), Chinese mentors group (MMG), mathematics FTs group (MFTG), Chinese FTs group (CFTG), displayed in Figure 1. MMG consists of a mathematics teacher educator and four mentors. MFTG consists of four pairs of FT-mentor in mathematics.

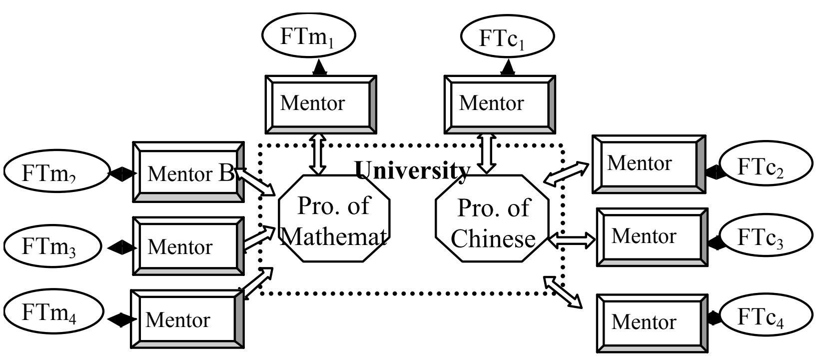

Figure 1: The School-University Partnership

The school-university partnership was designed to assist mentors in developing mentoring knowledge and skills, and then to enhance FTs' professional practice during the practicum. In developing the school-university partnership, there were four main considerations. First, the school to be recruited was dependent on the willing of the mentors and the FTs. Second, the school to be recruited at least consists of the mentors from mathematics and Chinese. Third, the school has a commitment to maximize the FTs' involvement in the community of mentors while at the same time minimizing the possible disruption this participation might cause the mentors and schools. Fourth, some kind of ancillary benefits and feedback for giving back to the school from the university when designing the mentoring program. The collaboration of school and university is depicted in Figure 2.

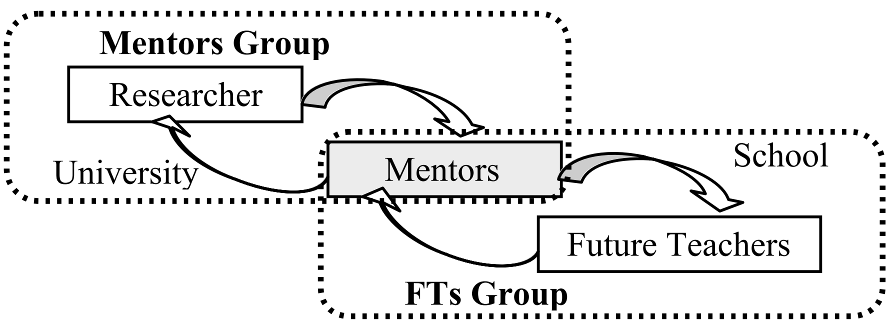

Figure 2: Collaborative Model of Mentoring in School-University

4.3 The Integrated Model of Mentoring

Due to the fact that FTs were to be a primary school teacher who teaches several subjects, but mathematics and Chinese are required subjects to be taught by a home-room teacher. To this end, an integrated model of the mentoring was developed in the study.

The model was called one-subject mentors with multiple-subjects future teachers (OSM-MFT). It means that each mentor was only trained to specialize in one subject by a teacher educator of the university, while a FT is trained in all subjects from two mentors who are interested in mathematics or Chinese, as in Figure 1.

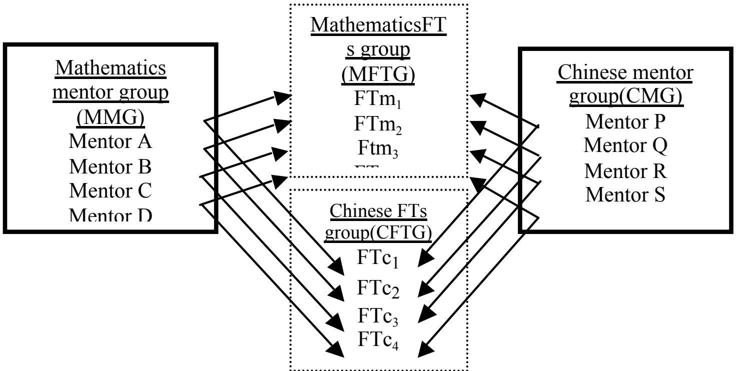

Figure 3: The OSM-MSFT Model Between Mentors and FTs

Figure 3 reveals was that each participant FT was mentored by a mentor in MMG and mentored by another mentor who is in CMG. Each FT in the mathematics group was mentored by a mentor from MMG and a mentor from CMG..

The integrated model took the critical constructivist perspective on mentoring, that knowledge is actively built by learners through the process of active thinking (Wang, & Odell, 2002). The teacher educators and the mentors were viewed as learners and generators of new knowledge and practices of mentoring. Likewise, the mentors and the FTs were also viewed as learners and generators of new knowledge, and they had to count on each other. The integrated model stressed mentors' active construction of mentoring knowledge through what they have leaned in practice and constant dialogue with teacher educators. There was a one-hour classroom observation on every Thursday morning and a follow-up three-hour mentoring group meeting in the afternoon throughout each phase of the mentoring program. Each mentor was required to immediately share with FTs the main ideas discussed in the MMG meeting.

Scheduling proved to be a challenge as two mentors needed to be available for each group of mentees: FTm1 and FTc1 in Figure 3 for example. FTm1 and FTc1 were arranged to present in mentor A classroom simultaneously to watch mentor A's lesson, and also appeared in Mentor P classroom simultaneously to watch Mentor P's Chinese lesson at other time. The mathematics class of these two mentors was arranged at the same time on the course schedule. It is the same for Chinese class. Both FTm1 and FTc1 always appeared altogether in the same classroom at the same time.

4.4 Four Phases of the Mentoring

Four mentors participating in the study had no experience in mentoring. To help them put their visions for mentoring into practice, the mentors were supported in four phases.

Phase 1

The first phase was two weeks long and involved providing mentors support with the concept of induction through mutual sharing amongst mentors the teacher educator. The mentors were provided with techniques to offer emotional support for interns to reduce psychological stresses caused by the conflicts between their personal lives and professional requirements. Each mentor took turns to report in public how the introduction of the intern to students and parent was accomplished in the first few days of the school year. Each FT was asked to report their feelings about how the introduction was handled by the mentor.

Phase 2

In the second phase, from week 3 to 6, each mentor was asked to teach several lessons for FTs in their own classroom. Before teaching each mentor would explain the purpose and method of the lesson so the FT could observe the lesson with greater understanding and purpose. In this way, each FT could see how their mentor taught a lesson. It was followed by a short conversation with the mentor concerning the relationship between the syllabus, the lesson plan, and the lesson actually taught. This phase provided the mentors an opportunity to support FTs on learning how to observe a lesson which was learner focused reinforced that the mentors had learned the teaching approach.

Phase 3

The third phase, from week 7 to 10, teacher educator supported the mentors and FTs as they worked together in preparing a lesson and a peer observation (called as LPPO). The process starts with the FT observing a mentor preparing a lesson and then observing the mentor teach the lesson. This was followed by other mentors' observation on how the mentor carried out the lesson, and then observing the mentor asking the intern a series of questions, such as explaining how well the lesson plan was carried out, how well the objectives she have achieved in the lesson, identifying the changes she made in the lesson compared to the lesson plan. During the third phase, other mentors not only learned from the mentor-intern relationship but also other mentor comments about the mentoring process, lesson plan and teaching, but also gave the mentor comments or suggestions on mentoring. Each mentor-intern pair took turns engaging in the activities of LPPO. The FT of each pair was asked to report what she learned in the activity of LPPO.

Phase 4

From weeks 11 – 14 each FT participated in teaching of classes. During this phase the mentor was a passive observer, assisting only as needed. The goal of this phase was to observe the impact of the mentoring on FT mathematics teacher performance. During this phase, each FT was evaluated by other FTs, mentors and a researcher. The evaluation of mathematics teaching consists of two aspects: teaching preparation and teaching behavior.

5. DATA COLLECTION AND ANALYSIS

Data collection consists of both qualitative and quantitative data. Pre- and post-tests were given to all participants. Mentors completed a self-assessment of the professional standards, and a survey regarding the workshop and mentoring practices. The summer workshop survey asked participants to rate the contents of the course.

Each FT's teaching was assessed according to the lesson preparation and teaching behavior. The indicators of lesson preparation include 7 items: understanding instructional objectives, structure of materials, mathematics content, readiness of preparation, activities building on students' pre-experience, adaptation of teaching activities, and lesson plan.

The effect of the integrated model of mentoring is organized at three levels in accordance with the model of Kirkpatrick and Kirkpatrick (2006). At the reaction level, the mentors were interviewed on the feedback of summer workshop and half-year school mentoring activities for measuring what they thought and felt about the program. At the learning level, pre-test and post-test were conducted aligned with self-assessment 5-scale questionnaire professional standards, to assess the extent to which mentors change attitudes, improve knowledge and skill. At the behavior level, classroom observation, interview, and mentors' mathematics journal were measured how mentors transferred their knowledge and skill in mentoring as a resulted of the mentoring program. Each mentor was also conducted individually with a semi-structure interview.

6. EFFECT OF THE INTEGRATED MODEL OF MENTORING

The effect of the integrated model of mentoring includes the participants' valued to the model and their reactions to the mentoring program.

6.1 Mentors' and FTs'Valued the Integrated Model

All mentors were committed to the integrated model because this model created the opportunity for them to learn a new pedagogy for teaching Chinese from their FTs who participated in the CMG. Conversely, the Chinese mentors have the same agreement. Mentors also mentioned that two FTs working with each mentor had greater potential to stimulate multiple perspectives than only one FT working with each mentor. The suggestion of the model the mentors made was that the two FTs worked with two same grade mentors since their concerns had the same focus.

For FTs, the integrated model afforded them rich professional learning. For instance, when creating a lesson plan FTs learned to create a strong lesson plan for effective teaching, including predicting potential responses from students and how to follow-up on those responses by preparing questions. The FTs learned to pay more attention to the sequence of the activities to be taught. They also learned that the sequence of the activities relied on the objectivities of the lesson, the context of the problems to be posed, the numbers involving in the problems, and students' prior knowledge.

6.2 The Effect on Mentor Learning

6.2.1 Reaction Level: Mentors' Satisfaction with the Course of Mentoring Program

The results show that all four of the mentors were satisfied with all topics covered during the summer workshop and half-year. The mentors had slightly less satisfaction with the lesson plan engaged in the school year ($\overline{M}$ =4.5) than in the summer workshop ($\overline{M}$ =4.25). Su made the comment on lesson plan as follows.

> ....What I learned in design of lesson plan in summer workshop was about the essential components, such as students' anticipated solutions, prior knowledge, objectives of the lesson, and key questions to be asked. Based on this experience, it helps me to move to observe how Juei worked with her

assigned FTs on planning a lesson and then wrote it into a lesson plan. I saw that Juei asked her FTs to read the textbook and search for relevant resources in advance. She asked them to make sure of the objective of the lesson and to be aware of the need of adaptation of the activities covered in the textbook.

6.2.2 *Learning level: Improvement of Mentors' knowledge of teaching and mentoring*

Regarding the knowledge of teaching, the percentages of pre- and post-test four mentors performed increase from 40% to 80%, from 53% to 80%, from 40% to 73%, from 40% to 67% respectively. The result indicates that the mentors enhanced their knowledge for teaching fractions because of what they learned in the program.

With regard to the conception of mentoring, initially, in their view of FTs' expectation for the role of mentors was to provide emotional and technical support. Learning to teach, in their view, was to be left FTs' own accumulation of teaching experience and lessons based on trial and error. Their lack of knowledge was clarified their responses to self-assessment questionnaire. Before entering the program, the mentors had no confidence in performing 7 items out of 16 items (termed as 7/16) of professional literacy, 18/34 items of mathematics teaching, and 22/36 items of mentoring practice, respectively. Through the process of mentoring, they gained more confidence in teaching and mentoring. The post program survey found that only 5 items; 2 items of teaching and 3 items of mentoring were not improved. The positive impact was note by Juei, who was pleased to her more awareness of problem-posing.

6.2.3 *Behavior Level: Transfer occurred in Mentoring FTs*

The mentors transferred their knowledge of teaching into their mentoring practice. The transfers of problem posing and lesson plan are presented here. The aspects the mentors attended to when working a lesson plan with FTs from Phase 2 to Phase 4 of the mentoring program are described in Table 1. Table 1 shows that the mentors expanded their perception of lesson plan and improved their ability to help FTs in writing a lesson plan. Comparing to Phase 2, two more aspects the mentors learned from the mentoring program on preparing a lesson were the scope and sequence of the mathematics contents and students' various anticipated solutions. They tried hard to ask FTs to put the possible key and follow-up questions on their own lesson plans.

Table1: Aspects of Lesson Plan the Mentors Attended to in Different Phases of Mentoring

Phase 2	*Phase 3*	*Phase 4*
Objectives of the lesson	Objectives of the lesson Objective of each activity	Objectives of the lesson Objective of each activity
--	Analysis of the scope and sequence of the content	Analysis of the scope and sequence of the content
Pupils' prior knowledge	Pupils' prior knowledge	Pupils' prior knowledge
Status of the lesson	Status of the lesson	Status of the lesson
Sequence of the activities	Sequence of the activities including the problems to be posed	Sequence of the activities including the problems to be posed
The setting	The setting	The setting
Instructor's activities	Instructor's activities with key & follow-up questions to be asked	Instructor's activities with key & follow-up questions to be asked
--	Students' activities including anticipating students' solutions	Students' activities including anticipating students' solutions

7. DISCUSSION

With reconceptualizing the meaning of a school-university partnership, the integrated model of mentoring provides some evidence for the crucial importance of the mentor in the development of the FTs' professional learning. It gives the view that simply placing FTs in school without adequate mentoring support would give FTs little chance to develop their classroom teaching skills and understanding. The teacher educators of a university offered the support with an integrated model of mentoring for mentors in school. However, there were several tensions and difficulties which emerged under the integrated model of mentoring.

Although the mentors and FTs agreed to participate, many commented that they were not given enough detail on the nature of the program. Initially the mentors showed hostility due to a belief that they now had additional work. They struggled with the additional work and the improvement of professional knowledge. However, the factors of additional work appeared not to play a significant part in influencing mentors choosing to take on the role. Gaining professional knowledge and professional confidence became an internal incentive. The difficulties mentors encountered in the integrated model included additional work, tight schedules, and lack of cooperation from FTs. Likewise, additional work and tight schedules were the difficulties for the FTs during practicum. The willingness of FTs participating in the integrated model of mentoring drastically decreased as time passed, since they have little opportunity to become an initial teacher in the school. Some of the FTs who planned to transit their profession to other occupation lacked professional engagement during practicum.

The finding of the study revealed the FTs' and Mentors' satisfaction with the course of mentors and FTs in the practicum through the integrated model of the collaboration of university and school. This indicates that the successful model has the following characteristics: (1) The partnership of university and school is based on a model of team-work between mentees and mentors, and teacher educators who supervise them. (2) We treated FTs not only as students but as members of the profession. (3) The integrated model is school-led in the sense that mentors in schools take the main responsibility for FTs and supervisors in university take the main responsibility for mentors.

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The Concept of Identity Positioning the Self within Research

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INTRODUCTION

This paper interrogates the concept of identity as it plays out within the research process. It engages general debates about the production of knowledge and, within that, more specific debates about reflexivity and the place of one's own subjectivity in the research process. Situated beyond past scientific pretentions, it attempts to take into account the place of emotions and unconscious interference both in relation to the researcher's own subjectivity and in relation to intersubjective relations between researcher and research participants, for understanding the practice of research. It begins with Lincoln and Denzin's (2000) vision of qualitative research as "simultaneously minimal, existential, authoethnographic, vulnerable, performative and critical" (p. 1048). The focus is on performing the self as researcher, both within the data gathering process and in the construction of research reports.

The performance of self as researcher is not a new theme of course, since writing oneself into the research is, if not celebrated or embraced as it is in much feminist research, it is at least condoned in mathematics education. Putting the researcher into the research is considered a way to move beyond subscribing to a particularly modernist set of assumptions informing conceptions of what it means to know and what it means to know others. This is a set of assumptions to the effect that researchers are able to put themselves in another's (participant's) place and know his or her circumstances and interests in exactly the same way as she or he (participant) would know them. Following on from those kind of understandings comes the belief that researchers will be able to produce "paradigmatic instances of the best knowledge possible, for everyone, in all circumstances" (Code, 1995, p. xi).

This way of thinking has come under interrogation from Foucault (1972) who has provided a critical analysis of how the particularly powerful modernist discourse determines who has access to the production, the distribution, and the legitimation of knowledge. The disruption of what Derrida (1976) has called the end of 'pure presence' has represented an immense challenge to researchers in mathematics education. For one thing, objectivity has been close to many a researcher's heart. Giving up control and mastery and the understanding that knowledge is made by the abstract, interchangeable individual (researcher), abstracted from the particularities of his or her circumstances, has forced us to think about a practice that would acknowledge researcher complicity in the research process. For another thing, it has required us to reassess concepts like reliability, generalisablity and validity that are part and parcel of the classical *episteme* of representation. To this end some have chosen to write themselves into the research—to make their core researcher self visible and voiced.

