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Regularity estimates for higher order elliptic systems on Reifenberg flat domains

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Abstract

Consider a higher order elliptic system

$$\begin{cases} D^\alpha (a_{ij}^{\alpha\beta}(x) D^\beta u^j) = D^\alpha f_i^\alpha & \text{in} \quad \Omega, \\ |u^i| + |Du^i| + \ldots + |D^{m_i-1} u^i| = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

for all $i = 1, \ldots, N$ with $N \in \mathbb{N}_+$, and all multi-indices $|\alpha| = m_i$, $|\beta| = m_j$ with $m_i \in \mathbb{N}_+$ for all $i = 1, \ldots, N$, and the standard summation notation is understood. We assume that the leading coefficients $a_{ij}^{\alpha\beta}(x)$ have small BMO norms and the domain $\Omega \subset \mathbb{R}^n$ is open, bounded and flat in the Reifenberg's sense. This article is to prove the regularity estimates of this system in weighted Lorentz spaces and in Lorentz–Morrey spaces. Our results require weak assumptions on the regularity of the coefficients $a_{ij}^{\alpha\beta}(x)$ and the boundary $\partial\Omega$, and they are new even for scalar higher order elliptic equations.

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Contents

1. Introduction

Consider the following higher order elliptic systems in the form:

$$\begin{cases} D^\alpha(a_{ij}^{\alpha\beta}(x)D^\beta u^j) = D^\alpha f_i^\alpha & \text{in} \quad \Omega, \\ |u^i| + |Du^i| + \ldots + |D^{m_i-1}u^i| = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

for all $i = 1, \ldots, N$ with $N \in \mathbb{N}_+$, and all multi-indices $|\alpha| = m_i$, $|\beta| = m_j$ with $m_i \in \mathbb{N}_+$ for all $i = 1, \ldots, N$, where Ω is an open, bounded subset on $\mathbb{R}^n$ and $\mathbf{f} = \{f_i^\alpha\}$ with $f_i^\alpha \in L^2(\Omega)$ for all $1 \le i \le N$ and multi-indices α with $|\alpha| = m_i$. (The standard summation notation on the elliptic system is understood).

In this article, we assume that the coefficients $a_{ij}^{\alpha\beta}(x)$ of the equation (1) are uniformly bounded and elliptic, i.e. there exist two positive constants $L, \nu > 0$ such that

$$|a_{ij}^{\alpha\beta}(x)| \le L, \tag{2}$$

and

$$\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} a_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta \ge \nu \sum_{i=1}^N \sum_{|\alpha|=m_i} |\xi_i^\alpha|^2, \quad \forall(\xi_i^\alpha). \tag{3}$$

We now give some notations:

- For every multi-index $m = (m_1, \ldots, m_N)$, we denote by $W^{m,p}(\Omega, \mathbb{R}^N)$ and $W_0^{m,p}(\Omega, \mathbb{R}^N)$ the cartesian products

$$W^{m_1,p}(\Omega) \times \ldots \times W^{m_N,p}(\Omega) \quad \text{and} \quad W_0^{m_1,p}(\Omega) \times \ldots \times W_0^{m_N,p}(\Omega),$$

 respectively.

- For $u^i \in W^{m_i,p}(\Omega)$ and $k \in \mathbb{N}$, we denote $D^k u^i = (D^\gamma u^i)_{|\gamma|=k}$ for $i = 1, \dots, N$.
- For $u = (u^1, \dots, u^N) \in W^{m,p}(\Omega, \mathbb{R}^N)$ we denote $D^m u = (D^{m_1} u^1, \dots, D^{m_N} u^N)$ and

$$\left|D^m u\right|^2 = \sum_{i=1}^N \left|D^{m_i} u^i\right|^2 = \sum_{i=1}^N \sum_{|\gamma|=m_i} \left|D^\gamma u^i\right|^2,$$

  for all multi-indices $m = (m_1, \dots, m_N)$.

We recall the definition of weak solutions to the system (1).

Definition 1.1. A function $u = (u^1, \dots, u^N) \in W_0^{m,2}\left(\Omega, \mathbb{R}^N\right)$ is said to be a (weak) solution to the system (1) if

$$\int_\Omega \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} a_{ij}^{\alpha\beta}(x) D^\beta u^j D^\alpha \varphi^i \, dx = \int_\Omega \sum_{i=1}^N \sum_{|\alpha|=m_i} f_i^\alpha D^\alpha \varphi^i \, dx, \tag{4}$$

for all $\varphi = \left(\varphi^1, \varphi^2, \dots, \varphi^N\right) \in W_0^{m,2}(\Omega, \mathbb{R}^N)$.

By applying the Lax–Milgram theorem, we can prove the following results.

Proposition 1.2. *The system* (1) *has a unique weak solution. Moreover, the following estimate holds true:*

$$\left\| \, |D^m u| \, \right\|_{L^2(\Omega)} \lesssim \| \, |\mathbf{f}| \, \|_{L^2(\Omega)} \, . \tag{5}$$

This paper is devoted to investigate the regularity results to the system (1) under the mild conditions on the coefficients and on the boundary of the domain. More precisely, we look for the conditions on the coefficients and on the boundary of the domain so that the following estimate holds true

$$\| |D^m u| \|_{\mathcal{F}} \lesssim \| |\mathbf{f}| \|_{\mathcal{F}}, \tag{6}$$

for some function spaces $\mathcal{F}$.

The regularity problem for elliptic equations is one of the most interesting topics and plays a very important role in the theory of partial differential equations. This topic has received a great deal of attention from many mathematicians. The regularity results concerning the second order elliptic equations were investigated in, for example, [2,3,11,5,9,15,16,25,21,20,26] and the references therein. Note that the second order elliptic equations are the particular cases of the system (1) corresponding to $N = 1$ and $m = 1$. We now give a brief summary of the progress so far in this direction of research (but the list is by no means exhaustive).

(i) An early result in this direction is due to N. Meyers (cf. [21]). He proved that the $W^{1,p}$ regularity estimate (6) is valid for p being close to 2 provided that the coefficient matrix A is uniformly bounded and elliptic, and Ω is a bounded domain (opened connected set).

(ii) If the coefficient matrix A is assumed to satisfy suitable continuity conditions and the domain Ω has some smoothness condition on its boundary then one can expect that (6) is valid on $\mathcal{F} \equiv L^p$ for some $p \in (1, \infty)$. See for example [22, Chapter 5] and [25].

(iii) In [15], the authors considered the quasilinear equations related to quasilinear elliptic operators $L = \operatorname{div} A(x, \nabla)$ on the regular domains which include the case of p-Laplacian $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. They obtained $W^{1,r}$ regularity for these equations with r being close to p. Moreover, a number of interesting results regarding comparison estimates and local difference estimates related to these equations were also obtained.

(iv) In [5], the authors introduced an effective approximation method to study the regularity of the general class of elliptic problems. Then they applied this method to study $W^{1,p}$ regularity for a nonlinear elliptic operator in divergence form. In particular case of the linear elliptic equation, this method gives an alternative approach to the classical one. In this work, the coefficients matrix A is assumed to be continuous or close to the identity matrix in a suitable sense.

(v) The regularity of Laplacian Dirichlet problems on the Lipchitz domains was obtained in [16]. The authors obtained a complete description of all Sobolev estimates including the $W^{1,p}$ regularity for these equations. More importantly, some counterexamples are also included to show that certain regularity estimates may fail if the boundary of the underlying domain is not smooth enough.

(vi) In [17,18], the author proves the regularity estimates of the second-order elliptic equations with VMO coefficients.

(vii) The $W^{1,p}$ regularity of the second-order elliptic equations and the second-order elliptic systems with non-smooth coefficients and Reifenberg flat domains were proved in [2,3].

(viii) Recently, the authors in [19] proved the global gradient estimates on the weighted Lorentz spaces for the standard weak solutions to quasilinear elliptic equations on Reifenberg flat domains. As a consequence, they obtained other regularity results in Lorentz–Morrey, Morrey, and Hölder spaces.

The regularity results for the higher order elliptic equations are less well-known. We list some of works related to the research direction.

(i) In [12], the authors introduced a unified treatment to prove regularity results for weak solutions to higher-order elliptic systems on a bounded domain in $\mathbb{R}^n$. This contains some important results concerned with the regularity problem of elliptic systems. Certain results on differentiability theory of weak solutions of the higher-order elliptic systems can be found in [13].

(ii) The work [8] is devoted to the L^p-theory of higher-order parabolic and elliptic systems in the whole space $\mathbb{R}^n$, on the half space $\mathbb{R}^n_+$ and on a bounded domain in $\mathbb{R}^n$. The leading coefficients of the systems are assumed to be merely measurable only in the time variable and have a small BMO norms with respect to the spatial variables.

(iii) The global regularity for higher order divergence elliptic equations on $\mathbb{R}^n$ was investigated in [28]. They made use of the Fefferman–Stein maximal functions to obtain the regularity on Orlicz spaces.

(iv) Recently, in [4], the authors established optimal gradient estimates in the Orlicz space for solutions of a nonhomogeneous elliptic equation of higher order with discontinuous coefficients on a nonsmooth domain. The regularity estimates of higher order elliptic equations on $\mathbb{R}^n$ with VMO-coefficients was investigated in [14].

The main aim of this paper is to prove the regularity estimates for the higher order elliptic systems (1) in the settings of weighted Lorentz spaces and Lorentz–Morrey spaces. We require neither smoothness conditions for the underlying domain Ω nor strong regularity conditions for the coefficients $\{a_{ij}^{\alpha\beta}\}$. This article only assumes some flatness condition on the domain and a small BMO norm condition on the coefficients. See Section 2. We note that neither the uniformly elliptic conditions (2) and (3) nor smoothness conditions guarantee the L^p-boundedness in the regularity estimates (5) even for the second-order elliptic case. See for example [21,16]. In this sense, our assumptions of the small BMO coefficients and the flatness domain are reasonable, and are flexible enough to cover a large class of elliptic systems. We refer to Section 2 for further comments on these two conditions.

We note that our approach in this paper is different from those in [28,8] which rely heavily on the sharp Fefferman–Stein maximal functions. The sharp Fefferman–Stein maximal functions approach may not be applicable to our setting due to the lack of the regularity of the coefficients and the smoothness of the domain. To overcome this problem, we employ the approach in [2,5,19] which relies on approximation estimates to the weak solutions, the Vitali type covering lemma and the Hardy–Littlewood maximal function.

Finally, it is worth noticing that the regularity estimates for the higher order elliptic equation and the second order elliptic systems, which are, respectively, particular cases of (1) corresponding to $i = j = 1$ and to $m_i = 1$ for all $i = 1, \dots, N$, were investigated in [4,3]. In comparison with [4], our article gives results for the more general setting of the systems of higher order elliptic equations which requires a number of new estimates such as approximation results in Section 3. Unlike [4], we work in the setting of the weighted Lorentz spaces and Lorentz–Morrey spaces instead of the Orlicz spaces as in [4]. Therefore, the results in this article are new even in the special case of a higher order elliptic equation. Compared with [3], our paper can be viewed as an extension to the higher order elliptic systems and to the more general setting of the weighted Lorentz spaces.

The organization of the paper is as follows. In Section 2, we give the assumptions for the system (1) and state the main results. Some approximation results are given in Section 3. Finally, Section 4 is devoted to the proofs of the main results.

Throughout the paper, we always use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We will write $A \lesssim B$ if there is a universal constant C so that $A \le CB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Assumptions and main results

2.1. Assumptions

We begin with some notations which will be used frequently in the sequel.

