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Analytic and arithmetic properties of the (Γ, χ)-automorphic reproducing kernel function and associated Hermite–Gauss series

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Dedicated to the memory of Professor Ahmed Intissar

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Abstract We consider the reproducing kernel function of the theta Bargmann–Fock Hilbert space associated with given full-rank lattice and pseudo-character, and we deal with some of its analytical and arithmetical properties. Specially, the distribution and the discreteness of its zeros are examined. The analytic sets of zeros of the theta Bargmann–Fock space inside a given fundamental cell is characterized and shown to be finite and of cardinal less or equal to its dimension. Moreover, we obtain some remarkable lattice sums by evaluating the so-called complex Hermite–Gauss coefficients. Some of them generalize some of the arithmetic identities given by Perelomov in the framework of coherent states for the specific case of von Neumann lattice. Such complex Hermite–Gauss coefficients are nontrivial examples of the so-called lattice's functions according the Serre terminology. The perfect use of the basic properties of the complex Hermite polynomials is crucial in this framework.

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1 Introduction

So far, the focus of studies in [5–7] has been the spectral properties of the so-called theta Bargmann–Fock Hilbert space $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ associated with given full rank-oriented lattice $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and pseudo-character χ (see also [2,10,11]). This space is defined to be the L^2-functional space of holomorphic (Γ, χ)-theta functions f on $\mathbb{C}$ of magnitude $\nu > 0$, provided that the functional equation

$$f(z+\gamma) = \chi(\gamma)e^{\nu(z+\frac{\gamma}{2})\overline{\gamma}} f(z) \tag{1.1}$$

holds for all $z \in \mathbb{C}$ and all $\gamma \in \Gamma$. The pseudo-character nature of χ, i.e.,

$$\chi(\gamma+\gamma') = \chi(\gamma)\chi(\gamma')e^{\frac{\nu}{2}(\gamma\overline{\gamma'}-\overline{\gamma}\gamma')}, \tag{1.2}$$

is required to ensure that $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is a nonzero vector space. It is also a necessary condition and implies in particular that the parameter ν is quantified in the sense that the imaginary part of $\nu S(\Gamma)$ belongs to $\pi\mathbb{Z}^+$, where $S(\Gamma) = \Im(\omega_1\overline{\omega_2})$ is the cell area of Γ. In this case, $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is moreover a reproducing kernel Hilbert space with respect to the hermitian product

$$\langle f, g\rangle_{\nu,\Gamma} = \int_{\Lambda(\Gamma)} f(z)\overline{g(z)}e^{-\nu|z|^2}dm(z). \tag{1.3}$$

Here $\Lambda(\Gamma)$ is any compact fundamental region representing $\mathbb{C}/\Gamma$ and $dm(z) = \mathrm{d}x dy$; $z = x + iy$, is the Lebesgue measure on $\mathbb{C}$. As pointed out in [5], the reproducing kernel function is given by the absolutely and uniformly convergent series on compact subsets of $\mathbb{C} \times \mathbb{C}$,

$$K^{\nu}_{\Gamma,\chi}(z,w) := \left(\frac{\nu}{\pi}\right)\sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\frac{\nu}{2}|\gamma|^2+\nu(z\overline{\gamma}-\overline{w}\gamma+z\overline{w})}. \tag{1.4}$$

Geometrically, the space $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ (under the assumption (1.2)) can be realized as the space of holomorphic sections over the complex torus $\mathbb{C}/\Gamma$ of the holomorphic line bundle $L = (\mathbb{C}\times\mathbb{C})/\Gamma$, constructed as the quotient of the trivial bundle over $\mathbb{C}$ by considering the Γ-action

$$\gamma(z; v) := \left(z+\gamma; \chi(\gamma)e^{\nu(z+\frac{\gamma}{2})\overline{\gamma}}\cdot v\right); \quad (z, v) \in \mathbb{C}\times\mathbb{C}.$$

Accordingly, the dimension of $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is known to be finite and given by the Pfaffian of the associated skew-symmetric form E. In our case, $E(z,w) := (\nu/\pi)\Im m(z\bar{w})$ and therefore

$$\dim \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \left(\frac{\nu}{\pi}\right) S(\Gamma). \tag{1.5}$$

Notice for instance that we have $\nu \geq \pi/S(\Gamma)$, otherwise $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is trivial. The proof of (1.5) can be handled using Riemann–Roch theorem which is well known in the theory of abelian varieties (see [2,3,9,12,14,16,19]). It can also be done à la Selberg [2,8,17] by determining the trace of the integral operator associated with (Γ, χ)-automorphic kernel function $K^{\nu}_{\Gamma,\chi}$. This is done in [5].

