Deficiency Indices and Symmetry of the Composite of Non-Commutative Higher Order Differential Operators

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Abstract

Let T_1 and T_2 be two formally symmetric differential operators defined on the Hilbert space $\mathcal{H}$ of square integrable functions on positive half-line. If T_1 and T_2 commute, then T_1T_2, their composite, is symmetric. The converse of this statement is not necessarily true. We have shown how to construct formally symmetric differential operators from the composites of two non-commutative formally symmetric differential operators. Similarly, it has been proved that a formally symmetric differential operator can be obtained from the composite of two non-commutative differential operators not both symmetric. Finally, by application of Rank-Nullity Theorem, the deficiency indices of T_1T_2 is equal to the sum of deficiency indices of T_1 and T_2 when the operators have closed ranges.

Key Words: Composite of Operators, Deficiency Indices, Differential Operators, Symmetric Operators.
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1 Introduction

We consider two differential equations of order $2n$th and $2m$th that are formally symmetric given by

$$\tau_1 y(x) = w^{-1}(x)\left(\sum_{k=0}^{n}(-1)^k(p_k(x)y^{(k)}(x))^{(k)}\right) \quad (1)$$

and

$$\tau_2 y(x) = w^{-1}(x)\left(\sum_{j=0}^{m}(-1)^j(q_j(x)y^{(j)})^j\right) \quad (2)$$

defined on $\mathcal{H}_w = \mathcal{L}^2_w([0,\infty))$. For simplicity of computations, we will assume that $w(x) = 1$ for all $x \in [0,\infty)$ so that we have the Hilbert space $\mathcal{H} = \mathcal{L}^2([0,\infty))$. Not all linear differential equations generate symmetric differential operators on $\mathcal{H}$. Walker [11] and Weidmann [13] had devised a way of constructing differential expressions that are formally symmetric. In line with the results in [11, 13], $\tau_1 y(x)$ and $\tau_2 y(x)$ generate formally symmetric differential operators on $\mathcal{H}$ while the differential operator generated by $\tau_1\tau_2 y(x)$ is not necessarily a symmetric operator. It is well documented in the literature that the composite of two commuting symmetric operators is symmetric as shown in the following Lemma.

Lemma 1.1 *Let T_1 and T_2 be two operators defined on a Hilbert space $\mathcal{H}$ and T_1T_2 be their composite. Suppose that T_1 and T_2 are symmetric operators with T_1 commuting with T_2, then T_1T_2 is symmetric.*

The converse of Lemma 1.1 is not necessarily true as we have discovered, at least, for unbounded operators. The necessary and sufficient conditions on $p_k(x)$ and $q_j(x)$ for (1) and (2) to commute were obtained by Amitsur, see the results in [1]. The problem of finding two differential operators defined on a Hilbert space of square integrable functions on a half line whose composite is symmetric and the operators are not symmetric nor commutative is equivalent to the problem of factorisation of linear differential equations. Factorisation of differential equations have largely been studied both in terms of differential theories and Galois theory. In summary, a linear differential equation is solvable (factorisable) in terms of its quadratures or Liouvillian extensions if and only if the connected components of the corresponding Galois group is tringularisable. This particular problem has a long standing history being a difficult problem in algebraic equation algorithims for factorising linear differential equations or polynomials at the level of resolutions of algebraic equations. For more details, see [6] and references therein.

In this particular paper, we are motivated to compute the deficiency indices of such composite and compare them with the sum of the deficiency indices of the minimal differential operators generated by τ_1 and τ_2 respectively. Similarly, we have focussed on when the composites of τ_1 and τ_2 can generate formally symmetric differential equations. We have shown here, how one can actually construct a symmetric differential operator from the composite of two non-commutative operators, one of which is symmetric and the other non-symmetric. Such operators are generated by $\tau_1 y(x)$ and $\tau_2 y(x)$.