For $r > 0$ and $x \in \mathbb{R}^n$, we denote:

- $B_r = \{y : |y| < r\}$, $B_r^+ = B_r \cap \{y = (y_1, \dots, y_n) : y_n > 0\}$ and $T_r = B_r \cap \{y = (y_1, \dots, y_n) : y_n = 0\}$;
- $B_r(x) = x + B_r$, $B_r^+(x) = x + B_r^+$ and $T_r(x) = x + T_r$;
- $\Omega_r = \Omega \cap B_r$, $\partial_w \Omega_r = \partial\Omega \cap B_r$ and $\Omega_r(x) = \Omega \cap B_r(x)$.

For a measurable function f on a measurable subset E in $\mathbb{R}^n$ we define

$$\overline{f}_E = \fint_E f = \frac{1}{|E|}\int_E f.$$

Throughout this paper, apart from (2) and (3) we additionally assume that the coefficients $a_{ij}^{\alpha\beta}$ satisfy the small BMO norm condition as follows.

Definition 2.1. Let $R, \delta > 0$. The coefficients $\{a_{ij}^{\alpha\beta}\}$ are said to satisfy the small (δ, R)-BMO condition if

$$\sup_{y\in\mathbb{R}^n, 0<r\le R} \fint_{B_r(y)} |a_{ij}^{\alpha\beta}(x) - \overline{a_{ij}^{\alpha\beta}}_{B_r(y)}|^2\, dx \le \delta^2, \tag{7}$$

for all i, j, α, β as in (1).

Remark 2.2. (a) The small (δ, R)-BMO condition (7) was firstly used in [2] to study the regularity of the second order elliptic equations. This condition allows the coefficients to be merely measurable in x and possibly not continuous in x. This is a good substitute to the VMO conditions in [9,14].

(b) Note that under the conditions (2), (3) and (7), it is easy to see that for any $\tau \in [1, \infty)$ there exists $\epsilon > 0$ so that

$$\sup_{y\in\mathbb{R}^n, 0<r\le R} \fint_{B_r(y)} |a_{ij}^{\alpha\beta}(x) - \overline{a_{ij}^{\alpha\beta}}_{B_r(y)}|^\tau\, dx \lesssim \delta^\epsilon,$$

for all i, j, α, β as in (1).

Concerning the underlying domain Ω, we do not assume any smoothness condition on Ω, but the following flatness condition.

Definition 2.3. Let $\delta, R > 0$. The domain Ω is said to be a (δ, R) Reifenberg flat domain if for every $x \in \partial\Omega$ and $0 < r \le R$, there exists a coordinate system depending on x and r, whose variables are denoted by $y = (y_1, \dots, y_n)$ such that in this new coordinate system x is the origin, and

$$B_r \cap \{y : y_n > \delta r\} \subset B_r \cap \Omega \subset \{y : y_n > -\delta r\}. \tag{8}$$

The concept of a (δ, R)-Reifenberg flatness domain was first introduced in [24]. This domain does not require any smoothness on the boundary of Ω, but sufficiently flat in the Reifenberg's sense. The Reifenberg flat domain includes domains with rough boundaries of fractal nature, and Lipschitz domains with small Lipschitz constants. See for example [24,7,23,27].

Fix $y \in \mathbb{R}^n$ and $r > 0$. Consider the re-scaled functions:

$$\tilde{u}^i(x) = \frac{u^i(rx+y)}{r^{m_i}}, \tilde{a}_{ij}^{\alpha\beta}(x) = a_{ij}^{\alpha\beta}(rx+y), \tilde{f}_i^\alpha(x) = f_i^\alpha(rx+y),$$

for all $i = 1, \dots, N$ and multi-indices α with $|\alpha| = m_i$. We set

$$\tilde{\Omega} = \left\{ \frac{x-y}{r} : x \in \Omega \right\}.$$

Then we deduce the following result immediately.

Lemma 2.4. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}\}$ satisfy (2), (3) and the small (δ, R)-BMO condition (7), and the domain Ω is a (δ, R) Reifenberg domain for some $\delta, R > 0$. If u is a weak solution to (1), then $\tilde{u}$ is a weak solution to the following problem:*

$$\begin{cases} D^\alpha(\tilde{a}_{ij}^{\alpha\beta}(x)D^\beta \tilde{u}^j) = D^\alpha \tilde{f}_i^\alpha & \text{in} \quad \tilde{\Omega}, \\ |\tilde{u}^i| + |D\tilde{u}^i| + \ldots + |D^{m_i-1}\tilde{u}^i| = 0 & \text{on} \quad \partial\tilde{\Omega}, \end{cases} \tag{9}$$

for all $i = 1, \ldots, N$ with $N \geq 1$. Moreover, the coefficients $\{\tilde{a}_{ij}^{\alpha\beta}\}$ satisfy (2), (3) and the small $(\delta, \frac{R}{r})$-BMO condition (7), and the domain $\tilde{\Omega}$ is a $(\delta, \frac{R}{r})$ Reifenberg flat domain.

Due to this result, in certain circumstances, we may assume that $R = 8$ or any fixed number.

2.2. State the main results

Let $1 \leq p < \infty$. A nonnegative locally integrable function w belongs to the *Muckenhoupt class* A_p, say $w \in A_p$, if there exists a positive constant C so that

$$[w]_{A_p} := \sup_{B:\text{balls}} \left(\fint_B w(x)dx \right) \left(\fint_B w^{-1/(p-1)}(x)dx \right)^{p-1} \leq C, \quad \text{if } 1 < p < \infty,$$

and

$$\fint_B w(x)dx \leq C \operatorname*{ess\text{-}inf}_{x\in B} w(x), \quad \text{if } p = 1,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. We say that $w \in A_\infty$ if $w \in A_p$ for some $p \in [1, \infty)$. We shall denote $w(E) := \int_E w(x)dx$ for any measurable set $E \subset \mathbb{R}^n$.

Lemma 2.5 *([10]). Let $w \in A_p$, $1 \leq p < \infty$. Then, there exist $\kappa_w > 0$, and a constant $C > 1$ such that for any ball B and any measurable subset $E \subset B$,*

$$C^{-1} \left(\frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^{\kappa_w}.$$

Let $w \in A_\infty$, $0 < p < \infty$ and $0 < q \leq \infty$. The weighted Lorentz space $L_w^{p,q}(\Omega)$ is defined as the set of all measurable functions f on Ω such that

$$\|f\|_{L_w^{p,q}(\Omega)} := \left\{ p \int_0^\infty \left[t^p w\left(\{x \in \Omega : |f(x)| > t\}\right) \right]^{q/p} \frac{dt}{t} \right\}^{1/q} < \infty$$

In the particular case $p = q$, the weighted Lorentz spaces $L^{p,p}_w(\Omega)$ coincide with the weighted Lebesgue spaces $L^p_w(\Omega)$ which is defined as all measurable functions f on Ω such that

$$\|f\|_{L^p_w(\Omega)} = \Big(\int_\Omega |f(x)|^p w(x)dx\Big)^{1/p}.$$

For $r > 0$, the Hardy–Littlewood maximal function $\mathcal{M}_r$ is defined by

$$\mathcal{M}_r f(x) = \sup_{Q \ni x} \Big(\frac{1}{|Q|}\int_Q |f(y)|^r\, dy\Big)^{1/r}, \ x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q containing x. When $r = 1$, we write $\mathcal{M}$ instead of $\mathcal{M}_1$.

We now record the following results concerning the weighted Lorentz estimates of the maximal functions. See [19, Lemma 3.11].

Lemma 2.6. *Let $r < p < \infty$, $0 < q \le \infty$ and $w \in A_{p/r}$. Then we have*

$$\|\mathcal{M}_r f\|_{L^{p,q}_w(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,q}_w(\mathbb{R}^n)}.$$

Our main results are formulated in the following two theorems. The first one gives a regularity estimate in the weighted Lorentz spaces. The later addresses a regularity estimate in the Lorentz–Morrey spaces.

Theorem 2.7. *Let $p \in (2, \infty)$ and let $w \in A_{p/2}$. Then there exist positive constants C and δ such that the following holds. If $f^\alpha_i \in L^{p,q}_w(\Omega)$ for all $|\alpha| = m_i$ and $i = 1, \dots, N$, the domain Ω is a (δ, R) Reifenberg flat domain, and the coefficients $\{a^{\alpha\beta}_{ij}\}$ satisfy (2), (3) and the small (δ, R)-BMO condition (7), then the system (1) has a unique weak solution $u \in W^{m,2}_0(\Omega, \mathbb{R}^N)$ satisfying the following estimate:*

$$\big\| \, |D^m u| \, \big\|_{L^{p,q}_w(\Omega)} \lesssim \| \, |\mathbf{f}| \, \|_{L^{p,q}_w(\Omega)} \, . \tag{10}$$

The estimate (10) is nothing, but implies the $W^{m,p}$ regularity estimates for the weak solution to the system (1), i.e,

$$\big\| \, |D^m u| \, \big\|_{L^p(\Omega)} \lesssim \| \, |\mathbf{f}| \, \|_{L^p(\Omega)} \, , \quad p > 2. \tag{11}$$

By the standard duality argument, it can be proved that (11) is valid for $1 < p < 2$, and hence, (11) holds true for all $1 < p < \infty$.

For $0 < p < \infty$, $0 < q \le \infty$ and $\lambda \in (0, n)$, the Lorentz–Morrey function space $L^{p,q;\lambda}(\Omega)$ is defined as the set of all measurable functions f such that

$$\|f\|_{L^{p,q;\lambda}(\Omega)} = \sup_{z \in \Omega} \sup_{0 < r \le \operatorname{diam} \Omega} r^{-\frac{\lambda}{p}} \|f\|_{L^{p,q}(B_r(z) \cap \Omega)}.$$

Theorem 2.8. *Let $p \in (2, \infty)$, $0 < q \leq \infty$ and $\lambda \in (0, n)$. Then there exist positive constants C and δ such that the following holds. If $f_i^\alpha \in L^{p,q;\lambda}(\Omega)$ for all $|\alpha| = m_i$ and $i = 1, \ldots, N$, the domain Ω is a (δ, R)-Reifenberg flat domain, and the coefficients $\{a_{ij}^{\alpha\beta}\}$ satisfy (2), (3) and the small (δ, R)-BMO condition (7), then the system (1) has a unique weak solution $u \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ satisfying the following estimate:*

$$\left\| |D^m u| \right\|_{L^{p,q;\lambda}(\Omega)} \lesssim \| |\mathbf{f}| \|_{L^{p,q;\lambda}(\Omega)} .$$

We end this section by the following two technical lemmas which play an important role in the sequel. The first one is known as a variant of the Vitali covering lemma in the weighted settings. The latter gives a characterization for functions in the weighted Lorentz spaces on a bounded domain.

