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MULTIPLIERS ON WEIGHTED GROUP ALGEBRAS

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This paper is dedicated to Professor VSK Assiamoua.

Abstract. Let G be a locally compact abelian group and let ω be a weight on G. We study the multipliers on the weighted group algebra $\mathcal{L}^1_\omega(G)$ which is the Banach space $L^1(G,\omega)$ endowed with a new convolution product $*_\omega$ which depends on the weight ω. We introduce a time-frequency shift like operators. We define a Fourier transform and we obtain among other results a convolution theorem and a Wendel like characterization of the studied multipliers.

1. Introduction

Multipliers are intensively studied since they are useful for instance in signal processing. In fact the concept of multiplier is ubiquitous in mathematics and its applications. They are extremely linked to pseudodifferential operators. In [9] the author indicated how pseudodifferential operators and Banach algebras can be utilized in mobile communications. From a theoretical point of view we refer to the source [4] for more details about multipliers on commutative Banach algebras.

Here we are interested in the multipliers on a certain large class of Banach algebras that contains the Beurling algebras. The study of multipliers on Beurling algebras started with the work of Gaudry [3]. Many references about the subject can be found in [2] where the author studied, among other topics, the multipliers on the Beurling algebras under the usual convolution product. Some recent publications about multipliers associated with locally compact groups are [1, 6, 7, 8]. In [5] the author defined on the weighted Lebesgue space $L^1_\omega(G)$ a new convolution product which has the particularity to depend on the weight ω. This new convolution generalizes the usual convolution. A Banach algebra is obtained under this new convolution and the author called it *the weighted group algebra*. In this article we denote this algebra by $\mathcal{L}^1_\omega(G)$. The amenability and the representations of $\mathcal{L}^1_\omega(G)$ had been mainly studied in [5].

Here we are interested in this algebra in another direction. Our main goal is to characterize the multipliers on this algebra. The main difficulties encountered come from the fact that the convolution depends on the weight. However the trigger came when we knew how to define the operators Γ^s_ω, the analogues of

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time-frequency shifts.
Our work is organized as follows. In the section 2, Beurling spaces and some results from [5] are recalled. In the section 3, we define the operators Γ_ω^s and we study some of their properties. In the section 4, we define and study the suitable notion of multiplier in our context. Finally in the section 5, we define a Fourier transform in $\mathcal{L}^1_\omega(G)$ and we characterize the multipliers in the Fourier/frequency domain.

2. Preliminaries

Let G be a locally compact abelian group equipped with its Haar measure dx. The neutral element of G is denoted by e. A *weight* on G is a continuous fonction $\omega : G \to (0, \infty)$ such that

$$\forall x, y \in G,\ \omega(xy) \leq \omega(x)\omega(y) \text{ and } \omega(e) = 1. \tag{2.1}$$

Consider the set

$$L^1_\omega(G) = \left\{ f : G \to \mathbb{C}\,,\ \int_G |f(x)|\omega(x)dx < \infty \right\}. \tag{2.2}$$

The mapping $f \to \|f\|_{1,\omega} = \displaystyle\int_G |f(x)|\omega(x)dx$ is a norm on $L^1_\omega(G)$. On $L^1_\omega(G)$ one defines the convolution product

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy. \tag{2.3}$$

We will often refer to this convolution product as the *usual convolution*. The space $(L^1_\omega(G), \|\cdot\|_{1,\omega}, *)$ is a Banach algebra called a *Beurling algebra*.
In [5] the author introduced a new convolution product on $L^1_\omega(G)$. The particularity of this convolution product is that it is linked to the weight ω. It is defined by

$$f *_\omega g(x) = \int_G f(y)g(y^{-1}x)\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy,\ f, g \in L^1_\omega(G). \tag{2.4}$$

When $\omega \equiv 1$ one recovers the usual convolution. Therefore results obtained here are generalizations of known results involving the usual convolution. In [5], it had been proved that the Banach space $L^1_\omega(G)$ is a Banach algebra under the convolution product $*_\omega$ and it has a bounded approximate identity. In this article we denote by $\mathcal{L}^1_\omega(G)$ this new Banach algebra. The rest of the paper is devoted to the study of the multipliers on $\mathcal{L}^1_\omega(G)$ following the path taken in [3, 2] where multipliers on $L^1_\omega(G)$ endowed with the usual convolution product had been studied.