In this paper I am attempting to understand identity and, specifically, what it is that structures the narrative experience. In that attempt I have two main objectives. One is a theoretical interest that involves examining the issue of subjectivity and how intersubjective negotiations take shape in relation to data gathering and the construction of research stories. Foucault's understandings of how subjects are produced within discourses and practices, Lacan's arguments about narratives of the self, and Zizek's related examination of how subjectivities are constructed across sites and time have all been highly influential. Their work tells us that self-conscious identifications and self-identity are not simple, given, presumed essences that naturally unfold but, rather, are produced in an ongoing process, through a range of influences, practices, experiences and relations that include social, schooling and psychodynamic factors. This brings up the issue of emotion and unconscious processes. I propose that a conceptual frame derived from this body of work offers a way of understanding a sense of self that is simultaneously present, prospective and retrospective, as well as rational and otherwise.

A second objective is to speculate what these understandings of the researcher's subjectivity tell us about the production of knowledge. Using data from my own research on girls in mathematics schooling, I place my 'self' under scrutiny as I explore the multiple layers of performing the art of research. This is the point where the interest moves from establishing truth onto an understanding of how meaning is produced and created and, specifically, in how these productions are influenced by fictions and fantasies. My purpose in doing this is to keep the research conversation going and specifically to accommodate the researchers' subjectivity, intersubjective negotiations, and the place of emotions and unconscious interference in these two, in performing the art of research.

CONFRONTING KNOWLEDGE PRODUCTION

We have come a long way from wholesale acceptance of the canons of truth and method of research. To date, albeit in small bites, the criteria for evaluating and interpreting educational research have been questioned and this has informed a revised thinking about the concepts of legitimacy and representation. More specifically, it has led to a reevaluation of the idea that researchers are able to capture lived experience—that they are able to speak on behalf of others. This heightened sense of awareness of the limits of research to explain social relations has crystallised into alternative research reporting approaches and new forms of expression. Steering a middle course between supporting long-held epistemological and ontological preoccupations that prop up the search for reality, and an effort to understand the conditions of knowledge production itself, research in the social sciences has scrutinised the place of the researcher in the research process. They have recognised the researcher's position of privilege in knowledge construction and transformed it into "to a more self-conscious approach to authorship and audience" (Coffey, 2003, p. 321).

Taking the lead from social science, scholars within mathematics education began to suggest that it is not enough to connect the researcher to the questions, methods, and conclusions of any research, but that such a relationship should be avowed and should be made transparent (see Burton, 1995, 2003; Cabral & Baldino, 2004). In writing the reflective self and research voice into research texts, contemporary work in the social sciences has emphasised the negotiation, physicality, and crafting of personal relationships within the research encounter. Driven by an epistemic responsibility to get perceptions 'right', the researcher seeks "the courage not to pretend to know what [she] does not know [and] the wisdom not to ignore its relevance" (Code, 1988, p, 191). Reflexivity, in these accounts, has become a methodological resource for authorising the researcher's self into the account.

> …the researcher-self has become a source of reflection and re-examination; to be written about, challenged and, in some instances celebrated. In more general terms, the personal narrative has developed as a significant preoccupation for many of those who espouse qualitative research strategies… There is an increasingly widespread assumption that personal narratives offer uniquely privileged data of the social world; personal narratives (re)present data that are grounded in both social contexts and biographical experiences. The personal narratives of the researcher have formed part of this movement, to be told, collected and (re)presented in the research and writing processes. (Coffey, 2003, p. 313)

Theoretical and methodological issues to do with the concept of the self and its textual visibility have been critiqued on a number of fronts (e.g., Adkins, 2003; Brown & England, 2004, 2005; Walkerdine, Lucey, & Melody, 2003). Such writers take pains to emphasise that there is no core self; instead the "self, like those of the research participants, is created as both fiction (in the Foucauldian sense) and fantasy" (Walkerdine et al., p. 180). It is an effect of the experience of interacting with social groups, cultures and institutions. One appropriates different 'selves' in relation to those interactions. In this line of thinking, giving the researcher a voice, as a methodological practice, resonates with Beck's (1992) notion of 'reflexive modernity', in which individuals seek out by strategic means a coherent life story within a fractured landscape. The claim that reflexive forms of action are demanded from contemporary life has been fiercely debated (e.g., Adkins, 2003; Skeggs, 2004; Walkerdine, 2003), not least because the reflexive self is based on a foundational conception of the human subject, and hence much too cognitive in nature (see Adkins, 2003).

The terms of the reflexive researcher debate centre around the tendency to believe that the addition of a researcher layer to the narrative has the effect of countering the effects of power, privilege, and perspective, and believing that it does this by "guarding against over-familiarity and the effects of context on the relationships that are formed in the field" (Coffey, 2003, p. 314). The claim that reflexivity has occurred is counterclaimed with the insistence that the insertion of one's self into the account fails to engage the very problem of narrating experience, neglecting to ask what is it that "conditions and structures the narrative impulse" (Pitt & Britzman, 2003, p. 756). As a version of the rational actor the reflexive self clearly does not have the effect of making relations between the researcher and participant transparent. The self tends to "move uncomfortably between the individual and the social or cultural without resolving, or satisfactorily exploring, the tensions inherent in this tussle" (Bibby, 2008, p. 37).

None of this is to suggest that the researcher should remain an invisible participant. Abandoning the practice of researcher reflexivity is not the objective here. Nevertheless, drawing attention to the implicatedness of the researcher in the production of knowledge primarily through the researcher's personal story, does not tell us the full story. In understanding the subjectivity of the researcher, the subjectivity of the participants, and the intersubjectivity of the two, out of which the research account is produced, other factors are crucially important. The place of emotions is a case in point. What needs to be emphasised here is that the concept of the authorial self, held in place so that the voice might surface, has been found wanting.

UNDERSTANDING IDENTITY/SUBJECTIVITY

In taking the authorial self to task, it is helpful to think of the subjectivity of the researcher as involving identifications, relationships and experiences, that are not in any way straightforward, but are rather, "mediated by multiple historical and contemporary factors, including social, schooling and psychodynamic relations" (McLeod & Yates, 2006, p. 38). What are being raised here are questions of a fundamental epistemological nature. The bad news is that the theories that we typically use in mathematics education do not tend to deal with such issues. We can't draw on a single theory to capture and explain subjectivity as a discursive constitution and to explain relations between positionings that work in contradictory, conflictual and emotional ways. The good news is that it is possible to work with a number of conceptual tools that allow us to deal with the complex interplay between hierarchies of social categories and the processes of self-formation that are at work in the practice of research.

One of the ways subjectivity has been explored in recent scholarship is through spatial metaphors that model research as a *space* that seeks to define and monitor subjectivities. Research constructs particular positionings for people and both creates and lends coherence to the understandings that those in the research process construct of themselves. Within the practices of research, researcher subjectivity is historically and situationally produced in relation to a range of constantly changing processes. In scholarship that draws upon these understandings (e.g., Blunt & Rose, 1994; Keith & Pile, 1993; Pink, 2001) the notion of a 'real' identity or 'true self' is an illusion. Pink (2001) elaborates that the "self is never fully defined in any absolute way,…it is only in specific social interactions that the…identity of any individual comes in to being *in relation to* the negotiations that it undertakes with other individuals" (p. 21).

We can draw on Foucault (e.g., 1984, 1988) to explore the dynamic self/social spatiality. For him, identity is historical and situationally produced; it exceeds singular definition precisely because it is always contingent and precarious. His concept of discursivity allows us to make connections between social process and individual biography. In Foucault's (1977) formulation, discursive spaces trace out what can be thought, said and done by providing people with a viewpoint of the social and natural worlds. They are, above all, knowledge producing systems (Walshaw, 2007). But describing how the subject is produced and regulated in multiple and contradictory discourses, is not the same as subjectivity—the condition of being a subject.

Understanding how this process operates for the researcher and researcher participants requires conceptualising how they live their subjectivity at the crossroads of a range of often competing discourses. In searching for a theory of the self that can offer a model of interpretation that extends beyond the historical and personal, I have found psychoanalytic theory particularly helpful. Arguably, psychoanalysis has many shortcomings, yet the theories of scholars, such as Lacan and Žižek, provide us with the tools for understanding the self in relation to social, cultural and psychic processes (Britzman, 1998; Ellsworth, 1997; Evans, 2000;

Felman, 1987; Jagodzinski, 2002; Pitt, 1998; Walkerdine, 1997; Walkerdine, Lucey, & Melody, 2002). Grosz (1995) maintains that psychoanalytic theories are "wide-ranging, philosophically sustained, incisive, and self-critical" (p. 191) and offer complex and well-developed theories of subjectivity.

Subjectivity, for Lacan, is not constituted by consciousness. Rather, conscious subjectivity is fraught and precarious. For him, the reduction of interpretation to conscious experience covers over the complexity in which researchers find themselves. Methodologically, the Lacanian understanding of the self highlights the difficulty in producing a research account that tries to avoid problems concerning speaking for others, even when the researcher exercises reflexivity about her relation to the research participants. If, as Lacan suggests, the unconscious is the place where our sense of self is developed and the place where we find out the kinds of interpretations that we can make (Lacan, 1977a, 1977b), what does that mean for the subjectivity of the researcher and, for that matter, the truthfulness of her research report? Is it possible to tap into unconscious levels of awareness? How can we deal with these issues systematically?

WORKING WITH SUBJECTIVITY

The discussion that follows focuses on two episodes taken from my own research practice (Walshaw, 2005, 2006a, 2006b). It focuses on the subjectivity of the researcher and the subjectivity of research participants. The ideas the examples embrace are used as a counterpoint to current thinking about researcher reflexivity and as a potential vantage point for highlighting the centrality of emotion in the research process. The analysis acknowledges Valero's (2004) argument that "the practices of 'practitioners' intermesh with the practices of 'researchers' and the role of the researcher evidences their mutual constitutive character" (p. 50). Drawing out instances from the two projects referred to above, I have tried to develop a coherent line of thinking that systematically deals with traces of recognition and misrecognition and in which issues of transference and defence come to the fore.

Understanding who I am and who you see

We start with an interview with a group of girls [aged 11] conducted in a committee room in the school's administration block before the lunch break during a regular day. The specific group under investigation comprised a cohort of four girls all of whom had attended in the first year of the *Girl Power* study a small urban school servicing a low socioeconomic population. The following year into the study, the girls all moved as Year 7 students to an Intermediate school for the next two years in the same locality. This is the customary practice in New Zealand where this study took place. The latter school's roll was approximately three times the size of their primary [elementary] school. Like the primary school, it attracted students from an ethnically mixed urban area.

The previous year I had spent three weeks observing and recording in the girls' mathematics classroom. I had interviewed them individually and had also interviewed their mothers. Now, another year on, I was seeking a group interview from them. The girls familiarised themselves with the audio recording equipment before the interview by asking each other questions and playing the recording back to the group. They had a lot of fun in doing this and as a consequence I prepared myself for a productive interview. The interview schedule dealt with questions about the classroom. I told them that what I was interested in the group interview were the students in the classroom —the boys and the girls. What do the students do and how do they behave?

Shanaia opened the conversation by saying:

> Shanaia Well, the boys, they're just like the most disgusting boys I've ever met on the earth 'cause you know last year at primary school the boys were a lot more behaved, but the ones in my class they're just disgusting, farting on peoples' desks, throwing bugs in your hair and doing everything.

This was not exactly what I had expected to hear. To be frank, I was taken aback, downright shocked, that a student would talk in this way to someone who, I imagined, they thought embodied respectability and authority. My classroom observations did not substantiate Shanaia's claim. We will consider this extract from the position of Shanaia, as research participant in a group situation, and also from the position of me, as researcher. The interview provided Shanaia with a power and a voice to oppose masculinities confronted in the classroom and to assert herself as "more mature and educationally focused than the boys" (Reay, 2001, p. 157). Through her words about what is 'normal' and 'not normal' gendered practice in the classroom, she produced an image from her previous classroom of the normal, conforming male student. Precisely because she was well aware from the study's Information Letter that I was interested in

Girl Power, it is possible to understand her response as produced in relation the popular media discourse of female power and to what she fantasises I wanted to hear. The fantasy is built around complex social processes, involving the public, parents, schools, and the media, and in particular, an obsession in the popular press with falling standards that have punctuated societal understandings of young persons' behaviour. It is easy to read the same critical assessment of young people's behaviour as "out of control and a threat to the moral order" (Lucey & Reay, 2000, p. 193) that is given an airing in the public arena.

Yet I am feeling most uncomfortable about the response. Shanaia has assigned an identity position to me to which I cannot identify. Perhaps her intention is to shock? I do not participate in a network of social discursive practices in which language such as 'farting' is typically used. Nor, do I imagine, do the teachers. Lacan's Symbolic identification places me in a particular positioning from where I am being observed by Shanaia and the rest of the group. That is to say, coming into this school as researcher has foregrounded a particular subjective position. Yet the self-as-researcher that has been designated for me through a cultural and hierarchical order, is merely a fabrication that exists in the space between the girls and me.

What images do I have of myself in this context? What images do I choose to identify with? Because I had no desire to set myself apart from the teachers at this school, I had taken steps to 'fit in', such as deliberately 'dressing down', 'talking the talk' of the teachers, and being discreet and unobtrusive in the classroom. It is the visual-spatial images (and the illusion) of my place in this school as 'fitting in' that represents what I would *like* to be at this school during this interview. There is a conflict in this image I hold of myself in that I am still the researcher in this interview and there is no escaping from the symbolic identification assigned to me. The Symbolic works with the Imaginary to inform my experience of self in this context. The two Lacanian registers worked together, shaping my conflicting experience, producing anxieties and defences about what I was hearing and about the direction that this interview might take. They also worked together to inform the kinds of interpretations I made about the contents of the interview and the 'truthful' account that I subsequently produced.

It is my contention that the fantasies, defences, and anxieties, operating to deal with self-image, conflict and contradiction in this episode, lend support to the notion that subjectivities are multiple and continually in motion. What does the notion of multiple subjectivities mean for the notion of reflexivity? In speaking about the researcher's multiple subjectivities and in taking account of emotions and non-rational processes, we go against the grain of speaking about the core self embodied in reflexive researcher accounts. In that the stories that the researcher and the participants tell are often not thought about and told through rational deliberation, the notion of reflexivity is seriously undermined. Unconscious processes on the part of the researcher, on the part of the participant, and within the space between them, will always intervene.

Understanding the self-in-conflict

The second instance is taken from the research on girls and mathematics (1999) in which Rachel is talking to me about what it is like to learning calculus for the first time in Mrs Southee's classroom. She had expressed an immediate, enthusiastic interest in participating in the research. Mrs Southee, too, had indicated Rachel that would "likely be considerable interest" to my research. Rachel presents as lively and fun-loving. Her liveliness contrasted with the 'sophistication' and 'poise' of the other girls in this class. She has an infectious laugh. "Giggly", is how Mrs Southee put it. Every mathematics lesson, she sat herself at the same desk in the middle bank of paired seating arrangements at the front of the classroom, alongside her friend Kate. As Year 10 students, the two of them were the only two 'extension' girls in this Year 12 class, and as such, is obliged to wear school uniform. I could not find myself completely in her giggly disposition, yet, as researcher, I could identify with being an 'exotic other' in her mathematics classroom. It is with regard to 'being different' in the mathematics classroom that I felt a powerful empathy with her story.

Rachel has just told me about her previous year's success with mathematics and how her achievement promoted her to this class. She explained:

> I just seem to be good at doing exams. I've got a lot of friends—they know the stuff in class and I could sit there and it goes right over my head. But I get into an exam and I'm surprisingly clear-headed and a lot of people just get stressed out about it and I don't. It doesn't worry me because I think if I go in there and I don't know it then I don't know it. There's nothing I can do about it so there's no point in worrying. But I did, I worked quite hard last year. I spent ages going through the pink *Mathematics Workbook* and I was going over and over and over it. Trig [Trigonometry] was the worst bit. I couldn't do trig last year, and then like two days before the exam I was looking at it

> and it finally clicked. I spent about six hours just on trig that day and right at the end I just got it, and my parents were trying to make me go to bed and, no, I'm really understanding this. I'm not giving up now. I just did a lot of study. Always read and do examples. Working out answers, checking them and making sure, and if I don't get it I go back and try and figure it out and if I still don't get it I get my brother to have a look at it or I ask someone at school the next day.

As researcher listening to her story, I have an understanding of Rachel's mathematical 'experience' as fixed and immutable. She is able and she is motivated to learn. I have in Grosz's (1990) words, "branded" her, with "the marks of a particular social law and organization, and through a particular constellation of desires and pleasures" (p. 65). I wanted to hear about her good fortune, and her achievements. I had deliberately chosen her as my 'case' in order to question the assumptions typically held about girls in mathematics. I wanted to provide evidence that research founded on those assumptions, while it claimed to tell the truth about girls, in fact regulated them and overlooked other important aspects of subjectification which cannot be contained within that discourse. An 'extension' student's story, I believed, would problematise normalized gender patterns in mathematics. Through her accomplishments she would reveal how it is possible to subvert the status quo and how to 'do gender in mathematics' differently.