Lemma 2.9 *([19]). Let Ω be a (δ, R) Reifenberg flat domain, $w \in A_\infty$ and $r < R/60$. Suppose that $E \subset F \subset \Omega$ are measurable and satisfy the following conditions:*

(a) *$w(E) < \epsilon w(B_r(y))$, for some $\epsilon \in (0, 1)$ and for every $y \in \Omega$;*
(b) *for any ball $B_\rho(y)$ with $\rho \in (0, 2r)$ and $y \in \Omega$, if $w(E \cap B_\rho(y)) \geq \epsilon w(B_\rho(y))$ then $\Omega \cap B_\rho(y) \subset F$.*

Then there exists $c := c(n, w)$ such that

$$w(E) \leq c\epsilon w(F).$$

Lemma 2.10 *([19]). Let $w \in A_\infty$ and f be a nonnegative measurable function in a bounded subset Ω. Let θ and $\lambda > 1$ be constants. Then for $0 < p, q < \infty$ we have*

$$f \in L_w^{p,q}(\Omega) \text{ if and only if } S := \sum_{k \geq 1} \lambda^{kq} w(\{x \in \Omega : g(x) > \theta\lambda^k\})^{q/p} < \infty;$$

moreover,

$$S \lesssim \|f\|^q_{L_w^{p,q}(\Omega)} \lesssim w(\Omega)^{q/p} + S.$$

For $0 < p < \infty$ and $q = \infty$, we have

$$T \lesssim \|f\|_{L_w^{p,\infty}(\Omega)} \lesssim w(\Omega)^{1/p} + T,$$

where

$$T = \sup_{k \geq 1} \lambda^k w(\{x \in \Omega : g(x) > \theta\lambda^k\})^{1/p}.$$

3. Approximation results

In this section, we assume that the underlying domain Ω is a $(\delta, 8)$ Reifenberg flat domain whereas the coefficients $\{a_{ij}^{\alpha\beta}\}$ satisfy (2), (3) and the small $(\delta, 8)$-BMO condition (7).

3.1. Interior estimates

We first restrict ourself to consider the case $B_{2\rho} \subset \Omega$ with $2\rho < R = 8$. Let $u = (u^1, \dots, u^N) \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1). We now consider the following Dirichlet problem

$$\begin{cases} D^\alpha (a_{ij}^{\alpha\beta}(x) D^\beta w^j) = 0 & \text{in} \quad B_{2\rho}, \\ |u^i - w^i| + \ldots + |D^{m_i-1}(u^i - w^i)| = 0 & \text{on} \quad \partial B_{2\rho}, \end{cases} \tag{12}$$

for all $i = 1, \dots, N$.

We then have the following higher integrability estimate for the weak solution to the system (12).

Proposition 3.1. *Let $w \in W^{m,2}(B_{2\rho}, \mathbb{R}^N)$ be a weak solution to the problem (12). Then there exists $\epsilon_0 > 0$ such that*

$$\Big(\int_{B_\rho} |D^m w|^{2+\epsilon_0} dx \Big)^{\frac{1}{2+\epsilon_0}} \lesssim \Big(\int_{B_{2\rho}} |D^m w|^2 dx \Big)^{\frac{1}{2}}. \tag{13}$$

To prove this proposition, we need the following auxiliary results concerning the Poincaré type inequality. See for example [12].

Lemma 3.2. *Let $u \in W^{k,p}(B_r(x_0))$ with $k \in \mathbb{N}_+$ and $p \in [1, \infty)$. There exists a unique polynomial $P(x)$ of degree at most $k-1$ depending on x_0, r, u such that there holds:*

(i) $\displaystyle\int_{B_r(x_0)} D^\alpha (u - P) dx = 0$ *for all α with $|\alpha| \le k-1$.*

(ii) *For every $0 \le s < t \le k$, there exists a constant $c = c(n, p, s, t, k)$ such that*

$$\sum_{|\gamma|=s} \int_{B_r(x_0)} |D^\gamma (u - P)|^p dx \le c r^{p(t-s)} \sum_{|\gamma|=t} \int_{B_r(x_0)} |D^\gamma (u - P)|^p dx.$$

Proof of Proposition 3.1. Let $P = (P_i)_{i=1}^N$ be polynomials associated to the weak solution w in the ball $B_{2\rho}$ as in Lemma 3.2. For $x \in B_{2\rho}$ and $r > 0$ such that $B_r(x) \subset B_{2\rho}$, we fix a function $\eta \in C_0^\infty(B_r(x))$ so that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{r/2}(x)$ and $|D^\alpha \eta| \lesssim r^{-|\alpha|}$ for all α with $|\alpha| \le \bar{m} := \max_i m_i$. Taking $\varphi^i = (w^i - P_i)\eta^{2\bar{m}} \in W_0^{m_i,2}(\Omega)$ as a test function, we then have, from Definition 1.1,

$$\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \int_{B_r(x)} a_{ij}^{\alpha\beta}(y) D^\beta w^j D^\alpha [(w^i - P_i)\eta^{2\bar{m}}] dy = 0.$$

Since $\deg P_i \le m_i - 1$ and $|D^\alpha \eta| \lesssim r^{-|\alpha|}$ for all α with $|\alpha| \le \bar{m}$, the identity above together with (2), (3), the product rule and Hölder's inequality implies that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{B_r(x)}|\eta^{\bar m}D^{m_i}w^i|^2dy\\
&\lesssim \sum_{i,j=1}^{N}\sum_{\gamma<\alpha}\sum_{|\beta|=m_j}\Big(\int_{B_r(x)}|\eta^{\bar m}D^{\beta}w^j|^2dy\Big)^{1/2}\Big(\int_{B_r(x)}|\eta^{2\bar m-m_i+|\gamma|}D^{\alpha-\gamma}\eta D^{\gamma}(w^i-P_i)|^2dy\Big)^{1/2}\\
&\lesssim \sum_{i,j=1}^{N}\sum_{\gamma<\alpha}\sum_{|\beta|=m_j}r^{-m_i+|\gamma|}\Big(\int_{B_r(x)}|\eta^{\bar m}D^{\beta}w^j|^2dy\Big)^{1/2}\Big(\int_{B_r(x)}|D^{\gamma}(w^i-P_i)|^2dy\Big)^{1/2}.
\end{aligned}
$$

We now apply Lemma 3.2 (ii) to conclude that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{B_r(x)}|\eta^{\bar m}D^{m_i}w^i|^2dy\\
&\qquad\lesssim \sum_{i,j=1}^{N}\sum_{|\gamma|=m_i-1}\sum_{|\beta|=m_j}r^{-1}\Big(\int_{B_r(x)}|\eta^{\bar m}D^{\beta}w^j|^2dy\Big)^{1/2}\Big(\int_{B_r(x)}|D^{\gamma}(w^i-P_i)|^2dy\Big)^{1/2}.
\end{aligned}
$$

This implies

$$
\sum_{i=1}^{N}\int_{B_r(x)}|\eta^{\bar m}D^{m_i}w^i|^2dy\lesssim\sum_{i=1}^{N}\sum_{|\gamma|=m_i-1}r^{-2}\int_{B_r(x)}|D^{\gamma}(w^i-P_i)|^2dy,
$$

and hence,

$$
\int_{B_{r/2}(x)}|D^m w|^2dy\lesssim\sum_{i=1}^{N}\sum_{|\gamma|=m_i-1}r^{-2}\int_{B_r(x)}|D^{\gamma}(w^i-P_i)|^2dy.
$$

Using Sobolev's embedding Theorem, we obtain

$$
\Big(\fint_{B_{r/2}(x)}|D^m w|^2dy\Big)^{1/2}\lesssim\sum_{|\gamma|=m_i-1}\Big(\fint_{B_r(x)}|D^{\gamma}(w-P)|^{\frac{2n}{n+2}}dy\Big)^{\frac{n+2}{2n}}+\Big(\fint_{B_r(x)}|D^m w|^{\frac{2n}{n+2}}dy\Big)^{\frac{n+2}{2n}}.
$$

Applying Lemma 3.2 again, we can dominate the first term on the right hand side by

$$
\begin{aligned}
\sum_{|\gamma|=m_i-1}\Big(\fint_{B_r(x)}|D^{\gamma}(w-P)|^{\frac{2n}{n+2}}dy\Big)^{\frac{n+2}{2n}}&\lesssim r\Big(\fint_{B_r(x)}|D^m w|^{\frac{2n}{n+2}}dy\Big)^{\frac{n+2}{2n}}\\
&\lesssim\Big(\fint_{B_r(x)}|D^m w|^{\frac{2n}{n+2}}dy\Big)^{\frac{n+2}{2n}},
\end{aligned}
$$

where in the last inequality we used the fact that $0<r<8$.

Therefore,

$$\Big(\fint_{B_{r/2}(x)} |D^m w|^2 dy\Big)^{1/2} \lesssim \Big(\fint_{B_r(x)} |D^m w|^{\frac{2n}{n+2}} dy\Big)^{\frac{n+2}{2n}}.$$

This along with Proposition 1.1 in [11, Chapter V] deduces the desired estimate. □

Let $w \in W^{m,2}(B_{2\rho}, \mathbb{R}^N)$ be a weak solution to the problem (12). We consider the following problem:

$$\begin{cases} D^\alpha(\overline{a_{ij}^{\alpha\beta}}_{B_\rho} D^\beta v^j) = 0 & \text{in} \quad B_\rho, \\ |v^i - w^i| + \ldots + |D^{m_i-1}(v^i - w^i)| = 0 & \text{on} \quad \partial B_\rho, \end{cases} \tag{14}$$

for all $i = 1, \ldots, N$.

Proposition 3.3. *Let v be a weak solution to the problem (14). Then we have*

$$\|\,|D^m v|\,\|_{L^\infty(B_{\rho/2})} \lesssim \Big(\fint_{B_\rho} |D^m v|^2\Big)^{1/2}, \tag{15}$$

and there exists $\epsilon_1 > 0$ such that

$$\Big(\fint_{B_\rho} |D^m(v-w)|^2\Big)^{1/2} \lesssim \delta^{\epsilon_1}\Big(\fint_{B_{2\rho}} |D^m w|^2\Big)^{1/2}. \tag{16}$$

Proof. We first prove (15). Let $P = (P_i)$ be polynomials associated to the weak solution v in the ball B_ρ as in Lemma 3.2. Then, $v - P$ is a weak solution to the problem

$$D^\alpha(\overline{a_{ij}^{\alpha\beta}}_{B_\rho} D^\beta v^j) = 0 \quad \text{in} \quad B_\rho.$$

From the inequality (3.21) in [13, p. 121], we get that

$$\sup_{B_{\rho/2}} |D^{m_j}(v^j - P_j)| \lesssim \sum_{i=1}^N \sum_{|\gamma| \le m_i - 1} \rho^{-m_i+|\gamma|}\Big(\fint_{B_\rho} |D^\gamma(v^i - P_i)|^2\Big)^{1/2},$$

or equivalently,

$$\sup_{B_{\rho/2}} |D^{m_j} v^j| \lesssim \sum_{i=1}^N \sum_{|\gamma| \le m_i - 1} \rho^{-m_i+|\gamma|}\Big(\fint_{B_\rho} |D^\gamma(v^i - P_i)|^2\Big)^{1/2}.$$

At this stage, using Poincaré inequality in Lemma 3.2, we obtain

$$\sup_{B_{\rho/2}} |D^{m_j} v^j| \lesssim \Big(\fint_{B_\rho} |D^m v|^2 \Big)^{1/2}.$$

This proves (15).