In the present paper, we will discuss some analytical and arithmetical properties of the reproducing kernel function $K^{\nu}_{\Gamma,\chi}(z, w)$. More precisely, we show in Theorem 2.1 that the set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ of zeros of $K^{\nu}_{\Gamma,\chi}$ is symmetric, not isolated and its distribution is uniform, in the sense that $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ consists of the Γ-translation of the zeros of $K^{\nu}_{\Gamma,\chi}(z, w)$ contained in a cartesian product of fundamental cells. The systematic study of the zero set of $K^{\nu}_{\Gamma,\chi}$ inside a cartesian product of fundamental cells leads to a characterization of the common zeros (analytic set) contained in a fundamental cell of all holomorphic functions belonging to the theta Bargmann–Fock Hilbert space $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. It is shown that this analytic set is finite and its cardinal is less or equal to $(\nu/\pi)\, S(\Gamma)$. In Theorem 2.3, we give an interesting power series representation of $K^{\nu}_{\Gamma,\chi}$ by means of the so-called complex Hermite–Gauss coefficients. Moreover, we establish the connection to the Poincaré series associated with the monomials. Determination of zeros of $K^{\nu}_{\Gamma,\chi}$ gives rise to interesting identities. More remarkable lattice sums are also proved in Theorem 2.4 by evaluating the complex Hermite–Gauss coefficients (the coefficients of $K^{\nu}_{\Gamma,\chi}(z, w)$ when expanded as power series), generalizing certain arithmetic identities obtained by Perelomov [15] in the framework of coherent states for the specific case of von Neumann lattice [20] and rediscovered later by Boon and Zak [1]. The proof we provide seems to be new, simpler, and more direct. Theorem 2.6 is devoted to show that the complex Hermite–Gauss coefficients $a^{p,q}_{m,n}(\Gamma|\nu, \chi)$ are lattice's functions in the sense of Serre [18, Chapter VII].

The rest of the paper is organized as follows.

- Section 2: Motivations and statement of main results.
- Section 3: On $K^{\nu}_{\Gamma,\chi}$ and proofs of Theorems 2.1 and 2.3.
- Section 4: Proofs of Theorems 2.4 and 2.6 related to the coefficients $a^{p,q}_{m,n}(\Gamma|\nu, \chi)$.
- Section 5: Concluding remarks.

2 Motivations and statement of main results

The present work is motivated by the fact that for the von Neumann lattice [20], the concrete description of $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ allows one to recover some arithmetic identities obtained by Perelomov [15] (see also [1]). In fact, for the specific case of $\nu = \pi/S(\Gamma)$ and χ being the "Weierstrass pseudo-character," $\chi(\gamma) = \chi_W(\gamma) = +1$ if $\gamma/2 \in \Gamma$ and $\chi(\gamma) = \chi_W(\gamma) = -1$ otherwise, the space $\mathcal{O}^{\nu}_{\Gamma,\chi_W}(\mathbb{C})$ is one-dimensional and is generated by the modified Weierstrass σ-function [7]

$$\widetilde{\sigma}_{\mu}(z; \Gamma) := e^{-\frac{1}{2}\mu z^2}\sigma(z; \Gamma) = z e^{-\frac{1}{2}\mu z^2} \prod_{\gamma\in\Gamma}{}' \left(1 - \frac{z}{\gamma}\right) e^{z/\gamma + \frac{1}{2}(z/\gamma)^2}, \tag{2.1}$$

where the prime in the product excludes the term with $\gamma = 0$, $\sigma(z;\Gamma)$ denotes the classical Weierstrass σ-function and $\mu = \mu(\Gamma)$ is an invariant of the lattice $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ given in terms of the Weierstrass zeta function $\zeta(z;\Gamma) = \sigma'(z;\Gamma)/\sigma(z;\Gamma)$ by

$$\mu = \mu(\Gamma) = \frac{2i}{S}\left(\zeta(\omega_1^*;\Gamma)\overline{\omega_2^*} - \zeta(\omega_2^*;\Gamma)\overline{\omega_1^*}\right) \tag{2.2}$$

with $\omega_\ell^* = 2\omega_\ell$; $\ell = 1, 2$. In such case, the expression of $K^\nu_{\Gamma,\chi_W}(z,w)$ reduces further to

$$K^\nu_{\Gamma,\chi_W}(z,w) = C\, e^{\frac{1}{2}(\mu z^2 + \overline{\mu}\,\overline{w}^2)}\sigma(z;\Gamma)\overline{\sigma(w;\Gamma)},$$

up to nonzero constant C. Therefore, one concludes easily that the zeros of K^ν_{Γ,χ_W} are localized in $(\Gamma\times\mathbb{C})\cup(\mathbb{C}\times\Gamma)$. As immediate consequence, one asserts that $K^\nu_{\Gamma,\chi_W}(0,0) = 0$, or equivalently

$$\sum_{\gamma\in\Gamma}\chi_W(\gamma)e^{-\frac{\nu}{2}|\gamma|^2} = 0. \tag{2.3}$$

Thus, one recovers the arithmetic identity obtained by Perelomov in [15, Eq. (47) with $k = 0$] (see also (15) and (15a) in [1]).

For arbitrary lattice, the lattice sum in (2.3) as well as more arithmetic identities can be derived by considering and evaluating the so-called complex Hermite–Gauss coefficients

$$a_{m,n}(\Gamma|\nu,\chi) := \sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\frac{\nu}{2}|\gamma|^2}H^\nu_{m,n}(\gamma;\overline{\gamma}), \tag{2.4}$$

where

$$H^\nu_{m,n}(z,\overline{z}) := (-1)^{m+n}e^{\nu|z|^2}\frac{\partial^{m+n}}{\partial\overline{z}^m\partial z^n}\left(e^{-\nu|z|^2}\right). \tag{2.5}$$

denote the weighted complex Hermite polynomials (see [4] and the references therein). This is possible thanks to the crucial observation that $K^\nu_{\Gamma,\chi}(z,w)$ can be expanded as in (2.7). Thus, the present paper is aimed to investigate and complete the study of the kernel function given by both (1.4) and (2.7) by considering further interesting basic properties of $K^\nu_{\Gamma,\chi}(z,w)$. Special attention is given to its zeros. Namely, we prove the following