As she began to tell me what mathematics is like for her this year, there was a sense that Rachel's self was a fabrication—a fiction (in the Foucauldian meaning), changing moment by moment within the structures of the discursive situation in which she is located. I found it difficult to understand that the self that she was telling me about mathematics this year, was the same self in the narrative a few moments previously.

> …Mrs S, she tends to go right over my head and I don't tend to ask questions from her because last time I did that she tried to explain and it just went, well, I sort of understood half when I asked the question and by the time she'd finished I understood none of it! I don't know. But I don't have a very good relationship with her, because we've had a few arguments in the past. My auntie works in the music block and she really likes Mrs S but, the guys, they know that I laugh really easily and they keep making me laugh in class and she just

> gets really frustrated with me because when I start laughing I can't stop and so she starts to get really angry at me. And apparently no one has ever heard her raise her voice before she met me. So it's a bit stressed there. I'm just trying very hard not to let the guys get to me now. Then I don't have to laugh.

Listening to her story I felt deeply dismayed. In my understanding, Rachel was a bright and capable student, caught up in practices and discourses that prevented her from succeeding in mathematics. I felt upset that she was the victim of surreptitious classroom practices that appeared to create a detrimental effect on her achievements and on her sense of self. I imagined in broaching the issue, she wanted me to know her pain; that she also wanted me to continue this line of conversation. But would pursuing this issue mean that I became caught up in situation which was beyond my powers or role to address? Who am I listening to her story? Who does she see me? I attempt to put my identity outside of myself; into the image of myself. Yet I cannot determine that image. Feeling wedged between a rock and a hardplace—between being impartial non-involved researcher, on the one hand, and caring about her wellbeing in mathematics, on the other—I opted for further clarification as a way of dealing with an uncomfortable experience.

> [MW: The boys who sit behind you?] Yea. Mostly, Blair and Richard, he's one of the bad ones as well.

> [MW: The girls in the class don't stir you up?] No. Because the only one I really talk to is Kate. Blair—he just likes really to get me in trouble and he has done for the last three years and he'll just keep on doing it and there's nothing I can do so I just try not to sit in front of him. And hope that he doesn't sit in the row behind me …

Rachel's story is full of contradictory mathematical experiences. It is told within the space that both of us share in interview and hence cannot escape the effects of her own desire to relate a coherent and compelling account that allows me, the listener, to attempt to understand. Thus at one level the story is a construction of a personal mathematical biography that develops, through a set of thematic clusters to do with success and peer and teacher-student conflict. And, at another level, the account registers disruptions and tensions that have the effect of undermining the coherent and cohesive story. In looking beyond the literal reading of what she said, her story evokes traces of other events and interpersonal relations that create a counter story to the one related to me at this moment in time. Together these two 'stories' open up important aspects of her subjectification as it relates to being a female senior mathematics student.

Rachel sees herself as simultaneously able and struggling in mathematics. I see her as victimised. What needs to be emphasised here is that between the identifications she, and others, like me, have of her, there will always be a divide. There is always a trace of mis-recognition that arises from the difference between how one party perceives itself and how the other party perceives it. As a consequence, Lacan maintains, the very existence of the subject consists of closing the gap between images received within the Symbolic and Imaginary realms. Both Rachel and I, during the course of the interview, worked independently at closing the gap. As Žižek (1989) as put it: The subject "put(s) his identity outside himself, so to speak, into the image of his double" (p. 104).

CONCLUSION

Research is about performing an art. It has a lot more to do with fictions and fantasies than we might suspect. In working towards a theoretical understanding of the researcher's self, issues of emotion and unconscious interference have come under scrutiny for the part they play in the subjectivity of the researcher, the researched and in the space they both share. It has been argued that the performance of self as researcher is about a discursive positioning that is constantly changing, in relation to the discourses and practices researchers find themselves within, and in relation to their intersubjective relations with the researched. 'Intersubjective relations' are not mean to convey simply those relations operating at the conscious and accessible level of awareness. They are intended to include the emotions and unconscious processes. In my formulation of researcher self, fictions and fantasies play a central part.

If it is axiomatic that non-rational connections get caught up in the research account, then where does this leave current accounts of reflexivity or the authorial self? I would suggest that accounts that write the researcher into the process or that practice reflexively speaking for others, promise more than they can deliver. An alternative that significantly enhances the practice of reflexivity and the practice of writing oneself into the research, is to begin with tools taken from psychoanalysis and to acknowledge the intrusion of the self in all research endeavours. In describing episodes taken from specific research encounters, I have provided a first steps approach at what this understanding might mean for methodology—how we might begin to confront, rather than slide over, the delicate issue of emotion within the research process. The approach offered a way to understand processes within the research encounter that give form to difficult, contradictory or conflicting experiences from the past, the present and even those anticipated in the future.

Subjectivity is the cornerstone of the research encounter. Centralising subjectivity in the research process means just that. It means that the researcher can never hope to be detached. Talking about researcher bias is not a particularly fruitful exercise and this is because the subjectivity of the researcher is always implicated in the complex and dual-pronged research encounter. The researcher self is always performed in and for others. Methodologically, the researcher can never truly know what she is seeking and why, because "the fictions of subject positions are not linked by rational connections, but by fantasies, by defences which prevent one position from spilling into another" (Walkerdine, Lucey, & Melody, 2003, p. 180). Our research accounts need to acknowledge that research is more than the elements of trust, doubt, humility, and power. It is about fictions and fantasies and the complicity and fragility of these in relation to others.

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The Notions and Roles of Theory in Mathematics Education Research

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The rationale for this Survey Team (ST), commanded by the International Program Committee of ICME 11, is that:

> Notions and concepts of theory play key roles in mathematics education research, as they do in any scholarly or scientific discipline. On closer inspection, the notion, concept, and nature of what is termed "theory" in such research are very varied indeed, as are the roles, uses and implications of theories employed in mathematics education research. In other words, the term "theory" does not have one universal meaning in our field. Moreover, concrete theories put to use with regard to mathematics education originate in several different disciplines, many of which are external to mathematics education research itself. The task of this ST is to identify, survey, and analyse different notions and roles of "theory" in mathematics education research, as well the origin, nature, uses, and implications of specific theories pertaining to different types of such research.

This task defines a very important problematique in mathematics education research but, even if this problematique is clear, its treatment is problematic. Of course, the investigation of this problematique can be different and we can produce different answers.

In this paper, to carry out this task, we will consider three levels corresponding to some questions. The first level is a preliminary interrogation about: how to do a survey? What are the data? What are the tools for doing this survey? What are the criteria? Are these criteria theoretical or empirical? Have we common or different tools for doing this task? What are our assumptions about this task? This level is a methodological level but it is too an epistemological one: our practice and assumptions of mathematics education research found what we do in order to achieve this task.

A second level is a results level. We produce different surveys and we identify and analyse different roles and functions of "theory" in mathematics education research. We must point out different results of these surveys and these results are depending on the data and tools used in this work.

A third level is a reflexive level. We want to compare our different methodologies and assumptions in doing this task. What are the different types of theory? What is a theory in mathematics education research? What is the role of theory in the autonomy and identity of mathematics education as a scientific domain?

These three levels organise our text in three parts and we conclude by some "open questions". We organized the ST from preliminary individual work. We prepared five papers and this common paper is the result of the collective work. Three of us tended to work especially in the first and second level, and the other two in the third level but this is just a trend. Sometimes for further developments we will make reference to these preliminary papers because we cannot include their full content in this paper.

1. FIRST LEVEL: DATA, METHODOLOGIES, TOOLS, ASSUMPTIONS

In this chapter, we will make explicit our different data, methodologies and tools. We want to note that some of these surveys are not exhaustive and the results depend on the choice of data, methodologies and tools for analysing these data.

Lerman, Herbst and Assude each analysed a sub-section of the research literature in the field of mathematics education. Each researcher developed a set of categories for that analysis, hence producing a theoretical framework in interaction with an empirical set. We will present here extracts from each of these three papers in which the authors describe the methods, motives and categories used in each of these papers.

Lerman surveyed how researchers in the mathematics education research community work with theories, both in terms of which theories and how they work with them. In carrying out the survey he sampled research carried out between 1991 and 2003 on 12 years of the publications in *Educational Studies in Mathematics* (*ESM*), *Journal for Research in Mathematics Education* (*JRME*), and *Proceedings of the International Group for the Psychology of Mathematics Education* (*PME*).

In his[1] research he developed a tool, in interaction with the data, for analysing a whole range of aspects of the research productions of the community as evidenced in a sampling of published articles. In this paper he focused on just two elements of the analysis, those of use of theory & orientation. By orientation he meant to theoretical or empirical inquiry; whether the theories used have changed over time; whether researchers revisit the theories used in their studies; the relationships established between the theoretical and the empirical; and the focus and methodology of the studies.

By 'theories' he intended learning theories, perhaps set in the context of philosophical orientations, perhaps informed by psychology, or sociology or other fields. It is our expectation that such theories guide the design of a research study and the analysis, or perhaps are used retrospectively as lenses through which to interpret a set of findings. This approach focuses on theories as resources to help towards the achievement of those desired outcomes.

Herbst wants to complement the contribution made in the chapter "Theory in mathematics education scholarship" (Silver and Herbst, 2007) with some data gathered from a superficial inspection of the 39 articles published in the *Journal for Research in Mathematics Education* from January 2005 to January 2008.

His main objective has been to describe whether and how authors of research articles use the word theory (or its cognates such as theorizing, theorization, theoretical) in relation to the pursuit of their research. One question has been

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[1] The project, entitled "The Production and Use of Theories of Teaching and Learning Mathematics" and funded by the Economic and Social Research Council in the UK, project No. R000 22 3610. The full text of the project proposal and the research papers published from it are at http://www.lsbu.ac.uk/~lermans/ESRCProjectHOMEPAGE.html

to describe the extent to which the articles in this corpus identify themselves as theory building, theory using, or otherwise make no appeal to theory. Subsidiary questions are, in the first case, whether the articles contribute to building local theories, middle range theories, or grand theories. In the second case, whether the articles use theory to describe, explain, predict, or prescribe practices, or whether they prescribe research operations. Simultaneously, he's been attentive to the particular practices aimed at by articles that use theory and by articles that build theory.

The methodology used for this survey included the following procedures. To constitute the corpus he extracted all research articles from all issues of JRME starting in January 2005—this means that he did not include editorials, brief reports, research commentaries, book reviews, telegraphic book reviews, or announcements in the sample. Other than that all articles were included, totalling as noted above 39 articles. JRME publishes 5 issues per year and each of those issues tends to include 3 articles. Once the text of each article was available electronically we produced three word searches after "theor," "framework," and "construct." He second-guessed the idea of looking only at places where authors had used the word "theory" and its cognates based on some of the reasons noted in Silver and Herbst (2007) that might propel people to shy away from its use.

The word search heuristic based on those three words (theor, framework, construct) was useful inasmuch as it allowed to find intellectual tools that researchers have used to do a number of operations in their work. He specifically attended to the operations of describing, explaining, and predicting phenomena, prescribing educational practices, and prescribing research operations as examples of the ways in which theory might help researchers connect research to practice and to the problems of practice. These tools are used the earlier work by Silver and Herbst (2007).

In this survey, theory assists the triadic relationships between research, problems, and practices. Drawing on the distinction between local theories (e.g., what levels of development exist in students' learning of fractions?), middle range theories (e.g., what is classroom mathematics instruction), or grand theories (e.g., what is the mathematics education field) he identified those articles that had a theoretical aim and noted what that aim was.

Assude wants to identify the roles and functions of "theory" in mathematics education research taking a corpus formed by the papers published in the review "Recherches en didactique des mathématiques". This review is an important tool for the researchers' community, especially the French speaking one: it is one of the main tools to disseminate the researchers' work in this domain in France (or among French speaking researchers).

Her data are formed by all the papers published in RDM between 2000 and 2006. RDM publishes 3 issues per year and 3 papers per issue or so. There are 59 papers, 8 in Spanish, 2 in English and 49 in French.

For analysing these data, she needs to precise what is theory in this context. In her opinion, theory in mathematics education deals with teaching and learning mathematics from two points of view. First a structural point of view: theory is an organised and coherent system of concepts and notions in the mathematics education field. Second a functional point of view: a theory is a system of tools that permit a "speculation" about some reality. This "speculation" is an active one because these tools can allow to observe, analyse, interpret a teaching and learning reality (or practices), and can produce new knowledge about this reality. According of this double point of view, she can take a theory as a tool and a theory as an object. Finally she will take other indicators like: internal /external theory in mathematics education if theory is produced or not within this domain; local/global theory if the theory concerns a study of a problem or a study of a domain; the effective theoretical elements used in the work; the functions of these elements (for example, a theory can be a tool to conceive a didactical engineering).

She will use this preliminary grid for analysing our data and she wants to point out that some functions and roles of theory are not specified of one theory, but different theories can assumed the same functions even if the knowledge produced by their uses are different.

Radford developed an analytical tool which can be applied to any of the theories that are used in mathematics education research. He presents the elements of the tool and then exemplifies it by the analysis of three theories; the theory of didactic situations; constructivism; and sociocultural theories. Radford will deal with the question of the types of theories used in mathematics education research (Radford's paper[2]). His goal is to contribute to clarify

[2] The full version of the paper ("Theories in Mathematics Education: A Brief Inquiry into their Conceptual Differences") can be retrieved from the Publication section of http://www.laurentian.ca/educ/lradford/

one of the two central themes around which our Survey Team revolves, namely the investigation of the notion of theory in mathematics education research, as stipulated in the appointing official letter. How will he proceed? He could proceed by giving a definition, T, of the term "theory" and by choosing some differentiating criteria c_1, c_2, etc. Theories, then, could be distinguished in terms of whether or not they include the criteria c_1, c_2, etc. Although interesting, he will take a different path. In the first part of his paper, he will focus on a few "well-known" theories in Mathematics Education (constructivism, theory of didactical situations, social cultural theory) and attempt to locate their differences at the theoretical level, that is, he will discuss their differences in terms of their theoretical stances.

Boero carries out a study of the relationship between key theories in the field and the ways in which external frameworks are drawn into the field. His analysis will be presented in the third level.

2. SECOND LEVEL: SOME RESULTS

In this level we are presenting some results of our surveys. Sometimes we use the results of the authors' works before the work in the ST.

2.1. *Uses of Theory and orientation: theory as a tool*

Lerman's analysis showed, for the period from 1990 to 2001, that 70.1% of all articles in ESM have an orientation towards the empirical, with a further 8.5% moving from the theoretical to the empirical, and 21.5% presenting theoretical papers. This changed little over those years. Most of the papers used theory (92.7%), and more than four-fifths (86.4%) were explicit about the theories they used in the research reported in the project. Again this has not varied across the years. Similarly, 86.2% of all articles in the journal JRME had an orientation towards the empirical, with a further 2.2% moving from the theoretical to the empirical, and 11.6% presenting theoretical papers. This changed little over the years. Most of the papers used theory (83.3%), with a relatively higher percentage of papers that did not use any theory, compared to the other two journals considered here. Three-quarters (75.4%) were explicit about the theories they were used in the research reported in the articles. Again this has not varied across the years. Finally, 84.5% of all papers in the PME proceedings had an orientation towards the empirical, with a further 6.8% moving from the theoretical to the empirical, and 8.8% staying in the theoretical. This has changed little over the years. Furthermore 89.9% of the papers used theory, with 10.1% not using any theory, and more than four-fifth (82.4%) were explicit about the theories they are using in the research reported in the article. Again this has not varied across the years.

Regarding the relationship between the theory and the empirical study, in 65.5% of articles in ESM the theory informs the empirical, in 2.3% the empirical informs the theoretical and in a further 4.0% we determined that the relationship is dialectical. 7.3% did not refer to a theory either explicitly or implicitly. In JRME, in 71.7% of articles the theory informs the empirical, in 0.7% the empirical informs the theoretical but there are no cases in which we determine that the relationship is dialectical. 16.7% did not refer to a theory either explicitly or implicitly. In PME proceedings, in 79.1% of articles the theory informs the empirical, in 4.7% the empirical informs the theoretical and in a further 0.7% we determine that the relationship is dialectical. 10.1% did not refer to a theory either explicitly or implicitly.

A result of this survey is that the uses of theory is important in mathematics education research but the empirical orientation prevails. The role of theory is especially a tool.

2.2. *Types of Theory: external or internal?*

In Lerman' analyses, some interesting changes have been depicted concerning the item 'theory type'. The predominant theories throughout the period examined for all three types of text were traditional psychological and mathematics theories, but there is an expanding range of theories used from other fields. The psycho-social theories, including re-emerging ones, and the sociological and socio-cultural theories are increasing. The predominant theories were external theories in mathematics education as a scientific domain.

This result is not verified in the Assude's analysis about papers published in the journal RDM. In this case, the predominant theories are internal theories in mathematics education research: these theories are constructed within this domain.

