Report

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and P^+ the set of dominant weights of $\mathfrak{g}$. The current algebra associated to $\mathfrak{g}$ is a subalgebra of the special maximal parabolic subalgebra of the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ associated to $\mathfrak{g}$. As a vector space it is isomorphic to $\mathfrak{g} \otimes \mathbf{C}[t]$, where $\mathbf{C}[t]$ is the polynomial ring in the indeterminate t and the Lie bracket is given in the obvious way. The study of the category of graded finite–dimensional representations of the current algebra has been of interest in recent years for a variety of reasons. Many interesting families of finite dimensional representations of current algebras arise naturally. Demazure modules, local Weyl modules, Kirillov-Reshetikhin modules, irreducible modules and their fusion prducts are few of them. We are interested in studying relations between these modules and their fusion prducts.

Given a positive integer ℓ and a dominant integral weight λ of $\mathfrak{g}$, we can define the Demazure modules of level ℓ (denoted by $D(\ell,\lambda)$), for the maximal parabolic subalgebra of $\widehat{\mathfrak{g}}$ and hence for current algebra. These modules were extensively studied in [3], [4], [5]. We note here, that [3], [4] work with a special family of Demazure modules and it has been generalized in [5]. The defining relations of this generalized Demazure modules of current algebra $\mathfrak{g}[t]$ are greatly simplified in [2]. Using this simplified presentation of Demazure modules, we are trying to prove that the fusion product of the Demazure modules of the same level again turns out to be a Demazure module. This generalizes a theorem of G. Fourier and P. Littelmann that the fusion product of the (special) family of Demazure modules of the same level again turns out to be a (special) Demazure module (see Theorem C, [4]). Set $\Gamma = \left\{\lambda \in P^+ : \quad \lambda = \sum_{i\in I} d_i s_i \omega_i\right\}$, where $d_i = \frac{2}{(\alpha_i,\alpha_i)}$. The following is already established in [6].

Theorem 1. Let $\ell \in \mathbb{N}$, $\lambda \in P^+$ and $\mu \in \Gamma$. Suppose that there exists $\mu_j \in \Gamma$, $p_j \in \mathbb{N}$, $1 \le j \le m$ such that $\mu = \mu_1 + \cdots + \mu_m$ and $\lambda(h_\Theta) \le \ell$. Then we have an isomorphism

$$D(\ell,\ell\mu+\lambda) \cong D(\ell,\ell\mu_1) * \cdots * D(\ell,\ell\mu_m) * D(\ell,\lambda) \cong D(\ell,\ell\mu_1) * \cdots * D(\ell,\ell\mu_m) * V(\lambda)$$

The special case when $\lambda = 0$, was proved by G. Fourier and P. Littelmann in [4]. I give two applications of theorem 1. As a first application, I provide some additional evidence for the generalization of the Schur positivity (see [6]), that was conjectured in [1] and as a second application I construct $\mathfrak{g}[t]$-module structure of the associated untwisted affine Kac-moody algebra module $V(\ell\Lambda_0 + \lambda)$ as a semi-infinite fusion product of finite dimensional $\mathfrak{g}[t]$-modules, that was conjectured in [4].

In join work with Prof. Vyjayanthi Chari and her students Peri Shereen, Jeffrey Wand, I am trying to prove the following general statement:

Theorem 2. Let $\ell \in \mathbb{N}$, $\lambda \in P^+$ and $\mu \in \Gamma$. Suppose that there exists $\mu_j \in \Gamma$, $p_j \in \mathbb{N}$, $1 \le j \le m$ such that $\mu = \mu_1 + \cdots + \mu_m$ and $\lambda \in P^+_\ell$, where $P^+_\ell = \{\nu \in P^+ : \nu(h_i) < d_i\ell,\ i = 1, \cdots, n\}$ Then we have an isomorphism

$$D(\ell,\ell\mu+\lambda) \cong D(\ell,\ell\mu_1) * \cdots * D(\ell,\ell\mu_m) * D(\ell,\lambda)$$

Note that this fusion product decomposition is similar to Steinberg tensor product theorem for positive characteristic. We already proved theorem 2 for general $\mathfrak{g}$ and for sufficiently large level ℓ and $\mathfrak{g} = A_n$, C_n or G_2 and general ℓ. We expect this to be true for any $\mathfrak{g}$ and ℓ.

Research In Progress. In particular, any Demazure module of level ℓ is a fusion product of $D(\ell, \ell\mu)$ and $D(\ell, \lambda)$, where $\mu \in \Gamma$ and $\lambda \in P_\ell^+$. The $\mathfrak{g}$-structure of $D(\ell, \ell\mu)$ for $\mu \in \Gamma$ is well understood. Thus to understand the $\mathfrak{g}$-structure of any Demazure module, it is enough to understand the $\mathfrak{g}$-structure of Demazure modules of the form $D(\ell, \lambda)$, where $\lambda \in P_\ell^+$. We are trying to compute $\mathfrak{g}$–character of these modules.

Preprint.

- R. Venkatesh. *Fusion product structure of Demazure modules,* arXiv:1311.2224.
- Vyjayanthi Chari, Peri Shereen, R. Venkatesh, Jeffrey Wand. *Demazure modules and its fusion product decomposition*, under preparation.

Students advised. Peri Shereen, Jeffrey Wand

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