Activity report
Research period: 2017-11-06 - 2017-12-17.

IMU-Simons African Fellowship Program

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1 Identification

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Start date of research period: 2017-11-06

End date of research period: 2017-12-17

Name of host Institution and host: Wilfrid Gangbo, PhD, Professor of Mathematics. Mathematics Department, UCLA. Los Angeles, CA 90095, USA.

2 Introduction

The research work has been devoted to PDE and ODE methods for constructively solving one version of the Monge–Kantorovich mass transfer problem.

During my stay, the research work was mainly oriented to the:

- determination of u the potential and a the transport density,
- numerical simulations for an optimal transfer plan s,
- creation of movies showing the mass transfert from pail of soil ("déblais") to excavation ("remblais") for some given functions.

3 Research program

The aim is to contribute to one of the most vibrant area of mathematics, optimal transportation and its application.

4 Material extracted from the paper of Evans-Gangbo

In [1], Evans and Gangbo demonstrate that a solution to the classical Monge–Kantorovich problem of optimally rearranging the measure $\mu^+ = f^+dx$ onto $\mu^- = f^-dy$ can be constructed by studying the p-Laplacian equation

$$-div(|Du_p|^{p-2}Du_p) = f^+ - f^-$$

in the limit as $p \longrightarrow \infty$. The idea is to show $u_p \longrightarrow u$ where u satisfies

$$|Du| \leq 1, \quad -div(aDu) = f^+ - f^-$$

for some density $a \geq 0$, and then to build a flow by solving an ODE involving a, Du, f^+ and f^-.
Evans and Gangbo [1] introduce some PDE and ODE methods for constructively

solving one version of the Monge-Kantorovich mass transfer problem. The basic issue is this. Given two nonnegative, summable functions $f^\pm$ on $\mathbb{R}^n$ satisfying the compatibility condition

$$\int_{\mathbb{R}^n} f^+dx = \int_{\mathbb{R}^n} f^-dy$$

they consider the corresponding measures $\mu^+ = f^+dx$, $\mu^- = f^-dy$, and ask how they can optimally rearrange μ^+ onto μ^-; that is,

$$f^+(x) = f^-(r(x))detDr(x) \qquad (x \in \mathbb{R}^n) \tag{1}$$

Denote by $\mathcal{A}$ the admissible class of smooth, one-to-one functions r satisfying (1). They then seek a mass transfer plan $s \in \mathcal{A}$ which is optimal in the sense that

$$I[s] = min_{r\in\mathcal{A}} I[r],$$

where

$$I[r] = \int_{\mathbb{R}^n} |x - r(x)| f^+(x)dx = \int_{\mathbb{R}^n} |x - r(x)| d\mu^+$$

This is a form of Monge's problem of the "déblais" and "remblais" (cf. Monge [2], Dupin [4], Appell [5]), dating from the early 1780's. The physical interpretation is that they are given a pile of soil or rubble (the "déblais"), with mass density f^+, which they wish to transport to an excavation or fill (the "remblais"), with mass density f^-. For a given transport scheme r, condition (1) is conservation of mass. Furthermore, as each particle of soil moves a distance $|x - r(x)|$, they can interpret $I[r]$ as the total work involved. They consequently are looking for a way to rearrange $\mu^+ = f^+dx$ onto $\mu^- = f^-dy$, which requires the least work. This optimization problem, and its many, variants and extensions (entailing for example more general measures on more general spaces, different cost functionals, etc.) has been intensively studied for over two hundred years. Some of the principal discoveries are in [1].

They set forth their hypotheses regarding the densities f^+, f^- and then obtain estimates, independent of p, on solutions of the corresponding p-Laplacian equations.

They henceforth suppose f^+, f^- are nonnegative, Lipschitz functions on $\mathbb{R}^n$ with compact support, satisfying the compatibility condition:

$$\int_{\mathbb{R}^n} f^+ dx = \int_{\mathbb{R}^n} f^- dy$$

Recall from Rademacher's Theorem that Du exists a.e. The next Theorem characterizes u.