Our results, on the sum of deficiency indices which have been proved by application of the Rank-Nullity Theorem, states that if T_1 and T_2 are differential operators such that T_1 and T_2 are closed densely defined operators with T_1 and T_2 having closed ranges in $\mathcal{H}$, then the deficiency indices of the composite operator T_1T_2 is equal to the sum of the individual deficiency indices of T_1 and T_2. For the constructed two second order differential operators, the deficiency indices of their composite achieve this in the limit point case with the spectrum of the selfadjoint operator extension as pure discrete. On the other hand, we have constructed a formula for obtaining a formally symmetric differential operator using the differential operators generated by $\tau_1 y(x)$ and $\tau_2 y(x)$ that are not necessarily commutative.

Deficiency indices results require proper understanding of the asymptotic behaviour of the solutions associated to these differential equations. It is, therefore, prudent to start by giving a result that shows the existence of the basis of solutions of the composite of τ_1 and τ_2 in relation to the bases of the solutions of $\tau_1 y(x) = 0$ and $\tau_2 y(x) = 0$ and for more details, see [6].

Lemma 1.2 *Let $\tau_1\tau_2 y(x) = 0$ be a homogeneous differential equation which is composite of $\tau_1 y(x)$ and $\tau_2 y(x)$ with $y(x) \in \mathcal{H} = \mathcal{L}^2([0,\infty))$. Then there exists a basis of solutions of the composite that can be determined from the bases of solutions of homogeneous equations $\tau_1 y(x) = 0$ and $\tau_2 y(x) = 0$.*

Proof Suppose that $\{y_1, y_2, \dots, y_{2m}\}$ is a basis of solutions of $\tau_2 y = 0$ and $\{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{2n}\}$ is the basis of solutions of $\tau_1 y = 0$, then their union is not necessarily the basis of solutions of $\tau_1\tau_2 y(x) = 0$, $x \in [0,\infty)$. The existence of the basis of solutions of the composite is obtained as follows. Let $\mathcal{K}_2(x,t)$ be the Green's function associated with $\tau_2 y = 0$, and $\mathcal{W}_2\{(y_1, y_2, \dots, y_{2m}\}(t)$ be the corresponding Wronskian determinant. Now define for a fixed $a \in [0,\infty)$ a function

$$y_{2m+s}(x) = \int_a^x \mathcal{K}_2(x,t)\tilde{y}_s(t)dt, \quad s = 1, 2, \dots, 2n.$$

Then the set $\{y_1, y_2, \dots, y_{2m}, y_{2m+1}, y_{2m+2}, \dots, y_{2m+2n}\}$ is a basis of solutions of the composite equation $\tau_1\tau_2 y(x) = 0$ for $x \in [0,\infty)$. In order to see that this is true, consider the equations $\tau_2 y_k(x) = 0$, $k = 1, 2, \dots, 2m$, $\tau_2 y_k(x) = \tilde{y}_k(x)$, $k = 2m+1, \dots, 2(n+m)$ and $\tau_1 \tilde{y}_k(x) = 0$, $k = 2m+1, \dots, 2(n+m)$. This implies that

$$\tau_1\tau_2 y_k(x) = \tau_1(\tau_2 y_k(x)) = \tau_1 \tilde{y}_k(x) = 0, \quad k = 2m+1, \dots, 2(n+m).$$

Therefore, for a set of constants $\{\beta_k : k = 1, 2, \dots, 2(n+m)\}$ with the following linear combination,

$$\beta_1 y_1(x) + \beta_2 y_2(x) + \cdots + \beta_{2m} y_{2m}(x) + \beta_{2m+1} y_{2m+1} + \cdots + \beta_{2(n+m)} y_{2(n+m)}(x) = 0,$$

successive application of τ_1 and τ_2 starting with τ_2 on both sides with the right assumptions of basis elements for solutions of each differential map, it follows that all the constants β_k, $k = 1, 2, \dots, 2(n+m)$ are zero.

