PROCEEDINGS

OF THE

INTERNATIONAL MATHEMATICAL CONGRESS

TORONTO, 1924

UNIVERSITY COLLEGE—MEETING PLACE OF SECTIONS I, II AND VI

PROCEEDINGS

OF THE

INTERNATIONAL MATHEMATICAL CONGRESS

HELD IN

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EDITED BY

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RESEARCH PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF TORONTO

WITH THE COLLABORATION OF AN EDITORIAL COMMITTEE

VOL. I

REPORT OF THE CONGRESS
LECTURES
COMMUNICATIONS TO SECTIONS I AND II

TORONTO:
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The Congress met in Toronto by invitation of the University of Toronto and the Royal Canadian Institute, its sessions being held in the buildings of the University. In its organization and the conduct of its proceedings it conformed to the regulations of the International Research Council, and the International Mathematical Union.

PREFATORY NOTE

IT was an unexpected and none too welcome task which fell to the lot of the undersigned when circumstances determined that he should edit the Proceedings of the Mathematical Congress of 1924. As it happened, however, the resident General Secretary, Professor J. L. Synge, was making his preparations to leave Toronto, and it was felt that the work of editing should be undertaken by one who was on the ground and who could see it through to the end.

From the very nature of the case one would expect a publication like the present to be little homogeneous in its character. This could hardly be otherwise in view of the multiplicity of the contributing authors, the number of the nationalities represented and the diversity of the problems handled. Individual authors have their predilections in regard to notation, here and there a national trend is in evidence, while the man in applied science does not always have the same idea as the pure mathematician with regard to the use of mathematical symbols.

We have not sought, then, to introduce any great degree of uniformity into the work as a whole. We have had to make concessions and accept compromises in the matter of terminology and notation. We have even on occasion passed a more or less serious discrepancy in notation where it would not cause misunderstanding on the part of the reader and where its correction would have entailed far-reaching changes with unnecessary cost and delay.

The important thing is that a paper should be understandable by those for whom it is intended. It is, of course, desirable that a uniform and self-consistent notation should hold at least within the individual papers. This too, however, is an ideal which has not always been attained.

In arranging the papers in the sections we have roughly grouped together ones on the same or cognate subjects, not so much because there is a unique arrangement in this sense as because it furnishes an alternative to the more stilted form of an alphabetical arrangement of authors. In this connection we have transferred a few papers from the sections in which they were actually presented to other sections in which they find themselves associated with papers of the group to which they more naturally belong.

Though the *procès verbal* of the meeting of the International Mathematical Union does not constitute a part of the Proceedings of the Congress we have included it as an Appendix, believing that it would have a certain interest for mathematicians and serve their convenience as a ready reference.

In the course of editing the Proceedings one has had occasion to consult many of his colleagues and collaborators with regard to matters linguistic as well as scientific. The list of these would be too long to enumerate here. I thank them all for their counsel and assistance. In particular, and I am sure that this will not be regarded as invidious, I desire to express my sincere thanks to my friend and colleague, Professor Chapelon, for his ever-ready willingness to give of his time and effort and for the effectiveness of his aid. I would also thank the Manager of the University of Toronto Press, Mr. R. J. Hamilton, for the good-will shown in taking on what must have been a heavy burden in view of his other commitments and too I would not forget my debt to those members of his staff whose co-operation has aided in bringing the work to completion.

J. C. FIELDS.

PART I: Report of the Congress

PART II: Lectures and Communications

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University of Kentucky. *Dr. Paul P. Boyd, Dr. Elizabeth Le Stourgeon.*
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University of Pittsburgh. *Professor K. D. Swartzel.*
University of South Carolina. *Professor J. B. Coleman.*
University of Texas. *Dr. Goldie P. Horton, Dr. R. L. Wilder.*
University of Utah. *Professor J. L. Gibson.*
University of Virginia. *Professor W. M. Thornton.*
University of Washington. *Professor E. T. Bell.*
University of Wisconsin. *Professors Arnold Dresden, E. B. Skinner.*
Vassar College. *Professors Henry S. White, Louise D. Cummings.*
Washington Academy of Sciences. *Professor Elihu Thomson.*
Wellesley College. *Professors Lennie P. Copeland, Clara E. Smith.*
Wells College. *Professor T. R. Hollcroft.*
West Virginia University. *Professors John Eiesland, Bird M. Turner.*
Western Electric Company. *Dr. Thornton C. Fry.*
Western Reserve University. *Professor W. G. Simon.*
Wisconsin Academy of Sciences, Arts and Letters. *Professors Arnold Dresden, E. B. Skinner.*
Wooster College. *Professor B. F. Yanney.*
Yale University. *Professor E. W. Brown.*
Yerkes Observatory. *Dr. Oliver J. Lee, Dr. G. Van Biesbroeck.*

LIST OF DELEGATES AND MEMBERS PRESENT IN TORONTO

Adams, Professor C. R., Brown University, Providence, R.I., U.S.A.
 Adams, Mrs.
Adams, Professor F. D., F.R.S., McGill University, Montreal, Canada.
d'Adhémar, le Vicomte Robert, Lambersart (Nord), France.
Alden, Dr. H. L., University of Virginia, Charlottesville, Virginia, U.S.A.
Alger, Philip L., General Electric Company, Schenectady, N.Y., U.S.A.
Allan, Professor F. B., University of Toronto, Toronto, Canada.
Allcut, Professor E. A., University of Toronto, Toronto, Canada.
Allen, Professor Frank, University of Manitoba, Winnipeg, Canada.
Allen, Miss Mildred, Mount Holyoke College, South Hadley, Mass., U.S.A.
Alvarez Ude, Professor D. José, Instituto Nacional de Ciencias, Madrid, Spain.
Ambrose, J. R. W., Chief Engineer, Toronto Terminal Railway Company, Toronto, Canada.
Anderson, Professor G. R., University of Toronto, Toronto, Canada.
Andrade, Professor Jules, University of Besançon, Besançon, France.
Anning, Professor Norman, University of Michigan, Ann Arbor, Mich., U.S.A.
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Ashley, Sir William, Professor at the University of Birmingham, Birmingham, England.
 Miss Alice Ashley.
 Miss Anne Ashley.
Aude, Professor H. T. R., Colgate University, Hamilton, N.Y., U.S.A.
Austin, Miss E. E., Professional Assistant, Meteorological Office, Air Ministry, London, England.

Babb, Professor M. J., University of Pennsylvania, Philadelphia, Pa., U.S.A.
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Baker, Professor (Emeritus) Alfred, University of Toronto, Toronto, Canada.
Bareis, Dr. Grace M., Ohio State University, Columbus, Ohio, U.S.A.
Barr, Archibald, F.R.S., Formerly Regius Professor of Civil Engineering and Mechanics in the University of Glasgow; Chairman of Barr and Stroud, Ltd., Glasgow, Scotland.
 Barr, Mrs.
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Barré, le Commandant, Ministry of War, Paris, France.
Bauer, Dr. Louis A., Director, Department of Terrestrial Magnetism, Carnegie Institution, Washington, D.C., U.S.A.
 Weeks, Mrs. R. W. (daughter).
Beare, Professor T. Hudson, F.R.S.E., University of Edinburgh, Edinburgh Scotland.
 Beare, Mrs.

Béland, The Honourable Dr. Henri S., Minister of Health, and Minister of Soldiers' Civil Re-establishment, Ottawa, Canada.
Bell, Professor E. T., University of Washington, Seattle, Wash., U.S.A.
Bernstein, Dr. B. A., University of California, Berkeley, Cal., U.S.A.
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Bragg, Lady.
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Fehr, Professor Dr. Henri, University of Geneva, Geneva, Switzerland.
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Fowler, Dr. R. H., F.R.S., Fellow of Trinity College and University Lecturer in Mathematics, Cambridge, England.
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Gerhardt, Dr. W. Frederick, Flight Research Branch, Flying Section, Engineering Division, Air Service, Dayton, Ohio, U.S.A.

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Gray, Professor J. G., University of Glasgow, Glasgow, Scotland.
Greenhill, Sir George, F.R.S., formerly Professor of Mathematics at the Artillery College (Woolwich), London, England.
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Hamilton, Miss M. M.
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Henderson, Lady
Henderson, J. P. Dominion Observatory, Ottawa, Canada.
Henderson, Robert, Second Vice-President and Actuary, Equitable Life Assurance Society, New York, N.Y., U.S.A.
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Mittag-Leffler, Professor Gustav, Founder of the Mittag-Leffler Institute for Mathematics, Djursholm, Stockholm, Sweden.

Montoriol, Professor E. P. G., École supérieure des Postes et Télégraphes, Paris, France.

Moore, Professor E. H., University of Chicago, Chicago, Ill., U.S.A.

Murnaghan, Professor F. D., Johns Hopkins University, Baltimore, Md., U.S.A.

Naess, Dr. Almar, Naval Academy, Horten, Norway.

Narishkina, Miss E., Assistant in the Physico-Mathematical Institute of the Academy of Sciences of Russia, Leningrad, Russia.

Nörlund, Professor N. E., Director of the Geophysical Institute, Copenhagen, Denmark.

Painlevé, Paul, President of the Chamber of Deputies, Former Premier of France and Minister of War, Member of the Academy of Sciences of Paris, Professor at the Sorbonne and at the École Polytechnique.

Picard, Émile, Member of the French Academy, Perpetual Secretary of the Academy of Sciences of Paris, Professor at the Sorbonne, and at the École Centrale des Arts et Manufactures, Paris, France.

Pineda, Professor P., Secretary of the Faculty of Sciences, University of Saragossa, Saragossa, Spain.

Pomey, Léon, Dr. ès Sciences, Ingénieur des Manufactures de l'État, Paris, France.

Pompéiu, Professor D., University of Bucharest, Bucharest, Roumania.

Pounder, Professor I. R., University of Toronto, Toronto, Canada.

Pupin, Professor M. I., Columbia University, New York, N.Y., U.S.A.

Putnam, Professor T. M., University of California, Berkeley, Cal., U.S.A.

Ricci, Professor G., Royal University of Padua, Padua, Italy.

Rider, Professor P. R., Washington University, St. Louis, Mo., U.S.A.

Rietz, Professor H. L., University of Iowa, Iowa City, Iowa, U.S.A.

Risser, René, Chef du Service de l'Actuariat du Ministère du Travail et de la Prévoyance Sociale, formerly Répétiteur at the École Polytechnique.

Ritt, J. H., Assistant Professor, Columbia University, New York, N.Y., U.S.A.

Rosenblatt, Professor Alfred, University of Cracow, Cracow, Poland.

Samsioe, A. F., Civil Engineer, Stockholm, Sweden.

Scatizzi, Professor Pio, Pontifical Gregorian University, Rome, Italy.

Smirnoff, Professor W. J., Institute of Bridges and Roads, Leningrad, Russia.

Snow, Dr. Chester, Physicist, Bureau of Standards, Washington, D.C., U.S.A.

Stedman, Wing-Commander E. W., Chief Aeronautical Engineer, Canadian Air Force, Ottawa, Canada.

Tamarkine, Professor J., University of Leningrad, Leningrad, Russia.

Touchard, Jacques, Electrical Engineer (École supérieure d'Électricité, Paris), Alexandria, Egypt.

Vanderlinden, Dr. H. L., Royal Observatory of Belgium, Uccle, Belgium.

Varopoulos, Th., Docteur ès Sciences (Paris), Athens, Greece.

Vasiliev, Professor A. V., University of Moscow, Moscow, Russia.

Venkoff, B., Meteorological Institute, Leningrad, Russia.

Villat, Henri, Professor at the University of Strasburg, Strasburg, France.

Volterra, The Honourable Senator Vito, President of the Royal National Academy of the Lincei, Professor in the Royal University of Rome, Rome, Italy.

Whitehead, Dr. T. T. Mathematical Master, Burnley Grammar School, Burnley. England.

Yamaga, Professor N., Imperial University of Tokyo, Tokyo, Japan.

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A. Delegates and Members present in Toronto.

B. Corresponding Members.

C. Relatives accompanying those listed under A.

PROCEEDINGS OF THE CONGRESS

PROCEEDINGS OF THE CONGRESS

Monday, August 11

The Opening Session of the Congress was held at 10 a.m. in Convocation Hall.

The Chair was taken by Professor J. C. Fields, Chairman of the Organizing Committee. The proceedings opened with an address by The Honourable Dr. Henri S. Béland, Minister of Health and Minister of Soldiers' Civil Re-establishment, who, on behalf of the Government of the Dominion of Canada, extended welcome to the Delegates and Members thus:

Mr. Chairman, Ladies and Gentlemen of the Congress:

Those of us who, a quarter of a century ago, felt that, at not too distant a date, some development would take place, the effect of which could be to concentrate the world's attention on Canada, scarcely dreamed of an event such as this.

We indeed entertained the hope of a sensational, transcending, commercial and economical achievement, something commensurate with the resources which, in a large measure, still lie dormant in the bosom of our land, of our forests, of our lakes and oceans.

Little did we cherish the belief that world-wide renowned scientists would honour Canada by selecting for their convention one of her most charming cities.

This, however, has come to pass and, as spoiled children of Fortune, we find that is only natural.

On this memorable occasion it is my privilege, a privilege which, I can assure you, I personally prize very highly, to extend to your distinguished assemblage a cordial welcome in the name of the Government of Canada.

Had not the Prime Minister been unavoidably engaged elsewhere, he would himself have relished the opportunity of greeting you.

It has undoubtedly occurred to you that Canada, being comparatively young, cannot boast of a long list of scientists, or offer for your appreciation many daring feats of engineering. The majesty of our mountains and streams has, however, invited some manifestations in this regard, which have given us reason to be proud.

They will, I trust, reveal to you a stubbornness of study, an exactitude of method, a boldness of conception, that are a credit to those amongst Canadians who have devoted their life to the advancement of the exact sciences.

Your activities, however, are not confined to our planet; they extend far beyond, indeed to other solar systems, to distances which impress our minds with a sentiment of the deepest awe.

You must, sometimes, be carried away into realms of abstraction, the access to which is denied ordinary mortals, and where the souls of Copernicus and Galileo, Descartes, Pascal, Leibnitz, Newton and Lagrange fraternize with yours in a communion of supreme delight.

Their work of genius is your work; you carry with you the tradition of all centuries, nursing it gently, adding to it as to a most precious heritage for the benefit of humanity.

I am perfectly aware that a serious accusation has been brought against Science. It has been whispered that, by its discoveries, Science has enhanced the desire and the power for destruction. What an odious calumny! Pasteur, in a remarkable passage, has refuted in advance the sordid allegation. The exclusive object of Science is the improvement of the conditions of humanity, morally, intellectually, economically and socially. Any diversion from that end is the difigurement of science's divine and celestial rule.

Your task is a noble task. It partakes of heroism and of genius; your altruistic endeavours should be the theme for praise and gratitude.

Let me express the hope that your sojourn in Canada may be an agreeable and a fruitful one, that you will carry with you an enduring and favourable impression of our composite population which, I assure you, is greatly honoured by your visit.

Changing to a language more familiar to many of those in the audience, a Canadian language, too, the Honourable Dr. Béland continued as follows:
Messieurs,

Si M. Frédéric Masson, chargé par ses collègues de l'Académie Française, de souhaiter la bienvenue à M. Henri Poincaré, l'un de vos maîtres, sentit le besoin de manifester sa timidité en présence du grand mathématicien, quel embarras ne doit pas être le mien en ce moment?

Je m'empresse d'ajouter toutefois qu'à ce sentiment de crainte bien naturel se mêle un plaisir très vif, car si la tâche est périlleuse de paraître devant les représentants autorisés des sciences exactes, elle porte avec elle un insigne honneur, celui de vous souhaiter la bienvenue sur le sol canadien au nom du Premier Ministre et du Gouvernement, et de vous exprimer tous nos regrets de n'avoir pu, comme à vos collègues de l'Association Britannique pour l'Avancement de la Science, vous faire les honneurs de la Capitale fédérale.

Connaissant cependant le généreux esprit d'hospitalité qui est une des caractéristiques de la ville de Toronto, j'imagine que vous ne regrettez pas trop amèrement l'ombre bienfaisante de la tour de nos imposants édifices parlementaires.

Vous chercherez vainement en Amérique les trésors de l'art antique. Dans quelques musées sans doute, dans certains salons privés, des toiles célèbres, même des groupes de sculpture pourraient inviter et charmer vos regards, mais nous devrons attendre l'oeuvre pieuse et persévérante des siècles avant de pouvoir présenter aux visiteurs de distinction les chefs-d'oeuvre d'architecture qui pullulent en Europe.

J'estime, d'ailleurs, qu'en venant vers le Canada vous n'y avez pas été attirés par le style de nos monuments, de nos cathédrales ou de nos édifices publics, encore que quelques-uns de ces derniers soient dignes d'attention, d'admiration peut-être.

Non, ce qui fait le charme de ce pays, ce qui le rend presqu'irrésistiblement attrayant, c'est sa beauté physique, ce sont ces hardis coups de pinceau et de ciseau du Créateur: montagnes, mers intérieures, rivières majestueuses, plaines immenses, forêts luxuriantes bien faites toutes pour captiver l'âme de savants comme vous. Nous sentons particulièrement en cette occasion de combien nous sommes redevables aux beautés et aux richesses naturelles du pays puisqu'elles vous ont amenés sur nos rives. Aux mathématiciens comme aux poètes, il faut de l'inspiration. Vous rechercherez la vôtre dans les champs riches et vastes qui s'offrent comme des fruits mûrs à cueillir, au génie minier, au génie mécanique, au génie civil, à la science de l'électricité.

On a dit de l'humanité qu'elle est une perpétuelle collaboration; cette vérité s'affirme et se démontre davantage chez les savants. En effet, chez vous, elle remonte très loin la collaboration; elle trouve ses manifestations jusque dans les siècles les plus reculés. Vous vous réclamez, et avec raison, de la famille de Pythagore et d'Euclide, de Copernic, et de Galilée, de Newton, de Leibnitz, de Lagrange, de Poincaré, et de Bertrand. Vous vous appliquez avec un zèle infatigable et un désintéressement trop peu connu dans les autres sphères de l'activité humaine, à enrichir, au bénéfice de l'humanité, l'héritage reçu de vos illustres prédécesseurs. Vous escaladez les cieux étoilés et vous passez avec la vitesse de la lumière à côté de nos soeurs planétaires, plongeant dans l'infini pour scruter les mondes, reconnaître les rapports et les mouvements, puis revenir à la terre, tels que des abeilles, chargés d'un butin scientifique précieux.

Vous passez prestement de l'abstrait à l'utilitaire. Et après avoir orné l'intelligence et émerveillé l'imagination votre sollicitude s'emploie à relever le niveau matériel par des découvertes, des améliorations et des innovations dont le résultat est une somme accrue de bien-être pour le genre humain.

Votre tâche est une noble tâche. Elle tient de l'héroïsme et du génie. Elle force l'admiration de tous ceux qu'intéresse le sort de l'humanité.

Le Gouvernement canadien forme des voeux pour que votre congrès soit à la fois agréable et fructueux; il espère que vous rapporterez une impression durable de votre passage au sein de notre population, laquelle s'honore grandement de votre présence.

The President of the University of Toronto, Sir Robert Falconer, K.C.M.G., in the name of the University welcomed the visitors with the following words:

Mr. Chairman, Ladies and Gentlemen:

I have much pleasure in welcoming, on behalf of the University of Toronto, the members of the International Mathematical Congress to the meetings which are about to be held in this place. This University and City are being honoured in having here the first meeting of the Congress that has been held on this side of the Atlantic. Never before have so many distinguished mathematicians and those interested in the applications of mathematics been gathered together in Canada. A very large number of the celebrated universities of the world are represented in this gathering, and many of the names on the programme are those of persons who have long been distinguished for their contributions to the advancement of the mathematical sciences. The variety of the nationalities is an indication of the breadth of science; the common purpose that brings you together is an indication of the unity of the sciences. Your visit to the Dominion of Canada, and especially to this University, will be a powerful impulse for the development of Mathematics. It will be an encouragement to many individual workers to have the opportunity of meeting others who have been foremost in the advancement of their subject. We hope that you will enjoy your visit socially and will make many acquaintances and friendships which will last long after you have returned to your homes.

The Chairman, speaking for the Royal Canadian Institute and the Organizing Committee, then said:

Ladies and Gentlemen:

For the first time an International Mathematical Congress meets in America, and a Canadian city has the privilege and honour of being chosen as the place of meeting. Mathematics on this continent has no such retrospect as in Europe. American mathematical achievements are of comparatively recent date, and Canada's place in the world of mathematics is a very modest one.

We in Canada derive our earlier scientific traditions from Great Britain. More recently we have begun to feel the influence of continental Europe. The founding of the Johns Hopkins University in 1876 marked a new era in the history of universities and science in America. It meant the recognition of the place of research in the university. It meant acceptance of the fact that one of the functions of a university is to train men for research. It meant that the professor was to be encouraged to engage in research. The founding of the American Journal of Mathematics in 1878 was a natural sequence to the founding of the university. This provided a means of publishing the output of American mathematicians and stimulated their productive activity. The character of this output was largely determined by the mathematical training furnished in the universities on this side of the Atlantic. The defective teaching of the calculus here made itself felt, more particularly on the side of analysis. During the first couple of decades of its existence, too, a considerable proportion of the more important papers which appeared in the Journal were of European origin.

Among the papers published in the early numbers of the Journal, we may here point out, was a series of communications on the solution of algebraic equations by Professor George Paxton Young of the University of Toronto. With these papers, we may say, that Canadian mathematics sprouted. The tree is not yet large. May its growth be stimulated by this Congress!

The gradual reform in the teaching of the calculus has had its influence in improving the quality and increasing the quantity of the American mathematical product. I know of no more striking illustration of the dependence of man on his environment than that afforded by the history of mathematics on this continent. In less than two generations, America has passed from near sterility in mathematics to a comparatively affluent productivity. This transition I would attribute primarily to the change in the teaching of the calculus.

The sterilizing of one or more generations by the false teaching of a subject is a tragedy of the first order. To those who have called a halt to such teaching, who have made correct texts possible, who have helped remedy the evil, we owe a debt of gratitude which it is not easy to repay. Such benefactors are in attendance at this Congress. We have them with us here to-day.

It would not be easy to differentiate between the influence of the Johns Hopkins University and that of immediate European contact on the development of the other great American universities. One might, however, indicate that the founding of Clark University in 1889, as a purely graduate institution of a highly specialized character, was a direct outcome of the spread of the Johns Hopkins idea.

More important than the event just mentioned, in promoting the growth of the research spirit in general and the progress of mathematics in particular in this part of the world, was the founding of the University of Chicago in 1892. Two years later the New York Mathematical Society extended the range of its activities by becoming the American Mathematical Society. This marked an epoch in the history of mathematics in America. An event of capital importance was the appearance of the first number of the Transactions of the Society in the year 1900.

In later years the evolution of the Annals of Mathematics illustrates the growing need of increased facilities for the publication of research work in mathematics. The founding of the Mathematical Association of America, too, is evidence of the spread of the mathematical interest.

May we hope that the present Congress will do much to further the cause of mathematics on this continent. As constituted it brings together the theoretical man and the applied scientist, the mathematician whose occupation it is to spin fine webs and elaborate beautiful configurations in the realm of the subjective and the applied man who takes all the risk of assuming that over against the subjective network prepared by the mathematician there is something corresponding in the external universe.

May we hope, too, that the Congress will not be without its influence on the layman to whom science must ultimately look for its material support and that on seeing the practical man standing side by side with the theoretical man, he will realize that practice cannot be divorced from theory and that the pure scientist does his share in contributing to the well-being of the community.

Ladies and Gentlemen, Delegates, Members of the Congress, I extend to you on behalf of the Royal Canadian Institute and the Organizing Committee a most cordial welcome to our country, Canada, and to the city of Toronto, Mesdames et Messieurs, je vous souhaite la bienvenue!

Replying on behalf of the delegates and members of the Congress, Professor Charles de la Vallée Poussin, President of the International Mathematical Union, responded thus:

Mesdames et Messieurs,

Le dernier Congrès International de Mathématiciens, le premier qui suivit la guerre, se réunit à Strasbourg. Le choix de cette ville avait une signification morale évidente. Quand en 1920 je vins dans cette ville pour participer au congrès, je retrouvai dans la métropole alsacienne les mêmes sentiments qui m'avaient émus l'année précèdente lors de l'inauguration de l'Université française de Strasbourg. Ce n'était pas seulement un congrès scientifique qui allait s'ouvrir; c'était un symbole et c'était une fête, celle de la délivrance de l'Alsace et aussi, comme je le disais alors, celle de la libération de la science que des mains sacrilèges avaient asservies trop lóngtemps à des dessins criminels.

Mais cela n'empèche que le congrès de Strasbourg fut avant tout une manifestation scientifique importante. Il réunit des adhérents de presque tous les pays et le très beau volume de documents que nous devons aux soins de M. Villat est le témoignage incontestable et durable de sa féconde activité. C'est à l'occasion de ce congrès que l'Union Mathématique Internationale, reçut son statut définitif et que j'eus moi-même l'honneur d'être choisi pour son président. Cet honneur, éphémère comme la plupart des grandeurs humaines, car il expire demain, m'appelle à prendre la parole dans cette belle réunion d'aujourd'hui, un peu comme le cygne qui, dit-on, ne chante qu'une seule fois et qui meurt après son premier discours.

C'est à l'Union Mathématique qu'il appartient en principe d'organiser les congrès internationaux et la détermination du siège du congrès actuel fut le premier point mis à l'ordre du jour dans la réunion de Strasbourg. Comme vous pouvez le lire dans le procès verbal de cette séance, rédigé par le Secrétaire-Général, M. Koenigs, membre de l'Institut de France, deux propositions avaient été présentées, l'une de tenir le congrès en Belgique et l'autre aux environs de New York.

Ce fut la proposition américaine qui l'emporta et, tout en étant belge moi-même, j'estime qu'il y a tout lieu de s'en réjouir.

Tout d'abord, aussi bien au point de vue moral qu'au point de vue géographique, Bruxelles est trop voisin de Strasbourg. Le choix de ce centre comme siège du congrès eût paru s'inspirer trop exclusivement des mêmes préoccupations. C'eût été sacrifier le caractère international et universel du congrès aux exigences d'un sentiment sans doute légitime, mais auquel le choix de Strasbourg avait donné pleine et entière satisfaction. J'ajouterai même que le choix de Bruxelles, loin de fortifier la leçon de choses donnée par le congrès de Strasbourg en aurait au contraire en la dédoublant émoussé le véritable sens. Après Strasbourg, il fallait affirmer avant tout le caractère international et exclusivement scientifique du congrès.

Or quel pays pouvait mieux satisfaire à ce *desideratum* que l'Amérique? Tenir le congrès dans le nouveau monde, c'était lui donner tout de suite une extension que les précédents n'avaient pas connue. C'était donner une légitime satisfaction aux désirs si naturels des savants américains. Eux qui étaient venus tant de fois et si nombreux en Europe avaient bien le droit de penser que leur tour était venu de recevoir les européens chez eux. C'était rendre un juste hommage à l'importance de l'oeuvre accomplie par la science américaine, déjà si riche dans le présent, et plus riche encore par les perspectives merveilleuses que l'avenir ouvre devant elle.

Et pour nous, habitants du vieux monde, n'était-ce pas un rajeunissement que de nous arracher en quelque sorte à nous-mêmes pour nous retremper au contact d'idées nouvelles et oublier, ne fût-ce qu'un jour, les difficultés qui nous étreignent là-bas? Nous qui avons traversé l'océan pour nous retrouver ici, regretterons-nous d'y respirer une atmosphère plus sereine? Vous dirai-je l'impression singulière, inattendue, un peu troublante même, que j'ai éprouvée moi-même en m'entendant, ici pour la première fois, désigner sous ce beau nom d'Européen dont nous pourrions être si fiers et qui nous confond tous ici les uns avec les autres à plus de mille lieues des rivages de la Méditerranée?

Rejouissons-nous donc d'être ici et que nos remercîments aillent tout d'abord au noble et grand pays dont la générosité a rendu ce congrès possible. Chacune des nations ici représentées tenait à coeur d'exprimer ses remercîments elle-même par la voix de ses délégués. Mais le succès du congrès a depassé les prévisions du comité. Le nombre des délégués est trop grand pour que l'on puisse leur accorder la parole à tous. Il a bien fallu supprimer la partie de l'ordre du jour consacrée à leurs réponses. Mais leurs sentiments sont les mêmes à tous, et ce sont aussi les mêmes que les miens, ils voudront bien me permettre en ce moment de les formuler en leur nom.

C'est donc en leur nom comme au mien que j'adresse l'expression de notre gratitude au gouvernement canadien, assez éclairé pour connaître l'importance de la science, assez généreux pour en être le Mécène, et qui a mis à la disposition du comité organisateur les sommes considérables exigées par les circonstances. Nous le remercions dans la personne de son représentant, M. le Ministre Béland, qui a bien voulu apporter lui-même dans cette assemblée le témoignage de sa très haute bienveillance. C'est en leur nom comme au mien que j'adresse mes remercîments au Royal Canadian Institute, à l'Université de Toronto, à son président, et, en particulier, à M. le professeur Fields. C'est lui qui après avoir pris l'initiative du transfert du congrès à Toronto, a accepté la présidence du comité organisateur et a assumé la plus grande partie de la lourde tâche qui devait en assurer le succès. Depuis la réunion du Conseil International de Recherches en 1922 à Bruxelles où ce projet s'est ébauché, M. Fields n'a plus connu de trêve ni de repos. Je ne vous dirai pas combien de fois il a traversé l'Atlantique, combien il a visité de pays et de villes, rencontré de mathématiciens, sollicité de collaborations. Je ne ferai pas le calcul de toutes les démarches qu'il a faites ni des kilomètres qu'il a parcourus, je craindrais en en proclamant le total de l'effrayer lui-même, car je suis sûr qu'il ne les a jamais comptés.

Mais le but est atteint, pleinement atteint, nous qui n'avons point été à la peine, nous allons être à l'honneur. Le Congrès s'annonce magnifique par le nombre de ses adhérents, par l'importance et l'intérêt des communications promises. C'est à M. Fields et à ses collaborateurs du Royal Canadian Institute que nous devons ce résultat. C'est à eux que nous devons d'être ici, dans ce décor si séduisant de monuments et de verdure qu'est l'Université de Toronto, dans ce cadre si bien approprié à nos besoins, où rien n'a éte épargné pour assurer le maximum de confort à nos personnes, le maximum de facilité et d'attraits à nos travaux.

Et voici maintenant que ce cadre enchanteur où tout semble jeune encore me ramène, bien loin par delà la guerre, aux années de ma jeunesse. Cette belle ville au bord de son lac immense évoque pour moi le souvenir d'une autre ville étendue, elle aussi, sur les rives d'un lac admirable, et je songe au tout premier congrès, celui de Zurich en 1897, comme celui-ci tout encadré de frais paysages et jeune aussi d'espérances.

Comme il serait facile et comme il serait vain d'insister sur les ressemblances et sur les contrastes que ce rapprochement suggère. Je ne veux en retenir que deux choses. D'abord je constate que l'objet d'un congrès mathématique a été défini, dès celui de Zurich, avec une précision parfaite et une netteté qui n'a pas été surpassée. Cet objet est resté le nôtre: relations personnelles entre les mathématiciens, rapports et conférences, organisation des congrès, bibliographie, etc. Ensuite je détache du discours de bienvenue que Hurwitz adressait aux congressistes les quelques pensées suivantes, parce qu'elles conviennent aujourd'hui encore et pourront servir de conclusion à mon discours.

Les grandes idées qui fécondent la science mathématique naissent dans l'isolement et le silence du cabinet; aucune autre science, sauf peut-être la philosophie, ne requiert au même degré la réflexion solitaire. Cependant les mathématiciens sont des hommes comme les autres et c'est pourquoi ils ont besoin de la société de leur semblables. Puisse une franche et cordiale confraternité animer nos réunions, charmer nos rapports mutuels; puissent les représentants des diverses nations, épris du même idéal, se sentir réconfortés par la conscience qu'unis dans leurs aspirations, ils travaillent au rapprochement des peuples et préparent la paix entre les nations.

Tel fut le sens des paroles d'Hurwitz. Hélas! La guerre éclatait dix-sept ans plus tard, ce qui prouve que la paix ni la guerre ne se décident dans les congrès de mathématiciens. Cependant les paroles de Hurwitz ne sont pas fausses et quelle que soit l'apparence de vanité que les événements leur donnent, je veux les faire miennes en ce moment, mais en les complétant par une dernière réflexion.

La paix que nous rêvons entre les peuples est celle même qui règne ici parmi nous. Celle-là repose sur la justice, la bonne foi et l'amour de la vérité. Les mathématiciens trouvent l'une de leur grande jouissance dans l'admiration du travail des autres et leur plus grand désir est de rendre à chacun ce qui lui est dû. C'est sans doute une chimère de vouloir que les nations soient aussi désintéressées que les mathématiciens, mais, quoi qu'on veuille, la paix avec tout ce qu'elle comporte pour l'humanité d'espoir de bonheur et de promesses de progrés, cette paix-là ne peut reposer que sur la justice. Puissent nos réunions être franches et cordiales et pouvoir servir à la fois de leçon et d'exemple au monde!

Professor G. Koenigs, assisted by Professor W. H. Young, read a provisional list of delegates to the Congress.

General Session

Following the Opening Session a General Session of the Congress was held for the election of Officers. On the nomination of Professor de la Vallée Poussin, Professor J. C. Fields was elected President of the Congress, and the following Vice-Presidents were elected: Professors B. Bydžovský, F. M. Da Costa Lobo, L. E. Dickson, Senator F. Faure, Professors H. Fehr, L. E. Phragmén, S. Pincherle, E. Schou, C. Servais, C. Størmer, W. van der Woude, W. H. Young, and S. Zaremba.

Professors J. L. Synge and L. V. King were elected General Secretaries of the Congress.

Following the General Session a group photograph of the members of the Congress was taken in front of the Physics Building.

2.30 p.m. Sections I, II, III (*a*), III (*b*), IV (*a*), IV (*b*), V, and VI, having been separately installed by the Introducers, papers were read and discussed.

4.30 p.m. The members of the Congress were entertained at a Garden Party at the York Club by Professor and Mrs. J. C. McLennan.

8.30 p.m. Professor Carl Størmer delivered his lecture on "Modern Norwegian Researches on the Aurora Borealis."

Tuesday, August 12

9.00 a.m. Sections I, II, III (*a*), III (*b*), IV (*a*), IV (*b*), V, and VI met separately. Papers were read and discussed.

2.30 p.m. Professor F. Severi delivered his lecture on "géométrie algébrique".

4.30 p.m. The members of the Congress were entertained at a Garden Party at Government House by His Honour Henry Cockshutt, Lieutenant-Governor of Ontario and Mrs. Cockshutt.

8.30 p.m. The members of the Congress were entertained at a Conversazione in Hart House by the University of Toronto and the Royal Canadian Institute.

Wednesday, August 13

9.00 a.m. Sections I, II, III (*a*), IV (*a*) and V met separately. Papers were read and discussed.

11.30 a.m. Professor É. Cartan delivered his lecture on "La théorie des groupes et les recherches récentes de géométrie différentielle".

3.00 p.m. The honorary degree of D.Sc. was conferred by the University of Toronto on the following delegates to, and members of, the Congress: Sir William Bragg, Professor Charles de la Vallée Poussin, Professor G. Koenigs, The Honourable Sir Charles A. Parsons, Professor F. Severi, Professor W. Stekloff. Following the conferment, the members of the Congress were entertained at a Garden Party given by the University of Toronto.

8.30 p.m. Professor W. H. Young delivered his lecture on "Some characteristic features of Twentieth Century pure mathematical research".

THURSDAY, AUGUST 14

The members of the Congress crossed to Niagara, where, on the invitation of the Hydro-Electric Power Commission of Ontario, they inspected the generating station at Queenston. They then proceeded to Niagara Falls, where they were entertained at Luncheon in the Clifton Inn as the guests of the Power Commission. After viewing the Falls, and taking the trip along the Gorge Route, the party returned by boat to Toronto.

FRIDAY, AUGUST 15

8.30 a.m. A General Assembly of the International Mathematical Union was held in Convocation Hall.

10.00 a.m. Sections I, II, IV (*a*), and V met separately. A joint session of Sections III (*a*) and III (*b*) was held. Papers were read and discussed.

2.00 p.m. Professor L. E. Dickson delivered his lecture "Outline of the theory to date of the arithmetics of algebras".

3.00 p.m. The joint session of Sections III (*a*) and III (*b*) was continued.

3.15 p.m. Professor S. Pincherle delivered his lecture on "Sulle operazioni funzionali lineari".

4.30 p.m. The members of the Congress were entertained at a Garden Party at the Grange by the Council of the Art Gallery.

7.30 p.m. The members of the Congress attended a soirée at the Hunt Club.

SATURDAY, AUGUST 16

9.00 a.m. Sections I, II, and V met separately. Sections III (*a*) and III (*b*) held a joint session. Sections IV (*a*) and IV (*b*) held a joint session. Papers were read and discussed.

2.00 p.m. Professor de la Vallée Poussin, on behalf of the members of the Congress, laid a wreath at the foot of the Soldiers' Memorial Tower.

2.30 p.m. Professor Le Roux delivered his lecture "Considérations sur une équation aux derivées partielles de la physique mathématique".

5.00 p.m. Closing Session in Convocation Hall.

The Chair was taken by Professor J. C. Fields, President of the Congress, who spoke as follows:

"A Congress such as this brings together many eminent men from many different lands, to benefit by the interchange of ideas and to come into personal contact. The policy of the present Congress was to accentuate more than has been done at previous Congresses the side of applied mathematics. I think that you will agree with me that the results justified the policy.

"We have been accustomed here in Canada to look to Europe for our intellectual inspiration, and the presence of so many distinguished scientists from the other side of the Atlantic cannot but act as a great stimulus to the intellectual activities of our country. We have been carrying on a campaign for some years past with the object of impressing our Governments and the people with the importance of science, for to the layman the scientist must ultimately look for the material resources necessary to the support of research. We have not been altogether unsuccessful in our efforts, and I am sure that the movement will be greatly helped by the meeting which has been in session during the past week. While we have gained by the presence of so many distinguished visitors, we will hope that they have profited by learning something of Canada. We would wish that a larger number could take the Western Excursion and inform themselves more fully in regard to the physical features of Canada and its natural resources.

"I wish to express to the following bodies and individuals the thanks of the Congress for their kind and generous assistance in making this Congress a success:

"To the Dominion Government, the Provincial Government of Ontario, and the Carnegie Corporation, for their generous financial assistance.

"To the President and the Board of Governors of the University of Toronto for the facilities for holding the Congress, and for their hospitality.

"To the Royal Canadian Institute, through which the gift of the Carnegie Corporation was conveyed.

"To the Board of Governors of the Royal Ontario Museum for their hospitality.

"To His Honour Henry Cockshutt, Lieutenant-Governor of Ontario, and Mrs. Cockshutt for their hospitality.

"To the Hydro-Electric Power Commission for their courtesy in showing the installation at Queenston to the members of the Congress, and also for their hospitality.

"To General C. H. Mitchell and his Committee, in particular to Mr. S. G. Bennett, for the organization of the Conversazione.

"To Professor and Mrs. McLennan for their hospitality.

"To the Council of the Art Gallery for its hospitality.

"To the Hunt Club, and in particular to Mr. William Greening.

"To those members of the Congress who suggested the laying of a wreath at the Soldiers' Tower.

"To Professor J. L. Synge, Secretary of the Organizing Committee, and Professor J. H. Parkin, Assistant Secretary, for their work in organizing the Congress.

"To Dr. W. G. Miller and Dean R. W. Brock for their work in connection with the Western Excursion.

"To Professor Wasteneys for his work in connection with publicity.

"To Professor Cockburn for arrangements with regard to meeting rooms and equipment.

"To Professor Allcut for the arrangements in connection with signs and messengers.

"To Mr. D. B. Hanna for his help in obtaining concessions in railway transportation rates.

"To Mr. Hamilton for efficient work done at the University Press."

Professor S. Pincherle, President of the International Mathematical Union, and Major P. A. MacMahon then spoke.

Professor J. L. Synge, General Secretary, reported on the membership of the Congress and read a telegram from Professor Dickstein of Warsaw conveying his wishes for the success of the Congress.

The following resolution was proposed by Professor J. L. Synge, seconded by Professor B. Bydžovský, and carried unanimously: "That this International Mathematical Congress assembled in Toronto hears with pleasure that the Royal Irish Academy contemplates the publication of a collected edition of the works of Sir William Rowan Hamilton".

The following resolution was proposed by Professor C. V. Raman, seconded by Professor G. A. Bliss, and carried by acclamation: "That the best thanks of the members of the International Mathematical Congress are due to Dr. J. C. Fields, and to those mentioned in his speech as having contributed to the success of the Congress, and that this vote of thanks be communicated to them".

The Session then closed.

8 p.m. A dinner for the members of the Congress and their wives was held in Hart House. Professor J. C. Fields presided.

On the night of August 17 a number of members of the Congress left Toronto on a transcontinental excursion to Vancouver and Victoria, which had been arranged through the courtesy of the Canadian National and the Canadian Pacific Railways for overseas members of the Congress and of the British Association for the Advancement of Science, whose Sessions, held also in the buildings of the University of Toronto, had in part coincided with those of the Congress. The excursion returned to Toronto on September 3. Other members took shorter trips before returning to their home countries.

AN INCIDENT OF COURTESY

Professor Charles de la Vallée Poussin on behalf of the members of the Congress, presenting a commemorative wreath to Sir Robert Falconer, President of the University of Toronto.

APPENDIX I.

MEETING OF THE INTERNATIONAL MATHEMATICAL UNION

PROCÈS VERBAL DE LA SÉANCE DU 15 AOÛT 1924 DE L'UNION INTERNATIONALE MATHÉMATIQUE

Le 15 Août 1924, à l'occasion du Congrès International de Mathématiques tenu à Toronto, s'est réunie, sous la Présidence de M. de la Vallée Poussin, Président de l'Union, l'Assemblée statutaire.

La réunion a eu lieu dans l'aula de l'Université de Toronto, mise aimablement à notre disposition. Les pays suivants, faisant partie de l'Union, étaient représentés: Belgique, Canada, Danemark, États Unis, France, Grande Bretagne, Hollande, Italie, Norwège, Pologne, Portugal, Suède, Suisse, Tchécoslovaquie.

Étaient, en outre présents plusieurs savants des pays suivants qui n'ont pas encore adhéré à l'Union: Espagne, Géorgie, Russie, Inde.

Le Secrétaire donne lecture du procès verbal de la séance du 20 Août 1920 tenue à Strasbourg, où fut créée l'Union Internationale Mathématique.

Ce procès verbal est adopté. Lecture est ensuite donnée par le Trésorier de l'État financier de l'Union. Les comptes du Trésorier sont approuvés. Le taux de la part contributive unitaire reste fixe à cent vingt-cinq francs.

Le Trésorier s'étant plaint du retard apporté par les pays adhérents au paiement de leurs cotisations, il est décidé que, par ses soins, les sommes dues seront réclamées aux retardataires.

Des remerciements sont votés au Trésorier pour sa gestion.

Il est ensuite procédé au renouvellement du Bureau, dans les conditions statutaires. D'après l'article 6 des Statuts, le Bureau de l'Union est élu pour huit ans, mais exceptionnellement, le mandat du Président et de trois Vice-Présidents (désignés par un tirage au sort) nommés à la fondation de l'Union, expire à la fin de la première Assemblée générale qui suit celle de leur élection.

Le sort désigne comme sortants MM. les Vice-Présidents Bianchi, Dickson et Larmor.

M. le Professeur Pincherle est élu Président.

MM. Bliss, Fehr et Holmgren sont élus Vice-Présidents.

Il convient de dire que dans ces divers votes, chaque pays affilié à l'Union a disposé d'un nombre de suffrages égal à celui que les Statuts lui assignent. Un savant de chaque pays, désigné par ses compatriotes, a déposé le nombre voulu de bulletins.

Ont été ensuite nommés Présidents d'Honneur, en outre de ceux déjà existants, MM. les Professeurs de la Vallée Poussin, Président sortant, Fields, Dickson et Mittag-Leffler.

L'Assemblée s'occupe ensuite de la question de la Bibliographie. Une Commission spéciale de la Bibliographie est instituée dont feront partie, outre le Président, MM. Archibald, Bortolotti, Fréchet, Van der Woude, Young. En ce qui concerne le choix du siège du futur Congrès en 1928, dont la disposition appartient à l'Assemblée, elle décide de reporter ce choix à l'année 1926 et de s'en remettre au Bureau pour ce soin.

En fin de séance, les États Unis déposent entre les mains du Président un vœu concernant l'intervention du Comité International de Recherches dans l'admission des Pays dans l'Union. Le Danemark, la Hollande, l'Italie, la Suède, la Norwège, et la Grande Bretagne s'associent à ce vœu.

Le Président se chargera de le transmettre au Bureau exécutif du Comité International de Recherches.

Le Secrétaire Général:
G. Koenigs.

Le Président:
de la Vallée Poussin.

État du Bureau de l'Union International Mathématique.

Présidents d'Honneur: MM. Lamb, Émile Picard, Volterra, de la Vallée Poussin, Dickson, Fields et Mittag-Leffler.

Président—M. Pincherle.

Vice-Présidents: MM. Appell, Young, Bliss, Fehr, Phragmén.

Secrétaire général—G. Koenigs.

Trésorier—M. Demoulin.

APPENDIX II.

GRANTS AND DONATIONS

ACKNOWLEDGMENT OF GRANTS AND DONATIONS

To finance the organization of the International Mathematical Congress of 1924 the Government of the Dominion of Canada and the Government of the Province of Ontario each made a grant of $25,000 and the Carnegie Corporation a grant of $5,000. Appreciation of the generous action of the two Governments and of the Carnegie Corporation in this connection was voiced by the members of the Congress in resolutions adopted at the closing meeting.

The grants referred to were later on supplemented by additional grants of $2,000 each in the case of the two Governments and by a further contribution of $1,500 from the Carnegie Corporation, to assist in publishing the Proceedings.

When the University Press began the work of printing the funds at the disposition of the Organizing Committee fell short of the estimated cost. The magnitude of the undertaking too kept growing as copy of papers presented at the Congress continued to come in long after its sessions were over. With the aid of the additional grants just mentioned and with contributions from other willing sources, however, the Committee was in a position to carry on and finish the task which it had undertaken.

The Committee desires to express its heartfelt thanks to all those whose financial assistance has enabled it to complete the publication of the Proceedings —to the Federal and Provincial Governments for their augmented grants, to the Carnegie Corporation for its increased contribution, to the Board of Governors of the University of Toronto for a grant of $2,000 and to all those firms and private citizens listed below whose generous donations have helped to make possible the completion and perpetuation of the work of the Congress as represented in the printed Proceedings.

List of Contributors

The Government of the Dominion of Canada	$27,000
The Government of the Province of Ontario	27,000
The Carnegie Corporation	6,500
The University of Toronto	2,000
Membership fees and subscriptions (to date)	2,565
Western Excursion Fund	1,080
The Imperial Oil Company	500
The T. Eaton Company	500
Wood, Gundy and Company	500
Mr. A. E. Ames	100
Mr. J. E. Atkinson	500
Mr. C. S. Blackwell	100
Mr. C. L. Burton	100
Sir Joseph Flavelle, Bart	500
Colonel A. E. Gooderham	500
Sir Edward Kemp	100
Dr. E. R. Wood	500

PART II

LIST OF LECTURES

Cartan, É: La théorie des groupes et les recherches récentes de géométrie différentielle.

Dickson, L. E.: Outline of the theory to date of the arithmetics of algebras.

Le Roux, J.: Considérations sur une équation aux dérivées partielles de la physique mathématique.

Pierpont, J.: Non-euclidian geometry from non-projective standpoint.

Pincherle, S.: Sulle operazioni funzionali lineari.

Severi, F.: La géométrie algébrique.

Størmer, C.: Modern Norwegian researches on the aurora borealis.

Young, W. H.: Some characteristic features of twentieth century pure mathematical research.

LIST OF SECTIONS AND ANALYSIS OF COMMUNICATIONS

Section	*Subjects*	*Papers*
I	Algebra, Theory of Numbers, Analysis	67
II	Geometry	36
III (*a*)	Mechanics, Physics,	51
III (*b*)	Astronomy, Geophysics	
IV (*a*)	Electrical, Mechanical, Civil and Mining Engineering	47
IV (*b*)	Aeronautics, Naval Architecture, Ballistics, Radiotelegraphy	
V	Statistics, Actuarial Science, Economics	24
VI	History, Philosophy, Didactics	16
		241

Total number of contributions, including lectures 249

LIST OF COMMUNICATIONS

SECTION I

Algebra, Theory of Numbers, Analysis

Andrade, J.: Problème proposé sur les équations fonctionelles.
Bell, E. T.: General class number relations whose degenerates involve indefinite forms.
Bernstein, B. A.: Modular representations of finite algebras.
Birkeland, R.: On the solution of quintic equations.
Bliss, G. A.: The transformation of Clebsch in the calculus of variations.
Coïalowitsch, B. M.: Sur les équations différentielles indéterminées.
Corral, J. I. del: Meditations on trigonometry.
Crelier, L. J.: Sur quelques équations intégrales simples.
Curtiss, D. R.: Rational processes for separating the real branches of a plane curve at a multiple point.
Delaunay, B.: Sur le nombre de représentations d'un nombre par une forme binaire cubique d'un discriminant négatif.
Dickson, L. E.: A new theory of linear transformations and pairs of bilinear forms.
Dickson, L. E.: Further development of the theory of arithmetics of algebras.
Drach, J.: Sur *l'intégration logique* des équations différentielles: applicatoins aux équations de la géométrie et de la mécanique.
Du Pasquier, L. G.: L'évolution du concept de nombre hypercomplexe entier.

Evans, G. C.: The Dirichlet problem for the general finitely connected region.
Fichtenholz, Gr. M.: Sur la notion de fermeture des systèmes de fonctions.
Fields, J. C.: A foundation for the theory of ideals.
Ford, W. B.: On determining the asymptotic developments of a given function.
Fréchet, M.: L'expression la plus générale de la «distance» sur une droite.
Fréchet, M.: Number of dimensions of an abstract set.
Fréchet, M.: Sur une représentation paramétrique intrinsèque de la courbe continue la plus générale.
Fueter, R.: Some applications of the theory of functions to the theory of numbers.
Gavriloff, A. F.: Sur l'intégration des équations des lignes géodésiques et d'un problème de la dynamique du point.
Glenn, O. E.: Differential combinants and associated parameters.
Glenn, O. E.: Theorems of finiteness in formal concomitant theory, modulo P.
Gunther, N.: Sur la résolution des systèmes d'équations, Rot $X=A$, Grad $\Phi=A$.
Gunther, N.: Sur un problème fondamental de l'hydrodynamique.
Gunther, N.: Quelques récents travaux de mathématiciens de Léningrad.
Haag, J.: Sur le problème des séquences.
Haag, J.: Sur un problème général de probabilités et ses diverses applications.
Hazlett, O. C.: On the arithmetic of a general associative algebra.
Hille, Einar: On the zeros of the functions defined by linear differential equations of the second order.
Hutchinson, J. I.: On the roots of the Riemann zeta function.
Kapteyn, W.: Expansion of functions in terms of Bernoullian polynomials.
Kössler, M.: On a generalization of Fabry's and Szász's theorems concerning the singularities of power series.
Krawtchouk, M.: Note sur l'interpolation généralisée.
Kryloff, N.: Sur quelques recherches dans le domaine de la théorie de l'interpolation et des quadratures, dites mécaniques.
Kryloff, N. et Tamarkine, J.: Sur une formule d'interpolation.
Lévy, A.: Sur une méthode de calcul des idéaux d'un corps du second degré.
MacMahon, P. A.: The expansion of determinants and permanents in terms of symmetric functions.
Miller, G. A.: Commutative conjugate cycles in subgroups of the holomorph of an Abelian group.
Murray, F. H.: The asymptotic distribution of the characteristic numbers for the self-adjoint linear partial differential equation of the second order.
Narishkina, E.: On the analogue of Bernoullian numbers in quadratic fields.
Øre, Ø.: A new method in the theory of algebraic numbers.
Petrovitch, M.: Correspondence entre la fonction et la fraction décimale.
Plancherel, M.: Sur les séries de fonctions orthogonales.
Pomey, L.: Sur les équations intégro-différentielles à plusieurs variables et leurs singularités.
Pomey, L.: Sur l'indicateur d'un nombre entier.
Prasad, Gorakh: On the numerical solution of integral equations.
Razmadzé, A.: Sur les solutions discontinues dans le calcul des variations.
Razmadzé, A.: Sur quelques formules de la moyenne.

Ritt, J. H.: Elementary functions and their inverses.

Rosebrugh, T. R.: A theorem on the determinants of substitutions involving products of variables which are themselves given by linear substitutions.

Scatizzi, P.: L'algebra delle derivate.

Shohat, J. A.: On the asymptotic properties of a certain class of Tchebycheff polynomials.

Sierpinski, W.: Les ensembles bien définis, non mesurables B.

Smirnoff, W. J.: Sur la théorie des groupes automorphes.

Stekloff, W.: Sur les problèmes de représentation des fonctions à l'aide de polynomes, du calcul approché des intégrales définies, du développement des fonctions en séries infinies suivant les polynomes, et de l'interpolation, considérés au point de vue des idées de Tchébycheff.

Tonelli, L.: Sul calcolo delle variazioni.

Touchard, J.: Sur certaines équations fonctionnelles.

Touchard, J.: Sur la théorie des différences.

Uspensky, J. V. and Venkoff, B.: On some new class-number relations.

Vandiver, H. S.: On the first case of Fermat's last theorem.

Varopoulos, Th.: Sur les valeurs exceptionnelles des fonctions multiformes.

Wilder, R. L.: On a certain type of connected set which cuts the plane.

Williams, W. L. G.: Formal modular invariants of forms in q variables.

Wolff, J.: On the sufficient conditions for analyticity of functions of a complex variable.

SECTION II

Geometry

Barrau, J. A.: Conditions for the intersection of linear spaces situated in a quadratic variety.

Bonnesen, T.: Recherches géométriques sur le problème isopérimétrique.

Bortolotti, E.: L'algebra geometrica ed i prodromi della geometria analytica in un manuscritto inedito di R. Bombelli.

Bydžovský, B.: Contribution à la théorie de la sextique à huit points doubles.

Coble, A. B.: The behaviour of the rational plane sextic and its related Cayley symmetroid under regular Cremona transformation.

Costa, A. R. da: L'enseignement des mathématiques doit être dirigé vers l'étude de la relativité.

Cummings, L. D.: Cyclic systems of six points in a binary correspondence.

Delaunay, B.: Sur la sphère vide.

Delaunay, N.: Sur les bases nouvelles de la théorie des systèmes articulés.

Demoulin, A.: Détermination des invariants différentiels et des invariants intégraux des surfaces pour le groupe conforme.

Demoulin, A.: La théorie des équations M et quelques-unes de ses applications à la géométrie.

Eiesland, J.: Quadratic flat-complexes in odd n-space and their singular spreads, flat-sphere transformation.

Errera, A.: Quelques remarques sur le problème des quatre couleurs.