Theorem 1. *(i) There exists function $a \in L^\infty(B(0,R))$ such that*

$$-div(aDu) = f \quad in\ B(0,R) \tag{2}$$

in the weak sense. In addition

$$|Du| \leq 1\ a.e.,\ \ a \geq 0\ a.e.$$

and for a.e. z,

$$a(z) > 0 \text{ implies } |Du(z)| = 1.$$

(ii) Furthermore,

$$\int_{B(O,R)} ufdz = \max_{|Dw|\leq 1\ a.e.} \int_{B(O,R)} wfdz.$$

We hereafter call u the potential and a the transport density.

For the proof of the theorem and for more details, see [1].

Recall that, Monge himself contributed the essential insight that an optimal transfer plan s should be in part determined by a potential u. More precisely, he deduced by heutistic, geometric arguments that if an optimal plan s exists, then there exists a scalar potential function u such that:

$$\frac{s(x)-x}{|s(x)-x|} = -Du(x) \qquad (x \in X) \tag{3}$$

where $X = supp(f^+)$.
In other words the direction that each particle of soil should move is determined as the (opposite of the) gradient Du of u. Observe that necessarily then

$$|Du| = 1 \text{ in } X. \tag{4}$$

The function a is the Lagrange multiplier for the constraint that $|Du| \leq 1$ a.e. We employ u and a to design an optimal mapping s, by solving for a.e. point x the ODE (cf. [1],[3]):

$$\begin{cases} \dot{T}(t,x) = \frac{-div(a(T(t,x)))\nabla u(T(t,x))}{(1-t)f^+(T(t,x))+tf^-(T(t,x))} & 0 \leq t \leq 1 \\ T(0,x) = x \end{cases} \tag{5}$$

where $u \in \mathcal{L} = \left\{ w : \mathbb{R}^n \to \mathbb{R} \mid \text{Lib}[w] = \sup\limits_{x\neq y} \frac{|w(x)-w(y)|}{|x-y|} \leq 1 \right\}$ and u maximizes

$$K[w] = \int_{\mathbb{R}^n} wfdz$$

During the stay, **my research focused on the numerical resolutions of these optimization and PDE problems** (determination of potential, transport density and optimal transfer plan $T(t, x)$, solution of the PDE (5)) for mass transport problem, so complex and very difficult to solve.

In the following section, we present some simulation results.

5 Simulations

For numerical simulations, first we start with dimension 1.

5.1 Dimension 1

Exemple 1 Let $f(x) = x$, $x \in [-R, R]$ with $R = 10$ (See Figure 1 for f^+ and f^- functions). We use the software Matlab [12] to compute the transport density a and

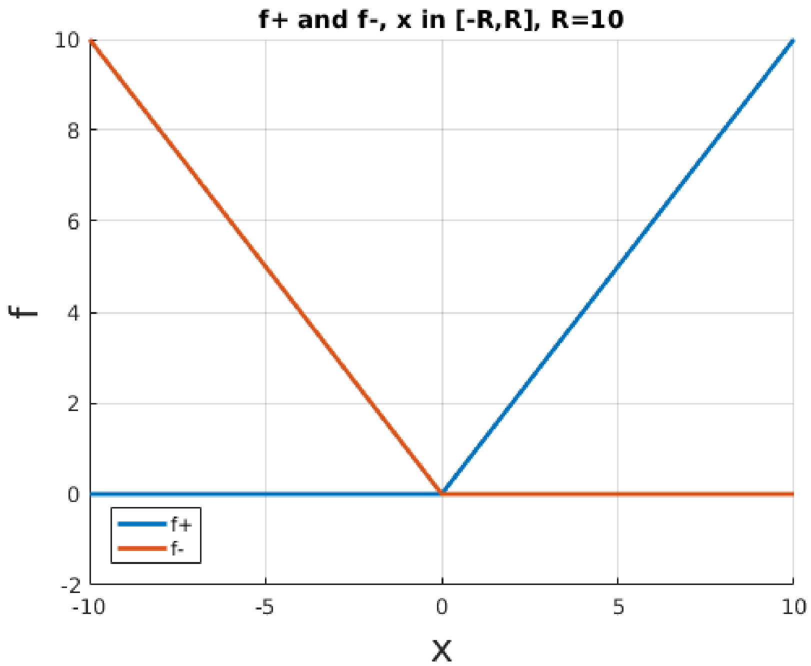

Fig. 1. Function f^+ and f^- for f(x)=x

the potential u ($\nabla u = 1$), and obtain (for the transport density and the optimal mapping):