Theorem 2.1 *Let $\nu > 0$ and χ any pseudo-character as in (1.2) on a given lattice Γ. Then, the set $\mathcal{Z}(K^\nu_{\Gamma,\chi})$ of zeros of $K^\nu_{\Gamma,\chi}$ is symmetric and is given by*

$$\mathcal{Z}(K^{\nu}_{\Gamma,\chi}) = \bigcup_{w\in\Lambda(\Gamma)} \Big(\big(\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) + \Gamma\big) \times \{w\} \Big) \cup \bigcup_{w\in\Lambda(\Gamma)} \Big(\{w\} \times (\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) + \Gamma) \Big),$$

where $\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})$ *denotes the set of zeros of the function* $\varphi_w(z) := K^{\nu}_{\Gamma,\chi}(z, w)$ *contained in an arbitrary fundamental cell* $\Lambda(\Gamma)$. *Moreover, the analytic set (the common zeros) of the theta Bargmann–Fock Hilbert space* $\bigcap_{f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})} \mathcal{Z}(f|_{\Lambda(\Gamma)})$ *reduces further to* $\Xi := \{w \in \Lambda(\Gamma);\ \varphi_w \equiv 0 \text{ on } \Lambda(\Gamma)\}$ *which is finite with cardinal less or equal to the dimension of* $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$.

Remark 2.2 The zero set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)})$ is uncountable and the distribution of such zeros is uniform in the sense that

$$\mathcal{Z}(K^{\nu}_{\Gamma,\chi}) = \mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)}) + \Gamma \times \Gamma.$$

Such set of zeros is not discrete if Ξ is not-trivial. Moreover, the analytic set of zeros corresponding to the functions $\varphi_w|_{\Lambda(\Gamma)}$; $w \in \Lambda(\Gamma)$, coincides with Ξ.

The second main theorem concerns the expansion in power series of $K^{\nu}_{\Gamma,\chi}$ and the connection to $\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)$, the periodization à la Poincaré of the monomials $e_m(z) := z^m$. The (Γ, χ)-Poincaré theta operator is defined by

$$\mathcal{P}^{\nu}_{\Gamma,\chi}(\psi)(z) := \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2 + \nu z\overline{\gamma}} \psi(z-\gamma), \tag{2.6}$$

provided that the series converges. More precisely, we assert

Theorem 2.3 *The* (Γ, χ)*-automorphic reproducing kernel function can be expanded in power series as follows:*

$$K^{\nu}_{\Gamma,\chi}(z, w) = \left(\frac{\nu}{\pi}\right) \sum_{m,n=0}^{\infty} (-1)^m a_{m,n}(\Gamma|\nu, \chi) \frac{z^n \overline{w}^m}{m!n!}, \tag{2.7}$$

where the coefficients $a_{m,n}(\Gamma|\nu, \chi)$ *are given by*

$$a_{m,n}(\Gamma|\nu, \chi) := \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma; \overline{\gamma}). \tag{2.8}$$

Moreover, its expression in terms of the automorphic functions obtained by periodization à la Poincaré of the monomials is given by

$$K^{\nu}_{\Gamma,\chi}(z, w) = \left(\frac{\nu}{\pi}\right) \sum_{m=0}^{\infty} \left[\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)\right](z) \frac{(\nu\overline{w})^m}{m!}. \tag{2.9}$$

The involved $a_{m,n}(\Gamma|\nu,\chi)$, called here complex Hermite–Gauss coefficients, possess interesting arithmetic properties. More generally, if we consider the quantities

$$a^{p,q}_{m,n}(\Gamma|\nu,\chi) := \sum_{\gamma\in\Gamma} \chi(\gamma)\gamma^p\overline{\gamma}^q e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma;\overline{\gamma}) \tag{2.10}$$

for given integers m, n, p, q, then we prove the following:

Theorem 2.4 *Assume that χ is real-valued. Then, the generalized complex Hermite–Gauss coefficients $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ vanish*

$$\sum_{\gamma\in\Gamma} \chi(\gamma)\gamma^p\overline{\gamma}^q e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma,\overline{\gamma}) = 0 \tag{2.11}$$

for every integers m, n, p, q such that $m + n + p + q$ is odd.

Remark 2.5 Under the assumption that χ is a real-valued pseudo-character, we have $a_{2m+1,2n}(\Gamma|\nu,\chi) = 0 = a_{2m,2n+1}(\Gamma|\nu,\chi)$. In particular, we have the arithmetic identities

$$\sum_{\gamma\in\Gamma} \chi(\gamma)\gamma^{2k+1} e^{-\frac{\nu}{2}|\gamma|^2} = 0 \tag{2.12}$$

and

$$\sum_{\gamma\in\Gamma} \chi(\gamma)\overline{\gamma}^{2k+1} e^{-\frac{\nu}{2}|\gamma|^2} = 0 \tag{2.13}$$

that follow by taking $p = q = 0$ in (2.11). These identities generalize those obtained in [1,15]. Accordingly, the expansion series (2.7) of the reproducing kernel function $K^{\nu}_{\Gamma,\chi}$ reduces further to the following:

$$\begin{aligned} K^{\nu}_{\Gamma,\chi}(z,w) = \left(\frac{\nu}{\pi}\right) \Bigg(& \sum_{m,n=0}^{\infty} a^{\nu}_{\Gamma,\chi}(2m,2n) \frac{z^{2n}\overline{w}^{2m}}{(2m)!(2n)!} \\ & - \sum_{m,n=0}^{\infty} a^{\nu}_{\Gamma,\chi}(2m+1,2n+1) \frac{z^{2n+1}\overline{w}^{2m+1}}{(2m+1)!(2n+1)!} \Bigg). \end{aligned} \tag{2.14}$$

The lattice sums $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ will be seen as functions in the lattice $\Gamma \in \mathcal{L}$, where $\mathcal{L}$ denotes the set of all lattices in $\mathbb{C}$. Hence, the next theorem shows that the complex Hermite–Gauss coefficients $a_{m,n}(\Gamma|\nu,\chi)$ are lattice's functions.