This difference has perhaps a link with the global project of building a new scientific field – mathematics education research – with some autonomy regarding to other neighbouring fields like psychology or sociology. Silver and Herbst (2007) show that David Johnson (the first editor of JRME) point out the lack of theory in mathematics education in 1980 and he suggests to the researchers:

> "first investigate the adaptability of various psychological theories… to the learning and teaching mathematics, and [only] in the event such adaptation is not feasible, move the creation of a new theory" (in Silver and Herbst 2007, p.43).

This position – adaptation to mathematics education of theories existing in other fields – is a common position yet now. The Herbst's analyses about 39 articles published in JRME from 2005 to 2008, confirm these results since they show that there is no paper dealing with the construction of a "grand theory" (e.g. what is the mathematics education field). But 10 articles are involved in theory making to produce a local or a middle range theory while 24 papers are involved only in theory using and 5 articles don't use theory. Here we can say again the predominant role of theory as a tool.

2.3. Functions of Theory as a Tool

The Assude's analyse (Assude's paper) identifies some functions of theory in the researchers' work (see table 6 for some examples of papers):

- *conception of didactical engineering or didactical device*: for example, theory can allow to define some didactical variables to produce a didactical engineering;
- *methodological development*: for example an a priori analysis is a methodology based on a theory;
- *didactical analysis*: the analysis can be very different according to the reality (an observation of a classroom, an observation of a pupil's work, a curriculum, etc.). Different operations as describing, explaining, interpreting, justifying can be identified;
- *definition of a research problematique*: some practical problems in the educational system are not research problems. It is necessary to transform these problems in a research problem (for example doing some hypothesis or doing some categorisations);
- *study of a research problem*: theory can be a tool for defining different steps in the study of a problem;
- *production of knowledge*: theory is a tool to identify some didactical phenomena, some new knowledge about some reality.

In Herbst's analysis about papers published in JRME, he identifies some functions of theory as a tool to describe, explain, prescribe and we precise the different object these functions are dealing with such as activity or curriculum. Silver and Herbst (2007) analyse the uses of theory in mathematics education scholarship and propose to consider theory as mediator between problems, practices and research. In this work, the authors identify some functions of theory in the role of mediator between:

- *research and problems:* interpretation results; analysing data; producing results of research on a problem; giving closure to the corpus of data to study a problem; transforming a commonsensical problem into a researchable problem; generator of researchable problem; organization of a corpus of research on a problem;
- *research and practice:* prescription; understanding; description; explanation; prediction; generalisation;
- *practice and problems:* solution to a problem of practice; comparison; designing new practices; justifying choices

There is a great variety of functions for theory as a tool and it concerns all researchers' activities. These functions are not specific to a particular theory.

2.4 – *Functions of a Theory as an Object*

We suppose that theory can have two roles: as a tool and as an object. We want to give explicit some of the functions of theory as an object. Theory is not something static but dynamic: the evolution of theories in a scientific field is a means to understand the evolution of this field.

Lerman looked at whether, after the research, the researchers have revisited the theory and modified it, expressed dissatisfaction with the theory, or expressed support for the theory as it stands, he concluded that authors may not revisit the theory at all; content to apply it in their study.

The role of theory as a tool is predominant but some works exists where theory making is one of the goals. In Herbst's survey, he distinguishes three types of theories: local theories (e.g., what levels of development exist in students' learning of fractions?), middle range theories (e.g., what is classroom mathematics instruction?), or grand theories (e.g., what is the mathematics education field?). Ten papers are concerned with theory building: 7 for local theories, 3 for middle range theories and none for grand theories.

In Assude's survey (Assude's paper), she identifies that some authors use a theory for putting to the test the theory or some concepts or relations in this theory. This "theory testing" is a way to produce new theoretical developments. These are some functions for this "theory testing":

- decontextualisation, transposition and generalisation of theory in other contexts;
- relations with contingency;
- new interpretations of a phenomena;
- verifying the domain of validity of a theory;

The development of a theory is one of the functions of theory as an object: sometimes there is just one theory, sometimes two or more theories exist, and the development of a local or middle range theory is done by articulating or juxtaposing some elements of different theories.

We can quote Silver and Herbst' work for complementing this list:

> "the role of theory [is] not so much as a mediator of relationships among practices, problems and research,(…) but rather (or also) as the collector, beneficiary, or target of that interplay in a fundamentally academic theory-making exercise".

Theory-making (especially internal theories) has a role in the constitution of a mathematics education research as a specific field with an identity different from other fields as psychology. This project of constitution is present in the beginnings of this domain in some countries: for example Brousseau'work was based in the piagetian psychology but it had a theoretical ambition to become relatively independent. This idea is developed in Silver & Herbst (2007) too and we are going to develop some ideas about the autonomy and identity of mathematics education research in the 3th part.

2.5. *Conceptual Differences about Theories in Mathematics Education*

In the Radford's analysis, his goal is to contribute to clarify one of the two central themes around which our Survey Team revolves, namely the investigation of the notion of theory in mathematics education research. His choice of theories has been guided by what may be termed their "historical impact" in the constitution of mathematics education as a research field. By "historical impact" he does not mean the amount of results that a certain theory produced in a certain span of time. Although important, what he has in mind here is rather something related to the foundational principles of a theory:

> The foundational principles of a theory determine the research questions and the way to tackle them within a certain research field, helping thereby to shape the form and determine the content of the research field itself.

For him, to ask the question about the types of theories in our field is to ask for their differences and, more importantly, for that what accounts for these differences. Our argument is that these differences are better understood in terms of theoretical suppositions. Sriraman and English (2006) argued that the variety of frameworks in mathematics education is directly related to differences in their epistemological perspectives. He wants to suggest that, in addition to the underpinning corresponding epistemologies, differences can also be captured by taking into account the cognitive and ontological principles that theories in mathematics education adopt.

Radford gives three examples in his paper for the survey: constructivism, the theory of didactic situations (TDS) and the sociocultural approaches. It is not possible to present here this work but we will take just an example.

For constructivism and the TDS the autonomy of the cognizing subject vis-à-vis the teacher is a prerequisite for knowledge acquisition. For sociocultural approaches, autonomy is not the prerequisite of knowledge acquisition. Autonomy is, in fact, its result. This is one of the central ideas of Vygotsky's concept of zone of proximal development.

The ontological principle of the sociocultural approaches is that knowledge is historically generated during the course of the mathematical activity of individuals. The epistemological principle of these approaches is that the production of knowledge does not respond to an adaptive drive but is embedded in historical-cultural forms of thinking entangled with a symbolic and material reality that provides the basis for interpreting, understanding and transforming the world of the individuals and the concepts and ideas they form about it (Radford, 1997). The cognitive principle of these approaches is that learning is the reaching of a culturally-objective piece of knowledge that the students attain through a social process of objectification mediated by signs, language, artifacts and social interaction as the students engage in cultural forms of reflecting and acting. Learning, from a sociocultural perspective, is the result of an active engagement and self-critical, reflexive, attitude towards what is being learned. Learning is also a process of transformation of existing knowledge. And perhaps more importantly, learning is a process of the formation of subjectivities, a process of agency and the constitution of the self (Radford, 2008b).

3. THIRD LEVEL: THEORIES, AUTONOMY, IDENTITY

In our different surveys, we have not used the same categories and methodologies. These choices depend on our research practices and our assumptions about what a theory is and which is the role of theory for giving autonomy and identity to mathematics education field. This level is a reflexive level. We choice here to think about the relationships between the uses of theories in mathematics education and the autonomy and identity of this field.

If mathematics education aims at growing as a scientific discipline, it must develop theoretical work in order to deal with teaching and learning problems in a systematic, scientific way. Now this is a rather obvious, widely shared position. The problem is that the ways of developing theoretical work, and its autonomy or dependence from theories elaborated in other disciplines, have been rather controversial since the birth of mathematics education as a scientific discipline, in the seventieths. We have seen above the differences of theories in terms of theoretical suppositions and we have seen that these theories are not completely independent from theories in other fields. Then what is the autonomy and identity of mathematics education field?

3.1. *Permeability and the illusion of a complete autonomy*

In Boero's reflexion, mathematics education as a scientific discipline should neither work in a completely autonomous, autarchic way, nor transpose paradigms and results of other disciplines in its specific field of investigation. According to him, we should look instead to the possibility of an autonomous specific theoretical work mainly intended as selection, adoption or re-elaboration of tools coming from other disciplines, possibly integrated with the construction of other tools needed according to the specificity of the content to be taught (Boero & Radnai Szendrei, 1998; Kilpatrick & Sierpinska, 1998).

Among the disciplines that could be relevant for scientific work in mathematics education (history of mathematics, epistemology, psychology, sociology, anthropology, etc.), Boero focus on the relationships with epistemology and psychology. This choice depends on three reasons: first, in his opinion these disciplines have played a major role in influencing important changes in the teaching of mathematics during the last century; second, they can assume a crucial role in the development of mathematics education as a scientific discipline because they concern the "what" and the "how" teachers teach and students learn; third, they challenge autonomy of mathematics education as a scientific discipline because research in our field cannot ignore the fact that many results of those disciplines concern mathematics as a paradigmatic subject.

Psychological and epistemological investigations do not work (as their main aim) for a better learning of mathematics and for a better understanding of what is learning and teaching mathematics. When they deal with mathematics, epistemological theories are aimed at describing and framing some aspects of that discipline; most psychological theories dealing with learning of mathematics try to describe, interpret and, possibly, predict learners' laboratory behaviour on a given area of paradigmatic mathematical tasks. However, in the reality of the school teaching of mathematics, what comes from mathematics, epistemology and psychology is filtered and frequently deformed when it meets the complex school culture (textbooks, materials, tradition, programs…). In general, processes in the noosphere are sensitive to external influences (coming from politics, culture, etc) but they develop with a relative autonomy and inertia. What is the role of mathematics educators in those processes?

Some members of the noosphere that have special responsibilities in teachers' preparation and curriculum development (in particular, researchers in mathematics education) frequently act as if some epistemological and psychological theories would carry the truth about what mathematics is, and how students learn it. Frequently they assume an important role in "transposing" those theories in the school system, in particular through teachers' training. Other mathematics educators adapt and interpret ideas coming from epistemology and psychology by trying to match them with existing teaching devices and habits.

Boero says that mathematics educators frequently adopt ideas coming from the exterior (in particular, epistemology and psychology) to promote more or less coherent and radical changes in the school teaching of mathematics. In most cases they do not move from the identification of teaching and learning problems to the choice of theoretical tools suitable for tackling them. In those cases we can say that mathematics education mainly develops as a subaltern discipline. On the other hand, mathematics educators can not (and should not) develop a completely autonomous and autarchic science (or technology) of the teaching of mathematics in school. This is an illusion for two reasons: on one side, teachers come from a given school or university mathematics culture and are embedded in a given cultural environment, and mathematics educators are prepared in given cultural institutions; thus it is not possible to ignore what teachers and mathematics educators know and think about the teaching and learning of mathematics, and their scientific preparation. On the other, if mathematics educators want to go beyond mere descriptions of what happens in the mathematics classroom they need to consider what mathematics is, and how mathematics is appropriated by student; thus they need to deal with scientific results coming from epistemology and psychology. The unavoidable reference to epistemology and psychology can be denied or underestimated, but in that case what usually happens is that implicit assumptions are made, or explicit assumptions are assumed as unquestionable truth.

We can think different positions to develop mathematics education as a relatively autonomous scientific discipline, i.e. a research space where tackle teaching and learning mathematics problems with its own theoretical tools as well as adapted theoretical tools coming from other disciplines, critically considering their potential and limits, and their consequences on the solution of those problems.

3.2. *Towards a relative autonomy: adaptation and development*

The first position is the use of theories existing in other fields but we need to adapt these tools: these adaptations is part of the field autonomy. Boero argues that the problem is what choices to make and how to move on from those choices, keeping into account the variety of results and perspectives provided, in particular, by epistemology and psychology. The task of mathematics educators is not to choose an epistemological position or a psychological theory as an "all purpose" and universal reference (each outstanding epistemological position being culturally situated, each psychological theory having a limited domain of validity). What mathematics educators can do is to identify important teaching and learning problems, consider different existing theories and try to understand the potential and limitations of the tools provided by those theories, possibly adapted to the specific problems in order to tackle them. However this statement is still vague for two reasons. First, to identify important teaching and learning problems requires some preliminary theoretical assumptions regarding the importance and nature of the concerned competence and the way to ascertain related learning difficulties. Second, it is necessary to adopt some preliminary keys (suggested by epistemological and psychological analyses) to avoid a disperse view of the whole panorama of the teaching and learning of mathematics. A dialectic process should be developed: our epistemological and psychological culture together with our knowledge of what happens in school suggest to consider specific educational problems; in order to tackle those problems we need to identify and adapt appropriate tools from epistemology and psychology (and, in some cases, history of mathematic, sociology, etc.). It may happen that such tools oblige us to re-formulate the original educational problems, or to identify further related problems. When dealing with specific mathematics teaching and learning problems, we must recognize that in many cases existing tools elaborated by epistemology, psychology, sociology, etc. need to be adapted and re-elaborated. Cobb (2006) says:

> Mathematics educators should view the various theoretical perspectives as sources of ideas to be appropriated and adapted to their purposes. Cobb (2006)

The proliferation of theories can be a problem. In his recent article (2006) Cobb outlines two criteria through which to facilitate a conversation concerning what researchers should do when faced by a proliferation of theoretical perspectives. His first criterion is to focus on the types of questions that can be asked within each perspective about "the learning and teaching of mathematics, and thus the nature of the phenomena that are investigated and the forms of knowledge produced." His second criterion is that of usefulness:

> The usefulness criterion focuses on the extent to which different theoretical perspectives might contribute to the collective enterprise of developing, testing, and revising designs for supporting learning. This second criterion reflects the view that the choice of theoretical perspective requires pragmatic justification whereas the first focuses on the questions asked and the phenomena investigated. (Cobb 2006)

3.3. *Towards a relative autonomy: production*

The second position is the production of a new specific theoretical tools for tackling the specific problems of mathematics education domain. In spite of the eclecticism in terms of theory adopted for research, given that the goal is usefulness, or what works, elsewhere Cobb argues strongly for the importance of theory, but in the sense of the production of theory as a key part of the job of the design scientist. He illustrates this in DiSessa and Cobb (2004) by offering one category of theory production, that of 'ontological innovation', seen as the production of new objects, emerging from design experiments, that then prove useful as objects for study. Interestingly, one of the two examples offered in that paper is a retrospective look at the early work Cobb carried out with Erna Yackel and Terry Wood, a long term project based firmly within a constructivist paradigm. Nevertheless, the notions of social norms and socio-mathematical norms are presented as examples of ontological innovations that emerged from those studies, which themselves are re-interpreted retrospectively as design experiments.

3.4. *Towards a relative autonomy: reorganisation*

The third position is the reorganisation of the theoretical field. This reorganisation can be done by different forms. One example of this reorganization, of a new trend has observed in the Fifth Congress of the European Society for Research in Mathematics Education (CERME-5, 2007). The European Society for Research in Mathematics Education organizes biannual conferences that are designed to encourage an exchange of ideas through thematic working groups. One of the recurring CERME working groups is the one devoted to theories in mathematics education. The goal of this working group was not just to understand differences, but to seek new forms of linking and connecting current theories. More specifically, the idea was to discuss and investigate theoretical and practical forms of *networking* theories. Most of the papers presented at the meetings of working group 11 appeared in volume 40(2) of the journal *ZDM - The International Journal on Mathematics Education*. As we mention in the commentary paper written for this ZDM issue (Radford, 2008a), this new trend consisting of investigating ways of connecting theories is explained to a large extent by the rapid contemporary growth of forms of communication, increasing international scientific cooperation, and the attenuation of political and economical barriers in some parts of the world, a clear example of which being, of course, the European Community.

This new trend is leading to an inquiry about the possibilities and limits of using several theories and approaches in mathematics education in a meaningful way. The papers presented at the conference provided an interesting array of possibilities.

Depending on the goal, connections may take several forms. Prediger, Bikner-Ahsbahs, and Arzarello (2008) identify some of them, like "comparing" and "contrasting" and define them as follows. In "comparing" the goal is finding out similarities and differences between theories, while in "contrasting" the goal is "stressing big differences". Cerulli, Georget, Maracci, Psycharis, & Trgalova (2008) is an example of comparing theories, while Rodríguez, Bosch, and Gascón (2008) is an example of contrasting theories. These forms of connectivity are distinguished from others like "coordinating" and "combining". In coordinating theories, elements from different theories are chosen and put together in a more or less harmonious way to investigate a certain research problem. Halverscheid's paper (2008) is a clear example of an attempt at coordinating theories, in that, the goal is to study a particular educational problem (the problem of modelling a physical situation) through the use of elements from two different theories (a modeling theory and a cognitive one). In combining theories, the chosen elements do not necessarily show the coherence that can be observed in coordinating connections. It is rather a "juxtaposition" of theories (Prediger, Bikner-Ahsbahs, and Arzarello's paper (2008)). Maracci (2008) and Bergsten (2008) furnish examples of combining theories.