Fubini, G.: Riassunto di alcune ricerche di geometria proiettivo-differenziale.
Godeaux, L. A.: Sur les involutions régulières d'ordre deux, appartenant à une surface irrégulière.
Haskell, M. W.: Curves autopolar with respect to a finite number of conics.
Janet, M.: Sur les systèmes linéaires d'hypersurfaces.
Koenigs, G.: Sur les mouvements à deux paramètres doublement décomposables.
MacLean, N. B.: On certain surfaces related covariantly to a given ruled surface.
Merlin, É.: Sur les lignes asymptotiques en géométrie infinitésimale.
Morley, F.: The condition that the curves of a net have a common point.
Murnaghan, F. D. The generalised Kronecker symbol.
Naess, A.: On the generalization of the vector product to S_n.
Naess, A.: Three theorems of analysis derived by the vector method as corollaries from a single proposition.
Ricci, G.: Contributo alla teoria delle varietà Riemanniane.
Servais, C.: Sur la géométrie du tétraèdre.
Servais, C.: Sur les lignes asymptotiques.
Sisam, C. H.: On surfaces whose asymptotic curves are cubics.
Sullivan, C. T.: The determination of surfaces characterized by a reducible directrix quadric.
Synge, J. L.: Normals and curvatures of a curve in a Riemannian manifold.

Thompson, D'Arcy W.: The repeating patterns of the regular polygons and their relation to the Archimedean bodies.
Torroja y Miret, D.: Sur la représentation des espaces pluridimensionels.
Tzitzéica, G.: Un nouveau problème sur les suites de Laplace.
van der Woude, W.: On the finiteness of the system of algebraic invariants of a binary form.
Weaver, J. H.: On a system of triangles related to a poristic system.
Wheeler, A. H.: Certain forms of the icosahedron and a method for deriving and designating higher polyhedra.

SECTION III

Mechanics, Physics, Astronomy, Geophysics

Andrade, J.: Balances spirales: frottements hydrostatiques et viscosités.
Andrade, J.: Le problème actuel des horloges élastiques—frottements et viscosités.
Barré, E.: Sur la propagation des ondes planes dans les milieux élastiques anisotropes.
Bauer, L. A.: The mathematical analysis of the earth's magnetic field.
Bezikowitsch, A.: Über relative Maxima des Newtonschen Potentials.
Bjerknes, V.: Solved and unsolved problems in dynamical meteorology.
Brillouin, L.: Les lois de l'élasticité en coordonnées quelconques.
Brown, E. W.: The orbit of the eighth satellite of Jupiter.
Buchanan, D.: Asymptotic solutions in the problem of three bodies.
Campbell, G. A.: A system of "definitive units" proposed for universal use.

Cartan, É.: Sur la stabilité ordinaire des ellipsoïdes de Jacobi.
Castro Bonel, H.: Sur quelques méthodes graphiques pour la détermination de la position géographique d'un dirigeable.
Chapman, S. and Whitehead, T. T.: The influence of electromagnetic induction within the earth upon terrestrial magnetic storms.
Chapman, S.: On the electrostatic potential energy of the calcite crystal.
Chazy, J.: Sur l'arrivée dans le système solaire d'un astre étranger.
Conway, A. W.: On the quantization of certain orbits.
Costa Lobo, F. M. da: New physical theories.
Dadourian, H. M.: On the fundamental principle of dynamics.
Dixon, A. C.: The theory of a thin elastic rectangular plate clamped at the edges.
Drach, J.: Sur le mouvement d'un solide pesant qui a un point fixe.
Eddington, A. S.: Absolute Rotation.
Fowler, R. H.: The equilibrium properties of gases at high (stellar) temperatures.
Gianfranceschi, G. Perturbations in the orbits of electrons.
Gillson, A. H. S.: The dynamical theory of tides in an ocean of varying depth.
Giorgi, G.: On the functional dependence of physical variables.
Gray, J. G.: Gyroscopic tops.
Greenhill, Sir G.: The top in two moves.
Haag, J.: Sur l'application des méthodes du calcul tensoriel à la théorie des moindres carrés.
Heck, N. H.: Velocity of sound in sea water.
Humphreys, W. J.: The effect of surface drag on surface winds.
Jacobsohn, S. J.: Note on the force equation of electrodynamics.
King, L. V.: On the direct numerical calculation of elliptic functions and integrals.
Kostitzin, V. A.: Sur quelques applications des équations intégrales au problème de l'hystérésis magnétique.
Levinson, H. C.: The gravitational field of n moving particles in the theory of relativity.
Losada y Puga, C. de: A short contribution to the kinetic theory of gases.
McEwen, G, F,: Calculation of the velocity of vertical ocean currents in the San Diego region from the accompanying temperature reduction below "normal" values.
Patterson, J.: The theory of the anemometer.
Plummer, H. C.: Note on the reduction of parallax plates.
Pomey, J. B.: Sur la nature des grandeurs électriques considérées en électrostatique.
Raman, C. V.: Theory of the structure of liquid surfaces.
Risser, R.: Au sujet des ondes d'émersion.
Shaw, Sir N.: The convective energy of saturated air in a natural environment.
Silberstein, L.: A finite world-radius and some of its cosmological implications.
Silberstein, L.: Quantum theory of photographic exposure.
Snow, C.: Alternating current distribution in cylindrical conductors.
Stekloff, W : Les recherches posthumes de Liapounoff sur les figures d'équilibre d'un liquide hétérogène en rotation.

St. John, C. E.: The red shift of the solar lines and relativity.

Swann, W. F. G.: A generalization of electrodynamics consistent with restricted relativity and affording a possible explanation of the earth's magnetic and gravitational fields, and the maintenance of the earth's charge.

Swann, W. F. G.: A new deduction of the electromagnetic equations.

Vanderlinden, H. L.: The gravitational field in a curved space of an electrical sphere in which the density of matter is variable.

Zaremba, S.: Sur un groupe de transformations qui se présentent en électrodynamique.

SECTION IV

Engineering

Angus, R. W.: Arithmetic solution of engineering problems with special reference to hydraulics.

Berry, W. J.: The influence of mathematics on the development of naval architecture.

Bjerknes, V.: The forces that lift aeroplanes.

Boyajian, A.: Physical interpretation of circular, hyperbolic and elliptic angles and their functions.

Bréguet, L.: Analyse des effets des pulsations du vent sur la résultante aérodynamique moyenne d'un planeur.

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LECTURES

LA THÉORIE DES GROUPES ET LES RECHERCHES RÉCENTES DE GÉOMÉTRIE DIFFÉRENTIELLE

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I

On sait, depuis F. Klein (Programme d'Erlangen) et S. Lie, le rôle important joué par la théorie des groupes dans la Géométrie. H. Poincaré a popularisé dans le grand public scientifique cette idée fondamentale que la notion de groupe est à la base des premières spéculations géométriques. La Géométrie élémentaire est au fond la théorie des invariants d'un certain groupe, le groupe des déplacements euclidiens; elle a en effet pour but l'étude des propriétés des figures qui se conservent par un déplacement arbitraire; dire que tous les déplacements forment un groupe, c'est justement exprimer en langage précis l'axiome d'après lequel deux figures égales à une troisième sont égales entre elles

La Géométrie projective a de même pour objet l'étude des propriétés des figures qui se conservent par le groupe des transformations homographiques, et on peut également assigner à la Géométrie affine, à la Géométrie conforme ou anallagmatique, etc., un groupe correspondant. Inversement tout groupe continu donne naissance à une discipline géométrique autonome.

Dans chacune de ces Géométries on attribue, pour la commodité du langage, à l'espace dans lequel les figures étudiées sont localisées les propriétés elles-mêmes du groupe correspondant, ou groupe *fondamental;* c'est ainsi qu'on est arrivé à dire: «l'espace euclidien»; «l'espace affine», etc, au lieu de «l'espace dans lequel on n'étudie que les propriétés des figures invariantes par le groupe euclidien, le groupe affine», etc. Chacun de ces espaces est *homogène*, en ce sens que ses propriétés restent inaltérées par une transformation du groupe fondamental correspondant.

Plusieurs années avant le Programme d'Erlangen, B. Riemann avait introduit, dans son mémoire célèbre: «*Ueber die Hypothesen welche der Geometrie zu Grunde liegen*», des espaces *non homogènes* au sens qui vient d'être donné à cette expression. Dans ces espaces le carré de la distance de deux points infiniment voisins était défini par une forme différentielle jusqu'à un certain point arbitraire, mais qu'en fait, on a toujours supposée quadratique. Ces espaces ont fait l'objet de nombreux et importants travaux, principalement en Italie. Mais ils ont surtout pris une importance considérable depuis que M. Einstein, par sa théorie de la relativité généralisée, a essayé, en identifiant notre Univers à un espace de Riemann, de réunir en une seule et même théorie la gravitation, l'optique et l'électromagnétisme. Le mouvement d'idées auquel cette théorie a donné naissance a conduit, par des généralisations importantes, à des espaces nouveaux; il suffira de citer les espaces de M. H. Weyl et les espaces de M. Eddington. Quel rôle la notion de groupe joue-t-elle, ou plutôt doit-elle jouer dans ce champ nouveau de la géométrie? Est-il possible de faire rentrer dans le cadre, suffisamment élargi, du programme d'Erlangen, toutes les géométries nouvelles et une infinité d'autres? C'est ce que je me propose d'examiner.

II

A première vue, la notion de groupe semble étrangère à la Géométrie des espaces de Riemann, car ils ne possèdent l'homogénéité d'aucun espace à groupe fondamental. Néanmoins, si un espace de Riemann ne possède pas une homogénéité absolue, il possède cependant une sorte d'homogénéité infinitésimale; au voisinage immédiat d'un point donné il est assimilable à un espace euclidien. Mais si deux petits morceaux voisins d'un espace de Riemann peuvent être assimilés chacun à un petit morceau d'espace euclidien, ces deux petits morceaux sont sans lien entre eux, ils ne peuvent pas, *sans convention nouvelle*, être regardés comme appartenant à un seul et même espace euclidien. Autrement dit, un espace de Riemann admet, au voisinage d'un point A, une rotation autour de ce point, mais une translation, même considérée dans les effets qu'elle produit sur une région très petite de l'espace, n'a pas de sens. Or, c'est le développement même de la théorie de la relativité, liée par l'obligation paradoxale d'interpréter dans et par un Univers non homogène les résultats de nombreuses expériences faites par des observateurs qui croyaient à l'homogénéité de cet Univers, qui permit de combler en partie le fossé qui séparait les espaces de Riemann de l'espace euclidien. Le premier pas dans cette voie fut l'oeuvre de M. Levi-Civita, par l'introduction de sa notion de *parallélisme*.

Voici comment, grâce à cette notion, les choses peuvent être présentées. On peut imaginer, en chaque point d'un espace de Riemann, un *espace euclidien* (fictif) *tangent* dont ce point et les points infiniment voisins font partie; la définition du parallélisme de M. Levi-Civita permet alors de raccorder en un seul les espaces euclidiens tangents en deux points infiniment voisins quelconques; autrement dit, elle confère à l'espace de Riemann une *connexion euclidienne.* Si l'on considère dans l'espace de Riemann une ligne continue AB, on peut raccorder de proche en proche en un seul les espaces euclidiens tangents aux différents points de AB; par suite aussi, aux infiniment petits près du second ordre, tous les points de l'espace de Riemann voisins de la ligne AB viendront, par cette espèce de *développement*, se localiser dans l'espace euclidien tangent en A. Le mot *développement* est mis là à dessein. Si en effet on applique le procédé qui vient d'être indiqué à une surface ordinaire, regardée comme un espace de Riemann à deux dimensions défini par le ds^2 de la surface, le raccord de proche en proche des plans (euclidiens) tangents à une ligne AB tracée sur la surface est identique au développement classique sur un plan de la développable circonscrite à la surface le long de AB.

Comme on le voit, la notion de parallélisme de M. Levi-Civita permet d'assimiler à un vrai espace euclidien, ou au moins à une portion de cet espace, toute la région d'un espace de Riemann qui avoisine un arc de courbe quelconque AB tracé dans l'espace donné. La différence essentielle qui subsiste encore entre un espace de Riemann et l'espace euclidien est la suivante: si l'on joint un point A à un point B par deux chemins différents, ACB, AC'B, et qu'on développe sur l'espace euclidien tangent en A les deux régions qui entourent ces deux chemins, on n'obtiendra dans les deux cas, pour le point B et le petit morceau d'espace qui l'entoure, ni la même position ni la même orientation. Autrement dit, le développement de l'espace euclidien tangent, quand on se déplace dans l'espace de Riemann, n'est pas *holonome.* Au lieu de dire que l'espace de Riemann est à connexion euclidienne, on peut dire que c'est un espace euclidien non holonome. Mais il est important de remarquer qu'il ne l'était pas par lui-même, je veux dire par son *seul* ds^2; il l'est *devenu par la définition du parallélisme de M. Levi-Civita.*

III

Cette manière d'envisager la notion de parallélisme est, je crois, celle qui va le mieux au fond des choses. Ce serait restreindre sa portée que de n'y voir, comme on l'a fait en général, qu'un procédé de comparaison des vecteurs issus de deux points infiniment voisins; il faut y voir au contraire un moyen d'introduire dans un espace de Riemann toute la gamme des déplacements de l'espace euclidien, du moins en ce qui concerne les effets qu'ils produisent dans une région infiniment petite de l'espace.

Le point de vue habituel permet la fondation de la Géométrie affine non holonome, parce que la notion de l'équipollence de deux vecteurs a un sens dans l'espace affine; le second point de vue seul permet la fondation de la Géométrie projective ou de la Géométrie conforme non holonomes, bien que la notion de vecteurs équipollents n'ait aucun sens dans l'espace projectif et que la notion elle-même de vecteur n'ait aucun sens dans l'espace conforme.

Pour définir par exemple un espace projectif non holonome (ou un espace à connexion projective) on imaginera, en chaque point d'un espace supposé initialement dénué de toute propriété géométrique, un espace projectif tangent, ainsi qu'une loi permettant le raccord en un seul des espaces projectifs tangents en deux points infiniment voisins. Cette loi permet alors le développement, sur l'espace projectif tangent en un point A, d'une ligne quelconque AB et de la région de l'espace donné avoisinant immédiatement cette ligne. Cette loi ne sera soumise à priori qu'aux restrictions habituelles en Géométrie différentielle (linéarité des composantes de la connexion projective par rapport aux différentielles des coordonées, existence de dérivées jusqu'à un certain ordre, etc.).

D'une manière générale, à tout groupe continu G correspond, dans la conception de M. Klein, une Géométrie holonome; dans la conception nouvelle, il lui correspond une infinité de Géométries non holonomes. La Géométrie des espaces de Riemann correspond au groupe des déplacements euclidiens, et *ce n'est même pas la plus générale de cette espèce,* car, un ds^2 étant donné, on peut imaginer une infinité de lois de parallélisme autres que celle de M. Levi-Civita; toutes sont également légitimes; nous verrons dans un instant ce qui différencie celle de M. Levi-Civita de toutes les autres. Les espaces de M. Weyl constituent de même une catégorie particulière des espaces non holonomes admettant pour groupe fondamental le groupe des déplacements et des similitudes; les espaces d'Eddington correspondent au groupe des transformations affines.

En résumé, dans les généralisations précédentes, l'idée directrice est la suivante. Dans un espace holonome, au sens de M. F. Klein, tout est commandé par le groupe fondamental et ses différents opérations. Ce sont ces opérations qui font de l'espace un tout organique. Dans les espaces non holonomes, ce sont encore les opérations du groupe qui sont un principe d'organisation, mais uniquement de proche en proche. C'est précisément en analysant ce que cette organisation a d'incomplet que nous allons arriver au rôle tout à fait nouveau que va jouer encore la notion de groupe dans les Géométries nouvelles.

IV

Prenons par exemple un espace de Riemann et considérons dans cet espace un contour fermé partant d'un point A. Développons de proche en proche, sur l'espace euclidien tangent en A, l'espace euclidien tangent aux différents points du contour. Le petit morceau d'espace qui entoure le point A prendra, suivant qu'on considère ce point comme point de départ ou point d'arrivée, deux positions différentes dans l'espace sur lequel se fait le développement, et on passera de la position finale à la position initiale par un certain déplacement euclidien, que nous dirons *associé* au contour fermé; c'est un déplacement, repétons-le, qui opère dans l'espace euclidien tangent en A; bien qu'il ait été défini par ses effets sur le point A et son voisinage, on peut évidemment l'appliquer à n'importe quelle figure (F) tracée dans l'espace euclidien tangent en A.

Considérons maintenant les différents contours fermés partant d'un point donné A. *Les différents déplacements euclidiens qui leur sont associés forment un groupe.*

Soient en effet deux contours fermés (C_1) et (C_2) partant de A. Soient D_1 et D_2 les déplacements qui leur sont associés; soit enfin (C) le contour fermé obtenu en décrivant successivement (C_1) et (C_2), et D le déplacement associé à (C). Une figure (F) tracée dans l'espace euclidien tangent en A prendra respectivement, après développement du contour (C_1) ou du contour (C_2), la position (F_1) ou la position (F_2); après développement du contour total (C), elle prendra une position (F') placée par rapport à (F_1) comme (F_2) est placée par rapport à (F); autrement dit le déplacement D qui amène (F') en (F) est la résultante du déplacement D_2 qui amène (F') en (F_1) et du déplacement D_1 qui amène (F_1) en (F). La relation

$$D = D_2 D_1$$

qui vient d'être obtenue montre bien que l'ensemble des déplacements associés aux contours fermés issus de A forme un groupe g.

Que se passerait-il si, au lieu du point A, on considérait au autre point A'? Imaginons qu'on relie ces deux points par un chemin arbitraire, mais donné ABA'; on peut raccorder de proche en proche, par ce chemin, l'espace euclidien tangent en A' à l'espace euclidien tangent en A. Dans cet espace euclidien unique il est facile de voir que le groupe g' associé à A' est *identique* au groupe g associé à A. Soit en effet (C) un contour fermé partant de A, et (C') le contour fermé $A'BA(C)ABA'$; soient respectivement D et D' les déplacements qui leur sont associés. Soit (F) une figure quelconque de l'espace euclidien tangent en A, (F_1) la position qu'elle prend après développement du contour (C). Les figures (F) et (F_1) peuvent être respectivement regardées comme résultant de deux figures (F') et (F'_1) de l'espace euclidien tangent en A' par le raccord fait le long du chemin $A'BA$. Par développement du contour fermé (C'), il est bien évident que la figure (F') vient en (F'_1); les deux déplacements D et D' sont donc identiques. A tout déplacement de g correspond donc un déplacement identique de g' et réciproquement.

En définitive, à l'espace de Riemann donné est associé un sous-groupe g déterminé du groupe G des déplacements euclidiens, sous-groupe qui peut se confondre avec le groupe G lui-même, mais qui peut aussi se réduire à la transformation identique; dans ce dernier cas il est bien évident que l'espace de Riemann est complètement holonome et ne diffère qu'en apparence de l'espace euclidien proprement dit. Il est naturel de donner au groupe g le nom de "groupe d'holonomie" de l'espace de Riemann.

Plus généralement, à tout espace non holonome de groupe fondamental G est associé un sous-groupe g de G qui est son groupe d'holonomie et qui ne se réduit à la transformation identique que si l'espace est parfaitement holonome.

Le groupe d'holonomie d'un espace mesure en quelque sorte le degré de non holonomie de cet espace, de même que le groupe de Galois d'une équation algébrique mesure en quelque sorte le degré d'irrationalité des racines de cette équation.

V

Avant d'indiquer les problèmes les plus intéressants que pose la notion du groupe d'holonomie, il ne sera pas inutile de faire une remarque relative aux transformations infinitésimales de ce groupe. Il est évident que parmi ces dernières se trouvent les transformations associées aux contours fermés *infiniment petits* (dans tous les sens) tracés dans l'espace non holonome donné. On peut démontrer rigoureusement que si toutes ces transformations étaient nulles, le groupe d'holonomie se réduirait à la transformation identique. Or les Géométries non holonomes les plus importantes dans les applications sont celles pour lesquelles les transformations infinitésimales associées aux contours fermés infiniment petits partant d'un point *laissent ce point invariant.* Comme je l'ai proposé, on peut convenir de dire que les espaces non holonomes correspondants sont *sans torsion.* Il en est ainsi des espaces de M. Weyl et des espaces d'Eddington. Il en est ainsi également des espaces de Riemann à parallélisme de Levi-Civita: on peut même caractériser complètement le parallélisme de M. Levi-Civita par *la condition de priver l'espace de toute torsion.*

On conçoit que l'absence de torsion ait sa répercussion sur la nature du groupe d'holonomie, ce groupe, dans le cas où il ne se réduit pas à la transformation identique, devant admettre une transformation infinitésimale non identique laissant invariant un point arbitraire.

VI

Je ne citerai que pour mémoire le problème de la détermination du groupe d'holonomie d'un espace non holonome donné à groupe fondamental G. Il peut être résolu complètement dès qu'on connaît tous les types de sous-groupes de G

Dans la théorie des équations algébriques, on sait qu'il existe toujours des équations algébriques admettant un groupe de Galois donné à l'avance. Il existe toujours d'une manière analogue des espaces non holonomes à groupe fondamental G admettant pour groupe d'holonomie un sous-groupe arbitrairement donné de G. Il existe par exemple des espaces à connexion euclidienne dont le groupe d'holonomie est le groupe des translations: mais ce ne sont pas des espaces de Riemann (à connexion de Levi-Civita). L'absence de torsion d'un espace de Riemann restreint en effet, comme nous l'avons dit plus haut, la nature des groupes d'holonomie possibles, et c'est un problème intéressant que de déterminer, pour chaque nombre de dimensions de l'espace, tous ces groupes d'holonomie. Je me contente d'indiquer la solution de ce problème pour $n=2$ et $n=3$. Les espaces de Riemann à deux dimensions qui ne sont pas holonomes ne peuvent admettre comme groupe d'holonomie que le groupe à trois paramètres de tous les déplacements. Quant aux espaces de Riemann à trois dimensions, le groupe d'holonomie peut être:

Soit le groupe à 6 paramètres de tous les déplacements (cas général);

Soit le groupe à 5 paramètres qui laisse invariante une direction isotrope fixe (ds^2 réductible à la forme $dz^2+2dxdy+H(x, y, z)dx^2$);

Soit le groupe à 3 paramètres qui laisse invariant un point fixe (ds^2 réductible à la forme $dz^2+z^2d\sigma^2$, où $d\sigma^2$ ne dépend que de deux variables x, y);

Soit le groupe à 3 paramètres qui laisse invariant un plan fixe ainsi que tous les plans parallèles (ds^2 réductible à la forme $dz^2+d\sigma^2$, ou, dans le cas où le plan est isotrope, $dz^2+2dxdy+H(x,z)dx^2$).

A côté des espaces de Riemann, deux autres catégories d'espaces non holonomes sont particulièrement intéressantes. Si au lieu de considérer un ds^2 donné, on considère une équation $ds^2=0$, il est possible d'une infinité de manières d'attribuer à l'espace une connexion conforme de manière que les lignes de longueur nulle de l'espace soient précisément les courbes définies par l'équation donnée. Parmi cette infinité de connexions conformes, il en est une privilégiée qui jouit de propriétés particulièrement simples et que j'ai proposé d'appeler normale; les espaces conformes non holonomes *normaux* jouent par rapport à la Géométrie conforme le rôle des espaces de Riemann avec parallélisme de Levi-Civita par rapport à la Géométrie euclidienne. En particulier les espaces non holonomes conformes normaux à trois dimensions sont caractérisés par la condition que la transformation conforme associée à un contour fermé infiniment petit partant d'un point A laisse invariant ce point, ainsi que toutes les directions qui en sont issues. On déduit facilement de là que les seuls groupes d'holonomie possibles des espaces conformes normaux à trois dimensions sont:

1. Le groupe conforme général à 10 paramètres;

2. Le sous-groupe invariant à 6 paramètres du groupe qui conserve une droite isotrope fixe; dans ce second cas l'équation qui donne les lignes de longueur nulle est réductible à la forme $dz^2+2dxdy+H(x,z)dx^2=0$, et l'on a, par une quadrature, un invariant intégral linéaire absolu des équations différentielles des lignes qui jouent le rôle des droites isotropes.

Une autre catégorie importante d'espaces non holonomes est liée à la considération des géodésiques d'un espace à connexion affine. Les équations différentielles de ces géodésiques sont d'une forme particulière, à savoir, pour $n=2$,

$$\frac{d^2y}{dx^2}=A+B\frac{dy}{dx}+C\left(\frac{dy}{dx}\right)^2+D\left(\frac{dy}{dx}\right)^3.$$

Cela posé, si on se donne à priori un système d'équations différentielles de cette forme, il existe une infinité de connexions projectives telles que l'espace projectif non holonome qu'elles définissent admette les courbes données pour géodésiques; mais, parmi toutes ces connexions projectives, il en est une privilégiée dite *normale.* Les espaces non holonomes projectifs normaux sont à un système différentiel donné de géodésiques ce que les espaces de Riemann sont à un ds^2 donné. Four $n=2$, ils sont caractérisés par la propriété que la transformation projective associée à un contour fermé infiniment petit partant d'un point A laisse invariant le point A, ainsi que toutes les droites issues de A. D'après cela, les seuls groupes d'holonomie possibles sont pour $n=2$:

1° Le groupe projectif général à 8 paramètres;

2° Le sous-groupe invariant à 6 paramètres du groupe qui laisse invariant un point fixe; dans ce cas l'équation différentielle des géodésiques, au lieu d'être de la forme générale indiquée ci-dessus, est réductible à la forme

$$\frac{d^2y}{dx^2}=A(x, y);$$

mais on peut, sans faire la réduction, obtenir par une quadrature un multiplicateur de Jacobi de cette équation.

Je citerai enfin, comme dernier exemple, le cas des espaces réels de Weyl à trois dimensions, en supposant le ds^2 défini positif. Si le groupe d' holonomie n'est pas un sous-groupe du groupe des déplacements, il est, soit le groupe de toutes les similitudes (cas général), soit le groupe des déplacements et des similitudes qui laissent invariante une direction fixe; dans ce cas les deux formes, quadratique et linéaire, qui définissent l'espace, sont:

$$ds^2=dz^2+H(x, y, z)\ (dx^2+dy^2),\quad \omega=-\frac{\partial \log H}{\partial z}dz.$$

VII

J'aborde une dernière question, extrêmement intéressante. On sait le rôle que joue la théorie des groupes comme principe de subordination dans les Géométries (holonomes) à groupe fondamental. La Géométrie élémentaire, par exemple, se subordonne à la Géométrie projective en ce sens que les propriétés euclidiennes d'une figure sont tout simplement les propriétés projectives de la figure plus complète formée par la figure donnée et le cercle imaginaire de l'infini; la Géométrie élémentaire est au fond un simple chapitre de la Géométrie projective, et cela tient à ce que le groupe fondamental de la première est un sous-groupe du groupe fondamental de la seconde. Il convient d'insister sur ce fait que l'espace projectif peut être, d'une infinité de manières différentes, regardé comme un espace métrique, car on peut y distinguer n'importe quelle conique non dégénérée qui sera susceptible d'y jouer le rôle du cercle imaginaire de l'infini, ou encore n'importe quelle quadrique non dégénérée (et alors on aura un espace non euclidien ou cayleyen).

D'une manière générale tout espace holonome à groupe fondamental G peut être regardé comme un espace holonome à groupe fondamental G' si G' est un sous-groupe de G.

Existe-t-il quelque chose d'analogue pour les espaces non holonomes? La réponse à cette question est facile et à peu près évidente:

Pour qu'un espace non holonome à groupe fondamental G puisse être regardé comme un espace non holonome à groupe fondamental G', il faut et il suffit que son groupe d'holonomie g soit G'ou un sous-groupe de G'.

En particulier un espace à connexion projective à trois dimensions ne peut qu'exceptionnellement être regardé comme un espace métrique; il faut et il suffit pour cela que son groupe d'holonomie laisse invariante soit une conique, auquel cas il sera en général un espace de H. Weyl, soit une quadrique.

Un espace de H. Weyl ne peut être regardé comme un espace de Riemann que si son groupe d'holonomie est le groupe des déplacements (sans homothétie) ou un de ses sous-groupes.

Sans vouloir multiplier les exemples, nous pouvons indiquer une application intéressante à la théorie de la relativité. Dans l'étude de l'Univers physique on peut porter son attention sur le côté projectif (défini par les trajectoires d'un point matériel abandonné à lui-même), ou sur le côté conforme (défini par les lois de la propagation de la lumière, lesquelles dépendent simplement d'une équation différentielle quadratique $ds^2=0$). Plaçons-nous d'abord au premier point de vue. Une première hypothèse est que le système différentiel qui définit les trajectoires mécaniques est de la forme particulière signalée plus haut, c'est-à-dire qu'elles peuvent être regardées comme les géodésiques d'un espace à 4 dimensions à connexion projective. La loi de la gravitation dans le vide d'Einstein peut alors s'exprimer ainsi: le groupe d'holonomie de l'Univers mécanique, considéré comme espace non holonome projectif normal à 4 dimensions, laisse invariante une quadrique (1[e] forme de la loi d'Einstein) ou une hyperquadrique (2[e] forme de la loi, avec constante cosmologique). Cela revient à dire, dans l'un et l'autre cas, que l'Univers est métrique, et sa métrique se déduit de la seule connaissance des trajectoires.

Si l'on se place au second point de vue, la seule connaissance des lois de propagation de la lumière, supposées définies par une équation de Monge quadratique, permet d'attribuer à l'Univers une connexion conforme normale bien déterminée. La loi de la gravitation dans le vide d'Einstein peut alors s'exprimer ainsi: le groupe d'holonomie de l'Univers optique, considéré comme espace non holonome conforme normal à 4 dimensions, laisse invariante une hypersphère de rayon nul (1^e forme de la loi d'Einstein), ou une hypersphère de rayon non nul (2^e forme de la loi avec constante cosmologique). Cela revient à dire, dans l'un et l'autre cas, que l'Univers est métrique, et sa métrique se déduit de la seule connaissance de la loi de propagation de la lumière.

Ajoutons enfin que les deux métriques d'Univers déduites, l'une des trajectoires mécaniques, l'autre des lois de propagation de la lumière, coincident.

VIII

Indiquons en terminant la relation qui existe entre la notion de groupe d'holonomie et la notion de classe d'un espace de Riemann. M. G. Ricci a désigné sous ce nom le plus petit entier k tel que l'espace de Riemann supposé à n dimensions puisse être réalisé par une variété convenablement choisie de l'espace euclidien à $n+k$ dimensions. M. J. A. Schouten a démontré que, dans certains cas très étendus, la classe était égale au nombre de paramètres dont dépend la position finale du corps de vecteurs issus d'un point A transporté par parallélisme le long d'un contour fermé partant de A. Il est à peu près évident que la classe n'est autre que l'ordre du groupe γ qui indique comment le groupe d'holonomie g de l'espace de Riemann transforme entre elles les directions (ou les points à l'infini). Il y aurait lieu de reprendre cette question et de voir si le théorème de M. J. A. Schouten est général, ou du moins dans quels cas il est vrai.

Citons encore pour mémoire le problème général de l'isomorphisme, holoédrique ou mériédrique, de deux espaces non holonomes à groupe fondamental donné.*

Signalons aussi des généralisations possibles obtenues en considérant des espaces (non holonomes) *non ponctuels*, par exemple engendrés par des *éléments* au sens de S. Lie. C'est ainsi qu'on peut, étant donnée une équation différentielle arbitraire,

$$\frac{d^2y}{dx^2}=f\left(x, y, \frac{dy}{dx}\right),$$

attribuer au plan (x, y), supposé initialement privé de toute propriété géométrique, une connexion projective telle que les géodésiques correspondantes soient les courbes intégrales de l'équation donnée: seulement l'élément générateur du plan ainsi doué d'une connexion projective est, non pas le point (x, y), mais l'élément $\left(x, y, \frac{dy}{dx}\right)$. On indiquerait facilement nombre d'autres problèmes d'Analyse susceptibles d'être géométrisés d'une manière analogue, et dans lesquels la théorie des groupes interviendrait aussi légitimement que dans les problèmes dont nous avons plus spécialement parlé.

Je ne puis enfin terminer sans signaler les remarquables recherches dans lesquelles M. H. Weyl a repris l'ancien problème philosophique de l'espace,

*Voir en particulier mon mémoire: *Les espaces à connexion conforme* (Ann. de la Soc. Polon. de Math., 1923, p. 171-221.)

traité autrefois par Helmholtz et Lie, pour l'adapter aux points de vue nouveaux introduits par la théorie de la relativité; la notion de groupe est, là encore, à la base de l'énoncé même du problème posé par M. H. Weyl. Mais je ne puis songer à entrer dans l'exposition, même sommaire, de cette importante question, qui exigerait à elle seule une conférence spéciale.

OUTLINE OF THE THEORY TO DATE OF THE ARITHMETICS OF ALGEBRAS

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Scope of the Lecture. Our purpose is to sketch in a broad way the leading features of the origin and development of a new branch of number theory which furnishes a fundamental generalization of the theory of algebraic numbers. Algebraic fields (Körper) are all very special cases of linear associative algebras, briefly called algebras. The integral quantities of any algebra will be so defined that they reduce to the classic integral algebraic numbers in the special case in which the algebra becomes an algebraic field.

For the sake of clearness, we shall not presuppose any acquaintance with the concept of algebras, but explain that concept and such of the results concerning the theory of algebras as are indispensable in the later discussion of the arithmetics of algebras.

Examples of Algebras. All complex numbers $x.1+yi$ form an algebra of order 2 with the basal units 1 and i. The quantities of which an algebra is composed may be numbers as in this example, or matrices as in the next example, or abstract elements.

A more typical algebra is that whose quantities are two-rowed square matrices

$$m=\begin{pmatrix}ab\\cd\end{pmatrix},\ \mu=\begin{pmatrix}\alpha\beta\\\gamma\delta\end{pmatrix},$$

etc., whose elements $a, b, \ldots$ are numbers of a specified kind, complex, or real, or rational. We define the sum and product of these two matrices to be

$$m+\mu=\begin{pmatrix}a+\alpha & b+\beta\\c+\gamma & d+\delta\end{pmatrix},\ m\mu=\begin{pmatrix}a\alpha+b\gamma & a\beta+b\delta\\c\alpha+d\gamma & c\beta+d\delta\end{pmatrix}.$$

Since the last matrix is altered in form by the interchange of the Roman and Greek letters, $m\mu$ is usually distinct from μm, so that multiplication of matrices is not always commutative. If k is any number, we call the matrix

$$\begin{pmatrix}ka & kb\\kc & kd\end{pmatrix}$$

the scalar product of the number k and the matrix m and denote it by km or mk. Consider the four special matrices

$$e=\begin{pmatrix}10\\00\end{pmatrix},\ f=\begin{pmatrix}01\\00\end{pmatrix},\ g=\begin{pmatrix}00\\10\end{pmatrix},\ h=\begin{pmatrix}00\\01\end{pmatrix}.$$

Then

$$m=\begin{pmatrix}a0\\00\end{pmatrix}+\dots+\begin{pmatrix}00\\0d\end{pmatrix}=ae+bf+cg+dh.$$

Hence e, f, g, h are basal units of our *total matric algebra* of order 4 of two-rowed square matrices.

In the definition of any algebra we employ three operations called addition, multiplication, and scalar multiplication, which are assumed to have properties entirely analogous to those holding for the foregoing three operations on matrices. This close relation between general algebras and matric algebras is explained by the theorem which states that any algebra of order n can be expressed concretely as an algebra of matrices with n or $n+1$ rows, although the latter is usually not the total matric algebra.

QUATERNIONS. The four special matrices

$$u=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ i=\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix},\ j=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ k=ij=\begin{pmatrix}0&\sqrt{-1}\\\sqrt{-1}&0\end{pmatrix}$$

satisfy the relations

$$i^2=j^2=k^2=-u,\ ij=k=-ji,\ ki=j=-ik,\ jk=i=-kj$$

and are the basal units of quaternions $xu+yi+zj+wk$. Since matrix u plays the rôle of unity in multiplication, it is usually denoted by 1.

Consider the algebra of all real quaternions

$$q=x+yi+zj+wk$$

with real coordinates x, y, z, w. Its *conjugate* is

$$q'=x-yi-zj-wk.$$

Each of the products qq', $q'q$ has the value

$$N=x^2+y^2+z^2+w^2,$$

which is called the *norm* of q. Let q be not zero, so that x, y, z, w are not all zero and hence $N\neq 0$. Then q evidently has the inverse

$$q^{-1}=\frac{1}{N}q',$$

which is a quaternion with the real coordinates $x/N, \dots, -w/N$. The equation $\xi q=r$ in real quaternions has the unique solution $\xi=rq^{-1}$, while the equation $q\eta=r$ has the unique solution $\eta=q^{-1}r$. Hence our algebra of all real quaternions is an example of a *division* algebra, in which the two kinds of division (except by zero) can always be performed uniquely.

The special quaternions $x+yi$ form an algebra of order 2 called a *sub-algebra* of the algebra of all quaternions. The $x+zj$ form another sub-algebra.

DEFINITIONS AND THEOREMS ON ALGEBRAS. A sub-algebra I of an algebra A is called *invariant* in A if the product taken in either order of every quantity of I and every quantity of A belongs to I. In case A has no invariant sub-algebra other than itself, A is called a *simple* algebra. It is known that every simple algebra can be expressed in a form such that its quantities are the matrices whose elements belong to a division algebra.

The square of the matrix $\begin{pmatrix} 01 \\ 00 \end{pmatrix}$ is the matrix zero all four of whose elements are zero. A quantity is called *nilpotent* if some power of it is zero. An algebra is called nilpotent if all of its quantities are nilpotent. A *semi-simple* algebra is one which has no nilpotent invariant sub-algebra.

An algebra A is said to be the *sum* of two sub-algebras B and C if every quantity of A can be expressed as a sum of a quantity of B and a quantity of C. If also the product in either order of every quantity of B and every quantity of C is zero, and if B and C have in common no quantity other than zero, then A is called the *direct sum* of B and C.

Every semi-simple algebra is either simple or is a direct sum of simple algebras, and conversely.

The principal theorem on algebras states that every algebra which is neither nilpotent nor semi-simple is the sum $N+S$ of its unique maximal nilpotent invariant sub-algebra N and a semi-simple sub-algebra S.

THE INTEGRAL QUATERNIONS OF LIPSCHITZ AND THOSE OF HURWITZ. In his book* of 1886, Lipschitz called a quaternion integral if and only if its four coordinates are ordinary integers. By very complicated discussions be obtained some interesting results. But his theory was not a real success, since his integral quaternions do not obey the essential laws of ordinary arithmetic. For example, there does not exist a greatest common left (or right) divisor of 2 and $q=1+i+j+k$, as shown by listing their divisors.

Hurwitz† overcame all such difficulties by developing a new, successful theory of the arithmetic of quaternions. Using postulates quoted below, he was led to define integral quaternions to be those whose four coordinates are either all ordinary integers or all halves of odd integers. He proved that the essential laws of ordinary arithmetic hold also for such integral quaternions. For example, there exists a greatest common left divisor of any two integral quaternions, that of 2 and the above q being 2 since q is the product of 2 by the integral quaternion $\frac{1}{2}q=\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$. Thus we do not now have the difficulty which we met under the definition by Lipschitz.

HURWITZ'S POSTULATES. Although Hurwitz stated his postulates only for the case of quaternions, it will prove convenient for later comparisons to formulate them for any rational algebra A whose quantities have rational coordinates

**Intersuchungen über die Summen von Quadraten*, Bonn, 1886; French translation in Jour. de Math., p. 4, t. 2, 1886, 393-439.

†Göttinger Nachrichten, 1896, 311-40. Amplified in his book, *Zahlentheorie der Quaternionen*, Berlin, 1919.

and obey the associative law of multiplication. We assume also that A has the modulus 1 which plays the role of unity in multiplication.

The integral quantities of A are defined to be the quantities belonging to a set of quantities satisfying the following four postulates:

C (closure): The sum, difference, and product of any two quantities of the set are also quantities of the set.

B (basis): The set has a finite basis (*i.e.*, it contains quantities $b_1, \ldots, b_k$ finite in number, such that every quantity of the set is a linear combination of the b's with ordinary integral coefficients).

U': The set contains 1 and the basal units of A.

M (maximal): The set is a maximal (*i.e.*, is not contained in a larger set having properties C, B, U').

Note that Lipschitz's integral quaternions with integral coordinates form a set having the properties C, B, U'. For example, they have the basis 1, i, j, k. This set is, however, not a maximal, being contained in the larger set of Hurwitz's integral quaternions. We saw that the latter maximal set has properties which are simpler and more desirable than those of the former non-maximal set. The superiority of a maximal set is illustrated also by the advantage of the set of all complex numbers over number systems containing only real numbers, or only positive real numbers, or only the primitive numbers 1, 2, 3, ...

Du Pasquier, a pupil of Hurwitz, published during the past fifteen years many papers* in which he replaced Hurwitz's postulate U' by the milder postulate U that the set contains 1. This replacement is an improvement, since all of the resulting postulates are invariant under every transformation of the basal units, while U' is evidently not invariant.

THE DEFINITIONS BY HURWITZ AND DU PASQUIER ARE UNSATISFACTORY. This fact will be illustrated for the special algebra having the two basal units 1 and e, where $e^2=0$. Under Du Pasquier's definition, any set of quantities with properties B and U has a basis of the form 1, $q=r+se$, where r and s are rational numbers and $s\neq 0$. Since q^2 must belong to the set by property C, and hence is equal to $a+bq$, where a and b are ordinary integers, we find that $r^2=a+br$, $2r=b$, whence $r^2=-a$. Thus r is an integer. We may therefore replace the initial basis 1, q by 1, $q-r=se$. Our set is evidently contained in the larger set with the basis 1, $\frac{1}{2}se$, which in turn is contained in the still larger set with the basis 1, $\frac{1}{4}se$, etc., where each such set has properties C, B, U. In other words, there does not exist a maximal set, so that the algebra does not possess integral quantities.

The same unfortunate conclusion results also from the definition by Hurwitz, which imposes the further condition that e shall belong to the set and hence that s be the reciprocal of an integer.

*Vierteljahrsschrift Naturf. Gesell. Zürich, 51 (1906), 55-129; 52 (1907), 243-8; 54 (1909), 116-48. L'Enseignement Math., 17 (1915), 340-3; 18 (1916), 201-60. Nouv. Ann. Math. (4), 18 (1918), 448-61. Bull. Soc. Math. France, 48 (1920), 109-32. Comptes Rendus du Congrès International des Mathématiciens, Strasbourg, 1920, 164-75.

Under the definition by either Hurwitz or Du Pasquier there exist no integral quantities in the great majority of algebras, in fact for any algebra which is not semi-simple.

THE NEW CONCEPTION OF INTEGRAL QUANTITIES. The lecturer has recently published* a satisfactory theory of the integral quantities of any rational algebra having a modulus 1. Let its basal units be $u_1, \ldots, u_n$. If $\xi_1, \ldots, \xi_n$ are variables ranging independently over all rational numbers, the quantity $q = \xi_1 u_1 + \ldots + \xi_n u_n$ is a root of a uniquely determined *rank equation* whose coefficients are polynominals in $\xi_1, \ldots, \xi_n$ with rational coefficients, the leading coefficient of the equation being unity, while q is not a root of an equation of smaller degree all of whose coefficients are such polynomials. For example, the quaternion $q = x + yi + zj + wk$ and its conjugate are roots of

$$\omega^2 - 2x\omega + (x^2 + y^2 + z^2 + w^2) = 0,$$

which is the rank equation of the algebra of rational quaternions if x, y, z, w are variables ranging independently over all real numbers.

The new definition of integral quantities employs postulates C, U, M and (in place of B).

R: For every quantity of the set, the coefficients of the rank equation are all ordinary integers.

As a first justification of this definition of the integral quantities of any rational algebra A having a modulus 1, note that when A is any algebraic field it is readily proved that its integral quantities coincide with the integral algebraic numbers of the field. In other words, the new theory is a direct generalization of the classic theory of algebraic numbers.

Second, when A is the algebra of rational quaternions, the new definition leads very simply to the desirable integral quaternions of Hurwitz.

Third, every algebra now has† integral quantities, whereas this was rarely true under the earlier definitions.

The final justification of the new conception of integral quantities of an algebra lies in the rich array of fundamental general theorems which have been developed under the new conception and will be summarized later on, whereas under the earlier conceptions no general theorem had been obtained.

OLD AND NEW CONCEPTIONS CONTRASTED IN AN EXAMPLE. We shall apply the new definition to the foregoing rational algebra having the basal units 1 and e, where $e^2 = 0$. For $x = a + be$, we evidently have $(x - a)^2 = 0$, which is the rank equation when a and b are variables ranging independently over all rational numbers. Its coefficients are ordinary integers if and only if a is an integer. Evidently the unique maximal set of quantities x having properties C, U, R is composed of all the $x = a + be$ in which a is an integer and b is a rational number.

Algebras and their Arithmetics, University of Chicago Press.

†Proved in the lecturer's paper, *Further development of the Theory of Arithmetics of Algebras*, these Proceedings.

These quantities x are therefore the integral quantities of the algebra. For any rational number k, the product of the integral quantities $u=1+ke$ and $1-ke$ is 1, whence each is called a *unit.* Let $a\neq 0$ and choose $k=-b/a$. Then $xu=a$. Such a product of x by a unit u is said to be *associated* with x. Associated quantities play equivalent roles in questions of divisibility. The integral quantities of our algebra are therefore associated with the ordinary integers a and may be replaced by the latter in questions of divisibility.

Contrast these simple and satisfactory results under the new conception with the unfortunate conclusion, under the conceptions of integral quantities held by Hurwitz and Du Pasquier, that this algebra has no integral quantities. Faced with the dilemma that no maximal set exists under his definition, Du Pasquier suggested that we omit the desirable postulate M, that the set be a maximal and hence define the integral quantities to be those of an arbitrarily chosen one of the infinitude of sets with the basal units 1 and se. But it has been definitely proved by the lecturer* that factorization into indecomposable integral quantities is then not unique and cannot be made unique by the introduction of ideals however defined. These insurmountable difficulties are in marked contrast with the simple conclusion, under the new conception, that the integral quantities are uniquely determined and are associated with ordinary integers.

It is worthy of notice that our set of integral quantities is the aggregate of the infinitude of non-maximal sets of Du Pasquier. Our satisfactory set may therefore be derived by a suitable enlargement of any one of his unsatisfactory sets. There are many instances in the history of mathematics where success has been achieved by the principle of enlargement; examples are the evolution of our number system, the introduction of ideals in the theory of algebraic numbers, and the enlargement of Lipschitz's unsatisfactory set of integral quaternions to Hurwitz's satisfactory set.

General theory of arithmetics of algebras. Let A be any rational associative algebra with a modulus 1. According to the principal theorem on algebras stated above, $A=S+N$, where N is the maximal nilpotent invariant sub-algebra of A, and S is a semi-simple sub-algebra. The fundamental theorem on arithmetics states that the arithmetic of A is associated with that of S in the sense that every integral quantity (whose determinant is not zero) of A is the product of an integral quantity of S by a unit. This theorem is illustrated by the foregoing example, in which N is composed of the components be and S is composed of the rational components a, so that the integral quantities of S are the ordinary integers. Another statement of this theorem is that in questions of divisibility we may suppress the bizarre nilpotent components belonging to N. This elimination of undesirable elements is fortunate both for the theory and for its applications.

We have therefore reduced the problem of the arithmetics of all algebras to that of semi-simple algebras S. We can further reduce the problem to the case of simple algebras. For, we saw that S is a direct sum of simple algebras

*Bull. Amer. Math. Soc., 28 (1922), 438-42; Jour. de Math. p. 9, t. 2 (1923), 281-326.

$S_1, S_2, \ldots$, so that each quantity σ of S is a sum of components $\sigma_1, \sigma_2, \ldots$, belonging to $S_1, S_2, \ldots$, respectively. It is an important theorem that if each σ_i is an integral quantity of S_i, then σ is one of S, and conversely. Moreover, the divisibility properties for S follow at once from those of the component algebras S_i.

We saw that the quantities of any simple algebra Σ can be expressed as matrices whose elements range independently over the same division algebra D. It can be proved that there is a unique set of integral quantities of Σ which contains* all matrices whose elements are ordinary integers, and that this set is composed of the matrices whose elements range independently over the integral quantities of D, and conversely.

Although we know the integral quantities of Σ as soon as we know those of D, it remains to deduce the divisibility properties of the former from the latter. This has been accomplished for the case of those division algebras D which have the property that its integral quantities possess a process of division yielding always a remainder whose norm is numerically less than the norm of the divisor. This property holds when D is the algebra of rational numbers, or one of numerous quadratic algebraic fields, or the algebra of rational quaternions, or certain algebras of generalized quaternions. For such a D, we have a theory of reduction and equivalence of matrices whose elements are integral quantities of D. The resulting theory is a direct generalization of the classic theory of matrices whose elements are ordinary integers, and then factorization into prime matrices is unique apart from unit factors. In our more general case, each matrix is a product of units and a matrix having only zeros outside the diagonal.

We have therefore reduced the study of arithmetics of all rational algebras to the case of simple algebras, *i.e.*, of the algebra of all matrices whose elements belong to a division algebra D, and have treated the arithmetic of the latter algebra when the integral quantities of D admit a process of division yielding a remainder of norm numerically less than that of the divisor. Such a process of division implies the existence of a right (or left) greatest common divisor. But the latter may exist even when that process of division is lacking†.

APPLICATIONS TO DIOPHANTINE EQUATIONS. The theory of algebraic numbers is applicable only to problems involving polynomials which contain only one variable or two variables homogeneously, so that the polynomial can be factored into linear functions. This serious limitation can often be removed by employing quantities of an algebra. For example, $N=x^2+y^2+z^2+w^2$ has as factors the quaternion $x+yi+zj+wk$ and its conjugate. By using integral quaternions we readily find all integral solutions of $N=uv$. Taking u as the sum and v as the difference of two unknowns, we deduce all ways of finding five integers the sum of whose squares is a square.

*If we omit this assumption, we find many sets of integral quantities. For example, let D be composed of the rational numbers. Every set of integral quantities of the resulting total rational matric algebra Σ can be transformed into the set of matrices whose elements are ordinary integers by a suitably chosen matrix with rational elements.

†Dickson, Amer. Jour. Math., 1923.

Various other Diophantine equations have been solved completely in integers for the first time by using the integral quantities of certain algebras of generalized quaternions.

Since the new theory of arithmetics of algebras solves completely certain types of Diophantine equations in any number of variables which were not solvable by any earlier method, it furnishes us with an effective new tool for the theory of numbers.

CONCLUSION. We have given a brief outline of the theory to date of arithmetics of algebras. This new branch of the theory of numbers is a far reaching generalization of the classic theory of integral algebraic numbers.

We have also made it clear why it was necessary to discard the earlier conceptions of the integral quantities of a general algebra and introduce a new conception of them.

The gradual enlargement of the conception of number from the primitive numbers used in counting to the system of all complex numbers and finally to its culmination in hypercomplex numbers (or quantities of any algebra) has its parallel in the growth of the concept integer, which was first restricted to the counting numbers, was greatly enriched in the last century by the study of integral algebraic numbers, and now finds its culmination in the integral quantities of any algebra.

CONSIDÉRATIONS SUR UNE ÉQUATION AUX DÉRIVÉES PARTIELLES DE LA PHYSIQUE MATHÉMATIQUE

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Parmi les équations aux dérivées partielles que la Physique mathématique propose à notre étude, l'une des plus simples et des plus intéressantes est celle de la propagation du Son:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=0.$$

Cette équation se retrouve non seulement dans l'étude du son, mais aussi dans celle des phénomènes électromagnétiques, et, en général, dans tous les cas de propagation par ondes en milieu homogène et isotrope.

L'intérêt spécial qui s'y attache tient, d'une part, à l'étendue de ses applications physiques, et, d'autre part, au fait que les théories mathématiques générales appliquées à cette équation donnent des résultats immédiatement représentables par des faits physiques concrets.

Les plus grands mathématiciens s'en sont occupés. Je citerai seulement les noms d'Euler, Poisson, Cauchy, Riemann, Volterra, Hadamard. Il y a sans doute peu de chose à ajouter aux résultats qu'ils ont obtenus. Je n'ai donc pas la prétention de vous exposer une théorie nouvelle, mais seulement d'analyser et de grouper les résultats déjà établis et de les rapprocher de certaines théories de la physique moderne.

Je m'excuse de rappeler d'abord quelques résultats relativement récents de la théorie des équations linéaires aux dérivées partielles.

Le premier problème qui se pose, en général, dans l'étude d'une équation ou d'un système d'équations aux dérivées partielles consiste à déterminer l'intégrale satisfaisant à des conditions données par les valeurs initiales ou par des conditions aux limites. Ce problème soulève des questions analytiques du plus haut intérêt et il semble que les plus grands progrès réalisés dans l'analyse moderne s'y rattachent d'une manière plus ou moins directe. Ainsi se trouve justifiée cette pensée souvent cités de Fourier: «L'étude approfondie des lois de la Nature est la source la plus féconde des découvertes mathématiques».

A ce premier problème, on peut en rattacher un autre non moins important: Reconnaitre les propriétés fondamentales des fonctions ainsi définies, déterminer leurs groupements naturels et trouver les transformations qui les relient entre elles.

Ce problème est du domaine de la théorie générale des groupes de transformations, étendue aux fonctions d'ensembles infinis, de lignes, de surfaces, ou d'une infinité discrète de variables indépendantes.

Ici encore, nous trouvons une synthèse des théories les plus nouvelles et les plus élevées de l'analyse.

Les caractéristiques et les ondes. Dans l'étude approfondie de tous les problèmes relatifs aux équations aux dérivées partielles, on rencontre un élément fondamental dont l'importance apparait de jour en jour plus grande: c'est l'ensemble des caractéristiques.

Les caractéristiques ont éte d'abord considérées par Monge, mais c'est Darboux qui semble en avoir le premier, pour les équations à deux variables indépendantes, mis en évidence les propriétés essentielles en analyse et en géométrie. M. Hadamard a éte ensuite conduit, par ses recherches sur la propagation des ondes, à reconnaitre le rôle essentiel des caractéristiques dans certains problèmes de physique mathématique.

Les caractéristiques se présentent normalement dans la discussion du problème de Cauchy pour les équations aux dérivées partielles. Dans les cas ordinaires, les intégrales se trouvent déterminées par leurs valeurs et celles de quelques unes de leurs dérivées sur une multiplicité d'un certain nombre de dimensions. Mais il y a des multiplicités pour lesquelles le problème est toujours impossible ou indéterminé: ce sont les multiplicités caractéristiques.

Dans le cas de l'équation des ondes, les multiplicités caractéristiques sont à trois dimensions, et chacune peut être définie par une relation entre les quatre variables x, y, z, t.

Soit, par exemple:

$$\phi(x, y, z, t)=0 \tag{2}$$

l'équation d'une multiplicité caractéristique; la fonction devra vérifier l'équation aux dérivées partielles suivante,

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2=0 \tag{3}$$

sur toute la multiplicité considérée.

Si l'on considère un ensemble de multiplicités caractéristiques dépendant au moins d'une constante arbitraire et définies par une équation

$$\phi(x, y, z, t)=\text{constante arbitraire}, \tag{4}$$

la fonction ϕ devra satisfaire identiquement à l'équation (3).

La détermination des caractéristiques se ramène donc à l'intégration d'une équation aux dérivées partielles du premier ordre.

Cette équation admet une intégrale complète définie par la relation suivante dépendant de quatre constantes,

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2-c^2(t-t_0)^2=0. \tag{5}$$

C'est l'intégrale à point singulier de M. Darboux ou le conoïde caractéristique de M. Hadamard.

Or, il suffit d'un simple coup d'œil pour reconnaitre, dans cette caractéristique à point singulier l'onde sphérique émise par le point fixe x_0, y_0, z_0 à l'époque t_0. Elle représente une sphère ayant pour centre le point considéré et dont le rayon croît proportionnellement à la différence $t-t_0$.