Theorem 2.6 *There exist specific ν and χ such that $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ are lattice's functions in sense that for every nonzero complex number λ, we have*

$$a^{p,q}_{m,n}(\lambda\Gamma|\nu_{\lambda\Gamma},\chi_{\lambda\Gamma}) = \frac{\lambda^p\overline{\lambda}^q}{\overline{\lambda}^m\lambda^n} a^{p,q}_{m,n}(\Gamma|\nu,\chi). \tag{2.15}$$

Remark 2.7 The generalized complex Hermite–Gauss coefficients $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ are nontrivial examples of the so-called lattice's functions according the Serre terminology [18, Chap. VII].

3 On $K^{\nu}_{\Gamma,\chi}$ and proofs of Theorems 2.1 and 2.3

We begin by discussing some properties of $K^{\nu}_{\Gamma,\chi}$ that are common between reproducing kernels. The following result is easy to obtain.

Proposition 3.1 *The z-function*

$$\sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\frac{\nu}{2}|\gamma|^2+\nu(z\overline{\gamma}-\overline{z}\gamma)}$$

is nonnegative real-valued function on $\mathbb{C}$*.*

Proof This follows from the fact that

$$K^{\nu}_{\Gamma,\chi}(z,z)=\left(\frac{\nu}{\pi}\right)\sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\frac{\nu}{2}|\gamma|^2+\nu(z\overline{\gamma}-\overline{z}\gamma+|z|^2)}, \tag{3.1}$$

and that the kernel function $K^{\nu}_{\Gamma,\chi}$ is nonnegative real-valued function on the diagonal of $\mathbb{C}\times\mathbb{C}$ for satisfying

$$K^{\nu}_{\Gamma,\chi}(z,z)=\int_{\Lambda(\Gamma)}|K^{\nu}_{\Gamma,\chi}(z,w)|^2e^{-\nu|w|^2}dm(w)\geq 0, \tag{3.2}$$

which is a particular case of

$$K^{\nu}_{\Gamma,\chi}(z,\xi)=\int_{\Lambda(\Gamma)}K^{\nu}_{\Gamma,\chi}(z,w)K^{\nu}_{\Gamma,\chi}(w,\xi)e^{-\nu|w|^2}dm(w)$$

for every fixed $\xi\in\mathbb{C}$. This follows by making use of the reproducing property [5]

$$f(z)=\int_{\Lambda(\Gamma)}K^{\nu}_{\Gamma,\chi}(z,w)f(w)e^{-\nu|w|^2}dm(w) \tag{3.3}$$

for every $f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. □

The proof of Theorem 2.1 is contained in Lemmas 3.2 and 3.3, Propositions 3.4 and 3.6, and Remarks 3.5 and 3.8.

Lemma 3.2 *The zeros set* $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ *of* $K^{\nu}_{\Gamma,\chi}$ *is symmetric.*

Proof The symmetry of $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ follows easily since $K^{\nu}_{\Gamma,\chi}(z,w)=\overline{K^{\nu}_{\Gamma,\chi}(w,z)}$. □

Lemma 3.3 *Let* $(z_0, w_0) \in \mathbb{C}\times\mathbb{C}$. *Then,* $K^{\nu}_{\Gamma,\chi}(z_0, w_0) = 0$ *if and only if* $(\{z_0\} + \Gamma)\times(\{w_0\} + \Gamma))$ *is contained in the zeros set* $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ *of* $K^{\nu}_{\Gamma,\chi}$. *Thus, we have*

$$\mathcal{Z}(K^{\nu}_{\Gamma,\chi}) = \mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)}) + \Gamma \times \Gamma. \tag{3.4}$$

Proof Making use of the Γ-bi-invariance property [5],

$$K^{\nu}_{\Gamma,\chi}(z+\gamma, w+\gamma') = \chi(\gamma)e^{\frac{\nu}{2}|\gamma|^2+\nu z\overline{\gamma}}K^{\nu}_{\Gamma,\chi}(z, w)\overline{\chi(\gamma')e^{\frac{\nu}{2}|\gamma'|^2+\nu\overline{w}\gamma'}} \tag{3.5}$$

that holds for every $z, w \in \mathbb{C}$ and $\gamma, \gamma' \in \Gamma$, it follows readily that the elements of the set $(\{z_0\} + \Gamma) \times (\{w_0\} + \Gamma)$ are also zeros of $K^{\nu}_{\Gamma,\chi}$ whenever $K^{\nu}_{\Gamma,\chi}(z_0, w_0) = 0$. Then, we can conclude since $(z, w) \in \mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ is equivalent to the existence of unique $(z_0, w_0) \in \Lambda(\Gamma) \times \Lambda(\Gamma)$ and unique $(\gamma_0, \gamma_0') \in \Gamma \times \Gamma$ such that $(z, w) = (z_0, w_0) + (\gamma_0, \gamma_0')$ and $(z_0, w_0) \in \mathcal{Z}(K^{\nu}_{\Gamma,\chi})$. □

The following result describes the set $\mathcal{Z}(\varphi_w)$ of all zeros of φ_w in terms of $\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})$ that denotes the set of zeros of φ_w inside a fundamental cell $\Lambda(\Gamma)$.