At least in principle, "comparing" and "contrasting" theories are always possible: given two mathematics education theories, it is possible to seek out their similarities and/or differences. In contrast, to "coordinate" or to "integrate" theories, which is another possible form of connection (Prediger, Bikner-Ahsbahs, and Arzarello's paper (2008), seems to be a more delicate task.

Connecting theories can, in sum, be accomplished at different levels (principles, methodology, research questions), with different levels of intensity. Sometimes the connection can be strong, sometimes weak. It is still too early to make prognostics of how this new trend will evolve.

What is clear, in contrast, is that the investigation of integration of theories and their differentiation is likely to lead to a better understanding of theories and richer solutions to practical and theoretical problems surrounding the teaching and learning of mathematics.

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Understanding "hidden rules": the challenge of becoming a competent member of a mathematics classroom

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INTRODUCTION

In his invited address to the Special Interest Group in Research in Mathematics Education at the annual meeting of the NCTM in 1979, Heinrich Bauersfeld spoke about "hidden dimensions in the so-called reality of a mathematics classroom" and argued for researching these dimensions. While suggesting the study of the interactive constitution of shared meanings in classrooms, he also reminded the audience of considering the impact of the institutional settings. Institutions "constitute norms and roles", "develop rituals in actions and in meanings", "tend to seclusion and self-sufficiency" and "even produce their own content – in this case, school mathematics" (Bauersfeld, 1980, pp. 35-36). Bauersfeld suggested that ethnomethodology and linguistics provide promising theoretical bases for a research agenda that addresses the hidden dimensions of mathematics classrooms.

Whether what is going on "below the surface" in mathematics classrooms remains not only hidden to the students and teachers, but also to the researcher, is a matter of methodology and theorizing. Since Bauersfeld gave his address, many researchers in mathematics education have come to investigate what he indicated by "the hidden dimensions in the so-called reality of a mathematics classroom" in order to understand how these afford or constrain students' access to mathematical knowledge. The most prominent theories employed in empirical classroom research to achieve this goal include Symbolic Interactionism and Phenomenology, in particular Ethnomethodology, as well as theories that are concerned with the social reproduction through schooling, such as those of Bourdieu and Bernstein. But also some theorizing or compilations of other theories that emerged from within mathematics education as a research domain addresses the problématique.

CONCEPTUALISING "HIDDEN RULES"

The following episode from a mathematics classroom illustrates some dimensions of what the title intends to indicate by "hidden rules". Meyer (2010) discusses some episodes from a 4th grade classroom in Germany in which the teacher intends to introduce the notions "parallel", "perpendicular" and "right angle". The terms are written on the board. After asking the students to freely associate what comes to their minds, a reproduction of a painting by Mondrian is shown to the students.

> „Teacher: Why do I fix such a picture on the blackboard? And why are these concepts written down on the blackboard? I have a reason to do so. Jonathan, it is your turn.
> Jonathan: Because the painter has done everything in parallel, perpendicular and in right angles.
> Teacher: You are right. You seem to know what parallel, perpendicular and right angle means. Maybe you can show it to us on the picture.
> Jonathan: Perpendicular is this here (points first at a vertical, afterwards at a horizontal line). Parallel is this here (points at two vertical lines). A right angle is this (pursues two lines he former would have called perpendicular)."
> (Meyer, 2010, p. 909)

Meyer, by drawing on Wittgenstein's notion of language-games, discusses the scene as an instance of establishing the "exemplaric use" of words in this classroom. He also points out that Jonathan must have been participating in practices of using the words "parallel", "perpendicular" and "right angle" in a similar language-game outside this classroom.
However, the episode shows that Jonathan had to know more than how to engage in the language game of ostensive definitions that employ visual recognition. For producing his positively sanctioned answer, Jonathan also had to understand the question as a prompt to associate the notions written on the board with the configuration of lines in the painting. Alternative replies that might have been produced without understanding the actual illocutionary act performed by the teacher's question, such as "because you like the painting", most likely would have been taken as an expression of sarcasm by the participants. In addition, Jonathan had to recognise that the teaching here is organised as a series of related questions to be answered or discussed by the students, and to have access to the criteria for producing an appropriate contribution to a description of a piece of art in a geometry lesson, in contrast, for example, to a discussion of the style of the painting in an arts lesson.

In this episode, different dimensions of "hidden rules" become visible. The (emergent) rules for using and producing mathematical signs and for a legitimate way of presenting an externalisation of one's thinking according to these rules (orally or in a written form), the rules of the pedagogical principle adopted by the teacher that account for the establishment of routines in communication, the rules that constitute the specificity of the school mathematical practice and its discourse in relation to other practices and their discourses, as well as the norms for favoured behaviour, aspirations and attitudes. As all these rules regulate how students relate to and gain access to different forms of mathematical knowledge, the challenge for the students is to acquire knowledge of these rules in order to develop the skills that are necessary for successful participation. This opens up the question of whether al students have equal access to these rules. There might be hidden principles in operation that account for the stratification of achievement because not all students gain equal access to the knowledge code.

At this point, a remark on terminology seems advisable. Because of lack of alternatives, in the heading the term *rule* is used as an umbrella term, referring to *norms, specific rules, routines* and *principles*. The term *norm* often refers to established standards to be achieved, sometimes also to typical patterns found in social actions. Typical patterns of actions that are carried out repetitiously and often are followed unconsciously might be classified as *routines* (or *rituals*). A *specific rule* combines a norm that consists of specific criteria with a regulation for achieving them. The term *principle* indicates an underlying invisible mechanism. According to this differentiation of meanings, specific rules, norms, and principles differ in stability, visibility, accessibility and relations to a wider system.

Specific rules

The notion of a *specific rule* is used here to suggest that it refers to norms that condense a set of specific criteria for an action in a regulation for achieving them. Many of such specific rules in a mathematics classroom are about the behaviour and the social organisation of the work. The rules might be unspoken, but if asked, many students would be able to express them: "When we work individually, the we are actually allowed to discuss with the students sitting next to us." "We can ask questions, when we get stuck in a task and the teacher will then come to our desk and help us." "When the teacher writes something on the board, we have to copy it into our notebooks." "We always have to write down the answer to a word problem as a full sentence." "The result of a calculation has to be double-underlined." "When the teacher says 'tell me more about this', she wants us to show how we calculated it." "She wants us to just work on the warm-up and get the answers for it, and then later she asks us for the answers so that we can correct ourselves." The students are more or less conscious of such rules. Consciousness opens up a space for tactical behaviour. And only if the students are consciously aware of the rules, they can intentionally not comply, which can then be interpreted as an act of resistance.

To the chagrin of many mathematics educators, teachers often introduce explicit specific rules for solving certain types of mathematical problems. In a comparative study of six year-8 mathematics classrooms[1], two of which were from Germany, Hong Kong and the United States respectively, Jablonka (2004) found examples of explicit guiding manuals for solving tasks in classrooms of all three countries. These included manuals for tackling word

[1] The study was part of the Learner's Perspective Study, see extranet.edfac.unimelb.edu.au/DSME/lps/)

problems and careful consideration of single steps conceptualised as rules in algebraic transformations. In a Hong Kong classroom, for example, the teacher introduced a six-step procedure for setting up equations in two variables (called "unknowns") in order to solve word problems: (1) *Examine*, (2) *Let* (short for „let *x* be, let *y* be…"), (3) *Form* (two equations), (4) *Solve*, (5) *Check*, (6) *Answer*. In line with others, Jablonka (2004) also found a preference on the side of the students for step-by-step manuals for solving tasks. Many students referred to a set of explicit and detailed rules as a good "explanation" by their teachers.

If criteria for actions are transformed into regulations for achieving them, the criteria remain implicit, and validation of the outcome can only be achieved through checking the correctness of the procedure, but not in relation to the criteria. The students will not be able to check the validity of their solutions in relation to the problems to be solved, and not get used to invent ways of solving unfamiliar problems. Consequently, such a focus on teaching explicit rules has been an ongoing concern of mathematics educators. The alternative typically consists in presenting a sequence of problems so that the students themselves can construct a general underlying meaning structure. As Ernest (2006, p. 75) points out, there remains an unresolvable tension between leaving the general principle implicit or rendering it explicit: "Thus the paradox is that general understanding is achieved through concrete particulars, and specific responses only may result from general statements."

However, there is a price to be paid for leaving the work of constructing more general mathematical meanings to the students in inquiry based mathematics classrooms. For example, Theule Lubienski's (2000) study in what has become called a reform mathematics classroom, shows that students did not equally make use of the open whole-class discussions. While high-socioeconomic status students were able to recognise the importance of looking for generalisations, lower-socioeconomic status students focused more on giving correct answers to specific, contextualized problems and could not fully appreciate the presentation of a diversity of ideas but preferred more teacher direction. Jablonka (2004) found that many of the lower-achieving students felt lost as soon as open-ended tasks were introduced that allow for different solution strategies. Teese (2000, p. 171) reports from a reform project in Victoria, Australia, in which an inquiry-based curriculum has been followed. The approach turned out to be of disadvantage for working class girls. This group was more successful in the traditional setting. Dowling (2009) shows how an investigative approach to school mathematics introduces new skills and "tricks". What makes such skills or tricks meaningful for the construction of new mathematical knowledge can only tacitly be decided on the grounds of previously acquired mathematical knowledge. Similar concerns can be raised about approaches that favour teaching mathematics through mathematical modelling.

Norms of classroom practice as (emerging) conventions

Emerging norms embody expectations and values that are supposed to be shared by the group about what is an appropriate contribution to the practice. These norms can be reconstructed from an observers' point of view by the fact that most participants show some signs of having adopted the expected actions at some stage. The reconstruction resembles an ethnographer's re-construction of the "folk-ways". Such an interpretation of classroom practice will be a hermeneutic, immanent one. But it is not done by the participants who are involved, except, perhaps, in the case of a breakdown of the smooth flow of co-ordinated actions. The "socio-mathematical" and "social" norms (e.g. Yackel & Cobb, 1996), the "didactical contract" (Brousseau, 1980), and some of the "meta-discursive rules" (e.g. Sfard, 2001) refer to these types of norms.

The focus in studies of classroom practice is often on the changing character of the norms when the construction of new mathematical knowledge is at issue. Voigt (1984) studies regularities in mathematics classroom interaction in relation to the learning behaviour of the students. He assumes that teacher and students are in the possession of unconscious practices or routines (Schütz & Luckmann, 1975) that help them to structure the process of constituting knowledge that eventually counts as shared knowledge. The notion of *routine* here refers to the fact that these interaction patterns are unconsciously accomplished, have the function of reducing the complexity of the situation, and yield a harmonising effect. Voigt (1984) analyses variations of a common whole class pattern of interaction in German mathematics classrooms that is called the *fragend-entwicklendes Unterrichts-gespräch* [questioning-developing instructional talk]. There are similar terms in other European countries, as for example the *onderwijs leergesprek* [classroom teaching talk] in Dutch. In classrooms from the U.S.A., "guided development" resembles a similar, perhaps more open form of such a pattern. Successful participation in this activity does not imply that all students share the mathematical meanings the teacher intended to constitute. The students might only have developed the competencies of how to participate in the interactive production of knowledge that is institutionalised. The pattern has been criticised as it affords acting according to the teacher's expectations. The students might spend much effort in finding out the implicit rules of the "didactical contract", which is constituted through mutual expectations and interpretations of "specific habits" of the teacher by the students and vice versa (Brousseau 1980, p. 180). This description of the didactical contract is reminiscent of the description of interpretive procedures described by ethnomethodology (e.g. Voigt, 1984, p. 23 ff.).

Voigt (1984, p. 22) observes that the functioning of the routines for the interactive construction of new knowledge is apparently contradictory. As there is no shared frame of reference from the outset, the teacher's initial question or task is necessarily ambivalent. But the task is reflexively bounded to its solution: Only retrospectively the official solution reduces the ambivalence of the question. The institutionalised solution constitutes the meaning of the task of which it is a consequence. Voigt (1984, p. 56) gives an example of classroom interaction, in which the routine is disturbed. The teacher asks the students to articulate "whether they can already notice something" [a pattern in the numbers written on the board]. The obligation is to bring about constructive contributions. In the example, a student complains: "What am I supposed to notice there?" The teacher replies: "What you are supposed to notice, this you have to know yourself. Björn [another student] can you notice anything?" The teacher evaluates the student's question as a violation of the obligation to try to answer his question that has to be assumed to make sense and be of (didactical) value.

That initial question necessarily has to be ambivalent in order to make the construction of new knowledge possible. The "funnel pattern" observed by Bauersfeld (see, e.g., Cobb & Bauersfeld, 1995) is a routine for narrowing down the scope of possible responses, without ever revealing what exactly the criteria for a valid contribution are. The pattern can be seen as the interactional manifestation of what Ernest (2006), from a semiotic perspective, refers to as the general-sepecific paradoxon (see above, the section on specific rules). Not all students are equally able to acquire and interpret the emerging expectations of what an appropriate contribution consists of.

Hidden principles related to a wider social context

The teachers and the students in a classroom are not free to redefine the practice of school mathematics. There are principles in operation that guarantee continuity of classroom practices. The teacher has an obligation to deliver the intended curriculum and reach a result that is defined by curriculum documents and assessment practices. Teaching is the mediation of the institutional culture by local personnel. Patterns of classroom interaction are functional in terms of the goals of the institution and are not accomplished at the initiative of the participants in a single classroom. One of these goals is channelling different groups of students into different career pipelines.

For analysing classrooms in relation to the institutional context, a layer of interpretation has to be introduced that goes beyond the reconstruction of the participants' interpretations (the individual students' learning) and beyond the reconstruction of classroom norms. The participants' ways of acting is then interpreted by using information and theories, which the participants (usually) are not aware of. This is to reveal the social function of what happens in classrooms caused by factors to which the students and teachers have no access. It is to re-construct those principles that function in covert ways and serve the interest of power in the social system, independently of the actors' intentions. Conceptualisation and investigation of these principles draws on structuralist and critical theories. This section outlines some issues and outcomes of research dealing with principles that account for unequal attainment.

RECONTEXTUALISATION, DISRUPTIONS AND DISCURSIVE GAPS

It has been argued from different perspectives that school mathematics differs fundamentally from other types of mathematics, especially from the practice of researching mathematicians. School mathematics has a distinct epistemological character, its own systems of symbolising and a knowledge structure that is different from other mathematical practices. The culture of the mathematics classroom brings about a specific type of mathematical knowledge and mathematical language (Steinbring, 1998). Anna Sfard (1998) proposes that mathematicians and mathematics educators' views of mathematical knowledge might even be incommensurable. The disparity between different institutionalised mathematical practices and the forms of knowledge developed in these practices can be seen as the "raison d'être" of the French "Antropological Theory of Didactics".

Mathematics classrooms belong to a special type of practice, that is, to pedagogical practices. In this classrooms are very different from other practices, in which mathematics is used and developed. Pedagogic discourse is achieved by a principle of recontextualising other discourses. Recontextualisation (e.g. Bernstein, 2000; Dowling, 2009) points to the transformation of discourses that are moved from one social context to another. The process brings about the subordination of one discourse under the principles of the other. Bernstein (2000, p. 33) sees pedagogic discourse as constructed by a recontextualising principle which selectively appropriates, relocates, refocuses and relates other discourses to constitute its own order. Hence, pedagogic discourse can never be identified with any of the discourses it has recontextualised. School mathematics commonly not only recontextualises academic mathematics, but also outside-school practices. As school mathematical discourse is not static, but changes according to some progression in the curriculum, learning in a mathematics classroom can be described as moving through a range of practices and their constituting discourses, in which students have to successfully participate. Many students get lost on the way.

PROBLEMS WITH "INTERMEDIARY DOMAINS"

A common strategy to overcome the discursive gap between everyday discourse that has been described as exhibiting a "horizontal knowledge structure", and school mathematical discourse that resembles a "vertical knowledge structure" (e.g. Bernstein 2000), is the construction of intermediary domains:

> "As Anna Sfard shows us, in discussing the limits of mathematical discourse, the differences in the 'meta-discursive' rules between everyday discourse and mathematical discourse require us to develop a well-defined intermediary between the two."
>
> (Umland & Hersh, 2006, p.9).

Dowling (e.g. 2009) has described these intermediary domains as a collection of everyday objects and events that are recontextualised from the perspective of mathematics. This collection constitutes the public domain of school mathematics. This domain only becomes "well-defined" through a process of institutionalisation.

A recontextualisation brings about a new focus and a change of perspective. There are certain aspects to be sought after and others have to be dismissed and a decision is made about what is considered significant and what is accidental. New meanings and new relationships between meanings are established and at the same time new forms of expressions are introduced as well as new rules for elaborating their internal coherence. These changes in focus and signification (in "socio-mathematical norms") are rarely made explicit, except, perhaps, in the case of a breakdown of the smooth flow normally guaranteed through the interactional routine.

The following example, where the rolling of a dice is involved, shows the difficulty of the transition from everyday to mathematical meanings.

T: And if I said now roll a number smaller than one?
S: … won't work!
T: But this is also an event. Indeed, as you have already said correctly, this event…
S: … won't work! … Won't work!
T: Yes. How would we now attach an adjective to this?
S: … certain …
S: … the uncertain event.
T: The uncertain? Let us just call it the impossible event. And now my question. What subset is that actually, if I speak about the impossible event?
S: That won't work at all!