Remark 1.3 This result can easily be generalised by mathematical induction to the composite of s homogeneous differential equations of even orders. Suppose that $\tau_r y(x) = 0$ is a homogeneous differential equation of order $2n_r$, $r = 1, 2, \dots, s$ and basis of solutions of each of the homogeneous equation is known, then by use of corresponding associated Green's function in a successive reverse of the composite $\tau_1\tau_2 \cdots \tau_s y(x) = 0$, the right basis of solutions of the composite of these s differential equations will be determined from the basis of solutions of the individual r homogeneous equations $\tau_r y(x) = 0$, $r = 1, 2, \dots, s$.

In order to solve the deficiency index problem of the composite as well as those of the operators themselves, one solves the equations $\tau_1\tau_2 y(x) = zy(x)$, $\tau_1 y(x) = zy(x)$ and $\tau_2 y(x) = zy(x)$ where z is a spectral parameter. Each of these equations can be converted to their first order form using either quasiderivatives as given by Walker [11] or any other known methods of reducing higher order differential equations to first order equations using vector valued functions which are lower order derivatives of the function $y(x)$ and for details, see [9, 10] and references therein.

Definition 1.4 Suppose that $y^{[k]} = y^{(k)}$ denotes the kth derivative of y with respect to x, $y^{[0]} = y$, then let $\{y^{[0]}, y^{[1]}, \dots, y^{[2(n+m)-1]}\}$ be set of vector valued functions that are square integrable, then one defines the maximal domain of the composite, $\mathcal{D}(T^*)$ by

$$\begin{aligned}\mathcal{D}(T^*) &= \Big\{y \in \mathcal{L}^2([0,\infty)) \mid y^{[0]}, y^{[1]}, \ldots, y^{[2(n+m)-1]} \text{ are absolutely} \\ &\quad \text{continuous, in } [0,\infty),\ T^*y \in \mathcal{L}^2([0,\infty))\Big\}, \\ &\quad T^*y = \tau_1\tau_2 y(x) \text{ for all } y \in \mathcal{D}(T^*).\end{aligned}$$

$D(T^*)$ is the maximum possible domain in $\mathcal{L}^2([0,\infty))$ for which the quasiderivatives make sense. T^* is the Hilbert adjoint of the minimal possible operator generated by $\tau_1\tau_2 y(x)$. Weidmann [13] had shown that if $\tau_1\tau_2 y(x)$ is symmetric then T^* is densely defined and closed. An operator defined by restricting the domain of maximal operator to only functions y with compact support is known as a pre-minimal operator. We can denote this operator by $\tilde{T}$ and its domain is defined by

$$\mathcal{D}(\tilde{T}) = \{y \in \mathcal{D}(T^*) \mid y \text{ has compact support in } (0,\infty)\}.$$

$\tilde{T}y = T^*y = \tau_1\tau_2 y(x)$, for all $y \in \mathcal{D}(\tilde{T})$. For unbounded domains, $\tilde{T}$ is symmetric, densely defined but not closed. The closure of pre-minimal operator $\tilde{T}$, $\bar{\tilde{T}}$, is the minimal operator generated by $\tau_1\tau_2 y(x)$ and will be denoted by T. Here, T is symmetric, densely defined and closed with $T \subset T^*$. The minimal and maximal operators generated by τ_1 and τ_2 can be defined analogously.

This paper is organised as follows: Section 1: Introduction, Section 2: Deficiency Indices of Composite Operators and Section 3: Symmetry of Non-Commutative Composites.

2 Deficiency Indices of Composite Operators

In this section, we have applied the Rank-Nullity Theorem to prove that the deficiency indices of the composite of two differential operators is equal to the sum of the deficiency indices of the individual operators when the operators are injective and have closed ranges. The main result of this section is given in Theorem 2.2.