We now take care of (16). From (3), we have

$$\fint_{B_\rho} |D^m (v-w)|^2 \lesssim \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_{B_\rho} \overline{a_{ij}^{\alpha\beta}}_{B_\rho} D^\alpha (v^i - w^i) D^\beta (v^j - w^j).$$

On the other hand, since $v - w \in W_0^{m,2}(B_\rho)$, from the definition of the weak solution to the problem (12) and (14) we get that

$$\begin{aligned}\int_{B_\rho} \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} & \overline{a_{ij}^{\alpha\beta}}_{B_\rho} D^\alpha v^i D^\beta (v^j - w^j) \\ &= \int_{B_\rho} \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} a_{ij}^{\alpha\beta}(x) D^\alpha w_i D^\beta (v^j - w^j) = 0.\end{aligned}$$

This along with the inequality above implies

$$\fint_{B_\rho} |D^m (v-w)|^2 \lesssim \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_{B_\rho} \Big(a_{ij}^{\alpha\beta}(x) - \overline{a_{ij}^{\alpha\beta}}_{B_\rho} \Big) D^\alpha w_i D^\beta (v_j - w_j).$$

Applying Hölder's inequality and Proposition 3.1, we get that

$$\begin{aligned}\fint_{B_\rho} |D^m (v-w)|^2 &\lesssim \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \Big(\fint_{B_2} |D^\beta (v^j - w^j)|^2 \Big)^{1/2} \\ &\quad \times \Big(\fint_{B_\rho} |D^\alpha w_i|^{2+\epsilon_0} \Big)^{\frac{1}{2+\epsilon_0}} \Big(\fint_{B_\rho} \Big| a_{ij}^{\alpha\beta}(x) - \overline{a_{ij}^{\alpha\beta}}_{B_\rho} \Big|^{\frac{2(2+\epsilon_0)}{\epsilon_0}} \Big)^{\frac{\epsilon_0}{2(2+\epsilon_0)}} \\ &\lesssim \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \delta^{\epsilon_1} \Big(\fint_{B_\rho} |D^\beta (v^j - w^j)|^2 \Big)^{1/2} \Big(\fint_{B_{2\rho}} |D^m w|^2 \Big)^{\frac{1}{2}},\end{aligned}$$

This yields that

$$\Big(\fint_{B_\rho} |D^m (v-w)|^2 \Big)^{\frac{1}{2}} \lesssim \delta^{\epsilon_1} \Big(\fint_{B_{2\rho}} |D^m w|^2 \Big)^{\frac{1}{2}}. \quad \square$$

Proposition 3.4. *For every $\epsilon > 0$, there exists δ such that the following holds. If $u \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ is a weak solution to the system (1) with*

$$\fint_{B_{2\rho}} |D^m u|^2 < 1, \tag{17}$$

and

$$\fint_{B_{2\rho}} |\mathbf{f}|^2 < \delta^2, \tag{18}$$

then there exists $v \in W^{m,\infty}(B_{\rho/2}, \mathbb{R}^N)$ such that

$$\| \, |D^m v| \, \|_{L^\infty(B_{\rho/2})} \leq c_0, \tag{19}$$

and

$$\fint_{B_{\rho/2}} |D^m(u-v)|^2 \leq \epsilon. \tag{20}$$

Proof. Let w, v be solutions to the problems (12) and (14), respectively. Then we have, by (15) and (16),

$$\begin{aligned}
&\| \, |D^m v| \, \|_{L^\infty(B_{\rho/2})} \\
&\quad \lesssim \Big(\fint_{B_{2\rho}} |D^m v|^2 \Big)^{1/2} \\
&\quad \lesssim \Big(\fint_{B_{2\rho}} |D^m u|^2 \Big)^{1/2} + \Big(\fint_{B_{2\rho}} |D^m(u-w)|^2 \Big)^{1/2} + \Big(\fint_{B_{2\rho}} |D^m(w-v)|^2 \Big)^{1/2} \\
&\quad \lesssim \Big(\fint_{B_{2\rho}} |D^m u|^2 \Big)^{1/2} + \Big(\fint_{B_{2\rho}} |D^m(u-w)|^2 \Big)^{1/2} + \Big(\fint_{B_{2\rho}} |D^m w|^2 \Big)^{1/2} \\
&\quad \lesssim \Big(\fint_{B_{2\rho}} |D^m u|^2 \Big)^{1/2} + \Big(\fint_{B_{2\rho}} |D^m(u-w)|^2 \Big)^{1/2}.
\end{aligned} \tag{21}$$

On the other hand, observe that $u - w$ solves the following problem

$$\begin{cases} D^\alpha(a_{ij}^{\alpha\beta}(x) D^\beta(u^j - w^j)) = D^\alpha f_i^\alpha & \text{in} \quad B_{2\rho}, \\ |u^i - w^i| + \ldots + |D^{m_i - 1}(u^i - w^i)| = 0 & \text{on} \quad \partial B_{2\rho}, \end{cases}$$

for all $i = 1, \ldots, N$.

By Proposition 1.2, one gets that

$$\Big(\fint_{B_{2\rho}} |D^m(u-w)|^2\Big)^{1/2} \lesssim \Big(\fint_{B_{2\rho}} |\mathbf{f}|^2\Big)^{1/2}. \tag{22}$$

Taking this, (21), (18) and (17) into account, we can dominate $\|D^m v\|_{L^\infty(B_1)}$ by

$$\| \, |D^m v| \, \|_{L^\infty(B_{\rho/2})} \lesssim \Big[\Big(\fint_{B_{2\rho}} |D^m u|^2\Big)^{1/2} + \Big(\fint_{B_{2\rho}} |\mathbf{f}|^2\Big)^{1/2}\Big] \lesssim (1+\delta).$$

We now move on to prove (20). Note that

$$\fint_{B_{\rho/2}} |D^m(u-v)|^2 \lesssim \fint_{B_{\rho/2}} |D^m(u-w)|^2 + \fint_{B_{\rho/2}} |D^m(w-v)|^2.$$

This estimate, in combination with (22) and (16), implies that

$$\Big(\fint_{B_{\rho/2}} |D^m(u-v)|^2\Big)^{1/2} \lesssim \delta^{\epsilon_1}\Big(\fint_{B_{2\rho}} |D^m w|^2\Big)^{1/2} + \Big(\fint_{B_{2\rho}} |\mathbf{f}|^2\Big)^{1/2}.$$

Observe from (21) and (22) that

$$\begin{aligned}\Big(\fint_{B_{2\rho}} |D^m w|^2\Big)^{1/2} &\lesssim \Big[\Big(\fint_{B_{2\rho}} |D^m u|^2\Big)^{1/2} + \Big(\fint_{B_{2\rho}} |\mathbf{f}|^2\Big)^{1/2}\Big] \\ &\lesssim (1+\delta).\end{aligned}$$

From these two estimates, we conclude that

$$\fint_{B_{\rho/2}} |D^m(u-v)|^2 \lesssim \delta^{\epsilon_1}(1+\delta) + \delta.$$

This completes the proof. □

3.2. Boundary estimates

We now localize our interest to consider the following case:

$$B_5^+ \subset \Omega_5 \subset \{x \in B_5 : x_n > -12\delta\}. \tag{23}$$

Let $u \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1). We now consider the following Dirichlet problem:

$$\begin{cases} D^\alpha(a_{ij}^{\alpha\beta}(x)D^\beta w^j) = 0 & \text{in} \quad \Omega_5, \\ |u^i - w^i| + \ldots + |D^{m_i-1}(u^i - w^i)| = 0 & \text{on} \quad \partial\Omega_5, \end{cases} \tag{24}$$

for all $i = 1, \ldots, N$.

Proposition 3.5. *Let $w \in W^{m,2}(\Omega_5, \mathbb{R}^N)$ be a weak solution to (24). Then there exists $\epsilon_0 > 0$ (without loss of generality, we may assume the same ϵ_0 as in Proposition 3.1) such that*

$$\Big(\int_{\Omega_4} |D^m w|^{2+\epsilon_0} dx\Big)^{\frac{1}{2+\epsilon_0}} \lesssim \Big(\int_{\Omega_5} |D^m w|^2 dx\Big)^{\frac{1}{2}}. \tag{25}$$

Proof. Let $\bar{w}$ be a zero extension of w from Ω_4 to B_4. By Proposition 1.1 in [11, Chapter V], it suffices to prove that for $y \in \Omega_4$ and $0 < r < 4$,

$$\Big(\fint_{\Omega_{4r/5}(y)} |D^m w|^2 dx\Big)^{1/2} \lesssim \Big(\fint_{B_r(y)} |D^m \overline{w}|^{\frac{2n}{n+2}} dx\Big)^{\frac{n+2}{2n}}. \tag{26}$$

Indeed, if $B_{5r/6}(y) \subset \Omega_5$ then arguing similarly to Proposition 3.1 we obtain

$$\Big(\fint_{\Omega_{4r/5}(y)} |D^m w|^2 dx\Big)^{1/2} \lesssim \Big(\fint_{B_{5r/6}(y)} |D^m \overline{w}|^{\frac{2n}{n+2}} dx\Big)^{\frac{n+2}{2n}}.$$

Moreover, (26) follows immediately if $B_{5r/6}(y) \subset \Omega_5^c$.

It remains to prove (26) in the case $B_{5r/6}(y) \cap \Omega_5^c \neq \emptyset$. To do this, we fix a function $\eta \in C_0^\infty(B_r(y))$ so that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{4r/5}(y)$ and $|D^\alpha \eta| \lesssim r^{-|\alpha|}$ for all α with $|\alpha| \le \bar{m} := \max_i m_i$. Taking $\varphi = \overline{w}\eta^{2\bar{m}} \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ as a test function, and then arguing similarly to Proposition 3.1, we obtain that

$$\Big(\fint_{\Omega_{4r/5}(y)} |D^m w|^2 dx\Big)^{1/2} \lesssim \sum_{j=1}^N \sum_{|\gamma| \le m_j - 1} r^{-m_j + |\gamma|} \Big(\fint_{B_r(y)} |D^\gamma \overline{w}^j|^2 dx\Big)^{\frac{1}{2}}.$$

This together with the Poincaré inequality near the boundary (see for example [1, Corollary 8.2.7]) yields

$$\Big(\fint_{\Omega_{4r/5}(y)} |D^m w|^2 dx\Big)^{1/2} \lesssim \sum_{j=1}^N \sum_{|\gamma| = m_j - 1} r^{-1} \Big(\fint_{B_r(y)} |D^\gamma \overline{w}^j|^2 dx\Big)^{\frac{1}{2}}.$$

Applying Sobolev's embedding Theorem, we can conclude that

$$\Big(\fint_{\Omega_{4r/5}(y)} |D^m w|^2 dx\Big)^{1/2} \lesssim \sum_{j=1}^N \sum_{|\gamma| = m_j - 1} \Big(\fint_{B_r(y)} |D^\gamma \overline{w}^j|^{\frac{2n}{n+2}} dx\Big)^{\frac{n+2}{2n}} + \Big(\fint_{B_r(y)} |D^m \overline{w}|^{\frac{2n}{n+2}} dx\Big)^{\frac{n+2}{2n}}.$$

Applying Poincaré inequality again for the first term on the right hand side of the inequality above, we get (26) as desired. □

Let w be a weak solution to the problem (24). We next consider the following system:

$$\begin{cases} D^\alpha(\overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta h^j) = 0 & \text{in} \quad \Omega_4, \\ |h^i - w^i| + \ldots + |D^{m_i-1}(h^i - w^i)| = 0 & \text{on} \quad \partial\Omega_4, \end{cases} \tag{27}$$

for all $i = 1, \ldots, N$.

Similarly to the proof of Proposition 3.3, we also obtain the following estimate.