(Transcript translated from Steinbring, 1998, p. 164)

The task for the students is to see the activity of rolling a dice from the perspective of probability theory using a set-theoretic notation. As they recognise rolling dice from playing games, they interpret the teacher's questions in terms of the discourse belonging to this everyday domain, and there is of course no expression for rolling a number smaller than one. However, in the next turn, the teacher uses specialised language, such as *event* and *subset*, while only the latter might be recognised as such. This is understood at least by one student as a hint that this is not about playing games, but about rolling a dice from the perspective of school mathematics. When subordinating one practice (rolling dice) to the principles of another (school mathematics) it is always ambiguous to what extend the subordinated practice remains relevant. And this issue is even more complicated if the principles of the other practice are not completely known to the recontextualisers, that is, to the students (see Gellert & Jablonka, 2009, for further discussion).

Empirical evidence suggests that the institutionalisation of segments from everyday discourse within school mathematical discourse has a tendency to allocate the everyday insertions to marginalised groups (see, for example, Boaler, 1994; Cooper and Dunne, 1999; Dowling, 1998). The recontextualisation of domestic practices in school mathematics serves as means of stratification of achievement.

EPISTEMOLOGICAL DISRUPTION

The discursive gaps are not restricted to the problem with the "intermediary domain". In the course of a year-8 lesson about algebra in a Hong Kong classroom from the study quoted in the previous section (Jablonka, 2004), a disruption of meaning of "exact solution" is visible. A student suggests using a ruler for measuring the coordinates of the point of intersection in a Cartesian graph in order to get an "exact answer" of a system of linear equations. He learns that this is "not very accurate".

> T: Okay. Continue with your work...everybody. It's difficult for you to look for the answer in question four...very difficult...very difficult.
> T: What shall I do if I want to find the exact answer?
> S: Use a ruler.
> T: Huh? I want a very...very accurate answer.
> S: Method of substitution.
> T: Yes. Method of substitution...or?
> S: Method of elimination.
> T: Yes. Good. I'm going to look for the lazy bones that have done nothing. [Teacher walks around]

What is the meaning of "exact answer"? It is obvious that the student's suggestion was not satisfactory because the teacher repeats his initially ambiguous question in a slightly different version. The students might conclude that there is a seamless transition from accuracy of measurement to mathematical exactness, but it is in fact an epistemological difference, a difference in the quality of how the knowledge is warranted.

A discussion of how, what here has been called an "epistemological disruption", is linked to the students' background is provided by Gellert (2008). The analysis contrasts an interactionist with a structuralist analysis of an episode from a classroom.

HOW TO GUESS THE ESSENTIAL THING: RECOGNITION AND REALISATION RULES

The following example may serve as an illustration that the learner must know both, what Bernstein (e.g. 2000) calls *recognition rules* and *realisation rules*. There is a little piece of text. It is a quote from a book:

> "They kept on running, even though they were tired.
> At eight o'clock we begin studying.
> They will soon stop working.
> Usually Anita gets her cleaning done on Friday."
> The original language version (in Swedish):
> ”De fortsatte springa fast de var trötta.
> Vi börjar studera klockan åtta.
> De slutar arbeta om en stund.
> Anita brukar städa på fredagarna."

What is this text about? Is there any relationship between these statements? Is there a storyline? Is this text coherent? What is the principle one has to know in order to construct a similar text?

The text is from a language course in Swedish for second language learners. The sentences are grouped together only for meta-textual reasons and there is no other relationship between the meanings. Hence the text hardly makes any sense in terms of everyday discourse. Discovering the meta-textual similarity is hard because in everyday contexts when using language, even if one is very competent, there is no need to be consciously aware of a distinction between meta-textual features and meaning. Knowing the context (a language course for foreigners) is necessary, but not sufficient. Command of the recognition rule is important for being able to locate classroom discourse, that is, to distinguish the specificity of this context. One has to pay attention to different things and one is positioned differently in relation to the others when one participates in a language course. But recognizing the specificity of the school discourse is not sufficient for successful participation. In addition one has to be able to utter ones thoughts in an appropriate way. Command of the realisation rule is important for the production of a legitimate contribution. Without command of the recognition rule, the problem is that one does not even know what it is that one does not understand. Without the realisation rule, one cannot participate. Recognition is a necessary condition for production.

Bernstein (e.g. 2000) deconstructs "invisible pedagogy" because of its differential effect stemming from the implicitness of the recontextualisation principle, which makes invisible the classificatory principle of the knowledge to be acquired and students do not have equal access to the recognition and realisation rules.

RESUMÉE

Not all the rules operating in mathematics classrooms are equally accessible to all students. Understanding or non-understanding shapes the control over participation and eventually determines who is included, excluded or marginalised. Teachers differ in the ways in which they provide access for the students to the organising principles of the discourse in ways that some practices are of advantage or disadvantage for distinct groups of students. In an ongoing study that involves classrooms from Canada, Germany and Sweden, the researchers collaborating in the project are concerned with the emergence of disparity in achievement in mathematics classrooms[2]. The project investigates the emergence of disparities from a theoretical perspective that examines their social construction in the context of the practices of the mathematics classroom while taking into account factors that might lead to the systematic exclusion of some students and to the success of others. The project seeks to identify and describe discursive and interactional mechanisms that can explain if and how structural elements can be found in classroom interactions. Hence, the questions asked include:

- How do teachers actually introduce students to the to the organising principles of the discourse? Are there distinct groups of

[2] See http://www.acadiau.ca/~cknippin/sd/index.html

students who benefit from these introductions? Who could benefit if this practice were different?
- At which moment in the course of a teaching unit or of a school year, on which occasion, do teachers provide an insight into the criteria along which the stratification of attainment within the mathematics classroom is achieved – if they do at all?
- What can the students articulate about the criteria?

As to the practice of teaching, describing the subtleties of the process might help to be more aware of it.

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Use of Mathematical Software for Teaching and Learning Mathematics

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ABSTRACT

The computer algebra systems (CAS) such as Mathematica, Maple, MuPAD, MathCAD, Derive, Maxima have potential to facilitate an active approach to learning, to allow students to become involved in discovery and to consolidate their own knowledge, thus developing conceptual and geometrical understanding and a deeper approach to learning. Emergence of such mathematical tools and its ability to deal with most of the undergraduate mathematics cannot be ignored by mathematics educators. Use of Computer Algebra Systems in mathematics teaching is in its infancy in India.

The main idea of this paper is to give introduction to computer algebra systems, its advantages and disadvantages in mathematics teaching. We include our experiment and experiences in Mumbai University, India, where an attempt was made to include CAS-based practicals at the final year under graduate mathematics course. However this experiment did not really work. We look at some of the reasons due to which this experiment did not work and the lessons we learned from this experiment. We also mention some of the challenges one faces in the deployment of CAS in teaching mathematics and some steps to be taken to overcome these challenges in India. Some of our experiences may also be useful to mathematics educators from other developing countries, which lack the necessary infrastructure and technical expertise to implement these ideas.

We believe that mathematics teaching can be made much more interesting, inventive and exploratory using CAS. We include a small module developed using a MuPAD Pro to support our claim. The role of teachers is very important in order to make the effective use of available mathematical tools.

1. INTRODUCTION

Few working in mathematics education today would be unaware of the growth in recent years of computer technologies for teaching, learning and research in mathematics. Calculating technology in mathematics has evolved from four-function calculators to scientific calculators to graphing calculators and now to computers with computer algebra system software. The use of CAS in education is still relatively rare but the growing body of research and the interest suggests that its extended use is imminent.

The underlying concepts and proofs of many mathematical concepts involve difficult and abstract ideas that present a mountainous obstacle to many students. Computer algebra systems offer both an opportunity and a challenge to present new approaches that assist students and teachers to develop better understanding of the concepts. They can be used to change the emphasis of learning and teaching of mathematical concepts away from techniques and routine symbolic manipulation towards higher-level cognitive skills that focus on concepts and problem solving. Two of the key indicators of deep learning and conceptual understanding are the ability to transfer knowledge learned in one task to another task and the ability to move between different representations of mathematical objects. Computer algebra systems are multiple representation systems and they have the ability to facilitate graphical, algebraic and numerical approaches to a most of the mathematical concepts. Most of the CAS also provide a high-level programming language which helps the users to prepare their own set of library files to suit their needs. CAS thus allow learners to discover rules, to make and test conjectures and to explore the relationship between different representations of functions and other mathematical objects using a blend of visual, symbolic and computational approaches. Students enjoy the power and versatility of computer algebra and are encouraged to become reflective, deep learners.

While use of CAS in many countries in teaching and learning mathematics have made a significant impact at University level, in India the progress and awareness of these technology has been really very slow. Mostly, it has been confined among the researchers and handful of university and college teachers in well established research institutes, IIT's and University Departments. In this article we look at the advantages and disadvantages of using these tools in teaching mathematics at undergraduate and postgraduate levels. We also look at some of the challenges and hurdles in using these tools in India and how to overcome them. We present a Mumbai chapter on use of these technology where an attempt was made to implement these tools at under graduate level partially. However, this has not really made an impact because of several hurdles.

2. HISTORICAL PERSPECTIVE

Computer algebra systems began to appear in the early 1970s, and evolved out of research into artificial intelligence. Pioneering work was conducted by the Nobel laureate Martin Veltman, who designed a program for symbolic mathematics, especially High Energy Physics in 1963. The first popular systems were Reduce, Derive, and Macsyma which are still commercially available. A free version of Macsyma called Maxima is actively being maintained. The current market leaders are Maple, Mathematica, MatLab, SciLab and MuPAD. These are commonly used by mathematicians, scientists, and engineers. Some computer algebra systems focus on a specific area of application; these are typically developed in academia and are free.

Here is a list of some of the most popular free and commercial mathematical software. More informations on these can be found on their respective websites.

Software	*Year of Start*	*Utility*
Mathematica*	1998	General purpose CAS
Maple*	1985	General purpose CAS
MuPAD*	1993	General purpose CAS
MatLab*	Late 1970	General purpose CAS
MathCAD*	1985	General purpose CAS
Magma*	1993	Arithmetic Geometry, Number Theory
SciLab	1994	General purpose CAS
Maxima	1998	General purpose CAS
YACAS	1999	General Purpose CAS
SAGE	2005	Algebra and Geometry Experimentation
Macaulay2	1995	Commutative Algebra, Algebraic Geometry
GAP	1986	Group Theory, Discrete Math
GP/PARI	1985	Number Theory
Kash/Kant	2005	Algebraic Number Theory
Octave	1993	Numerical computations, Matlab-like
Singular	1997	Commutative Algebra, Algebraic Geometry
CoCoA	1995	Polynomial Calculation
Gnuplot	1986	Plotting software
Dynamic Solver	2002	Differential Equation
R	1993	Statistics

Here star (*) ones are commercial software and remaining are free software. Note that the above list is not complete still and there may be many more mathematical software.

3. INTRODUCTION TO CAS

Computer algebra systems (CAS) are special kind of mathematical applications providing users means for doing symbolic, algebraic and graphical manipulations with computers. This means that instead of only counting with numbers, computer algebra systems can also manipulate symbols and, when possible, carry out complex calculations exactly. These systems can be roughly divided into two main categories: special purpose systems and general purpose systems. Special purpose systems usually deal with some specialized branch of mathematics, viz. dynamical solver for differential equations, singular for algebra and algebraic geometry, KASH for algebraic number theory, gap for group theory, magma for number theory, CoCoA, Macauly2 for commutative algebra/ algebraic geometry, Octave for numerical computations etc. General purpose system, on the other hand, usually try to cover as many mathematical areas as possible. This generality makes general purpose systems ideal for open learning environments in most cases.

Most CAS allow the user to write sequential programs for complex tasks, and have all features of high-level programming languages. CAS also have most of the features of numerical systems for visualization of 2D and 3D-plots, numerical computations and animations. It is therefore an ideal tool for directing learning towards multiple-linked representations of mathematical concepts. Through carefully designed activities students can investigate the links between different representations of objects, recognize their common properties and begin to construct their personal structures of mathematical knowledge. Student activities have to be designed with very detailed cognitive steps in mind. Appropriate teacher intervention will usually be required to ensure that the students follow through the required learning stages, in particular, the reflective thinking.

A typical student approach to problem solving is to find a suitable worked out example to mimic and then carry out the computation. Clearly this strategy is limited by the extent of the students' memory bank of similar problems and inhibits flexible thinking. A better approach is to consider alternatives, experiment, conjecture and test, then analyze the results. A computer algebra system can be a major factor in developing an exploratory approach to learning mathematics and, in particular, investigating problems from multiple representational perspectives. Using CAS to produce graphs, carry out calculus operations or perform repetitive calculations, students can be encouraged to make and test conjectures, to consider alternative solutions and to tackle open-ended problems. Removing the burden of manipulation and computation allows students to spend the more time on these other activities. This approach can make the study of mathematics more enjoyable, more relevant and more rewarding to it. At present most of their time is spent practicing routine skills. Perhaps it is not surprising that students view mathematics as a collection of formulae (to be memorized) and to do maths is to compute. If more routine computation is done on a computer more time is available for concentrating on concepts, motivation, applications and investigations.

With the traditional undergraduate curriculum, students do not often regard themselves as active participants in mathematical exploration. Rather they are passive recipients of a body of knowledge, comprising definitions, rules and algorithms. Computers offer a number of didactic advantages that can be exploited to promote a more active approach to learning. Students can become involved in the discovery and understanding process, no longer viewing mathematics as simply receiving and remembering algorithms and formulae. The power of computer algebra goes beyond routine computation. It has the potential to facilitate an active approach to learning, allowing students to become involved in discovery and constructing their own knowledge, thus developing conceptual understanding and a deeper approach to learning.

We include a sample output using MuPAD Pro 3.1, which explains the geometricmeaning of Lagrange multipliers to solve constrained optimization problem.

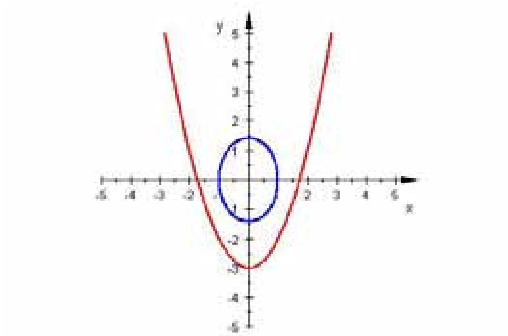
Example 1. Use the method of Lagrange Multipliers to maximize/minimize

$y - x^2$ ubjected to $y^2 + x^2 = 2$

For convenience let $f(x,y) = y - x^2$ and $g(x,y) = 2x^2 + y^2 - 2$. Geometrically, the maximum/minimum of the above problem occur where ever the gradient of $f(x,y) = y - x^2$ and gradient of $g(x,y) = 2x^2 + y^2 - 2$ are parallel. This is same as, the level curves of f and b have common tangents at these points.
Using MuPAD animation, we can show that there are four points on the ellipse $g(x,y) = 0$ at which this happens.

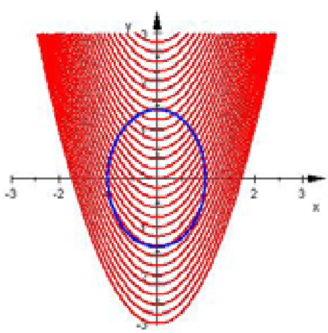
```
>>f:=(x,y)-> y-x^2//to define the function f;
>>g:=(x,y)->2*x^2+y^2-2//to define the function g
>>pf:=plot::Implicit2d(f(x,y)=c,x=-5..5,y=-5..5,
        c=-3..3,Color=RGB::Red,Frames=100,LineWidth=0.5)
>>pg:=plot::Implicit2d(g(x,y)=0,x=-3..3,y=-3..3,Color=RGB::Blue,Line
Width=0.75)
>>plot(pf,pg,Scaling=Constrained);
```

The output is shown in the figure below

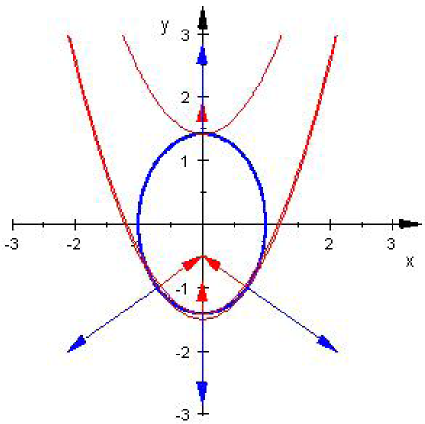


When we animate the graph we see that there are four points at which the level curves of f and g have common tangents. This is shown in the next figure.

```
>>plot(plot::Implicit2d(f(x,y)=c/5,x=-3..3,y=-   3..3,Color=RGB::Red,Lin
eWidth=0.5,
VisibleAfter = c)c=-15..15,pg,Scaling=Constrained)
```



Now we can plot (using MuPAD) the gradient at the point at which the level curves f and g have common tangent. Look at the Figure below. We are suppressing the MuPAD codes which produced this figure.