Before the main results of this section, we define the deficiency indices of a differential operator as given in [5]. For any operator $A : \mathcal{H} \to \mathcal{H}$, let $\mathcal{N}(A^* - zI)$ and $\mathcal{N}(A^* + zI)$ be null spaces of $A^* - zI$ and $A^* + zI$ respectively for some complex number z with $\text{Im}z > 0$. Define for the operator A a pair of indices (N_+, N_-), where $N_+ = \dim\mathcal{N}(A^* + zI)$ and $N_- = \dim\mathcal{N}(A^* - zI)$. The pair (N_+, N_-) is known as the deficiency indices of A and will be denoted by def$A$.

Theorem 2.1 *Let $\mathcal{H} = \mathcal{L}^2([0,\infty))$ be the Hilbert space of square integrable functions defined on $[0,\infty)$ with subspace Γ closed and assume that T_1 and T_2 are minimal differential operators generated by (1) and (2) respectively. Suppose that T_1 and T_2 are injective with their respective ranges closed. Then*

(i) T_1T_2 is closed and densely defined.

(ii) $\mathcal{R}(T_1T_2)$, the range of T_1T_2, is closed.

Proof

(i) To show that T_1T_2 is closed, we need to show that the graph of T_1T_2 is closed. T_1 and T_2 are differential operators, thus linear and since they are injective by assumption, it follows that $\ker(T_1) = \ker(T_2) = \{0\}$. T_1T_2 is an injective linear operator and therefore, $\ker(T_1T_2) = \{0\}$. Now assume that $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of square integrable functions on $\mathcal{H}$ such that $f_n(x) \to f(x)$ as $n \to \infty$. By Closed Graph Theorem if $T_1T_2 : \mathcal{H} \to \mathcal{H}$ is closed and $\mathcal{D}(T_1T_2) = \mathcal{H}$ then T_1T_2 is bounded. That is not true in this case since T_1T_2 is unbounded and $\mathcal{D}(T_1T_2) \neq \mathcal{H}$. The operators T_1 and T_2 are closed by construction and since their ranges are closed by assumption, it implies that their exists a closed subspace of $\mathcal{H} \times \mathcal{H}$, say Γ, such that

$$\Gamma = \{(u, T_1T_2u) \mid u \in \mathcal{D}(T_1T_2)\}.$$

By injectivity and linearity of T_1T_2, Γ is graph of T_1T_2 since for any $(0, g) \in \Gamma$, it follows that $g = 0$. For any $f \in \mathcal{D}(T_1T_2)$, there is at most one g such that $(f, g) \in \Gamma$ and $g = T_1T_2f$, $f = f(x)$ and $g = g(x)$. The graph of T_1T_2 is closed and the operator T_1T_2 is closed as required.

Now consider $\mathcal{D}(T_1T_2)$ which can be expressed as

$$\mathcal{D}(T_1T_2) = \mathcal{D}(T_2) \cap T_2^{-1}\mathcal{D}(T_1).$$

Since $\mathcal{D}(T_1)$ and $\mathcal{D}(T_1)$ are dense in $\mathcal{H}$ by construction, $T_2^{-1}\mathcal{D}(T_1)$ is a dense subspace of $\mathcal{H}$ and thus by Baires Category Theorem, the intersection of $\mathcal{D}(T_2)$ and $T_2^{-1}\mathcal{D}(T_1)$ is dense in $\mathcal{H}$. The operator T_1T_2 is densely defined.

(ii) The closure of $\mathcal{R}(T_1T_2)$ follows immediately by Closed-Range Theorem since the operator T_1T_2 is closed and densely defined on $\mathcal{H}$.