Proposition 3.4 *We have*

$$\mathcal{Z}(\varphi_w) = \mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) + \Gamma.$$

Moreover, the number of zeros of φ_w *contained in any fundamental cell* $\Lambda(\Gamma)$ *is constant and independent of* w. *More exactly, we have*

$$\#(\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})) = \dim \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}).$$

Proof The first assertion is clear by means of Lemma 3.3. We need only to determine the number of zeros of the holomorphic function $\varphi_w := K^{\nu}_{\Gamma,\chi}(\cdot, w)$ inside $\Lambda(\Gamma)$. By a small displacement of $\Lambda(\Gamma)$, say $u + \Lambda(\Gamma)$ for certain $u \in \mathbb{C}$, we are free to assume that $f \neq 0$ along the border $\partial\Lambda(\Gamma)$ of a fundamental region $\Lambda(\Gamma)$. Without loss of generality, we can assume that $\partial\Lambda(\Gamma)$ is a piecewise smooth path that runs around each zero of φ_w in $\Lambda(\Gamma)$ exactly one time and that the summits are the origin O, ω_1, $\omega_1 + \omega_2$, and ω_2. Let denote by $\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})$ the set zeros of φ_w inside such $\Lambda(\Gamma)$. Hence, by applying the Cauchy's argument principle to φ_w, we get

$$\begin{aligned}\#(\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})) &= \frac{1}{2i\pi}\oint_{\partial\Lambda(\Gamma)}\frac{\varphi_w'(z)}{\varphi_w(z)}dz = \frac{1}{2i\pi}\left(\int_O^{\omega_1} + \int_{\omega_1}^{\omega_1+\omega_2} + \int_{\omega_1+\omega_2}^{\omega_2} + \int_{\omega_2}^{0}\right)\\ &= \frac{1}{2i\pi}\int_0^1\left[\omega_1\left(\frac{\varphi_w'(t\omega_1)}{\varphi_w(t\omega_1)} - \frac{\varphi_w'(t\omega_1+\omega_2)}{\varphi_w(t\omega_1+\omega_2)}\right)\right.\\ &\quad\left.+\,\omega_2\left(\frac{\varphi_w'(\omega_1+t\omega_2)}{\varphi_w(\omega_1+t\omega_2)} - \frac{\varphi_w'(t\omega_2)}{\varphi_w(t\omega_2)}\right)\right]dt.\end{aligned}$$

Now, according to the well-established fact

$$\frac{f'(z+\gamma)}{f(z+\gamma)} - \frac{f'(z)}{f(z)} = \nu\overline{\gamma}$$

provided that $f(z+\gamma)\neq 0$, $\gamma\in\Gamma$, and valid for every f satisfying the functional equation (1.1), we can prove the following

$$\#(\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})) = \frac{1}{2i\pi}\int_0^1 \nu\,(\omega_1\overline{\omega_2}-\omega_2\overline{\omega_1})\,dt = \left(\frac{\nu}{\pi}\right)\Im m(\omega_1\overline{\omega_2}) = \left(\frac{\nu}{\pi}\right)S(\Gamma).$$

This completes the proof. □

Remark 3.5 The assertion of Proposition 3.4 is valid for any $f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. Notice also that Proposition 3.4 shows that $\mathcal{Z}(\varphi_w)\times\{w\}$ is contained in $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ for every arbitrary fixed $w\in\mathbb{C}$. This can be used to show that $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ is in particular not discrete.

Proposition 3.6 *We have* $\mathcal{Z}(K^{\nu}_{\Gamma,\chi}) = \mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)}) + \Gamma\times\Gamma$ *and the set*

$$\Xi = \{w\in\Lambda(\Gamma)\ \textit{such that}\ \varphi_w\equiv 0\ \textit{on}\ \Lambda(\Gamma)\} \tag{3.6}$$

is finite with $\#(\Xi)\leqslant \dim\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \left(\frac{\nu}{\pi}\right)S(\Gamma)$. *Moreover, we have*

$$\Xi = \bigcap_{f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})}\mathcal{Z}(f) = \bigcap_{w\in\mathbb{C}}\mathcal{Z}(\varphi_w) = \bigcap_{w\in\Lambda(\Gamma)}\mathcal{Z}(\varphi_w). \tag{3.7}$$

Proof The distribution of zeros of $K^{\nu}_{\Gamma,\chi}$ is uniform in the sense that they are completely determined by translating those localized in $\Lambda(\Gamma)\times\Lambda(\Gamma)$. This is contained in 3.3. Accordingly, we will concentrate on $\mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)})$. Recall that from Proposition 3.4, the number of zeros in $\Lambda(\Gamma)$ of every nonzero f belonging to $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is constant and is given by

$$\#(\mathcal{Z}(f|_{\Lambda(\Gamma)})) = \dim\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \left(\frac{\nu}{\pi}\right)S(\Gamma). \tag{3.8}$$

Thus, from the definition of Ξ given through (3.6), we have $\varphi_w(z)=0$ for all z in $\Lambda(\Gamma)$ and $w\in\Xi$. Therefore,

$$f(w) = \int_{\Lambda(\Gamma)} K^{\nu}_{\Gamma,\chi}(w,z)f(z)e^{-\nu|z|^2}dm(z) = \int_{\Lambda(\Gamma)}\overline{\varphi_w(z)}f(z)e^{-\nu|z|^2}dm(z) = 0.$$