Une autre caractéristique est représentée par l'équation

$$\alpha x+\beta y+\gamma z-ct=\text{const.}$$

où l'on suppose α, β, γ liés par la relation

$$\alpha^2+\beta^2+\gamma^2=1.$$

Elle définit un plan qui se meut avec la vitesse constante c en restant parallèle à une direction fixe. C'est l'onde plane.

Les deux intégrales que nous venons d'envisager constituent des intégrales complètes de l'équation aux dérivées partielles des caractéristiques. Toutes les intégrales peuvent se déduire de l'une ou l'autre d'entre elles par l'application de la méthode des enveloppes. C'est une propriété bien connue des équations aux dérivées partielles du premier ordre.

Les bicaractéristiques et les rayons. On sait quel'intégration d'une équation aux dérivées partielles du premier ordre conduit à la considération d'un système d'équations différentielles ordinaires dont la solution définit un ensemble de valeurs des variables considérées en fonction d'un seul paramètre. J'ai donné à ces ensembles le nom de lignes caractéristiques. M. Hadamard les appelle les bicaractéristiques et les rattache à la considération d'un fait physique: le rayon de propagation.

Considérons le cas général d'une équation aux dérivées partielles de la forme suivante à quatre variables x_1, x_2, x_3, x_4:

$$D(\phi)=\Sigma A_{ik}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_k}=0, \qquad (A_{ik}=A_{ki}), \tag{6}$$

où l'on suppose que les coefficients dépendent seulement des variables x_i. Les équations différentielles des bicaractéristiques sont

$$\frac{dx_1}{\dfrac{\partial D}{\partial\left(\dfrac{\partial\phi}{\partial x_1}\right)}}=\frac{dx_2}{\dfrac{\partial D}{\partial\left(\dfrac{\partial\phi}{\partial x_2}\right)}}=\ldots=\frac{-d\left(\dfrac{\partial\phi}{\partial x_k}\right)}{\dfrac{\partial D}{\partial x_k}}. \tag{7}$$

Dans le cas particulier de l'équation (3) et de l'intégrale à point singulier (5), elles donnent

$$\frac{dx}{x-x_0}=\frac{dy}{y-y_0}=\frac{dz}{z-z_0}=\frac{dt}{t-t_0}. \tag{8}$$

Ce sont des droites parcourues d'un mouvement rectiligne et uniforme. En tenant compte de l'équation (5) on trouve que la vitesse de ce parcours est égale à C. Quelle que soit d'ailleurs la caractéristique dont on part, on trouve toujours le même ensemble de bicaractéristiques.

Ce résultat devient évident si l'on rapproche les équations différentielles (6) des équations canoniques hamiltoniennes de la mécanique.

Désignons par du la valeur commune des rapports (6), u étant une nouvelle variable indépendante, et posons:

$$\frac{dx_i}{du} = x_i'.$$

Soit Δ le discriminant de la forme quadratique (6); construisons la forme quadratique adjointe

$$(9) \qquad 2T = -\frac{1}{\Delta} \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} & x_1' \\ A_{21} & A_{22} & A_{23} & A_{24} & x_2' \\ A_{31} & A_{32} & A_{33} & A_{34} & x_3' \\ A_{41} & A_{42} & A_{43} & A_{44} & x_4' \\ x_1' & x_2' & x_3' & x_4' & 0 \end{vmatrix}.$$

Les bicaractéristiques cherchées seront définies, indépendamment de la caractéristique ϕ par les équations de Lagrange

$$(10) \qquad \frac{d}{du}\left(\frac{\partial T}{\partial x_i'}\right) - \left(\frac{\partial T}{\partial x_i}\right) = 0.$$

Les rayons de propagation se présentent ainsi comme les géodésiques d'une forme quadratique de différentielles; mais la condition $D(\phi)=0$ entraîne aussi $T=0$.

Ce sont des géodésiques de longueur généralisée nulle.

Les très belles recherches de M. Hadamard contiennent des remarques du plus haut intérêt sur les propriétés géométriques et physiques qui se rattachent à la considération des bicaractéristiques.

Les multiplicités caractéristiques sont engendrées par les ensembles de bicaractéristiques, groupées entre elles de manière à admettre une enveloppe.

En particulier, les ensembles convergents engendrent les caractéristiques à point singulier (ondes émanées d'un centre fixe) et les ensembles parallèles engendrent les ondes planes.

Singularités des intégrales analytiques. Les multiplicités caractéristiques se rencontrent encore lorsqu'on cherche les ensembles de singularités accidentelles des intégrales analytiques. J'appelle singularités accidentelles celles qui dépendent des conditions initiales, celles qui ne sont pas déterminées uniquement par la forme analytique des coefficients de l'équation.

Les ensembles de points singuliers accidentels des intégrales analytiques constituent des multiplicités caractéristiques.

Par exemple, une discontinuité instantanée qui se produit en un point se propage dans le milieu suivant les ondes représentées par les caractéristiques à point singulier. Si le milieu est homogène et isotrope, on retrouve les ondes sphériques dont nous avons déjà parlé.

Lorsqu'on veut étudier non seulement la nature des multiplicités singulières, mais encore la forme des singularités, on est amené, avec M. Hadamard à considérer les fonctions analogues à celle qui figure au premier membre de l'équation (5).

Posons

$$\phi = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 - c^2(t-t_0)^2.$$

Cette fonction, égalée à zéro, définit une multiplicité caractéristique. Elle satisfait à l'équation aux dérivées partielles

$$D(\phi) = 4\phi \tag{11}$$

où l'on a posé

$$D(\phi) = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 - c^2\left(\frac{\partial\phi}{\partial t}\right)^2.$$

M. Hadamard, par une généralisation naturelle, a été amené à considérer, pour les équations les plus générales, les fonctions qui satisfont à une équation de la forme (11).

Si nous posons $\phi = u^2$ nous trouvons pour définir la fonction u l'équation

$$D(u) = 1 \tag{12}$$

analogue à celle qui intervient dans les recherches sur les transformations des formes quadratiques de différentielles. Nous retrouverons ces fonctions en étudiant les intégrales à pôle mobile.

Du rôle des caractéristiques dans l'intégration des équations linéaires. Dans la détermination des intégrales satisfaisant à des conditions initiales données, nous sommes encore amenés à faire intervenir les caractéristiques.

La caractéristique à point singulier joue naturellement un rôle spécial, et c'est celle qui fournit immédiatement les résultats les plus simples. Les recherches de MM. Hadamard, Volterra et d'Adhémar se rapportent à l'emploi de cette caractéristique et constituent une généralisation de la méthode de Riemann et de Darboux pour les équations à deux variables indépendantes. Mais il est possible de généraliser ces résultats.

La connaissance d'une seule intégrale complète de l'équation aux dérivées partielles des caractéristiques permet d'obtenir, par le procédé des enveloppes toutes les caractéristiques de l'équation. Il était donc naturel de rechercher les résultats que pouvait fournir pour l'intégration de l'équation aux dérivées partielles linéaires du second ordre, la connaissance d'une intégrale complète des caractéristiques.

Je me suis occupé de cette question dans des mémoires déjà anciens. Je me permets de rappeler quelques-uns des résultats.

Soit $\phi(x, y, z, t, \alpha, \beta, \gamma) = 0$ une équation définissant une intégrale complète des caractéristiques, α, β, γ étant des constantes arbitraires.

Regardant α, β, γ comme des coordonnées et x, y, z, t comme des paramètres, l'équation considérée définit une surface variant avec les paramètres

x, y, z, t. Cette surface variable, ou une portion de la surface, complétée s'il y a lieu par d'autres surfaces fixes, limite un domaine Δ des variables α, β, γ.

Soit $u(x, y, z, t, \alpha, \beta, \gamma)$ une intégrale de l'équation aux dérivées partielles du second ordre considérée dépendant des paramètres α, β, γ et satisfaisant sur la caractéristique ϕ à une certaine condition aux limites que j'ai définie dans un mémoire du Journal de Mathématiques (1900).

L'intégrale triple suivante, où $f(\alpha\beta\gamma)$ désigne une fonction arbitraire

$$V=\iiint f(\alpha, \beta, \gamma)u(x, y, z, t, \alpha, \beta, \gamma)d\alpha d\beta d\gamma,$$

étendue au domaine Δ, représente une nouvelle solution de l'équation aux dérivées partielles proposée.

On peut également envisager des intégrales simples ou doubles étendues de la même façon à des domaines à limites variables.

Lorsqu'il s'agit de déterminer la fonction arbitraire $f(\alpha, \beta, \gamma)$ de manière à satisfaire à des conditions initiales données, on a à résoudre un nouveau type d'équations intégrales à limites variables, généralisant, pour les intégrales multiples les équations ordinaires à intégrales simples que j'ai rencontrées à l'occasion d'une étude sur les questions à deux variables indépendantes.

L'étude de ces équations intégrales parait susceptible d'offrir aux mathématiciens un sujet de recherches extrêmement vaste et que je soupçonne très fécond. Parmi les résultats que j'ai obtenus, je signale la généralisation curieuse de la dérivée par un résidu d'intégrale multiple permettant de passer d'une fonction de point à une fonction de droite ou de plan, ou inversement.

Signification physique du résultat. La signification physique des résultats précédents est immédiate. Les caractéristiques définissent en quelque sorte les caractères géométriques de l'onde. L'intégrale $u(x, y, z, t, \alpha, \beta, \gamma)$ correspond à la valeur ou à l'amplitude de l'ondulation, aux différents instants et aux différents points de l'onde considérée. Représenter une intégrale par une somme d'autres intégrales relatives à une famille de caractéristiques, c'est considérer l'amplitude correspondante comme la résultante d'ondulations relatives aux formes d'ondes représentées par ces caractéristiques. Par exemple, on peut envisager une onde sphérique comme résultant de la superposition d'une infinité d'ondes planes infiniment petites ou inversement.

Une intégrale complète de l'équation des caractéristiques représente une famille complète d'ondes, susceptible de fournir par superposition les différentes ondulations correspondant à l'équation aux dérivées partielles considérées.

Application de la théorie des caractéristiques au cas d'une onde engendrée par une source ponctuelle qui se déplace, par rapport aux axes de coordonnées, d'un mouvement rectiligne et uniforme.

1° *Caractéristique à ligne singulière.* L'application de la théorie des caractéristiques au cas d'une onde engendrée par une source ponctuelle qui se déplace, par rapport aux axes de coordonnées, d'un mouvement rectiligne et uniforme, fournit des résultats qui m'ont parus très intéressants parce qu'ils fournissent une interprétation physique immédiate et extrêmement simple des formules de la théorie de la relativité.

Dans le cas d'une source fixe, l'onde est représentée par l'intégrale à point singulier. Dans le cas d'une source ponctuelle mobile, nous aurons à considérer une intégrale à ligne singulière. Mais cette intégrale s'obtiendra normalement conformément à la théorie générale des équations aux dérivées partielles, par l'application de la théorie des enveloppes à l'intégrale à point singulier.

Appliquons cette méthode au cas du mouvement rectiligne uniforme.

Je suppose que la source décrit l'axe ox. Soient

$$x_0 = vt_0, \quad y_0 = 0, \quad z_0 = 0$$

les coordonnées de la source.

Je considère la fonction

$$\phi = (x - vt_0)^2 + y^2 + z^2 - c^2(t - t_0)^2$$

satisfaisant à l'équation (11).

Égalons à zéro la dérivée par rapport à t_0

$$0 = -v(x - vt_0) + c^2(t - t_0). \tag{13}$$

L'élimination de t_0 entre cette équation et la précédente donne

$$\phi = \frac{(x - vt)^2}{1 - \frac{v^2}{c^2}} + y^2 + z^2.$$

La caractéristique singulière est imaginaire lorsque l'on a $v < c$.

Les équations $\phi = \text{const.}$ définissent des ellipsoïdes aplatis dans le rapport de Lorentz, et qui se déplacent d'un mouvement de translation uniforme avec la vitesse v.

Pour $v = 0$ on a des sphères concentriques.

Ce résultat m'a paru particulièrement intéressant par sa connexion avec la théorie de la relativité.

En cherchant la valeur de t_0 qui annule l'équation dérivée (13), on obtient un résultat qui rappelle encore dans une certaine mesure les formules de Lorentz.

On a

$$t_0 = \frac{t - \frac{vx}{c^2}}{1 - \frac{v^2}{c^2}}.$$

Les formules de transformation de Lorentz donnent

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

On a donc

$$t' = t_0 \sqrt{1 - \frac{v^2}{c^2}}.$$

On s'explique ainsi pourquoi les formules de Lorentz s'introduisent naturellement dans l'étude des phénomènes ondulatoires engendrés par des sources mobiles, lorsque ces ondulations sont régies par l'équation aux dérivées partielles de la propagation du son. Voigt y avait été conduit par ses propres recherches bien avant qu'eût été proposée en physique l'hypothèse de FitzGerald et Lorentz.

Intégration. L'intégration de l'équation pour le cas d'un pôle mobile va mettre en évidence le rôle extrèmement important de la fonction ϕ.

Cette intégration peut s'effectuer de la manière la plus simple par la transformation de Lorentz qui ramène le problème de la source mobile au cas du problème de la source fixe. Il faudra, naturellement, pour l'interprétation des résultats, revenir aux variables primitives. La transformation de Lorentz dans ce problème doit être considérée comme une simple transformation de variables analytiques.

En vue de généralisations possibles, il n'est pourtant pas inutile d'examiner comment le problème se présente au point de vue de la théorie des équations aux dérivées partielles, indépendamment de toute question de transformation qui conserve la forme de l'équation.

Nous déterminons d'abord la fonction ϕ satisfaisant à l'équation (11) ou la fonction $\sqrt{\phi}$ vérifiant l'équation (12) et admettant une ligne singulière définie par le mouvement de la source mobile.

On cherchera ensuite une intégrale de la forme

$$V = \frac{U}{\sqrt{\phi}}$$

où U désigne une fonction régulière dans le voisinage de la ligne singulière et présentant, dans le voisinage de chaque point de cette ligne, les caractères de symétrie analogues à ceux que nous allons trouver dans le cas du mouvement rectiligne et uniforme.

Dans le cas d'une source fixe située à l'origine des coordonnées, les intégrales qui représentent des ondes isotropes issues de cette source s'obtiennent par la méthode de Poisson en posant

$$r = \sqrt{x^2 + y^2 + z^2}$$

et en cherchant les solutions de l'équation des ondes qui dépendent seulement de r et de t.

On est ainsi conduit à l'équation suivante:

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{dV}{\partial r} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0,$$

qui se ramène à la forme

$$\frac{\partial^2 (rV)}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 (rV)}{\partial t^2} = 0,$$

et donne

$$V = \frac{f(r-ct)+\phi(r+ct)}{r},$$

f et ϕ désignant des fonctions arbitraires.

Dans le cas de la source mobile, décrivant Ox d'un mouvement rectiligne et uniforme, nous posons de même

$$r = \sqrt{\frac{(x-vt)^2}{1-\frac{v^2}{c^2}}+y^2+z^2},$$

$$u = \frac{ct-\frac{vx}{c}}{\sqrt{1-\frac{v^2}{c^2}}},$$

et nous cherchons les solutions qui dépendent seulement de r et de u.

Nous avons l'intégrale à deux fonctions arbitraires f_1 et f_2:

$$V = \frac{f_1(r-u)+f_2(r+u)}{r}.$$

Les fonctions arbitraires considérées dépendent des deux arguments caractéristiques $r-u$ et $r+u$ qui correspondent à deux séries d'ondes: les ondes positives $r-u=\text{const.}$ qui se dilatent en s'écartant du pôle, et les ondes négatives $r+u=\text{const.}$ qui se contractent en se rapprochant de ce point. Les unes correspondent à une sorte d'émission et les autres à une sorte d'absorption. Le même pôle peut correspondre à la fois à une émission et à une absorption, mais on peut concevoir aussi des pôles uniquement positifs et des pôles uniquement négatifs. L'emploi de ces termes ne suppose, bien entendu, aucune corrélation nécessaire avec les phénomènes électriques du même nom.

Lorsque les fonctions arbitraires f_1 et f_2 sont des fonctions périodiques ou des sommes de fonctions périodiques, la valeur moyenne de l'intégrale V est inversement proportionnelle r.

Je donne à r le nom de module de l'intégrale.

Interférence de deux ondes concentriques de signes contraires.

Considérons un *doublet* constitué par la juxtaposition d'un pôle positif et d'un pôle négatif correspondant à des ondes périodiques égales.

Soit, par exemple:

$$V = \frac{A}{r}\left\{\sin\frac{2\pi(r-u+\alpha)}{\lambda}+\sin\frac{2\pi(r+u+\beta)}{\lambda}\right\},$$

où A, α, β, λ désignent des constantes.

Cette intégrale peut être mise sous la forme suivante:

$$V=\frac{2A}{r}\sin\frac{2\pi\left(r+\dfrac{\alpha+\beta}{2}\right)}{\lambda}\cos\frac{2\pi\left(u+\dfrac{\beta-\alpha}{2}\right)}{\lambda}.$$

Elle s'annule identiquement, quel que soit t pour

$$r=\frac{n\lambda}{2}-\frac{\alpha+\beta}{2}, \tag{13}$$

n désignant un nombre entier quelconque.

Les lieux géométriques définis par cette équation sont des ellipsoïdes aplatis, entraînés avec la source dans un mouvement de translation uniforme, et déjà rencontrés à propos des caractéristiques à ligne singulière.

Les ellipsoïdes correspondant à l'équation (13) constituent des nappes d'interférence de l'onde double définie par les valeurs de V. Ces nappes sont stationnaires dans un système d'axes de directions fixes entraînés avec la source.

C'est pourquoi j'ai désigné, en général, sous le nom d'ellipsoïdes d'interférence les ellipsoïdes définis par les valeurs constantes du module.

Quant au paramètre u, je l'ai désigné sous le nom de *paramètre de rayonnement* dans mon mémoire sur la Relativité restreinte et la géométrie des systèmes ondulatoires.

Ondes d'interférence et ondes de progression. Les multiplicités $r\pm u=$const. sont des variétés caractéristiques. Si, dans une équation de cette forme, on suppose $u=$const., on a l'équation d'un ellipsoïde; si, au contraire, on y suppose $t=$const. on a l'équation d'une sphère. On peut regarder ainsi les mêmes ondes comme progressant par ellipsoïdes concentriques entraînés avec la source ou par sphères ayant pour centres les positions successives de la source. Je donne aux ondes ellipsoïdales le nom d'*ondes d'interférence,* et aux ondes sphériques celui d'*ondes de progression.*

La considération des deux séries d'ondes s'impose. Mais l'existence du dénominateur r dans l'expression de V indique l'importance spéciale qui s'attache aux ondes d'interférence.

Je ne puis, malheureusement, entrer dans l'étude détaillée des propriétés géométriques qui interviennent dans l'étude des ondes. Je dois renvoyer pour ce sujet à mon mémoire déjà cité.

Il y a cependant des propriétés essentielles que je dois signaler.

Observation sur la valeur du module. L'expression du module donne lieu à une première observation très importante.

La valeur du module en un point M de l'espace, à une époque t ne dépend que de la position de la source par rapport au point M au même instant. Tout se passe exactement comme si la source était fixe, sauf toutefois que l'onde est aplatie au lieu d'être sphérique. Il est impossible de reconnaitre si cet aplatissement provient d'une anisotropie du milieu ou simplement d'un déplacement de la source dans un milieu isotrope.

L'étude des phénomènes d'interférence et de réflexion conduit aux mêmes conséquences. J'en indiquerai tout à l'heure les résultats principaux. J'observe dès maintenant, toutefois, que la distinction entre l'onde d'interférence et l'onde de progression pourrait servir à déceler le mouvement de la source. Mais cette distinction ne pourrait être faite que par l'observation d'un phénomène extérieur au système entraîné, c'est-à-dire par la détermination du temps sidéral, ou, plus exactement, du temps canonique de la gravitation.

Interférence et réflexion. Considérons un système d'ondes planes. Si le milieu est isotrope par rapport à un système d'axes $Oxyz$, les *rayons*, c'est-à-dire les bicaractéristiques sont dirigés suivant des droites perpendiculaires aux plans d'onde.

Si maintenant; nous rapportons la même ondulation à un autre système O, x, y, z, en translation euclidienne par rapport au premier, les plans d'onde relatifs au second système ne sont pas parallèles aux plans du premier. La direction des rayons stationnaires dans le second système changera également. Il existe toutefois une relation extrêmement simple entre les plans d'onde et les rayons stationnaires rapportés au système mobile O, x, y, z.

Si l'on imaginait une source entraînée dans le mouvement de ce système, il correspondrait à cette source une famille d'ellipsoïdes d'interférence. La relation entre la direction du rayon et celle du plan d'onde peut se traduire ainsi:

La direction du rayon est conjuguée de celle du plan d'onde par rapport à l'ellipsoïde d'interférence.

L'expression de direction conjuguée est entendue ici dans le sens qu'on donne à ce mot dans la théorie des surfaces du second degré. Elle correspond, en fait, à l'expression de *conormale* introduite par M. d'Adhémar dans la théorie des équations linéaires aux dérivées partielles.

Prenons maintenant deux trains d'ondes planes périodiques de directions différentes. Si ces deux trains ont la même période dans l'un des systèmes d'axes, ils auront en général des périodes différentes dans le second. S'ils peuvent interférer dans l'un, ils ne pourront pas interférer dans l'autre. S'ils interfèrent dans le système O, x, y, z à module isotrope, il y aura une série de nappes planes d'interférence, et les plans de ces nappes, parallèles entre eux sont parallèles à l'une des directions de plans bissecteurs des plans d'ondes.

S'ils interfèrent dans le système O, x, y, z, à module ellipsoïdal, les plans des nappes d'interférence sont encore parallèles entre eux, mais ils ne sont plus parallèles à un plan bissecteur. La direction des nappes peut être définie de la manière suivante:

Par l'intersection de deux plans d'ondes, on peut mener deux plans conjugués harmoniques, par rapport aux deux premiers et conjugués en direction par rapport à l'ellipsoïde d'interférence. Les nappes d'interférence sont parallèles à l'un de ces plans conjugués.

Les lois de la réflexion plane se déduisent de l'interférence.

La réflexion sur une surface plane est toujours accompagnée de phénomènes dynamiques plus ou moins intenses dépendant de la nature matérielle de cette surface. Dans la mesure où l'on peut négliger ces phénomènes, l'onde incidente et l'onde réfléchie admettent pour plan d'interférence le plan de réflexion. Cette propriété détermine l'une quand on connait l'autre.

En tenant compte des conditions d'interférence que nous avons établies, on trouve, pour la relation entre le rayon incident et le rayon réfléchi la propriété suivante:

Menons au point d'incidence la droite conjuguée en direction du plan de réflexion par rapport à l'ellipsoïde d'interférence (pseudo-normale ou conormale). Le rayon incident, le rayon réfléchi et la pseudo-normale sont situés dans un même plan, et les deux rayons sont conjugués harmoniques par rapport à la pseudo-normale et au plan de réflexion.

Si l'on considère un système entraîné dans un mouvement de translation et une source liée à ce système, les phénomènes observés d'interférence et de réflexion seront semblables à ceux qui seraient observés dans un système fixe, sauf la variation résultant de l'aplatissement de l'onde entraînée. Mais si l'on ne sait pas *a priori* que le milieu dans lequel se déplace le système est rigoureusement homogène et isotrope, les phénomènes considérés ne permettent de rien conclure en ce qui concerne le déplacement du système.

L'extension de la méthode au cas des milieux hétérogènes peut s'effectuer en suivant la voie indiquée par M. Hadamard, à condition que la propagation de l'onde soit représentée par les solutions d'une équation linéaire aux dérivées partielles du second ordre à quatre variables indépendantes.

Dans un domaine suffisamment restreint, les résultats précédents restent applicables en première approximation.

Observation sur la théorie de la gravitation. Parmi les intégrales de l'équation des ondes dans le cas d'un pôle mobile, il y en a une de la forme

$$V=\frac{A}{r}$$

où r désigne le module elliptique. Si l'on considère à la fois plusieurs sources animées de vitesses quelconques, on aura une intégrale de la forme

$$V=\sum\frac{A_i}{r_i}.$$

Cette expression donne lieu à une remarque très importante: la valeur de la fonction V en un point M à un instant donné ne dépend que des positions et des vitesses des sources mobiles du même instant.

Dans les anciennes théories concernant la propagation de la gravitation, on admettait généralement que l'action d'un corps A sur un corps B à un instant donné devait dépendre de la position occupée par le corps A à une époque antérieure, ce qui conduirait à la considération des potentiels retardés. Or ici, nous obtenons une expression où n'interviennent que des positions simultanées des sources mobiles.

La valeur de l'intégrale, en fait, peut être considérée comme la résultante des ondulations produites par les actions antérieures des sources, mais le résultat remarquable, c'est que cette valeur résultante ne dépend que des positions actuelles simultanées et des vitesses.

Nous ne devons pas d'ailleurs oublier que les mouvements ont été supposés rectilignes et uniformes et le milieu rigoureusement homogène et isotrope.

Aussi, de ce résultat, je ne veux retenir qu'un enseignement: c'est la possibilité de concevoir l'apparence instantanée de la propagation de la gravitation, bien qu'en fait la vitesse de cette propagation soit finie. C'est un fait analogue à celui d'une intégrale curviligne dont la valeur ne dépend que des limites.

Si nous connaissions en toute rigueur la constitution du milieu, nous pourrions en déduire les formes précises des solutions possibles; mais il n'est pas vraisemblable que l'on puisse réduire le problème de la gravitation à la simple résolution d'une équation linéaire aux dérivées partielles du second ordre.

Je n'ai pas à formuler d'hypothèses. J'expose simplement les résultats que peut fournir l'étude d'une équation aux dérivées partielles.

J'ai cité, en commençant, une pensée profonde de Fourier sur la fécondité pour les mathématiciens de l'étude des lois naturelles. Je pourrais terminer en rappelant l'opinion de Galilée sur la nécessité de l'emploi des mathématiques comme langue essentielle et fondamentale de l'expression de ces lois. Nous nous trouvons ici en face d'une manifestation de la solidarité admirable de toutes les branches des connaissances humaines.

Mais j'ajoute que le mot de Shakespeare reste toujours vrai: nos systèmes philosophiques ne donnent qu'une image fragmentaire et incomplète de l'immensité de l'univers:

> There are more things in heaven and earth, Horatio
> Than are dreamt of in your philosophy.

NON-EUCLIDEAN GEOMETRY FROM NON-PROJECTIVE STANDPOINT

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1. There are three ways of developing the elementary parts of non-euclidean geometry:

(*1*) The synthetic method of the founders Lobatschewsky and Bolyai, which in more recent days has been refined and extended by Hilbert and many others.

(*2*) The method of projective geometry in which the fundamental notions of distance and angle are defined as cross ratios relative to a conic or quadric surface. Klein has been the great protagonist of this school.

(*3*) The method of differential geometry inaugurated by Riemann and Beltrami.

So far as the writer knows this third method has been employed only in a fragmentary way to develop those subjects which in Euclidean geometry are found in treatises on analytic geometry. In a paper* which I gave last year at the annual meeting of the Mathematical Association of America an attempt was made to show how readily this method of approach lent itself to an elementary treatment of what I may call non-euclidean analytic geometry. In that paper I confined myself to elliptic geometry; here I will treat them together but more particularly the hyperbolic geometry.

2. I begin by assuming that *e*-geometry† has been established with all requisite rigour, and on this we shall build the elliptic and hyperbolic geometries. The method is analogous to that employed in general arithmetic. Everyone knows the difficulties which beset our efforts to establish in a rigorous manner the real number system. But this once effected, other number systems as quaternions may be established with comparative ease.

Let x_1, x_2, x_3 be the rectangular coordinates of ordinary analytic geometry. Then in *e*-geometry the element of arc is defined by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{1}$$

and the angle θ between two curves meeting at the point x by

$$\cos\theta = \sum_a \frac{dx_a}{ds}\frac{\delta x_a}{\delta s}, \qquad (a=1,\ 2,\ 3.) \tag{2}$$

*The Amer. Math. Monthly, vol. XXX (1923) and vol. XXXI (1924).

†*e*-geometry, *E*-geometry, *H*-geometry are abbreviations for euclidean, elliptic and hyperbolic geometry.

Straight lines or e-straights are such that

$$\delta \int ds = 0. \tag{3}$$

To get a non-euclidean geometry we keep $x_1 x_2 x_3$ the same, but define ds by

$$ds^2 = \Sigma a_{ij} dx_i dx_j, \qquad (i, j = 1, 2, 3), \tag{4}$$

where $a_{ij} = a_{ji}$ are functions of the x. The angle θ we define by

$$\cos \theta = \sum a_{ij} \frac{dx_i}{ds} \frac{\delta x_j}{\delta s}. \tag{5}$$

Straight lines or "straights" in this geometry are such that (3) holds, ds having the value (4).

How many geometries are defined by (4)? Obviously there are an infinite number of differential forms (4) which define the same geometry, at least in regions of not too great extent. For if we change the variables x to x' in a 1 to 1 manner, curves in x space go over into curves in x' space so that corresponding arcs have the same length, straights go over into straights and angles are unaltered. Hence figures are unaltered in size and shape in their respective metrics. Riemann showed that by a proper change of variables (4) can be reduced to

$$ds = \frac{d\sigma}{1 \pm \dfrac{r^2}{4R^2}} \quad ; \quad r^2 = x_1^2 + x_2^2 + x_3^2, \; d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2. \tag{6}$$

If we set

$$\lambda = 4R^2 - r^2, \qquad \mu = 4R^2 + r^2, \tag{7}$$

formula (6) gives

$$ds = \frac{4R^2}{\mu} d\sigma \qquad \text{(elliptic geometry)}, \tag{8}$$

or

$$ds = \frac{4R^2}{\lambda} d\sigma \qquad \text{(hyperbolic geometry)}, \tag{9}$$

according as we take the $+$ or $-$ sign in (6).

The two geometries defined by (8), (9) have much in common and much that is different. In the first place (8) and (9) show at once in connection with (5) that the measure of the angle under which two curves cut is the same in these two geometries as in e-geometry. Secondly we observe that (8) and (9) interchange on replacing R by iR; we must expect this duality in our analytical formulae. Thirdly we can show at once, using (3), that e-straights through the origin O are also straights in E and H-geometry.

On the other hand (8) and (9) reveal a great difference in these geometries. In fact (8) holds for all points of space while (9) breaks down on the sphere $\lambda = 0$ or

$$G = x_1^2 + x_2^2 + x_3^2 - 4R^2 = 0, \tag{10}$$

and ds becomes even negative for points without. For this reason only points within G are regarded as real in H-geometry. This G-sphere plays a dominant role in both geometries; we call it the *fundamental sphere.* Associated with G is the imaginary sphere $\mu = 0$ or

$$G' = x_1^2 + x_2^2 + x_3^2 + 4R^2 = 0, \tag{11}$$

which is also important. We observe that they interchange on replacing R with iR.

Another difference is the following. Let ρ be the length of the segment OP of a straight through O. In E-geometry

$$\rho_E = \int_o^r \frac{dr}{1 + \frac{r^2}{4R^2}} = 2R \tan^{-1} \frac{r}{2R}, \tag{12}$$

while in H-geometry

$$\rho_H = 2R \tanh^{-1} \frac{r}{2R}. \tag{13}$$

Thus the length of the entire straight through O in E-geometry is $2\pi R$. On the other hand in H-geometry, when P approaches a point of G, $\rho_H \doteq \infty$. Hence all points in H-geometry are at an infinite distance from any point of the fundamental sphere. In e-geometry the introduction of imaginary points has proved indispensable, *e.g.*, the circular points at infinity. In non-euclidean geometry the same is true. As an example a straight in H-geometry through O cuts the sphere G' in two imaginary points for which $r^2 = -4R^2$. For these points (13) gives

$$\rho = \pm \frac{\pi i R}{2}. \tag{14}$$

Thus on each of these straights there are points whose distance from O in H-measure is given by (14). This is a particular case of a very important property.

3. Let us consider briefly some of the simple facts of plane E-geometry, which unroll with hardly any effort from the e-geometry on a sphere. As we deal here only with points x in the plane we drop x_3 from the foregoing formulae. If we introduce the variables

$$z_1 = \frac{4R^2 x_1}{\mu}, \qquad z_2 = \frac{4R^2 x_2}{\mu}, \qquad z_3 = R\frac{\lambda}{\mu}, \tag{15}$$

we find x is the stereographic projection of the point z on the sphere

$$z_1^2 + z_2^2 + z_3^2 = R, \tag{16}$$

while

$$ds^2 = dz_1^2 + dz_2^2 + dz_3^2.$$

This shows that when x describes a curve of length s in E-measure, z describes a curve on the sphere of equal length in e-measure. Thus to E-straights in the x plane correspond geodesics on the sphere by virtue of (3), that is, great circles. Any such great circle is determined by a plane through the centre of (16) whose equation is, say

$$A_1z_1+A_2z_2+A_3z_3=0.$$

Replacing the z's by (15) we obtain

$$A_3(x_1^2+x_2^2)-4R(A_1x_1+A_2x_2)=4A_3R^2.$$

Thus E-straights are from the standpoint of e-geometry, circles cutting the fundamental G circle in diametral points, and the imaginary G' circle orthogonally. To any two curves in the x plane meeting under the angle θ in E-measure correspond two curves on the sphere (16) meeting under the same angle θ in e-measure. Thus to a triangle in the E-plane corresponds a triangle on the sphere having respectively equal sides and angles. The trigonometry of the E-plane is thus identical with ordinary trigonometry on a sphere of radius R. In particular the sum of the angles of an E triangle is $>180°$. Finally since figures may be moved about freely without distortion on the sphere the same holds in plane E-geometry. Hence the length of all E straights is $2\pi R$ since this is the length in E measure of a straight through O. On the sphere two points determine uniquely a great circle unless they lie on a diameter, hence in E-geometry two points determine but one straight unless they lie on a diameter of G. Similarly two E-straights do not cut in one point, but in two.

To avoid this anomaly one has defined a *restricted E-geometry* by imposing the conditions that to diametral points on the sphere (16) shall correspond but one point in the x plane, viz., that one of the two points which lies within the fundamental G circle. The two end points of a diameter of G are regarded as identical points. There are no points outside G. In this geometry which we may denote by E^* two points determine uniquely a straight; two straights intersect in a single point. The length of any E^* straight is πR instead of $2\pi R$ as in E-geometry.

A peculiarity of E^*-geometry is illustrated by the accompanying figures: 1, 2, 3. The circle $ABCD$ in Fig. 1 is moved toward the right. When it meets G in Fig. 2, the two points L, M are identical with their diametral points $L'M'$ and when the circle reaches O again the figure has been turned through 180° as indicated in Fig. 3. This is analogous to the twisted band of Möbius.

We mention one other feature which is of great importance in the following. On the sphere (16) the locus of the points whose distance from a point A is $\pi R/2$ is a great circle, the polar of A. Thus in the E-plane the locus of all points at a distance $\pi R/2$ from A is an E-straight a, the polar of A, also A is the pole of a. In E-geometry a straight has two poles, in E^*-geometry only one. The three E-straights $z_1=0$, $z_2=0$, $z_3=0$ form a right polar triangle; each side is the polar of the opposite vertex. We may use this as a triangle of reference. From a point z let us drop E-perpendiculars on the three sides of this triangle. If $\delta_1, \delta_2, \delta_3$ are the lengths in E-measure of these perpendiculars we find

$$z_k=R.\sin\ \delta_k/R, \qquad (k=1,\ 2,\ 3),$$

which gives us another geometric interpretation of the z's, analogous to the homogeneous coordinates of projective geometry.

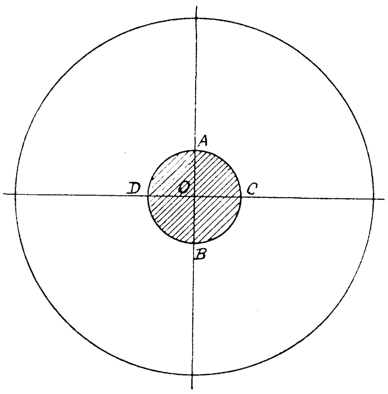


Fig. 1

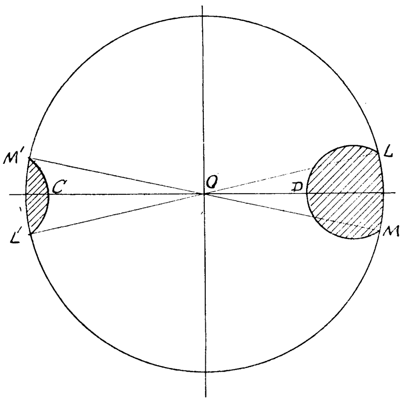


Fig. 2

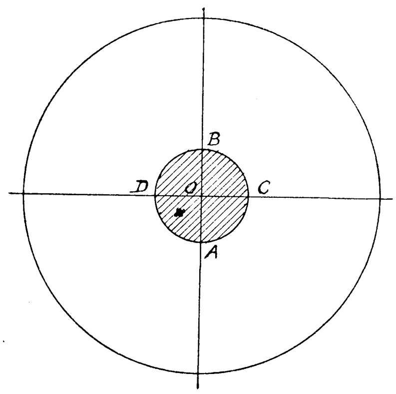


Fig. 3

4. We now leave these very instructive particulars and turn to hyperbolic geometry. First a word about plane H-geometry. If we replace R by iR the sphere (16) used for stereographic projection becomes imaginary. Perhaps for this reason one has introduced new variables so that (9) becomes

$$ds^2 = \frac{R^2(dy_1^2 + dy_2^2)}{y_1^2},$$

which defines the element of arc on a pseudosphere. This was first done by Beltrami and later by v. Escherich with considerable success, but it introduces unnecessary difficulties and one loses the close relations which exist between E and H-geometry. We shall therefore introduce variables analogous to those of (15), and as we propose to treat space and not the plane we need four, viz.:

$$\zeta_a = \frac{4R^2 x_a}{\lambda}, \quad (a = 1, 2, 3), \quad \zeta_4 = \frac{R\mu}{\lambda} > 0. \tag{17}$$

The metric is that of (9). We shall find it convenient to use the abbreviations

$$[ab]=a_1b_1+\ldots\ldots\ldots+a_4b_4,\quad [a^2]=a_1^2+\ldots\ldots+a_4^2,$$
$$\{ab\}=a_1b_1+a_2b_2+a_3b_3-a_4b_4,\quad \{a^2\}=a_1^2+a_2^2+a_3^2-a_4^2.$$

We now find for all points x for which $\lambda\neq 0$, *i.e.*, points not on the fundamental G-sphere (10), that

$$\{\zeta^2\}=-R^2, \tag{18}$$

which, if we like, may be regarded as a hyperboloid in 4-way e-space, but we shall not urge this interpretation. We find also that the element of arc (9) satisfies

$$ds^2=\{d\zeta^2\}. \tag{19}$$

Let the straight joining O with the point $P(x_1, x_2, x_3)$ have the direction cosines l_1, l_2, l_3. The length ρ of the segment OP is given by (13). We find that

$$\zeta_a=Rl_a \sinh \rho/R,\quad (a=1, 2, 3),\quad \zeta_4=R\cosh \rho/R. \tag{20}$$

Let us first consider the locus defined by

$$[Az]=0, \tag{21}$$

which we call an *H-plane*. Setting (17) in this equation we get

$$A_4r^4+4R(A_1x_1+A_2x_2+A_3x_3)+4A_4R^2=0; \tag{22}$$

or

$$\sum_a\left(x_a+\frac{4RA_a}{A_4}\right)^2=\frac{4R^2}{A_4^2}\{A^2\},\quad A_4\neq 0. \tag{23}$$

Thus the H-plane (21) is in e-geometry a sphere cutting the fundamental G-sphere orthogonally. Since only the ratios of the A's in (21) are important we may suppose

$$\{A^2\}=R^2. \tag{24}$$

In this case we say (21) is in *normal form.* In case $A_4=0$, (21) reduces to $A_1z_1+A_2z_2+A_3z_3=0$ or in x coordinates $A_1x_1+A_2x_2+A_3x_3=0$. Hence H-planes through O are also e-planes. To the four H-planes $\zeta_1=0, \ldots, \zeta_4=0$ correspond the three coordinate planes $x_1=0$, $x_2=0$, $x_3=0$ and the imaginary G' sphere. They form a tetrahedron which we shall always denote by τ. Let us now see what H-straights are. If we perform the variations indicated in (3), using the ds of (19), we get four differential equations

$$\frac{d^2\zeta_k}{ds^2}=\frac{\zeta_k}{R},\quad (k=1,\ldots,4), \tag{25}$$

whose integrals are

$$\zeta_k=a_k\cosh\frac{s}{R}+b_k\sinh\frac{s}{R}. \tag{26}$$

The 8 constants of integration must be so chosen that ζ satisfies (18). When

$s=0$, $\zeta_k=a_k$, hence $\{a^2\}=-R^2$. To satisfy (18) we impose the further conditions

$$\{b^2\}=R^2, \quad \{ab\}=0. \tag{27}$$

With these conditions (26) are the parameter equations of an H-straight. We easily find that every H-straight is the intersection of two H-planes of the type (21) and conversely, as in e-geometry. This is our reason for calling (21) a plane. We have thus the important result: H-straights are e-circles cutting the fundamental G-sphere orthogonally and the imaginary G' sphere in diametral points. Just the opposite is true in E-geometry as we should expect, since these spheres like the two metrics (8), (9) interchange on replacing R by iR. A peculiarity of plane H-geometry is illustrated in the accompanying Fig. 4 which we suppose, to fix our ideas, lies in the x_1x_2 plane. G is the fundamental circle; e-circles cutting G orthogonally are H-straights. Straights as a, b which meet on G, *i.e.*, at infinity, are said to be *parallel*. Through a point A we can draw two parallels b, c to a given straight a. On the other hand there are an infinity of

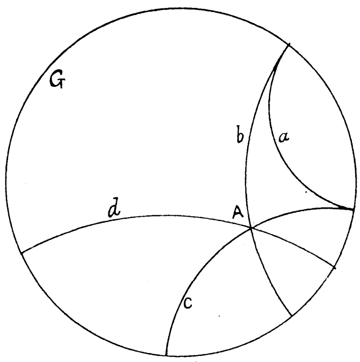


Fig. 4

straights through A which do not meet a given line, as for example a. Such a line is d, we observe that the parallels b, c are the limiting positions of lines through A which do not meet a.

Returning to our main theme, let us multiply (26) by $a_1, \ldots, a_4$; we get

$$\{a\zeta\} = \{a^2\}\cosh\frac{s}{R} + \{ab\}\sinh\frac{s}{R},$$

or

$$\cosh\frac{s}{R} = \frac{\{a\zeta\}}{-R^2}. \tag{28}$$

Similarly, multiplying (26) by $b_1, \ldots, b_4$ gives

$$\sinh\frac{s}{R} = \frac{\{b\zeta\}}{-R^2}. \tag{29}$$

The relation (28) is of utmost importance.

We have derived (25) and (26) on the supposition that the quantities involved are real. If we allow imaginary values we may regard (26) as defining an imaginary straight and (28) as defining the distance between the (imaginary) points a, ζ. The ambiguity of this definition is not disturbing as s is usually real or purely imaginary. Points whose coordinates are of the form $b_k = i\beta_k$, the β's real, are of extreme importance in H-geometry. Such points are, *e.g.*, the intersections of a straight through O with the imaginary G' sphere, *i.e.*, the H-plane $\zeta_4 = 0$. We saw in (14) that the distance of any point of this plane from O is $s = \pi i R/2$, agreeing with (28).

Let us generalize and say two points a, ζ are *associate* when

$$\{a\zeta\} = 0, \tag{30}$$

or, by (28), when their distance apart is $\pi i R/2$. The locus of all points ζ associated with a is thus an H-plane. We call this plane the *polar* of a and a the pole of the plane (30). If a is real a_4 is real and (23) shows that the polar of a is imaginary, since $\{a^2\} = -R^2$; on the other hand if a is imaginary, *i.e.* if its coordinates have the form $i\alpha_k$, the polar of a is real.

With this in mind let us return to (25). If we set $s = \pi i R/2$ we get $\zeta_k = ib_k$. The relations (27) therefore mean that the H-straight passes through a and the associate point whose coordinates are ib_k. The tetrahedron τ mentioned above is a polar tetrahedron; each of its faces is the polar of the opposite vertex.

Let us now consider the angle θ between two H-planes

$$[A\zeta] = 0, \quad [B\zeta] = 0, \tag{31}$$

which we will suppose are in normal form. To these planes correspond e-spheres whose equations are of the type (22). The angle under which these spheres cut is also θ in e-measure as we saw in § 2. Then from analytic geometry

$$\cos\theta = \frac{\{AB\}}{R^2}. \tag{32}$$

The two planes (31) are orthogonal when

$$\{AB\} = 0 \tag{33}$$

and in this relation we may note it is not necessary that the H-planes (31) should be in normal form.

Let b, c be two points in the plane $\alpha = \{a\zeta\} = 0$. The plane ω through a, b, c has the equation

$$\omega = \begin{vmatrix} \zeta_1 \ldots\ldots \zeta_4 \\ a_1 \ldots\ldots a_4 \\ b_1 \ldots\ldots b_4 \\ c_1 \ldots\ldots c_4 \end{vmatrix} = A_1\zeta_1 + \ldots + A_4\zeta_4 = 0.$$

This plane is perpendicular to α if $\{aA\} = 0$, by (33). But $\{aA\} = a_1A_1 + \ldots + a_4A_4$ is the development of the above determinant when we replace the ζ's by the a's; hence $\{aA\} = 0$ and ω is orthogonal to α. If we keep a, b fixed and let c vary in the plane α, we see that all planes through the join of a, b cut the α plane orthogonally. Hence the important theorem: The H-straight joining a point with any point of its polar is perpendicular to this plane, or all straights perpendicular to an H-plane meet in the pole of this plane. This is illustrated by the τ tetrahedron; we see now all its faces cut orthogonally.

Let $\alpha=[A\zeta]=0$ be a plane in normal form so that $\{A^2\}=R^2$. If we set $A_1=ia_1$ $A_2=ia_2$, $A_3=ia_3$, $A_4=-ia_4$ we have $\{a^2\}=-R^2$ and hence a_1, a_2, a_3, a_4 are the coordinates of a point a. Then $[A\zeta]=0$ becomes $\{a\zeta\}=0$, which shows that a is the pole of α. We can now find the length δ of the H-perpendicular p dropped from a point ζ on the plane α. For let p cut α in the point c, then a is the associate of c on p and (29) gives

$$\sinh\frac{\delta}{R}=\frac{\{a\zeta\}}{iR^2}=\frac{[A\zeta]}{R^2}. \tag{34}$$

Hence if δ_k is the length of the H-perpendicular let fall from the point ζ on the face $\zeta_k=0$ of the τ tetrahedron we get at once

$$\zeta_j=R\sinh\frac{\delta_j}{R},\quad j=1,2,3;\quad \zeta_4=\frac{R}{i}\sinh\frac{\delta_4}{R}. \tag{35}$$

5. As an application of these formulae let us show how easily the formulae of H-trigonometry may be deduced.

Let ABC be a triangle in the x_1x_2 plane whose opposite sides have the lengths a, b, c in H-measure. Let A coincide with the origin O and OC with the $+x$ axis as in Fig. 5. We shall show later that any triangle may be moved into this position without altering any of its dimensions.

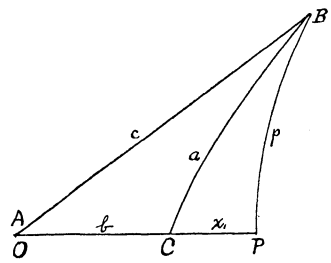


Fig. 5

Let the ζ coordinates of the vertex B be b_1, b_2, b_3 and c_1, c_2, c_3 those of C. Then by (20)

$$b_1=R\sinh\frac{c}{R}\cos A,\qquad b_2=R\sinh\frac{c}{R}\sin A,\qquad b_3=R\cosh\frac{c}{R},$$

$$c_1=R\sinh\frac{b}{R},\qquad c_2=0,\qquad c_3=R\cosh\frac{b}{R}.$$

These in combination with (28) give

$$\cosh\frac{a}{R}=\frac{\{bc\}}{-R^2}=\cosh\frac{b}{R}\cosh\frac{c}{R}-\sinh\frac{b}{R}\sinh\frac{c}{R}\cos A. \tag{36}$$

By cyclic permutation we obtain the corresponding formulae for $\cosh b/R$, $\cosh c/R$.

To get the sine formula we drop an H-perpendicular of length p on the side OC, or the line $\zeta_2=0$, or in normal form $R\zeta_1=0$. Then, by (34),

$$\sinh\frac{p}{R}=\frac{Rb_2}{R^2}=\sinh\frac{c}{R}\sin A.$$

This being true for any right triangle is true for CBP. Hence $\sinh\frac{p}{R}=\sinh\frac{a}{R}\sin C.$

Hence

$$\sinh\frac{a}{R}\sin C=\sinh\frac{c}{R}\sin A.$$

As these are entirely general we get as usual

$$\sinh\frac{a}{R}:\sinh\frac{b}{R}:\sinh\frac{c}{R}=\sin A:\sin B:\sin C. \tag{37}$$

The third fundamental formula is

$$\sinh\frac{a}{R}\cos B=\sinh\frac{c}{R}\cosh\frac{b}{R}-\cosh\frac{c}{R}\sinh\frac{b}{R}\cos A. \tag{38}$$

This is obtained from (36) in precisely the same manner as the corresponding formulae in ordinary spherical trigonometry. From (36), (37), (38) all the formulae of H-trigonometry may be obtained by following the steps employed in any elementary treatise on spherical trigonometry. We observe that we have merely to replace the sides a, b, c in such formulae by a/iR, b/iR, c/iR, to convert a formula of e-spherical trigonometry into one of H-trigonometry.

6. In e-geometry we assume the existence of rigid geometric figures, *i.e.*, figures that can be moved about freely without altering their size or shape. This fact is characterized by the existence of continuous point transformations which leave ds unaltered. We extend this to H-geometry. To simplify our analysis it will be convenient to set

$$z_j=\zeta_j,\ j=1,2,3;\quad z_4=i\zeta_4. \tag{39}$$

Then the z coordinates satisfy the relation

$$[z^2]=-R^2, \tag{40}$$

while ds^2 becomes

$$ds^2=[dz^2]. \tag{41}$$

Let us effect a linear transformation of the z's

$$z_k'=a_{k1}z_1+\ldots+a_{k4}z_4=\sum_a a_{ka}z_a,\quad (a,\ k=1,\ldots,4). \tag{42}$$

If the determinant A of the $a_{k\alpha}$ is $\neq 0$, (42) defines a one to one transformation of the point z to z' and conversely, provided the relation (40) is valid for z'. Now

$$[z'^2] = \sum_k z_k'^2 = \sum_k \sum_\alpha a_{k\alpha} z_\alpha \sum_\beta a_{k\beta} z_\beta = \sum_{\alpha,\beta} z_\alpha z_\beta \sum_k a_{k\alpha} a_{k\beta}.$$

Hence if the a's satisfy the so-called orthogonal relations

$$(43) \qquad \begin{cases} \sum_k a_{k\alpha} a_{k\beta} = 1 \ (\alpha=\beta), \\ \qquad\qquad = 0 \ (\alpha \neq \beta), \end{cases}$$

the condition (40) is satisfied for the z's, that is $z'_1, \ldots, z'_4$ are indeed the coordinates of a point z'.

By virtue of (43) we find $A = \pm 1$; if we take $A = +1$ we find $a_{k\alpha} = A_{k\alpha}$ the minor of $a_{k\alpha}$; also

$$(44) \qquad \begin{cases} \sum_k a_{\alpha k} a_{\beta k} = 1 \ (\alpha=\beta), \\ \qquad\qquad = 0 \ (\alpha \neq \beta). \end{cases}$$

We must subject the a's to another condition. When the point ζ is real, z_1, z_2, z_3 are real and z_4 is imaginary. Thus, as we wish the transformation (42) to convert a real point ζ into a real point ζ', we will take the a's according to the scheme

(45)

	$z_1=\zeta_1$	$z_2=\zeta_2$	$z_3=\zeta_3$	$z_4=i\zeta_4$
$z_1'=\zeta_1'$	C_{11}	C_{12}	C_{13}	iC_{14}
$z_2'=\zeta_2'$	C_{21}	C_{22}	C_{23}	iC_{24}
$z_3'=\zeta_3'$	C_{31}	C_{32}	C_{33}	iC_{34}
$z_4'=i\zeta_4'$	$-iC_{41}$	$-iC_{42}$	$-iC_{43}$	C_{44}.

Here the C's are real and the elements of this table satisfy the orthogonal relations by rows and by columns. We take $C_{44} > 0$.

Let (45) transform the two near-by points z, $z+dz$ whose distance apart is ds into the points z', $z'+dz'$ whose distance apart is ds'. Since the coefficients a in (42) are constants we see that the dz's transform the same as the z's. Since the orthogonal relations (43) now hold we see that $ds'^2 = ds^2$. Thus the linear orthogonal transformation (45) leaves all distances unaltered and hence also all angles. This is further confirmed by applying (45) to (28); we find at once that

$$\{a'\zeta'\} = \{a\zeta\}.$$

Obviously the transformation (45) does not transform a point ζ within the fundamental G sphere to one without it or on it. It is now not difficult to show that the tetrahedron τ can be made to coincide with any other tetrahedron τ' of the same character. In particular any triangle can be brought into the special position employed in § 5.

Let us briefly mention a few special cases of (45).

Example 1. $C_{44}=1$, the other C's in the last row and column $=0$. This defines a rotation about O in the e-sense.

Example 2.

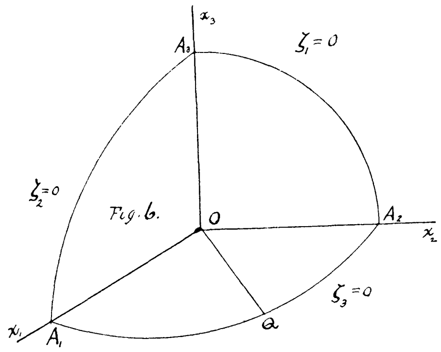
(46)

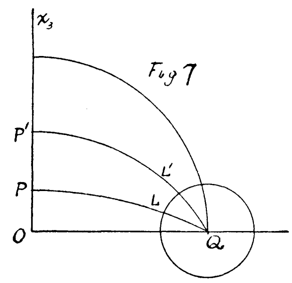
	ζ_1	ζ_2	ζ_3	$i\zeta_4$
ζ_1'	1	0	0	0
ζ_2'	0	1	0	0
ζ_3'	0	0	$\cos i\theta$	$\sin i\theta$
$i\zeta_4'$	0	0	$-\sin i\theta$	$\cos i\theta$,

that is:

$$\zeta_1' = \zeta_1,\ \zeta_2' = \zeta_2,\ \zeta_3' = \zeta_3 \cosh\theta - \zeta_4 \sinh\theta,\ \zeta_4' = -\zeta_3 \sinh\theta + \zeta_4 \cosh\theta.$$

The Fig. 6 represents the tetrahedron τ. A plane $\alpha = \zeta_1 - g\zeta_2 = 0$ through the x_3 axis is unaltered. A plane $\beta = \zeta_3 - h\zeta_4 = 0$ through the edge A_1A_2 is rotated about it as an axis. Thus points on the x_3 axis are shifted along it, in such a manner, however, that points within G remain in it. The intersection of $\alpha = 0$, $\beta = 0$ is an H-straight as QP, Fig. 7, perpendicular to the x_3 axis. A point L on it is moved to L', while P goes to P'. As distances are unaltered $\overline{LP} = \overline{L'P'}$ in





H-measure. The points L at a given distance from the x_3 axis form an H-circle C whose centre is Q. If we rotate the plane α about the x_3 axis, the locus of C is a sort of torus. The equation of this surface is

$$A(\zeta_1^2 + \zeta_2^2) + B(\zeta_3^2 - \zeta_4^2) = 0, \tag{47}$$

for $\zeta_1^2 + \zeta_2^2$ is unaltered by the rotation about the x_3 axis, that is, by the following:

(48)

	ζ_1	ζ_2	ζ_3	$i\zeta_4$
ζ_1'	$\cos\theta$	$\sin\theta$	0	0
ζ_2'	$-\sin\theta$	$\cos\theta$	0	0
ζ_3'	0	0	1	0
$i\zeta_4'$	0	0	0	1,

and $\zeta_3^2 - \zeta_4^2$ is unaltered for (46). If we apply both (46) and (48) a point on the surface (47) describes a screwlike motion upon it. This surface is the analogue of the celebrated *Clifford Surface*. The rectilinear generators or Clifford parallels are here imaginary.