Theorem 2.2 *Assume all the conditions in Theorem 2.1 above are satisfied. Moreover, assume that* $\dim \mathcal{R}(T_2)^{\perp}$ *is finite, then*

$$\dim \mathcal{R}(T_1T_2)^{\perp} = \dim \mathcal{R}(T_1)^{\perp} = \dim \mathcal{R}(T_2)^{\perp}.$$

Proof

Assume that T_1^* and T_2^* are Hilbert adjoints of T_1 and T_2 respectively. Then by definition, $\ker((T_1T_2)^*)$ is a subspace of $\mathcal{D}((T_1T_2)^*)$ defined by

$$\mathcal{U} = \ker((T_1T_2)^*) = \{u \in \mathcal{D}((T_1T_2)^*) \mid (T_1T_2)^*u = 0\}.$$

Suppose that $\tilde{T_1^*} = T_1^* \mid_{\mathcal{U}}$, it follows by application of Rank-Nullity Theorem that

$$\dim\ker((T_1T_2)^*) = \dim\ker(\tilde{T_1^*}) + \dim\tilde{T_1^*}(\mathcal{U}) \tag{3}$$

and since $\ker(\tilde{T_1^*}) = \ker(T_1^*)$ equation (3) becomes

$$\dim\ker((T_1T_2)^*) = \dim\ker(T_1^*) + \dim\tilde{T_1^*}(\mathcal{U}). \tag{4}$$

T_1^* is an injective linear operator, therefore, there exists no $h \in \mathcal{D}((T_1T_2)^*)$ such that $T_1^*h \neq 0$ but $T_2^*h = 0$. Thus, $\dim\tilde{T_1^*}(\mathcal{U}) = \dim\ker(T_2^*)$. (4) finally becomes

$$\dim\ker((T_1T_2)^*) = \dim\ker(T_1^*) + \dim\ker(T_2^*).$$

The proof is now complete by noting that $\ker((T_1T_2)^*) = \mathcal{R}(T_1T_2)^{\perp}$.

Corollary 2.3 *Suppose that T_1 and T_2 are minimal differential operators generated by (1) and (2) respectively and let T_1T_2 be their composite. Moreover, assume that all the conditions in Theorems 2.1 and 2.2 are satisfied, then*

$$def(T_1T_2) = defT_1 + defT_2.$$

Here, defT is the defect index of the operator T.

Proof Note that by construction $\mathcal{R}(T_1T_2)^{\perp} = \mathcal{N}((T_1T_2)^*)$ and by definition $\text{def}T_1T_2 = \dim \mathcal{R}(T_1T_2)^{\perp}$. The rest of the proof are now immediate from proofs of Theorems 2.1 and 2.2.

The injectivity of the operator T_1 and by extension of T_1^* is necessary since otherwise there will exist a vector $v \in \mathcal{D}((T_1T_2)^*)$ such that $T_1^*v \neq 0$ but $T_2^*v = 0$ forcing

$$\tilde{T_1^*}\mathcal{U} = T_1^*\mathcal{U} \subseteq \ker(T_2^*).$$

This will therefore lead to the famous Rank-Nullity inequality given by

$$\dim\ker((T_1)^*) \leq \dim\ker((T_1T_2)^*) \leq \dim\ker(T_1^*) + \dim\ker(T_2^*).$$

3 Symmetry of Non-Commutative Composites

In this section we have shown how to obtain a formally symmetric differential operator from the composites of non-commuting operators generated by (1) and (2). Similarly, it has been shown that a symmetric operator can be derived from the composite of two non-commutative operarors with one symmetric and the other non-symmetric. Finally, by application of Levinson's theorem to the case of the composite of two non-commutative second order differential operators, the results of Theorem 2.2 have been verified for limit point case. We start by stating without proof the modified version of Levinson's theorem and for more details, you can see the results in [4, 8] and the references stated therein.