Consequently

$$\Xi\subset\bigcap_{f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})}\mathcal{Z}(f)\subset\bigcap_{w\in\mathbb{C}}\mathcal{Z}(\varphi_w)\subset\bigcap_{w\in\Lambda(\Gamma),\varphi_w\neq 0}\mathcal{Z}(\varphi_w)\subset\mathcal{Z}(f) \tag{3.9}$$

for all $w\in\Lambda(\Gamma)$ and for all $f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. Notice that the set $\{w\in\Lambda(\Gamma),\varphi_w\neq 0\}$ is nontrivial since $K^{\nu}_{\Gamma,\chi}$ is not identically zero on $\mathbb{C}\times\mathbb{C}$. This entails in particular that Ξ is finite with

$$0\leqslant\#(\Xi)\leqslant\dim\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \left(\frac{\nu}{\pi}\right)S(\Gamma). \tag{3.10}$$

To prove the inverse inclusion, let $z_0 \in \bigcap_{w\in\Lambda(\Gamma)} \mathcal{Z}(\varphi_w)$, then $\varphi_w(z_0) = 0$ for all $w \in \Lambda(\Gamma)$. Therefore, by Hermitian symmetry of the kernel function we see that φ_{z_0} is identically zero on $\Lambda(\Gamma)$. This implies that $\varphi_{z_0} \equiv 0$ on $\Lambda(\Gamma)$, since φ_{z_0} admits an infinite set of zeros in $\Lambda(\Gamma)$ which shows that $z_0 \in \Xi$. In conclusion, we get (3.7). This completes the proof. □

Corollary 3.7 *The analytic sets of* $\left((\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)|_{\Lambda(\Gamma)}\right)_m$ *and* $\left(\varphi_w|_{\Lambda(\Gamma)}\right)_{w\in\Lambda(\Gamma)}$ *coincide and are given by*

$$\bigcap_{m\in\mathbb{N}} \mathcal{Z}(\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)|_{\Lambda(\Gamma)}) = \bigcap_{w\in\Lambda(\Gamma)} \mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) = \Xi = \{w \in \Lambda(\Gamma);\ \varphi_w \equiv 0\}. \tag{3.11}$$

Proof We have already shown that $\bigcap_{w\in\mathbb{C}} \mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) \subset \mathcal{Z}(f)$, for all f in $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ (see (3.9)). This holds also true for the specific functions $f = \mathcal{P}^{\nu}_{\Gamma,\chi}(e_m) \in \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. The converse

$$\bigcap_{m\in\mathbb{N}} \mathcal{Z}(\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)|_{\Lambda(\Gamma)}) \subset \bigcap_{w\in\mathbb{C}} \mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) = \Xi \tag{3.12}$$

follows making use of (2.9) in Theorem 2.3 and (3.7) in Proposition 3.6. Indeed, if

$$z_0 \in \bigcap_{m\in\mathbb{N}} \mathcal{Z}(\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)|_{\Lambda(\Gamma)}),$$

then we have

$$\varphi_w(z_0) = K^{\nu}_{\Gamma,\chi}(z_0, w) = \sum_{m\in\mathbb{N}} \mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)(z_0)\frac{\overline{w}^m}{m!} = 0 \tag{3.13}$$

for every $w \in \mathbb{C}$. □

Remark 3.8 The fact that Ξ is a finite set implies that

$$\bigcup_{w\in\Lambda(\Gamma)} \left(\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) \times \{w\}\right) \tag{3.14}$$

is a uncountable set of zeros of $K^{\nu}_{\Gamma,\chi}$. The zeros set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)})$ admits then condensation points by means of the compactness of $\overline{\Lambda(\Gamma) \times \Lambda(\Gamma)}$. Moreover, it is not discrete if Ξ is not-trivial. Indeed, we have $K^{\nu}_{\Gamma,\chi}(w_0, w) = \varphi_w(w_0) = 0$ for every $w_0 \in \Xi$ and $w \in \mathbb{C}$.

We conclude this section by giving a proof for Theorem 2.3.

Proof of Theorem 2.3 The proof of (2.7) lies essentially in the fact that in (1.4) we recognize the exponential generating function [4]

$$e^{\nu(a\gamma+b\overline{\gamma}-ab)} = \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m!n!} H^{\nu}_{m,n}(\gamma;\overline{\gamma}) \tag{3.15}$$

with $a = -\overline{w}$ and $b = z$. To prove (2.9), we recall that $\left[\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)\right](z)$ is given through

$$\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)(z) := \sum_{\gamma\in\Gamma} \chi(\gamma)(z-\gamma)^m e^{-\frac{\nu}{2}|\gamma|^2+\nu z\overline{\gamma}}. \tag{3.16}$$

Thus, making use of the generating function [4]

$$\nu^n(\overline{\xi}-z)^n e^{\nu\xi z} = \sum_{m=0}^{\infty} \frac{z^m}{m!} H^{\nu}_{m,n}(\xi;\overline{\xi}) \tag{3.17}$$

with $\xi = \overline{\gamma}$, one obtains

$$\left[\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)\right](z) = \frac{(-1)^m}{\nu^m} \sum_{n=0}^{\infty} a_{m,n}(\Gamma|\nu,\chi)\frac{z^n}{n!}. \tag{3.18}$$

Therefore, the relationship (2.9) follows easily from (2.7) and (3.18). The proof of Theorem 2.3 is completed. □

Remark 3.9 As a particular case of (2.9) we have

$$K^{\nu}_{\Gamma,\chi}(z,0) = \left(\frac{\nu}{\pi}\right)\left[\mathcal{P}^{\nu}_{\Gamma,\chi}(e_0)\right](z). \tag{3.19}$$

4 Proofs of Theorems 2.4 and 2.6

Before giving the proof of Theorem 2.4, we begin by discussing some of its immediate consequences. In fact, under the assumption that the pseudo-character is real-valued, we get

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{2m+1,2n}(\gamma;\overline{\gamma}) = 0 \tag{4.1}$$

as well as

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{2m,2n+1}(\gamma;\overline{\gamma}) = 0. \tag{4.2}$$

This follows by setting $p = q = 0$ in (2.11). Changing γ by $-\gamma$ and next taking the complex conjugate, keeping in mind the fact $\overline{H^{\nu}_{m,n}(z;\overline{z})} = H^{\nu}_{n,m}(z;\overline{z})$ in (4.1) gives rise to the lattice sum (4.2).