SULLE OPERAZIONI FUNZIONALI LINEARI

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1. Non è, al certo, in questa riunione in cui convengono tanti eccellenti matematici dalle varie parti del mondo, che un vecchio e modesto lavoratore può recare idee nuove e feconde: il silenzio sarebbe stato più consigliabile per me, se l'accoglienza tanto cordiale fatta alla delegazione di cui sono il più anziano, non mi avesse fatto un dovere di prendere la parola. Ma la stessa Vostra cortesia, che non mi permette di rinunziare ad intrattenervi, vorrà scusare se, dopo le dotte conferenze che abbiamo avuto la ventura di udire, io non potrò esporvi che una semplice *causerie*, e se, rassegnato fin d'ora a scorgere sulle Vostre labbra il sorriso di compatimento che accoglie ciò che può sembrare sintomo d'involuzione senile, rimetto a nuovo qui, dinanzi a voi, alcune mie vecchie idee.

2. Da molti anni—non ardisco dire quanti—ho considerato le funzioni di un insieme lineare ben determinato, come fossero gli elementi o punti di uno spazio, cui ho dato il nome di *funzionale*, ed ho considerato quelle operazioni a carattere lineare, che eseguite su codeste funzioni, riproducono gli elementi di quell'insieme o di un altro insieme lineare. Le denominazioni di spazio funzionale, di operazioni funzionali, sono state poi generalmente adottate, ed i concetti che vi si connettono sono stati sviluppati in varie direzioni; ma, mentre io mi occupavo del lato algoritmico o qualitativo di codesti concetti, maestri della scienza ne sviluppavano i lati quantitativi: si ricordino la teoria delle funzioni di linee del *Volterra*, cui il compianto *Arzelà* ha portato contributi di considerevole importanza; il calcolo di composizione dovuto pure al *Volterra*, le belle sue applicazioni alla teoria delle equazioni integrali ed integro differenziali e per conseguenza a numerose questioni di fisica matematica; le interessanti ricerche iniziate da *Hadamard* e sapientemente continuate da *Fréchet*; gli studi di *Hilbert* e della sua scuola; il recente libro di *P. Lévy*, ecc.

3. Ma il punto di vista al quale mi sono posto, e che direi *qualitativo*,—si potrebbe quasi dire *geometrico*—ha avuto minore successo: ed infatti meno dirette e meno immediate ne sono le applicazioni. Pur tuttavia, esso non è privo d'interesse: numerosi problemi si riattaccano pure ad esso, e fra altri, ve n'è uno che, a mio parere, lo rende degno dell'attenzione dei matematici, ed è che esso lascia intravedere un metodo atto ad avviare ad una graduale classificazione delle trascendenti analitiche, questione quanto mai interessante, ma per la quale mancano per ora gli instrumenti opportuni.

4. Consideriamo uno spazio S di funzioni, e l'insieme delle operazioni lineari—cioè a dire distributive rispetto all'addizione—che, applicate a questo

spazio, lo riproducono: codeste operazioni, alla lor volta, costituiscono un insieme lineare avente carattere di gruppo. Si viene così a generalizzare, in modo concreto, la teoria delle omografie, estendendole ad uno spazio ad infinite dimensioni e la generalizzazione dei problemi classici sulle omografie si presenta spontanea; ad es.:

(*a*) Data un'operazione A, ammette essa radici? esistono cioè elementi α dello spazio funzionale, per i quali sia $A(\alpha)=0$?

(*b*) Date due operazioni A, B, si consideri il fascio di operazioni $A-kB$: esistono valori di k, e quali, tali che $A-kB$ ammetta radici?

Come caso particolare, questo problema contiene quello della risoluzione dell'equazione integrale di seconda specie omogenea di *Fredholm*, cioè la ricerca dei numeri caratteristici e delle funzioni caratteristiche di un'operazione integrale.

(*c*) Riduzione di un'omografia alla sua forma canonica, cioè ricerca della base dello spazio S rispetto ad un'operazione data.

5. Lo studio delle omografie così generalizzate dà luogo ad una osservazione importante, ch'io ho publicata nel 1897 e che è stata ritrovata, una dozzina d'anni più tardi, da *Hellinger* e *Toeplitz*: questa osservazione è la seguente.

Una omografia applicata ad uno spazio S ad un numero finito di dimensioni, se non ammette una inversa unica, è degenere; ciò significa:

1° che essa ammette radici,

2° che essa trasforma lo spazio su cui essa opera in una parte dello spazio medesimo, cioè in un sottospazio di S non coincidente con S. Or bene, quando l'omografia opera su uno spazio ad un numero infinito di dimensioni, queste due proprietà non sono più conseguenza l'una dell'altra: oltre alla degenerescenza di cui si è ora parlato, si hanno operazioni che ammettono radici, pur riproducendo l'intero spazio S; è la degenerescenza parziale di prima specie: mentre altre non ammettono radici, ma trasformano S in un suo sottospazio: è questa la degenerescenza parziale di seconda specie.

6. E' facile dare esempi, non meno semplici che suggestivi, del fatto ora ricordato. Sia S lo spazio funzionale delle serie di potenze, di cui

$$1,\ x,\ x^2,\ \ldots,\ x^n,\ \ldots \tag{1}$$

può dirsi *la base*. Una operazione lineare P sia definita colle a_n non nulle, da

$$P(1)=a_0,\quad P(x^n)=a_n x^n - x^{n-1},\quad (n=1,2,3,\ \ldots);$$

questa operazione, applicata alla serie

$$\alpha=\sum h_n x^n$$

dà

$$P(\alpha)=h_0a_0-h_1+(h_1a_1-h_2)x+\ \ldots\ +(h_na_n-h_{n+1})x^n+\ \ldots,$$

e qui i coefficienti $h_1, h_2, \ldots$ si possono determinare in modo che sia $P(\alpha)=0$: l'operazione P ammette dunque radici nello spazio S; ma d'altra parte ogni elemento di S può essere rappresentato da $P(\alpha)$: l'operazione P riproduce dunque tutto lo spazio S.

Si definisca invece, nel medesimo spazio, colle a_n non nulle, un'operazione Q da

$$Q(1) = a_0 - x, \quad Q(x^n) = a_n x^n - x^{n+1}, \quad (n = 1, 2, 3, \ldots)$$

si avrà:

$$Q(\alpha) = \sum k_n x^n, \quad k_n = h_n a_n - a_{n-1},$$

e qui si può avere $Q(\alpha) = 0$ solo se α è identicamente nullo; mentre d'altra parte, sotto ovvie condizioni di convergenza, è

$$\sum k_n a_0 a_1 \ldots a_n = 0;$$

per modo che le serie $Q(\alpha)$ danno soltanto una parte dello spazio S.

7. L'operazione D di derivazione, nello spazio S ora considerata, è operazione degenere della prima specie poichè ammette come radice la costante, mentre trasforma in sè tutto S; invece, nello spazio funzionale la cui base è

$$\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, , \ldots, \tag{2}$$

è un'operazione degenere della seconda specie, poichè essa genera il sotto-spazio

$$\frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, , , \ldots,$$

mentre non vi ha radici. Infine, in uno spazio avente come base

$$e^{a_1 x}, e^{a_2 x}, \ldots e^{a_n x}, \ldots \tag{3}$$

dove $a_1, a_2, \ldots a_n, \ldots$ sono numeri fra loro diversi, l'operazione D non è degenere, e gli elementi (3) ne danno senz'altro la forma canonica.

8. Fra le operazioni lineari che si applicano ad uno spazio S determinato e riproducono lo spazio stesso, si può fissare l'attenzione su una di esse in particolare, sia H, che si dirà *operazione principale*, indi classificare le altre a seconda dei loro rapporti con quella. In primo luogo, si possono considerare le operazioni permutabili con H, cioè quelle per le quali è

$$AH - HA = 0; \tag{4}$$

indi, il primo membro di (4) può considerarsi come una nuova operazione, che dà, per così dire, lo *scarto* della permutabilità di A rispetto ad H. Indicando questo scarto con A'_H, e detto questo di *indice* 1, si potranno considerare gli scarti di indici 2, 3, ...

$$A''_H = A'_H H - HA'_H = AH^2 - 2HAH + H^2A,$$
$$A'''_H = AH^3 - 3HAH^2 + 3H^2AH - H^3A$$

e così via. E mentre $A'_H = 0$ caratterizza la permutabilità ordinaria con H, si potrà dire che le operazioni per le quali è $A_H^{(n)} = 0$ hanno, rispetto ad H, una permutabilità di ordine n.

Lo scarto ha, nel Calcolo delle operazioni, una analogia formale notevole colla derivazione nel Calcolo delle funzioni: così, lo scarto di un prodotto AB di operazioni è dato dalla regola

$$(AB)'_H = A'_H B + A B'_H$$

e per gli scarti d'ordine superiore vale la legge evidente

$$(AB)''_H = A''_H B + 2A'_H B'_H + A B''_H,$$

$$(AB)'''_H = A'''_H B + 3A''_H B'_H + 3A'_H B''_H + A B'''_H,$$

ecc.

9. Sia S uno spazio funzionale tale che il prodotto di due suoi elementi appartenga ancora ad S. Non si può davvero immaginare, in S, un'operazione più semplice della moltiplicazione degli elementi generici di S per un suo elemento fisso α. In particolare, se S è lo spazio, già considerato, delle serie di potenze, si può assumere x come moltiplicatore, ed ogni altra operazione di moltiplicazione sarà permutabile con questa. Lo scarto della permutabilità di un'operazione A sarà allora dato da

$$A(x\alpha) - xA(\alpha),$$

α essendo un elemento qualsiasi in S: questo scarto verrà indicato semplicemente con A', e con $A^{(n)}$ lo scarto d'ordine n. Or bene, la considerazione di questi scarti dà luogo ad una formula assai notevole. Se, essendo α e β due elementi di S, si pone

$$A(\alpha\beta) = \lambda_0\alpha + \lambda_1 D\alpha + \lambda_2 D^2\alpha + \ . \ . \ .,$$

si trova che i coefficienti $\lambda_0, \lambda_1, \lambda_2, \ . \ . \ .$ dipendono esclusivamente da β, e che è precisamente

$$\lambda_0 = \beta,\ \lambda_1 = A'(\beta),\ \lambda_2 = \frac{1}{2!}A''(\beta),\ . \ . \ . \ \lambda_n = \frac{1}{n!}A^{(n)}(\beta),\ . \ . \ .;$$

talchè vale la formula

$$A(\alpha\beta) = \sum_{n=1}^{\infty} \frac{1}{n!} A^{(n)}(\beta) D^n \alpha \tag{5}$$

e per questa serie (che nel Calcolo delle operazioni funzionali ha un ufficio analogo alla serie di *Taylor* nella teoria delle funzioni), esiste in ogni caso un dominio funzionale di convergenza.

Un'espressione differenziale lineare, i cui coefficienti siano serie di potenze, gode, rispetto alla moltiplicazione per x, della permutabilità di ordine $m+1$ se m è il suo ordine; essendo A una tale espressione, si ha dunque $A^{(m+1)} = 0$. Inoltre lo scarto, per una tale espressione, si costruisce mediante una regola che è formalmente quella stessa della derivazione dei polinomi, gli esponenti essendo sostituiti dagli indici di derivazione. Se A è una simile espressione, la formula (5) si riduce ad un numero finito di termini: ed il caso particolare della (5) così ottenuto risale a *D'Alembert*.

10. Alla moltiplicazione per x sostituiamo ora, come operazione principale in S, la derivazione; indichiamo con A'_D lo scarto della operazione A dalla permutabilità con D. La permutabilità propriamente detta, cioè la $A'_D=0$, si ha per le espressioni differenziali lineari (anche d'ordine infinito, in ispecie per le espressioni lineari alle differenze) a coefficienti costanti; la permutabilità d'ordine $m+1$, $A_D^{(m+1)}=0$, si ha per le espressioni differenziali lineari d'ordine finito od infinito, aventi per coefficienti polinomi di ordine non superiore ad m.

11. Fin qui abbiamo considerato operazioni lineari che riproducono lo spazio su cui operano, o parti di esso. Ma si possono anche studiare operazioni lineari che trasformino uno spazio funzionale S in un altro T: le omografie di T sono allora le trasformate mediante una tale operazione, sia B, delle omografie di S. A questo punto di vista, occupiamoci di una di queste trasformazioni B delle omografie di S; è a questo punto di vista, che ho studiato una di queste trasformazioni B, la più nota e la più interessante: è la trasformazione di *Laplace*, studiata pure da *Abel*, e di cui *Borel* ha fatto uso con tanta efficacia nella teoria moderna delle funzioni. Questa operazione ha, come proprietà caratteristica, quella di far passare dalle operazioni permutabili di un dato ordine rispetto alla moltiplicazione, alle operazioni permutabili dello stesso ordine rispetto alla derivazione. Mi sia permesso di trattenermi un istante sul caso più semplice, quello della permutabilità propriamente detta.

12. Consideriamo come spazio S quello delle funzioni intere *di tipo esponenziale* (secondo l'espressione di *Pòlya*) di una variabile t, come spazio T quello delle serie di potenze di $\frac{1}{x}$, senza termine costante. Indico con ϕ, ψ, . . . gli elementi di S, con f, g . . . gli elementi di T. L'operazione B—trasformazione di *Laplace-Abel-Borel*—è data da:

$$\int_0^\infty e^{-xt}\phi(t)dt=f(x), \tag{6}$$

o brevemente

$$B(\phi)=f;$$

ϕ è la funzione generatrice, f la sua determinante. La moltiplicazione per t, eseguita in S, si muta, in T, nella derivazione cambiata di segno,—D. Sia $\alpha(-D)$ un polinomio razionale intero a coefficienti costanti in—D, e sia la equazione in g

$$\alpha(-D)\{g\}=f; \tag{7}$$

questa equazione è risoluta da

$$g(x)=\int \bar{e}^{xt}\frac{\phi(t)}{\alpha(t)}dt; \tag{8}$$

qui il cammino d'integrazione è formato da una linea che, partendo da 0, tende all'infinito secondo un azimut determinato θ, tracciata però in modo da evitare le radici di $\alpha(t)$. L'espressione (8) rappresenta un ramo ad un valore di funzione analitica in un semipiano contenente il semiasse reale positivo e limitato da una perpendicolare al raggio di azimut—θ; variando il cammino di integrazione senza oltrepassare le radici di $\alpha(t)$, si ottiene il prolungamento analitico di quel ramo; oltrepassando le radici, si aggiungeranno termini esponenziali, integrali dell'-equazione

$$(9) \qquad \alpha(-D)=0$$

priva di secondo membro. Di questa $g(x)$, la teoria delle funzioni determinanti dà, in modo semplicissimo, lo sviluppo asintotico.

13. Al caso elementare che ho così considerato si può riattaccare, a quanto mi sembra, un avviamento ad una graduale classificazione di numerose trascendenti.

Sia $f(x)$ un elemento dello spazio T, $\phi(t)$ la sua generatrice, $\rho(t)$ una funzione razionale qualsiasi, escludendo solo, ciò che non ha nulla di essenziale, che $t=0$ ne sia un polo. La

$$g(x)=\int_0^{\infty e^{i\theta}} \rho(t)\phi(t)e^{-kt}dt$$

dà il risultato di un'operazione $A\ (f)$, eseguita su f e trasformata mediante B della moltiplicazione per $\rho(t)$: risultato che è una trascendente il cui carattere essenziale è facile a stabilirsi: esso dipende infatti esclusivamente dalla funzione

$$(10) \qquad \lambda(x,\ z)=\int_0^{\infty e^{i\theta}} \frac{\phi(t)e^{-xt}dt}{t-z},$$

poichè $g(x)$ non è altro che una combinazione lineare a coefficienti costanti di $\lambda(x,\ z)$ ed eventualmente delle sue derivate rispetto a z, calcolate pei valori di z che sono i poli di $\rho(t)$.

Ora, quando $\phi(t)$ varia nello spazio S, una classe di trascendenti è costituita nello spazio T, e questa classe riconosce a capofila, per così dire, quella trascendente che si ha per $\phi(t)=1$, cioè quella in cui $\phi(t)$ è l'elemento più semplice in S; e questa trascendente capofila è

$$(11) \qquad l(x,\ z)=\int_0^{\infty} \frac{e^{-xt}dt}{t-z},$$

trascendente ben nota, poichè essa non è altro che

$$e^{-zx}li(e^{zx}),$$

dove li è il logaritmo integrale. Le varie $\lambda(x,\ z)$ hanno un comportamento analitico che offre con quello della (11) la più grande analogia. Le trascendenti di questa categoria forniscono gli integrali delle equazioni lineari differenziali, a coefficienti costanti, d'ordine finito od infinito; in particolare, si hanno soluzioni delle equazioni alle differenze finite che sono in stretta relazione colle soluzioni, dette principali, che Nörlund ha considerato nei suoi recenti ed importanti lavori.

14. Ho accennato così al caso più semplice della utilizzazione della trasformazione B nel calcolo funzionale qualitativo; ma questo caso si può generalizzare in varie direzioni, di cui voglio indicare alcune.

Anzitutto, si può sostituire, alle generatrici del precedente spazio S, classi più generali di funzioni: analitiche lungo una striscia comprendente il cammino di integrazione, ed anche non analitiche. La funzione determinante è tuttavia analitica, regolare in un semipiano determinato se la generatrice ha un andamento asintotico di carattere esponenziale; essa può ancora venire assoggettata alle operazioni di tipo A_ρ: in particolare, si può dare la soluzione di equazioni lineari differenziali o alle differenze a coefficienti costanti il cui secondo membro sia una tale funzione determinante, e decomporre questa soluzione in termini della forma

$$\lambda(x, z_i)$$

e

$$\frac{\partial^r \lambda(x, z_i)}{\partial z_i^r}.$$

Oppure, si può generalizzare l'operazione eseguita su $\phi(t)$; considerare, ad esempio, in luogo della moltiplicazione per $\rho(t)$, l'applicazione a $\phi(t)$ di una espressione differenziale lineare normale e dedurne le soluzioni della trasformata mediante B: una questione di questo genere risale a *Poincaré*.

15. Ma, rimanendo alla considerazione della moltiplicazione operata sulla funzione generatrice, si prenda come moltiplicatore la potenza t^α, α essendo un esponente a parte reale > -1. La trasformata mediante B di questa semplice moltiplicazione è, per ogni elemento $f(x)$ di T, una operazione che, per tutti i motivi che guidano nell'applicazione del principio di permanenza, definisce la derivata d'indice α di $f(x)$, moltiplicata per $e^{\pi i \alpha}$. La proprietà $D^\alpha D^\beta = D^{\alpha+\beta}$ viene da sè soddisfatta, ed una relazione che si deduce immediatamente da

$$e^{\pi i \alpha} D^\alpha f(x) = \int_0^\infty e^{-xt} t^\alpha \phi(t) dt \tag{12}$$

permette di estendere la definizione agli indici α la cui parte reale è < -1. Vi è eccezione solo per i valori interi negativi di α; si vede però che il secondo membro d (12) definisce una funzione meromorfa di α che ha, per $\alpha = -1, -2, -3, \ldots$ poli di prim'ordine. Se ci riferiamo ancora all'elemento più semplice di T, ad $\frac{1}{x}$, l'operazione (12) dà come risultato

$$\frac{e^{\pi i \alpha}\Gamma(\alpha+1)}{x^{\alpha+1}}$$

Le operazioni $D^\alpha(-1)^\alpha$ così definite formano un gruppo ad un parametro, e non è senza interesse il notare la operazione infinitesima di questo gruppo,

$$\int_0^\infty e^{-xt}\phi(t) \log t \, dt$$

16. Questo caso suggerisce lo studio della trasformate mediante B di moltiplicazioni in cui il moltiplicatore abbia forma meno semplici. Un tale studio è stato tentato solo in qualche caso speciale: se il moltiplicatore è una irrazionale quadratica della forma $(t+\sqrt{1+t^2})^n$, si ottiene una classe di operazioni che, applicate agli elementi di T, danno una categoria di trascendenti che ricono-

scono come capofila le funzioni di *Bessel*, cioè che si riducono a queste se l'elemento di T cui si applicano è la semplice $\frac{1}{x}$; se l'irrazionale è della forma

$$\left((1+t^2)(1+k^2t^2)\right)^{\frac{n}{2}},$$

n essendo intero, si ha una categoria di trascendenti che hanno proprietà analoghe e di cui le capofila danno una notevole generalizzazione delle Besseliane.

Uno studio generale in questo ordine di idee è ancora da farsi, ma non mi pare dubbia la sua utilità in un tentativo per una classificazione delle trascendenti.

17. Concludo questa conversazione, che forse ha abusato della Vostra cortesia, riprendendo per un istante l'operazione che, applicata allo spazio S delle serie di potenze, era definita da

$$Q(1)=a_0-x,\ Q(x^n)=a_nx^n-x^{n+1},\ (n=1,2,3,\ .\ .\ .),$$

ammettendo, per fissare le idee, che sia

$$\lim_{n\to\infty} a_n=a.$$

L'operazione Q, applicata ad una serie di potenze, avente un raggio di convergenza $r>|a|$, dà come risultato una serie avente il medesimo cerchio di convergenza; ma se è

$$Q(\Sigma h_nx^n)=\Sigma k_nx^n,$$

si ha

$$\Sigma k_na_0a_1\ .\ .\ .\ a_{n-1}=0, \tag{13}$$

la serie del primo membro essendo per altro assolutamente convergente: per modo che, come si è già osservato, la operazione Q, nello spazio S_a delle serie di potenze convergenti in cerchi di raggio $>|a|$, è degenere della seconda specie. Se Ψ è un elemento di S_a, $Q^{-1}(\Psi)$ sarà dunque alla sua volta elemento di S_a soltanto sotto la condizione che i coefficienti k_n di Ψ verifichino l'uguaglianza (13). Senza di ciò, l'espressione di $Q^{-1}(\Psi)$, cioè, la soluzione dell'equazione.

$$Q(\omega)=\Psi \tag{14}$$

può ottenersi soltanto coll'*aggiunzione* di un elemento non appartenente ad S_a, e precisamente coll'aggiunzione della funzione.

$$\theta_a(x)=\Sigma\frac{x^n}{a_0a_1\ .\ .\ .\ a_n}, \tag{15}$$

di cui $|a|$ è esattamente il raggio di convergenza.

Tutte le funzioni $Q^{-1}(\Psi)$, qualunque sia l'elemento Ψ per il quale la (13) non sia soddisfatta, hanno sulla circonferenza di raggio $|a|$ la stessa singolarità, nel senso che è

(16) $$Q^{-1}(\psi)=k\theta_a(x)+\psi_1(x)$$

dove $\psi_1(x)$ è un elemento di S_a, e k una costante.

L'analogia della (13) colla condizione di divisibilità di una serie di potenze per $x-a$ è manifesta; la condizione (13) si riduce appunto, per $a_0=a_1= \ldots =a_n \ldots =a$ alla condizione $\sum k_n a^n=0$ di divisibilità; il termine $k\theta_a(x)$ in (16) isola le singolarità di $Q^{-1}(\psi)$ sulla circonferenza, come il termine $(x-a)^{-1}$ isola il polo nel quoziente $\frac{\psi(x)}{x-a}$, ed infine la costante k ha l'ufficio stesso del residuo.

Non abuserò più oltre della Vostra pazienza; mi basti aggiungere che questa analogia colla divisibilità può proseguirsi assai oltre, permettendo, in casi notevoli, di separare e di isolare le singolarità in varie classi di funzioni analitiche.

MODERN NORWEGIAN RESEARCHES ON THE AURORA BOREALIS

BY PROFESSOR CARL STØRMER,
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I have the honour in this lecture to give you a short report on the Norwegian researches on the *aurora borealis* during the last thirty years.*

Here in Canada you ought to be well acquainted with these remarkable phenomena whose real nature has been a great mystery until the last few years. I think it is not too pretentious to say that most of this mystery has been elucidated by the Norwegian researches. One of the reasons for the success of these researches is the very favourable situation of Norway for the observation of the aurora. If you look at a chart of the frequency of the aurora, you will see lines passing through the regions where the frequency is the same and the thickest of them, corresponding to the maximum frequency, lies over northern Norway. Here at the Alten Fjord you have the well known observation place Bossekop where numerous expeditions have studied the aurora and where the conditions for studying it are also most excellent. The great extent of Norway north and south has also been most important for studying the different types of aurora and their occurrence and movements during magnetic storms.

The interest in these remarkable phenomena has been most lively ever since the middle ages. Thus we find in the old saga Kongespeilet (The Kings Mirror) of the thirteenth century a description of the aurora with an attempt at explanation remarkable for that time.

In modern times, during the second part of the last century, Sophus Tromholt spent most of his life collecting all Norwegian observations regarding the aurora from the earliest times up to 1878. This has been published since his death by Professor Schroeter at the cost of the Videnskabsselskabet and the Nansenfondet in Christiania.

The new epoch in the researches on the aurora began with the beautiful experiments of my late colleague Professor Kristian Birkeland. It was just at the time of the discovery of the Röntgen rays, and physicists all over the world were interested in experimenting with cathode rays. Birkeland made a series of experiments with these rays in magnetic fields. He discovered an interesting phenomenon which he called the suction of cathode rays towards a magnetic

*See *Les aurores boréales, Conférence faite à la Sorbonne le* 14 *décembre* 1923 par Carl Størmer. Livre du Cinquantenaire de la Société Française de Physique, Paris, 1925.

pole. A magnetic pole has an effect on a beam of parallel cathode rays analogous to that of a lens upon a beam of light, namely to make them converge towards a point.

This phenomenon led him in 1896 to the idea that the aurora borealis was due to a similar effect of the earth's magnetism on cathode rays coming from the sun. In order to test his hypothesis, Birkeland exposed a small spherical electromagnet to a stream of cathode rays and the result was most promising. In 1900 he describes his experiment in the following words:

"If the little globe was unmagnetized, only the half of it that was exposed to cathode rays was shining with an evenly distributed light. As soon as the globe was rendered magnetic the rays were thrown away from the surface of the globe except at certain places near the magnetic poles (Fig. 1). Both near the north pole and near the south pole the rays come down to the little earth-model in inclined striated wedges of light, and these wedges can be seen outside the globe up to 5 centimetres from its surface. These luminous wedges strike the surface of the model and produce two luminous narrow bands, one near the north pole and one near the south pole. Each of these bands stretches along the latitude of 70° from a point directly opposite the cathode and along the afternoon and night side of the earth-model, the cathode being considered as the sun. No corresponding light is seen on the morning and afternoon side of the globe."

During the following years Birkeland made a long series of similar experiments, which showed a striking resemblance to the aurora belts.

Birkeland has published the results of his experiments on cathode rays and of his extended studies of magnetic storms in two large volumes,* a work which certainly will be of great importance for future researches.

His later experiments were made with a large glass reservoir having a volume of about 1000 liters. In Fig. 2 are shown a series of photographs of the artificial aurora belts at the poles of the magnetic globe. When the magnetization is weaker, the phenomena are quite different.

At the beginning of 1903 I became extremely interested in Birkeland's experiments and in his theory of the aurora; knowing that the phenomenon of the suction of cathode rays towards a magnetic pole had been mathematically treated by Poincaré, I thought it might be worth while to find by mathematical methods the trajectories of electric corpuscles in the magnetic field of the earth, hoping in this way to find again not only the details of Birkeland's experiments but also the principal features of aurora and of magnetic storms.

When one has to solve a problem as difficult as that of the aurora borealis, it is nearly hopeless to begin with the most general assumptions. The only reasonable way, it seemed to me, was to start from a series of simplifying hypotheses and to attempt to solve the mathematical problem in this ideal case. Afterwards, when the first simplified problem was solved, one could go on to

*See *The norwegian aurora polaris expedition* 1902-1903, New York, Longmans Green & Co., 1913.

solve the aurora problem under more general assumptions approaching more nearly the real conditions in nature.

As a simplifying hypothesis I first consider the motions of the earth and the sun as negligible, so that only their relative positions come into consideration. It is furthermore assumed that the force acting on the corpuscles is only the earth's magnetism, and for simplicity the magnetic field is considered to be that of a uniformly magnetized sphere,—or, what amounts to the same thing, to be that of an elementary magnet at the earth's centre with its magnetic axis coinciding with that of the earth. Thus we neglect in this first approximation the mutual action between the electric corpuscles as well as the action of the sun's magnetic field.

Nevertheless it is surprising how many details of the auroral phenomena we can explain by these simplified hypotheses.

The mathematical treatment and solution of the resulting problem, viz., to find the trajectories of electric corpuscles in the field of an elementary magnet, is itself a very heavy task. It is impossible in this short lecture to give a more detailed report on my mathematical researches on this problem during the years 1903-1907. Only a few of the more striking results will be mentioned. The differential equations which determine the trajectories are herewith given:

$$\frac{d^2x}{ds^2} = \frac{1}{r^3}\left[3yz\frac{dz}{ds} - (3z^2-r^2)\frac{dy}{ds}\right],$$

$$\frac{d^2y}{ds^2} = \frac{1}{r^3}\left[(3z^2-r^2)\frac{dx}{ds} - 3xz\frac{dz}{ds}\right],$$

$$\frac{d^2z}{ds^2} = \frac{1}{r^3}\left[3xz\frac{dy}{ds} - 3yz\frac{dx}{ds}\right].$$

The elementary magnet (Fig. 3) is at the origin of the system of coordinates and the unit of length C is chosen equal to the square root of $\frac{M}{H\rho}$. M is the moment of the magnet and $H\rho$ is a product characteristic of the corpuscles. The arc of the trajectory is s, which is chosen as independent variable.

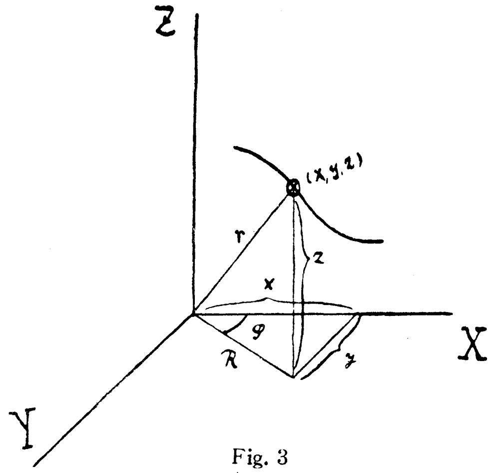


Fig. 3

If in place of x and y one introduces the polar coordinates R and ϕ where $x=R\cos\phi$, $y=R\sin\phi$ it will be found that it is possible to integrate one of the resulting equations. This introduces a constant of integration γ, and we are led to the transformed system:

$$R^2\frac{d\phi}{ds}=2\gamma+\frac{R^2}{r^3}$$

and

$$\frac{d^2R}{ds^2}=\tfrac{1}{2}\frac{\partial Q}{\partial R},\quad \frac{d^2z}{ds^2}=\tfrac{1}{2}\frac{\partial Q}{\partial z},\quad \left(\frac{dR}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=Q,$$

where

$$Q=1-\left[\frac{2\gamma}{R}+\frac{R}{r^3}\right]^2.$$

This system not only can be interpreted in a very useful mechanical manner, but it is also very convenient for numerical integration.

We will only give a geometrical consequence of the equation for the angle ϕ. It is easily seen that $R\frac{d\phi}{ds}$ is the sine of a certain angle θ and thus we obtain the formula:

$$\sin\theta=\frac{2\gamma}{R}+\frac{R}{r^3}.$$

Now $\sin\theta$ along the trajectory is confined to the interval -1 to $+1$. The trajectory corresponding to a given value of γ must then be limited to the region of space where

$$-1\leqq\frac{2\gamma}{R}+\frac{R}{r^3}\leqq 1.$$

To each value of γ thus corresponds a region Q_γ to which all trajectories corresponding to this value of γ are confined.

In Fig. 4 is given a series of regions Q_γ. The upper row represents their sections cut by a plane through the magnetic axis. The regions are described by the white parts when rotated about the magnetic axis.

From this consequence of the differential equations which I already found in 1904 we can make an interesting application to the aurora. In this case the earth is very small compared with the dimensions of the regions Q_γ and thus we see that these regions strike the earth only in the polar regions, this corresponding to the fact that the aurora generally is confined to these regions.

To obtain further information about the trajectories a mechanical interpretation of the system in R and z, followed by a general discussion of the trajectories directly from the differential equations themselves, was made. In this discussion I found a most valuable aid in the methods of numerical integration of differential equations. In fact the mathematical methods gave the qualitative results, and the numerical methods gave quantitative results which could be directly applied to nature.

More than 5000 hours were spent on numerical calculations and in this extended work I received most valuable assistance from several young students of mathematics; the results were illustrated by wire models giving the shape in space of the calculated trajectories.*

The trajectories in the magnetic equatorial plane can be found exactly by elliptic functions. In space the trajectories must be found by numerical integration.

In Fig. 5 is shown a wire model of trajectories issuing from two separate points towards the elementary magnet placed in the centre of the little sphere. Fig. 6 shows the coincidence between this calculation and one of Birkeland's experiments.

For application to the *aurora borealis* the orbits which strike the little sphere are of special interest. The nearer the orbits come the more they take the spiral shape like the one resembling the geodetic lines on a narrow cone of revolution (Fig. 7). The orbits approach the elementary magnet in a corkscrew spiral of ever narrowing windings, reach a minimum distance from the magnet, then recede. The more this minimum distance approaches zero, the more the trajectory approaches an orbit of a special kind passing through the elementary magnet, an orbit which I have termed the "trajectory through the origin".

In Fig. 8 we have a series of trajectories of this kind corresponding to cathode rays coming from the sun and reaching the earth in the north polar regions. A corresponding wire-model of the trajectories in the neighbourhood of the magnetic globe is given in Fig. 9. It is apparent that the points of precipitation are located on the afternoon and night side, corresponding to Birkeland's experiments.

These points of precipitation lie about the magnetic axis in a sort of spiral the form of which is in full agreement with experiment.

Also another series of Birkeland's experiments can be easily explained by the mathematical theory. For instance, the remarkable form of the luminous wedges of light going down towards the magnetic poles of the sphere are demonstrated (Fig. 10); this corresponds to the characteristic form of the region Q_γ to which the trajectories are confined.

A further application to the aurora of the numerical results on the trajectories yields a limitation towards the magnetic axis. Corpuscles coming from the sun cannot strike the earth too near the axis. Thus we get auroral zones in the form of belts around the magnetic axis, the northern ones of which correspond to the zone of maximum frequency of the aurora borealis.

The formation of auroral rays and of auroral curtains are also easily explained, but for the details I must refer to my published papers.

Since the magnetic axis of the earth is rotating with the earth, the relative positions of this axis to the sun are ever changing. This gives a continual varia-

*See *Sur les trajectoires des corpuscules électrisés dans l'espace*, etc. Archives des Sciences Physiques et Naturelles, Genève 1907 and *Résultats des calculs numériques*, etc., I, II, III, Videnskabsselskabets skrifter 1913, Christiania.

tion of the initial conditions for the different orbits and corresponding fluctuations in the auroral phenomena.

This is especially so in the case of auroral curtains; the special conditions conducive to these curtains rapidly pass away and thus cause the short duration of these beautiful displays.

In spite of this fine harmony between theory and observation the agreement is incomplete in one respect: that is, the dimensions of the calculated and the observed auroral belts are not in accord, if the cathode rays are causing the aurora. Here the fact that we have neglected the mutual action of the corpuscles on each other is probably the cause.

For the cathode rays the angular radius of the auroral zone proves to be about 3° instead of 23°; and during magnetic storms the angular distance of the aurora from the magnetic axis can be even much greater, say to 30° or 40°.

To take into account this mutual action of streams of cathode rays is a very difficult problem, especially because these streams move in the earth's magnetic field where the problem of finding the trajectory of a single corpuscle is already so difficult.

I have only succeeded in obtaining one interesting result, and this is the following: If one assumes, corresponding to Birkeland's experiments and the theory of the trajectories, that a temporary corpuscular ring may exist in the earth's magnetic equator far out in space, the magnetic action of such a ring will draw the aurora away from the magnetic axis. A calculation I undertook in 1911 showed that a corpuscular ring can draw the auroral belt, even for cathode rays, from its theoretical position down to the real position without exercising greater magnetic effects on the earth than about a thousandth part of the earth's magnetism. If the ring is stronger, however, the aurora can be drawn even much farther away from the magnetic axis, and then the magnetic action of the ring on the earth will be of the order of that which we observe during magnetic storms. This is in accordance with the fact that during such storms the aurora is seen much farther south than usually.

It is very interesting that Carlheim Gyllensköld* and Ad. Schmidt† have both been led to assume the existence of such a corpuscular ring through their studies of great magnetic storms, what Birkeland had already found, and noted in his above mentioned work: *The Norwegian aurora polaris expedition 1902-1903.*

You see that the theoretical researches on the aurora are still in their infancy, and that one has a series of most difficult and attractive problems for future researches.

Now I must pass to the other phase of my own researches, which began in 1909 and which are still in progress. This is the study of the aurora by means of photography, especially the exact determination of its height and position in space. I was naturally led to these researches by my theoretical studies. It was necessary to try to confirm the theoretical results by observations, and

*See *Norrskenet och solens atmosfär*, Populae astronomisk tidsskrift. Stockholm, 1920, p. 124.

†See Encyclopädie der Mathematischen Wissenschaften, Bd. VI, 1, 10 Ad. Schmidt: *Erdmagnetismus*, p. 394.

the only reliable method seemed to me to be the photographic. When I began the work in 1909, the results obtained hitherto by this method were very poor. Only a single photograph with relatively short exposure, 7 seconds, had been obtained in 1892 by Brendel in Bossekop. I systematically tried a series of objectives and plates on auroras and found that a little "kinolens" with aperture 25 mm. and focal distance 50 mm., together with plates "Lumière, *etiquette violette*" solved the problem. I succeeded in taking good pictures of strong auroras with an exposure of less than one second.

In 1910 and 1913, I made expeditions to Bossekop in order to photograph the aurora and determine its height and position by simultaneous photographs from two stations connected by telephone. The method is very practical, but requires much work afterwards in measuring and calculating the photograms. Bossekop is situated in the northern part of Norway at about 70° latitude.

During the last expedition in particular I obtained a great many excellent photograms with a base of about 27 kilometers. On each plate was photographed at the same time as the aurora the face of a watch, and thus the exact time and the exposure could be seen later, on the negative, a very useful arrangement.

Fig. 11 shows an arch with the constellation *Cygnus* in the background The arch was faint and quiet which allowed a relatively long exposure. Height about 120 km.

In Fig. 12, we have a series of bands with *Vega* as reference star. Altitude about 105 km.

In Fig. 13 is an arch where curtains begin to develop to the right. Altitude of inferior border about 100 km.

Other curtains are seen in Fig. 14, with *Vega* in the background. The aurora descends to about 90 km.

Very beautiful draperies together with *Venus* are seen in Fig. 15. Altitude of lower border about 100 km. The draperies stretch to a distance of 700 km. from Bossekop towards the west.

In Figs. 16 and 17 are shown two stages of the development of an auroral curtain, each with an exposure of a few seconds. The altitude of the lower border is between 95 and 100 km. and the vertical extension between 15 and 20 km.

Lastly Fig. 18 shows a remarkable aurora which has the form of a luminous surface; height between 90 and 100 km: great horizontal extension and a very small vertical one.

The results of all the determinations of height are seen in Fig. 19, each altitude being marked by a dot. One sees how well marked is the lower limit of about 87 to 90 km. The maximum altitude is about 350 km.

The next year 1914, Krogness and Vegard made a great number of determinations of height in the same region, and by the same method. Their results were essentially the same. They have also made extended studies of the distribution of intensity with height for the different forms of the aurora.

Both my own report of the expedition of 1913 and the results obtained by Krogness and Vegard in 1914 are published in *Geofysiske publikationer*, Christiania (Vol. I).

After the extensive work done in Bossekop, it was of the greatest interest to study and measure the altitude of the aurora in other places also, especially further south. For this purpose I have since 1911 had a series of stations in southern Norway, where several hundred photograms have been obtained. Fig. 20 shows some of the base lines, the lengths varying between 27 and 258 km. There follow some interesting single pictures of auroras taken from these stations.

In Fig. 21 is a corona of the night of March 7 to 8, 1918. The constellation of the *Great Bear* is seen in the background.

During the night of March 22 to 23, 1920, an exceptionally great aurora was seen over Europe, Canada and the United States. I had 7 stations in action and more than 600 photographs were secured, among these about 40 coronas. Some pictures of coronas taken on that night are given here. Fig. 22 was taken early in the evening, the display being reddish in colour.

Another shown in Fig. 23, was yellow green. This was taken at 2 o'clock in the morning; finally a very beautiful one of blue colour is represented in Fig. 24. Of this blue corona, which occurred about 4 o'clock in the morning, I secured twelve very successful pictures well adapted to determine the position of the point of radiation.

Now we shall pass to the photograms and the determination of height.

Very fine rays observed March 4, 1920, are seen in Fig. 25. The lower border of these rays was at a height of 106 to 111 km.; the summits reached about 300 km.

During the remarkable aurora on the night of March 22 to 23, 1920, a great number of interesting photograms were taken both from two and from three stations simultaneously.

Fig. 26 shows one taken from 3 stations, which gives a good idea of the exactitude attained. A discussion of this case* has shown that the relative error in the determination of height probably does not exceed two per cent.

On this night we obtained highly interesting records of the altitude of auroral rays.

For instance, the rays shown in Fig. 27, photographed with a base of 66 km., had summits which surpassed 600 km. altitude.

A later picture of the same rays is seen in Fig. 28. (The stars are marked by arrows).

In Fig. 29 are shown rays to the west of Scotland photographed from Christiania, Oscarsborg and Horten simultaneously: foot 400 km., summit 600 km.

In Fig. 30 we have rays whose summits reached the enormous altitude of 750 km.

*See *Notes relatives aux aurores boréales*, Geofysiske publikationer, Christiania, 1922, Vol. II, no. 8.

On the same night I also measured aurora curtains down to 83 km. (Fig. 31). The reference stars were *Castor* and *Pollux*, and the base was 64 km.

All the photographs collected during the last 11 years from my stations in southern Norway are now measured and auroral heights and distances calculated. They will shortly be published in an extended report.* I shall here give some of the preliminary results:

As regards the distribution of the calculated heights (Fig. 32); a comparison with those obtained from my Bossekop expedition of 1913 shows that the auroral rays seen in southern Norway are very much higher than at Bossekop. On the other hand the lower limit of auroral curtains is lower in southern Norway than in Bossekop.

Fig. 33 exhibits the geographical distribution of the arches over Scandinavia.

The time is already so far advanced that I must pass rapidly over the later work on the aurora, that is to say, the photographing of the auroral spectrum and the possible artificial reproduction of it. As you know, my colleague, Professor Vegard has succeeded in photographing with great dispersion the spectrum of the aurora in northern Norway. He has obtained about 35 lines, of which all have been identified with lines of nitrogen except five, including the well known green aurora line. No traces of hydrogen and helium lines were found; but Vegard has not as yet photographed auroras of a height greater than about 160 km. The facts just mentioned led Vegard to a new hypothesis regarding the upper air, a hypothesis which he tried to verify last winter in the laboratory of Kammerlingh Onnes at Leyden. His experiments were most interesting, and he found that the spectrum of solid nitrogen when bombarded by cathode rays was of the same type as the characteristic auroral spectrum.

Vegard thought that he had by these experiments confirmed his hypothesis about the upper atmosphere, but recent experiments by Professor McLennan on solid nitrogen have thrown some doubt on this conclusion, and judgment must be reserved until more data are available.†

At all events Vegard has here opened up a new field of research which will certainly give rise to many new and valuable suggestions for solving the remaining mysteries of the aurora borealis.

I hope that the researches which have been carried on with so much ardour by Norwegian scientists, may also stimulate interest in these remarkable auroral phenomena here in Canada which is so admirably situated for an extended study of these beautiful displays.

**Resultats des mesures photogrammétriques des aurores boréales observés dans la Norvège méridionale de 1911 à 1922*, Geofysiske publikationer, Oslo, 1926, Vol. IV, No. 7.

†"Nature", Vol. 115, March 14th, 1925, p. 382 and Vegard's paper in Skrifter utgit av Det Norske Videnskabsakademi i Oslo, Math. Naturv. Klasse No. 9, 1925.

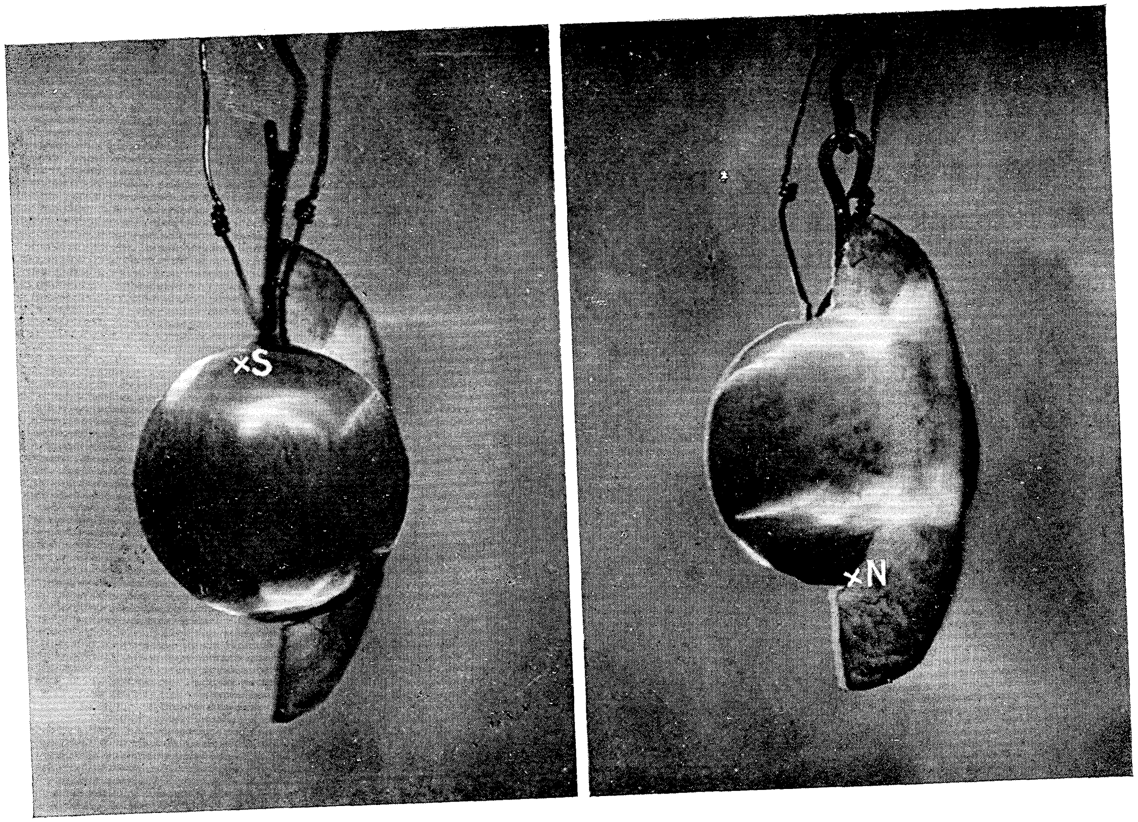


Fig. 1

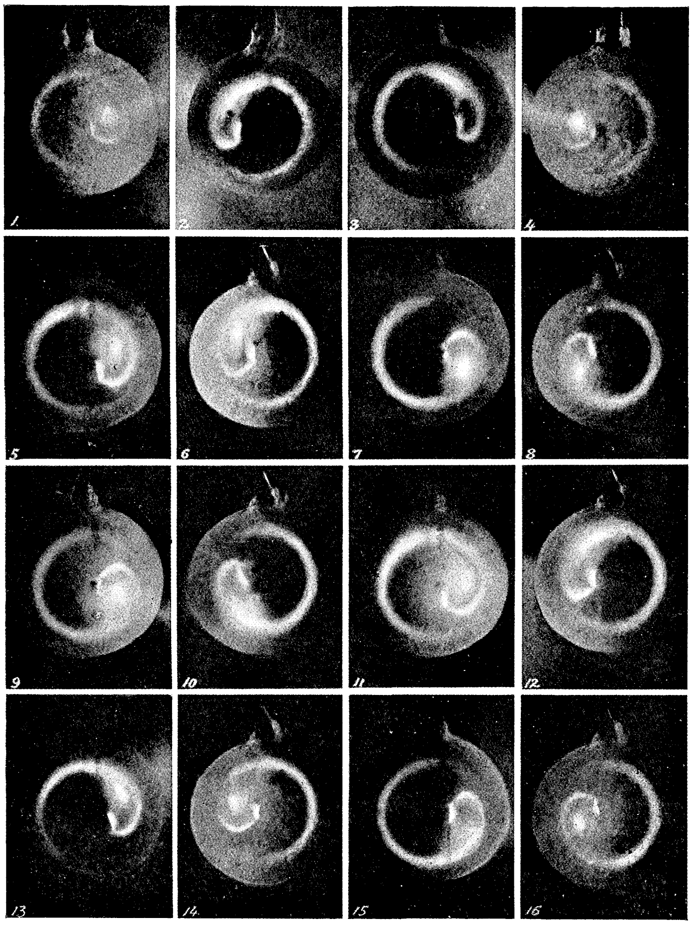
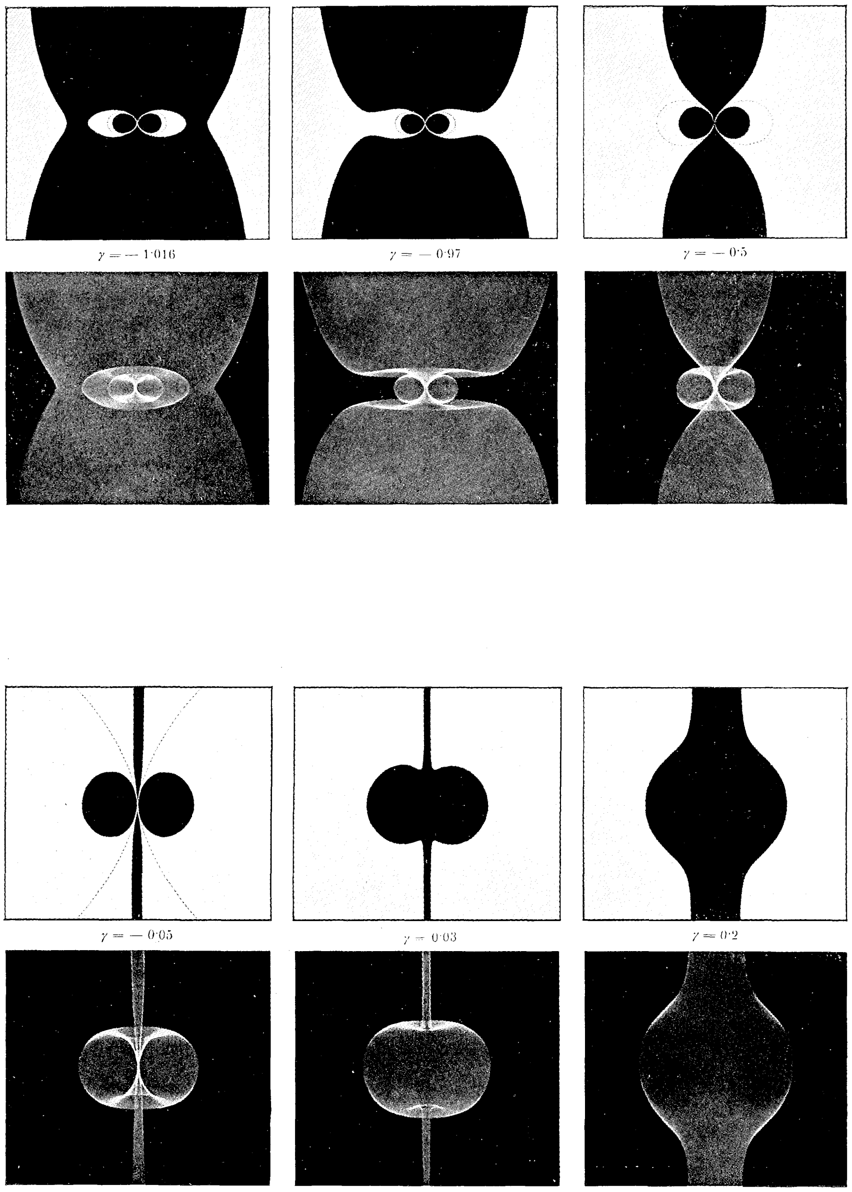