Proposition 3.6. *Let h be a weak solution to the problem (27). Then there exists $\epsilon_1 > 0$ (without loss of generality, we may assume the same ϵ_1 as in Proposition 3.3) such that*

$$\Big(\fint_{\Omega_4} |D^m(h-w)|^2 \Big)^{1/2} \lesssim \delta^{\epsilon_1} \Big(\fint_{\Omega_5} |D^m w|^2 \Big)^{1/2}. \tag{28}$$

The main difference between Proposition 3.6 and Proposition 3.3 is that in Proposition 3.6 we could not expect the L^∞ norm of $D^m h$ to be bounded up to the boundary of Ω due to the irregularity of the underlying domain Ω. To overcome this trouble, we consider another problem

$$\begin{cases} D^\alpha(\overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta h^j) = 0 & \text{in} \quad \Omega_4, \\ |h^i| + \ldots + |D^{m_i-1} h^i| = 0 & \text{on} \quad \partial_w\Omega_4, \end{cases} \tag{29}$$

for $i = 1, \ldots, N$, and its limited problem

$$\begin{cases} D^\alpha(\overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta v^j) = 0 & \text{in} \quad B_4^+, \\ |v^i| + \ldots + |D^{m_i-1} v^i| = 0 & \text{on} \quad T_4, \end{cases} \tag{30}$$

for $i = 1, \ldots, N$

Definition 3.7. (a) A function $h = (h^1, \ldots, h^N) \in W^{m,2}\left(\Omega_4, \mathbb{R}^N\right)$ is said to be a weak solution to the problem (29) if

$$\int_\Omega \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta h^j D^\alpha \varphi^i \, dx = 0,$$

for all $\varphi \in W_0^{m,2}(\Omega_4, \mathbb{R}^N)$ and the zero extension $\bar{h}$ of h from Ω_4 to B_4 is in $W^{m,2}(B_4, \mathbb{R}^N)$.

(b) A function $v = (v^1, \ldots, v^N) \in W^{m,2}\left(B_4^+, \mathbb{R}^N\right)$ is said to be a weak solution to the problem (30) if

$$\int_\Omega \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta v^j D^\alpha \varphi^i \, dx = 0,$$

for all $\varphi \in W_0^{m,2}(B_4^+, \mathbb{R}^N)$ and the zero extension $\bar{v}$ of h from B_4^+ to B_4 is in $W^{m,2}(B_4, \mathbb{R}^N)$.

Note that if h solves the system (27), then h also solves the system (29).
We then prove the following result.

Proposition 3.8. *For every $\epsilon > 0$, there exists δ such that the following holds. If h is a weak solution to the problem (29) along with (23) and*

$$\fint_{\Omega_4} |D^m h|^2 \lesssim 1, \tag{31}$$

then there exists $v \in W^{m,\infty}(B_2^+, \mathbb{R}^N) \cap W^{m,2}\left(B_4^+, \mathbb{R}^N\right)$ solving the problem (30) with

$$\fint_{B_2^+} |D^m v|^2 \lesssim 1 \tag{32}$$

such that

$$\fint_{\Omega_2} |D^m (h - \bar{v})|^2 \leq \epsilon, \tag{33}$$

where $\bar{v}$ is the zero extension of v to B_4.

Proof. We will argue by contradiction as in [2]. Assume, to the contrary, that there exist an $\epsilon > 0$, a sequence of domains $\{\Omega_k\}$, and a sequence of functions $\{h_k\} \subset W^{m,2}(\Omega^k, \mathbb{R}^N)$ such that

$$B_5^+ \subset \Omega_5^k \subset \{x \in B_5 : x_n > -\frac{12}{k}\}, \tag{34}$$

$$\begin{cases} D^\alpha(\overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta h_k^j) = 0 & \text{in} \quad \Omega_4^k, \\ |h_k^i| + \ldots + |D^{m_i - 1} h_k^i| = 0 & \text{on} \quad \partial_w \Omega_4^k, \end{cases} \tag{35}$$

for $i = 1, \ldots, N$, and

$$\fint_{\Omega_4^k} |D^m h_k|^2 \lesssim 1. \tag{36}$$

But, we have

$$\fint_{\Omega_2} |D^\gamma (h_k - \bar{v})|^2 > \epsilon, \tag{37}$$

where v is a weak solution to the following problem and $\bar{v}$ is its zero extension to B_4

$$\begin{cases} D^\alpha(\overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta v^j) = 0 & \text{in} \quad B_4^+, \\ |v^i| + \ldots + |D^{m_i - 1} v^i| = 0 & \text{on} \quad T_4, \end{cases} \tag{38}$$

for $i = 1, \dots, N$, with

$$\fint_{B_2^+} |D^m v|^2 \lesssim 1 \tag{39}$$

From (36) and Poincaré inequality, it follows that there exist $h_0 \in W^{m,2}(B_4^+, \mathbb{R}^N)$, and a subsequence, which will still be denoted by $\{h_k\}$, so that for each $j = 1, \dots, N$,

$$D^i h_k^j \to D^i h_0^j \text{ in } L^2(B_4^+),\ i = 0, \dots, m_j - 1,$$

and

$$D^{m_j} h_k^j \to D^{m_j} h_0^j \text{ weakly in } L^2(B_4^+).$$

Therefore, this and (36) imply that

$$\fint_{B_4^+} |D^m h_0|^2 \lesssim 1.$$

Moreover, from (35), we have

$$\int_{B_4^+} \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta h_k^j D^\alpha \varphi^i \, dx = 0,$$

for all $\varphi \in W_0^{m,2}(B_4^+, \mathbb{R}^N)$.

Passing to the limit $k \to \infty$, we get that

$$\int_{B_4^+} \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \overline{a_{ij}^{\alpha\beta}}_{B_4} D^\beta h_0^j D^\alpha \varphi^i \, dx = 0, \tag{40}$$

for all $\varphi \in W_0^{m,2}(B_4^+, \mathbb{R}^N)$.

We will prove that

$$D^i h_0^j = 0 \text{ on } T_4,$$

for all $i = 0, \dots, m_j - 1$ and all $j = 1, \dots, N$. Indeed, we adapt the method of Browder and Minty used for the second order-elliptic equations in [2, p. 1301] to our situation. For every $j = 1, \dots, N$, and every multi-index $|\gamma| \le m_j - 1$, we fix any small $\theta > 0$ and take $x' \in T_4$. We set $s_k = \min\{s : (x', \theta - s) \in \partial\Omega_4^k\}$ for each $k \in \mathbb{N}$. Then, $0 < s_k < \theta + 12/k$ and $D^i h_k^j(x', \theta - s_k) = 0$ for all $k \in \mathbb{N}$. Without loss of generality, we may assume that $D^\gamma h_k^j \in [C^1(\overline{B_4})]^n$ ($h_k^j = 0$ in $B_4 \setminus \Omega_4^k$). Then we have

$$
\begin{aligned}
|D^\gamma h_k^j(x',\theta)| &= |D^\gamma h_k^j(x',\theta) - D^\gamma h_k^j(x',\theta - s_k)| = s_k \left| \int_0^1 \frac{\partial}{\partial x_n} D^\gamma h_k^j(x', \theta - (1-s)s_k) ds \right| \\
&\le (\theta + 12/k) \int_0^1 |D^{|\gamma|+1} h_k^j(x', \theta - (1-s)s_k)| ds.
\end{aligned}
$$

Hence,

$$
|D^\gamma h_k^j(x',\theta)|^2 \le (\theta + 12/k)^2 \int_0^1 |D^{|\gamma|+1} h_k^j(x', \theta - (1-s)s_k)|^2 ds.
$$

Integrating this over T_4 and using Poincaré inequality, we find that

$$
\begin{aligned}
\int_{T_4} |D^\gamma h_k^j(x',\theta)|^2 dx' &\le (\theta + 12/k)^2 \int_{T_4} \int_0^1 |D^{|\gamma|+1} h_k^j(x', \theta - (1-s)s_k)|^2 ds dx' \\
&\le (\theta + 12/k)^2 \int_{B_4} |D^{|\gamma|+1} h_k^j|^2 dx \\
&\le C(\theta + 12/k)^2 \int_{B_4} |D^{m_j} h_k^j|^2 dx \\
&\le C(\theta + 12/k)^2.
\end{aligned}
$$

Letting $k \to \infty$ and $\theta \to 0$, we get that

$$
\int_{T_4} |D^\gamma h_0^j(x', 0)|^2 dx' = 0,
$$

for all $j = 1, \dots, N$ and $|\gamma| \le m_j - 1$.

Therefore,

$$
D^i h_0^j = 0 \text{ on } T_4,
$$

for all $i = 0, \dots, m_j - 1$ and all $j = 1, \dots, N$.

Gathering this with (40) we conclude that h_0 is a weak solution to the problem (30).

Let $\bar{h}_k$, $\bar{h}_0$ be zero extensions of h_k, h_0 to B_4, respectively. The argument above show that there exist $H_0 \in W^{m,2}(B_4, \mathbb{R}^N)$, and a subsequence, which will still be denoted by $\{\bar{h}_k\}$, so that for each $j = 1, \dots, N$,

$$
D^i \bar{h}_k^j \to D^i H_0^j \text{ in } L^2(B_4), i = 0, \dots, m_j - 1,
$$

and

$$D^{m_j}\bar{h}_k^j \to D^{m_j}H_0^j \text{ weakly in } L^2(B_4).$$

This together with (34) implies $H_0 = 0$ on $B_4 \cap \{x_n < 0\}$. As a consequence, $\bar{h}_0 \equiv H_0$ in B_4. Hence, for each $j = 1, \dots, N$,

$$D^i\bar{h}_k^j \to D^i\bar{h}_0^j \text{ in } L^2(B_4),\ i = 0, \dots, m_j - 1, \tag{41}$$

and

$$D^{m_j}\bar{h}_k^j \to D^{m_j}\bar{h}_0^j \text{ weakly in } L^2(B_4). \tag{42}$$

We now claim that

$$D^m\bar{h}_k \to D^m\bar{h}_0 \text{ in } L^2(B_2).$$

Indeed, take $\phi \in C_0^\infty(B_4)$ so that $0 \le \phi \le 1$, $\phi = 1$ in B_2, and $|D^\alpha\phi| \lesssim 1$ for all $|\alpha| \le \bar{m} := \max_i m_i$. Fix $i \in \{1, \dots, N\}$. Define the test functions $\varphi_k = \phi^{2\bar{m}}(\bar{h}_k - \bar{h}_0) \in W_0^{m,2}(\Omega_4^k, \mathbb{R}^N)$ for every $k \in \mathbb{N}_+$. We then have, by (3),

$$\begin{aligned}
&\int_{B_2}\sum_{i=1}^N\sum_{|\alpha|=m_i}|D^\beta(\bar{h}_k^j - \bar{h}_0^j)|^2dx\\
&\lesssim \int_{B_4}\sum_{i,j=1}^N\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\overline{a_{ij}^{\alpha\beta}}_{B_4}D^\beta(\bar{h}_k^j - \bar{h}_0^j)D^\alpha(\bar{h}_k^j - \bar{h}_0^j)\phi^{2\bar{m}}dx\\
&= \int_{B_4}\sum_{i,j=1}^N\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\overline{a_{ij}^{\alpha\beta}}_{B_4}D^\beta\bar{h}_k^jD^\alpha(\bar{h}_k^j - \bar{h}_0^j)\phi^{2\bar{m}}dx\\
&\quad - \int_{B_4}\sum_{i,j=1}^N\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\overline{a_{ij}^{\alpha\beta}}_{B_4}D^\beta\bar{h}_0^jD^\alpha(\bar{h}_k^j - \bar{h}_0^j)\phi^{2\bar{m}}dx\\
&:= I_1(k) + I_2(k).
\end{aligned} \tag{43}$$

Due to (42), we have $I_2(k) \to 0$ as $k \to \infty$.