3. The operators Γ_ω^s

For a fixed weight ω, consider the operator Γ_ω^s, $s \in G$, defined by

$$\Gamma_\omega^s f(x) = \frac{\tau_s M_\omega f(x)}{\omega(x)} \tag{3.1}$$

where M_ω is the multiplication operator defined by

$$(M_\omega f)(x) = \omega(x)f(x),\ f \in \mathcal{L}^1_\omega(G) \tag{3.2}$$

and τ_s is the translation operator defined by

$$(\tau_s f)(x) = f(s^{-1}x),\ f \in \mathcal{L}^1_\omega(G). \tag{3.3}$$

Naturally, $\Gamma^e_\omega f = f$.

Proposition 3.1. *The operator Γ^s_ω is a linear isometry from $\mathcal{L}^1_\omega(G)$ into $\mathcal{L}^1_\omega(G)$.*

Proof. The linearity is obvious. Now let $f \in \mathcal{L}^1_\omega(G)$. Then

$$\begin{aligned}
\|\Gamma^s_\omega f\|_{1,\omega} &= \int_G |\Gamma^s_\omega f(x)|\omega(x)dx \\
&= \int_G |f(s^{-1}x)\frac{\omega(s^{-1}x)}{\omega(x)}|\omega(x)dx \\
&= \int_G |f(s^{-1}x)|\omega(s^{-1}x)dx \\
&= \int_G |f(x)|\omega(x)dx \\
&= \|f\|_{1,\omega}.
\end{aligned}$$

$\square$

Proposition 3.2. *If $f \in \mathcal{L}^1_\omega(G)$ then the function $s \longmapsto \Gamma^s_\omega f$ is continuous from G into $\mathcal{L}^1_\omega(G)$.*

Proof. Let $\mathcal{C}_c(G)$ stands for the set of compact supported continuous functions on G. It is well-known that $\mathcal{C}_c(G)$ is dense in $\mathcal{L}^1_\omega(G)$ under the norm $\|\cdot\|_{1,\omega}$. Therefore we will first show the proposition for $g \in \mathcal{C}_c(G)$ and then deduce the general case by density.

Let $\varepsilon > 0$. Consider $g \in \mathcal{C}_c(G)$ and set $C_1 = \text{supp}(g)$. Pick a compact neighborhood C_2 of the neutral element e. Set $C = C_1 \cup C_2 \cup (C_1C_2)$. We have for $s \in C_2$,

$$\|\Gamma^s_\omega g - g\|_{1,\omega} = \int_C |\Gamma^s_\omega g(x) - g(x)|\omega(x)dx = \int_C |g(s^{-1}x)\omega(s^{-1}x) - g(x)\omega(x)|dx.$$

However $g\omega$ is uniformly continuous, therefore there exists a neighborhood U of e, which we may assume to be contained in C_2, such that

$$\forall s \in U,\ |(g\omega)(s^{-1}x) - (g\omega)(x)| < \frac{\varepsilon}{|C|}$$

where $|C|$ is the measure of the compact set C. Then for $s \in U$, one has

$$\|\Gamma^s_\omega g - g\|_{1,\omega} = \int_C |(g\omega)(s^{-1}x) - (g\omega)(x)|dx < \frac{\varepsilon|C|}{|C|} = \varepsilon.$$

Now let us show the claim for $f \in \mathcal{L}^1_\omega(G)$. Let K be a compact neighborhood of e. Since $\mathcal{C}_c(G)$ is dense in $\mathcal{L}^1_\omega(G)$ there exists $g \in \mathcal{C}_c(G)$ such that $\|f - g\|_{1,\omega} < \frac{\varepsilon}{3}$. There exists a compact neighborhood V of e, which we may assume to be contained in K, such that $\|\Gamma_\omega^s g - g\|_{1,\omega} < \dfrac{\varepsilon}{3}$ for all $s \in V$. Then, for $s \in V$, we have

$$\begin{aligned}
\|\Gamma_\omega^s f - f\|_{1,\omega} &\leqslant \|\Gamma_\omega^s f - \Gamma_\omega^s g\|_{1,\omega} + \|\Gamma_\omega^s g - g\|_{1,\omega} + \|f-g\|_{1,\omega} \\
&< \|f-g\|_{1,\omega} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