Fig. 2



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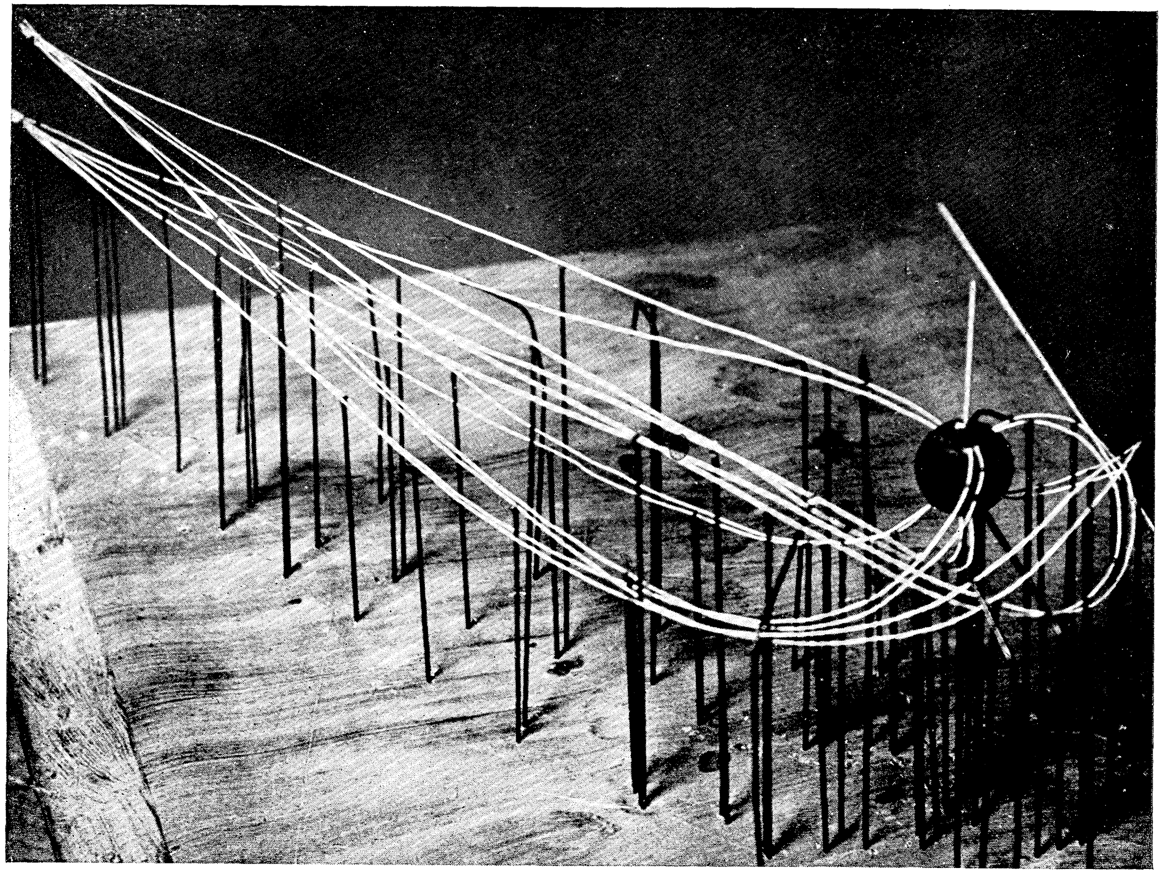
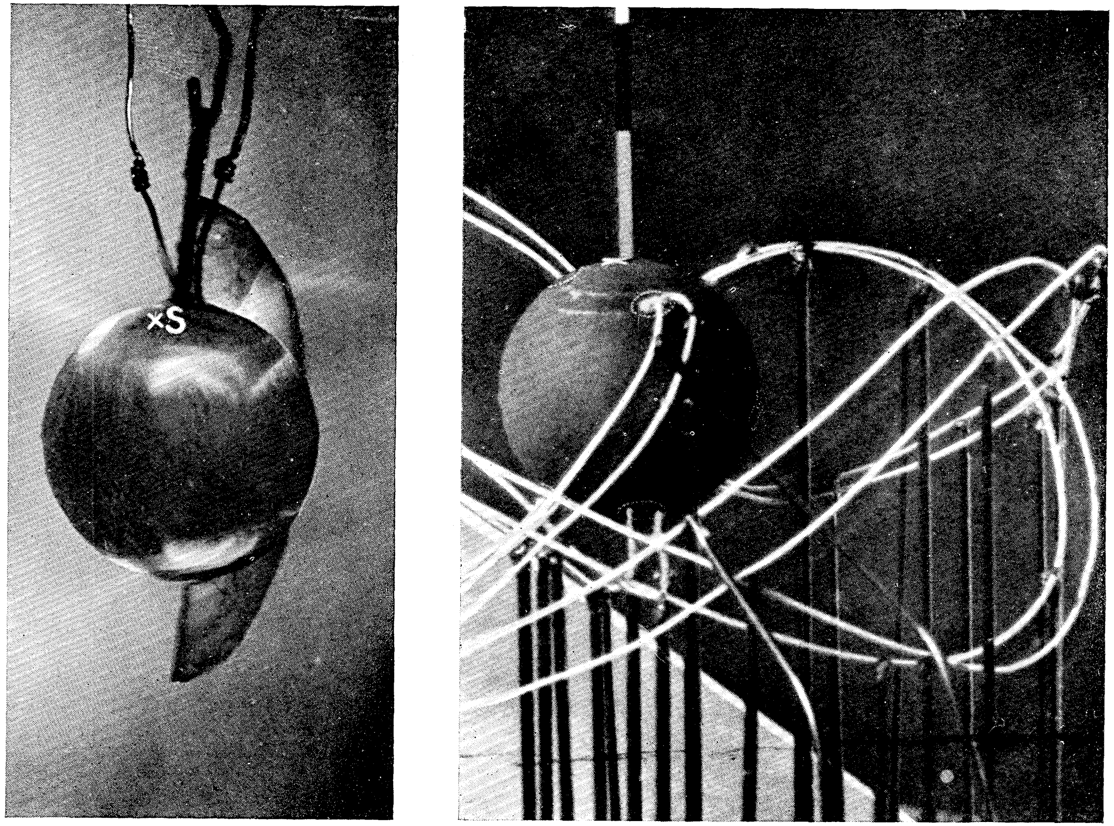


Fig 5



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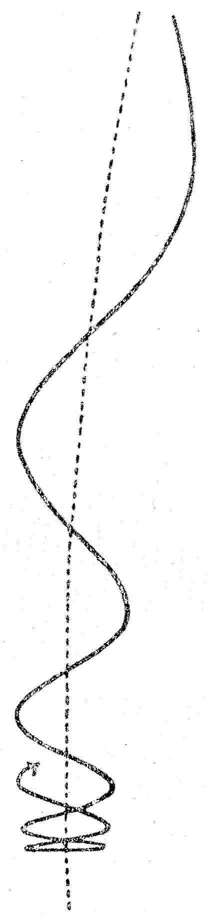
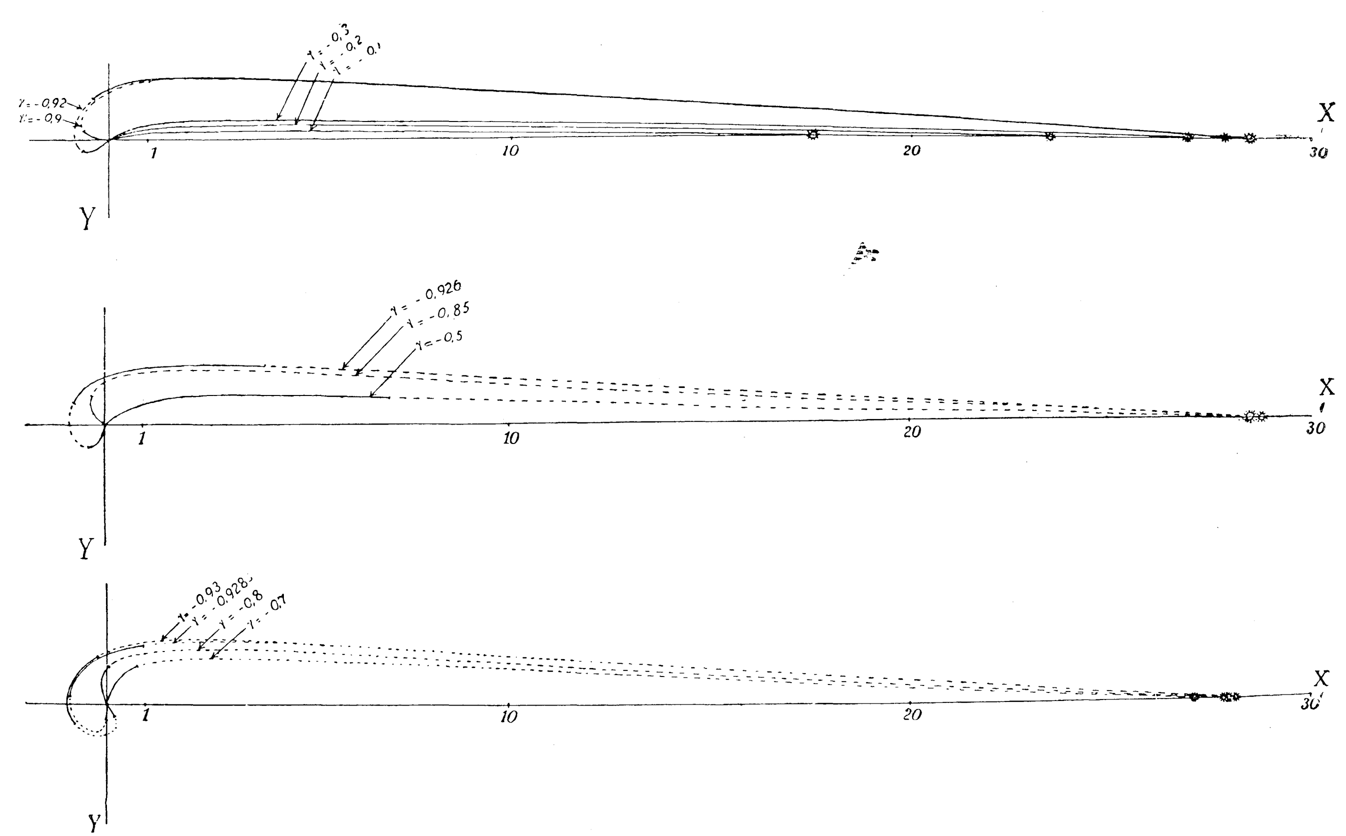


Fig. 7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γ = -0,928
γ = -0,8
γ = -0,7
X
Y
1
10
20
30

Fig. 8

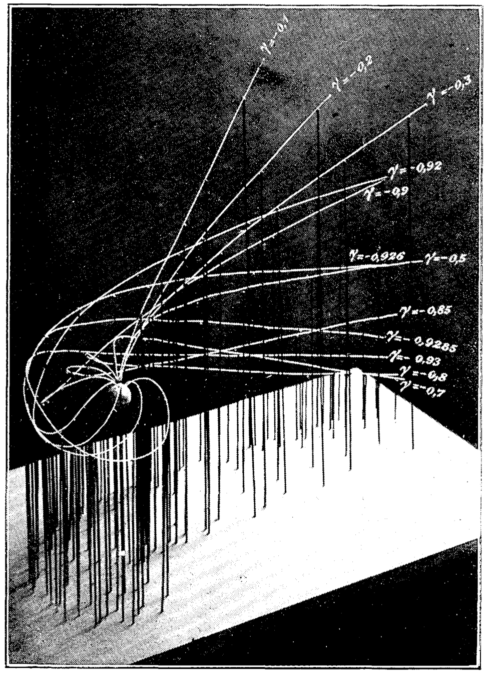


Fig. 9

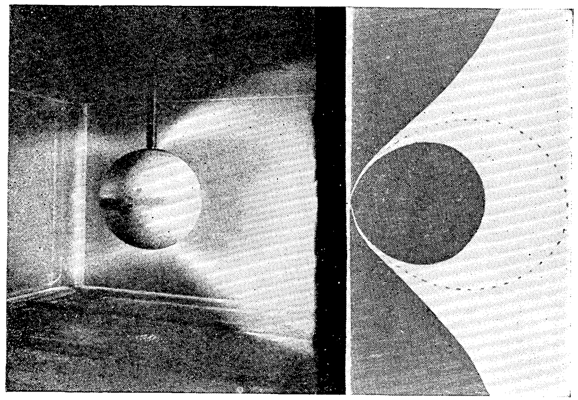
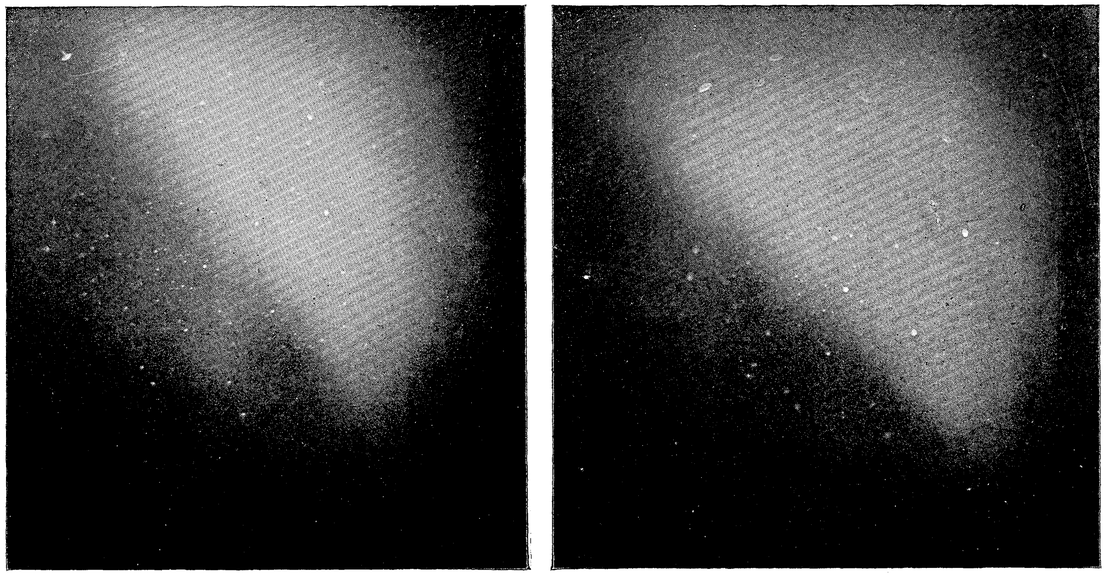


Fig. 10



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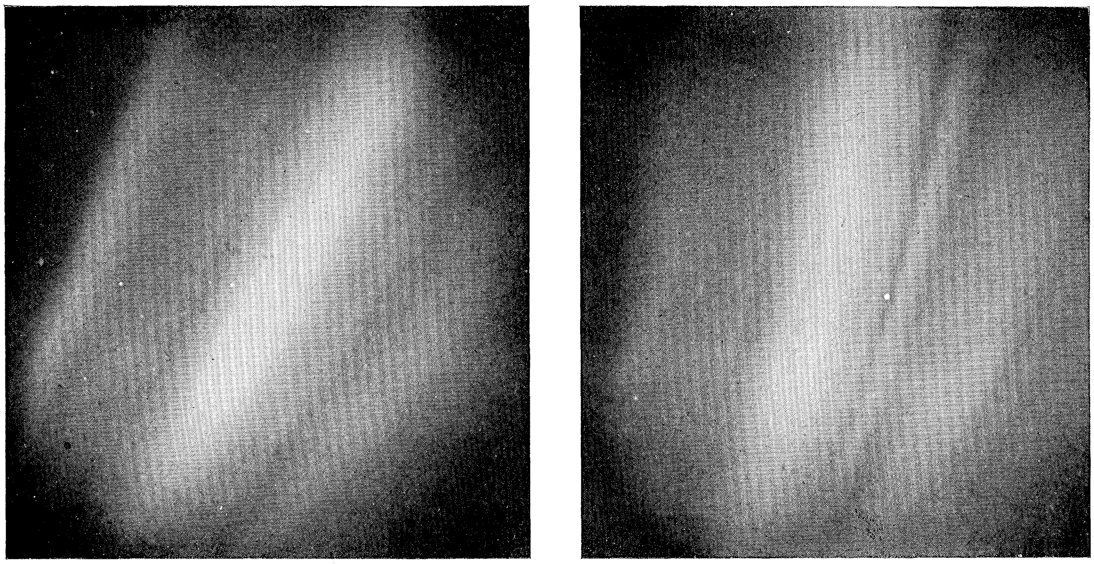

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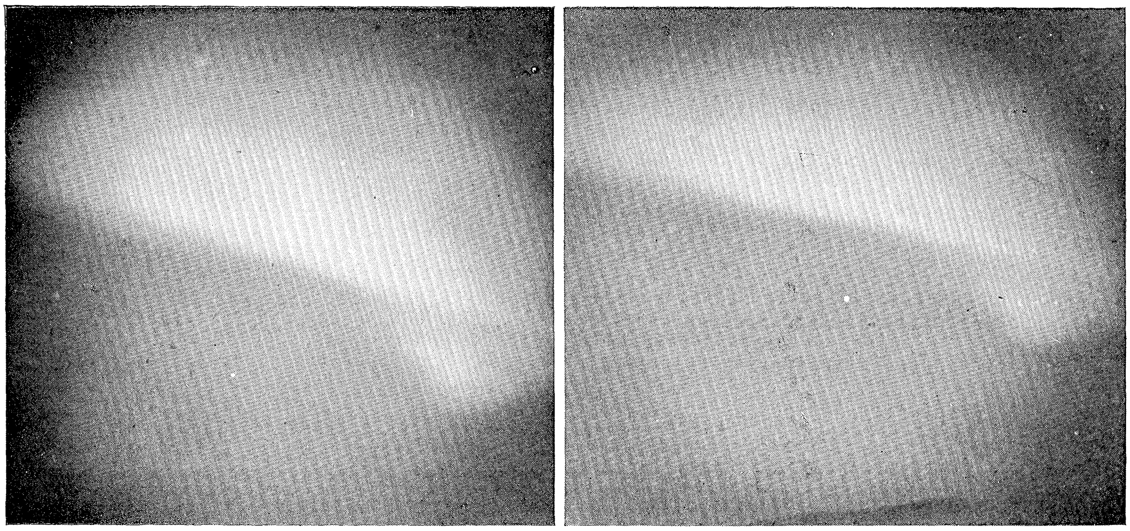
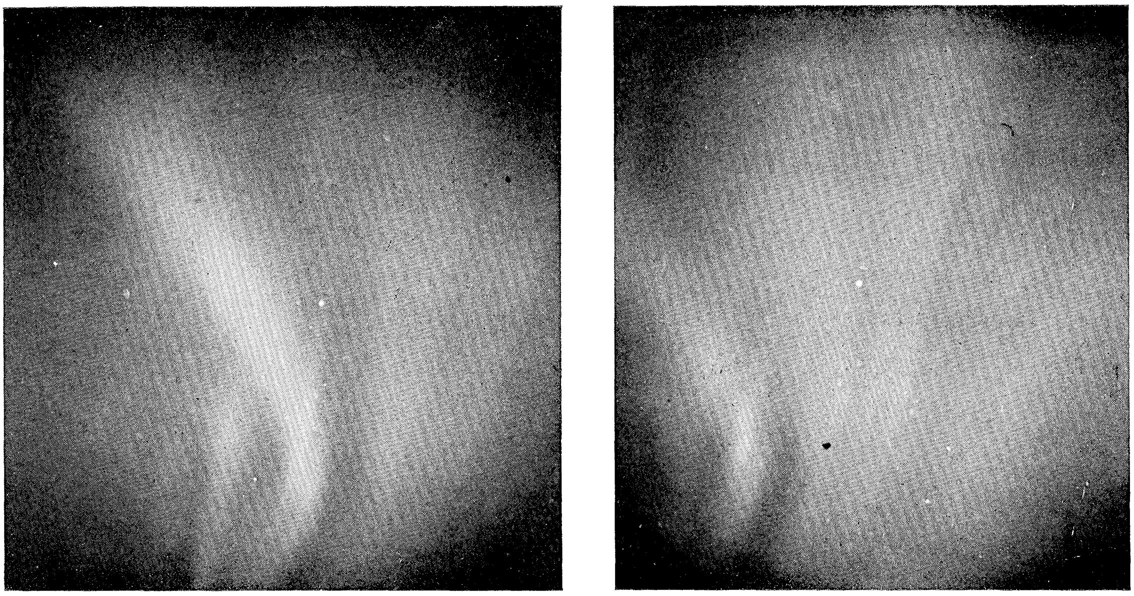


Fig. 13



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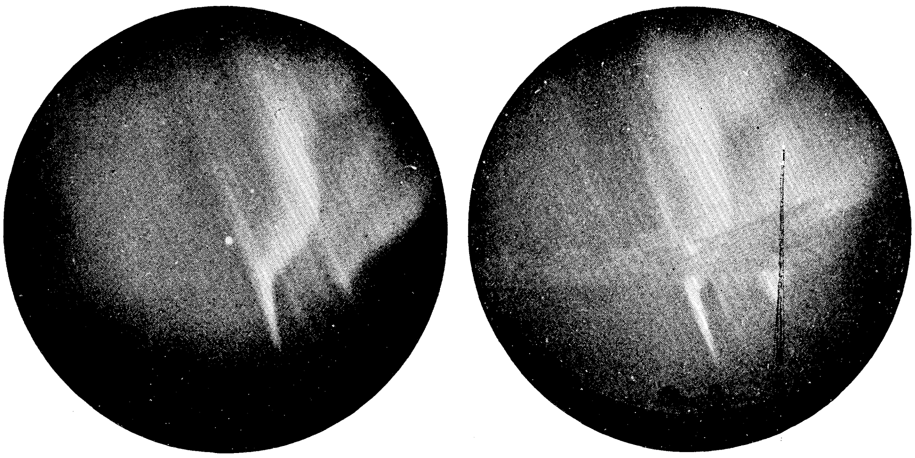


Fig. 15

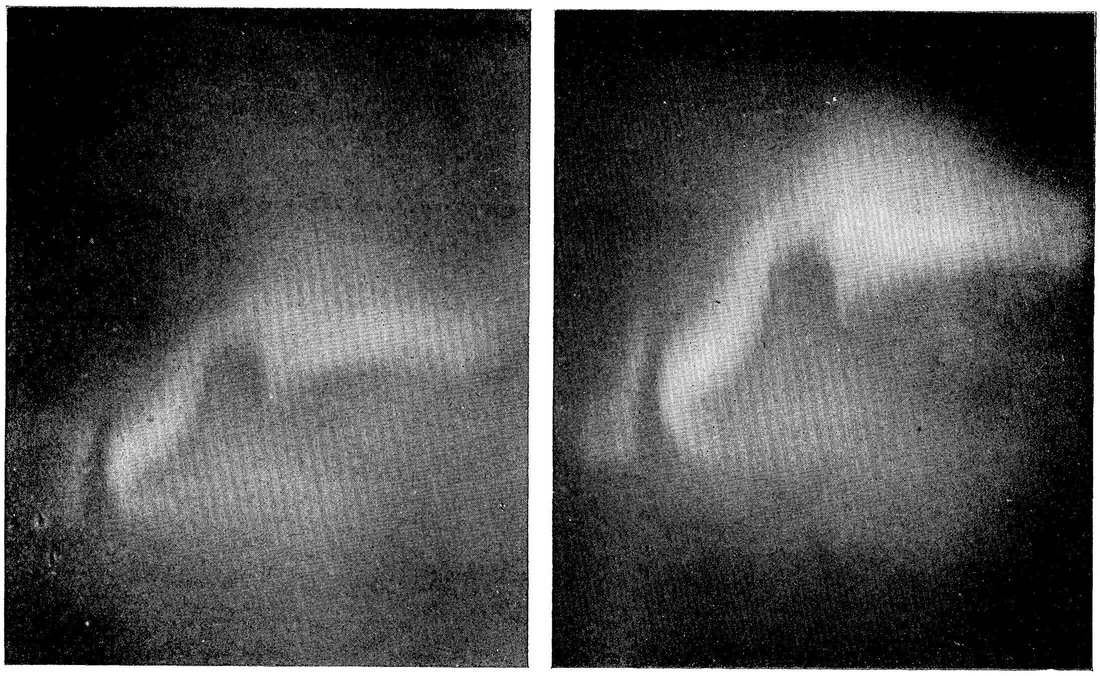
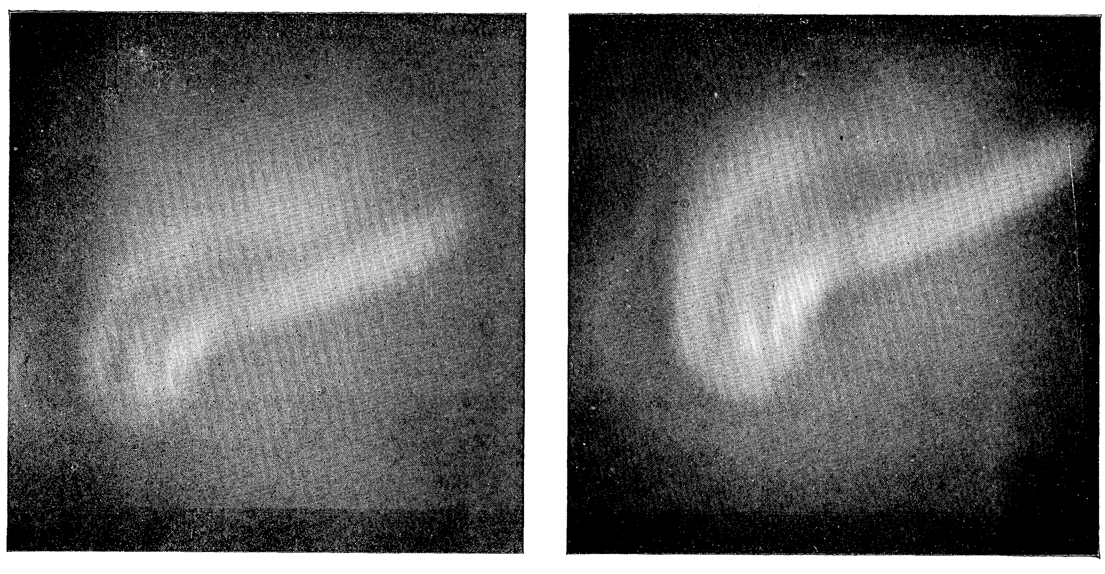


Fig. 16



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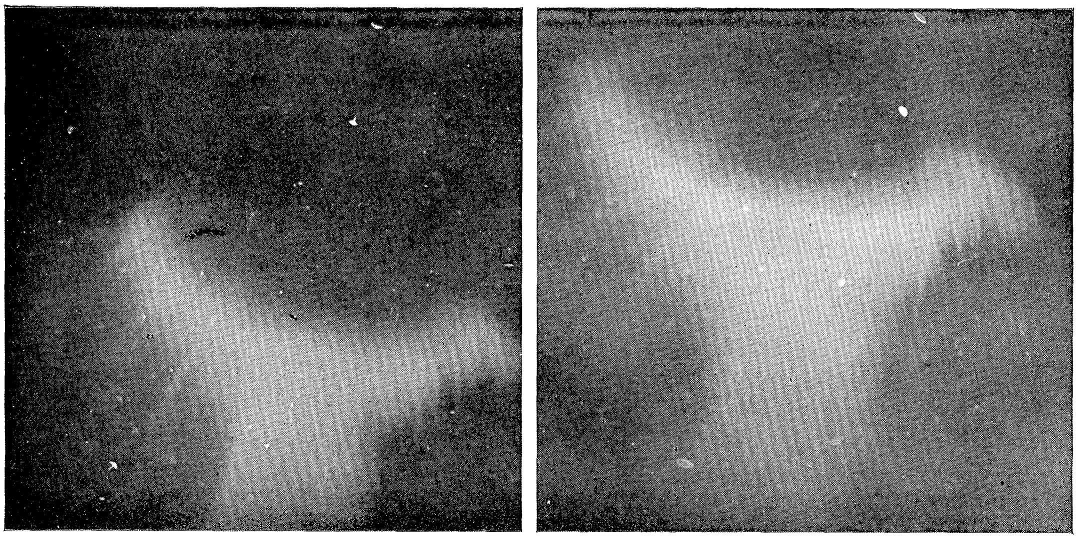

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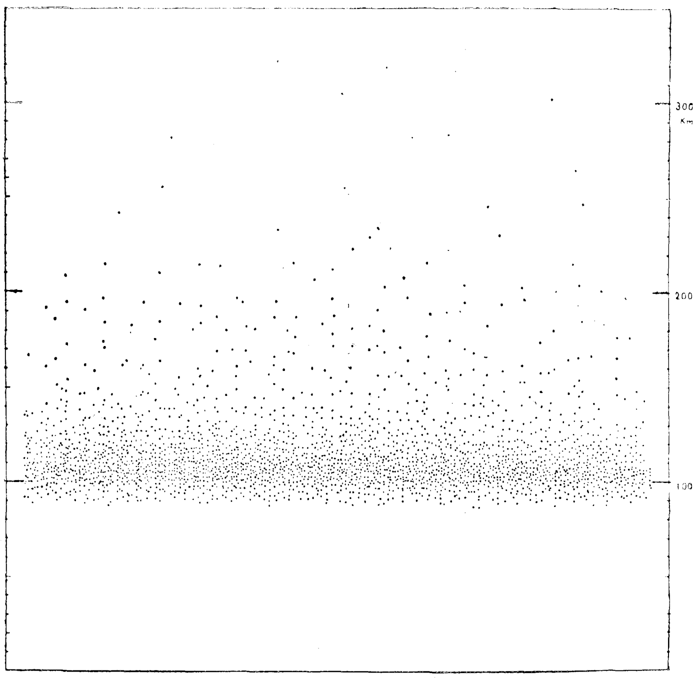

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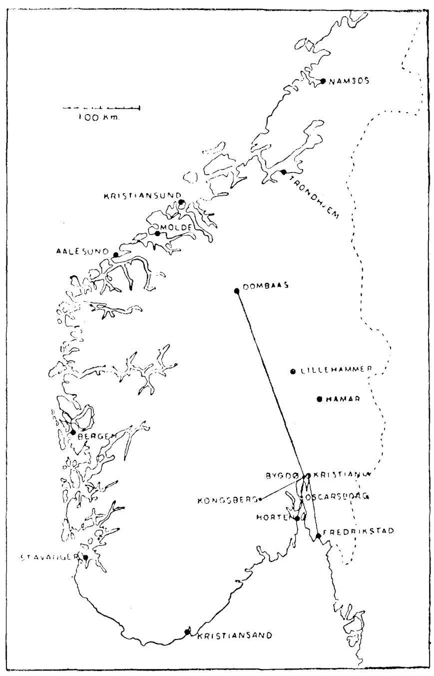
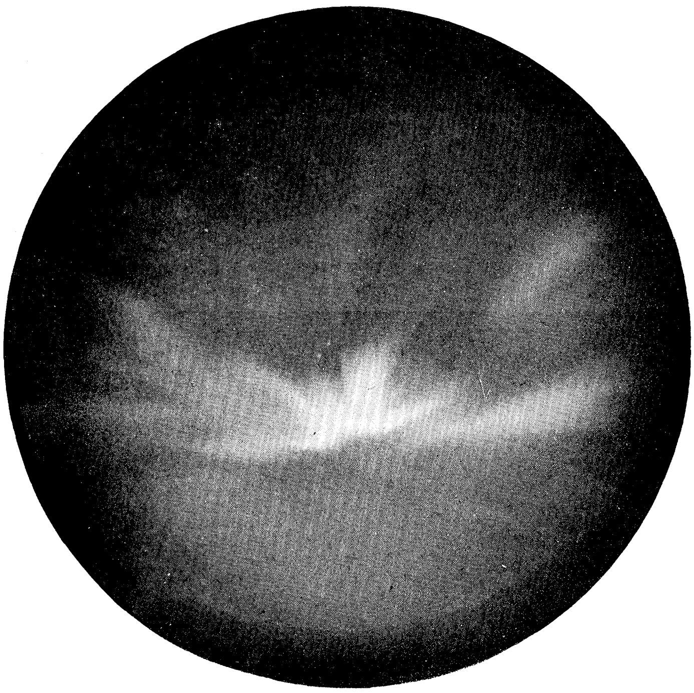


Fig. 20

Fig. 21



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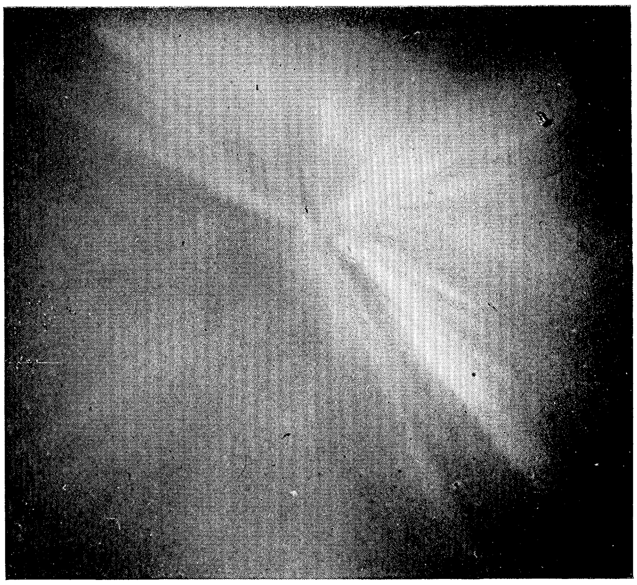


Fig. 23

Fig. 24

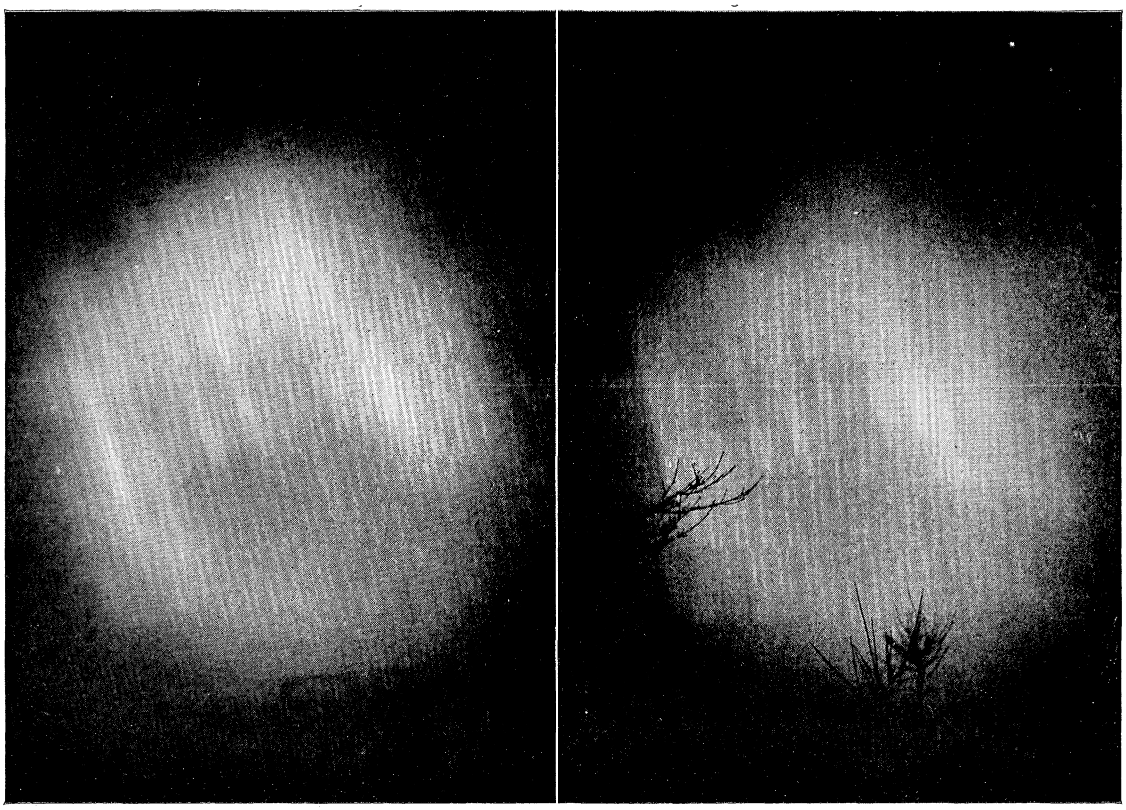
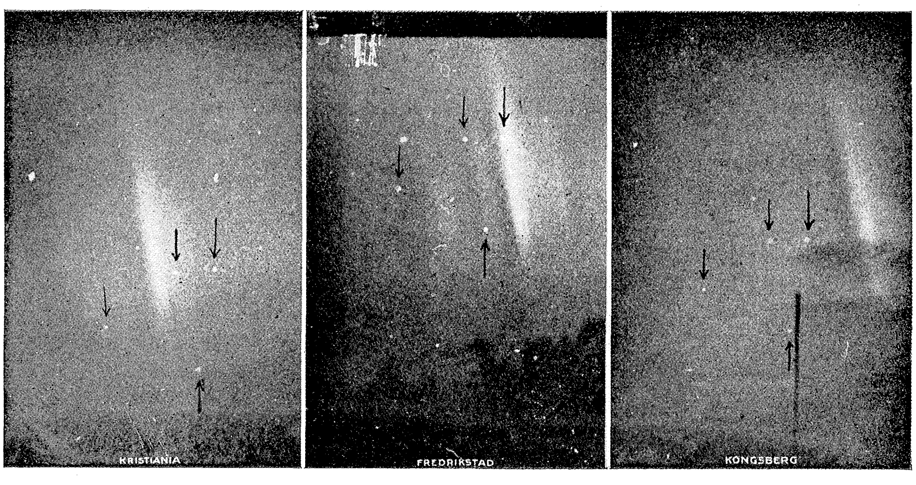


Fig 25



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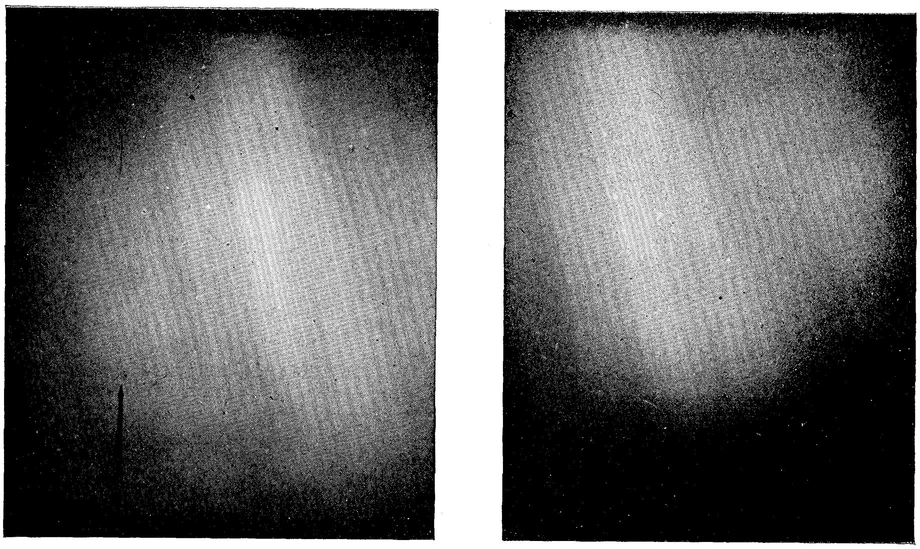

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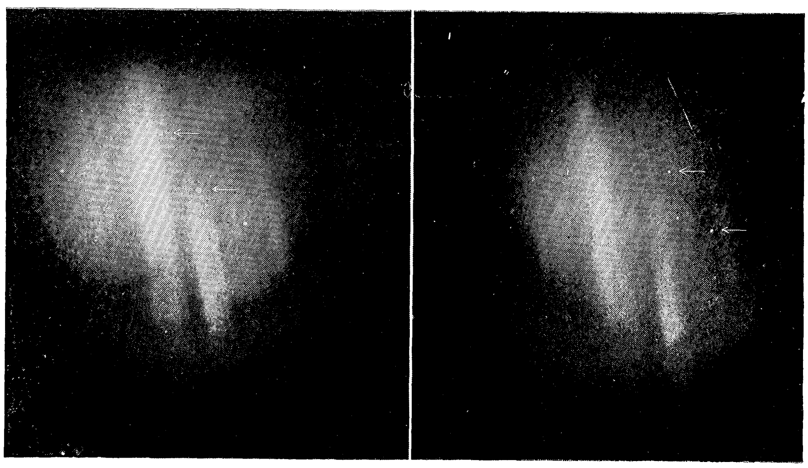

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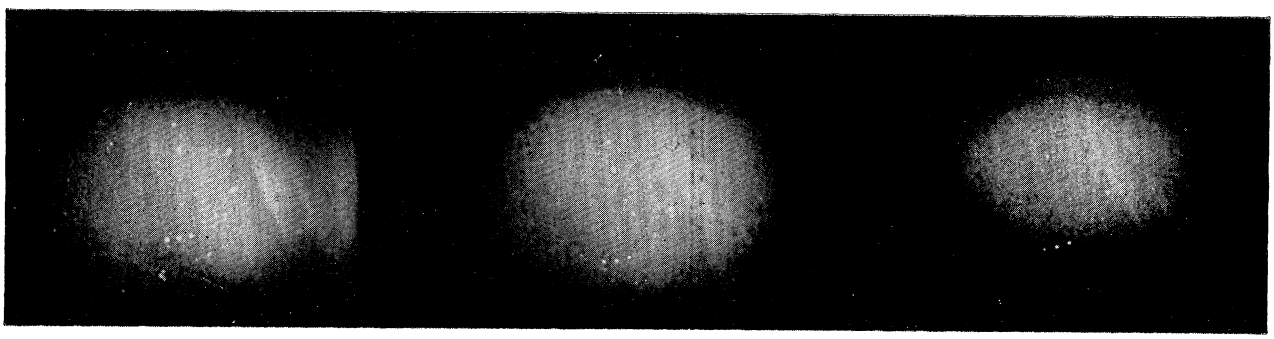

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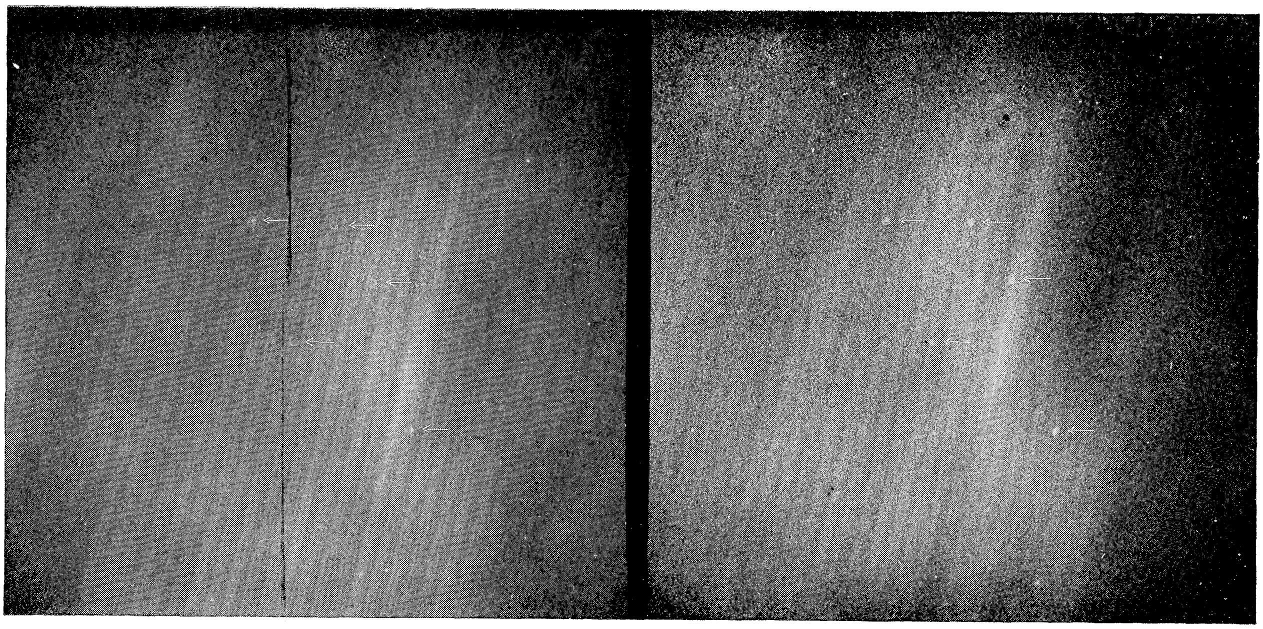

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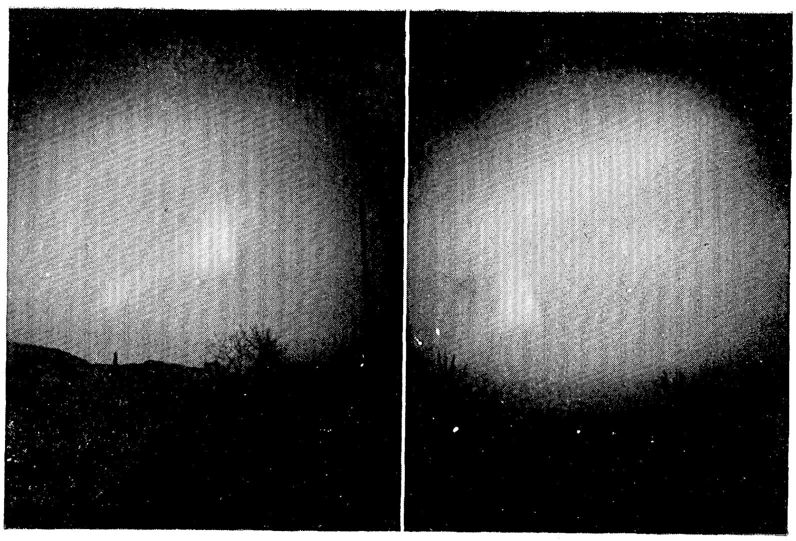

Fig. 31

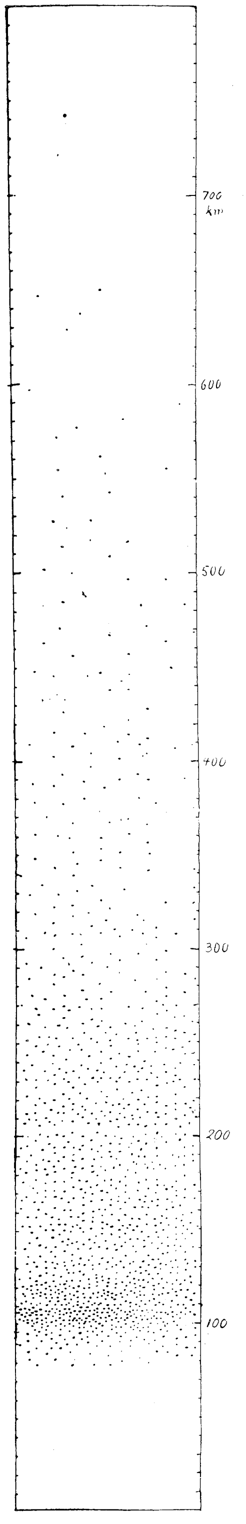

Fig 32

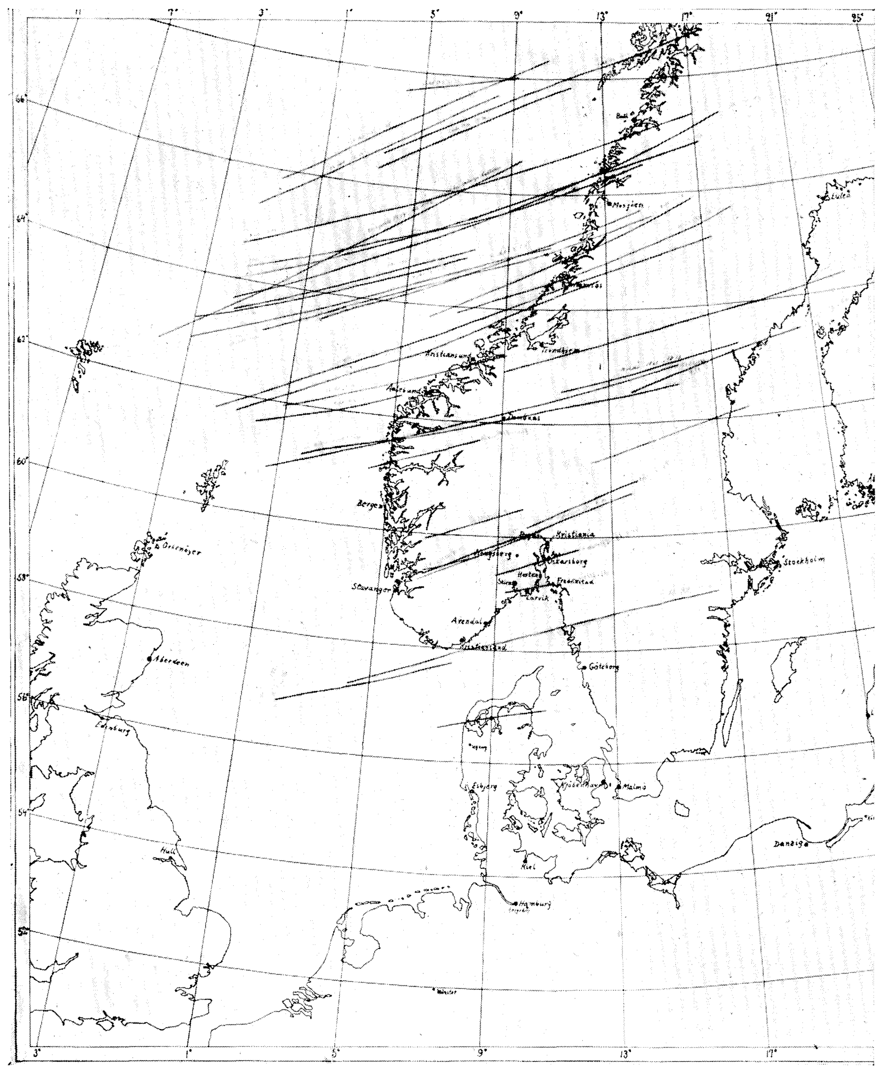

Fig. 33

LA GÉOMÉTRIE ALGÉBRIQUE

PAR M. FRANCESCO SEVERI,
Recteur de l'Université de Rome, Rome, Italie.

Je vais vous parler aujourd'hui de la géométrie algébrique, une des branches caractéristiques de la mathématique italienne.

Je me bornerai à un aperçu des progrès réalisés dans les vingt dernières années et des problèmes les plus importants qui se rapportent à cette branche et qui demeurent toujours en suspens.

Il faut d'abord que je rappelle la distinction des surfaces algébriques en régulières et irrégulières, qui se présente lorsqu'on étend aux surfaces la notion de *genre* des courbes.

Soit une courbe algébrique plane, d'ordre n, ayant seulement des points doubles, ce qui n'est pas restrictif au point de vue des transformations birationnelles. Il est alors bien connu que les courbes d'ordre $n-3$, passant par les points doubles de la courbe donnée, sont invariantes vis-à-vis des transformations birationnelles. Leur nombre est, d'après Clebsch, le genre p de la courbe, que Riemann avait auparavant considéré comme un caractère de connexion de la riemannienne correspondante.

Pour calculer p, on n'a qu'à faire la différence entre le nombre des paramètres dont dépend une courbe d'ordre $n-3$ et le nombre des conditions de passage par les d points doubles. On trouve que ce dernier nombre est exactement d, c'est-à-dire que les conditions susdites sont indépendantes.

Eh bien, le concept de genre peut se transporter aux surfaces, d'après Clebsch et Noether, de la façon suivante:

Soit F une surface algébrique d'ordre n, de l'espace à trois dimensions, douée de singularités ordinaires (ligne double et points triples); ce qui n'est point restrictif. Les surfaces d'ordre $n-4$ passant par la courbe double de F sont invariantes vis-à-vis des transformations birationnelles et leur nombre est le genre de la surface.

Mais une différence inattendue se présente ici, dans la comparaison avec le genre des courbes.

Lorsqu'on va calculer le genre de la surface, il faut, en premier lieu, chercher le nombre des conditions imposées par la ligne double aux surfaces d'un ordre l donné, qui doivent la contenir. C'est la formule de postulation relative à cette ligne double. Malheureusement, la formule n'est valable que pour des valeurs suffisamment grandes de l'ordre l, et il peut se faire que la limite de validité soit plus grande que $n-4$.

Lorsque cette limite ne dépasse pas $n-4$, le genre de la surface est donné par la valeur purement arithmétique calculée avec la formule de postulation; dans le cas opposé, on trouve que le genre est supérieur à cette valeur. Toutefois, dans chacun des deux cas, cette valeur est elle-même invariante vis-à-vis des transformations birationnelles. On l'appelle pourtant le *genre arithmétique* p_a de la surface, tandis qu'on appelle *genre géométrique* p_g le genre défini auparavant.

Les surfaces régulières sont celles pour lesquelles les deux genres sont égaux; les surfaces irrégulières les autres. La différence, non négative, p_g-p_a s'appelle l'irrégularité de la surface. Cette irrégularité peut-elle même se considérer, à un certain point de vue, comme analogue au genre des courbes.

Rappelons-nous en effet que le genre d'une courbe peut être envisagé soit comme le nombre des intégrales abéliennes de première espèce attachées à la courbe, soit comme la dimension maxima d'un système continu de séries linéaires complètes ayant un ordre donné. Une série linéaire complète sur la courbe est constituée par tous les groupes de niveau de la fonction rationnelle la plus générale qui a un groupe donné de pôles. Si n est le nombre des pôles (compté chacun avec son ordre) le système continu de tous les groupes de n points de la courbe, se partage en plusieurs séries linéaires distinctes, dépendant d'un nombre de paramètres qui, pour $n \geqq p$, atteint le maximum p.

Or, lorsqu'on cherche à étendre aux surfaces ces propriétés des courbes, on trouve des difficultés très sérieuses.

L'extension, plus aisée, des intégrales abéliennes de première espèce, conduit aux intégrales doubles partout finies sur la surface. C'est l'extension donnée depuis longtemps par Clebsch. Le nombre de ces intégrales est égal au genre géométrique p_g.

Mais il y a une autre extension possible, qui est bien plus cachée: c'est l'extension donnée par M. Picard. Il s'agit, là, des intégrales de différentielles totales ou intégrales simples, partout finies sur la surface.

Dès 1884, dans la théorie des fonctions algébriques de deux variables, la question de caractériser les surfaces possédant des intégrales finies de Picard était à l'ordre du jour. Pendant la période de 1884 à 1904, la théorie des intégrales simples de première, deuxième et troisième espèce, s'était perfectionnée, surtout par les travaux de MM. Picard, Poincaré et Humbert. Pendant la même période, les géomètres italiens avaient cherché à pénétrer au fond de la distinction entre les surfaces régulières et les surfaces irrégulières. On avait ainsi trouvé, d'après Castelnuovo, Enriques et moi-même, des exemples de surfaces irrégulières contenant des systèmes continus non linéaires de courbes, c'est-à-dire des systèmes ne pouvant pas être regardés comme des systèmes de courbes de niveau d'une fonction rationnelle, et M. Enriques avait aussi démontré que tout système continu complet sur une surface régulière est linéaire. On possédait enfin des exemples de surfaces irrégulières admettant des intégrales simples de première espèce.

Tout cela amenait à la prévision que les surfaces irrégulières, les surfaces possédant des intégrales picardiennes de première espèce et celles qui possèdent des systèmes continus non linéaires étaient une seule et même classe.

C'est ce qu'on a démontré en 1904-1905. On a alors trouvé que le nombre des intégrales simples de première espèce est égal à l'irrégularité de la surface et au maximum du nombre des paramètres dont peut dépendre un système linéaire variable dans un système continu. Mais pour dépasser les difficultés qui avaient arrêté la théorie à ce point, il y fallaient des éléments conceptuels nouveaux.

J'ai ouvert la succession très rapide des travaux qui ont conduit au théorème fondamental, en introduisant, en 1904, le concept de série caractéristique d'un système continu (série coupée sur une courbe du système par les courbes infiniment voisines) et le concept de fonction rationnelle résidue, définie par une intégrale abélienne de seconde espèce dépendant rationnellement d'un paramètre.

En conséquence, j'ai pu démontrer, en premier lieu, que les surfaces possédant des intégrales simples de première espèce sont irrégulières. Tout de suite, M. Enriques avait pu, à son tour, en déduire* que la série caractéristique d'un système continu complet est complète, d'où découlait immédiatement la réciproque de mon théorème. Ensuite, en 1905, M. Castelnuovo et moi, nous sommes arrivés, indépendamment l'un de l'autre†, aux parties ultérieures du théorème fondamental. Un peu plus tard, M. Picard considérait les mêmes questions à l'aide d'une certaine équation différentielle à points critiques fixes et présentait tout cela sous un jour nouveau. Et, plus tard encore, en 1910, M. Poincaré reprenait le problème *ab imis* et il en donnait une nouvelle solution dans de profonds travaux qui sont parmi les derniers écrits par le grand mathématicien.

Du théorème fondamental, combiné avec un ancien résultat de M. Picard, découle que le double de l'irrégularité est égal, soit au nombre des intégrales simples de seconde espèce, soit au nombre des cycles linéaires indépendants qu'on peut tracer sur la variété riemannienne à quatre dimensions, attachée à la surface.