We now take care of $I_1(k)$. To do this, we write

$$\begin{aligned}
I_1(k) &= \int_{\Omega_4^k}\sum_{i,j=1}^N\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\overline{a_{ij}^{\alpha\beta}}_{B_4}D^\beta\bar{h}_k^jD^\alpha[\phi^{2\bar{m}}(\bar{h}_k^i - \bar{h}_0^i)]dx\\
&\quad - \sum_{|\gamma|+|\eta|=m_i, |\gamma|<m_i}\frac{\alpha!}{\gamma!\eta!}\int_{B_4}\sum_{i,j=1}^N\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\overline{a_{ij}^{\alpha\beta}}_{B_4}D^\beta\bar{h}_k^jD^\gamma(\bar{h}_k^j - \bar{h}_0^j)D^\eta\phi^{2\bar{m}}dx\\
&:= I_{11}(k) + I_{12}(k).
\end{aligned}$$

Since $\phi^{2\bar{m}}(\bar{h}_k - \bar{h}_0) \in W_0^{m,2}(\Omega_4^k, \mathbb{R}^N)$, $I_{11}(k) = 0$.

Using Hölder's inequality, (2) and (36), we have

$$|I_{12}(k)| \lesssim \Big(\int_{B_4} |D^m \bar{h}_k|^2 dx\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{N} \sum_{|\gamma|\le m_j-1} \int_{B_4} |D^\gamma (\bar{h}_k^j - \bar{h}_0^j)|^2 dx\Big)^{\frac{1}{2}}$$
$$\lesssim \Big(\sum_{j=1}^{N} \sum_{|\gamma|\le m_j-1} \int_{B_4} |D^\gamma (\bar{h}_k^j - \bar{h}_0^j)|^2 dx\Big)^{\frac{1}{2}}.$$

This, in combination with (42), yields $I_{12}(k) \to 0$ as $k \to \infty$. From the estimates of $I_{11}(k)$, $I_{12}(k)$, $I_2(k)$ and (43), we imply that

$$D^m \bar{h}_k \to D^m \bar{h}_0 \text{ in } L^2(B_2)$$

This contradicts (37) by taking $v = h_0$ and k to be sufficiently large. □

Proposition 3.9. *For every $\epsilon > 0$, there exists δ such that the following holds. If $u \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ is a weak solution to the system (1) with*

$$\fint_{\Omega_5} |D^m u|^2 < 1, \tag{44}$$

and

$$\fint_{\Omega_5} |\mathbf{f}|^2 < \delta^2, \tag{45}$$

then there exists $v \in W^{m,\infty}(B_2^+, \mathbb{R}^N) \cap W^{m,2}\left(B_4^+, \mathbb{R}^N\right)$ solving (30) such that

$$\sup_{B_1^+} |D^m v| \le c_0, \tag{46}$$

and

$$\fint_{\Omega_2} |D^m (u - \bar{v})|^2 \le \epsilon, \tag{47}$$

where $\bar{v}$ is a zero extension of v to Ω_4.

Proof. Let w, h be weak solutions to the problems (24) and (27), respectively. We then have, by Proposition 3.6,

$$\fint_{\Omega_4} |D^m h|^2 \lesssim \fint_{\Omega_4} |D^m (h - w)|^2 + \fint_{\Omega_4} |D^m (w - u)|^2 + \fint_{\Omega_4} |D^m u|^2$$

$$\lesssim \delta^{2\epsilon_1} \fint_{\Omega_5} |D^m w|^2 + \fint_{\Omega_5} |D^m(w-u)|^2 + \fint_{\Omega_5} |D^m u|^2$$

$$\lesssim \fint_{\Omega_5} |D^m(w-u)|^2 + \fint_{\Omega_5} |D^m u|^2,$$

provided that δ is sufficiently small ($\delta < 1$ is enough).

Similarly to (22), we have

$$\fint_{\Omega_5} |D^m(w-u)|^2 \lesssim \fint_{\Omega_5} |\mathbf{f}|^2. \tag{48}$$

As a consequence,

$$\fint_{\Omega_4} |D^m h|^2 \lesssim \fint_{\Omega_4} |D^m u|^2 + \fint_{\Omega_4} |\mathbf{f}|^2 \lesssim 1. \tag{49}$$

Hence, by Proposition 3.8, there exists $v \in W^{m,\infty}(B_2^+, \mathbb{R}^N) \cap W^{m,2}(B_4^+, \mathbb{R}^N)$ solving the problem (30) with

$$\fint_{B_2^+} |D^m v|^2 \lesssim 1 \tag{50}$$

such that

$$\fint_{\Omega_2} |D^m(h-\bar{v})|^2 \le \tilde{\epsilon}. \tag{51}$$

Similarly to the inequality (3.21) in [13, p. 121], we can prove that

$$\| \, |D^m v| \, \|_{L^\infty(B_1^+)} \lesssim \sum_{i=1}^{N} \sum_{|\gamma| \le m_i - 1} \rho^{-m_i + |\gamma|} \Big(\fint_{B_2} |D^\gamma \bar{v}|^2 \Big)^{1/2}.$$

Then applying the Poincaré inequality, we obtain that

$$\| \, |D^m v| \, \|_{L^\infty(B_1^+)} \lesssim \Big(\fint_{B_2^+} |D^m v|^2 \Big)^{1/2} \lesssim 1.$$

In order to obtain the desired estimate (47), using Proposition 3.6, (48) and (51) we arrive at

$$\begin{aligned}
\int_{\Omega_2} |D^m(u-\overline{v})|^2 &\lesssim \int_{\Omega_2} |D^m(h-\overline{v})|^2 + \int_{\Omega_2} |D^m(h-w)|^2 + \int_{\Omega_1} |D^m(w-u)|^2 \\
&\lesssim \delta^{\epsilon_1} \int_{\Omega_5} |w|^2 + \int_{\Omega_5} |\mathbf{f}|^2 + \tilde{\epsilon} \\
&\lesssim \delta^{\epsilon_1} \left(\int_{\Omega_5} |w-u|^2 + \int_{\Omega_5} |u|^2 \right) + \int_{\Omega_5} |\mathbf{f}|^2 + \tilde{\epsilon} \\
&\lesssim \delta^{\epsilon_1} (\delta + 1) + \delta + \tilde{\epsilon}.
\end{aligned}$$

This completes our proof. □

4. Regularity estimates

This section is devoted to prove Theorem 2.7 and Theorem 2.8.

We need the following technical results.

Proposition 4.1. *For $w \in A_\infty$, there exists a positive constant $\lambda_0 > 0$ so that the following holds true. For any $\epsilon > 0$ there exists δ such that if Ω is a $(\delta, 8)$ Reifenberg flat domain, the coefficients $\{a_{ij}^{\alpha\beta}\}$ satisfy (2), (3) and the small $(\delta, 8)$-BMO condition (7), and $u \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ is a weak solution to the system (1), and if, for some $y \in \Omega$,*

$$w\left[B_{1/10}(y) \cap \{x : \mathcal{M}_2(|D^m u|)(x) > \lambda_0\}\right] \geq \epsilon w(B_{1/10}(y)), \tag{52}$$

then

$$\Omega_{1/10}(y) \subset \{x \in \Omega : \mathcal{M}_2(|D^m u|)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}_2(|\mathbf{f}|\chi_\Omega)(x) > \delta\}. \tag{53}$$

Proof. By Lemmas 2.5, it suffices to prove this proposition for the unweighted case $w \equiv 1$. To do this we argue by contradiction. Assume that

$$|B_{1/10}(y) \cap \{x : \mathcal{M}_2(|D^m u|)(x) > \lambda_0\}| \geq \epsilon |B_{1/10}(y)|, \tag{54}$$

but there is $x_0 \in \Omega_{1/10}(y)$ so that $x_0 \notin \{x \in \Omega : \mathcal{M}_2(|D^m u|)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}_2(|\mathbf{f}|\chi_\Omega)(x) > \delta\}$. Hence, for any $r > 0$ we have

$$\fint_{B_r(x_0)} |D^m u|^2 \leq 1, \quad \text{and} \quad \fint_{B_r(x_0)} |\mathbf{f}\chi_\Omega|^2 \leq \delta^2. \tag{55}$$

Observe that for any $x \in B_{1/10}(y)$ we have

$$\mathcal{M}_2(|D^m u|)(x) \leq \max\left\{\mathcal{M}_2(|D^m u|\chi_{B_{1/5}(y)})(x), 3^n\right\}. \tag{56}$$

We now consider the following 2 cases: $B_{2/5}(y) \cap \Omega^c \neq \emptyset$ and $B_{2/5}(y) \subset \Omega$.

Case 1: $B_{2/5}(y) \cap \Omega^c \neq \emptyset$**.**

Fix some $y_0 \in \partial\Omega \cap B_{2/5}(y)$. It is easy to see that

$$B_{1/5}(y) \subset B_1(y_0) \subset B_{20}(y_0) \subset B_{40}(x_0).$$

This together with (54) implies that

$$\fint_{B_{20}(y_0)} |D^m u|^2 \le 2^n, \quad \text{and} \quad \fint_{B_{20}(y_0)} |\mathbf{f}\chi_\Omega|^2 \le 2^n \delta^2. \tag{57}$$

Moreover, from the definition of the (δ, R) Reifenberg flat domain, it can be seen that there exists a coordinate system, whose variables are denoted by $z = (z_1, \dots, z_n)$ with the origin at some interior point of Ω such that in this new coordinate system $y_0 = (0, \dots, 0, -6\delta)$ and

$$B_5^+ \subset \Omega \cap B_5 \subset B_5 \cap \{z : z_n > -12\delta\}. \tag{58}$$

For $\delta < 1$, it is obviously that in this coordinate system, we have $B_5 \subset B_{20}(y_0)$. Taking this, (57) and (58) into account and applying Proposition 3.9, we can find a function $v \in W^{m,\infty}(B_2, \mathbb{R}^N)$ so that

$$\| \, |D^m v| \, \|_{L^\infty(B_{1/5}(y))} \le \| \, |D^m v| \, \|_{L^\infty(B_1)} \le C_1,$$

and

$$\fint_{B_{1/5}(y))} |D^m(u-v)|^2 \, dx \le C \fint_{B_1} |D^m(u-v)|^2 \, dx \le \widetilde{\epsilon},$$

where $\widetilde{\epsilon}$ is a positive small constant which will be fixed later.

On the other hand, by (54) we have

$$\mathcal{M}_2(|D^m u|)(x) \le \mathcal{M}_2(|D^m v|\chi_{B_{1/5}(y)})(x) + \mathcal{M}_2(|D^m(u-v)|\chi_{B_{1/5}(y)})(x).$$

By choosing $\lambda_0 = 2(C_1 + 3^n)$, we obtain that

$$\begin{aligned}
\{x \in B_{1/10}(y) : \mathcal{M}_2(|D^m u|)(x) > \lambda_0\} &\le \{x \in B_{1/10}(y) : \mathcal{M}_2(|D^m(u-v)|\chi_{B_{1/5}(y)})(x) > \lambda_0/2\} \\
&\le \frac{C}{\lambda_0^2} \int_{Q_{1/5}(y)} |D^m(u-v)|^2 \, dx \\
&\le C_3 \widetilde{\epsilon} |Q_{1/10}(y)|.
\end{aligned}$$

We now choose δ so that $C_3\widetilde{\epsilon} < \epsilon$. Hence,

$$\{x \in B_{1/10}(y) : \mathcal{M}_2(|\nabla u|)(x) > \lambda_0\} \le \epsilon |B_{1/10}(y)|,$$

which is a contradiction. This completes our proof of this case.

Case 2: $B_{2/5}(y) \subset \Omega$.

In this situation, we can repeat the argument above in which we make use of Proposition 3.4 instead of Proposition 3.9. We omit details here. □

By using the rescaling-argument, we have the following result.