Proof of Theorem 2.4 Under the assumption that χ is real-valued, we have $\chi(-\gamma) = \overline{\chi(\gamma)} = \chi(\gamma)$. Therefore, the symmetry

$$a^{p,q}_{m,n}(\Gamma|\nu,\chi) = (-1)^{m+n+p+q} a^{p,q}_{m,n}(\Gamma|\nu,\chi) \tag{4.3}$$

follows easily from the definition of $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ combined with the well-established fact

$$H^{\nu}_{m,n}(-\xi,-\overline{\xi}) = (-1)^{m+n} H^{\nu}_{m,n}(\xi,\overline{\xi}),$$

being indeed

$$\begin{aligned} a^{p,q}_{m,n}(\Gamma|\nu,\chi) &= \sum_{\gamma\in\Gamma} \chi(-\gamma)(-\gamma)^q(-\overline{\gamma})^p e^{-\frac{\nu}{2}|-\gamma|^2} H^{\nu}_{m,n}(-\gamma,-\overline{\gamma}) \\ &= (-1)^{m+n+p+q} a^{p,q}_{m,n}(\Gamma|\nu,\chi). \end{aligned}$$

This entails in particular that $a^{p,q}_{m,n}(\Gamma|\nu,\chi) = 0$ whenever $m+n+p+q$ is odd. □

Proof of Theorem 2.6 To prove (2.15), let $\nu = \nu_\Gamma$ and $\chi = \chi_\Gamma$ be such that

$$\nu_{\lambda\Gamma} = |\lambda|^{-2}\nu_\Gamma \quad \text{and} \quad \chi_{\lambda\Gamma}(\lambda\gamma) = \chi_\Gamma(\gamma) \tag{4.4}$$

for every complex $\lambda \in \mathbb{C}\setminus\{0\}$. Therefore

$$e^{-\frac{\nu_{\lambda\Gamma}}{2}|\lambda\gamma|^2} = e^{-\frac{\nu_\Gamma}{2}|\gamma|^2}. \tag{4.5}$$

Moreover, using the explicit expression of the complex Hermite polynomials [4],

$$H^{\nu}_{m,n}(\xi,\overline{\xi}) = m!n!\nu^{m+n} \sum_{k=0}^{min(m,n)} \frac{(-1)^k}{\nu^k k!} \frac{\xi^{m-k}}{(m-k)!} \frac{\overline{\xi}^{n-k}}{(n-k)!},$$

we get

$$H^{\nu_{\lambda\Gamma}}_{m,n}(\lambda\gamma;\overline{\lambda\gamma}) = \frac{1}{\lambda^n\overline{\lambda}^m} H^{\nu_\Gamma}_{m,n}(\gamma;\overline{\gamma}). \tag{4.6}$$

Hence, by inserting (4.4), (4.5), and (4.6) in the definition of $a^{p,q}_{m,n}(\lambda\Gamma|\nu_{\lambda\Gamma},\chi_{\lambda\Gamma})$, we obtain

$$a^{p,q}_{m,n}(\lambda\Gamma|\nu_{\lambda\Gamma},\chi_{\lambda\Gamma}) = \lambda^{p-n}\overline{\lambda}^{q-m} a^{p,q}_{m,n}(\Gamma|\nu,\chi).$$

□

Remark 4.1 The key tool in establishing (2.15) is the hypothesis (4.4). The existence of such lattice's functions $\nu = \nu_\Gamma$ and $\chi = \chi_\Gamma$ is assured by considering for example

$$\nu = \nu_\Gamma = \frac{2i\pi(\overline{\omega_1}^k\omega_2^k - \omega_1^k\overline{\omega_2}^k)}{\overline{\omega_1}^{k+1}\omega_2^{k+1} - \omega_1^{k+1}\overline{\omega_2}^{k+1}}$$

and the Weierstrass pseudo-character

$$\chi_\Gamma(\gamma) = \begin{cases} +1 & \text{if} \quad \gamma/2 \in \Gamma \\ -1 & \text{if} \quad \gamma/2 \notin \Gamma. \end{cases}$$

Such ν and χ are not unique.

We conclude this section by proving further interesting properties satisfied by the coefficients $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$.