Lemma 3.1 *Let $Y'(x,z) = [\Lambda(x,z) + R(x,z)]Y(x,z)$ be a first order differential system of order $2n$ operator. Suppose that $\Lambda(x,z) = diag(\lambda_k(x,z))$, $k = 1,2,\ldots,2n$ with the sign of $Re(\lambda_j(x,z) - \lambda_s(x,z))$ constant modulo integrable terms, $j \neq s$ and $\|R(x,z)\| \in \mathcal{L}^1([0,\infty))$, then the solutions of the system have asymptotic form*

$$y_k(x,z) = (e_k(x,z) + R_{kk}(x,z)) \cdot \exp\int_0^x \lambda_k(t,z)dt.$$

Here, $e_k(x,z)$ is the kth normalised unit vector and $R_{kk}(x,z)$ tends to zero analytically as $x \to \infty$.

In order to obtain the spectral results of the selfadjoint operator extension, H, of the operator T_1T_2 , the deficiency indices of the composite operator $\text{def}T_1T_2$ must be equal, that is, (l,l), $n \leq l \leq 2n$, for a composite operator of order $2n$. The description of H can be found in the following references [3, 8, 12, 13]. Suppose that an operator $A : \mathcal{H} \to \mathcal{H}$ is a closed densely defined symmetric operator with def$A = (l, l)$, then by von Neumann theorems [12], A has selfadjoint operator extension, H, whose domain can be described by

$$D(H) = D(A) \oplus \mathcal{N}(A^* + zI) \oplus \mathcal{N}(A^* - zI).$$

The same construction can always be achieved using Cayley transforms of A via the symmetric extensions of the corresponding Cayley unitary operators.

We now prove the main result of this section.

Theorem 3.2 *Let $\{T_1, T_2, \ldots, T_n\}$ be a set of formally symmetric even higher order differential operators defined on $\mathcal{H}$ that are not pairwise commutative. Then the differential operator $T = T_1 T_2 \cdots T_{n-1} T_n T_{n-1} \cdots T_2 T_1$ as well as αT for any real α are formally symmetric.*

Proof Note that $T_k^* = T_k$, for all $k = 1, 2, \ldots, n$. Then it follows that

$$\begin{aligned} T^* &= (T_1 T_2 \cdots T_{n-1} T_n T_{n-1} \cdots T_2 T_1)^* \\ &= T_1^* T_2^* \cdots T_{n-1}^* T_n^* T_{n-1}^* \cdots T_2^* T_1^* \\ &= T_1 T_2 \cdots T_{n-1} T_n T_{n-1} \cdots T_2 T_1 \\ &= T. \end{aligned}$$

Note that any real constant multiple of symmetric operator is symmetric. This clears the proof.

The results of Theorem 3.2 in the case of two formally symmetric operators that are non-commutative lead to an interesting case where a symmetric composite can be constructed from two operators that are symmetric and non-symmetric.

Theorem 3.3 *Let T_1 and T_2 be formally symmetric differential operators of order $2n$ and $2l$, $n, l = 0, 1, 2, 3, \ldots$, that do not formally commute. Then the operators $A = T_1 T_2$, $B = T_2 T_1$ are not necessarily symmetric but $T_1 B$ and $T_2 A$ are symmetric.*

Proof The non-symmetry of A and B are immediate from Lemma 1.1 unless the leading coefficients of their respective differential equations satisfy the conditions given in [1]. From the assumptions, it is clear that

$$T_1^* = T_1, \quad T_2^* = T_2, \quad \text{and} \quad T_1 T_2 \neq T_2 T_1.$$

Now if we consider the operator $T_1 B = T_1 T_2 T_1$ then this operator is formally symmetric since

$$(T_1 B)^* = B^* T_1^* = (T_2 T_1)^* T_1^* = T_1^* T_2^* T_1^* = T_1 T_2 T_1 = T_1 B.$$

The proof for the symmetry of $T_2 A$ is done in a similar way.

In the next example, we will show how to obtain a formally symmetric differential equation from two non-comutative formally differential equations.