Proposition 4.2. *For $w \in A_\infty$, there exist a positive constant $\lambda_0 > 0$ so that the following holds true. For any $\epsilon > 0$ there exists δ such that if Ω is a (δ, R) Reifenberg flat domain, the coefficients $\{a_{ij}^{\alpha\beta}\}$ satisfy (2), (3) and the small (δ, R)-BMO condition (7), and $u \in W_0^{m,2}(\Omega, \mathbb{R}^N)$ is a weak solution to the system (1), and if, for some $B_r(y)$, $y \in \Omega$ and $r < R/60$,*

$$w\left[B_r(y) \cap \{x : \mathcal{M}_2(|D^m u|)(x) > \lambda_0\}\right] \geq \epsilon w(B_r(y)), \tag{59}$$

then

$$\Omega_r(y) \subset \{x : \mathcal{M}_2(|D^m u|)(x) > 1\} \cup \{x : \mathcal{M}_2(|\mathbf{f}|\chi_\Omega)(x) > \delta\}. \tag{60}$$

Proof. Consider the re-scaled functions:

$$\tilde{u}^i(x) = \frac{u^i(rx+y)}{r^{m_i}}, \tilde{a}_{ij}^{\alpha\beta}(x) = a_{ij}^{\alpha\beta}(rx+y), \tilde{f}_i^\alpha(x) = f_i^\alpha(rx+y),$$

for all $i = 1, \dots, N$ and multi-indices α with $|\alpha| = m_i$. We set $\tilde{\Omega} = \{\frac{x-y}{r} : x \in \Omega\}$. Then we can check that $\bar{\Omega}$ is a $(\delta, 8)$ Reifenberg flat domain. At this stage, applying Proposition 4.1 to the system (1) with $\tilde{u}$, $\tilde{f}_i^\alpha$, $\tilde{a}_{ij}^{\alpha\beta}$ and the domain $\tilde{\Omega}$ replacing u, f_i^α, $a_{ij}^{\alpha\beta}$ and the domain Ω, respectively, we finish the proof. □

We are ready to give the proof of the main results.

Proof of Theorem 2.7. We first prove that if $|\mathbf{f}| \in L_w^{p,q}(\Omega)$ then $|\mathbf{f}| \in L^2(\Omega)$ with $p \in (2, \infty)$, $0 < q \leq \infty$ and $w \in A_{p/2}$. Indeed, from the definition of the weighted spaces $L_w^{p,q}(\Omega)$, we have, for $2 < p < \infty$, $0 < q \leq \infty$, $w \in A_\infty$,

$$L_w^{p,q}(\Omega) \subset L_w^{p,\infty}(\Omega) \subset L_w^{p-\sigma}(\Omega), \text{ for all } \sigma > 0 \text{ with } p - \sigma > 1. \tag{61}$$

Since $w \in A_{p/2}$, there exists $\sigma > 0$ such that $w \in A_{\frac{p-\sigma}{2}}$. See for example [10]. From this and (61), it suffices to prove that $L_w^s(\Omega) \subset L^2(\Omega)$ with $s > 2$ and $w \in A_{s/2}$. This follows from the following chain of inequalities

$$\begin{aligned}\Big(\int_\Omega |g(x)|^2 dx\Big)^{1/2} &= \Big(\int_\Omega |g(x)|^2 w^{2/s}(x) w^{-2/s}(x)\Big)^{1/2} \\ &\leq \Big(\int_\Omega |g(x)|^s w(x) dx\Big)^{1/s} \Big(\int_\Omega w(x)^{-\frac{2}{s-2}} dx\Big)^{\frac{s-2}{2s}} \\ &\leq \|g\|_{L_w^s(\Omega)} [w]_{A_{s/2}}^{1/s} \Big(\frac{w(\Omega)}{|\Omega|}\Big)^{-1/s},\end{aligned}$$

where in the second step we used Hölder's inequality.

We now turn to the proof of the theorem. Let δ, ϵ and λ_0 be as in Proposition 4.2. For $K > 0$, we define $u_K = \frac{u}{K}$ and $\mathbf{f}_K = \frac{\mathbf{f}}{K}$. We set

$$E = \{x \in \Omega : \mathcal{M}_2(|D^m u_K|)(x) > \lambda_0\}$$

and

$$F = \{x \in \Omega : \mathcal{M}_2(|D^m u_K|)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}_2(|\mathbf{f}_K|)(x) > \delta\}.$$

By (5), we have

$$\begin{aligned}|E| &\le \frac{c}{\lambda_0^2}\int_\Omega |D^m u_K|^2 = \frac{c}{(K\lambda_0)^2}\int_\Omega |D^m u|^2 \\ &\le \frac{A_1}{(K\lambda_0)^2}\int_\Omega |\mathbf{f}|^2.\end{aligned}$$

Let B_0 be a fixed ball so that $\Omega \subset B_0$. For a fixed $r = R/60$, from Lemma 2.5, there exists a constant A_2 so that

$$w(\Omega) \le w(B_0) \le w(2B_0) \le A_2 w(B_r(y)), \tag{62}$$

for all $y \in \Omega$.

Taking $K = \left[\frac{A_1}{\bar\epsilon |B_0| \lambda_0^2}\int_\Omega |\mathbf{f}|^2\right]^{1/2}$, where $\bar\epsilon$ is a constant will be fixed later, then we have

$$|E| \le \bar\epsilon |B_0|.$$

This, in combination with Lemma 2.5 and (62), implies that

$$w(E) \le A_3 \bar\epsilon^{\kappa_w} w(B_0) \le A_2 A_3 \bar\epsilon^{\kappa_w} w(B_r(y)), \quad \forall y \in \Omega,$$

for some constant $A_3 > 0$.

Taking $\bar\epsilon$ so that $A_2 A_3 \bar\epsilon^{\kappa_w} < \epsilon$, we get that

$$w(E) \le \epsilon w(B_r(y)), \quad \forall y \in \Omega,$$

which satisfies the condition (a) in Lemma 2.9. The condition (b) in Lemma 2.9 follows immediately from Proposition 4.2. Hence, applying Proposition 4.2 iteratively, we get that following estimate:

$$\begin{aligned} & w\Big(\{x \in \Omega : \mathcal{M}_2(|D^m u_K|)(x) > \lambda_0^k\}\Big)^{q/p} \\ &\le \epsilon_1^{qk/p} w\Big(\{x \in \Omega : \mathcal{M}_2(|D^m u_K|)(x) > 1\}\Big)^{q/p} \\ &\quad + \sum_{\ell=1}^{k} \epsilon_1^{q\ell/p} w\Big(\{x \in \Omega : \mathcal{M}_2(|\mathbf{f}_K|)(x) > \delta\lambda_0^{k-\ell}\}\Big)^{q/p},\end{aligned}$$

for $k = 1, 2, \ldots$, where $\epsilon_1 = c(n, w)\epsilon$.

Hence, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\lambda_0^{kq} w\Big(\{x\in\Omega : \mathcal{M}_2(|D^m u_K|)(x) > \lambda_0^k\}\Big)^{q/p}\\
&\quad\le \sum_{k=1}^{\infty}\epsilon_1^{qk/p}\lambda_0^{kq} w\Big(\{x\in\Omega : \mathcal{M}_2(|D^m u_K|)(x) > 1\}\Big)^{q/p}\\
&\qquad+\sum_{k=1}^{\infty}\lambda_0^{kq}\sum_{\ell=1}^{k}\epsilon_1^{q\ell/p} w\Big(\{x\in\Omega : \mathcal{M}_2(|\mathbf{f}_K|)(x) > \delta\lambda_0^{k-\ell}\}\Big)^{q/p}.
\end{aligned}$$

Taking ϵ small enough so that $\epsilon_1\lambda_0^p < 1$ and then applying Lemma 2.10, we get that

$$\|\mathcal{M}_2(|D^m u_K|)\|_{L^{p,q}_w(\Omega)} \lesssim \|\mathbf{f}_K\|_{L^{p,q}_w(\Omega)} + w(\Omega)^{1/p}.$$

This together with Lemma 2.6 yields

$$\||D^m u_K|\|_{L^{p,q}_w(\Omega)} \lesssim \|\mathbf{f}_K\|_{L^{p,q}_w(\Omega)} + w(\Omega)^{1/p}.$$

This implies

$$\||D^m u|\|_{L^{p,q}_w(\Omega)} \lesssim \|\mathbf{f}\|_{L^{p,q}_w(\Omega)} + K w(\Omega)^{1/p}.$$

On the other hand, we have

$$\begin{aligned}
K &= \left[\frac{A_1}{\bar{\epsilon}|B_0|\lambda_0^2}\int_{\Omega}|\mathbf{f}|^2\right]^{1/2} \lesssim \Big(\fint_{B_0}|\mathbf{f}\chi_\Omega|^2\Big)^{1/2}\\
&\lesssim \inf_{x\in\Omega}\mathcal{M}_2(|\mathbf{f}\chi_\Omega|)(x).
\end{aligned}$$

Hence, for $\sigma > 0$ such that $w \in A_{\frac{p-\sigma}{2}}$, we have

$$\begin{aligned}
\||D^m u|\|_{L^{p,q}_w(\Omega)} &\lesssim \|\mathbf{f}\|_{L^{p,q}_w(\Omega)} + w(\Omega)^{1/p}\inf_{x\in\Omega}\mathcal{M}_2(|\mathbf{f}\chi_\Omega|)(x)\\
&\lesssim \|\mathbf{f}\|_{L^{p,q}_w(\Omega)} + w(\Omega)^{\frac{1}{p-\sigma}}\inf_{x\in\Omega}\mathcal{M}_2(|\mathbf{f}\chi_\Omega|)(x)\\
&\lesssim \|\mathbf{f}\|_{L^{p,q}_w(\Omega)} + \|\mathcal{M}_2(|\mathbf{f}\chi_\Omega|)\|_{L^{p-\sigma}_w}\\
&\lesssim \|\mathbf{f}\|_{L^{p,q}_w(\Omega)} + \||\mathbf{f}|\|_{L^{p-\sigma}_w(\Omega)}\\
&\lesssim \|\mathbf{f}\|_{L^{p,q}_w(\Omega)},
\end{aligned}$$

where in the last inequality we used (61). This completes the proof. $\square$

We now give the proof of Theorem 2.8.

Proof of Theorem 2.8. Fix $x_0 \in \Omega$ and $r > 0$. According to Proposition 2 in [6] that for any $\sigma \in (0,1)$, $(\mathcal{M}(\chi_{B_r(x_0)}))^\sigma \in A_1$.

Choose $\sigma \in (\lambda/n, 1)$. Then for $p \in (2,\infty)$, we have

$$(\mathcal{M}(\chi_{B_r(x_0)}))^\sigma \in A_1 \subset A_{p/2}.$$

Hence, Theorem 2.7 tells us that there exists δ so that

$$\| \, |D^m u| \, \|_{L^{p,q}_w(\Omega)} \lesssim \| \, |\mathbf{f}| \, \|_{L^{p,q}_w(\Omega)}, \quad w(x) = (\mathcal{M}(\chi_{B_r(x_0)})(x))^\sigma. \tag{63}$$

We now set

$$S_k(B_r(x_0)) = \begin{cases} B_{2^k r}(x_0) \backslash B_{2^{k-1} r}(x_0), & k \ge 1, \\ B_r(x_0), & k = 0. \end{cases}$$

We now consider two cases:

Case 1: $q/p \ge 1$.