A côté des questions se rapportant aux intégrales simples de première et de seconde espèce, il y en a d'extrêmement intéressantes relatives aux intégrales simples de troisième espèce. Le théorème fondamental est dû ici à M. Picard qui a démontré, en 1901, l'existence d'un certain entier ρ attaché à la surface F, tel que si l'on prend sur F, ρ courbes irréductibles, on ne peut pas trouver une intégrale simple de troisième espèce ayant ces courbes seules comme courbes logarithmiques, tandis qu'on peut toujours construire une intégrale de troisième espèce devenant infinie logarithmiquement le long des courbes susdites et d'une courbe ultérieure arbitrairement choisie.

*Avec une démonstration qui a toujours besoin d'être complétée. Toutefois, on peut certainement affirmer que toute surface contient des systèmes continus ayant la série caractéristique complète, et cela suffit au but. Voir ma note dans les Rendiconti dei Lincei, t. XXX (1921), p. 297.

†Cependant M. Castelnuovo fait usage, dans un point essentiel de sa démonstration, du théorème que j'avais donné comme fondement de la mienne.

En 1905, j'ai trouvé la signification géométrique de ce théorème en introduisant la notion de courbes algébriquement liées. On dit que deux courbes de F sont algébriquement équivalentes lorsqu'elles appartiennent à un même système continu,* et que plusieurs courbes de F sont algébriquement dépendantes lorsqu'une combinaison linéaire à coefficients entiers positifs de certaines d'entre elles est équivalente à une combinaison semblable des courbes restantes.

Eh bien, j'ai démontré que la condition nécessaire et suffisante pour que plusieurs courbes de F soient algébriquement dépendantes, est qu'elles puissent être envisagées, par elles seules, comme des courbes logarithmiques d'une intégrale simple de troisième espèce. Ce résultat combiné avec le théorème de M. Picard, améne à la résolution générale du *problème de la base* pour les courbes de F, problème que j'avais déjà posé et résolu pour des classes particulières de surfaces.

Sur F, on peut donc fixer ρ courbes algébriquement indépendantes, de sorte que toute autre courbe de F dépende des courbes fixées. L'ensemble de ces courbes constitue une base des courbes de F. M. Poincaré est parvenu, plus tard, de son côté, au même résultat et il a appelées primitives les courbes d'une base.

Il serait très bon de construire cette théorie par des méthodes purement algébrico-géométriques, qui lui sont plus propres. Il faudra démontrer tout d'abord que les fonctions rationnelles d'un ordre donné, rationnellement déterminées sur une courbe renfermant un paramètre rationnel, se distribuent en un nombre fini de systèmes continus.

La considération de la base m'a permis soit de prouver que toute intégrale simple attachée à une surface régulière se réduit à une combinaison algébrico-logarithmique, soit de construire une sorte d'algèbre ayant pour éléments les courbes de F. J'ai développé cette algèbre dans plusieurs travaux. Avant tout, en considérant les intersections des courbes de la base deux à deux, on forme avec les ρ^2 entiers ainsi obtenus, un déterminant d'ordre ρ qui s'appelle le discriminant. Pour que ρ courbes forment une base, il faut et il suffit que le discriminant soit différent de zéro. Les discriminants des différentes bases ont le même signe; et les bases qui correspondent à la plus petite valeur absolue du discriminant sont des bases intermédiaires, par lesquelles il faut passer pour arriver à une base minima, c'est-à-dire à une base dont on déduit toute courbe de F, avec les seules opérations rationnelles d'addition et de soustraction.

Une base minima contient généralement plus que ρ courbes. Pour l'avoir, il faut en effet ajouter à une base intermédiaire un certain nombre $\sigma-1$ de courbes, σ étant un nouveau caractère de la surface, qui est le maximum du nombre des courbes algébriquement distinctes que l'on peut obtenir par l'opération de division d'une courbe par un entier.

M. Lefschetz a donné dans ces dernières années des interprétations singulièrement intéressantes de la théorie de la base en établissant des liens très étroits entre cette théorie et les résultats de Poincaré sur l'Analysis situs.

*Quelquefois, pour que la définition ne soit pas fictive, il faut d'abord ajouter aux deux courbes une même courbe convenable.

A la division d'une courbe par un entier et au groupe fini abélien d'opérations qui s'y rattache, correspond la notion de torsion des cycles des différentes dimensions tracés sur la riemannienne V attachée à F et celle de groupe de torsion.

L'équivalence algébrique de deux courbes algébriques tracées sur F revient à une homologie entre les cycles à deux dimensions images des deux courbes, de sorte que l'existence d'une base sur F, découle de l'existence d'un ordre fini de connexion à deux dimensions de la riemannienne V.

Je ne puis pas m'arrêter sur les belles propriétés des intégrales doubles et surtout des intégrales doubles de seconde espèce, introduites par M. Picard, ainsi que sur leurs rapports avec la base. Tout cela se trouve exposé d'une façon suggestive dans le petit Traité que M. Lefschetz a fait paraître cette année. Je me bornerai à rappeler le remarquable théorème de M. Lefschetz, suivant lequel les cycles à deux dimensions de la riemannienne V, qui correspondent aux courbes algébriques de la surface F, sont caractérisés par la propriété que la période de toute intégrale double de première espèce le long d'un tel cycle est nulle.

Quant aux intégrales doubles de première espèce qui sont nulles sur tout cycle à deux dimensions de V, je croyais avoir démontré qu'elles se réduisent à des constantes; mais cette démonstration a besoin d'être achevée. Dans ce champ, on se meut très péniblement à cause de la non maturité de la théorie des fonctions analytiques de deux variables. Des connaissances plus profondes d'Analysis situs sont aussi nécessaires. Peut-être serait-il bon de chercher à tirer quelque parti de la considération des intégrales comme fonctions de lignes ou de surfaces. Mais il faudrait auparavant développer la théorie des fonctions de lignes dans le champ complexe.

J'aurais maintenant à m'arrêter sur quelques autres chapitres les plus importants de géométrie algébrique, mais comme je ne dois pas abuser de votre amabilité, je me bornerai à donner très rapidement une liste de problèmes qui attendent leur solution et que j'estime essentiels pour le progrès de cette branche de géométrie.

Au sujet des intégrales de première espèce: Interprétation géométrique directe de l'identité par laquelle M. Picard exprime la condition d'existence des différentielles totales de 1ère espèce.

Au sujet des intégrales simples de deuxième espèce: Interprétation de la condition analytique pour qu'une intégrale de deuxième espèce se réduise à une fonction rationnelle. Conséquences relatives au théorème de Riemann-Roch sur une surface. Il s'agit là du nombre des paramètres renfermés dans la fonction rationnelle la plus générale ayant une courbe polaire donnée. Dans ce théorème énoncé par Noether, démontré par M. Castelnuovo et perfectionné par moi, on considère une valeur virtuelle et une valeur effective de ce nombre. La valeur effective n'est jamais plus petite que la valeur virtuelle, et dans le cas général, les deux valeurs coïncident.

On doit chercher une signification de leur différence, valable en tous cas. C'est une question très importante.

Au sujet des intégrales simples de troisième espèce: Etant donnée sur la surface une courbe C variable dans un système continu, construire toutes les intégrales de troisième espèce qui deviennent infinies logarithmiquement le long de C et d'une autre courbe indéterminée de même ordre. Cela revient à donner une construction transcendante du système continu complet déterminé par C.

Au sujet des systèmes continus de courbes algébriques dans l'espace: Donner les conditions pour qu'une courbe réductible soit la limite d'une courbe irréductible. J'ai résolu récemment la question analogue pour les courbes planes et pour certains cas de dégénérescence des courbes de l'espace.

J'en ai déduit aussi une démonstration algébrique très simple du théorème d'existence de Riemann qu'on faisait découler du problème de Dirichlet. C'est une question fondamentale pour la classification des courbes.

Au sujet des conditions de rationalité des involutions d'un espace linéaire. C'est à M. Castelnuovo qu'on doit le remarquable théorème concernant la rationalité des involutions planes. Il y a ici une anomalie des plus frappantes, qui ne sont pas rares dans le domaine de l'algèbre. En effet, les involutions algébriques sont toujours rationnelles sur la droite (Lüroth) et dans le plan; non plus dans l'espace. M. Enriques a donné le premier exemple d'une involution irrationnelle de l'espace.

On demande les conditions de rationalité des involutions spatiales. La question est aussi liée au problème de la rationalité de la variété générale à trois dimensions de l'espace à quatre dimensions. A première vue, on est tenté de regarder ce problème comme très aisé. C'est une apparence bien trompeuse!

Au sujet des variétés à plusieurs dimensions. Démontrer l'invariance du genre arithmétique vis-à-vis des transformations birationnelles et trouver la relation entre ce genre et les nombres des intégrales de première espèce, simples, doubles, triples . . . , attachées à la variété. J'ai résolu ces questions pour les variétés à trois dimensions et j'ai donné par induction, la relation générale qu'on doit démontrer.

Tout ce que j'ai dit jusqu'ici, malheureusement, ne peut que donner une idée assez imparfaite de l'esprit qui domine notre géométrie: esprit largement synthétique, loin de tout exclusivisme de méthodes si dangereux dans la science. On tient toujours en vue le but principal, qui est d'éclaircir la théorie des fonctions par l'intuition géométrique et de viser les propriétés fonctionnelles audessus du symbolisme qui, quoique instrument nécessaire de nos recherches, ne doit jamais constituer leur but final.

Nous avons abandonné le purisme des géomètres de la première moitié du XIX^{e} siècle et nous sommes ainsi très loin du jour où Steiner demandait à Weierstrass de l'aider à écrire l'équation de sa célèbre surface!

Nous devons cet esprit à nos maitres italiens Cremona, Betti, Bertini, Veronese, Segre, aux savants allemands Riemann, Clebsch, Klein, Brill et Noether, au danois Zeuthen, aux anglais Cayley, Sylvester et Salmon, et aux travaux, si profondément géométriques dans leur esprit, des analystes français, de Galois à Poincaré, à Picard, à Painlevé, à Humbert.

Comme le peu que j'ai pu faire dans la science est le fruit de l'enseignement savant et passionné de mon maitre direct, Corrado Segre, que la mort nous a prématurément ravi, le 18 mai dernier, qu'il me soit permis d'envoyer à son souvenir les hommages du disciple affectionné et reconnaissant et ceux, bien plus hauts, du Congrès.

SOME CHARACTERISTIC FEATURES OF TWENTIETH CENTURY PURE MATHEMATICAL RESEARCH

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1. If I have undertaken to try and give some indications as to the nature of the progress of mathematical science in the first quarter of the present century, it is not because I underrate the difficulty of the task, or because I can hope, in such a résumé, to do justice to the remarkable work accomplished in this period by my brother mathematicians, even in the region of research with which I shall more especially deal. A lecture on so wide a theme constitutes to a certain extent a new departure, and, if I have ventured to attempt it, it is because my example may be followed in succeeding Congresses. Such an attempt at abstraction within a science which is concerned with abstractions, and with abstractions of abstractions, seems indeed peculiarly appropriate and the more necessary that, even after more than a quarter of a century of Congresses, we mathematicians still resemble the workers at the Tower of Babel, trying to construct one and the same edifice, but speaking different mathematical tongues.

I can only regret that the time at my disposal on this occasion has not permitted me to carry out the process to its proper conclusion and in a manner worthy of the theme.

2. Among the characteristic features of 20th century research are those bound up with the Encyclopaedic movement, inaugurated at the first Mathematical Congress in 1896, a movement which has done much to remedy the state of things just referred to. It has tempted workers to move over into neighbouring fields, and, by attempting to define the frontier of existing knowledge, it has given a tremendous impulse to mathematical research. But we have a right to expect even more. It should be the task of Encyclopaedic workers also to present the subject-matter in such a form as to reduce the material of knowledge to as few simple and easily understood principles, concepts and properties as possible, and the encouragement of efforts in this direction should be one of the aims of each succeeding Mathematical Congress.

3. Another feature of mathematical research in our century, equally furthered by the Encyclopaedic Movement, has been the interest shown in the application of Pure Mathematics to the Applied Sciences and to the Industries; an interest which has been felt in the Pure Mathematical camp itself as well as outside it. It is unnecessary to emphasize this point, for a mere glance at

the Programme of Sessions of Sections of the present Congress suffices to show how tangible has been this progress in the present century.

The whole attitude of the world of intellectuals towards mathematics has changed, and it is for the most part only within the ranks of the mathematicians themselves that any pessimism exists as to the future of Mathematical Science. Biologists and chemists, equally with physicists and engineers, claim the help of the mathematician, yea, clamour for it, while training in accurate mathematical thinking, as distinct from mathematical calculation, has been demanded from their intending pupils by some of the most eminent Historians and professors of Linguistic, not to speak of other more closely affiliated Arts subjects, such as Philosophy and Economics.

I do not mean by this that there are no survivals of the old order of things. For example, in England there has always been, and still is, a school of engineers, bitterly opposed to all but the rudiments of Pure Mathematics. But against such an isolated fact we may set the brilliant success of the French and Italian engineers, all of whom are trained on highly advanced mathematical lines, while at Zurich, even in the beginning of the century, Hurwitz, who was nothing if not a Pure Mathematician, was called on successfully by his engineering colleagues for the solution of an engineering problem involving a question of dynamical stability, a problem which he only succeeded in solving by employing well nigh all the resources of the Analysis of the day. We may in this connection refer to the problems which present themselves in Hydrodynamics, interesting, as they do, not only Hydraulic engineers but workers in the theory of Ballistic and Aeroplane theory. These problems, many of which are mainly three-dimensional, require a mathematical apparatus which we do not yet possess, owing to there not having yet been found a proper analogon in space for Conformal Representation in the plane.

And we may allude here to the difficulties, essentially of a mathematical character, connected with the Paradox of d'Alembert, where it has been usual to make certain assumptions with respect to the conditions satisfied at infinity, and to the yet wider problem of reconciling theory with practice in the investigation of the motion of liquids, where what is at issue appears to be the difference between regarding the viscosity σ as finite until after integration of the equations, and then making it approach zero, or putting it equal to zero before proceeding to integrate.

This is perhaps the place to lay stress on a circumstance which is not always realized. The question nowhere arises in Pure Mathematics whether there is anything in Nature corresponding even approximately to a mathematical concept. The discoveries of the Quantum Theory, hypotheses as to the nature of matter, the finiteness, if we regard that as demonstrated, of the Time-universe in its spatial aspect, do not and cannot render nugatory or valueless the concepts of differential coefficient, limit, integral and the like, though these employ the notion of infinity, any more than the fact that -1 does not possess a square root can be said to vitiate the applications to science of theorems obtained by means of the complex variable. Infinite integrals, Euclidian Geometry, lose none of their potency, none of their usefulness. The reason why they have come to be employed is that they facilitate mathematical reasoning, they are useful tools, enormously more useful than tools easier to construct. The Calculus of Finite Differences cannot possibly displace the Infinitesimal Calculus, whatever be the progress of the human mind. None of the arguments, therefore, used by the mere working engineer against Mathematics have any value whatever. At most they can be employed to discourage the attempt to give to a man who is not to be a chief, to a mere artisan, in fact, knowledge which he is, and always will be, incapable of appreciating.

4. Confining ourselves now more particularly to the advance of Pure Mathematics itself, one other tendency should be noted, not wholly unconnected with the Encyclopaedic Movement, though preceding it. In proportion as knowledge of mathematical theories has increased, the interest in purely formal work has diminished even in England, which may perhaps be said to have been its last refuge. It has begun to be understood all over the world that a mathematician is only a calculator when he must be. He is by nature a creator, a poet, not an artisan, an architect, not a mere builder. The interest in a result for its own sake has diminished accordingly; if it is obtained by the application of a known method, presents no particular beauty, or elucidates no new or difficult theory, if it does not appreciably add to the equipment of the human mind, it attracts little or no attention, and is fortunately not always accepted for publication. There is a tendency even to reduce the labour spent on mere problems for examination purposes in England and France, though it may be said *en passant* that these have in the past occasionally served the progress of learning.

In the main, then, *Theory not Theorems*, *Principles not Formulae*, are what matter. There is some hope now that even in the higher classes of schools, even of our less progressive English ones, the master will not content himself with setting examples and explaining how they are to be done, and that in this way school mathematics will repel fewer of those gifted minds to whom calculation for its own sake does not appeal.

If among Pure Mathematicians only a few have been able to follow closely the work of the Logical School, of which Peano is the most illustrious member, an increasingly large number have interested themselves in the profound examination, which appears to have taken its rise in Italy, of the Foundations and Axioms of Mathematics. Time, however, does not permit me to go further into this interesting subject at present. It is, however, inevitable that from time to time much energy should be spent in underpinning, where necessary, our mathematical edifice. This movement is, in fact, only a part of a larger movement which has consisted in careful scrutiny of the legitimacy of our fundamental analytical and geometrical processes.

5. One of the remarkable things about the work of the 20th century has been the way in which mathematicians have been led to go back to the work of the early 19th century. Cauchy's theorems have been utilized in all sorts of unexpected directions, notably in Theory of Numbers, as you will have heard in Professor Fueter's communication. In the Theory of Functions of a Complex Variable itself, for example in the Theory of Integral Functions, it has

proved a most potent weapon. Contour Integration also has now taken its place among the weapons of the Applied Mathematician.

Still more remarkable is the renewed activity centering round the idea of Group. The Theory of Relativity may be said to have taken its rise in the recognition of the fact that the Lorentz-Maxwell equations of Electro-magnetic Theory remained unaltered for a group of transformations in which the space and time variables were interchanged. Cartan's work involves the use of what might be called a Tangent Group in connection with Levi-Cività's theory of parallelism in n-dimensional space, the object being the application of the group theory to Riemannian space. Moreover Galois Theory, as applied to Algebraic Equations, has achieved notable triumphs in the domain of Differential in the hands of Drach and Vessiot.

6. In Geometry the passage from the 19th to the 20th century is marked by the closing of purely Algebraic Geometry and its resuscitation in the hands of the Analysts, while Pure Geometry as such may be said to have almost disappeared as a living independent entity. Simultaneously the step has been taken of passing from one to two dimensions, or rather from curves to surfaces. Severi has obtained notable results in a domain which had occupied the attention of Castelnuovo and Enriques in papers which marked an epoch; in Severi's work the most highly specialized analytical tools are employed.

Geometry has also developed by ultra-analytical means, by means of the concepts of the Theory of Sets of Points. The notion of Group here again plays a very prominent part, but I wish more particularly to refer not to Analysis Situs, properly so called, or the determination or the minimum number of hypotheses which permit of a geometrical entity possessing a particular property, I wish to say a few words on the definition of the terms *curve* and *surface* and the associated concepts of the *length of a curve* and the *area of a surface*. Minkowsky, in order to deal with the difficult notion of the area of surface, conceived the idea of defining it by means of the concept of volume. The corresponding definition for the length of a curve would be that obtained as follows: Describe round every point of a plane curve a circle of radius r, find the area of the part of the plane covered over by these circles, divide it by $2r$ and then make r approach zero, this should be the length of the curve.

I tested this definition many years ago and found that it led to the most paradoxical conclusions. It naturally suggested, however, a definition of a curve which my wife and I gave in our book on the Theory of Sets of Points. The conclusion is that a curve defined as we defined it will not in general have a length. This does not mean, however, that that definition is to be finally rejected as giving a generalization of the naïve notion of a curve. Obviously it would be more general than a Jordan curve, and would require to have a different cognomen attached.

7. Let us for a moment consider what are the chief features of a generalization. One of the most striking questions is precisely the retention or rejection of the previous name. The concept in its original form involved certain properties or characteristics, some of which in the generalization are lost necessarily, inevitably, while others are retained. According as one set or another of these

properties are dropped you will get different generalizations, and when there are two generalizations it may be a matter of taste which has the right to inherit the name, or whether both shall retain it and be distinguished by a second name added.

Sometimes, however, as in the case of fractional integration, in which, among the moderns, Pincherle has interested himself, no finality seems possible; a definition suitable for one kind of function does not do for another. Sometimes it will happen that a generalization is attempted, but fruitlessly, only because a property discovered later was not intuitively perceived. This was, for instance, the case with an attempt of Lebesgue's to deal with the repeated differential coefficients of integrals, where later on success crowned the efforts of Tonelli, Fubini and myself.

Let us now return to the consideration of curves and surfaces. As I pointed out a moment ago, the definition which defines curve by means of a two-dimensional and surface in terms of a three-dimensional entity is in itself legitimate as a mode of generalization of our intuitive notions, though it sacrifices the property of possessing a length and an area. In dealing with length we have to take a definition of curve involving order, and, as I remarked in a communication to the Congress at Strasbourg, the step from curves to surfaces naturally involves the definition of surface in terms of double order. It is the ignoring of this fact that led to the shipwreck of Minkowski's definition of area, and also that of Lebesgue, which, though quite different, equally lacked the notion of order.

While on the subject of curves and surfaces I am tempted to refer to the work of Janiskewski, cut off in the prime of life. Had he lived, the quality of the small quantity of work he has left, suggests he would have done much to extend our knowledge of curves in space, and the limits of curves other than Jordan curves. But neither he, nor others who have been working in this domain, appear to have exhausted the subject of curves, while very little indeed is known of the topology of highly generalized surfaces, other than results already classical. Difficulties arise even at the threshold, when, for example, we seek to generalize for higher space the conditions under which the ordinary formula for a volume holds good. The word *topology* at once suggests the great name of Henri Poincaré, still better known for his researches in other domains and at an earlier date, but who was active to the middle of our period.

I should like also here to refer to Bianchi's great work in Differential Geometry, continued right through the century, and to the very interesting work of Fubini in the previously imperfectly exploited domain of Projective Differential Geometry. The former is careful to limit to a minimum the hypotheses required in his theorems. Bianchi belongs in this way to the modern school of thought. Fubini, on the other hand, illustrates the tendency nowadays of Geometry to merge into Analysis.

The idea even of a Jordan curve was by no means a clear one at the end of the last century, and has in the course of the present twenty-four years been considerably cleared up. Nevertheless we have to be careful in glancing over the literature not to fall into misconceptions owing to the fact that, though

Jordan himself defined a curve merely as the locus of points (x, y) whose coordinates x and y are continuous functions of a continuous variable t, or in more geometrical language points which are in continuous correspondence with the points of a straight line, some writers use the expression "Jordan curve" where others use "simple Jordan curve" when the correspondence is (1, 1), so that the curve has no multiple points.

Already at the end of the 19th century it had been perceived that without some such restriction the locus might fill up the whole interior of a square, and could not, therefore, without offending our intuition, be called a curve at all. This was due to Peano, and was followed by discussion by different authors, including E. H. Moore, of curves which, without themselves filling up any region, took up so much room in the plane that they must be regarded as possessing positive area.

The question then arose whether the boundary of a simply connected region was always a Jordan curve, and after Osgood had shown that this was certainly not the case, and several proofs, none absolutely satisfactory, had been given of Jordan's theorem that a simple Jordan curve divides the plane into two parts, the converse of this theorem was found by Schoenflies, who gave conditions under which the boundary of a simply connected region is a simple Jordan curve. Finally in this connection my wife and I gave in a note in the Comptes Rendus a discussion and a classification of the different kinds of singular points which the boundary of a simply connected region may present, and whose appearance may prevent the boundary from having the property characteristic of a simple Jordan curve. Such a singular point in its most exaggerated form is what we called a "sticklepoint," which is such that the points of the region and of its complementary region are packed so close together that they stick out of the singular point much as the quills of a porcupine do out of its body, no angle, however small, being free of points of both regions.

8. The consideration of what would have been called at the beginning of the century "funny curves and surfaces" leads up to and is closely connected with the conception of the nature of a continuous function which was almost completely foreign to the 19th century, and this, in spite of the fact that Bolzano, Weierstrass and Cellerier had devised functions which either did not, or all but did not, possess differential coefficients anywhere.

And when we pass from continuous to discontinuous functions we have evidence of a perfectly amazing state of ignorance twenty-five years ago. Baire has told me that his statement made to eminent mathematicians of that day of there being a distinction between continuity with respect to each of two variables separately and continuity with respect to the pair of variables was received with absolute incredulity. The idea of a function as being defined by a (1, 1)-correspondence between a stretch on the x-axis and a stretch on the y-axis, or, more generally, of an ideal table such that it gave corresponding to each value of x one, and only one, value of y, was barely grasped, and, when grasped, not utilized.

Yet for the discussion of such functions the ideas of George Cantor were available.

We now know, as I pointed out at the Congress at Rome, that there is complete symmetry at every point as regards the limits on the left and on the right of a point in the case of a function of a single variable x, except at a countable set of points. In the case of a function of two variables there is what my wife and I have called *complete crystalline symmetry* except at a set of points lying on a countably infinite set of monotone curves, with a similar statement for higher space.

By the expression "complete crystalline symmetry" we mean that all possible limiting values of the function in the neighbourhood of the point can be obtained by passing along sequences tangent at the point to a given direction arbitrarily chosen, and it must be borne in mind that the exceptional set is independent of the choice of that direction.

9. The question of the limiting values of $f(x)$ at a point suggests the further question of the differential properties of the function. That a continuous function need not have a differential coefficient even at a single point of the interval considered was known, and the concept of differential coefficient had been already replaced by the manifold of derivates. Isolated properties of derivates were to be found in the works of Du Bois Reymond, Dini and Scheeffer. It remained for our century to show that a certain symmetry exists also with respect to the derivates of a function. It is only at points forming a set of content zero that we do not have either a finite differential coefficient, or else both upper derivates positively infinite and both lower derivates negatively infinite. An infinite differential coefficient is only possible at a set of content zero. Moreover it has been shown that these results, which are due to Lusin, Denjoy and my wife, and are to be found in the newest treatise on the real variable by Professor Hobson of Cambridge, are the same whether the function to be derived be continuous or not.

10. If it be asked what possible applications there can possibly be of these considerations outside the realm of abstract thought, I am tempted to reply to you in the words of Jean Perrin, describing the dance of the golden molecules in an emulsion as seen through the ultra-microscope:

"The tangles of the trajectory are so numerous and so rapid that it is impossible to follow them, and the trajectory recorded is infinitely more simple and shorter than the real path. Again the mean apparent velocity of a grain in a given time varies madly in magnitude and in direction without tending towards a definite limit when the time of observation is diminished. We see this in a simple way by noting the positions of a grain on a bright background minute by minute, then, for example, every 5 seconds, and, still better, by photographing every twentieth of a second, as has been done by Victor Henri, Conandon or de Broglie, so as to cinematograph the motion. In the same way it is quite impossible to fix a tangent, even approximately, at any point of the trajectory. This is a case where it is really natural to think of those continuous functions which have no differential coefficients, imagined by mathematicians, and regarded quite wrongly as simple mathematical curiosities. Nature suggests them just as much as she does the functions which have differential coefficients."

11. But the obtaining of these results far transcends the powers of Analysis, using the term even in the most modern sense consistent with the exclusion of the Theory of Sets of Points (*théorie des ensembles*). In this theory we have an instrument far more delicate than any analytical machinery, and available when all such machinery fails. The first earnest of its power was the utilization by Hurwitz of the non-countability of the continuum in proving a theorem in Analysis, and this half a century ago. But it has been reserved for the twentieth century to present it in all its power and to show the far-reaching purposes to which it can be put. The school of French mathematicians at the turn of the century may justly claim to have given the first impulse. Baire in his discussion of discontinuous functions, Borel in his definition of the content of an open set, and Lebesgue in his adaptation of this notion to the service of the theory of integration. But the work has proceeded far beyond the point at which they left it, and the Theory of Functions of a Real Variable which has emerged has now a beauty and a fascination and a certain completeness of its own which is continually drawing to it fresh votaries.

Moreover the influence of this theory is felt in every branch of Analysis; it has given an impulse even to the Theory of Functions of a Complex Variable; it has revolutionized the Calculus of Variations and created the new Theory of Functionals. It has also created a new Geometry, while at the same time it has served to curb our intuition, by letting us perceive infinite possibilities which no intuition could grasp.

At the same time, by enabling us, for example, to visualize a plurality of limits, it has rendered it possible for us to simplify and correct old and abandoned attempts at demonstration in Analysis, while it has at the same time introduced the profoundest changes into its language.

12. But the most momentous step of all is the extension of the notion of integration. The work of Riemann, though epoch-making, had left the theory of integration in a *cul-de-sac*. The definition of an integral that Riemann found before him was in terms of a summation, in other words of an analytical expression, which was shown to have an unique limit when the function to be integrated had a form prescribed beforehand. The step taken by Riemann was equivalent to the securing that a function had an integral whenever this analytical expression had an unique limit, and he expressed in an obscure, not easily understood, form the condition that this should be the case. In the language of the present day this condition is equivalent to the requirement that the points of discontinuity of the function, which in general form an open set, should form a set of zero content.

Darboux gave a precise and easily intelligible form to Riemann's definition, and virtually obtained, without being aware of it, the Lebesgue integral of functions now recognizable as the upper and lower semi-continuous functions of Baire. These *integrals by excess and by defect* formed the starting-point of my own work on the subject. In the meanwhile Lebesgue had published his well-known *thèse* in which he introduced the integral which, if I am not mistaken, I was the first to call the *Lebesgue integral*; and I am sure that no one here present will grudge to the distinguished French mathematician, whose absence we deplore, the honours that have accrued to him from his discovery, and the consequences that he drew from it by his bold employment of transfinite numbers. The qualifying cognomen, however, as characterizing the most general absolutely convergent integral when the integrator is a single variable or set of variables, is doomed to disappear by reason of the very inevitableness and naturalness of the generalization, as we now see it, and the finality with which its introduction has closed the chain of such operations in the bounded field. Indeed, every bounded converging succession of functions, whatever be the nature of the limiting function, with the single obvious limitation that the functions are mathematically definable, is integrable term-by-term with the new definition.

This important theorem of Lebesgue's has enormously simplified modern analysis. As an immediate corollary we see that a bounded differential coefficient may be integrated in the new sense, and that its integral is the primitive function.

13. The definition of the new integral given by Lebesgue himself is open to certain objections of a pedagogic nature, and the same reproach may be made to the first of the definitions given by myself. Both of these involve as a basis the notion of the content of a set of points. Moreover a one-dimensional integral is the content of a plane set. I ought here to remind my hearers that I have always found it convenient to drop the term *measure* employed by Lebesgue and Borel for open sets, and to speak of the *content* whether the set be open or closed.

Now this concept of content requires for its justification an existence theorem which is precisely the same as that subsequently required in the case of the integral. The content of a set of points is in point of fact the integral of a function which is unity at all the points of the set and zero elsewhere.

The definition which I now usually employ, which not only enables us to establish the whole theory most easily and concisely, but also leads to the extension to integration with respect to a function of bounded variation, and this, whatever the number of independent variables, is based on the method of monotone sequences. For this purpose I classify all functions mathematically definable beginning with functions which are constant in each of a finite number of stretches, that is, intervals, filling up the whole segment which is the range of the independent variable. The values of the function at the points of division, that is the end points of the intervals, are conveniently chosen to be in the one case always greater than, and in the other case always less than, the values at the neighbouring points on each side. Such functions are a particular case of the *upper and lower semi-continuous functions* invented by Baire, and they generate the general upper and lower semi-continuous functions as limits of monotone sequences of these simple functions. It is unnecessary to weary you with the complete system of classification of which this is the commencement. We are able to show that, for the purpose of the integration of bounded functions, we need not go beyond functions which I have called *ul* and *lu* functions, an *ul*-function being defined as the limit of a monotone decreasing sequence of lower-semi-continuous functions, and a *lu*-function as the limit of a monotone ascending sequence of upper semi-continuous functions. In accordance with the definition of absolutely convergent integral, one more monotone sequence is in the first instance required in the case of an unbounded function; this complication is, however, removable, and we have only to define the integrals of bounded and unbounded (positive) *ul* and *lu*-functions.

This is immediate, as soon as the theorem of consistency has been proved. Beyond these functions we do not need to go, for we easily show that *any bounded function mathematically definable, and any unbounded function possessing a Lebesgue integral, can be enclosed between a lu and an ul function both having he same integral, which these accordingly share with the function.*

14. The advantages of the method of monotone sequences are not merely methodic, two may be mentioned.

The first is that, given a function possessing a Lebesgue integral, we see intuitively that a function of almost any convenient simple type can be found which is such that its integral differs from that of the given function by a quantity as small as we please, or, which is the same, such that the integral of the difference of the two functions is as small as we please. The auxiliary function may, for instance, be constant in stretches, or it may be continuous.

Secondly, it is equally intuitive that we can find two functions, one upper semi-continuous and not greater than the given function, the other lower semi-continuous and not less than the given function, whose integrals differ by a quantity as small as we please.

By means of the first of these properties we are able to prove in a line almost all the principal theorems with regard to the integration of oscillating successions of functions. From the second of these properties we obtain, again almost intuitively, all the results obtained by Lebesgue by the use of transfinite numbers. The integrals of these two auxiliary semi-continuous functions are indeed effectively identical with the *fonctions majorante et minorante* of M. de la Vallée Poussin.

These proofs hold almost word for word when the integration is made with respect to a positive and completely monotone increasing function; hence, a function of bounded variation being defined as the difference of two such functions, it follows that the whole theory holds for integration with respect to a function of bounded variation, and this whatever may be the number of independent variables.

15. I have dwelt for a moment on this mode of presentation instead of attempting to give an account of the rival methods of M. Lebesgue, M. de la Vallée Poussin and others, because these latter are still much better known, as having formed the subject of lectures which have appeared in book form.

The theory of content then appears as a special case of the theory of integration, capable of being treated simultaneously with the latter, or, if desired, before it in point of time, but not as a logical preliminary to the general theory. On the other hand, it is inevitable that, the theory once established, my first form of definition, adopted by Signor Tonelli in his very interesting researches in the Calculus of Variations, should have its place, as a new theorem in the theory to be used, whenever found desirable.

The idea of attaching a number to a set of points, a number not necessarily the content of the set, arises also naturally, and we thus come to the great and growing Theory of Functionals, which took its earliest impulse, if I am not mistaken, in the work of Pincherle, and is now being prosecuted on somewhat different lines by M. Frechet and others. M. Frechet's researches are interesting from another point of view, as involving the concept of sets of elements which have not the property that every infinite set of elements has a limiting element, a concept which appears in embryo in the writings of Arzelà.

16. Before passing to the application of the Lebesgue integral to the field of trigonometrical series, which was the point of departure of Riemann in his treatment of integrals, it will be desirable to occupy ourselves with another branch of modern research, that of the theory of series.

Euler's bold treatment of series without any regard to their convergence led in his skilful hands to correct results. He was followed by not a few, among others by applied mathematicians. In modern times the attitude of the electrician, Oliver Heaviside, was remarkable; he was frankly antagonistic to the use of convergent series, and he used frequently to say, when he came to a divergent series: "This series is fortunately divergent, and so its treatment will be simple." Physicists for the rest have troubled little about the convergence of their series, claiming that Mathematics is merely a convenient tool, and that they could be certain that any results they obtained by means of it were correct, provided they only kept the physical interpretation before their minds.

One of the characteristic features of 20th century research has been the recognition of the underlying mathematical principles responsible for the correctness of the result when the method of proof employed was apparently fallacious. This, of course, must not be taken to mean that all the results of earlier writers obtained by incorrect reasoning on series are right. It has happened frequently enough in the case of inferior mathematicians that the results were false. But the attitude of Heaviside has been justified. It is sometimes convenient to transform a convergent series into a divergent one, because the sum of the latter can be more easily obtained. To the word *sum* we must now, however, attach a generalized meaning. We are here in fact on the brink of far-reaching generalizations. One of the greatest impulses to work of this kind was given by the Italian mathematician, Cesaro. Instead of defining the sum of a series as the limit of the sum s_n of n terms, we may define it, following Cesaro, as the limit of the arithmetic mean S_n of the sum $s_1, s_2, \ldots, s_n$. If S_n does not have an unique limit, we may repeat the process on the S's, and so on as far as we please. And the theorem of consistency holds good. This idea, essentially that of Cesaro, has been modified in a number of ways, and particularly in such a manner as to yield continuously varying methods of summation. We, of course, get in this way $\frac{1}{2}$ as the sum of the series $1-1+1- \ldots$, which is Euler's result. There are, of course, divergent series to which this method will not apply, but which yield to analogous methods of procedure.

17. One of the most remarkable cases where Cesaro summation is effective is in the theory of Trigonometrical series, when the coefficients increase like a power of n. We now pass, therefore, to the application of what we have been

saying to such series and to the utilization of the Lebesgue integral in that theory. This is the more desirable because we shall be able in this way to illustrate the striking progress effected in Analysis by the introduction of the Lebesgue integral.

Fourier was led specially to consider those harmonic trigonometrical series

$$\sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

whose coefficients are expressible as integrals in what we call *the Fourier form*, involving a function f, called *the function associated with the series*. It is evident that the generalization of the concept of integration automatically enlarges the class of such series, and it may be added that the necessary and sufficient condition that the sum of the squares of these coefficients should converge is that the square of $f(x)$ should be integrable in the sense of Lebesgue. The sum of the series is then, to a constant factor *près*, the integral of the square of $f(x)$ over the interval of periodicity. This theorem, known as the Theorem of Parseval and its converse, was one of the first great triumphs of the Lebesgue integral.

The theorem is the more remarkable in view of results in the same order of ideas obtained by myself later. Although various attempts made to generalize this theorem had failed, it was possible, I found, to generalize both halves of the theorem, but only in such a manner as to make them not converses of one another. It is a *sufficient* condition for the series

$$\Sigma\{|a_n|^{1+p}+|b_n|^{1+p}\}$$

to converge, that the integral $\int |f(x)|\}^{1+1/p}\, dx$ exist, p being an integer. The the sum of series is then expressible as a repeated integral involving $f(x)$. But this condition is not *necessary*.

On the other hand, the convergence of the series

$$\Sigma\{|a_n|^{1+1/p}+|b_n|^{1+1/p}\}$$

is a *sufficient* condition that the integral $\int |f(x)|^{1+p} dx$ should exist in the Lebesgue sense, but this condition is not *necessary*.

I obtained these results in the first instance by the method of multiplication of Fourier constants, and then, for greater security, devised a method by inequalities. Quite recently Hausdorf, employing the latter method, and using my inequalities, has succeeded in removing the limitation that p should be an integer, and I see that a joint pupil of Hardy and my own alludes to the completed theorem in an abstract of a paper recently presented to the L.M.S. as the Theorem of Hausdorf. In a recent number of the Fortschritte, however, it is called the Theorem of Young-Hausdorf.

§18. Many years back I gave as my reason for working at the details of the theory of Fourier Series my conviction that this was the best way of approaching the more general theory of series of Normal Functions, a conviction which has been justified since. I have been interested to see that in a paper still more recent than that of Hausdorf, F. Riesz has now succeeded in proving the whole theorem for series of Normal Functions.

19. We come now to the utilization of the concept of Cesaro convergence, and I give at once one of the most striking theorems, namely, that *the well-known conditions for the convergence in an interval inside the interval of periodicity are the same for the nth derived series of a Fourier series as for the Fourier series, provided only we sum the series (Cn) and regard as the associated function of the derived series in the interval in question the nth differential coefficient of the function associated with the Fourier series.*

20. In the prosecution of these researches I was led to a still further generalization of Fourier series, which justifies once more the introduction of the Lebesgue integral. I had been led to call the derived series of a Fourier series "Restricted Fourier Series," but I have since found it more convenient to use the term somewhat differently, in such a manner as to emphasize the fact that the Trigonometrical series about to be described involve a generalization of Fourier series of a marked character, such as cannot be secured by any further extension of the concept of integral.

The matter will be best understood if I revert to the fundamental properties of Fourier series as discussed by Riemann. He showed that, if the integrals in terms of which the coefficients are expressed are integrals in accordance with his definition, these coefficients a_n and b_n converge to zero. The first great step in the generalized theory rendered possible by the use of the Lebesgue integral was taken by Lebesgue, when he proved that this property of the coefficients holds good. This is indeed obvious in the light of one of the intuitive consequences of our definition of the integral by the method of Monotone Sequences.

It is also a necessary and sufficient condition for a trigonometrical series to be a Fourier Series, that the integrated series

$$\sum \left\{ \frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx \right\}$$

should converge to a Lebesgue integral throughout the whole closed interval of periodicity, so that the first of these statements follows from the second.

Now suppose we assume a given trigonometrical series to converge to an integral only in an interval, or a set of intervals, inside the interval of periodicity, so that the coefficients are no longer expressible in the Fourier form, and the condition as to the convergence of the coefficients to zero no longer holds good of itself. Suppose, however, we make the further hypothesis that this last condition holds good, we then have what I call *an ordinary Restricted Fourier Series*, or *R-F-series*. Such a series I have found to possess all the properties of a Fourier series which do not directly or indirectly involve the expression of the coefficients in the Fourier form.

We are thus able in particular to prove that if, and only if, the coefficients converge to zero, do trigonometrical series having the Fourier form, with the limitation that the integrals are improper integrals, behave like ordinary Fourier series; and we deduce without any difficulty the order of Cesaro convergence needed in the contrary case to take the place of ordinary convergence.

21. Before leaving the subject of the definition of the Lebesgue integral of which we have just shown the importance from the point of view of Fourier series, I ought I think, to make an analogy which inevitably suggests itself to anyone who has worked seriously with the concept. The introduction of the Lebesgue integral involves a revolution in Analysis. It marks a great epoch. The gain can only be compared with that secured at each successive stage in the generalization of the concept of number. Each stage involved an economy of labour. We were enabled to perform our calculations without inquiring whether particular restrictions were effectively satisfied. So in the present instance we have no longer to inquire, for example, whether a series of continuous functions which converges boundedly has for sum a function possessing an integral according to Riemann, in order to be able formally to integrate it term-by-term. We know it must have an integral according to Lebesgue and that is sufficient for our purpose. With suitable modification the same is true for a series of positive functions which converges except at a set of content zero. And we may remark that, in this connection, *convergence except at a set of content zero*, takes the place of *convergence everywhere* in the older theory.

22. Finally, as a further example of the usefulness of the Theory of Sets of Points and of Cesaro Convergence in the theory of Fourier Series in the Lebesgue sense, we may quote the necessary and sufficient condition that a trigonometrical series should be a Fourier series, and the necessary and sufficient condition that it should be the Fourier series of a function of which a given power, the $(1+p)-$th, is known to be summable, conditions applicable also to R-F-series. The former condition is that $\int_E f_n(x)\,dx$ tends doubly to zero, where $f_n(x)$ denotes the nth Cesaro partial summation, and integration is over any set of points E, whose content, as well as the index n, tend to zero. The latter condition is that $\int |f_n(x)|^{1+p}dx$ should be bounded.

23. It will be noticed that, in our account so far of the modern theory of integration, we have given no place to non-absolutely convergent integrals. In other words, we have confined our definition, following Lebesgue, to functions which remain integrable according to the definition when they are made everywhere positive by changing the sign, where necessary. This is the case in particular for all bounded functions (mathematically definable), but in general ceases to be the case when the function is unbounded.

In this more general case great progress has been made, when the integration is with respect to a single variable. The generalization has, in fact, proceeded so far as to involve the loss of some of the more familiar properties of an integral. Here the name of Denjoy stands out, though some of us have gone still farther. Every finite differential coefficient possesses an integral according to Denjoy, and his integral retains some of the more elementary properties of ordinary integrals.

Practically nothing, however, has been done when the integration is to be taken with respect to several variables. This suggests a large unexplored field of research. It appears to be a necessary preliminary to such researches that further progress should be made in the theory of plane sets of points, to say nothing of sets in higher space, sets such as those referred to previously, which lie on a countably infinite number of monotone curves, appearing to form one of a chain of entities to be taken into consideration.

Needless to say change of order of integration is not, as in the case of a Lebesgue integral, usually allowable, and the investigation of sufficient conditions is one of the first desiderata. In the same connection the problem arises to determine when a double non-absolutely-convergent integral is obtainable by repeated integration.

24. We have not yet referred to the advance made in the theory of integrals with infinite limits; methods analogous to those of Cesaro for the treatment of series have been shown to be applicable to divergent integrals. This was *a priori* assured, but the details of the theory are very interesting, and it still needs systematic elaboration and presentation.

25. I cannot even give a passing mention of the progress made in the setting forth of rules under which the processes of differentiation and of proceeding to the limit under the integral sign are allowable. The movement here has had an immense influence on the handling of Contour Integrals in the Theory of Functions of a Complex Variable. Nor can I do more than refer to the immense progress made in the Theory of Functions of two or more complex variables, alike from the older and the more modern standpoints.

My time is at an end, I am conscious, as I close, of the extremely imperfect way in which I have been able to carry out my original intention. I can only ask you to accept this imperfect attempt as an earnest of better to come.

COMMUNICATIONS

SECTION I

ALGEBRA, THEORY OF NUMBERS, ANALYSIS

FURTHER DEVELOPMENT OF THE THEORY OF ARITHMETICS OF ALGEBRAS

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1. Introduction

The writer recently* gave a new conception of integral elements of a rational associative algebra A having a modulus 1, which avoids the serious objections against all earlier conceptions.

The integral elements of A are defined to be the elements which belong to a set S of elements having the following four properties:

C (closure): The sum, difference and product of any two elements of S are also elements of S.

R (rank equation†): For every element of S, the coefficients of the rank equation are all ordinary integers.

U (unity): The set contains the modulus 1.

M (maximal): The set is a maximal (*i.e.*, is not contained in a larger set having properties C, R, U).

It is proved in §2 for the first time that there exists a set of integral elements in any rational algebra.

The above conception of integral elements may be extended to algebras over an algebraic field (or any field for which the notion of integer is defined). In particular, quaternions over any quadratic field are investigated in §§ 4-9.

2. Existence of integral elements in any algebra

Theorem. *In any rational algebra A having a modulus, there exists a maximal set of elements having properties C, R, U.*

First, let A be semi-simple. We can choose‡ new basal units $u_1 = 1, u_2, \ldots, u_n$ of A such that the new constants of multiplication γ_{ijk} are all integers. With this simplification we shall prove the existence of a maximal set. The latter remains a maximal set when its elements are expressed in terms of the

Algebras and Their Arithmetics, University of Chicago Press.

†The element x whose coordinates ξ_i are independent variables of the field of rational numbers is a root of a uniquely determined rank equation in which the leading coefficient is unity, while the remaining coefficients are polynomials in $\xi_1, \ldots, \xi_n$ with rational coefficients. Also, x is not a root of an equation of lower degree all of whose coefficients are such polynomials.

‡*Algebras and Their Arithmetics*, p. 160, bottom.

initial basal units, since properties C, R, U are invariant under transformation of the basal units.

Consider the set I of all linear functions of $u_1, \ldots, u_n$ with integral coefficients. Evidently I has properties U and C (since the γ_{ijk} are integers). It has property R since the rank function $R(w)$ has the same irreducible factors as the characteristic determinant whose coefficients are integers (and leading coefficient is unity) when the element is in I, and the same is true also of its irreducible factors by Gauss's lemma, and hence for a product $R(w)$ of powers of them.

Let S be any set of elements (of A) having properties C and R and containing $u_1, \ldots, u_n$; one such S is I. Every element x of S can be expressed* in the form

$$x = \sum_{s=1}^{n} \frac{d_s}{d} u_s,$$

where d and every d_s is an integer, while d is a function $\neq 0$ of the γ's only and is independent of x. Let q_s be the quotient and r_s the remainder when d_s is divided by d, whence

$$d_s = dq_s + r_s, \qquad o \leqq r_s \leqq d-1.$$

Then

$$x = q + r,\ q = \Sigma q_s u_s,\ r = \Sigma \frac{r_s}{d} u_s,$$

where q is in I. Hence S is derived from I by annexing one or more of the d^n elements r. Since there is only a finite number of such annexations, there is only a finite number of sets S having properties C and R and containing $u_1, \ldots, u_n$. Hence there is a maximal S' of such sets. It is a maximal of all sets Σ having properties C, R, U, since a Σ which contains S' contains $u_1, \ldots, u_n$. and is one of the preceding sets S.

Second, let A be not semi-simple. By the principal theorem on algebras, $A = S + N$, where N is the maximal nilpotent invariant sub-algebra of A, and S is a semi-simple sub-algebra having a modulus. Let $[\sigma]$ be a maximal set of elements σ of S having properties C, R, U. When ν ranges over N, all sums $\sigma+\nu$ form a maximal set Σ (of A) having properties C, R, U. To show that Σ has property C, let σ and σ' be any elements of $[\sigma]$, so that $\sigma \pm \sigma'$ and $\sigma\sigma'$ are elements of $[\sigma]$. Evidently $\sigma + \nu \pm (\sigma' + \nu')$ is in Σ; also

$$(\sigma+\nu)\ (\sigma'+\nu') = \sigma\sigma' + \nu_1,\ \nu_1 = \sigma\nu' + \nu\sigma' + \nu\nu',$$

so that ν_1 is the invariant sub-algebra N of A. The modulus of A is of the form $\sigma_0 + \nu_0$; hence, if s is any element of S, $(\sigma_0+\nu_0)s = s$, whence $\nu_0 s$ is zero, being in N. From $\sigma_0 s = s$ and the similar result $s\sigma_0 = s$, we see that σ_0 is the modulus of S. Hence Σ has property U. It has also property R since† the rank function of the general element $s+\nu$ of A is independent of N and has integral coefficients when the rank function for S of element s has integral coefficients.

**Ibid.*, pp. 161-2.

†*Ibid.*, p. 186.

3. Algebraic integers, basis.

Let m be an integer $\neq 1$ without a square factor >1. Write $\mu=\sqrt{m}$ and denote by $R(\mu)$ the field composed of all $a+b\mu$ in which a and b are rational numbers. We call $a+b\mu$ an *algebraic integer* if the quadratic equation having it and $a-b\mu$ as roots has integral coefficients and unity as leading coefficient.

Consider a set S such that the sum and difference of any two of its elements are also elements of S. Then S is said to have a *basis* $b_1, \ldots, b_n$ if each b_i belongs to S and if every element of S is expressible as a linear function of $b_1, \ldots, b_n$ with integral coefficients.

It is well known* that the algebraic integers of the field $R(\mu)$ have the basis 1 and θ, where $\theta=\mu$ if $m\equiv 2$ or $m\equiv 3 \pmod 4$, while $\theta=\frac{1}{2}(1+\mu)$ if $m\equiv 1 \pmod 4$.

QUATERNIONS OVER A QUADRATIC FIELD

4. Definitions, Rank equation

Let A_μ be the algebra of quaternions

$$q=\sigma+\xi i+\eta j+\zeta k, \tag{1}$$

whose *coordinates* σ, ξ, η, ζ range independently over the field $R(\mu)$, $\mu=\sqrt{m}$, every number of which is assumed to be commutative with every q. We may regard A_μ as a rational algebra A with the eight basal units

$$1,\ \mu,\ i,\ \mu i=i\mu,\ j,\ \mu j=j\mu,\ k,\ \mu k=k\mu,$$

and having rational coordinates.

The product in either order of q and its *conjugate*

$$q'=\sigma-\xi i-\eta j-\zeta k$$

is the *norm* $N(q)=\sigma^2+\xi^2+\eta^2+\zeta^2$ of q. Thus q and q' are roots of

$$w^2-2\sigma w+N(q)=0. \tag{2}$$

We may write $2\sigma=a+b\mu$, $N(q)=c+d\mu$, where a, b, c, d are rational. The product of (2) by

$$w^2-(a-b\mu)w+c-d\mu=0 \tag{3}$$

is the quartic equation

$$(w^2-aw+c)^2-m(bw-d)^2=0, \tag{4}$$

whose coefficients are rational and which has the root q. It is the rank equation of A. For, if the latter be of degree <4, the special quaternion $q=f\mu+i$, in which f is rational and not zero, would satisfy an equation of type

$$C\equiv x+yq+zq^2+wq^3=0$$

with rational coefficients. Here

$$q^2=f^2m-1+2f\mu i,\ q^3=(f^3m-3f)\mu+(3f^2m-1)i.$$

*Cf. *Algebras and Their Arithmetics*, pp. 128-130.

The part of C free of i and the part involving i are separately zero. In each such part, the component free of μ and that involving μ are both zero. Hence $x=z=0$,

$$y+w(3f^2m-1)=0, \quad yf+w(f^3m-3f)=0.$$

The determinant of the coefficients of y and w is not zero if $f\neq 0, f^2\neq -1/m$, and then $y=w=0$, $C\equiv 0$. Thus (4) *is the rank equation of algebra A*.

5. Property R.

Let q be such that the coefficients of (4) are all integers, whence its roots are algebraic integers. Sums and products of these roots in pairs give the coefficients of (2) and (3), which are therefore algebraic integers.

Conversely, let the coefficients of (2) be algebraic integers. Then the same is true of (3), and the four roots of (4) are algebraic integers. Thus the coefficients of (4) are algebraic integers and at the same time are rational numbers; hence they are integers.

In other words, a set S of quaternions q has property R if and only if the coefficients of the quadratic (2) are algebraic integers for every q in S.

We seek all maximal sets S of quaternions over the field $R(\mu)=R(\theta)$ which have properties C and R and contain $1, i, j, k, \theta$. Such a set S will be called a *normal* set of integral elements. In iq, jq, kq (which belong to S), the terms free of i, j, k are $-\xi, -\eta, -\zeta$ respectively. Thus by (2),

$$2\sigma=A+B\theta, \quad 2\xi=C+D\theta, \quad 2\eta=E+F\theta, \quad 2\zeta=G+H\theta \tag{5}$$

are algebraic integers, whence $A, \ldots, H$ are ordinary integers. We obtain from q another element of S by subtracting the products of $1, i, j, k, \theta, \theta i, \theta j, \theta k$ by integers. Hence we may replace an odd value of $A, \ldots, H$ by 1 and an even value by 0.

For $m\equiv 2$ or 3 (mod 4), $\theta=\sqrt{m}$ and $N(q)$ is an algebraic integer, as required by property R and (2), if and only if

$$A^2+C^2+E^2+G^2+m(B^2+D^2+F^2+H^2)\equiv 0 \pmod 4, \tag{6}$$

$$AB+CD+EF+GH\equiv 0 \pmod 2. \tag{7}$$

6. Case $m\equiv 2$ (mod 4)

Then $A+C+E+G$ is even by (6).

First, suppose that A, C, E, G are all even in every element q of S. If also B, D, F, H are all even in every q, S is

(8) the set I with basis $1, i, j, k, \theta, \theta i, \theta j, \theta k$,

which is in the larger set (15). In the contrary case, we use iq, jq, or kq in place of q, if necessary, and have B odd. Treating only one of three similar cases arising from one by cyclic permutations of i, j, k, we may take also D odd. If every such q has also F and H odd, S is derived from I by annexing $\theta\rho$, where

(9) $$\rho=\tfrac{1}{2}(1+i+j+k),$$

whence S is in the larger set (15). Since $B+D+F+H$ is even by (6), there remains only the case $B\equiv D\equiv 1$, $F\equiv H\equiv 0$ (mod 2), whence we may take $q=\frac{1}{2}\theta(1+i)$. Then S contains the set S_1 with the basis q, $qj=\frac{1}{2}\theta(j+k)$, θj, θ, 1, i, j, k. After subtracting products of these by integers from any element of S, we get $r=\frac{1}{2}\theta(B+Fj)$, where $B+F$ is even. If B and F are odd, the set S contains

$$qr=\tfrac{1}{4}m(B+Bi+Fj+Fk)=\tfrac{1}{2}A'+\ldots,$$

where A' is odd, contrary to hypothesis. Hence B and F are even and r is in S_1. But $S\equiv S_1$ is in the larger set (15). Hence there exists no maximal set in this first case.

Second, let S contain an element q in which A, C, E, G are not all even. Since iq, jq, kq are in S and have $-C$, $-E$, $-G$ in place of the A in q, S contains an element having A odd, and hence one or all of C, E, G odd. Treating only one of three similar cases, we permute i, j, k cyclically if necessary and take C odd. Hence we may take $A=C=1$. Then $E+G$ is even.