Lemma 4.2 *For any pseudo-character χ and every integers m, n, p, q, we have*

$$\overline{a^{p,q}_{m,n}(\Gamma|\nu,\chi)} = (-1)^{m+n+p+q} a^{p,q}_{m,n}(\Gamma|\nu,\chi). \tag{4.7}$$

Moreover, the generalized complex Hermite–Gauss coefficients $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ satisfy the following recurrence formulas:

$$\nu a^{p,q+1}_{m,n}(\Gamma|\nu,\chi) = a^{p,q}_{m+1,n}(\Gamma|\nu,\chi) + \nu n a^{p,q}_{m,n-1}(\Gamma|\nu,\chi), \tag{4.8}$$

and

$$\nu a^{p+1,q}_{m,n}(\Gamma|\nu,\chi) = a^{p,q}_{m,n+1}(\Gamma|\nu,\chi) + \nu m a^{p,q}_{m-1,n}(\Gamma|\nu,\chi). \tag{4.9}$$

Proof The proof of the symmetry relationship (4.7) is similar to the one provided for (4.3). We need to use the fact $\overline{\chi(\gamma)} = \chi(-\gamma)$ combined with $H^{\nu}_{m,n}(-\xi,-\overline{\xi}) = (-1)^{m+n}\overline{H^{\nu}_{n,m}(\xi,\overline{\xi})}$. The recurrence formula (4.8) is an immediate consequence of the three-term recurrence formula

$$H^{\nu}_{m+1,n}(\xi,\overline{\xi}) = \nu\overline{\xi} H^{\nu}_{m,n}(\xi,\overline{\xi}) - \nu n H^{\nu}_{m,n-1}(\xi,\overline{\xi})$$

satisfied by the complex Hermite polynomials [4], while (4.9) follows easily from (4.8) by taking the complex conjugate, taking into account (4.7). □

5 Concluding remarks

For the data of any (Γ,χ,ν) (not necessary satisfying the pseudo-character property (1.2)), we can define the quantities $a_{m,n}(\Gamma|\nu,\chi)$ and in particular

$$a_{0,0}(\Gamma|\nu,\chi) := \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2}.$$

We claim that

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} = 0 \Longleftrightarrow \nu = \frac{\pi}{S(\Gamma)}$$

for any given real-valued χ. This will be a characterization of von Neumann lattices. In fact, from numerical observation, we assert that $\sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\frac{\nu}{2}|\gamma|^2}$ is positive when $\nu > \pi/S(\Gamma)$, vanishes when $\nu = \pi/S(\Gamma)$, and negative otherwise. When (1.2) is satisfied, the space $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is nontrivial and $a_{0,0}(\Gamma|\nu,\chi)$ coincides with $K^{\nu}_{\Gamma,\chi}(0,0)$. Moreover, we have $\nu = k\pi/S(\Gamma)$ for $k = 1, 2, \ldots$.

By numerical approach, it seems that the lattice sum

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} = \sum_{m,n=-\infty}^{+\infty} e^{\frac{-\nu}{2}(m^2+n^2)} =: f(\nu)$$

viewed as function in $\nu \in \mathbb{R}^{*+}$ and associated with $\Gamma = \mathbb{Z} + i\mathbb{Z}$ and $\chi \equiv 1$ is increasing with upper bound equal to 1 and with no lower bound.

Values of ν	Evaluation
0.001	− 911.395647437
0.01	− 100
0.1	− 9.99993971929989999314227390629644
1	0
1.25	0.3608381973529082048783912510840871280853
1.5	0.5852075679820198541304883766877041518512
1.75	0.7276806878628701459775102868169982969575
2	0.8196872998200458995950539646962870101812
3	0.9637440634268266347151459429784258329968
4	0.9925162797522077475130738581932461342102
8	0.9999860505819289371809872750653816298146
8×10	0.9999999999999999999999999999999999999999

The values in the above table are calculated using the following Mathematica code

```
NSum[(-1)^(i+j+i*j)*Exp[(-\[Nu]*Pi)/2*(i^2+j^2)],{i,-Infinity ,Infinity },
{j,-Infinity ,Infinity },NSumTerms->30, Method->"AlternatingSigns",
WorkingPrecision ->40]
```

From the summation formulas

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} = 0,$$

when $\nu = \pi/S(\Gamma)$, and

$$\sum_{\gamma\in\Gamma} \chi(\gamma)\gamma^p \overline{\gamma}^q e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma,\overline{\gamma}) = 0$$

for real-valued χ and integers m, n, p, q such that $m + n + p + q$ is odd (see (2.11) in Theorem 2.4), one can drive nice identities involving special functions such as Jacobi theta functions (see for example [13]). The first summation taken in the case of $\Gamma = \mathbb{Z} + i\mathbb{Z}$ gives rise to the following identity:

$$\vartheta_2(0, e^{-2\nu})^2 - \vartheta_3(0, e^{-2\nu})^2 - 2\vartheta_2(0, e^{-2\nu})\vartheta_3(0, e^{-2\nu}) = 0. \tag{5.1}$$

Indeed, by splitting such summation over (i) both m and n are odd, (ii) both m and n are even and lastly, (iii) m and n are of different parity, and next using the following equalities:

$$\begin{aligned}
\text{(i)} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{\frac{-\nu}{2}\left((2m+1)^2+(2n+1)^2\right)} &= \nu^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \vartheta_2(0, e^{-2\nu})\vartheta_3\left(\frac{\pi}{2}, e^{-\frac{\pi^2}{2\nu}}\right) \\
&= \vartheta_2(0, e^{-2\nu})^2. \\
\text{(ii)} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{\frac{-\nu}{2}\left((2m)^2+(2n)^2\right)} &= \vartheta_3(0, e^{-2\nu})^2. \\
\text{(iii)} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{-\nu\left((2m)^2+(2n+1)^2\right)} &= \vartheta_2(0, e^{-2\nu})\vartheta_3(0, e^{-2\nu}).
\end{aligned}$$

The proof of the identity (5.1) requires detailed knowledge of the relationships between Jacobi theta functions. The proof of the more complicated identities of the type (2.11) by direct methods is evidently nontrivial. This is, word by word, the same remark asserted by Perelomov in [15].

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