Example 3.4
Suppose that

$$\tau_0 y(x) = (-1)^k y^{(2k)}(x) + p(x)y(x), \ \tau_1 y(x) = (-1)^k y^{(2k)}(x).$$

Then by letting $\tau_2 y(x) = \tau_0\tau_1 y(x) = y^{(4k)} + (-1)^k p(x) y^{(2k)}(x)$, we can easily obtain $6k$th order formally symmetric differential equation given by

$$\tau y(x) = \tau_1\tau_0\tau_1 y(x) = \tau_1\tau_2 y(x) = (-1)^k y^{(6k)}(x) + (p(x)y^{(2k)})^{(2k)},$$

which is a formally symmetric differential equation expressed as a composite of two non-commutative differential equations with one symmetric and the other one not.

In the next example, we consider simple cases and derive the necessary and sufficient growth and decay conditions on the coefficients of the derived differential operators for the sum of their deficiency indices to satisfy the results of Theorem 2.2.

Example 3.5
Consider the differential equations $\tau_0 = -q(x)y$, $\tau_1 y = -y''$ and $\tau_2 y = q(x)y''$. Then $\tau_0\tau_1 y = \tau_2 y$ is not symmetric, but $\tau_1\tau_2 y = -(q(x)y'')''$ is symmetric and any real scalar multiple of $\tau_1\tau_2 y$ will be symmetric too.

By scaling the differential expression $\tau_1 y$ by non-zero real scalar α, we can consider two differential equations $\tau_1 = -\alpha y''$, $y = y(x)$, where α is a constant, then $\tau_1 y$ is formally symmetric by construction. Similarly, define a second order differential equation $\tau_2 y = q(x)y''$ then $\tau_2 y$ is not symmetric and its Hilbert adjoint is given by $\tau_2^* y = qy'' + 2q'y' + qy$. Besides $\tau_1\tau_2 y \neq \tau_2\tau_1 y$. It is easy to confirm that $\tau_1\tau_2 y$ is symmetric since

$$\tau_1\tau_2 y = \tau_2^*\tau_1 y = -\alpha(qy^{(iv)} + 2q'y''' + q''y'').$$

Here, $q = q(x)$ is assumed to be continuously differentiable and unbounded. We proceed to analyse the deficiency indices of the minimal differential operator generated by $\tau_1\tau_2 y(x)$ and the spectral properties of the selfadjoint operator extension of the minimal operator via asymptotic integration. Assume that z is a spectral parameter and we solve the equation $\tau_1\tau_2 y(x) = zy(x)$. The following assumptions on growth and decay conditions are made

$$|\, q(x) \,| \to \infty, \ \text{as } x \to \infty, \qquad q^{-1}q' \in \mathcal{L}^2, \qquad (q^{-1}q')^2, q^{-1}q'' \in \mathcal{L}^1 \quad (5)$$

Once the fourth order differential equation has been converted to its first order system, the associated characteristic polynomial of the equation $\tau_1\tau_2 y = -\alpha(q(x)y'')'' = zy$ when equated to zero gives $\alpha q(x)\lambda^4 + z = 0$. Assume that $\alpha q(x) < 0$ for all $x \in [0,\infty)$, then we have four eigenvalues of the form $\lambda_{1\pm} = \pm \,|\, \frac{z}{\alpha q} \,|^{\frac{1}{4}}$ and $\lambda_{2\pm} = \pm i \,|\, \frac{z}{\alpha q} \,|^{\frac{1}{4}}$. From results of Behncke [2], the eigenvalues $\lambda_{1\pm}(x,z)$ need no establishment of uniform dichotomy condition since the pair of eigenvalues will lead to one square integrable function and another non-square integrable function irrespective of the uniform dichotomy condition. This is because of the different signs of their real parts. Off the real line, that is, if $z = z_0 + i\eta$, z_0 and η real, for some small $\eta > 0$, by binomial expansion, the uniform dichotomy condition will be again satisfied for $\lambda_{2\pm}(x,z)$ since the correction term, that is, given by $\mid \frac{\partial \mathcal{P}(x,\lambda,x)}{\partial \lambda} \mid^{-1} \approx \frac{\eta}{|q|^{\frac{1}{4}}}$ and has different signs for each of the pair $\lambda_{2\pm}(x,z)$ as the real part . The determination of the deficiency indices will greatly depend on the integrability of $\mid q \mid^{-\frac{1}{4}}$. If $\mid q \mid^{-\frac{1}{4}}$ is integrable then $y_{1-}(x,z)$ and all the two solutions $y_{2\pm}(x,z)$ will all be uniformly square integrable. Now assume that the minimal differential operators generated by $\tau_1 y(x) = -\alpha y''$ and $\tau_2 y(x) = q(x)y''$ are T_1 and T_2 respectively, then $\mathrm{def}(T_1T_2) = (3,3)$. Any selfadjoint operator extension of T_1T_2 has discrete spectrum at most.