□

Proposition 3.3. *The Banach algebra $\mathcal{L}_\omega^1(G)$ is without order.*

Proof. Let $f \in \mathcal{L}_\omega^1(G)$ be such that $\forall g \in \mathcal{L}_\omega^1(G)$, $f *_\omega g = 0$. Then we have $(f\omega) * (g\omega) = 0$. Since $(L^1(G), *)$ is without order, we have $f\omega = 0$. Thus $f = 0$ since $\omega > 0$. □

Proposition 3.4. *Let $s \in G$ and $f, g \in \mathcal{L}_\omega^1(G)$. Then*

$$\Gamma_\omega^s (f *_\omega g) = f *_\omega \Gamma_\omega^s g = \Gamma_\omega^s f *_\omega g. \tag{3.4}$$

Proof.

$$\begin{aligned}
\Gamma_\omega^s (f *_\omega g)(x) &= \frac{\omega(s^{-1}x)}{\omega(x)} (f *_\omega g)(s^{-1}x) \\
&= \frac{\omega(s^{-1}x)}{\omega(x)} \int_G f(y) g(y^{-1}s^{-1}x) \frac{\omega(y)\omega(y^{-1}s^{-1}x)}{\omega(s^{-1}x)} dy \\
&= \int_G f(y) g(y^{-1}s^{-1}x) \frac{\omega(y)\omega(y^{-1}s^{-1}x)}{\omega(x)} dy \\
&= \int_G f(y) g(s^{-1}y^{-1}x) \frac{\omega(y)\omega(s^{-1}y^{-1}x)}{\omega(x)} [\frac{\omega(y^{-1}x)}{\omega(y^{-1}x)}] dy \\
&= \int_G f(y) [g(s^{-1}y^{-1}x) \frac{\omega(s^{-1}y^{-1}x)}{\omega(y^{-1}x)}] \frac{\omega(y)\omega(y^{-1}x)}{\omega(x)} dy \\
&= \int_G f(y) \Gamma_\omega^s g(y^{-1}x) \frac{\omega(y)\omega(y^{-1}x)}{\omega(x)} dy \\
&= f *_\omega \Gamma_\omega^s g(x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
f *_\omega \Gamma^s_\omega g(x) &= \int_G f(y)\Gamma^s_\omega g(y^{-1}x)\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy \\
&= \int_G f(y)[g(s^{-1}y^{-1}x)\frac{\omega(s^{-1}y^{-1}x)}{\omega(y^{-1}x)}]\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy \\
&= \int_G f(s^{-1}y)g(y^{-1}x)\frac{\omega(y^{-1}x)}{\omega(sy^{-1}x)}\frac{\omega(s^{-1}y)\omega(sy^{-1}x)}{\omega(x)}dy \\
&= \int_G f(s^{-1}y)g(y^{-1}x)\frac{\omega(y^{-1}x)\omega(s^{-1}y)}{\omega(x)}dy \\
&= \int_G [f(s^{-1}y)\frac{\omega(s^{-1}y)}{\omega(y)}]g(y^{-1}x)\frac{\omega(y^{-1}x)\omega(y)}{\omega(x)}dy \\
&= \int_G \Gamma^s_\omega f(y)g(y^{-1}x)\frac{\omega(y^{-1}x)\omega(y)}{\omega(x)}dy \\
&= \Gamma^s_\omega f *_\omega g(x).
\end{aligned}$$

Thus

$$\Gamma^s_\omega(f *_\omega g) = f *_\omega \Gamma^s_\omega g = \Gamma^s_\omega f *_\omega g.$$