$$a = 50 - \frac{x^2}{2}$$

$$\begin{cases} T(t,x) = \sqrt{100 + \frac{x^2-100}{1-t}} & 0 \le t \le \hat{t} \\ T(t,x) = -\sqrt{100 - \frac{x^2}{t}} & t \ge \hat{t} \end{cases}$$

with $\hat{t} = \frac{x^2}{100}$.
The optimal mapping is represented in Figure 2, and the transport density in Figure

3, satisfying the border conditions:

$$a(10) = a(-10) = 0 \qquad \text{and} \qquad T(0,x) = x$$

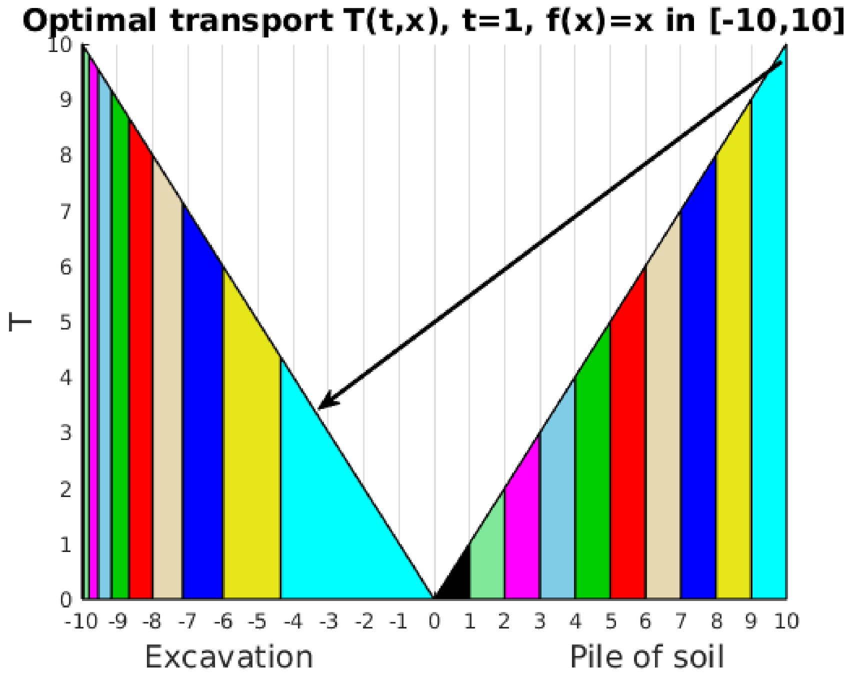

Fig. 2. Pile of soil to excavation for function $f(x) = x$

We can see in Figure 2, the mass transfer from pail of soil to excavation (cyan, red, yelow, blue, etc.).
We give in separate files, some movies showing the transport for a given pile of soil or rubble (the "déblais"), with mass density f^+, to an excavation or fill (the "remblais"), with mass density f^-.

Exemple 2 For $f(x) = x^3$, we obtain with computations on Matlab:

$$a = -\frac{x^4}{4} + 2500$$

$$\begin{cases} T(t,x) = (10000 + \frac{x^4 - 10000}{1-t})^{1/4} & 0 \leq t \leq \hat{t} \\ T(t,x) = -(10000 - \frac{x^4}{t})^{1/4} & t \geq \hat{t} \end{cases}$$

with $\hat{t} = \frac{x^4}{10000}$. The transport density and the optimal mapping are represented respectively in Figures 4 and 5.