Suppose that $\mid q \mid^{-\frac{1}{4}}$ is not integrable, then $y_{1-}(x,z)$ and $y_{2+}(x,z)$ will be square integrable with $y_{2-}(x,z)$ losing its square integrability as $\eta \to 0^+$. Thus $\mathrm{def}(T_1T_2) = (2,2)$. In this case, the solution that loses its square integrability, contributes to absolutely continuous spectrum of the selfadjoint operator extension of T_1T_2. Thus the absolutely continuous spectrum of the selfadjoint operator extension of T_1T_2 is a subset of $[\frac{1}{\alpha \underline{q}}, \infty)$, $\underline{q} = \liminf q(x)$, of spectral multiplicity one.

Analysis of the deficiency indices of the individual operators are done in a similar way. The eigenvalues of T_1 are $\pm i \mid \frac{z}{\alpha} \mid^{\frac{1}{2}}$ whenever α is greater than zero leading to $\mathrm{def}T_1 = (2,2)$. In this particular case, the absolutely continuous spectrum for the associated selfadjoint extension can only be obtained within the intervals $[-\alpha^{-1}, \alpha^{-1}]$ of spectral multiplicity one as $\eta \to 0^+$. Meanwhile, if $\alpha < 0$ then $\mathrm{def}T_1 = (1,1)$ since the eigenvalues are given by $\pm \mid \frac{z}{\alpha} \mid^{\frac{1}{2}}$. In this case, only discrete spectrum can be realised.

In the case of the operator T_2, if $q(x) < 0$, for all $x \in [0,\infty)$ then the eigenvalues are given by $\pm i \mid \frac{z}{q} \mid^{\frac{1}{2}}$. Therefore, for $\mid q \mid^{-\frac{1}{2}}$ integrable, $\mathrm{def}T_2 = (2,2)$. On the other hand if $\mid q \mid^{-\frac{1}{2}}$ is not integrable, then $\mathrm{def}T_2 = (1,1)$. Similarly, if $q(x) > 0$, the eigenvalues are given by $\pm \mid \frac{z}{q} \mid^{\frac{1}{2}}$ and $\mathrm{def}T_2 = (1,1)$. Note, here, that T_2 is non-symmetric and no symmetric extension nor selfadjoint extension.

In the results of Example 3.5, we can formulate the following theorem which conforms to the results of Theorem 2.2.

Theorem 3.4 *Let T_1 and T_2 be minimal differential operators generated by $\tau_1 y = -\alpha y''$ and $\tau_2 y = q(x)y''$ respectively on $\mathcal{H}$. Suppose that $\alpha < 0$, $\mid q \mid^{-\frac{1}{4}}$ and growth and decay conditions in (5) satisfied. Then*

$$\mathit{def}(T_1T_2) = \mathit{def}T_1 + \mathit{def}T_2 = (2,2),$$

and the spectrum of selfadjoint operator extension of T_1T_2 is at most discrete.

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