□

Proposition 3.5. *Let $s, s' \in G$. Then*

$$\Gamma^s_\omega \Gamma^{s'}_\omega = \Gamma^{ss'}_\omega. \tag{3.5}$$

Proof. Let $f, g \in \mathcal{L}^1_\omega(G)$. Then

$$\begin{aligned}
[f *_\omega \Gamma^s_\omega \Gamma^{s'}_\omega g](x) &= [\Gamma^s_\omega f *_\omega \Gamma^{s'}_\omega g](x) \quad \text{(Proposition 3.4)} \\
&= \int_G \Gamma^s_\omega f(y)\Gamma^{s'}_\omega g(y^{-1}x)\frac{\omega(y^{-1}x)\omega(y)}{\omega(x)}dy \\
&= \int_G [f(s^{-1}y)\frac{\omega(s^{-1}y)}{\omega(y)}][g(s'^{-1}y^{-1}x)\frac{\omega(s'^{-1}y^{-1}x)}{\omega(y^{-1}x)}]\frac{\omega(y^{-1}x)\omega(y)}{\omega(x)}dy \\
&= \int_G \frac{f(s^{-1}y)\omega(s^{-1}y)g(s'^{-1}y^{-1}x)\omega(s'^{-1}y^{-1}x)}{\omega(x)}dy \\
&= \int_G \frac{f(y)\omega(y)g(s'^{-1}s^{-1}y^{-1}x)\omega(s'^{-1}s^{-1}y^{-1}x)}{\omega(x)}dy \\
&= \int_G f(y)g((ss')^{-1}y^{-1}x)\frac{\omega(y)\omega((ss')^{-1}y^{-1}x)}{\omega(x)}dy \\
&= \int_G [f(y)g((ss')^{-1}y^{-1}x)\frac{\omega((ss')^{-1}y^{-1}x)}{\omega(y^{-1}x)}]\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy \\
&= \int_G [f(y)\Gamma^{ss'}_\omega g(y^{-1}x)]\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy \\
&= f *_\omega \Gamma^{ss'}_\omega g(x)
\end{aligned}$$

and by Proposition 3.3 we have $\Gamma^s_\omega \Gamma^{s'}_\omega = \Gamma^{ss'}_\omega$. □

Proposition 3.6. *Let $\omega \geq 1$. If $f, g \in \mathcal{L}^1_\omega(G)$ then $f *_\omega g \in L^1(G)$ and*

$$\|f *_\omega g\|_1 \leq \|f\|_{1,\omega}\|g\|_{1,\omega}.$$

Proof. Let $f, g \in \mathcal{L}^1_\omega(G)$.

$$\begin{aligned}
\int_G |f *_\omega g(x)|dx &\leqslant \int_G\int_G |f(y)g(y^{-1}x)|\omega(y^{-1}x)\omega(y)dydx \\
&= \int_G\int_G |f(y)g(y^{-1}x)|\omega(y^{-1}x)\omega(y)dxdy \text{ (Fubini's theorem)} \\
&= (\int_G |f(y)|\omega(y)dy)(\int_G |g(y^{-1}x)|\omega(y^{-1}x)dx) \\
&= (\int_G |f(y)|\omega(y)dy)(\int_G |g(x)|\omega(x)dx) \text{ (invariance of the Haar measure)} \\
&= \|f\|_{1,\omega}\|g\|_{1,\omega} < \infty.
\end{aligned}$$

Thus $f *_\omega g \in L^1(G)$. Moreover $\|f *_\omega g\|_1 \leq \|f\|_{1,\omega}\|g\|_{1,\omega}$. □

4. Multipliers on $\mathcal{L}^1_\omega(G)$

Definition 4.1. A map $T : \mathcal{L}^1_\omega(G) \to \mathcal{L}^1_\omega(G)$ is called a *multiplier* if T is linear, bounded and commutes with the operators Γ^s_ω, $s \in G$.

We denote by $M(\mathcal{L}^1_\omega(G))$ the set of all such multipliers.

Proposition 4.2. *$T \in M(\mathcal{L}^1_\omega(G))$ if and only if $\forall f, g \in \mathcal{L}^1_\omega(G)$,*

$$T(f *_\omega g) = Tf *_\omega g = f *_\omega Tg.$$

Proof. (1) Let us suppose that $\forall f, g \in \mathcal{L}^1_\omega(G)$, $T(f *_\omega g) = Tf *_\omega g = f *_\omega Tg$.

- Let $f, h, g \in \mathcal{L}^1_\omega(G)$ and $\alpha, \beta \in \mathbb{C}$. We have

$$\begin{aligned}
h *_\omega T(\alpha f + \beta g) &= Th *_\omega (\alpha f + \beta g) \\
&= \alpha Th *_\omega f + \beta Th *_\omega g \\
&= \alpha h *_\omega Tf + \beta h *_\omega Tg \\
&= h *_\omega (\alpha Tf + \beta Tg).
\end{aligned}$$

Since h is arbitrary and the algebra $\mathcal{L}^1_\omega(G)$ is without order then one has

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg.$$

Therefore T is linear.