5.2 Dimension 2

We calculate the approximate solution v on a mesh where each index point (i,j) (x_i, y_j) is located on the axis x by its position $x_i = i\Delta x$ and on the y axis by $y_j = j\Delta y$. The approximate solution to the nodes of the mesh will be noted:

$$v(x_i, y_j) = v(i\Delta x, j\Delta y) = v_{i,j}; \qquad v(x_i, y_{j+1}) = v(i\Delta x, (j+1)\Delta y) = v_{i,j+1}$$

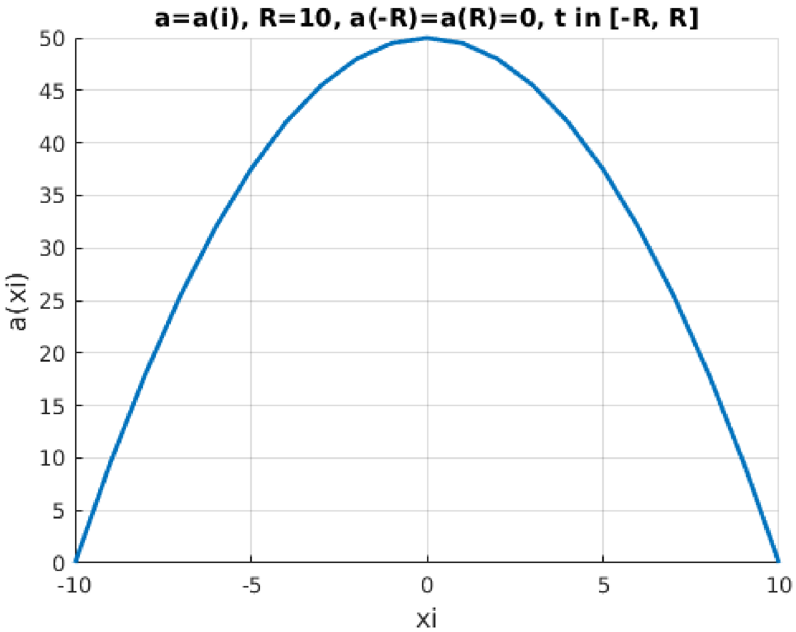

Fig. 3. Transport density function a(x) for the function $f(x) = x$

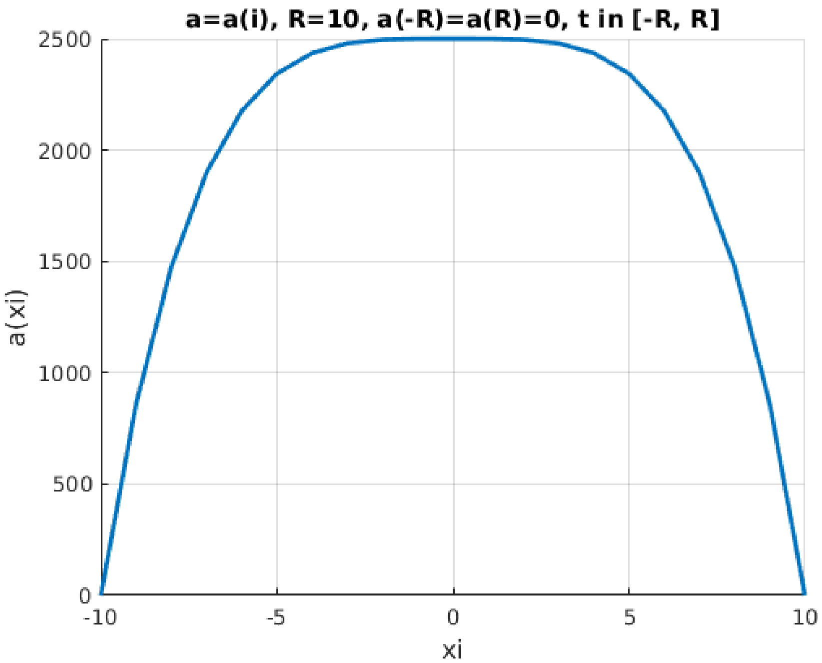

Fig. 4. Transport density function a(x) for function $f(x) = x^3$

and

$$v(x_{i+1}, y_j) = v((i+1)\Delta x, j\Delta y) = v_{i+1,j}; \qquad v(x_{i-1}, y_j) = v((i-1)\Delta x, j\Delta y) = v_{i-1,j}$$

