ODEs
Exam 1 **(50 points)**

Instructions: Show all of your work. *No credit will be awarded for an answer without the necessary work.*

1. **(21 points)** Solve the following initial value problems:
   (a) $x' = 2 + t$, $x(0) = 1$.
   (b) $x' = 2x + t$, $x(0) = 1$.
   (c) $x' = (2 + t)x$, $x(0) = 1$.

2. **(2 points)** Write the scalar ODE $x''' + 2x'' - 3x' - x = 0$ as a first-order system.

3. **(12 points)** Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $u^{(0)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find the general solution of $u' = Au$ and solve the initial value problem $u' = Au$, $u(0) = u^{(0)}$.

4. **(8 points)** Let $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$. You are given that A has a single eigenvalue, $\lambda = -2$, of algebraic multiplicity 2 and geometric multiplicity 1. An eigenvector corresponding to λ is $x = (1, 1)$. Use this information to find the general solution of $u' = Au$.

5. **(2 points)**
   (a) Consider the following initial value problem: $\frac{dx}{dt} + x = \frac{1}{1-t^2}$, $x(0) = 1$. Can you determine the interval of existence of the solution without solving the IVP? If so, what is it? Briefly explain your reasoning.
   (b) Answer the same questions for the IVP $\frac{dx}{dt} + x^2 = \frac{1}{1-t^2}$, $x(0) = 1$.

6. **(2 points)** Consider the ODE $x'' - x = 0$. Each of the following functions is a solution of the ODE: $x_1(t) = e^t$, $x_2(t) = e^{-t}$, $x_3(t) = \cosh(t)$, $x_4(t) = \sinh(t)$ (you do *not* have to verify that these are solutions).
   (a) Explain how you know, without doing any calculations, that $\{x_1, x_2, x_3, x_4\}$ is linearly dependent.
   (b) How many linearly independent solutions are required to write the general solution of the given ODE?

7. **(3 points)** Consider the three functions $u^{(1)}(t) = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$, $u^{(2)}(t) = \begin{bmatrix} t^2 \\ 2t \\ 2 \end{bmatrix}$, $u^{(3)}(t) = \begin{bmatrix} t^3 \\ 3t^2 \\ 6t \end{bmatrix}$. Let $W(t) = [u^{(1)}(t)|u^{(2)}(t)|u^{(3)}(t)]$ be the Wronskian matrix.
   (a) Prove that $W(0)$ is singular and and $W(1)$ is nonsingular.
   (b) Explain why this implies that $u^{(1)}, u^{(2)}, u^{(3)}$ are linearly independent functions.
   (c) What else can you conclude from part (a)? Explain.