- Let $f, h \in \mathcal{L}^1_\omega(G)$ and let (f_n) be a sequence in $\mathcal{L}^1_\omega(G)$ such that $f_n \longrightarrow f$ and $Tf_n \longrightarrow h$. Pick an arbitrary element g in $\mathcal{L}^1_\omega(G)$. Then

$$\begin{aligned}
\|g *_\omega h - g *_\omega Tf\|_{1,\omega} &\leqslant \|g *_\omega h - g *_\omega Tf_n\|_{1,\omega} + \|g *_\omega Tf_n - g *_\omega Tf\|_{1,\omega} \\
&\leqslant \|g\|_{1,\omega}\|h - Tf_n\|_{1,\omega} + \|Tg *_\omega f_n - Tg *_\omega f\|_{1,\omega} \\
&\leqslant \|g\|_{1,\omega}\|h - Tf_n\|_{1,\omega} + \|Tg\|_{1,\omega}\|f_n - f\|_{1,\omega}.
\end{aligned}$$

If we tend n to ∞, then $g *_\omega h - g *_\omega Tf = 0$, this implies $g *_\omega (h - Tf) = 0$. However $\mathcal{L}^1_\omega(G)$ is without order, therefore $Tf = h$. Thus T is continuous by the closed graph theorem.

• Let us show that T commutes with the operator Γ^s_ω. Let $f, g \in \mathcal{L}^1_\omega(G)$. Then

$$\begin{aligned} T(\Gamma^s_\omega f) *_\omega g &= T(\Gamma^s_\omega f *_\omega g) \\ &= T(f *_\omega \Gamma^s_\omega g) \\ &= Tf *_\omega \Gamma^s_\omega g \\ &= (\Gamma^s_\omega Tf) *_\omega g. \end{aligned}$$

Since $\mathcal{L}^1_\omega(G)$ is without order, we have $T\Gamma^s_\omega = \Gamma^s_\omega T$.

(2) In the converse let us suppose that $T \in M(\mathcal{L}^1_\omega(G))$. Let $\xi \in L^\infty(G)$. Then the mapping $f \longmapsto \int_G Tf(x)\xi(x)dx$ is a continuous linear functional on $\mathcal{L}^1_\omega(G)$ because

$$\left|\int_G Tf(x)\xi(x)dx\right| \leq \int_G |Tf(x)\xi(x)|dx \leq \left\|\frac{\xi}{\omega}\right\|_{\infty,\omega} \|Tf\|_{1,\omega} \leq \left\|\frac{\xi}{\omega}\right\|_{\infty,\omega} \|T\|\|f\|_{1,\omega}.$$

Thus, there exists a function $\varphi \in \mathcal{L}^\infty_\omega(G)$ such that

$$\int_G Tf(x)\xi(x)dx = \int_G f(x)\varphi(x)\omega(x)dx, \forall f \in \mathcal{L}^1_\omega(G).$$

Now let $f, g \in \mathcal{L}^1_\omega(G)$. We have

$$\begin{aligned} \int_G [Tf *_\omega g](x)\xi(x)dx &= \int_G \int_G Tf(s^{-1}x)g(s)\xi(x)\frac{\omega(s^{-1}x)\omega(s)}{\omega(x)}dsdx \\ &= \int_G \int_G [Tf(s^{-1}x)\frac{\omega(s^{-1}x)}{\omega(x)}]g(s)\xi(x)\omega(s)dsdx \\ &= \int_G \int_G \Gamma^s_\omega Tf(x)\xi(x)g(s)\omega(s)dsdx \\ &= \int_G \int_G T\Gamma^s_\omega f(x)\xi(x)g(s)\omega(s)dsdx \\ &= \int_G \int_G \Gamma^s_\omega f(x)\varphi(x)\omega(x)g(s)\omega(s)dsdx \\ &= \int_G \int_G f(s^{-1}x)\left[\frac{\omega(s^{-1}x)}{\omega(x)}\right]\varphi(x)\omega(x)g(s)\omega(s)dsdx \\ &= \int_G \int_G \left[f(s^{-1}x)g(s)\frac{\omega(s^{-1}x)\omega(s)}{\omega(x)}ds\right]\varphi(x)\omega(x)dx \\ &= \int_G [f *_\omega g](x)\varphi(x)\omega(x)dx \\ &= \int_G T[f *_\omega g](x)\xi(x)dx. \end{aligned}$$