We seek an approximation of the first derivatives in space $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$. We can also express, if necessary, the second derivatives $\frac{\partial^2 v}{\partial x^2}$ and $\frac{\partial^2 v}{\partial y^2}$.
Thus, we obtain an approximation of the first derivatives $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ by finite differ-

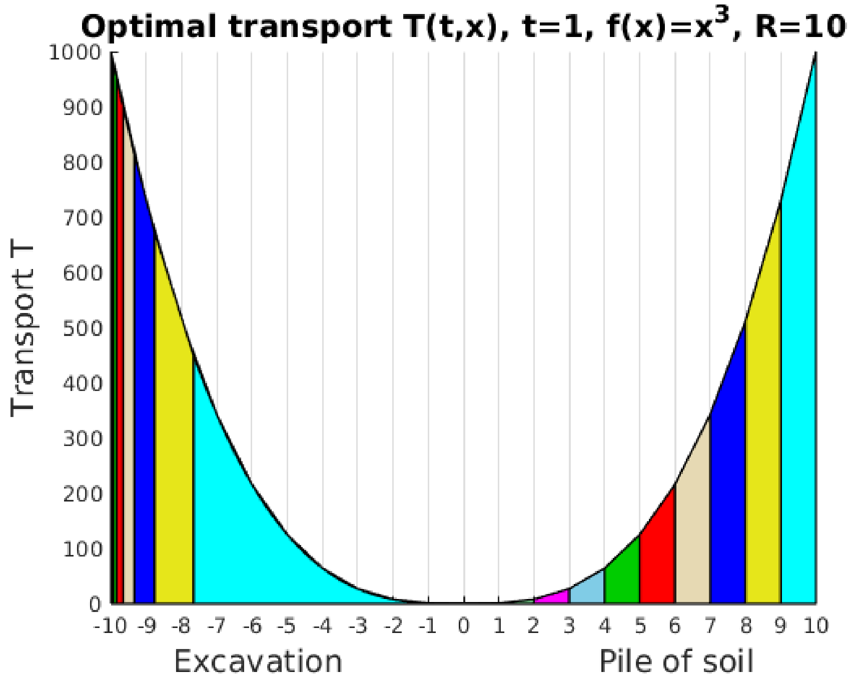

Fig. 5. Pile of soil to excavation for function $f(x) = x^3$

ences to order 1: $\left.\dfrac{\partial v}{\partial x}\right|_i^j \approx \dfrac{v_{i+1,j} - v_{i,j}}{\Delta x}$ and $\left.\dfrac{\partial v}{\partial y}\right|_i^j \approx \dfrac{v_{i,j+1} - v_{i,j}}{\Delta y}$. The approximation of the second derivatives $\frac{\partial^2 v}{\partial x^2}$ and $\frac{\partial^2 v}{\partial y^2}$ by finite differences to order 2 can be also given by: $\left.\dfrac{\partial^2 v}{\partial x^2}\right|_i^j \approx \dfrac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta x^2}$ and $\left.\dfrac{\partial^2 v}{\partial y^2}\right|_i^j \approx \dfrac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta y^2}$.

Finite difference scheme for equation (2): By finite differences, the scheme of the equation (2) gives the following problem:

$$\begin{array}{ll} -div(v) = f & \text{in } B =]0,1[*]0,1[\\ v(0,y) = v(1,y) = v(x,0) = v(x,1) = 0 & \end{array} \tag{6}$$

with $v = a\nabla u$.
On a mesh of n_x points following x and n_y points following y (Figure 6), the centered finite difference discretization of the equation (6) is written:

$$\frac{v_{i+1,j} - v_{i,j}}{dx} + \frac{v_{i,j+1} - v_{i,j}}{dy} = -f_{i,j} \qquad \forall i = 1, ..., n_x - 1 \ \ \forall j = 1, ..., n_y - 1 \tag{7}$$