(i) Let also $E=G=0$. Then $B+D$ is even by (7) and $F+H$ is odd by (6). Since we may take iq in place of q, we may apply the substitution (signs apart)

(10) $$(\sigma\xi)\ (\eta\zeta):\ (AC)\ (BD)\ (EG)\ (FH),$$

and hence take $H=0$, $F=1$. Since $D=B+2p$, S contains

$$l=q-p\theta i=\tfrac{1}{2}(1+B\theta)\ (1+i)+\tfrac{1}{2}\theta j,$$

$$\theta l-\tfrac{1}{2}m[B(1+i)+j]=\tfrac{1}{2}\theta(1+i)\equiv t_1,\ \phi\equiv t_1 j=\tfrac{1}{2}\theta(j+k),$$

$$L\equiv l-Bt_1=\tfrac{1}{2}(1+i+\theta j),\ s=kL+t_1=\tfrac{1}{2}(j+k+\theta).$$

Hence S contains

(11) the set T with basis t_1, ϕ, L, s, 1, i, j, k.

It is easily verified that this set T has the closure property C. To prove that T is a maximal, annex any new element q. By subtracting in turn products of ϕ, L, t_1, s by integers, we may assume that the coefficients H, F, D, B of $\frac{1}{2}\theta k$, $\frac{1}{2}\theta j$, $\frac{1}{2}\theta i$, $\frac{1}{2}\theta$ are all zero. Then, by (6), $A^2+C^2+E^2+G^2\equiv 0$ (mod 4), whence A, C, E, G are either all even and q would be in T, contrary to hypothesis, or all odd and then we may take $q=\rho$. Then the enlarged set contains

(12) $$\rho t_1=\tfrac{1}{2}\theta(i+j)=t_1+t_2-\theta,\ t_1\rho=\tfrac{1}{2}\theta(i+k)=t_1+t_3-\theta,$$

and hence contains

(13) $$t_2=\tfrac{1}{2}\theta(1+j),\ t_3=\tfrac{1}{2}\theta(1+k),$$

(14) $$Lt_2=\tfrac{1}{4}\theta(1+i+j+k-\theta+\theta j),$$

the double of whose scalar part is $\frac{1}{2}(\theta-m)$, which is not an algebraic integer. Hence T is a maximal set.

(ii) Let $E=G=1$. Then either (6) or (7) states merely that $B+D+F+H$ is even. If B, D, F, H are all even, we may take $q=\rho$. In the contrary case, we may take $B=1$ since we may employ iq [cf. (10)], jq or kq in place of q. Permuting i, j, k cyclically if necessary, we have $D=1$. Then if F and H are odd, S contains $q=(1+\theta)\rho$ and θq and hence also $\theta\rho$ and ρ. If F and H are even, S contains $\rho+t_1$. Since we employed cyclic permutations of i, j, k, we conclude that the initial set may be derived from the set (8) by annexing one or more of ρ, $\rho+t_1$, $\rho+t_2$, $\rho+t_3$, and hence is contained in

(15) the set Σ with basis $\rho, i, j, k, \theta, t_1, t_2, t_3$,

obtained by annexing all of them. This set has the closure property and is a maximal. For, if we annex l in (i) and therefore L, we annex (14).

The general element of Σ is

$$X=x_0\rho+x_1i+x_2j+x_3k+x_4\theta+x_5t_1+x_6t_2+x_7t_3=\alpha+\alpha_1i+\alpha_2j+\alpha_3k,$$

where the x's are integers. Write

$$\xi_1=x_0+2x_1,\ \xi_2=x_0+2x_2,\ \xi_3=x_0+2x_3,\ \xi_4=2x_4+x_5+x_6+x_7,\ ,\ m=2n.$$

Then

$$\alpha=\tfrac{1}{2}x_0+\tfrac{1}{2}\xi_4\theta,\ \alpha_s=\tfrac{1}{2}\xi_s+\tfrac{1}{2}\theta x_{4+s}\ (s=1,\ 2,\ 3),\ N(X)=U+V\theta,$$

$$4U=x_0^2+\xi_1^2+\xi_2^2+\xi_3^2+2n(x_5^2+x_6^2+x_7^2+\xi_4^2),\quad 2V=x_0\xi_4+\xi_1x_5+\xi_2x_6+\xi_3x_7.$$

Hence if $n>0$, U is never negative and is zero only when $X=0$.

If $n>0$, $X^2=-1$ implies* $N(X)=+1$, $X'+X=0$, $\alpha=0$, $x_0=0$, $\xi_4=0$.

Then if $n>1$, whence $n\geqq 3$, $4U=4$ implies $x_5=x_6=x_7=0$, two of ξ_1^2, ξ_2^2, ξ_3^2 are. zero and one is 4, whence $X=\pm i$, $\pm j$, $\pm k$. Hence if $n>1$, Σ is not equivalent to the set T defined by (11).

But if $n=1$, $m=2$, the sets Σ and T are equivalent. Write

$$I=i,\ J=\phi,\ K=IJ=\tfrac{1}{2}\theta(k-j).$$

Then

$$I^2=J^2=K^2=-1,\ \theta k=J+K,\ \theta j=J-K,$$

$$t_1=\tfrac{1}{2}\theta(1+I),\ s=\tfrac{1}{2}\theta(1+J),\ s-j=\tfrac{1}{2}\theta(1+K),\ L+K=\tfrac{1}{2}(1+I+J+K).$$

THEOREM. *If $m\equiv 2$ (mod 4), the only maximal sets are T and Σ defined by* (11) *and* (15) *and two sets derived from T by permuting i, j, k cyclically. If $m=2$, these four sets are equivalent. But if $m>2$, T and Σ are not equivalent.*

7. CASE $m\equiv 1$ (MOD 4)

The algebraic integers have the basis 1 and $\theta=\frac{1}{2}(1+\sqrt{m})$. Thus $\theta^2=\theta+n$, where $n=\frac{1}{4}(m-1)$. Then

$$N(q)=\tfrac{1}{4}(U+V\theta),$$

(16) $$U=A^2+C^2+E^2+G^2+n(B^2+D^2+F^2+H^2)\equiv 0 \pmod 4,$$

(17) $$V=2(AB+CD+EF+GH)+B^2+D^2+F^2+H^2\equiv 0 \pmod 4.$$

*Also if n is negative.

THEOREM. *If n is even, the only maximal set Σ is composed of all elements $Q+\theta Q_1$, where Q and Q_1 range over the Hurwitz integral quaternions with the basis ρ, i, j, k. For n odd there are only two maximal sets: Σ_1, which has the basis*

(18)
$$q_1=\tfrac{1}{2}(1+i)+\tfrac{1}{2}\theta(1+j),\ q_2=\tfrac{1}{2}(1+j)+\tfrac{1}{2}\theta(1+k),\ q_3=\tfrac{1}{2}(1+k)+\tfrac{1}{2}\theta(1+i),\ \theta,\ \rho,\ i,\ j,\ k,$$

and Σ_2, which is derived from Σ_1 by interchanging i with j and changing the sign of k (and this leaves invariant the multiplication table of quaternions).

By (17), $B^2+D^2+F^2+H^2$ is even, whence, by (16), $A+C+E+G$ is even. If A, C, E, G are all even, they may be taken to be zero; then, by (17), B, D, F, H are all even or all odd, whence $q=\theta Q$, where Q is a Hurwitz integral quaternion. All such q's belong to Σ and to Σ_1, since

(19) $\theta\rho=q_1+q_2+q_3-1-\theta-\rho,\quad \theta i=2q_3-1-k-\theta,\quad \ldots,\quad \theta k=2q_2-1-j-\theta.$

A set S which contains a q in which A, C, E, G are not all even contains a q in which A is odd (§6, second case). Treating only one of three similar cases arising from one by a cyclic permutation of i, j, k, we may assume that S contains a q having A and C odd. We may take $A=C=1$.

(I) Let also $E=G=1$. Then (17) is equivalent to

$$(B+1)^2+(D+1)^2+(F+1)^2+(H+1)^2\equiv 0 \pmod 4.$$

Hence $B+1, \ldots, H+1$ are all even or all odd, whence B, D, F, H are all even or all odd. We may therefore take $q=\rho$ or $(1+\theta)\rho$.

(II) Let $A=C=1$, $E=G=0$. Then (16), (17) become

$$n(B^2+D^2+F^2+H^2)\equiv 2,\ 2(B+D)+B^2+D^2+F^2+H^2\equiv 0 \pmod 4.$$

The first shows that n is not divisible by 4. If $n\equiv 2 \pmod 4$, the first shows that $B^2+D^2+F^2+H^2$ is odd, and contradicts the second. Hence case (II) arises only when n is odd and then

$$B^2+D^2+F^2+H^2\equiv 2 \pmod 4,\quad B+D=\text{odd},\quad F+H=\text{odd}.$$

In view of (10), we may take $H=0$, $F=1$, whence

$$q=\tfrac{1}{2}+\tfrac{1}{2}B\theta+(\tfrac{1}{2}+\tfrac{1}{2}D\theta)i+\tfrac{1}{2}\theta j,\quad B+D=\text{odd}.$$

If $B=1$, $D=0$, q becomes q_1 in (18). Hence the set S contains

$$(k-i)q_1+(1+\theta)i=\rho.$$

If $B=0$, $D=1$, q becomes R:

(20) $$R=\tfrac{1}{2}(1+i)+\tfrac{1}{2}\theta(i+j),\quad (i+j)R+1+\theta+k=\rho.$$

In either case, S contains ρ.

Consider first a set S which contains ρ and hence all $Q+\theta Q_1$, where Q and Q_1 are Hurwitz integral quaternions. If possible, let S contain a further element q. Replacing q by $q+\rho$ if necessary, we may take $A=1$. As before (I), we may take also $C=1$. Case (I) is excluded since q was assumed to be not of the form

$Q+\theta Q_1$. Also case (II) is excluded if n is even. Hence if n is even, S coincides with Σ of the theorem and Σ is a maximal set.

Next, let n be odd. Then by (II), q is either q_1 or R in (18) or (20). A set S which contains ρ and q_1 contains

$$z=\theta q_1-q_1+\tfrac{1}{2}(1-n)(1+j)=\tfrac{1}{2}(\theta+\theta i-i+j),\quad -q_2=iq_1-z-i-\theta k,$$

and $q_3=iz+\theta$. Thus S contains the set Σ_1. The latter has the closure property.

To prove that Σ_1 is a maximal set, annex any element q. By subtracting products of q_1, q_2, q_3, ρ by integers, we get a q having $F=H=D=A=0$. Then B is even by (17) and we may take $B=0$. Then (16) becomes $C^2+E^2+G^2\equiv 0 \pmod 4$, whence C, E, G are all even and may be taken to be zero. Then $q=0$.

For n odd, consider a set S containing ρ and R, defined by (20). It contains

$$w=\theta R-R-\tfrac{1}{2}(n-1)\ \ (i+j)+1=\tfrac{1}{2}(1+j)+\tfrac{1}{2}\theta(1+i).$$

Interchange i with j and change the sign of k. From w we get q_1; from iR we get $j-q_2$; from ρ we get $\rho-k$. The new set has $q_2+k-kq_1=q_3$ and hence coincides with Σ_1.

Finally, consider a set S having $(1+\theta)\rho$. If n is odd, S contains $\theta(1+\theta)\rho$ and hence ρ, a case treated above. If n is even, and S contains an element q not in Σ, we may take $A=1$ by adding $(1+\theta)\rho$ to q if necessary. As before (I), we may take also $C=1$. Since case (II) is excluded, and the elements in (I) lie in Σ, we have a contradiction. Hence S is in Σ.

8. Case $m\equiv 3 \pmod 4$

The algebraic integers have the basis 1, $\theta=\sqrt{m}$.

If A, C, E, G are all even in every element q of S,

$$\theta q=\tfrac{1}{2}(mB+A\theta)+\tfrac{1}{2}(mD+C\theta)i+\tfrac{1}{2}(mF+E\theta)j+\tfrac{1}{2}(mH+G\theta)k$$

shows that $mB, \ldots, mH$ are all even. Hence B, D, F, H are all even and S is the set I in (8) and is contained in the larger set (21).

In the contrary case, S contains a quaternion q in which $A=1$

First, let C, E, G be all even. We may take them to be zero. Then B is even by (7); take $B=0$. Then (6) is equivalent to $D^2+F^2+H^2\equiv 1 \pmod 4$, whence two of D, F, H are even and one is odd. We treat only one of three similar cases arising from one by the cyclic permutation of i, j, k, and hence take $D=1$, $F=H=0$. Then

$$q=\tfrac{1}{2}+\tfrac{1}{2}\theta i,\quad r\equiv iq=qi=\tfrac{1}{2}i-\tfrac{1}{2}\theta,\quad s\equiv qj=\tfrac{1}{2}j+\theta k,\quad t\equiv kq=\tfrac{1}{2}k+\tfrac{1}{2}\theta j.$$

Consider

(21) the set Σ with the basis $q, r, s, t, 1, i, j, k$.

It possesses the closure property.

To prove that Σ is a maximal set, annex a new q. After subtracting the products of r, q, t, s by integers, we may take B, D, F, H all zero. By (6), A, C, E, G are all integers (whereas q is not in Σ) or all halves of odd integers.

Hence we may take $q=\rho$. But the double of the scalar part of $r\rho$ is $-\frac{1}{2}(1+\theta)$ which is not an algebraic integer.

From this maximal set Σ, we obtain two more by permuting i, j, k cyclically.

Second, let C, E, G be not all even. If necessary we permute i, j, k cyclically and have C odd. We may take $A=C=1$.

(i) Let also $E=G=1$. By (6), $B^2+D^2+F^2+H^2$ is divisible by 4, so that B, D, F, H are all even or all odd, and we may take $q=\rho$ or $(1+\theta)\rho$.

(ii) Let $E=G=0$. By (7), $B+D$ is even. By (6),

$$B^2+D^2+F^2+H^2 \equiv 2 \pmod 4.$$

If B is even, D is even and F and H are odd; thus q becomes

$$q_1=\tfrac{1}{2}(1+i)+\tfrac{1}{2}\theta(j+k).$$

If B is odd, D is odd and F and H are even, whence

$$q_2=\tfrac{1}{2}(1+\theta)(1+i).$$

(iii) Let one of E and G be odd and the other even. Since we may use iq instead of q, we may by (10) interchange E with G and take $E=1$, $G=0$. By (6), $B^2+D^2+F^2+H^2 \equiv 3 \pmod 4$. By (7), $B+D+F$ is even. Hence H is is odd. Then $B^2+D^2+F^2 \equiv 2 \pmod 4$. Thus two of B, D, F are odd and one is even. According as the even one is F, D, or B, we get

$$q_3=(\tfrac{1}{2}+\tfrac{1}{2}\theta)(1+i)+\tfrac{1}{2}j+\tfrac{1}{2}\theta k,\quad q_4=(\tfrac{1}{2}+\tfrac{1}{2}\theta)(1+j)+\tfrac{1}{2}i+\tfrac{1}{2}\theta k,$$
$$q_5=\tfrac{1}{2}+\tfrac{1}{2}\theta k+(\tfrac{1}{2}+\tfrac{1}{2}\theta)(i+j).$$

In $q_5 j$ and $q_4 j$ permute i, k, j cyclically; we get

$$\tfrac{1}{2}i-\tfrac{1}{2}\theta k+(\tfrac{1}{2}+\tfrac{1}{2}\theta)(j-1)=q_4-1-\theta-\theta k,\quad (\tfrac{1}{2}+\tfrac{1}{2}\theta)(i-1)+\tfrac{1}{2}j-\tfrac{1}{2}\theta k=q_3-1-\theta-\theta k.$$

It therefore suffices to consider a set having q_3 and hence also

$$r\equiv q_3-q_3 i-1-\theta=\tfrac{1}{2}(j+k)+\tfrac{1}{2}\theta(k-j),\ jr+1=q_2.$$

Hence case (iii) has been reduced to q_2 of case (ii).

Consider a set S containing q_2 and therefore also

$$\pi=q_2 j=kq_2=\tfrac{1}{2}(1+\theta)(j+k).$$

Annex to S a new q. Subtracting products of q_2 and π by integers, we make the coefficients of $\frac{1}{2}\theta i$ and $\frac{1}{2}\theta k$ zero; thus take $D=H=0$ in q. Then, by (6) and (7),

$$AB+EF=\text{even},\ A^2+C^2+E^2+G^2+3(B^2+F^2)\equiv 0 \pmod 4. \tag{22}$$

(I) Let $F\equiv 0$, $A\equiv 1 \pmod 2$. By (22), $B\equiv 0$, $C\equiv E\equiv G\equiv 1 \pmod 2$ Thus we may take $q=\rho$. Since

$$q_2\rho=\tfrac{1}{2}(1+\theta)(i+k),\quad q_2\rho i=\tfrac{1}{2}(1+\theta)(j-1),$$

S lies in a set Σ_1 having the basis

$$q_2,\ f=\tfrac{1}{2}(1+\theta)(1+k),\ g=\tfrac{1}{2}(1+\theta)(1+j),\ \theta,\ \rho,\ i,\ j,\ k. \tag{23}$$

It is easily verified that Σ_1 has the closure property. To prove it is a maximal

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set, annex q. Subtracting products of q_2, f, g, ρ by integers, we may take $D=F=H=A=0$. Then $C^2+E^2+G^2-B^2\equiv 0 \pmod 4$ by (6). If $B\equiv 0$, then $C\equiv E\equiv G\equiv 0 \pmod 2$, $q\equiv 0$. Hence $B=1$, and two of C, E, G are 0 and one is 1. Permuting i, j, k cyclically (which leaves Σ_1 unaltered), we may take $C=1$, $E=G=0$. Then

$$q=\tfrac{1}{2}(\theta+i),\ qf=\tfrac{1}{4}(\theta+m)(1+k)+\tfrac{1}{4}(1+\theta)(i-j),$$

the double of whose scalar part is $\frac{1}{2}(\theta+m)$ and is not an algebraic integer.

(II) Let $F\equiv 0$, $A\equiv 0 \pmod 2$. By (22), $C^2+E^2+G^2-B^2\equiv 0 \pmod 4$. As just noted, $B=1$ and two of C, E, G are 0 and one is 1. We interchange j with k and change the sign of i and note that q_2 and π are essentially unaltered, while E and G are interchanged and F and H; hence we may drop the case in which $E=1$. If $G=1$, $C=E=0$, then

$$q=\tfrac{1}{2}\theta+\tfrac{1}{2}k,\ qq_2=\tfrac{1}{4}(\theta+m)(1+i)+\tfrac{1}{4}(1+\theta)(k+j),$$

the double of whose scalar part is $\frac{1}{2}(\theta+m)$ and is not an algebraic integer. Hence $C=1$, $E=G=0$, $q=\frac{1}{2}\theta+\frac{1}{2}i$, $q_2-q=\frac{1}{2}+\frac{1}{2}\theta i$, which has been treated at the beginning of §8.

(III) Let $F\equiv A\equiv 1 \pmod 2$. By (22), $B\equiv E \pmod 2$ and $C^2+G^2\equiv 0 \pmod 4$, whence C and G are even,

$$q=\tfrac{1}{2}+\tfrac{1}{2}B\theta+\tfrac{1}{2}Bj+\tfrac{1}{2}\theta j.$$

If $B=0$, we permute i, k, j cyclically and get $\frac{1}{2}+\frac{1}{2}\theta i$, treated earlier. If $B=1$, q is g in (23). The set has

$$f=\pi-g+1+\theta,\ q_2f=\tfrac{1}{2}(1+m+2\theta)(\rho-j),\ \theta(\rho-j),\ \theta\rho,\ q_2+\pi=\rho+\rho\theta,\ \rho.$$

Hence we have the set Σ_1 in case (I).

(IV) Let $F\equiv 1$, $A\equiv 0 \pmod 2$. By (22), $E\equiv 0$, $C^2+G^2\equiv B^2+1 \pmod 4$. If $B\equiv C\equiv 0$, then $G\equiv 1$, $q=\frac{1}{2}\theta j+\frac{1}{2}k$, $jq+\theta=\frac{1}{2}i+\frac{1}{2}\theta$. Subtracting this from q_2 we get $\frac{1}{2}(1+\theta i)$, treated in (21). If $B\equiv 0$, $C\equiv 1$, then $G\equiv 0$, $q=\frac{1}{2}i+\frac{1}{2}\theta j$, $q+qk+j=\frac{1}{2}(1+\theta)(i+j)$. From this and q_2, we get f, treated in (III). Finally let $B\equiv 1$, whence $C\equiv G\equiv 1$,

$$q=\tfrac{1}{2}(i+k+\theta+\theta j),\ r=\pi-q+\theta+i=\tfrac{1}{2}(i+j+\theta+\theta k).$$

Consider the set Σ_2 with the basis $q, r, q_2, \theta, 1, i, j, k$. It has the closure property.

To prove Σ_2 is not a maximal set annex a new Q. By subtracting products of q, r, q_2 by integers, we may take $D=F=H=0$. By (22), AB is even. If $A=1$, $B=0$ and $1+C^2+E^2+G^2\equiv 0 \pmod 4$, whence $C=E=G=1$, $Q=\rho$. Now ρ extends Σ_2 to the larger set Σ_1 of case (I). Next, let $A=0$, whence $C^2+E^2+G^2\equiv B^2 \pmod 4$. If $B\equiv 0$, then $C\equiv E\equiv G\equiv 0$, $Q\equiv 0$. Hence $B\equiv 1$ and two of C, E, G are $\equiv 0$ and one is $\equiv 1$. We may interchange j with k without altering Σ_2 and hence take $Q=\frac{1}{2}\theta+\frac{1}{2}i$ or $\frac{1}{2}\theta+\frac{1}{2}k$. In the former case we obtain the maximal set Σ in (21). The second case is excluded by (II).

It remains to consider a set S not containing q_2 such that the two sets derived from S by cyclic permutations of i, j, k do not contain q_2. If every element q of S having $A=1$ is of type (i), and hence is ρ or $(1+\theta)\rho$, S is not a maximal, but is contained in Σ_1 of (I). Hence S contains q_1 of (ii) and therefore $t_1=jq_1$. The sub-set S_1 with the basis q_1, t_1, θj, θ, 1, i, j, k has the closure property. It is not maximal, being extended by ρ to set Σ_1 of (I). To S_1 annex any element q. Since we may add the products of q_1 and t_1 by integers, we may take $A=1$ in q, and assume that E and G are not both even. Interchanging j and k if necessary (which leaves S_1 unaltered), we may take $A=E=1$. If also $C=1$, q is of type (i) or (iii), of which the latter is excluded by the hypothesis on q_2. But $(1+\theta)\rho=q_1+t_1+k+\theta$ is in S_1, while ρ extends S_1 to a set containing

$$t_1+\rho-j+\theta=q_2.$$

Hence $C=0$. The cyclic substitution (i, k, j) replaces q by q' having $A'=A=1$, $C'=E=1$, $G'=C=0$, so that q' is of type (ii) or (iii) and hence is q_1. The inverse substitution (i, j, k) replaces q_1 by

$$q=\tfrac{1}{2}(1+j)+\tfrac{1}{2}\theta(k+i).$$

The enlarged set contains $q-t_1=\frac{1}{2}(1+\theta)(1+k)$. Applying (i, j, k), we get a set containing q_2.

THEOREM. *For $m\equiv 3$ (mod 4), there are just four maximal sets: Σ and Σ_1 defined by* (21) *and* (23), *and the two sets derived from Σ by cyclic permutations of i, j, k.*

The general element of Σ_1 is

$$X=x_0\rho+x_1i+x_2j+x_3k+x_4\theta+x_5q_2+x_6g+x_7f=\alpha+\alpha_1i+\alpha_2j+\alpha_3k,$$

where the x's are integers. Write

$$\xi_0=x_0+x_5+x_6+x_7,\ \xi_s=x_0+2x_s+x_{4+s}(s=1,\,2,\,3),\ \xi_4=2x_4+x_5+x_6+x_7.$$

Then

$$\alpha=\tfrac{1}{2}(\xi_0+\xi_4\theta),\ \alpha_s=\tfrac{1}{2}(\xi_s+x_{4+s}\theta),\ N(X)=U+V\theta$$

$$4U=\xi_0^2+\xi_1^2+\xi_2^2+\xi^2_3+m(\xi_4^2+x_5^2+x_6^2+x_7^2),\ 2V=\xi_0\xi_4+\xi_1x_5+\xi_2x_6+\xi_3x_7.$$

It follows readily as at the end of §6 that, if $m\geqq 3$, $X^2=-1$ only when $X=\pm i,\ \pm j,\ \pm k$. Hence if $m>0$, Σ and Σ_1 are not equivalent.

9. DIVISION ALGEBRAS

Consider the algebra Q of quaternions over the quadratic field $R(\theta)$. It is called a division algebra if a product is zero only when one of the factors is zero. *If Q is a division algebra, then $N(z)=zz'$ is zero only when $z=0$, and conversely.* For, if the converse is false, Q is not a division algebra and hence contains elements x and y, each not zero, whose product is zero. Then the product of their norms (which are algebraic numbers) is zero, so that one of the norms is zero, contrary to hypothesis.

Since a sum of squares of real numbers is zero only when all those numbers are zero, the converse shows that Q is a division algebra when θ is real.

THEOREM. *If m is a negative integer, the algebra Q of all quaternions over the field $R(\sqrt{m})$ is a division algebra if and only if $m\equiv 1$ (mod 8).*

First, let $m\equiv 2$ or $m\equiv 3$ (mod 4), so that 1 and $\theta=\sqrt{m}$ form a basis of the algebraic integers of the field. Write $m=-M$, $X=\theta+xi+yj+zk$, where x, y, z are integers. Then $N(X)=0$ if $M=x^2+y^2+z^2$. Since $M\equiv 2$ or 1 (mod 4), this equation has integral solutions by the well known theorem that every positive integer M not of the form $8l+7$ or $4l$ is a sum of three integral squares.

Second, let $m\equiv 1$ (mod 4). Then 1 and $\theta=\frac{1}{2}(1+\sqrt{m})$ form a basis of the algebraic integers of the field. Thus

$$\theta^2=\theta+n,\ n=\tfrac{1}{4}(m-1).$$

For n odd, consider $X=\rho-\theta+xi+yj+zk$. Then

$$4N(X)=1+4n+(1+2x)^2+(1+2y)^2+(1+2z)^2.$$

By hypothesis, $n=-(2l+1)$. By the theorem just quoted, $-1-4n=8l+3$ is a sum of three integral squares, necessarily all odd. Hence there exist integers x, y, z for which $N(X)=0$.

Finally, let n be even, so that $m\equiv 1$ (mod 8). We shall prove that -1 is not of the form $\alpha^2+\beta^2$, where α and β are in $R(\theta)$, and conclude* that Q is a division algebra. Suppose that $-1=\alpha^2+\beta^2$. The product of any number of $R(\theta)$ by a suitably chosen integer is known to be an algebraic integer. Hence

$$d\alpha=x+y\theta,\quad d\beta=z+w\theta,$$

where d, x, y, z, w are integers whose g.c.d. is 1. Thus $-d^2=d^2(\alpha^2+\beta^2)$ is equivalent to the pair of equations

$$-d^2=x^2+z^2+n(y^2+w^2),\quad 2xy+2zw+y^2+w^2=0.$$

Since y^2+w^2 and n are both even, $d^2+x^2+z^2$ must be divisible by 4. Hence d, x, z are all even. Then the second equation shows that y^2+w^2 is divisible by 4, so that y and w are even. Thus d, x, z, y, w have the common factor 2, contrary to hypothesis.

COROLLARY. *Every normal set of integral elements of any algebra of quaternions over an imaginary quadratic field contains an element, not zero, whose norm is zero, with the exception of the set Σ in §7 composed of the elements $q+\theta q_1$, where q and q_1 range over all Hurwitz integral quaternions, and $\theta=\frac{1}{2}(1+\sqrt{m})$,* $m\equiv 1$ (mod 8).

For, in the first case of the proof of the theorem, any normal set contains θ, i, j, k and hence X. In the second case, it may be assumed (§7) to contain also ρ and therefore X.

Such elements $X\neq 0$ of norm zero can not be chosen as b in a division process† (if one exists) $a=qb+c$, such that the algebraic norm of the quaternion norm of c is less than that of b.

**Algebras and Their Arithmetics*, p. 67.
†*Ibid*, p. 149.

ON THE ARITHMETIC OF A GENERAL ASSOCIATIVE ALGEBRA*

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1. Introduction. In his most recent book†, Dickson defined an integer for an associative algebra, having a modulus designated by 1, over the field of rational numbers, and from this definition has developed a beautiful theory of such integers. Unlike the earlier definitions of Lipschitz‡, Hurwitz§ and Du Pasquier||, this definition leads to unique factorization into primes for the classic algebras in two and three units¶. There he says** that each element of a set of elements of such an algebra A shall be called an integer if the set has the following four properties:

R: For every element of the set, the coefficients of the rank equation are all integers.

C: The set is closed under addition, subtraction and multiplication.

U: The set contains the modulus 1.

M: The set is a maximal (*i.e.*, it is not contained in a larger set having properties R, C and U).

Dickson showed†† that, if an algebra is an algebraic field, its unique maximal set of integers (according to the above definition) is composed of all the integral algebraic numbers of the field. Moreover, he proved‡‡ that the study of integers

*After having sent my abstract to the Secretary of the Congress, I discovered that Professor Dickson was presenting to the Congress a paper in which he proved the same results as those obtained in my paper. My work, however, was done quite independently; in fact, my rough draft is dated May 30, 1924. The coincidence of our results does not seem surprising in view of the fact that they seem the natural and inevitable ones after those obtained in Dickson's admirable book, *Algebras and their Arithmetics*.

†*Algebras and their Arithmetics*, University of Chicago Press (1923). This will be referred to by the letter A.

‡*Untersuchungen über die Summen von Quadraten* (Bonn, 1886); French translation in Jour. de Math., Sér. 4, Tome II (1886), pp. 393-439.

§Göttinger Nachrichten (1896), pp. 311-340; *Vorlesungen über die Zahlentheorie der Quaternionen* (Berlin, 1919).

||Vierteljahrsschrift Naturf. Gesell. Zürich, LIV (1909), 116-148; L'Enseignement Math., XVII (1915), 340-343; XVIII (1916), 201-260.

¶*A New Simple Theory of Hypercomplex Integers*, Jour. de Math., Sér. 9, Tome II (1923), pp. 281-326, especially pp. 301-306. This article will be referred to by the letter J.

**A, pp. 141-142.

††A, p. 143.

‡‡A, pp. 156-187.

for the general associative algebra over the field R of rational numbers reduces to the study of integers for division algebras over the field R.

After the aforementioned work on integers for algebras over the field of rational numbers, he proves† that the arithmetic of any linear associative algebra the coordinates of whose numbers range over all complex numbers is associated with the arithmetic of a direct sum of algebras each with a single unit, and that any number whose norm is not zero decomposes into primes uniquely. Although I agree thoroughly with the results, yet it seems to me that the proofs (insofar as they involve the notion of a set of integers for a complex algebra) are open to objection on purely aesthetic grounds.

His definition of integers of an algebra A over the field of complex numbers does not seem to be the one which is most natural or most artistic. For, since an integer of an algebra over the field of *rational* numbers is said to have the property R when the coefficients of the rank equation are *rational* integers, then when we are dealing with an algebra over any field F, it seems only consistent so to define an integer of this algebra that it shall be said to have the property R when the coefficients of the rank equation are integers of this *same* field, F. Although he formulates a definition of a set of integers only for the case when the algebra is over the field of rational numbers, yet he appears to use the same definition for the algebras over the field of complex numbers‡ as evidenced by his remarks near the beginning of the discussion in each case.

For example, let the algebra A be the totality C of all ordinary complex numbers of the form $x+iy$ when x and y are rational. Then we can regard A as an algebra of two units, 1 and i, over the field of rational numbers or as an algebra of one unit, 1, over the field of complex numbers of the above form $x+iy$. If we use a definition analogous to Dickson's and hence require that the coefficients of the rank equation be *rational* integers in each case, then we discover that, in the first case, we get the familiar complex integers; whereas, in the second case, we get merely the rational integers. But, for aesthetic reasons, we should consider it a desideratum so to define integers for the field C that we get exactly the same set of integers, whether we regard C as an algebra of two units over the field of rational numbers or as an algebra of one unit over the field C itself.

In view of these two reasons, we propose the following

DEFINITION. Let A be an algebra over any field F. Then each element of a set of elements of A shall be called an integer of A over F if the set has the properties R^*, C, U and M, where R^* is defined thus,—

R^*: For every element of the set, the coefficients of the rank equation are algebraic integers of the field F.

If A is any algebra of order n over an algebraic field F of order k, then the algebra A can be regarded as an algebra, A^*, of order nk over the field R of rational numbers. We readily prove that a set of integers of A over F possessing properties R^*, C and U coincides with a set of integers of A^* over R possessing properties R, C and U; and conversely. Furthermore, it then follows that the

†A, pp. 175-187; J, pp. 316-319.

‡A, p. 184, lines 8-11 from the bottom; J, p. 316, lines 19-21.

study of integers for the general associative algebra over the field F reduces to the study of integers for division algebras over the field F. Moreover, since a division algebra over a field F is equivalent, in a suitable extension of F, to a simple matric algebra and since Dickson has shown that a simple matric algebra has a maximal set of integral elements, it follows readily that a division algebra over F has a maximal set of integral elements. Accordingly, any associative algebra over F has a maximal set of integral elements.

2. RELATION OF PROPERTY R^* TO R. Consider an algebra A over an algebraic field F. (Throughout this paper it will be understood that we are considering only associative algebras.) The totality of all numbers of A over F form an algebra, A^*, over the field R of rational numbers†.

First, we shall consider a set of numbers of A over F possessing property R^*. Every such number is a root of an algebraic equation, $\rho(\omega)=0$, called the rank equation, where the coefficient of the highest power of ω is unity and where the coefficients of the other powers of ω are algebraic integers of F. Let the distinct conjugates of $\rho(\omega)$—obtained by replacing the set of coefficients of $\rho(\omega)$ by the sets of their conjugates, respectively, be denoted by $\rho^{(o)}=\rho$, $\rho^{(1)}$, $\rho^{(2)}$, ... Then $\bar{\rho}=\pi_i\,\rho^{(i)}$ is a polynomial in ω which is unaltered if the roots of the defining equation for the field F are subjected to any substitution among themselves, and thus the coefficients of the various powers of ω in $\bar{\rho}$ are rational numbers. Since, moreover, the coefficients of the powers of ω in ρ and in $\pi_i\,\rho^{(i)}$ where $i>0$ are algebraic integers, then so also are the coefficients in $\bar{\rho}$. Hence the coefficients in $\bar{\rho}$ are rational integers, and property R is satisfied.

Conversely, if x is a number of A^* which satisfies R, it is a root of an algebraic equation, $\bar{\rho}(\omega)=0$, with rational integral coefficients. In the field F, x will satisfy an algebraic equation and we shall understand that $\rho(\omega)=0$ is such an equation of lowest degree. Thus ρ must be a factor of $\bar{\rho}$; for if not, then by dividing $\bar{\rho}$ by ρ we should obtain an equation with coefficients in F which is satisfied by x and which is of lower degree than ρ. Hence, by Gauss' lemma, the coefficients of $\rho(\omega)$ are algebraic integers of F. Thus property R^* is satisfied, and we have the following

THEOREM 1. *Let A be an associative algebra over an algebraic field F which can be regarded as the algebra A^* over the field R of rational numbers. Then a set of integers of A over F which possesses properties R^*, C and U coincides with a set of integers of A^* over R; and conversely. If one set is maximal, the other is, also.*

3. DICKSON'S THEOREMS PROVED UNDER NEW DEFINITION. Let A be any algebra over a field F. Then by a well-known theorem‡, $A=S+N$, where S is a semi-simple sub-algebra and N is a maximal nilpotent invariant sub-algebra of A. For convenience, we shall assume that the elements of the algebra are chosen as e_i $(i=1, 2, \ldots, s)$ and E_j $(j=s+1, \ldots, n)$ where the e_i are elements of S and E_j are elements of N. We shall use a similar notation for the constants

†This fundamental theorem is due to Wedderburn (Proc. London Math. Soc., ser. 2, vol. 6 (1907), pp. 100-109).

‡See Dickson, A, p. 125.

of multiplication. For, since S is a sub-algebra of A, every product e_ie_j contains only e_k and therefore we may write

$$e_ie_j=\sum_k \gamma_{ijk}\, e_k.$$

Similarly, since N is an invariant sub-algebra, every product e_iE_j and every product E_je_i involves only E_k and thus we may write

$$e_iE_j=\sum_k \Gamma_{ijk}E_k,\ E_je_i=\sum_k \Gamma_{jik}E_k.$$

Finally, since N is a sub-algebra, we may write

$$E_iE_j=\sum_k \Gamma_{ijk}E_k.$$

We shall write the general number of the algebra, $Z=x+X=\sum x_ie_i+\sum X_jE_j$.

With this notation, the first characteristic determinant of the general number becomes

$$\delta(\omega)\equiv\begin{vmatrix} \begin{matrix}\sum_i \gamma_{ijk}x_i-\delta_{jk}\omega \\ (j,\ k=1,\ \ldots,\ s)\end{matrix} & \begin{matrix}\sum_i \Gamma_{ijk}X_i \\ \begin{cases} j=1,\ \ldots,\ s \\ k=s+1,\ \ldots,\ n\end{cases}\end{matrix} \\ 0 & \begin{matrix}\sum_i \Gamma_{ijk}x_i+\sum_i \Gamma_{ijk}X_i-\delta_{jk}\omega \\ (j,\ k=s+1,\ \ldots,\ n)\end{matrix}\end{vmatrix}=D_sD_n$$

where

$$D_s=\left|\sum_{i=1}^{s} \gamma_{ijk}x_i-\delta_{jk}\,\omega\right| \quad (j,\ k=1,\ \ldots,\ n)$$

and

$$D_n=\left|\sum_{i=1}^{s} \Gamma_{ijk}x_i+\sum_{i=s+1}^{n} \Gamma_{ijk}X_i-\delta_{jk}\omega\right| \quad (j,\ k=s+1,\ \ldots,\ n)$$

We shall determine a number $Y=\sum_k Y_kE_k$ of N such that $Z(1+Y)=x$.

Now $$Z(1+Y)=\sum_t\,(\sum_{i,k} x_iY_k\Gamma_{ikt}+\sum_{j,k} X_jY_k\Gamma_{jkt}+X_t)E_t+x$$

where i ranges over subscripts $1,\ \ldots,\ s$ pertaining to S and where j, k and t range over those pertaining to N. Hence

$$Z(1+Y)=x$$

if and only if

$$\left|\sum_i x_i\,\Gamma_{ikt}+\sum_j X_j\,\Gamma_{jkt}\right|\neq 0.$$

But this determinant is none other than the value of D_n when $\omega=0$; and the norm of X is the value of $\delta(\omega)=D_s(\omega)D_n(\omega)$ when $\omega=0$. Hence, for numbers Z of algebra A having non-zero norm, there exists a number Y of the sub-algebra N such that $Z(1+Y)=x$, where x is the part of Z in S. For any number Z having its $x=1$, we see that such a Z is a unit and therefore the foregoing $1+Y$ is a unit. Hence we have

THEOREM 2. *If $A = S + N$ is any associative algebra over a field F, then its arithmetic is associated with the arithmetic of its semi-simple sub-algebra, S.*

Here, as in Dickson's work*, the arithmetic of an algebra A is said to be *associated with the arithmetic of a sub-algebra B* if the integral elements of A whose determinant is not zero are associated with the various integral elements of B, in the sense that, if a is any integral element of A, then there is an integral element, b, of B such that $a = ubu'$ where u and u' are units of A. The foregoing proof is essentially the one given by Dickson†, and is repeated here merely because a casual reader of Dickson's proof might get the erroneous impression that the nomalized basal units of the algebra A are needed in the proof.

In view of Gauss' lemma, Dickson's proofs of his group of theorems‡ on the arithmetic for an associative algebra A § over the field R of rational numbers carry over to an associative algebra A over an algebraic field F; or, they are seen to hold for an algebra over a general algebraic field in virtue of the above Theorem 1. Hence, as in the case of the arithmetics of general associative algebras over the field of rational numbers*, we find that we have

THEOREM 3. *The problem of the arithmetics of all associative algebras over an algebraic field F reduces to the case of simple algebras over F and finally to the special case of division algebras.*

4. SETS OF INTEGERS FOR EQUIVALENT ALGEBRAS. In view of Theorem 3, we shall restrict our attention, for the moment, to division algebras over an algebraic field F. More especially, since any division algebra over a field F is expressible as the direct product of a field and a quadrate division algebra (*i.e.*, a division algebra which is equivalent to a simple matric algebra over a suitably enlarged field), we may confine our attention to quadrate algebras.

Moreover, since a simple matric algebra is known|| to have a maximal set of integral elements, we shall consider two algebras D and M which satisfy the following conditions:

(*a*) D and M are equivalent in a suitable field, K;

(*b*) M is known to have a maximal set of integers;

(*c*) D and M are defined over F_1 and F_2, sub-fields of K;

(*d*) M has a finite basis of order m.

Then we have the following

LEMMA. *If ϵ_i $(i = 1, \ldots, m)$ form a basis of order m for a set of integers of a linear algebra M over the field F_2, then any number of the form*

$$d = \sum_i a_{\,i} \epsilon_i$$

*A, p. 144.

†A, pp. 175-187.

‡See A, §93, Theorems 1, 2, 3; §94, Theorem; §§95-97. Note, however, that any proof (such as §98) which involves the h.c.f. may be expected to fail.

§A, p. 157. Although the arithmetic of A is proved to have a basis essentially as in §95, yet the order of this basis is not proved to be the same as the order of A. Moreover, instead of §96, we use Theorem 1 of the present paper.

||A, p. 174; J, p. 310.

satisfies an algebraic equation in which the coefficient of the highest power of the variable is unity, and in which the remaining coefficients are algebraic integers of the field F_1, provided that the a_i are algebraic integers of any over-field, K, of F_1 and F_2.

For, if $\phi(\xi)=0$ is the characteristic equation for the algebra M, then $\phi(\xi)=0$ is satisfied by every number d described in the lemma. Moreover, the coefficient of the highest power of ξ is unity and the remaining coefficients are polynomials in the constants of multiplication and in the a_i. But since the ϵ_i form a finite basis for the set of integers of M, then (by property C) the constants of multiplication will be algebraic integers of F_2 if the ϵ_i are taken as basis units for the algebra. Hence, since the a_i are integers in K, so also are the coefficients of the various powers of ξ in $\phi(\xi)$. Then, by the same line of argument as we used in proving Theorem 1, the above lemma follows at once.

If we now assume that D is a quadrate division algebra of order n^2 over a field F and that M is a simple matric algebra over R, then the conditions for the lemma are satisfied for $m=n^2$. Moreover, since D and M are equivalent in a suitable enlargement of F, there is a transformation of the form

$$d_i=\sum_j a_{ij}\,\epsilon_j$$

where the d_i range over n^2 linearly independent elements of D and where the ϵ_j range over the basis of M. If there is any a_{ij} which is not an algebraic integer of K, then (by a well-known theorem) there is some rational integer ρ such that ρa_{ij} is an integer of K. If ρ_i is the least common multiple of all such ρ's for the a_{ij} where i is fixed, then $D_i=\rho_i d_i$ is a linear combination of the ϵ with coefficients which are algebraic integers in K. Hence we may assume the original a_{ij} are all integers in K, and thus apply the lemma.

But the totality of all numbers of the form $\sum_i x_i d_i$—where the x_i are integers of F—is closed under addition and subtraction. Under multiplication we may get numbers of D outside this set, conceivably; but it is, nevertheless, clear that the product of two numbers of the set is a number to which the lemma applies and which will, accordingly, possess property R^*. Hence by adjoining to the original set all numbers that arise in the processes of multiplication, addition and subtraction we obtain a set S of numbers of D possessing the properties R^*, C and U and which is of order n^2. Thus we have

THEOREM 4. *Any associative division algebra D over any algebraic field F has a set of integers possessing properties R^*, C and U and whose order is the same as the order of D.*

Note that the above proof applies if M is any associative algebra known to have a set whose order is equal to the order of the algebra M, and if D is any algebra equivalent to D in a suitably enlarged field.

5. EXISTENCE OF A MAXIMAL SET. If the algebra M in section 4 is known to have a maximal set of integers possessing properties R^*, C and U, and if the ϵ_i form a basis for this set, then the set S' of integers of D obtained by Theorem 4 form a maximal set for D. For if S' is not maximal, then S' is contained in a larger set, $\bar{S}'$. But, if we apply Theorem 4 to the algebras D and M where D

is now the algebra known to have a set of integers satisfying R^*, C and U, namely S', this leads back to the original set, S, of M. Similarly, $\bar{S}'$ would lead to a set of integers containing S. Since this is a contradiction, S' is a maximal.

Moreover, since the Lemma of section 4 holds even if the ϵ_i are not the basis of a set of integers of M, Theorem 4 is true when D is any algebra equivalent to an algebra M—in a suitably enlarged field—when M has a maximal set of integers possessing properties R^*, C and U. But Dickson has proved* that any associative algebra over the field of ordinary complex numbers has a maximal set of integers satisfying properties R, C and U, and his proof holds without any essential changes if we replace R by R^*. Hence, by the same argument as used in the special case above, we have

THEOREM 5. *If A is any associative algebra over an algebraic field, then A has a maximal set of integers possessing properties R^*, C and U.*

*J, p. 316.

L'ÉVOLUTION DU CONCEPT DE NOMBRE HYPERCOMPLEXE ENTIER

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1. Le concept de nombre complexe entier a subi une évolution intéressante et assez rapide. Elle paraît avoir abouti aujourd'hui à un stade définitif, après six transformations que je vais brièvement exposer.

Pour construire une arithnomie, le premier pas, indispensable, consiste à définir le nombre complexe entier, l'objet essentiel de toute théorie des nombres étant l'étude des relations entre «nombres entiers». On sait aujourd'hui que cette définition est beaucoup plus importante qu'on ne le pensait, car, suivant la manière dont elle est posée, l'arithnomie qui en résulte peut être simple ou compliquée, régulière ou irrégulière.

2. Ce sont les *Recherches arithmétiques* de Gauss qui ont ouvert la voie*. Gauss a défini le nombre complexe entier comme suit: Un nombre complexe ordinaire $a+bi$, où $i^2=-1$, est dit *entier*, si chacune de ses deux coordonnées, a et b, est un nombre entier ordinaire; le complexe $a+bi$ est dit *non-entier* dans le cas contraire. Telle est la base de l'arithmétique généralisée gaussienne. On sait quelle abondante moisson de résultats nouveaux et importants est venue de ce fait enrichir plusieurs domaines des sciences mathématiques.

Parti de la notion de *nombre naturel*, c'est à dire entier et positif, le développement avait déjà incorporé le nombre négatif dans l'arithnomie, et l'arithmétique généralisée de Gauss constitue bien le premier stade de l'évolution qui nous occupe.

3. Je passe au deuxième stade. Au lieu de se borner aux nombres complexes ordinaires x_0+x_1i, on étendit les raisonnements aux nombres complexes généraux, appelés aussi nombres hypercomplexes, de la forme

$$(1)\qquad \begin{cases} x=x_0e_0+x_1e_1+x_2e_2+\dots+x_ne_n=\sum\limits_{\lambda=0}^{n} x_\lambda e_\lambda, \\ y=y_0e_0+y_1e_1+y_2e_2+\dots+y_ne_n=\sum\limits_{\lambda=0}^{n} y_\lambda e_\lambda, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

*C. F. Gauss, *Disquisitiones arithmeticae*, Leipzig 1801. Trad. française A. Ch. M. Poullet-Deslisle, Paris 1807. Gauss' Werke, Bd. I, Göttingen 1870.

où les x_λ, y_λ, etc., sont des nombres réels quelconques dits *les coordonnées* du nombre hypercomplexe envisagé x, y, etc., et les e_λ des symboles dits *les unités relatives* du système. Sur ces nombres complexes généraux, je suppose définies:

1) L'égalité de deux complexes par l'égalité des coordonnées correspondantes. L'équation unique $x=y$ entraîne les $n+1$ équations entre nombres réels

$$x_\lambda = y_\lambda, \qquad (\lambda = 0, 1, 2, \ldots, n).$$

2) L'addition de deux complexes par l'addition des coordonnées correspondantes:

$$x+y=\sum_{\lambda=0}^{n}(x_\lambda+y_\lambda)e_\lambda.$$

Il en résulte de suite que l'addition est associative et commutative et que son opération inverse la soustraction, toujours possible et univoque, se fait par la soustraction des coordonnées correspondantes.

3) La multiplication d'un complexe x par un nombre réel r; on multiplie par r chacune des coordonnées du complexe; ainsi

$$rx=\sum_{\lambda=0}^{n} r x_\lambda e_\lambda.$$

Quand r est un nombre entier ordinaire, cette multiplication est une conséquence immédiate de la définition de l'addition.

4) La multiplication des nombres complexes entre eux. Pour que le produit xy ou yx soit de nouveau un complexe appartenant au même sytème, il faut et il suffit que tous les produits $e_i e_k$ de deux unités relatives, e_i et e_k, s'expriment en fonction des mêmes unités relatives:

$$e_i e_k = c_{ik0}\, e_0 + c_{ik1}\, e_1 + \ldots + c_{ikn} e_n = \sum_{\lambda=0}^{n} c_{ik\lambda} e_\lambda, \quad (i,\ k,\ = 0, 1, 2, \ldots, n).$$

Au sujet des $(n+1)^3$ constantes c_{ikl} qui figurent dans ces $(n+1)^2$ équations de définition, et qui à priori peuvent être des nombres réels quelconques, nous ferons l'hypothèse qu'elles remplissent les conditions nécessaires et suffisantes pour que:

a) la multiplication qu'elles définissent soit associative et distributive par rapport à l'addition;

b) le système des nombres complexes ainsi définis contienne comme sous-groupe les nombres réels;

c) dans le cas particulier où les coordonnées x_λ sont telles que le complexe x rentre dans le dit sous-groupe, la multiplication des complexes se confonde avec la multiplication des nombres réels.
Il s'en suit que l'opération inverse de la multiplication, la division, est en général possible et univoque.

Enfin, je supposerai dans ce qui va suivre que les coordonnées x_λ appartiennent toutes à un même corps de nombres K.

Définition. J'appelle *système SD* tout système de nombres complexes généraux où ces conditions sont remplies*.

Dans le domaine ainsi délimité, les complexes x forment, pour chaque système *SD*, un corps de nombres dont il s'agit de faire l'arithnomie. J'envisagerai le cas le plus simple: celui où toutes les coordonnées sont des nombres rationnels ordinaires; le complexe x est alors lui-même dit *rationnel.* L'ensemble des nombres hypercomplexes rationnels constitue le corps $\{R\}$ sur lequel je vais raisonner.

4. Pour faire l'arithnomie du corps $\{R\}$, on se base sur la définition suivante que j'appelle la définition lipschitzienne et qui paraît, à première vue, une généralisation toute naturelle de celle de Gauss.

Définition lipschitzienne. Un complexe rationnel est dit entier, si ses n coordonnées x_λ sont toutes des nombres entiers ordinaires; il est réputé non-entier, dès que l'une au moins de ses n coordonnées n'est pas un nombre entier ordinaire.

C'est donc uniquement la nature des coordonnées qui décide si le complexe x est entier ou non. La définition lipschitzienne ne tient pas compte des propriétés intrinsèques du système de nombres envisagé. Mais à côté de sa simplicité, elle a le grand avantage d'être toujours applicable et toujours univoque. C'est en se basant sur elle qu'on a construit les premières arithmétiques généralisées.

5. Je passe au troisième stade de l'évolution. Il fut inauguré par une découverte de Lipschitz† relative aux quaternions de W. R. Hamilton, c'est-à-dire aux nombres tétracomplexes

$$\alpha = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3, \tag{2}$$

où

$$i_1^2 = i_2^2 = i_3^2 = -1\,;\; i_1 i_2 = i_3 = -i_2 i_1\,;\; i_2 i_3 = i_1 = -i_3 i_2\,;\; i_3 i_1 = i_2 = -i_1 i_3.$$

Lipschitz fit l'arithmétique des quaternions en la basant sur la définition suivante: un quaternion α est dit entier, si chacune de ses quatre coordonnées a_λ est un nombre entier ordinaire; α est dit non-entier dans le cas contraire. Lipschitz sut tourner les difficultés provenant de la non-commutativité de la multiplication, en construisant deux arithmétiques parallèles: une «arithnomie à gauche» et une «arithnomie à droite». Il découvrit alors que les théorèmes classiques relatifs au plus grand commun diviseur, à la décomposition en facteurs premiers, à la théorie des congruences, etc., présentent des exceptions étonnantes. J'en citerai un seul exemple. La décomposition d'un quaternion entier donné α en facteurs premiers,

$$\alpha = \pi \rho \sigma \ldots \psi,$$

*L. G. DuPasquier, *Sur les nombres complexes généraux.* Comptes Rendus du Congrès International des Mathématiciens, Strasbourg (septembre 1920), p. 164-175. Toulouse, 1921.

†R. Lipschitz, *Untersuchungen über die Summen von Quadraten,* Bonn, 1886. Trad. franç. F. Molk: *Recherches sur la transformation, par des substitutions réelles, d'une somme de deux ou de trois carrés en elle-même.* Jour. de Math., IV^e série, t. II, 1886, p. 373-439.

est toujours possible. Elle est aussi univoque dans le sens de l'arithmétique classique, une fois qu'on a fixé l'ordre de succession des facteurs de $N(\alpha)$, excepté cependant quand cette norme est divisible par 4. Dans ce cas la décomposition de α en facteurs premiers, quoiqu'on ait arrêté l'ordre de succession des facteurs de sa norme, est encore possible de 24 manières essentiellement différentes. Lipschitz trouva plusieurs anomalies tout aussi déconcertantes.

Vint alors la découverte d'A. Hurwitz*. Il a pu démontrer que toutes ces exceptions inattendues tiennent à la définition du quaternion entier; elles disparaissent comme par enchantement, si l'on base l'arithnomie sur une autre définition, *la définition hurwitzienne*: Un quaternion est dit entier, s'il est de la forme

$$\frac{n_0}{2}+\left(n_1+\frac{n_0}{2}\right)i_1+\left(n_2+\frac{n_0}{2}\right)i_2+\left(n_3+\frac{n_0}{2}\right)i_3, \tag{3}$$

où les quatre n_λ sont des nombres entiers ordinaires, arbitrairement choisis. En particulier, les 16 quaternions $u=\pm\frac{1}{2}\pm\frac{1}{2}i_1\pm\frac{1}{2}i_2\pm\frac{1}{2}i_3$, qui sont non-entiers dans le sens de Lipschitz, sont entiers dans le sens de Hurwitz. Ce sont même des unités, puisque $N(u)=(\pm\frac{1}{2})^2+(\pm\frac{1}{2})^2+(\pm\frac{1}{2})^2+(\pm\frac{1}{2})^2=1$.

6. Pour appliquer la définition hurwitzienne à d'autres systèmes de nombres hypercomplexes, il faut introduire les propriétés suivantes.

1) *La propriété* **C**: Les nombres hypercomplexes «entiers» doivent former un ensemble clos à l'égard des trois premières opérations rationnelles; en d'autres termes, ils doivent se reproduire par addition, soustraction et multiplication, ou encore: ils doivent former un domaine d'intégrité $[I]$.