Since ξ has been chosen arbitrarily, we conclude that $T(f *_\omega g) = Tf *_\omega g$. By the commutativity of $*_\omega$ one has

$$T(f *_\omega g) = Tf *_\omega g = f *_\omega Tg.$$

□

Proposition 4.3. *(1) If $T, T' \in M(\mathcal{L}^1_\omega(G))$ then $TT' \in M(\mathcal{L}^1_\omega(G))$.*
(2) If $T \in M(\mathcal{L}^1_\omega(G))$ is bijective then $T^{-1} \in M(\mathcal{L}^1_\omega(G))$.

Proof. (1) Assume that T and T' are multipliers. Since they are linear, bounded and commute with Γ^s_ω, so is TT'.
(2) Assume that $T \in M(\mathcal{L}^1_\omega(G))$ is bijective. Let $f, g \in \mathcal{L}^1_\omega(G)$.

$$\begin{aligned}
T^{-1}f *_\omega g &= T^{-1}T[T^{-1}f *_\omega g] \\
&= T^{-1}[TT^{-1}f *_\omega g] \\
&= T^{-1}[f *_\omega g] \\
&= T^{-1}[f *_\omega (TT^{-1}g)] \\
&= T^{-1}T[f *_\omega (T^{-1}g)] \\
&= f *_\omega T^{-1}g.
\end{aligned}$$

Thus $T^{-1} \in M(\mathcal{L}^1_\omega(G))$.

□

5. Fourier transform on $\mathcal{L}^1_\omega(G)$

We denote by $\widehat{G}$ the dual group of G, that is the set of continuous characters of the locally compact commutative group G.

Definition 5.1. The Fourier transform of a function $f \in \mathcal{L}^1_\omega(G)$ is given by

$$\hat{f}(\gamma) = \mathcal{F}_\omega(f)(\gamma) = \int_G \overline{\gamma(x)} f(x)\omega(x)dx, \quad \gamma \in \hat{G}. \tag{5.1}$$

We prove the following result.

Proposition 5.2. *If $f, g \in \mathcal{L}^1_\omega(G)$ then*

$$\mathcal{F}_\omega(f *_\omega g) = \mathcal{F}_\omega(f)\mathcal{F}_\omega(g). \tag{5.2}$$

Proof.

$$\begin{aligned}
\mathcal{F}_\omega(f *_\omega g) &= \int_G \overline{\gamma(x)} f *_\omega g(x)\omega(x)dx \\
&= \int_G \int_G \overline{\gamma(x)} f(xy^{-1}) g(y)\omega(xy^{-1})\omega(y)dydx \\
&= \int_G \int_G \overline{\gamma(xy)} f(x)g(y)\omega(x)\omega(y)dxdy \\
&= \int_G \int_G \overline{\gamma(x)}\,\overline{\gamma(y)} f(x)g(y)\omega(x)\omega(y)dxdy \\
&= \left(\int_G \overline{\gamma(x)} f(x)\omega(x)dx\right)\left(\int_G \overline{\gamma(y)} g(y)\omega(y)dy\right) \\
&= \mathcal{F}_\omega(f)\mathcal{F}_\omega(g).
\end{aligned}$$

Thus $\mathcal{F}_\omega(f *_\omega g) = \mathcal{F}_\omega(f)\mathcal{F}_\omega(g)$. □

Following [5], we denote by $M(G,\omega)$ the space of all complex-valued regular Borel measures m on G such that $\|m\| = \int_G \omega(x)d|m|(x) < \infty$. The mapping $m \mapsto \|m\|$ is a norm on $M(G,\omega)$. For $m,n \in M(G,\omega)$, set

$$m *_\omega n(g) = \int_G gd(m *_\omega n) = \int_G g(xy)\frac{\omega(x)\omega(y)}{\omega(xy)}dm(x)dn(y), \tag{5.3}$$

with $g \in C_0(G,\omega^{-1})$ where $C_0(G,\omega^{-1})$ is the topological dual of $M(G,\omega)$. Under the convolution $*_\omega$ the space $M(G,\omega)$ is a unital Banach algebra [5, Theorem 5.1]. The Fourier-Stieltjes transform of a measure $m \in M(G,\omega)$ is defined by

$$\mathcal{F}_\omega(m)(\gamma) = \widehat{m}(\gamma) = \int_G \overline{\gamma(x)}\omega(x)dm(x), \quad \gamma \in \widehat{G}. \tag{5.4}$$

The following result extends the convolution theorem to measures.