with the boundary conditions: $v_{1,j} = v_{n_x,j} = 0 \ \forall j = 1, ..., n_y$ and $v_{i,1} = v_{i,n_y} = 0 \ \forall i = 1, ..., N_x$. The space discretization steps are equidistant and verify $dx = \frac{1}{n_x - 1}$ and $dy = \frac{1}{n_y - 1}$.
This scheme leads to a matrix system of $n = n_x \times n_y$ unknown $v_{i,j}$. And to write this system in matrix form $\mathcal{A}X = \mathcal{B}$, we must transform the matrix of unknowns $v_{i,j}$ into an unknown vector X_k. For this we number the unknowns by lines, i.e. we perform the index transformation (i, j) to the mono-index $k = i + (j - 1)n_x$. With this change of index, the difference equation (7) is written:

$$av_{k+1} - cv_k + bv_{k+n_x+1} = -f_k$$

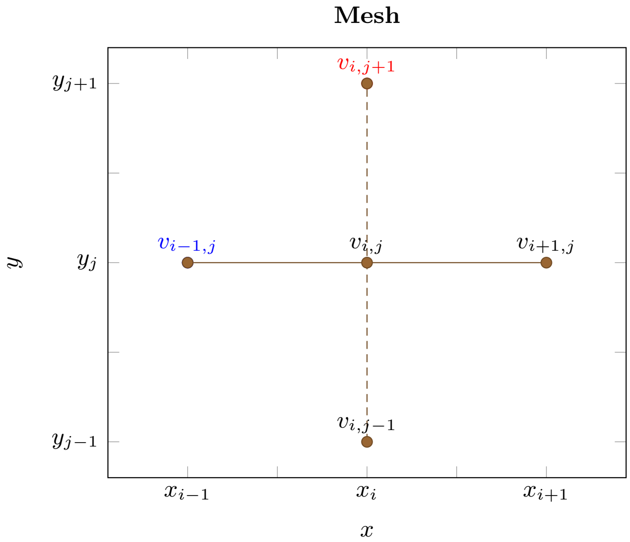

Fig. 6. Finite difference discretization for equation (2)

for all the internal nodes $k = i + (j-1)n_x$ with $1 < i < n_x$, $1 < j < n_y$ and $a = \frac{1}{dx}$, $b = \frac{1}{dy}$ and c=a+b=$\frac{1}{dx} + \frac{1}{dy}$.
The boundary conditions are written $v_k = 0$ for the boundary nodes $k = 1+(j-1)n_x$, $k = n_x + (j-1)n_x$ with $1 < j < n_y$ and $k = i$, $k = i + (n_y - 1)n_x$ with $1 < i < n_x$.
For the numerical experimentation with Matlab[12], we calculate the matrix $\mathcal{A}$ and the second member $\mathcal{B}$ on a finite difference mesh of $n_x \times n_y$ points for a function f defined at nodes (i,j) of the mesh. A sparse matrix data structure is used to store only non-null elements (given the matrix structure $\mathcal{A}$). For this, we store the non-zero coefficients of $\mathcal{A}$ in a vector U, and their indices (i,j) in two other vectors. To use this data structure with Matlab, we initialize $\mathcal{A}$ with the Matlab `spalloc` function, instead of using the `zeros(n,n)` function which creates a square array of $\mathtt{n}^2$ elements ($\mathtt{n} = n_x \times n_y$). For the second member $\mathcal{B}$ we transform the f matrix of values of the nodes of the mesh into a column vector of dimension n with the function Matlab `reshape`. This process makes it possible to solve the problem on a desktop computer.

Exemple: For $f(x,y) = x - y$ (see Figure 7), where $n_x = n_y = 50$ and $B =]0,1[*]0,1[$ satisfying the compatibility condition:

$$\int_{\mathbb{R}^n} f^+ dx = \int_{\mathbb{R}^n} f^- dy$$

we obtain with computations on Matlab, the 3D visualization (Figure 8) of the computed solution $v = a\nabla u$, with u the potential and a the transport density.

6 Exchanges-interactation

The stay at UCLA was an occasion to discuss with professors and postdoc working in PDE, computers sciences, namely: Alpar Richard Meszaros, Chenchen Mou, etc. I attend also to 4 seminars, two at the Institute of Pure and Applied Mathematics (IPAM) and two in the math department.