We have

$$\begin{aligned} \| |D^m u| \|^p_{L^{p,q}(\Omega \cap B_r(x_0))} &= \left\{ p \int_0^\infty \left[t^p \int_{\{x \in \Omega \cap B_r(x_0) : |D^m u(x)| > t\}} dx \right]^{q/p} \frac{dt}{t} \right\}^{p/q} \\ &= \left\{ p \int_0^\infty \left[t^p \int_{\{x \in \Omega : |D^m u(x)| > t\}} \chi_{B_r(x_0)} dx \right]^{q/p} \frac{dt}{t} \right\}^{1/q} \\ &\le \left\{ p \int_0^\infty \left[t^p \int_{\{x \in \Omega : |D^m u(x)| > t\}} (\mathcal{M}(\chi_{B_r(x_0)})(x))^\sigma dx \right]^{q/p} \frac{dt}{t} \right\}^{p/q}. \end{aligned}$$

This, in combination with (63), gives

$$\begin{aligned} &\| |D^m u| \|^p_{L^{p,q}(\Omega \cap B_r(x_0))} \\ &\le \left\{ p \int_0^\infty \left[t^p \int_{\{x \in \Omega : |\mathbf{f}(x)| > t\}} (\mathcal{M}(\chi_{B_r(x_0)})(x))^\sigma dx \right]^{q/p} \frac{dt}{t} \right\}^{p/q} \\ &\le \sum_{k=0}^\infty \left\{ p \int_0^\infty \left[t^p \int_{\{x \in \Omega \cap S_k(B_r(x_0)) : |\mathbf{f}(x)| > t\}} (\mathcal{M}(\chi_{B_r(x_0)})(x))^\sigma dx \right]^{q/p} \frac{dt}{t} \right\}^{p/q}. \end{aligned} \tag{64}$$

A simple calculation shows that

$$(\mathcal{M}(\chi_{B_r(x_0)})(x))^\sigma \le \begin{cases} 1, & x \in S_0(B_r(x_0)) \\ 2^{-\sigma n(k-1)}, & x \in S_k(B_r(x_0)), k \ge 1. \end{cases} \tag{65}$$

Inserting this into (64), we obtain

$$\begin{aligned} &\||D^m u|\|^p_{L^{p,q}(\Omega \cap B_r(x_0))} \\ &\quad \le \left\{ p \int_0^\infty \left(t^p |\{x \in \Omega \cap B_r(x_0) : |\mathbf{f}(x)| > t\}|\right)^{q/p} \frac{dt}{t} \right\}^{p/q} \\ &\quad + \sum_{k=1}^\infty 2^{-\sigma n(k-1)} \left\{ p \int_0^\infty \left(t^p |\{x \in \Omega \cap B_{2^k r}(x_0) : |\mathbf{f}(x)| > t\}|\right)^{q/p} \frac{dt}{t} \right\}^{p/q}. \end{aligned}$$

This implies that

$$\begin{aligned} &r^{-\lambda} \||D^m u|\|^p_{L^{p,q}(\Omega \cap B_r(x_0))} \\ &\quad \le r^{-\lambda} \left\{ p \int_0^\infty \left(t^p |\{x \in \Omega \cap B_r(x_0) : |\mathbf{f}(x)| > t\}|\right)^{q/p} \frac{dt}{t} \right\}^{p/q} \\ &\quad + \sum_{k=1}^\infty 2^{-\sigma n(k-1)} r^{-\lambda} \left\{ p \int_0^\infty \left(t^p |\{x \in \Omega \cap B_{2^k r}(x_0) : |\mathbf{f}(x)| > t\}|\right)^{q/p} \frac{dt}{t} \right\}^{p/q} \\ &\quad \le \||\mathbf{f}|\|^p_{L^{p,q;\lambda}(\Omega)} + \sum_{k=1}^\infty 2^{-k(n\sigma - \lambda)} \||\mathbf{f}|\|^p_{L^{p,q;\lambda}(\Omega)} \\ &\quad \lesssim \||\mathbf{f}|\|^p_{L^{p,q;\lambda}(\Omega)}. \end{aligned}$$

Hence

$$\||D^m u|\|_{L^{p,q;\lambda}(\Omega)} \lesssim \||\mathbf{f}|\|_{L^{p,q;\lambda}(\Omega)}.$$

Case 2: $q/p < 1$**.**
Using the inequality

$$\left(\sum_{k=0}^\infty a_k\right)^s \le \sum_{k=0}^\infty a_k^s, \quad \text{for all } a_k \ge 0 \text{ and } 0 < s < 1,$$

and (63), we then argue similarly to the Case 1 to obtain

$$
\begin{aligned}
\||D^m u|\|^q_{L^{p,q}(\Omega\cap B_r(x_0))} &\le p\int_0^\infty \left[t^p \int_{\{x\in\Omega:|D^m u(x)|>t\}} (\mathcal{M}(\chi_{B_r(x_0)})(x))^\sigma dx\right]^{q/p} \frac{dt}{t}\\
&\le p\int_0^\infty \left[t^p \int_{\{x\in\Omega:|\mathbf{f}(x)|>t\}} (\mathcal{M}(\chi_{B_r(x_0)})(x))^\sigma dx\right]^{q/p} \frac{dt}{t}\\
&\le \sum_{k=0}^\infty p\int_0^\infty \left[t^p \int_{\{x\in\Omega\cap S_k(B_r(x_0)):|\mathbf{f}(x)|>t\}} (\mathcal{M}(\chi_{B_r(x_0)})(x))^\sigma dx\right]^{q/p} \frac{dt}{t}.
\end{aligned}
$$

Using (65), we obtain

$$
\begin{aligned}
\||D^m u|\|^q_{L^{p,q}(\Omega\cap B_r(x_0))} &\le p\int_0^\infty \left(t^p|\{x\in\Omega\cap B_r(x_0):|\mathbf{f}(x)|>t\}|\right)^{q/p}\frac{dt}{t}\\
&\quad+\sum_{k=1}^\infty 2^{-\sigma nq(k-1)/p}\,p\int_0^\infty \left(t^p|\{x\in\Omega\cap B_{2^k r}(x_0):|\mathbf{f}(x)|>t\}|\right)^{q/p}\frac{dt}{t}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&r^{-\frac{\lambda q}{p}}\||D^m u|\|^q_{L^{p,q}(\Omega\cap B_r(x_0))}\\
&\quad\le r^{-\frac{\lambda q}{p}}\,p\int_0^\infty \left(t^p|\{x\in\Omega\cap B_r(x_0):|\mathbf{f}(x)|>t\}|\right)^{q/p}\frac{dt}{t}\\
&\qquad+\sum_{k=1}^\infty 2^{-\sigma nq(k-1)/p}r^{-\frac{\lambda q}{p}}\,p\int_0^\infty \left(t^p|\{x\in\Omega\cap B_{2^k r}(x_0):|\mathbf{f}(x)|>t\}|\right)^{q/p}\frac{dt}{t}\\
&\quad\le \||\mathbf{f}|\|^q_{L^{p,q;\lambda}(\Omega)}+\sum_{k=1}^\infty 2^{-kq(n\sigma-\lambda)/p}\||\mathbf{f}|\|^q_{L^{p,q;\lambda}(\Omega)}\\
&\quad\lesssim \||\mathbf{f}|\|^q_{L^{p,q;\lambda}(\Omega)}.
\end{aligned}
$$

Hence we obtain

$$
\||D^m u|\|_{L^{p,q;\lambda}(\Omega)}\lesssim \||\mathbf{f}|\|_{L^{p,q;\lambda}(\Omega)}. \quad\square
$$

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References

[1] D.R. Adams, L.I. Hedberg, Function Spaces and Potential Theory, Grundlehren der Mathematischen Wissenschaften, vol. 314, Springer-Verlag, Berlin, 1996.
[2] S-S. Byun, L. Wang, Elliptic equations with BMO coefficients in Reifenberg domains, Comm. Pure Appl. Math. 57 (10) (2004) 1283–1310.
[3] S-S. Byun, L. Wang, Gradient estimates for elliptic systems in non-smooth domains, Math. Ann. 341 (2008) 629–650.
[4] S-S. Byun, S. Ryu, Gradient estimates for higher order elliptic equations on nonsmooth domains, J. Differential Equations 250 (2011) 243–263.
[5] L.A. Caffarelli, I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form, Comm. Pure Appl. Math. 51 (1998) 1–21.
[6] R.R. Coifman, R. Rochberg, Another characterization of BMO, Proc. Amer. Math. Soc. 79 (1980) 249–254.
[7] G. David, T. Toro, A generalization of Reifenberg's theorem in $\mathbb{R}^3$, Geom. Funct. Anal. 18 (4) (2008) 1168–1235.
[8] H. Dong, D. Kim, On the L^p-solvability of higher order parabolic and elliptic systems with BMO coefficients, Arch. Ration. Mech. Anal. 199 (2011) 889–941.
[9] G. Di Fazio, L^p estimates for divergence form elliptic equations with discontinuous coefficients, Boll. Unione Mat. Ital. VII. Ser. A 10 (2) (1996) 409–420.
[10] J. Duoandikoetxea, Fourier Analysis, Grad. Stud. Math., vol. 29, American Math. Soc., Providence, 2000.
[11] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Annals of Mathematics Studies, vol. 105, Princeton University Press, Princeton, NJ, 1983.
[12] M. Giaquinta, G. Modica, Regularity results for some classes of higher order nonlinear elliptic systems, J. Reine Angew. Math. 311/312 (1979) 145–169.
[13] E. Giusti, Regolarità parziale delle soluzioni di sistemi ellittici quasi-lineari di ordine arbitrario, Ann. Sc. Norm. Super. Pisa (3) 23 (1969) 115–141.
[14] H. Heck, M. Hieber, Maximal L^p-regularity for elliptic operators with VMO-coefficients, J. Evol. Equ. 3 (2) (2003) 332–359.
[15] T. Iwaniec, C. Sbordone, Weak minima of variational integrals, J. Reine Angew. Math. 454 (1994) 143–161.
[16] D. Jerison, C. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal. 130 (1995) 161–219.
[17] N.V. Krylov, Parabolic and elliptic equations with VMO coefficients, Comm. Partial Differential Equations 32 (2007) 453–475.
[18] N.V. Krylov, Second-order elliptic equations with variably partially VMO coefficients, J. Funct. Anal. 257 (2009) 1695–1712.
[19] T. Mengesha, N.C. Phuc, Global estimates for quasilinear elliptic equations on Reifenberg flat domains, Arch. Ration. Mech. Anal. 203 (2012) 189–216.
[20] T. Mengesha, N.C. Phuc, Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains, J. Differential Equations 250 (2011) 2485–2507.
[21] N. Meyers, An L^p estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat. III. Ser. 17 (1963) 189–206.
[22] C.B. Morrey, Multiple Integrals in the Calculus of Variations, Grundlehren Math. Wiss., vol. 130, Springer, New York, 1966.
[23] M. Parviainen, Global higher integrability for parabolic quasiminimizers in nonsmooth domains, Calc. Var. Partial Differential Equations 31 (2008) 75–98.
[24] E. Reifenberg, Solutions of the plateau problem for m-dimensional surfaces of varying topological type, Acta Math. 104 (1960) 1–92.
[25] C.G. Simader, On Dirichlet Boundary Value Problem. An L^p Theory Based on a Generalization of Garding's Inequality, Lecture Notes in Math., vol. 268, Springer, Berlin, 1972.
[26] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126–150.

[27] T. Toro, Doubling and flatness: geometry of measures, Notices Amer. Math. Soc. 44 (1997) 1087–1094.
[28] L. Wang, F. Yao, Global regularity for higher order divergence elliptic and parabolic equations, J. Funct. Anal. 266 (2014) 792–813.