Theorem 5.3. *If $m,n \in M(G,\omega)$ then $\mathcal{F}_\omega(m *_\omega n) = \mathcal{F}_\omega(m)\mathcal{F}_\omega(n)$.*

Proof.

$$\begin{aligned}
\widehat{m *_\omega n}(\gamma) &= \int_G \overline{\gamma(x)}\omega(x)d(m *_\omega n)(x) \\
&= \int_G \int_G \overline{\gamma(yx)}\omega(yx)\frac{\omega(y)\omega(x)}{\omega(yx)}dm(y)dn(x) \\
&= \int_G \int_G \overline{\gamma(y)}\,\overline{\gamma(x)}\omega(y)\omega(x)dm(y)dn(x) \\
&= \left(\int_G \overline{\gamma(y)}\,\omega(y)dm(y)\right)\left(\int_G \overline{\gamma(x)}\omega(x)dn(x)\right) \\
&= \widehat{m}(\gamma)\widehat{n}(\gamma).
\end{aligned}$$

□

The following theorem gives a characterization of a multiplier in the Fourier domain.

Proposition 5.4. *$T \in M(\mathcal{L}^1_\omega(G))$ if and only if there exists a unique complex function $\mathfrak{P}$ defined on $\widehat{G}$ such that $\widehat{Tf} = \mathfrak{P}\widehat{f}$.*

Proof. • Assume that $T \in M(\mathcal{L}^1_\omega(G))$. Then

$$\forall f, g \in \mathcal{L}^1_\omega(G),\ T(f *_\omega g) = Tf *_\omega g.$$

Since G is commutative then the convolution $*_\omega$ is commutative in $\mathcal{L}^1_\omega(G)$. Therefore

$$Tf *_\omega g = T(f *_\omega g) = T(g *_\omega f) = Tg *_\omega f.$$

Taking the Fourier transform we obtain

$$\widehat{Tf}\widehat{g} = \widehat{Tg}\widehat{f}.$$

For each character $\gamma \in \widehat{G}$, let us choose g in $\mathcal{L}^1_\omega(G)$ such that $\widehat{g}(\gamma) \neq 0$. Now, define $\mathfrak{P}$ by $\mathfrak{P}(\gamma) = \dfrac{\widehat{Tg}(\gamma)}{\widehat{g}(\gamma)}$ (this definition does not depend on the choice of g because of the relation $\widehat{Tf}\widehat{g} = \widehat{Tg}\widehat{f}$). Then we have $\widehat{Tf}(\gamma) = \mathfrak{P}(\gamma)\widehat{f}(\gamma)$. Therefore $\widehat{Tf} = \mathfrak{P}\widehat{f}$.

Let's show the unicity of $\mathfrak{P}$. Let $\mathfrak{Q}$ be a second complex function on $\widehat{G}$ such that $\widehat{Tf} = \mathfrak{Q}\widehat{f} = \mathfrak{P}\widehat{f}$ for each $f \in \mathcal{L}^1_\omega(G)$. Then $(\mathfrak{P} - \mathfrak{Q})\widehat{f} = 0$ for each $f \in \mathcal{L}^1_\omega(G)$. Hence $\mathfrak{P} = \mathfrak{Q}$.

• Assume that $\widehat{Tf} = \mathfrak{P}\widehat{f}$ for all $f \in \mathcal{L}^1_\omega(G)$. For $f, g \in \mathcal{L}^1_\omega(G)$, we know that $f *_\omega g$ belongs $\mathcal{L}^1_\omega(G)$. Applying the hypothesis one has $\widehat{T(f *_\omega g)} = \mathfrak{P}\widehat{(f *_\omega g)} = \mathfrak{P}\widehat{f}\widehat{g} = \widehat{Tf}\widehat{g} = \widehat{Tf *_\omega g}$. It follows that $T(f *_\omega g) = Tf *_\omega g$. Hence $T \in M(\mathcal{L}^1_\omega(G))$. □

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