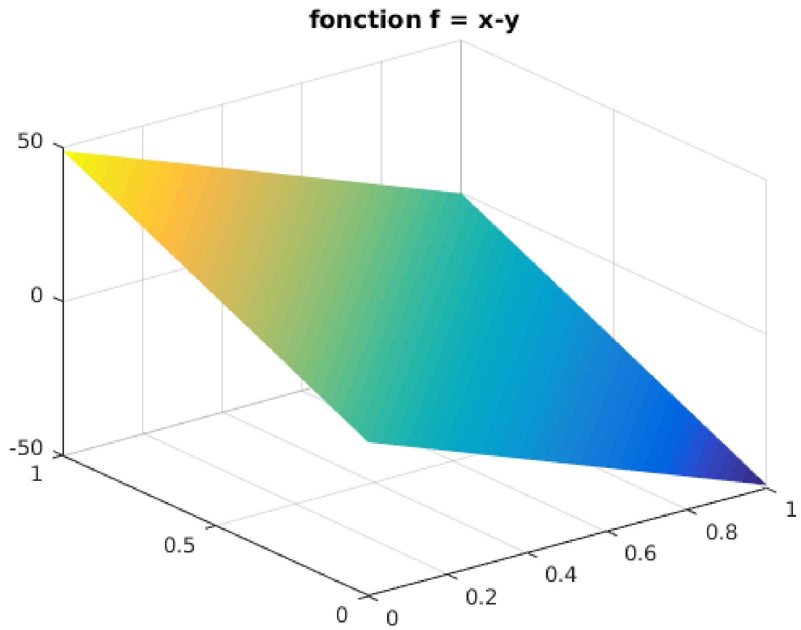

Fig. 7. Function $f(x, y) = x - y$

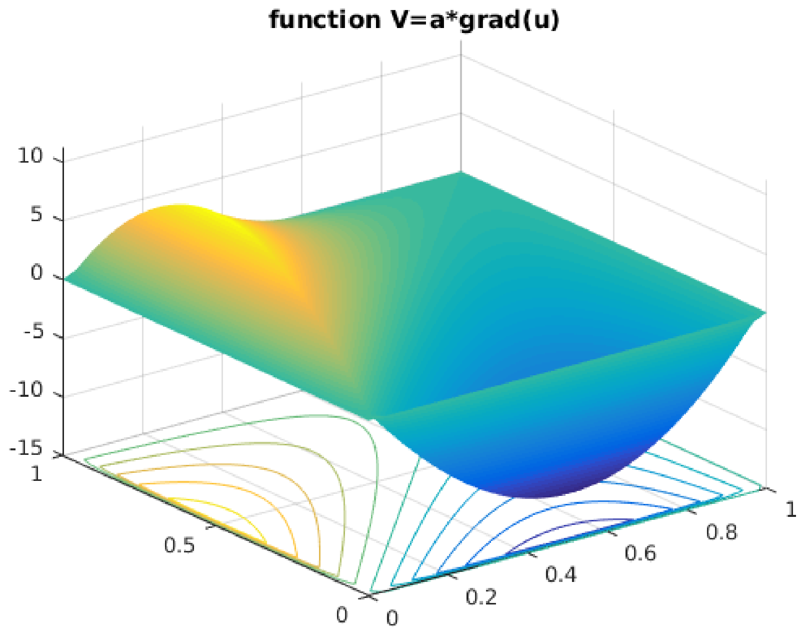

Fig. 8. Function $v = a\nabla u$ for function $f(x, y) = x - y$

7 Conclusion and futur works

As planning in the beginning of this stay, the rest of the work will rely on:

- optimal mapping for dimension 2, using equation (5).
- writing the article for dimensions 1 and 2,

which are still in progress.

Acknowledgments

I would like to thank the IMU-Simons African Fellowship Program for the support of this work. I also thank Professor Wilfrid Gangbo for the hosting at Mathematics Department, UCLA.

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Enclosed documents :

1. financial report,
2. accommodationn invoice,
3. bus tickets,
4. boarding tickets,
5. pictures (working with Professor Wilfrid Gangbo while doing my research, at the campus of UCLA, etc.),
6. others original invoices (foods, etc.).

Others documents are also available (movies of simulations, etc.).

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