All the steps to solve the above problem can be performed using MuPAD and it can be shown that there are four points and at which gradient f and g are parallel. We are not including the MuPAD codes for the analytic solution of the above problem, as we wanted to bring out the geometric behind the problem.

4. ADVANTAGES OF USING CAS

1. Helps develop visual/geometrical understanding.
2. CAS can help to increase the value of the knowledge and degree of interest of students.
3. Can explore concepts before "hand skills" to do so are available.
4. Can explore realistic problems.
5. CAS help to increase student motivation and improve students attitudes towards Mathematics.
6. Due to the potential interactivity of these tools, students are able to attain a higher level of abstraction in mathematical problem-solving something which clearly represents a significant didactic accomplishment.
7. Allows students to concentrate on problem formulation and solution analysis.

8. Easy to give math demos and advanced mathematical ideas can be introduced very easily and concretely.
9. Users having knowledge of some programming language (C, C++, Pascal, Fortran) have greater advantage and can prepare their own library function which are suited to their needs.
10. It will help teachers to develop innovative, challenging and exploratory teaching modules.
11. Researchers do not need to spend more time on tedious computations rather they can spend more time in analyzing and the computation part can be easily be done using these tools.
12. When CAS are not used, the teacher tends to be the sole center of attention whereas, when they are used, there is an observable increase in student participation, autonomous activity and interaction among students, hereby making the process of acquiring and constructing mathematical knowledge more student-centred.
13..Enhances job opportunities for students.
14..People from other disciplines not having sound mathematical knowledge can very easily solve mathematical problems which they come across.

The benefits of using CAS in mathematics teaching is enormous and almost every conference on technology for mathematics advocates this. For more detailed discussion one can refer to ((Albano G., Desiderio M. 2002) , (Artigue M. 2001)}, , (Bertemes J. 2006), (Bohm J. CAME 2007), (Mackie D. 1996), (Majewski M., 2004), Westermann T. 2000), (Yearwood, J.U. A. , 1996) etc.)

5. DRAWBACKS OF USING CAS

In spite of so many benefits of using CAS there are some drawback, that is why many people advocate against its use and raises some concerns. Through our experiences and discussion with teachers and students, we are listing some of the drawbacks of using CAS.

1. Students tend to use CAS blindly and they do not bother about thevalidity of answer obtained through CAS.
2. Most often students try to use CAS as an advanced calculator and refuse to learn concepts.

3. Decline of students' paper-and-pen skills.
4. Difficulties in evaluation of a course taught using CAS.
5. Greater time needed for class preparation.
6. Lack of familiarity with the computer and CAS.
7. Fear of making syntactical errors in class.
8. Lack of administrative recognition of increasing teaching load.
9. CAS syntax can be an unreasonable burden on students.
10. The course can be victimized by equipment failure or inadequate equipment.
11. Students' algebraic manipulation skills will deteriorate if they are allowed to rely on computer algebra but that these skills are an essential foundation for mathematics.
12. CAS at time can produce meaningless expressions.
13. Using CAS can potentially prevent students from making the proper connections between the techniques used and their mental approach to Mathematics.

6. CHALLENGES AND HOW TO OVERCOME THOSE

There are several challenges if we want to implement CAS-based mathematics teaching in India. However, these challenges can be over come. We list some of the major challenges which we/will come across in order to employment of such tools in mathematics teaching.

6.1 *Challenges and Difficulties*

1. Availability of computers in the laboratory and to teachers and students is still a distant dream.
2. Most of the CAS are too costly and hence not affordable to college students and teachers.
3. Classrooms are not equipped with relevant hardwares which is required to integrate teaching using CAS.
4. Teachers are not having proper computer literacy and knowledge of CAS.
5. Many teachers are not willing to move from traditional teaching style to CAS-based teaching wherever necessary.
6. Unavailability of innovative and exploratory teaching modules.
7. Courses are not designed properly. It does not give space, time and opportunity of exploring the subject using CAS.

6.2 Overcoming these challenges and difficulties

1. All colleges/institutes to have proper computer labs and to give students enough opportunities to explore.
2. Use of free mathematical software like Scilab, Maxima, octave etc. to be encouraged.
3. Development of similar software may be initiated and encouraged.
4. Classrooms should be equipped with relevant hardwares.
5. A series of teacher-training programmes throughout the country may be initiated in order to make them aware of such tools.
6. Innovative teaching modules and projects be prepared which make students and teachers realize that these tools are not merely advanced calculators but can be used to solve a very complex problems and help them to experiment and explore (one of the vital aspects of learning).
7. Courses may be redesigned to encourage the use of CAS and also provide time for its use.
8. Students must be allowed sufficient time to learn the language and features of CAS before using it to enhance their learning.
9. In recent years most of the students have knowledge of some programming language which will be very useful in order to experiment and explore not just existing inbuilt function in CAS but can create their own need based functions. This also fosters creativity.
10. CAS should not be used as a black-box in the beginning of introduction of a mathematical concept. Till the topic is not learnt properly, CAS should be used as a white-box. Once the topic is thoroughly learnt then it can be used as a black-box. Black-box/White-box principle (Buchberger B.,1990) is very useful for developing innovative teaching modules using CAS.

We believe that the most appropriate approach involves using programming and CAS together to allow students to create the specific necessary functions that will allow them to solve the problems involved in the subject matter under study.

7. CAS AND TEACHERS

It goes without saying that the classroom teacher is the key to the successful introduction of new methods and new technologies. Of course, it is possible for the student to come across these independently in the case of CAS. With the increasing speed of technological development, it is crucial that teachers keep themselves informed so that they are in a position to make valid judgments and adapt their teaching accordingly.

Teachers, of course, have a crucial role in students learning (with or without CAS). Integrating CAS into teaching changes many aspects of class-room practice which teachers will make on the basis of their prior teaching styles and their beliefs about mathematics and how it should be taught. While using CAS to solve problems, students sometime make silly mistakes which produces a totally irrelevant output.

Teacher support and appropriate intervention is crucial to correct such mistakes. Judging the right amount of help at the right time is a skill acquired through experience. Computer algebra system use in mathematics teaching and learning is in its infancy. Nevertheless there are many teachers and education-alists who have integrated CAS into their teaching or conducted research into student understanding with CAS or who have led curriculum/assessment projects involving CAS use.

8. USE OF CAS-- A MUMBAI CHAPTER

Use of computer algebra systems (CAS) at the University of Mumbai was initiated in late 1990's by means of workshops integrated with refresher courses for degree and engineering college teachers at the Department of Mathematics, University of Mumbai. Initially teachers who attended refresher courses were made aware of some of the mathematical tools mainly mathematica, WinPlot and MuPAD for teaching mathematics. Because those days only few computers were available in computer lab, occasionally they were given hands-on practices in groups. It was in the year 2003, a three days workshop on use of MuPAD 2.5 Lite and other related free mathematical software was held for Mumbai University degree college teachers teaching mathematics, keeping in mind to encourage the use of some of mathematical tools in mathematics teaching at college level. Teaching modules for few mathematical topics in analysis, multi-variable calculus, linear algebra were prepared to help the participants. In the beginning there were some concerns that many teachers may oppose this move, however after attending the workshops all were very happy and very keen to use them. About 100 teachers participated in this workshop very enthusiastically. The participants were also given hands-practices in different groups. All the teachers were very happy to see the kind of innovations and motivations that can be inculcated in teaching mathematics using these tools. Board of studies of Mathematics of Mumbai University then recommended that it will be compulsory for the final year students of mathematics to include printouts of solutions of two problems using MuPAD or any other mathematical software in each of the four papers in their syllabus. The main idea behind this endeavor was to expose the teachers and the students to some of these tools which will help in understanding and visualizing many mathematical concepts.

However, we believe that this has not worked properly. There are number of reasons behind this:

1. Most of colleges did not have required ambiance for teacher to integrate the CAS with their teaching.
2. College computer laboratory was also not available for this purpose in most of the colleges.
3. Most of the teachers themselves did not have access to computers at their college and their residence.
4. Teachers did not take interest in exploring and experimenting with these tools themselves and did not encourage their students to experiment these tools.
5. There were no follow-up workshops any further.
6. MuPAD Lite 2.5 is no longer freely available.

Due to the above difficulties in most of the cases students were just reproducing same solutions again and again and the original idea in our opinion got defeated. Few workshops in Mumbai at University Institute of Chemical Technology (2005), Indian Institute of Technology (2006 and 2007) were held to make the teachers, research scholars aware of the some of these technologies however there are not enough.

With the insights provided by this experience, we can improve the strategy/methodology of deployment of CAS in Mathematics teaching and move forward. We believe that the situation now has improved considerably. Most of the colleges do have relevant hardware, good computer laboratory where students and teachers can experiment, explore and discover using some of the CAS. Thus, if proper guidance is provided, these technology can make an impact and mathematics learning can become much more interesting and enjoyable.

9. WHERE DO WE STAND?

Many foreign universities have fully integrated CAS into mathematics teaching for several university degrees, to the extent that their use is no longer considered to be novel or innovative, but rather something common place in such courses. CAS-based mathematics teaching at the undergraduate and postgraduate level has not been explored in India much. Therefore, there is a lot of scope for improvements. In recent years many government funding agencies have provided financial support to setup computer laboratories in colleges and to acquire useful software. Therefore, we believe that the situation now is far more conducive than what it was few years back, in order to make these tools as a part of our curriculum and make teaching and learning process much more interesting, insightful and make students involved.

The authors had opportunity to interact with many young college teachers and students of undergraduate and postgraduate level of various universities who were very enthusiastic to learn these tools and incorporate them into their teaching. This makes the implementation of CAS-based mathematics teaching and learning much easier. What we need is to create proper awareness of these tools among the teachers by holding workshops and training programmes at various places. One of the good things about all these tools is that they have a very good inbuilt documentation, tutorials which make the learning much easier. Already tonnes of tutorials, lessons are available on the web which can be used for self-learning.

10. CONCLUSIONS

A computer algebra system is a tool not a self-contained learning package or encyclopaedia of mathematical knowledge. It is the way in which it is presented to and used by students that determines its ability to influence learning. Much emphasis these days is placed on student-centered learning and less on the teaching but teaching and learning are equally important. It is necessary to first understand the learning process and then design teaching and learning activities to achieve these. Only then will students become deep learners.

Our accumulated experience reveals that CAS are computer tools which are easy to use and useful in both pure and applied mathematics courses. Use of CAS in the teaching of Mathematics should be channelized to maximize the opportunities offered by CAS technologies. Optimal use should be aimed at improving student motivation, autonomy and achieving participatory and student-centered learning. One powerful idea involves combining CAS resources with the flexibility of a programming language.

There are many implications of using computers in the teaching and learning of mathematics at university. As students often point out to us it is very exciting, enjoyable and productive to use computers in class. They are keen to use computers, so the environment becomes more conducive for learning. Students' natural curiosity can be utilized to its fullest potential because they are keen to explore and discover.

Irrespective of the software packages used, it is important to remember that the software should support the learning and curriculum and can not substitute good teaching. Traditional teaching methods must be supported with modern tools for problem-solving. It does not imply a reduction in the standard of education or of necessary subjects, but it is vital that the curriculum is carefully considered and that passive teaching is replaced in favour of new methods which promote active participation of students.

In order to make the CAS based mathematics teaching reality, we must take some of the following measures:

1. To develop methodology for teaching mathematics with CAS.
2. To develop strategies to implement teaching methodologies.
3. To produce innovative teaching modules using CAS.
4. To organize regular workshops, training programmes for mathematics teachers.
5. To redesign the course curriculum.
6. A lot of research is needed to understand the students attitude and psychology of learning mathematics using CAS.

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Video Study of Mathematics Teaching in Chile

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ABSTRACT

A detailed characterization of school mathematics teaching patterns using the videos that teachers recorded themselves as part of the 2005 Chilean national teacher assessment program is presented. This is a new compulsory teacher assessment program, but in 2005 teachers volunteered to participate in a first version of the program. This research work includes the analysis of more than 700 forty minute video recordings, both from elementary and high school classes, from different regions of the country, containing lessons on the algebra, geometry, data and numbers strands. One 4 minute slice of each video was studied. The slice was randomly selected from the following moments of the class: first 4 minutes, from minute 10 to 14, from minute 20 to 24, from minute 30 to 34 and the last 4 minutes. More than one hundred variables were codified by independent coders. The codification methodology was successful and stable across different coders, making the analysis huge amounts of data possible. The main findings were: very little autonomous student participation (only one mathematical question made by students per 40 minutes class), teachers neither present nor discuss any proofs, no use of information technology, almost no use of textbooks, almost no explicit use of metaphors.

INTRODUCTION

This research work intended to study mathematics teaching practices in primary and secondary school education in Chile: how mathematics is actually being taught in the classrooms. With that information at hand, comparisons with other countries could be made, but also, and more important, recommendations for public policy towards improving the quality of pre-service and in-service teacher training could be made in order to improve the teachers classroom practices and in turn, their students achievements.

Considerable amounts of data were already available, consisting of hundreds of video recordings made for the 2005 Chilean Teachers Professional Assessment (Manzi, 2007). Mathematics teachers (as well as teachers of other disciplines) that chose to be evaluated in 2005 had to have a video of one of their classes recorded as part of the assessment requirements. These videos had been stored by the Ministry of Education after having been used for their original purpose. We could get a hold of a total of 720 of those videos, so that they could be analyzed in search for a characterization or description of the Chilean teaching practices in Mathematics.

The analysis was chosen to focus on didactic aspects such as the modality of student working arrangement (whole class work, small group work, or individual work), degree of student participation in class and level in which the teacher encourage it (Mathematics questions asked, for instance), motivational strategies and didactic strategies particular to Mathematics, such that proofs and use of mathematical metaphors (Lakoff and Nuñez, 2000; Richland, Holyoak and Stigler, 2004; National Mathematics Advisory Panel for multiple representations, 2008). The class dynamics was also quantified, attempting to quantify and find characterizations of the beginning, middle and ending of the lectures. Finally, the use of different types of technological aids, such as board, computers, text books, paper and scissors, etc, was quantified.

Theoretical and methodologically, only objective measurements, that cold be consistently repeated by a second observer, were made to account for the different aspects to be measured. Also, to be able to work with the large amount of videos we had at hand, for each one only a 4 minute (or sometimes a 2 minute) slice was randomly selected viewed, corresponding to either the first 4 minutes of the class, minute 10 through minute 14, minute

20 through minute 24, minute 30 through minute 34, or minute 40 through minute 44, given that in Chile there are 45 minute lectures.

The working hypothesis was essentially that there would be rather scarce use of proofs and metaphors, and that the percentage of use of modern technology such as computers would be low. On the other hand, polls conducted with teachers report that they, in a great percentage, use the textbooks that the Ministry gives for free to all children in public schools (Ministerio de Educación de Chile, 2002; Universidad de Chile, 2006). However as we will see, the results were appalling: no evidence of proofs, metaphors or computers, and the textbooks were almost not used at all. Also sadly surprising was the fact that student participation, in the way of (mathematics) questions asked, was extremely low: an average of one student question in the whole lecture time.

As expected also, some patterns emerged in the way that there was a clear characterization of the three thirds of the lecture, and the primary and highs school teachers have very distinctive didactic differences.

The fact that the short class slices would allow us to gather relevant and statistically significant information about the pedagogical practices of mathematics teachers was also a working hypothesis that we could confirm within the study. Examples in the literature that that encouraged this approach are, for instance, the SPAFF methodology that is able to predict accurately future behavior of married couples by watching 3 minute videos of them before their marriage (Gottman, 2000; Coan and Gottman, 2007) and the Ambady and Rosenthal study of college teacher assessment by students (Ambady and Rosenthal, 1993; Gladwell, 2005).

THEORETICAL FRAMEWORK

There are various theoretical approaches that can be adopted to design and analyze a study of the teaching/learning situation in the school classroom. The one chosen for this study had into consideration that the material to be studied consisted on hundreds of video recordings already made. And these videos consisted of one 45 minute class per teacher, and were made at the teacher's request, without any research consideration in mind. Having that sort of material does not really allow for questions involving why some teaching strategy is used, which would probably involve interviews with the teachers after the lecture, or studying in depth the strategy to teach some topics, which might require having whole sequences of lectures, for instance. Instead, the material forced us to search for and choose a very pragmatic theoretical framework, one in which one uses very objective information to give an account of what is globally happening in the classrooms over the country, without really being able to describe or analyse each separate class.

Such a framework had already been used by previous video studies, such as the TIMSS Video Studies (TIMSS, 1999; LessonLab, 2003; Hiebert, Gallimore, Garnier, Givvin, Hollingsworth and Jacobs, 2003; Stigler and Hiebert, 2004) . We have adopted it, and, essentially, within it we try to draw conclusions about what is going on in the classroom situation based only on very objective evidence. Evidence that does not depends on anyone's opinion, but on objective measurements such as number of mathematic questions that the teacher asks, or the amount of time the teacher spends writing on the black (ore white) board. The variables or indicators to be used also enjoy the quality of being repeatable, that is to say, independent of the particular observer. The idea is that if patterns are obtained using such variables, then the conclusions drawn can become very solid evidence of what is happening countrywide, even though probably no judgement or assessment can be made of individual teacher practices, because relevant though more subjective factors can be important when you come to judge an individual class. For such a judgement a different epistemological lens might very likely be necessary (Tuminaro and Redish, 2007; Redish, 2003; Díaz, 2006), and the material available for this study would probably be non suitable for the study.

RESEARCH METHODOLOGY

As described earlier, the class videos had already been recorded without a research study in mind. Therefore our research methodology did not start by dealing with the design of the recordings, as it usually happens with this type of study, but rather with the design of the means for obtaining the most information we could from the already available material.

The videos were available on tape, therefore the first step was to encode them into a digital format that would be both, compact and easy to reproduce so that more than one observer could view the same video at the same time, in different computers. The digital video format XviD was chosen, both for being open source and an efficient equivalent of the DivX standard. Furthermore there is plenty of computer software that is able to play this format, and under a variety of platforms (Windows, Mac, Linux, for instance).

As the videos were encoded, they were given a filename that consisted of the national ID number of the teacher of the class, which was among the few extra information we had available for each recording. It must be pointed out that the ID number is somehow correlated to the age of people, younger people typically has higher ID numbers, therefore this number gave us indications of the approximate age group of the teachers. The videos were classified also by the educational level they belonged to, that is to say, middle school or high school. 78.8% of the tapes were of middle school level and only 21.2% from high school. There were no tapes from elementary school.

In parallel to this encoding, research was performed to find out the variables and different classifications already used in other video studies, and a large number of variables (around 200) were chosen as candidates to be measured in the videos. Later, in an iteration process which used an increasing number of randomly selected videos, from 10 initially to finish with 100, the suitability of variables was tested, and in parallel assistants were trained on the video coding task. Those variables which did not fit the repeatability criterion (no statistical difference was to be found between the values measured by both researchers who recorded those variables), or those that were not able to be measured because they belonged to categories simply absent in the group of videos (for instance, metaphors were not found, therefore variables that distinguishing different types of metaphors did not make any sense for this sample) were dropped, and we were left with a group of 120 variables.

We saw no practical need for using specialized software, such as "Transana", to record the data, because, as we will mention later, only 4 minutes or less were coded for each video, and for those lengths it is sufficiently easy to find the sought information without the need of special software. Instead, the set of variables were divided into two Excel forms, and two researchers were in charge of collecting the data to be filled in each of the forms. Thus, each variable was recorded by two people for each video.

The number of videos to be analyzed was considerably large: 720. Viewing them all in full was impracticable considering that it was estimated that for accurately recording each of the variables to be measured the material would have to be replayed about 20 times, which would give about 15 viewing hours per video, per form and per reviewer. The decision was then made that only a short sample of each video would be processed.

There is plenty of encouraging experience on such time sampling of videos. For instance, Gottman (Gottman, 2000; Coan and Gottman, 2007) has developed a methodology named SPAFF with which he has shown that by recording a 3 minute video of an engaged couple he can predict with great accuracy future (several decades, in fact) behavior of the married couple, including things like if they will stay together or will get divorced. There is also the classic study by Ambady and Rosenthal (Ambady and Rosenthal, 1993), which shows that the students evaluation after viewing a 10 second video of a university professor class proved to be statistically equivalent to what they would write after a semester attending the class. On the other hand, there are studies that claim that much longer observation periods are needed to judge teaching practices. For instance, Shimizu and Yoshinori (Shimizu and Yoshinori, 2003) advocate for a whole sequence of 10 lectures as a minimum unit to study patterns of pedagogical practices. They state that, for instance, homework assignments play a linking role between lectures, and would have a significant educational role.

In our case the videos were pre recorded and we did not have access to any sequence of classes, and we chose to review 4 minute slices from all the recordings available. In fact, we chose to do some checking of the hypothesis that even smaller times might be good enough, and in several videos, slices only 2 minutes long were viewed.

The segments were chosen so that they would be well distributed over the lecture time: one starting at the beginning, one at minute 10, one at minute 20, one at minute 30 and one at minute 40.

Results and analysis

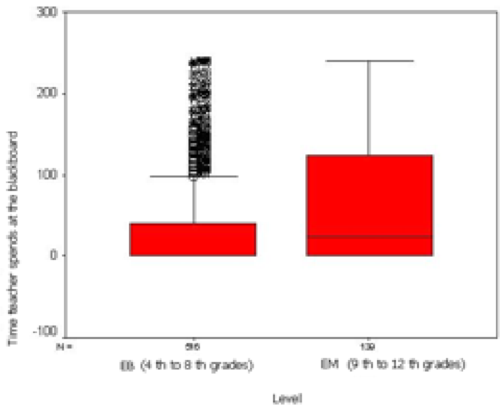
About the general statistical validity of the choices made, both of variables and of short slices, the data obtained is encouraging. From both, the preliminary tests and the final coding, the data shows that the different researchers coding the same variables obtained statistically similar patterns, thus validating the stability and repeatability hypothesis.

Furthermore, there was no statistical difference between the outcomes of the variables measuring total time of an event, for the 2 minute slices and the 4 minute slices (the 2 minute slice times properly scaled, naturally).

However variables counting number of events did not always show the same behavior independent on the slice size. In fact, when the events counted were rather short ones, the variables behaved the same, but for events essentially longer, the behaviors differed significantly.

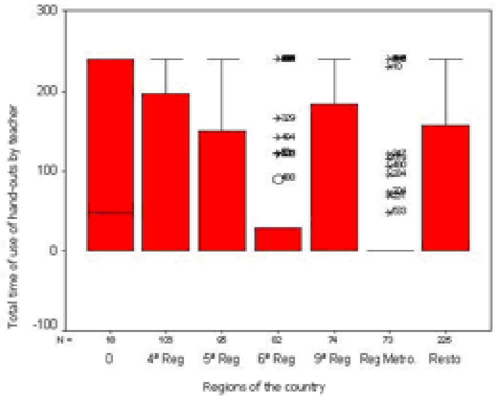
Let us now review the data gathered and the findings that seem to derive from it. Let us first concentrate in variables or groups of them which give us information that one could perhaps think that "common sense" would tell us that the results could not be different from the ones suggested by the data. We think that even if that is the case, having hard empirical evidence of the facts, justifies including this information as relevant results of the study.

The following graph shows the amount of time in seconds, within the 4 minute slice reviewed, that teachers from middle school (EB) and high school (EM) spent at the blackboard. It can be seen that high school teachers spent considerably more time at the board.



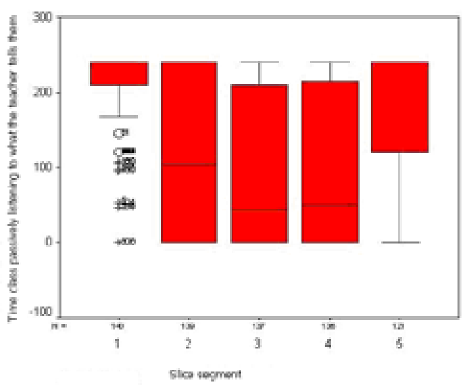
We also gathered similar data regarding the fact that high school teachers spend more time than their middle school peers writing mathematics, that they have their students spend less time in activities involving paper, scissors, cartons, etc, that their students spend more time solving mathematics problems, and that they make less eye contact with the students. All this tells us that there are clear differences between didactic strategies of middle school and high school teachers.

Looking for regional differences in didactic strategies, we found that there is a significant difference between the teachers from the "Región Metropolitana", which is the region where Santiago, the Capital City, is located (and which, incidentally, concentrates more than 1/3 of Chile's total population) and the rest of the country. For instance, the following figure shows the amount of time (within the 4 minute slices) spent working on teacher hand-outs. It is obvious from the graph that the use of hand-outs by teachers at Región Metropolitana is neglectible compared with teachers in other regions.



Similar data shows that teachers from Región Metropolitana approach the students a lot less than the rest (to check and supervise their work), but they ask their students considerably more mathematical questions (and therefore the number of answers to mathematical questions by the students is higher).

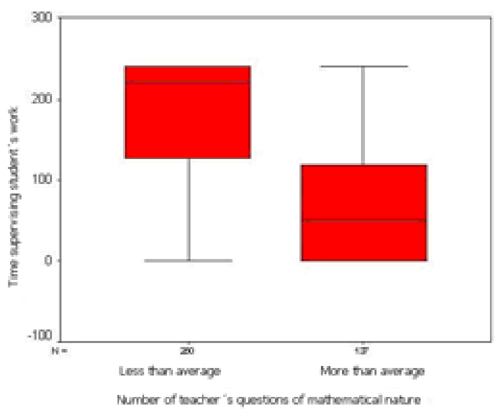
Regarding the different sections of the class, meaning beginning, middle and ending, as could be guessed in advance, there are measurements that differentiate them, for instance, next graph shows that in segment 1 (0-4 minutes) and 5 (from minute 40) there is no independent student work, and most of the time the whole class is passively listening to what the teacher tells them.



Some more "very obvious" information of a similar sort as the previous one can also be extracted from the data. Given the obvious nature, we only mention it here, without giving the associated numbers. Teachers walk more around the students seats supervising their work when they are not lecturing to the whole class. Teachers approach more the students to check their work when they are working with concrete materials (cutting and pasting papers, or drawing pictures). There is a different didactic pattern in the geometry strand: as opposed to all the rest of the strands (algebra, chance, numbers), the teacher shows more objects, or draw more pictures, and the students spend more time cutting and pasting papers.

When looking at what happens in the intermediate slices (number 2: from minute 10, number 3: from minute 20, and number 4: from minute 30) some predictable, although not necessarily desirable, correlations appear.

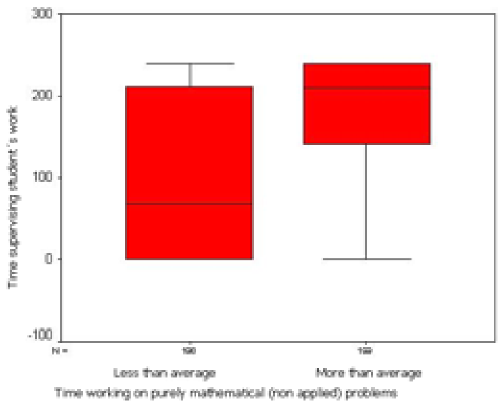
The following graph shows that when a teacher asks fewer questions of mathematical nature (fewer than the mean) to their students, he o she spends considerably more time supervising student work than the teachers that ask more (mathematics) question than the average:



Thus, teachers seem to be divided in two disjoint groups marking clearly differentiated didactic patterns: the ones who ask math questions and the ones that supervise student work.

It can also be seen from the data that, for instance, when teachers asked more than the average mathematical questions to the students, they also called more students to work on the blackboard.

Another phenomenon found when observing slices 2, 3 and 4, is that when students spend more time than the mean working on purely mathematical (non applied) problems, they spend less time doing activities with paper and scissors (not really surprising), but the teachers in turn spend more time supervising their work. The following graph illustrates this last statement, by presenting teacher's time supervising students for the cases of less and more than average student work on purely mathematical questions.



From segments 2, 3 and 4, it can also be observed that younger teachers seem to use some different strategies than older ones. Teachers with ID less than the average (less than 6.915.638) ask more questions of mathematical nature, walk around the students more and bring the students to the blackboard more than the ones with ID number greater than the mean. It must be recalled that ID number is correlated with age (higher ID numbers usually mean younger people). Of course, it must be pointed out that 82% of the teachers with smaller ID number were middle school teachers, and only 68% of the ones with higher ID numbers taught in middle schools. This might then be tied to teaching level rather than age.

Now, let us look at what is probably the most important, but also most disappointing, information that can be extracted from the data in this study:

The students ask very few question of mathematical nature in class. Indeed there is, as a group (not each student individually), at most one math question per class.

Other sources of bad news:

- There was very scarce use of textbooks in class.
- There was no evidence of ICT usage in class. No computers or educational software at all in the observed segment of any class.
- There were no mathematical proofs, or evidence of deductive reasoning in the observed videos.
- There were no mathematical metaphors to be observed in class.

One encouraging one, to finish:

- Teachers did not make mathematical mistakes. There were actually a few conceptual errors that were corrected right away, but that was.

DISCUSSION AND CONCLUSIONS

The conclusions of this study are of two types. One has to do with the validation of some methodological aspects, and the other with findings coming from the observed variable.

About methodology, we could conclude that our coding was stable and reliable. Since different coders watching the same slices of the same videos produced statistically similar patterns, then the information we are getting from the variables does not depend on the observer and our results can in a way be called objective, or at least repeatable.

It is also a conclusion belonging to the methodology realm that the short segments we chose to be viewed in the videos, actually provide relevant information. It has already been pointed out that the information provided by variables measuring total time duration of events does not degrade considerably when we consider 2 minutes slices instead of the longer 4 minute slices. And the same is true with counting variables, provided that the episodes being counted are short. For variables counting appearances of longer episodes, the quality of the information deteriorates when halving the temporal size of the slice. It easy to make up an explanation for the phenomenon, longer events have higher probability not to happen entirely during shorter observation period.

In terms of the information provided by the measured variables, as it has been mentioned before, there are some conclusions that might seem rather obvious, like the one saying that there is no independent student work at the beginning or ending of the class. Obviously we too share the view that those are not pioneering findings, but their inclusion here has the virtue of documenting that this is happening, and also quantifying it.

In many other cases, apart from the reasons given above for including some "common sense" conclusions, it is also true that those facts do not have to be that obvious. For instance, the fact that high school teachers spend more time at the blackboard than their middle school colleagues, might be something we would expect, but not necessarily something we desire. It might be beneficial for high school students to solve problems by themselves in class (a considerable part of the lecture time, at least) and have their teachers walking about the classroom checking on the student's work and helping them with it, instead of spending that much time lecturing from the board. Also, even though we expected to see that emerge from the data, there is no real reason why the use of concrete material in high school lectures should be scarce. Leaving aside the obvious example of the geometry strand, the teaching of probability, strand which is present in high school curriculum, can greatly benefit from playing al sort of chance games which usually involve concrete materials (under the form of cards or dice, for instance). It could also be arguable that it should be natural to spend less time writing mathematics while teaching in middle school than in high school.

In any case, justifiable or not, it is clear from the data that there are didactic differences between middle school and high school.

Somehow puzzling is the data showing that teachers from Región Metropolitana (which, as we said, concentrates a large amount of Chile's population, more than one third of it, and by far larger than any other region of the country) seem to have a more conservative teaching style. They seem not to give hand-outs to the students, then approach the kids a lot less than their colleagues in other regions to check on their work. However, they ask more questions of mathematical nature, and in turn, get more answers from the students.

Among the data describing slices 2, 3 and 4, there are also some obvious things, for which the major importance of measuring them is to have quantified evidence that these things happen, but here there are also some others where it could be argued that even though they were to be expected, they are not necessarily desirable.

For instance, there is no reason why the supervision of student work should mainly be only when they are using concrete material, as it was found.

Also, there is no need for teachers not to be able to supervise student's work and at the same address the whole class, or even ask math questions to the students. The use of technology, and/or games can make it easy for the teacher to do both things at the same time.

There is also no obvious reason why the concrete material should be used more frequently, as found, while working in applied problems. Such materials, and later the mental representation of them, can often help greatly the solving of problems of more mathematical or theoretical nature (National Mathematics Advisory Panel, 2008, Chapter 4: Report of the Task Group on Learning Processes), especially if the right metaphors are used to approach them.

Clearly, the most alarming finding of this study is that relevant student participation, seem to be neglectible. An average of a little less of a question of mathematical nature coming from the students per class is obviously much too little. The origin of this must be investigated in some other way. Being so few, our data does not allow us to correlate the number of student math questions to any other variable that we might think is a cause, such as time spent by the teacher lecturing from the front of the classroom.

To explain the scarce evidence of textbook usage, international evidence might have the answer. According to a NSF study in the United States (Banilower, Boyd, Pasley and Weiss, 2006), the probability of a teacher using a textbook is extremely sensitive to the hours spent on professional development training for the use of the materials. And even though Chilean state makes the great effort of giving every child from public schools a textbook, no teacher training for the book usage is offered. Also, the state policy of constantly opening competitive biddings for these textbooks, makes it very likely that a textbook for a given level change every 2 or 3 years, and it is hard for teachers to be constantly readjusting their lectures to ever-changing textbooks.

For the absence of ICT material, and of proofs in the lectures, some people argue that the videos we used were for teacher assessment, and the teachers probably wanted to record a class that would be evaluated as an excellent one, and they probably did not feel confident enough either with the use of technology, or discussing proofs. If that is true, and in "normal" classes there might be a bit more proofs and use of technology, the argument also would say that teachers are far from feeling confident with these two things, and so there is still a big problem with them. Similar arguments apply to lack of textbook usage and this sort of argumentation.

This line of argument tells us that we should not feel too happy about the absence of conceptual flaws in the classes that were reviewed. They were carefully prepared, and it might be expected that each teacher taught in a subject with which he or she felt rather confident.

Finally, some other international studies (Richland, Holyoak and Stigler, 2004) have found some limited use of them. Perhaps our criteria for defining and finding metaphors have to be reviewed and a search for them have to be performed again.

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