2) *La propriété* **B**: Le domaine d'intégrité $[I]$ doit posséder une base finie; en d'autres termes, il doit exister dans $[I]$ un nombre fini d'éléments, $t_1, t_2, \ldots, t_r$, tels que tout nombre hypercomplexe entier puisse se mettre sous la forme

$$m_1t_1+m_2t_2+m_3t_3+\ldots+m_rt_r, \tag{4}$$

où les m_λ représentent des nombres entiers ordinaires. Inversement, si l'on fait parcourir aux m_λ, indépendamment les uns des autres, l'ensemble des nombres entiers de $-\infty$ à $+\infty$, l'expression (4) doit fournir tous les éléments du domaine d'intégrité en question, et seulement ceux-là. Les complexes t_λ peuvent ainsi engendrer chaque élément de $[I]$, par les seules opérations de l'addition et de la soustraction répétées un nombre fini de fois.

3) *La propriété* **U**: Le domaine envisagé doit contenir toutes les unités relatives, $e_0, e_1, e_2, \ldots, e_n$.

Définition. J'appelle domaine *holoïde*, et je désigne par $[H]$, tout ensemble jouissant de ces trois propriétés B, C, U (à condition, bien entendu, que ses éléments appartiennent à un système SD; voir ci-dessus §3).

Pour caractériser les nombres entiers, il faut encore une propriété, c'est

*A. Hurwitz, *Ueber die Zahlentheorie der Quaternionen.* Göttinger Nachrichten, Math. Phys. Klasse, 1896, Heft 4. Hurwitz a développé les idées exposées dansce mémoire et donné les démonstrations dans son livre *Vorlesungen über die Zahlentheorie der Quaternionen*, Berlin, 1919.

4) *La propriété M dans le sens strict*: Le domaine holoïde en question doit être *maximal*; en d'autres termes, il ne doit pas exister, dans le corps de nombres envisagé $\{R\}$, un autre domaine holoïde contenant tous les éléments de $[H]$ plus encore d'autres éléments non contenus dans $[H]$. Désignons par $[M]$ un ensemble jouissant de ces quatre propriétés. La définition hurwitzienne est alors la suivante.

Définition hurwitzienne: Sont réputés «entiers» les nombres hypercomplexes contenus dans le domaine holoïde $[M]$ caractérisé par les quatre propriétés B, C, U, M; sont dits «non-entiers» tous les complexes qui ne sont pas contenus dans ce domaine $[M]$.

On voit bien le progrès réalisé. Dans le deuxième stade, seule la nature des coordonnées décidait de la qualité du complexe; dans le troisième stade, la définition est moins superficielle; on tient compte des propriétés intrinsèques du système SD, il faut un domaine holoïde maximal. Peu importe que les coordonnées soient entières ou fractionnaires.

7. Je passe au quatrième stade de l'évolution. Un progrès important le caractérise: La propriété U est remplacée par

5) *La propriété* U_1: Le domaine envisagé doit contenir le nombre 1; il n'est pas nécessaire qu'il contienne toutes les unités relatives $e_0, e_1, \ldots, e_n$. Deux raisons m'ont guidé* pour substituer à la propriété hurwitzienne U la propriété nouvelle U_1: 1° Elle est beaucoup moins limitative, moins éliminatoire; 2° elle est *invariante* en regard de toute transformation linéaire opérée sur les unités relatives, ce qui n'est nullement le cas de la propriété U. En termes plus précis: Si, à la place des unités relatives $e_0, e_1, e_2, \ldots, e_n$, on en prend d'autres, $\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_n$, qui soient des combinaisons linéaires des premières,

$$\epsilon_\lambda = m_{\lambda 0}e_0 + m_{\lambda 1}e_1 + \ldots + m_{\lambda n}e_n, \quad (\lambda = 0, 1, 2, \ldots, n),$$

où les m_{ik} sont des nombres rationnels arbitraires, on passe du domaine $[e_0, e_1, e_2, \ldots, e_n]$ à un nouveau domaine $[\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_n]$, lequel 1) est toujours de nouveau un domaine d'intégrité (propriété C); 2) possède toujours de nouveau une base finie (propriété B); 3) est toujours de nouveau maximal (propriété M); 4) contient toujours de nouveau le nombre 1 (propriété U_1); mais en général, il ne contient plus toutes les unités relatives $\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_n$.

La propriété U n'a lieu en effet que si les coefficients de la substitution sont des nombres entiers ordinaires de déterminant ± 1. On voit par ces considérations l'importance qu'il y avait, pour une étude systématique, à substituer la propriété U_1 à la propriété U.

Ma première définition disait donc: Sont réputés «entiers» les nombres hypercomplexes contenus dans le domaine $[M]$ jouissant des propriétés B, C, M et U_1; sont dits «non-entiers» les complexes généraux qui ne sont pas contenus dans ce domaine $[M]$.

*L. G. DuPasquier, *Ueber holoïde Systeme von Düotettarionen*, Vierteljahrsschrift d. naturf. Gesellsch. in Zürich, Jahrg. 54, p. 116-148, Zürich, 1909.

8. L'étude de divers systèmes SD me montra que la définition du nombre complexe entier peut, dans certains cas, être plurivoque, c'est-à-dire que l'opération qui consiste à départager le corps $\{R\}$ en «entiers» et «non-entiers» peut le plus souvent se faire de plusieurs manières, d'une infinité de manières même, lorsqu'on se base sur les quatre propriétés invariantes B, C, M, U_1. Cela se produit quand il existe dans le même corps de nombres toute une série (qui peut être illimitée) de domaines d'intégrité,

$$[M_1],\ [M_2],\ [M_3]\ ,\ldots,\ [M_\lambda],\ \ldots$$

dont chacun jouit des quatre propriétés B, C, M et U_1. On est alors amené à dire: un nombre hypercomplexe est dit «entier par rapport au domaine $[M_\lambda]$», s'il y est contenu et que ce domaine satisfasse aux conditions voulues. Dans la définition du complexe entier s'introduit ainsi une idée de relativité. Les nombres hypercomplexes rationnels envisagés se subdivisent non pas en deux groupes, comme les nombres rationnels ordinaires, mais en trois, savoir 1) les complexes «absolument entiers»: ceux qui se trouvent dans tous les domaines $[M_\lambda]$; 2) les complexes «absolument fractionnaires»: ceux qui ne se trouvent dans aucun des domaines $[M_\lambda]$; 3) les complexes «conditionnellement entiers»: ceux qui se trouvent dans un ou dans plusieurs des domaines $[M_\lambda]$, mais non dans tous*.

9. Je passe au cinquième stade de l'évolution. Il commença quand je fis la découverte que les définitions précédentes ne sont pas toujours applicables, parcequ'il existe une infinité de systèmes SD où le corps $\{R\}$ est dépourvu de domaine holoïde maximal*. Le plus souvent même, les propriétés B, C, U_1 ensemble excluent la propriété M. Le cas le plus simple de ce genre est fourni par les nombres bicomplexes $a+b\omega$, avec $\omega^2=0$. Ni la définition hurwitzienne ni ma première définition ne sont applicables dans ce cas†, et il se trouve que la définition lipschitzienne, comme il faut s'y attendre, conduit à une arithnomie irrégulière où la décomposition d'un complexe «entier» en facteurs premiers peut ne pas être univoque. Par exemple

$$p^4+12p^3\omega=(p+3\omega)^4=p^3(p+12\omega)=p^2(p+5\omega)(p+7\omega)=p(p+3\omega)(p+4\omega)(p+5\omega)$$

et ces quatre factorisations sont essentiellement différentes. Même la théorie des idéaux est impuissante à remédier à cet inconvénient. La même chose peut se dire des systèmes de nombres hypercomplexes

$$a_0+a_1\omega_1+a_2\omega_2+\ldots+a_n\omega_n \tag{5}$$

où

$$\omega_\lambda{}^2=0,\ \omega_\lambda\omega_\mu=\omega_\mu\omega_\lambda=0,\quad (\lambda,\ \mu,\ =1,\ 2,\ 3,\ \ldots,\ n),$$

*L. G. DuPasquier, *Sur l'arithmétique des nombres hypercomplexes.* L'Enseignement Mathématique t. 18, 1916, p. 201-259. *Sur la théorie des nombres hypercomplexes à coordonnées rationnelles.* Bul. Soc. Math., France, t. 48, 1920, p. 109-132.

†L. G. DuPasquier, *Sur les nombres complexes de deuxième et de troisième espèce.* Nouv. Ann. Math. IV^e série, t. 18, 1918, p. 448-461.

et de beaucoup d'autres systèmes encore. Deux possibilités s'ouvrent alors au mathématicien.

Oubien α), il accepte, faute de mieux, la définition lipschitzienne et l'arithnomie plus compliquée qui en résulte, quitte à rechercher les nouvelles lois de la factorisation et les propriétés qui restent invariantes en regard d'une transformation linéaire. Ce qui est plus compliqué est parfois aussi plus intéressant; on trouve là des liens remarquables avec l'arithnomie additive.

Oubien β), il tient à tout prix à l'unicité de la factorisation. Dans ce cas, il doit rechercher une autre définition du nombre complexe «entier».

10. J'essayai alors, dans le cas des nombres bicomplexes $a+b\omega$, avec $\omega^2=0$, de dire que seule la première coordonnée, a, est importante et décisive. Le nombre $a+b\omega$ serait donc réputé «entier», si sa première coordonnée, a, était un nombre entier ordinaire, quelle que soit la seconde coordonnée, b. Cela reviendrait à faire abstraction de b au point de vue arithnomique, ou à introduire $R_1\omega$ comme deuxième unité relative, où R_1 représenterait l'ensemble de tous les nombres rationnels. On arrive de cette façon à une arithmétique équivalente, à tous les égards, à la classique théorie des nombres et qu'on peut appeler holoédriquement isomorphe avec elle.

Des considérations tout analogues s'appliquent aux nombres hypercomplexes (5). Un tel nombre sera réputé «entier», si sa première coordonnée, a_0, est un nombre entier ordinaire et les autres coordonnées, a_1, a_2, a_3, $\ldots$, a_n, des nombres rationnels quelconques, entiers ou fractionnaires. C'est ma *définition subsidiaire* du nombre hypercomplexe entier. Elle est en effet bien de caractère subsidiaire, car elle entre en jeu uniquement lorsque la première définition est inapplicable, c'est-à-dire quand le corps $\{R\}$ du système SD est dépourvu de domaine holoïde maximal. Dans ces cas, la première coordonnée est en quelque sorte la tête de l'être mathématique envisagé, les autres coordonnées en sont la queue, une terminaison sans aucune importance pour les lois arithnomiques. On a «la coordonnée-tête» et «les coordonnées-terminaison». On arrive de cette manière à une arithnomie banale, en ce sens qu'elle ne diffère pas, en fait, de la classique théorie des nombres. En adoptant cette définition subsidiaire, on est ramené à une arithnomie holoédriquement isomorphe avec celle dont les éléments sont les nombres entiers ordinaires.

A remarquer qu'il en est ainsi dès que l'on renonce au postulat d'une base finie. Quand on abandonne la propriété B, on n'est plus conduit à quelque chose d'essentiellement nouveau au point de vue arithnomique, dans le domaine des systèmes SD.

11. Dans d'autres systèmes de nombres hypercomplexes, il faut considérer non seulement la première coordonnée, mais *les deux premières*, pour décider si le nombre hypercomplexe est entier ou non-entier. On a un être mathématique bicéphale. C'est le cas par exemple dans le système des nombres tétracomplexes

$$a=a_0+a_1i+a_2e_2+a_3e_3, \tag{6}$$

avec

$$i^2 = -1, \quad e_2^2 = e_3^2 = e_2e_3 = e_3e_2 = 0, \quad ie_2 = e_2i = e_3, \quad ie_3 = e_3i = -e_2.$$

Dans ce système SD, le domaine holoïde le plus général du corps $\{R\}$ a la base

$$[t_1, t_2, t_3, t_4], \text{ où } t_1 = 1, \quad t_2 = \delta e_2, \quad t_3 = \beta e_2 + \gamma e_3, \quad t_4 = \lambda e_3 + ai.$$

Ici $a > 0$ est un nombre naturel; β, γ, δ, λ, représentent des nombres rationnels satisfaisant aux conditions suivantes:

$$\gamma\delta \neq 0, \quad \frac{a\delta}{\gamma} = m_1, \quad \frac{a\beta}{\gamma} = m_2, \quad \frac{2a\lambda}{\delta} = m_3, \quad \frac{a(\beta^2+\gamma^2)}{\gamma\delta} = m_4,$$

où les quatre m_λ représentent des nombres entiers ordinaires.

On déduit facilement de ce théorème que le corps $\{R\}$ en question ne contient aucun domaine holoïde maximal. On peut dès lors adopter la définition subsidiaire et dire: Un nombre tétracomplexe (6) est dit «entier», si ses deux premières coordonnées ,a_0 et a_1, sont des nombres entiers ordinaires, les autres coordonnées, a_2 et a_3, étant rationnelles quelconques. Cette définition aboutit de nouveau à une banalité, je veux dire à une arithnomie holoédriquement isomorphe avec celle des nombres complexes ordinaires $a+bi$. Les deux dernières coordonnées, a_2 et a_3, forment une terminaison sans importance arithnomique.

Plus généralement, des considérations tout analogues peuvent s'appliquer aux hypercomplexes

$$\alpha = a_0 + a_1i + a_2\omega_2 + a_3\omega_3 + \ldots + a_r\omega_r,$$

avec

$$i^2 = -1, \; \omega_\lambda^2 = 0, \; i\omega_\lambda = \omega_\lambda i = \omega_\lambda\omega_\mu = 0, \quad (\lambda, \mu, = 2, 3, \ldots, r).$$

En vertu de la définition subsidiaire, que l'on peut appliquer ici, un tel hypercomplexe sera dit «entier», si ses deux premières coordonnées, a_0 et a_1, sont des nombres entiers ordinaires; peu importent les autres coordonnées $a_2, \ldots, a_r$, rationnelles quelconques.

On voit immédiatement comment cette théorie des coordonnées-tête et coordonnées-terminaison est généralisable. On pourrait par exemple former un système de nombres hypercomplexes à $n+4$ unités relatives, en ajoutant au système des quaternions (*v.* §**5**) une «terminaison»

$$+a_4\omega_4 + a_5\omega_5 + \ldots + a_{n+4}\omega_{n+4},$$

avec

$$\omega_\lambda^2 = \omega_\lambda\omega_\mu = 0, \; i_\nu\omega_\lambda = \omega_\lambda i_\nu = 0, \; (\lambda, \mu, = 4, 5, \ldots, n+4; \; \nu = 1, 2, 3).$$

Le corps $\{R\}$ étant ici aussi dépourvu de domaine holoïde maximal, on serait amené à appliquer ma définition subsidiaire et à considérer les *quatre* premières coordonnées comme seules importantes et la terminaison comme négligeable au point de vue arithnomique. On aboutirait ainsi à une arithnomie holoédriquement isomorphe avec celle des quaternions hamiltoniens.

12. On voit par les considérations précédentes à quoi aboutit le cinquième stade de l'évolution qui nous occupe. Ma découverte de corps de nombres dépourvus de domaine holoïde maximal pose un dilemme, si l'on ne veut pas d'emblée renoncer à toute arithnomie de ces corps-là, parceque pour eux une bonne définition du complexe «entier» n'existe pas. Voici le dilemme:

Ou bien α), on accepte la définition lipschitzienne, faute de mieux, et l'on recherche les nouvelles lois de la factorisation, du plus grand commun diviseur, etc., plus compliquées que dans la théorie classique;

Ou bien β), on accepte ma définition subsidiaire avec la théorie des coordonnées-tête et coordonnées-terminaison, définition qui n'aboutit pas à des arithnomies essentiellement nouvelles.

Si l'on se résigne à cette seconde possibilité, on est obligé d'avoir deux définitions du complexe entier, à savoir:

1) La définition principale, applicable aux systèmes *SD* dont le corps $\{R\}$ possède au moins un domaine holoïde maximal.

2) La définition subsidiaire, applicable aux systèmes *SD* dont le corps $\{R\}$ est dépourvu de domaine holoïde maximal.

Il est naturellement préférable d'avoir une seule et même définition, applicable à tous les systèmes *SD*. C'est à cela qu'est arrivée l'évolution du concept de nombre hypercomplexe entier dans son *sixième stade.*

13. En adoptant ce que j'appelais ci-dessus la définition subsidiaire de l'hypercomplexe «entier», on abandonne la propriété B (*v.* §**6**). Cela entraîne une modification de la notion de maximalité d'un domaine d'intégrité. En effet, étant dépourvu de base finie, il ne constituera plus un domaine holoïde. Ce sera un domaine d'intégrité qui ne jouira plus que des propriétés C et U_1. Il sera dit *maximal*, s'il n'existe pas, dans le corps de nombres envisagé, un autre domaine d'intégrité $[I']$ jouissant aussi des propriétés C et U_1 et contenant tous les éléments de $[I]$, plus encore d'autres éléments non contenus dans $[I]$. C'est «la propriété $\boldsymbol{M}$ dans le sens large.»

Or, les propriétés C, M et U_1 sont à elles seules manifestement insuffisantes pour caractériser les nombres entiers. Preuve en soit le corps $\{R\}$ des nombres rationnels ordinaires. Il jouit des trois propriétés C, M, U_1. Mais envisager tout nombre rationnel a/b comme «nombre entier» entraînerait que tout nombre «entier» serait divisible par n'importe quel autre nombre «entier». Cette seule proposition résumerait déjà tous les théorèmes de divisibilité et la théorie du plus grand commun diviseur. Le concept de nombre premier tomberait et avec lui tomberait aussi la théorie de la factorisation. Bref, il n'y aurait plus à proprement parler d'arithnomie. Il faut donc ajouter aux propriétés C, M et U_1 encore quelque postulat restrictif.

J'ai pensé tout naturellement à *la norme* des nombres complexes généraux, car, chose remarquable, dans tous les systèmes *SD* étudiés à l'aide des propriétés B, C, U_1 et M, il se trouvait que la norme était toujours un nombre entier ordinaire. Désignons ce fait par la lettre N, initiale de norme.*

*Pour représenter les autres propriétés fondamentales des nombres entiers, on a également choisi les initiales, B, C, U, M, R, des termes caractéristiques correspondants: Base finie, Clos, Unités relatives, Maximalité, Rang.

6) *La propriété* ***N***: Tous les nombres hypercomplexes du domaine en question doivent avoir pour norme un nombre entier ordinaire.

Cette propriété est invariante en regard de toute transformation linéaire des unités relatives.

La coexistence des quatre propriétés B, C, M et U_1 entraîne la propriété N; on a dans ce cas un domaine holoïde maximal, et ma première définition du nombre complexe entier est à adopter, éventuellement la définition hurwitzienne. Mais la coexistence des quatre propriétés C, M, U_1 et N n'entraîne pas la propriété B; on a dans ce cas un corps $\{R\}$ dépourvu de domaine holoïde maximal, et ma définition subsidiaire du complexe entier est à adopter (*v.* §**10**, §**11**). Dans ces deux cas, les propriétés C, M, U_1 et N ont lieu, que ce soit avec ou sans la propriété B. Dès lors, ne pourrait-on pas définir d'une manière générale les nombres hypercomplexes entiers à l'aide des quatre propriétés invariantes C, M, U_1 et N? Voilà la question qui se posait tout naturellement et qui m'a amené à essayer une
deuxième définition de l'hypercomplexe entier: Un nombre hypercomplexe rationnel $x = \sum_{\lambda=0}^{n} x_\lambda e_\lambda$ est réputé «entier», s'il fait partie d'un domaine contenu dans le corps envisagé et jouissant dans ce corps des propriétés C, M, U_1 et N.

14. M. L. E. Dickson* fait encore un pas de plus et consacre par là le sixième stade de l'évolution qui nous occupe. On sait que tout nombre hypercomplexe x d'un système *SD* (*v.* §**3**) satisfait à une équation dite *équation au rang*. C'est l'équation, univoquement déterminée, de degré minimum, admettant x comme racine et dont les coefficients sont des nombres rationnels, le coefficient du terme le plus élevé étant égal à 1. L'existence de cette équation résulte immédiatement du fait que les hypercomplexes $1, x, x^2, x^3, \ldots, x^{n+1}$, ne sont pas linéairement indépendants et que, par conséquent, quelque combinaison linéaire, à coefficients rationnels non tous nuls, des dits hypercomplexes doit être identiquement égale à zéro. Le premier membre de l'équation au rang divise le premier membre de chacune des deux "équations caractéristiques" de l'hypercomplexe x.

Pour certaines investigations théoriques, il est avantageux de remplacer la propriété N par une propriété plus profonde, plus restrictive aussi, qui se rapporte précisément à l'équation au rang. Appelons cette propriété, avec M. Dickson*.

7) *La propriété* ***R***: Pour tout nombre hypercomplexe appartenant au domaine considéré, les coefficients de son équation au rang doivent tous être des nombres entiers ordinaires.

Cette propriété R est également invariante en regard de toute transformation linéaire opérée sur les unités relatives.

La propriété R entraîne la propriété N, puisque la norme d'un nombre

*Voir L. E. Dickson, *A new simple Theory of hypercomplex Integers*, dans le Jour. de Math. série IX^e, 1923, t. II, fasc. 3, p. 281-326, en particulier §6, p. 292.

hypercomplexe n'est pas autre chose que le dernier terme de son équation au rang. On voit aussi par là que la réciproque n'a pas nécessairement lieu. Dans bien des cas, il y a donc avantage à baser les recherches arithnomiques sur la définition suivante que j'appellerai *la définition dicksonienne*: Un nombre hypercomplexe $x=\sum_{\lambda=0}^{n} x_\lambda e_\lambda$ est réputé «entier», s'il fait partie d'un domaine qui jouit, dans le corps envisagé, des propriétés C, U_1, M et R.

15. Dans le mémoire susmentionné, M. Dickson introduit la notion d'arithnomies associées et formule ainsi d'une manière générale ce que j'ai exposé ci-dessus à propos des coordonnées-tête et coordonnées-terminaison. Voici ses définitions. La notation «x_0 crochets», $[x_0]$, est un symbole abrégé pour dire ceci: un nombre hypercomplexe rationnel

(1) $$x=x_0e_0+x_1e_1+x_2e_2+\ldots+x_ne_n$$

est dit «entier» dès que sa première coordonnée, x_0, est un nombre entier ordinaire, les autres coordonnées, $x_1, x_2, \ldots, x_n$, étant rationnelles quelconques. Dans ce cas, on dira que la théorie classique des nombres constitue «*l'arithnomie associée* avec celle de ces nombres hypercomplexes x.»

De même, la notation «x_0, x_1, crochets», $[x_0, x_1]$, est une façon condensée d'écrire ceci: Le nombre hypercomplexe x défini par (1) est réputé «entier» dès que ses deux premières coordonnées, x_0 et x_1, sont des entiers ordinaires, les autres coordonnées, $x_2, \ldots, x_n$, étant rationnelles quelconques. On dira dans ce cas que l'arithnomie des nombres complexes de Gauss, $a+bi$, est *associée* avec celle de ces nombres hypercomplexes x. Signification tout analogue des crochets $[x_0, x_1, x_2]$, et ainsi de suite dans le cas général.

Il convient d'ajouter que pour M. Dickson, le dilemme mentionné au §**12** ci-dessus n'existe pas. Selon lui, la notion de complexe «entier» implique la factorisation univoque; c'est même une *conditio sine qua non*, et il faut poser les définitions en conséquence. M. Dickson rejette donc à priori toute définition qui ne satisfait pas à ce postulat. C'est aussi selon nous le point de vue qu'il convient de prendre quand on a affaire à un système SD. L'avenir montrera si l'on doit encore faire de ce postulat une *conditio sine qua non* de la notion de complexe entier, lorsqu'on sera un peu mieux orienté dans le champ des nombres hypercomplexes ne constituant pas des systèmes SD (*v.* §3).

16. D'après ce qui précède, on a **7** postulats à disposition pour caractériser et définir les nombres hypercomplexes entiers. Ce sont les propriétés

$$B,\ C,\ M,\ N,\ U,\ U_1,\ R.$$

Une seule n'est pas invariante, c'est la propriété hurwitzienne U (*v.* §**7**). Les 6 autres sont invariantes en regard de n'importe quelle transformation linéaire opérée sur les unités relatives.

L'évolution du concept de complexe «entier» a abouti à la distinction entre deux grandes catégories de systèmes SD.

a) La première catégorie comprend ceux où les six propriétés linéairement invariantes, et même les sept propriétés ci-dessus énumérées, sont compatibles. Elles coëxistent alors (p. ex. les complexes gaussiens $a+bi$, les quaternions, etc.). Les complexes entiers constituent des domaines holoïdes, caractérisés par l'existence d'une base finie (propriété B). On y rencontre des arithnomies essentiellement nouvelles; la factorisation y est toujours univoque dans le sens de l'arithmétique classique, si l'on recourt, au besoin, à la théorie des idéaux. Pour le résultat, il semble indifférent de postuler, à côté des propriétés C, U_1 et M qui sont de rigueur, encore la propriété B, comme je le faisais primitivement, ou la propriété N ou la propriété R, comme le propose M. Dickson.

b) La deuxième catégorie comprend les systèmes *SD* où les règles de la multiplication sont telles que le corps $\{R\}$ y est dépourvu de domaine holoïde maximal. Les sept propriétés ci-dessus énumérées y sont incompatibles entre elles; en particulier, les propriétés B, C et U_1 ensemble excluent la propriété M dans le sens stricte. Les nombres hypercomplexes «entiers» ne constituent plus des domaines holoïdes, si l'on maintient l'unicité de la décomposition en facteurs premiers. On y arrive seulement à des arithnomies associées avec celles de la première catégorie.

En résumé, les seuls systèmes *SD* pouvant conduire à des arithnomies nouvelles paraissent être ceux de la première catégorie; il faut que le corps $\{R\}$ possède au moins un domaine holoïde maximal. S'il en est ainsi, les deux définitions: à l'aide des propriétés B, C, U_1, M ou de C, U_1, M, R, conduisent aux mêmes résultats; l'une entraîne l'autre.

17. Pour certaines recherches, il y a avantage à admettre que les coordonnées x_λ, au lieu de rester réelles, parcourent le champ des nombres complexes ordinaires $a+bi$. On voit immédiatement qu'un tel système *SD* à n unités relatives et à coordonnées x_λ complexes ordinaires n'est autre chose qu'un système d'hypercomplexes à coordonnées réelles, mais à $2n$ unités relatives, si l'on pose $ie_\lambda = e_\lambda i$ ($\lambda = 0, 1, 2, \ldots, n-1$). Par analogie avec ce qui se passe dans d'autres domaines, notamment en algèbre, il est à prévoir que certaines propositions arithmétiques se simplifieront. C'est ce qui a lieu effectivement. La simplification est même presque trop grande, puisqu'on peut démontrer le théorème suivant, dû à M. Dickson: L'arithnomie d'un système *SD*, quand les coordonnées parcourent chacune le champ de tous les nombres complexes ordinaires, est associée avec l'arithnomie d'une somme directe d'autres systèmes dont chacun a une seule unité relative. Tout nombre hypercomplexe «entier» de norme non nulle peut s'y décomposer en facteurs premiers ou irréductibles, et cette décomposition est univoque dans le sens arithmétique.

Résumé

18. Il ressort des développements ci-dessus que l'évolution du concept de nombre hypercomplexe «entier» peut se résumer très succinctement comme suit.

Le premier stade est caractérisé par l'arithmétique généralisée de Gauss. Les éléments sont les nombres complexes ordinaires $a+bi$.

Dans *le deuxième stade,* on a la définition lipschitzienne. Les éléments sont les nombres complexes généraux. C'est la nature des coordonnées qui décide seule de la qualité du complexe, quelles que soient la structure ou les propriétés intrinsèques du système. La définition est univoque et toujours applicable.

Le troisième stade est amené par la découverte de Lipschitz qu'il existe des arithnomies irrégulières. Définition hurwitzienne de l'hypercomplexe entier, à l'aide du domaine holoïde maximal, c'est-à-dire du domaine caractérisé par les propriétés B, C, U, M.

Quatrième stade. Ma première définition du nombre hypercomplexe entier, à l'aide des propriétés B, C, M et U_1, est basée sur des propriétés invariantes en regard de toute transformation linéaire opérée sur les unités relatives. Découverte que cette définition n'est pas toujours univoque. L'idée de relativité s'introduit dans l'arithnomie. Les trois classes de nombres hypercomplexes rationnels.

Le cinquième stade fut amené par ma découverte que, le plus souvent, les propriétés B, C, M et U_1 sont incompatibles entre elles. La propriété de maximalité dans le sens large. Ma définition subsidiaire du nombre hypercomplexe entier. Inconvénients de l'abandon de la propriété B. Coordonnées-tête et coordonnées-terminaison.

Sixième stade. Introduction de la propriété N. Le nombre hypercomplexe entier défini à l'aide des propriétés C, M, N et U_1. Définition dicksonienne à l'aide des propriétés C, M, R et U_1. La notion d'arithnomies associées. Les deux grandes catégories de systèmes SD, suivant que les sept propriétés B, C, M, N, R, U et U_1 sont compatibles entre elles ou non.

MODULAR REPRESENTATIONS OF FINITE ALGEBRAS

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1. INTRODUCTION

Consider any algebra*. It may be characterized as a system consisting of certain classes of elements and certain operations or relations among the elements. Let K be a class of a finite number n of elements. We may denote these elements by the symbols $0, 1, \ldots, n-1$. Any *binary* operation $\oplus$ in K, *i.e.*, any rule by means of which two K-elements x, y are combined into a single element $x \oplus y$, may be defined by a table of the form†

(i)

$\oplus$	0	1	....	$n-1$
0	e_{00}	e_{01}	....	$e_{0(n-1)}$
1	e_{10}	e_{11}	....	$e_{1(n-1)}$
.	.....	.....	.....	.....
.	.....	.....	.....	.....
$n-1$	$e_{(n-1)0}$	$e_{(n-1)1}$	...	$e_{(n-1)(n-1)}$.

Any *dyadic* relation R in K, *i.e.*, any rule which states for any two K-elements x, y whether or not the proposition xRy ("x has the relation R to y") is true or false, may be defined by a table of the form‡

(ii)

R	0	1	...	$n-1$
0	$\pm$	$\pm$	...	$\pm$
1	$\pm$	$\pm$	...	$\pm$
.	.....	.....	.....	.....
.	.....	.....	.....	.....
$n-1$	$\pm$	$\pm$	...	$\pm$,

*The term *algebra* is here used in the widest sense—in the sense of any mathematical system.

†To obtain $x \oplus y$, take x from the labels at the left of the table and y from the labels at the top. Thus, $0 \oplus 1 = e_{01}$.

‡To determine whether xRy holds or not, take x from the labels at the left of the table and y from the labels at the top.

I have taken the $\pm$ notation of the table from an unpublished paper of H. M. Sheffer's, entitled *Notational Relativity*. Compare the dot-blank notation used by Schröder in his *Algebra der Logik*, Bd. III.

where the sign $+$ denotes that xRy is true, the sign $-$ that xRy is false. It is my purpose to obtain convenient concrete representations of any finite $\oplus$-table or R-table, hence of any finite algebra, and to indicate the value of these representations to the theory of postulate-sets. The representations are *modular*, and generalizations of the arithmetic representations which I obtained elsewhere* for binary operations and dyadic relations in classes of two and three elements.

2. Fundamental Theorem

My representations are based on the concept *the least positive residue modulo n* of an integer a, *i.e.*, the least positive integer obtained from a by rejecting multiples of n. I shall denote the least positive residue modulo n of a by

$$a \bmod n.$$

If

$$a \bmod n = b \bmod n,$$

I shall write more briefly

$$a = b \bmod n,$$

or shall speak of *the equation* (*modulo n*) $a=b$.

If the coefficients of a polynomial in one or more variables are all least positive residues of some modulus, I shall call the polynomial *modular*. A polynomial will be called a *polynomial modulo n*, if the coefficients and the values of the variables are all among the least positive residues of n; if each of the variables ranges over the complete system of k-residues $0, 1, \ldots, k-1$ $(k \leqq n)$, the polynomial will be said to be of *range k*. By the notation

$$a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \bmod n$$

a modular polynomial will always be understood. My theory of representation of finite algebras rests on the following theorem concerning a polynomial modulo n.

Theorem 1. *Let*

$$e_0, e_1, \ldots, e_{n-1}$$

be any set of least positive n-residues; if and only if n is prime, a polynomial modulo n of range n

$$f(x) \equiv a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \bmod n$$

can be found such that

$$f(0) = e_0, f(1) = e_1, \ldots, f(n-1) = e_{n-1}.$$

*See *Complete sets of representations of two-element algebras*, Bull. Amer. Math. Soc., vol. 30 (1924), p. 24, and *Representation of three-element algebras*, Amer. Jour. Math., vol. XLVI (1924), p. 110. These papers will be referred to hereafter as Paper I and Paper II respectively.

For consider the equations (modulo n)

$$\begin{cases} e_0 = a_0, \\ e_1 = a_0+a_1+a_2+\dots\dots+a_{n-1}, \\ e_2 = a_0+2a_1+2^2a_2+\dots\dots+2^{n-1}a_{n-1}, \\ \dots\dots\dots\dots\dots\dots \\ e_{n-1} = a_0+(n-1)a_1+(n-1)^2a_2+\dots+(n-1)^{n-1}a_{n-1}. \end{cases}$$

Let Δ_n be the determinant of the coefficients of the a's:

$$\Delta_n = \begin{vmatrix} 1 & 0 & 0 & \dots\dots & 0 \\ 1 & 1 & 1 & \dots\dots & 1 \\ 1 & 2 & 2^2 & \dots\dots & 2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{vmatrix}.$$

Then the a's are determined if and only if

$$\Delta_n \not\equiv 0 \bmod n.$$

But for any c

$$\begin{vmatrix} 1 & c & c^2 & \dots\dots & c^{n-1} \\ 1 & c+1 & (c+1)^2 & \dots\dots & (c+1)^{n-1} \\ 1 & c+2 & (c+2)^2 & \dots\dots & (c+2)^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & c+\overline{n-1} & (c+\overline{n-1})^2 & \dots & (c+\overline{n-1})^{n-1} \end{vmatrix} = 1!\,2!\dots(n-1)!,$$

and if n is prime, and only then,

$$1!\,2!\,\dots(n-1)! \not\equiv 0 \bmod n.$$

COROLLARY 1. *If n is prime, a polynomial modulo n of range n equivalent to the sequence of least positive n-residues.*

$$e_0, e_1, \dots, e_{n-1}$$

is given by

$$a_0+a_1x+\dots+a_{n-1}x^{n-1} \bmod n,$$

where (modulo n)

$$a_0=e_0,\ a_1=\frac{\begin{vmatrix} e_1-e_0 & 1 & \dots\dots & 1 \\ e_2-e_0 & 2^2 & \dots\dots & 2^{n-1} \\ \dots & \dots & \dots & \dots \\ e_{n-1}-e_0 & (n-1)^2 & \dots & (n-1)^{n-1} \end{vmatrix}}{\Delta_n},\ \dots,\ a_{n-1}=\frac{\begin{vmatrix} 1 & 1 & \dots\dots & e_1-e_0 \\ 2 & 2^2 & \dots\dots & e_2-e_0 \\ \dots & \dots & \dots & \dots \\ n-1 & (n-1)^2 & \dots & e_{n-1}-e_0 \end{vmatrix}}{\Delta_n},$$

$$\Delta_n = \begin{vmatrix} 1 & 1 & \dots\dots & 1 \\ 2 & 2^2 & \dots\dots & 2^{n-1} \\ \dots & \dots & \dots & \dots \\ n-1 & (n-1)^2 & \dots\dots & (n-1)^{n-1} \end{vmatrix}.$$

COROLLARY 2. *If n is composite, there exists no polynomial modulo n of range n equivalent to a sequence of least positive n-residues or there exists more than one such polynomial according as the determinants*

$$\begin{vmatrix} e_1-e_0 & 1 & \cdots & 1 \\ e_2-e_0 & 2^2 & \cdots & 2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ e_{n-1}-e_0 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{vmatrix}, \ldots, \begin{vmatrix} 1 & 1 & \cdots & e_1-e_0 \\ 2 & 2^2 & \cdots & e_2-e_0 \\ \cdots & \cdots & \cdots & \cdots \\ n-1 & (n-1)^2 & \cdots & e_{n-1}-e_0 \end{vmatrix}$$

all vanish or do not all vanish modulo n.

COROLLARY 3. *If and only if n is prime, the polynomials modulo n of range n,*

$$a_0+a_1x+\ldots+a_{n-1}\,x^{n-1} \ mod\ n,$$

are all different for the n^n different selections of a_i.

3. CLASS-CLOSING BINARY OPERATIONS

If $\oplus$ be an operation in a class K such that $x\oplus y$ is in K, I shall call $\oplus$ a K-*closing** operation. If the number of elements in K be a prime p, we have the following theorem for the representation of class-closing binary operations.

THEOREM 2. *If p be prime, each of the $(p^p)^p$ operations $x\oplus y$ defined by the tables*

(i)

$\oplus$	0	1	$\ldots\ldots$	$p-1$
0	e_{00}	e_{01}	$\ldots\ldots$	$e_{0(p-1)}$
1	e_{10}	e_{11}	$\ldots\ldots$	$e_{1(p-1)}$
.	. . . . . .	. . . . . .	. . . . . .	. . . . . .
.	. . . . . .	. . . . . .	. . . . . .	. . . . . .
$p-1$	$e_{(p-1)0}$	$e_{(p-1)1}$	$\ldots\ldots$	$e_{(p-1)(p-1)}$

$(e_{ij}=0,\ 1,\ \ldots,\ p-1),$

is equivalent to a polynomial modulo p of range p of the form

$$f_0(x)+f_1(x)y+\ldots+f_{p-1}\,(x)y^{p-1} \ mod\ p, \qquad \text{(ii)}$$

where $f_i(x)$ are polynomials modulo p of range p of the form

$$a_0+a_1x+\ldots+a_{p-1}\,x^{p-1} \ mod\ p. \qquad \text{(iii)}$$

This theorem is true because, first, every function (ii) determines a $\oplus$-table (i), and, secondly, there are (Theorem 1, Corollary 3) p^p distinct functions (iii).

A modular polynomial equivalent to a $\oplus$-table will be called the *function* of the $\oplus$-table.

For the representation of class-closing binary operations in a class consisting of a *composite* number of elements we have

*A term due to Sheffer.

THEOREM 3. *If n be composite, each of the operations $\oplus$ in a class K defined by the tables*

$$\text{(i)}\qquad \begin{array}{c|llll} \oplus & 0 & 1 & \cdots\cdots & n-1 \\ \hline 0 & e_{00} & e_{01} & \cdots\cdots & e_{0(n-1)} \\ 1 & e_{10} & e_{11} & \cdots\cdots & e_{1(n-1)} \\ \cdot & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & \cdots & \cdots & \cdots \\ n-1 & e_{(n-1)0} & e_{(n-1)1} & \cdots & e_{(n-1)(n-1)} \end{array} \qquad (e_{ij}=0,\ 1,\ \ldots,\ n-1),$$

is equivalent to a polynomial modulo p of range n of the form

$$f_0(x)+f_1(x)y+\ldots+f_{p-1}\ (x)y^{p-1} \ mod\ p,$$

where p is a prime exceeding n, and where the $f_i(x)$ are polynomials modulo p of range n.

For let K' be the class consisting of the complete system of least positive p-residues $0, 1, \ldots, n-1, \ldots, p-1$. If x' be any K'-element, there is a K-element x such that $x'=x \bmod n$. Let $\oplus'$ be an operation in K' such that $x'\oplus y'=x\oplus y$. If $f(x', y')$ denote the polynomial modulo p of range p which is the function of $\oplus'$, then when x', y' range over the least positive n-residues $0, 1, \ldots, n-1, f(x', y')$ will be equivalent to table (i).

4. BINARY OPERATIONS NOT CLASS-CLOSING

Consider any table defining an operation $\oplus$ in a class K. If $\oplus$ is not class-closing, there exist some values of x, y for which $x\oplus y=\xi$, an element not in K, and hence meaningless as far as K is concerned. Let $\oplus'$ be an operation obtained from $\oplus$ by substituting K-elements for all the ξ's. Let $\oplus''$ be the operation obtained from $\oplus$ by replacing the ξ's by 0 and the non-ξ's by K-elements not 0. Then (Theorems 2, 3) if $f(x, y)$ and $\phi(x, y)$ be the respective functions of $\oplus'$ and $\oplus''$, the function

$$f(x, y)+0/\phi(x, y)$$

will be equivalent to the operation $\oplus$. We therefore have for the representation of any finite binary operation which is not class-closing

THEOREM 4. *Every finite operation $x\oplus y$ which is not class-closing is equivalent to a function of the form*

$$f(x, y)+0/\phi(x, y),$$

where $f(x, y)$ and $\phi(x, y)$ are modular polynomials.

5. DYADIC RELATIONS

Let R be any finite dyadic relation defined by an R-table. Let $\oplus$ be the operation obtained by substituting in the R-table the elements 0, 1 for the signs $+$, $-$ respectively. Then (Theorems 2, 3) if $f(x, y)$ be the function of the operation $\oplus$, the equation $f(x, y)=0$ will be equivalent to the relation R. Hence for the representation of any dyadic relation in a finite class we have

THEOREM 5. *Any dyadic relation defined by a table of the form*

(i)

R	0	1	$\dots$	$n-1$
0	$\pm$	$\pm$	$\dots$	$\pm$
1	$\pm$	$\pm$	$\dots$	$\pm$
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$
$n-1$	$\pm$	$\pm$	$\dots$	$\pm$

is equivalent to an equation of the form

(ii) $$f(x, y)=0 \bmod p,$$

where p is a prime and $f(x, y)$ a modular polynomial of range n.

A *modular* equation (ii) equivalent to a given R-table, will be called the *equation* of the R-table.

6. ILLUSTRATIVE EXAMPLES

The calculation of functions of $\oplus$-tables and equations of R-tables will be illustrated by the following examples.

EXAMPLE 1. To find the function of

(i)

	0	1
0	0	1
1	1	1

By Theorem 2, the function of (i) is of the form

(ii) $$f(x, y)\equiv f_0(x)+f_1(x)y \bmod 2,$$

where $f_0(x), f_1(x)$ are polynomials modulo 2 of degree 1. By Theorem 1, Corollary 1, if $\phi(x)$ is a polynomial modulo 2 of range 2 and degree 1 of the form

$$a_0+a_1x \bmod 2,$$

then

(iii) $$a_0=\phi(0),\ a_1=\phi(1)-\phi(0)=\phi(0)+\phi(1) \bmod 2$$

and

(iv) $$\phi(x)=\phi(0)+[\phi(0)+\phi(1)]x \bmod 2.$$

Applying (iii) to (ii) regarded as a function of y alone, we have (modulo 2)

(v) $$f_0(x)=f(x, 0),\ f_1(x)=f(x, 0)+f(x, 1).$$

Applying (iv) to (v), we get (modulo 2)

(vi) $$\begin{cases} f_0(x)=f(0, 0)+[f(0, 0)+f(1, 0)]x=x \\ f_1(x)=f(0, 0)+[f(0, 0)+f(1, 0)]x+f(0, 1)+[f(0, 1)+f(1, 1)]x=1+x. \end{cases}$$

Substituting (vi) in (ii), we have for the function of (i) the polynomial

(vii) $$x+(1+x)y \bmod 2^*.$$

In general, the function of the operation

(I)

	0	1
0	e_{00}	e_{01}
1	e_{10}	e_{11}

$(e_{ij}=0,\ 1)$,

will be found to be

(II) $$e_{00}+(e_{00}+e_{10})x+[e_{00}+e_{01}+(e_{00}+e_{01}+e_{10}+e_{11})x]y \bmod 2.$$

EXAMPLE 2. The function of table

(i)

	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

will be of the form

(ii) $$f(x,\ y)\equiv f_0(x)+f_1(x)y+f_2(x)y^2 \bmod 3,$$

where the $f_i(x)$ are polynomials modulo 3 of degree 2. By Theorem 1, Corollary 1, if $\phi(x)$ is a polynomial modulo 3 of range 3 and degree 2 of the form

$$a_0+a_1x+a_2x^2 \bmod 3,$$

then (modulo 3)

(iii) $$\begin{cases} a_0= \phi(0), \\ a_1 = \dfrac{\begin{vmatrix} \phi(1)-\phi(0) & 1 \\ \phi(2)-\phi(0) & 2^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 2^2 \end{vmatrix}} = 2\phi(1)+\phi(2), \\ a_2 = \dfrac{\begin{vmatrix} 1 & \phi(1)-\phi(0) \\ 2 & \phi(2)-\phi(0) \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 2^2 \end{vmatrix}} = 2\,[\phi(0)+\phi(1)+\phi(2)]. \end{cases}$$

Hence

(iv) $$\phi(x)=\phi(0)+[2\phi(1)+\phi(2)]x+2[\phi(0)+\phi(1)+\phi(2)]x^2 \bmod 3.$$

Therefore we get (modulo 3)

*Table (i) in example 1 defines the logical sum of two elements in a two-element boolean algebra. Function (vii) is thus an arithmetic representation of this boolean operation. Compare Paper I, pp. 26, 29.

$$\text{(v)}\quad \begin{cases} f_0(x)=f(x,0)=f(0,0)+[2f(1,0)+f(2,0)]x+2[f(0,0)+f(1,0)+f(2,0)]x^2=x, \\ f_1(x)=2f(x,1)+f(x,2)=2, \\ f_2(x)=2[f(x,0)+f(x,1)+f(x,2)]=0. \end{cases}$$

Therefore the function of (i) is

$$\text{(vi)}\qquad x+2y \bmod 3^*.$$

In general, the function of the operation

(I)

	0	1	2
0	e_{00}	e_{01}	e_{02}
1	e_{10}	e_{11}	e_{12}
2	e_{20}	e_{21}	e_{22}

will be found to be

$$\text{(II)}\qquad f_0(x)+f_1(x)y+f_2(x)y^2 \bmod 3,$$

where (modulo 3)

$$\text{(III)}\quad \begin{cases} f_0(x)=e_{00}+(2e_{10}+e_{20})x+2(e_{00}+e_{10}+e_{20})x^2, \\ f_1(x)=2e_{01}+e_{02}+(e_{11}+2e_{12}+2e_{21}+e_{22})x+(e_{00}+2e_{02}+e_{11}+2e_{12}+e_{21}+2e_{22})x^2, \\ f_2(x)=2(e_{00}+e_{01}+e_{02})+(e_{10}+e_{11}+e_{12}+2e_{20}+2e_{21}+2e_{22})x \\ \qquad +(e_{00}+e_{01}+e_{02}+e_{10}+e_{11}+e_{12}+e_{20}+e_{21}+e_{22})x^2. \end{cases}$$

EXAMPLE 3. To find the function of†

(i)

$\oplus$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3 .

Since the number of elements is composite, consider (Theorem 3) the table

(ii)

$\oplus'$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	0
2	0	2	2	0	0
3	0	3	0	3	0
4	0	0	0	0	0 ,

where $x'\oplus'4=x\oplus 0$, $4\oplus' y'=0\oplus y$. If $\phi(x)$ is a polynomial modulo 5 of range 5 and degree 4, we get with the help of Theorem 1, Corollary 1,

*Table (i) or function (vi) of example 2 defines the operation of subtraction modulo 3. Compare Paper II, p. 112. In general, $x-y \bmod n=x+(n-1)y \bmod n$.

†Table (i) of example 3 defines the logical product in a boolean algebra consisting of the elements 0, 1, 2, 3.

(iii) $\phi(x)=\phi(0)+[4\phi(1)+2\phi(2)+3\phi(3)+\phi(4)]x$
$+[4\phi(1)+\phi(2)+\phi(3)+4\phi(4)]x^2+[4\phi(1)+3\phi(2)+2\phi(3)+\phi(4)]x^3$
$+4[\phi(0)+\phi(1)+\phi(2)+\phi(3)+\phi(4)]x^4 \bmod 5.$

Hence the function of (i) is

(iv) $(x^2+x^3)y+(x+x^2+x^3+x^4)y^2+(x+x^2+2x^3+2x^4)y^3+(x^2+2x^3+x^4)y^4 \bmod 5,$

where the range of x, y is 0, 1, 2, 3.

EXAMPLE 4. Find the function of*

(i) (ξ not in class).

$\oplus$	0	1	2
0	ξ	1	1
1	ξ	0	2
2	ξ	2	0

Here the operation $\oplus$ is not class-closing. Consider the tables

(ii)

	0	1	2
0	1	1	1
1	1	0	2
2	1	2	0,

(iii)

	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2,

where (ii) is (i) in which the ξ's are replaced by 1, and (iii) is (i) in which the ξ's are replaced by 0 and the non-ξ's by elements not 0 as indicated. The functions of (ii) and (iii) are respectively

(iv) $1+2xy \bmod 3,$ (v) y.

Therefore (Theorem 4) the function of (i) is

(vi) $1+2xy+0/y \bmod 3.$

EXAMPLE 5. To find the equation of

(i)

R	0	1	2
0	−	+	+
1	−	−	+
2	−	−	−.

Consider the $\oplus$-table

(ii)

$\oplus$	0	1	2
0	1	0	0
1	1	1	0
2	1	1	1

*Table (i) in example 4 defines the operation $1-x/y \bmod 3$. See Paper II, p. 114. In general, for a prime p, $1-x/y \bmod p=1+(p-1)xy^{p-2}+0/y \bmod p$.

got from (i) by replacing the $+$signs by 0 and the $-$ signs by 1. The function of (ii) is

(iii) $$1+(x+x^2)y+(2+2x)y^2 \bmod 3.$$

Then (Theorem 5) the equation of (i) is

(iv) $$1+(x+x^2)y+(2+2x)y^2=0 \bmod 3^*.$$

7. Application to postulate-theory†

The following considerations will indicate the value of my theory of representation to postulate-theory.

1. Since $f(x, y) \bmod p$ and $f(x, y)=0 \bmod p$ can be interpreted geometrically, we have that *any finite non-quantitative algebra*, in particular *any finite boolean algebra, can be represented arithmetically and geometrically.*

2. In obtaining the necessary proof-systems establishing consistency and independence of a set of postulates, no *method* has hitherto been in use. *My modular representations open a way for a method in obtaining consistency and independence systems by reducing the problem to that of finding a modular polynomial whose coefficients satisfy certain conditions.*

3. By reflecting on the nature of the equations $z=f(x, y)$ and $f(x, y)=0$, *we can find the answers to the questions: Just what is an operation? What a relation? Is there a connection between operations and relations, and just what is this connection?*

4. Since all finite operations are expressible in terms of the pair of operations modular addition and modular multiplication, and since modular multiplication is definable in terms of modular addition, we see that *all operations in a given finite class are definable in terms of the single operation of modular addition.*

5. Wiener has shown‡ that a *field* can be defined in terms of the single operation $x@y\equiv 1-x/y$. For p prime, it can be verified that the operation $1-x/y \bmod p=(p-1)xy^{p-2}+0/y \bmod p$ satisfies Wiener's postulates for a field. Hence, *any finite operation can be defined in terms of the single operation $1-x/y$ mod p.*

6. A problem of supreme importance in postulate-theory is to determine if two given sets of postulates are equivalent. If the system concerned is finite, *this problem is solved by determining if the two postulate-sets produce the same $\oplus$-tables and R-tables.*

*Table (i) in example 5 is the serial relation *less than* among the elements 0, 1, 2. Equation (iv) is the equation of this relation. Compare Paper II, p. 113.

†For other applications (confined to classes of two and three elements) see Paper I, pp. 29, 30; Paper II, pp. 114, 115, 116.

‡Trans. Amer. Math. Soc., vol. 21 (1920), p. 237.

SUR L'INDICATEUR D'UN NOMBRE ENTIER

PAR M. LÉON POMEY,
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Nous allons donner ici un théorème qui nous a servi de base pour démontrer certaines propositions de Théorie des Nombres relatives aux nombres premiers et publiées récemment aux Comptes Rendus de l'Académie des Sciences de Paris (T. 178, 17 Mars 1924, p. 987). Ce théorème, que l'on pourrait d'ailleurs facilement étendre encore, s'énonce ainsi:

Théorème.—L'entier p étant impair, si l'entier $\frac{p-1}{m}$ *l'est aussi quand m est*>1, *et si l'indicateur* $\phi(p)$ *de p est un multiple de* $\frac{p-1}{m}$ (*m étant l'un des nombres* 1, 2, 4, 6, 8, 10, 12), *le nombre p est premier, sauf pour les valeurs singulières* $p=9$, 91 *et* 8911, *qui sont des nombres composés faisant exception.*

Voici une des diverses manières, dont ceci peut être démontré:

Remarquons d'abord d'une manière générale que si $\phi(p)$ contient 2^λ en facteur, le nombre impair $p=a^\alpha . b^\beta . c^\gamma \ldots$ ne peut avoir plus de λ facteurs premiers $a, b, c, \ldots$ Cela résulte clairement de l'expression de $\phi(p)$

$$\phi(p)=a^{\alpha-1}b^{\beta-1}c^{\gamma-1}\ldots(a-1)\ (b-1)\ (c-1)\ldots,$$

dans laquelle chacun des nombres $a-1$, $b-1$, ... est divisible par une puissance de 2 au moins égale à l'unité. Cela prouve également que $\phi(p)$ est toujours pair.

Cela posé, $\frac{p-1}{m}$ étant supposé impair, $\phi(p)$ qui en est un multiple sera l'un des nombres: $K\frac{p-1}{m}$, où K est un nombre *pair* $\leq m \leq 12$.

Pour ne pas allonger inutilement cette communication, nous nous bornerons ici au cas de $m=10$, qui est un des plus complets et qui présente le plus grand nombre d'irrégularités. La démonstration serait analogue et d'ailleurs plus simple pour les autres valeurs de m (2, 4, 6, 8, 12).

Nous allons donc démontrer d'abord que *si* $\frac{p-1}{10}$ *est impair,* $\phi(p)$ *ne peut être de la forme* $K\frac{p-1}{10}$ *à moins que p ne soit premier et par suite alors K égal à* 10 (*sauf pour les valeurs exceptionnelles* $p=91$ *et* 8911).

En effet $\phi(p)$ étant toujours pair et $<p-1$, K doit être pair et égal à 2, 4, 6, 8 ou 10.

1° si $K=10$, $\phi(p)=p-1$ et p est bien premier.

2° si $K=2$ ou 6, $\phi(p)$ étant divisible seulement par 2 et non par 4, p ne peut contenir qu'un seul facteur premier a, soit $p=a^\alpha$. D'où

$$a^{\alpha-1}(a-1)=K\left(\frac{a^\alpha-1}{10}\right).$$

ou

$$(10-K)a^\alpha-10a^{\alpha-1}+K=0$$

ou

$$a^{\alpha-1}[(10-K)a-10]+K=0.$$

Or cherchons quand on aura: $(10-K)a-10\geq 0$ ou $a\geq \dfrac{10}{10-K}$. Cela aura lieu quand $K=2$ pour $a\geq \dfrac{10}{8}=\dfrac{5}{4}$, donc pour toute valeur de $a>1$; et quand $K=6$, pour $a\geq \dfrac{10}{4}=2+\dfrac{1}{2}$, donc pour $a\geq 3$.

Donc l'égalité ci-dessus est bien impossible, à moins que $\alpha=1$ et $p=a$,

3° Si $K=4$, p aura 2 facteurs premiers au plus:

S'il en a un, a, le raisonnement précédent montre que α doit encore être $=1$, puisqu'on a toujours $a\geq \dfrac{10}{10-4}=\dfrac{10}{6}=\dfrac{5}{3}=1+\dfrac{2}{3}$ ou $a\geq 2$. *Si p en a* 2, soit $p=a^\alpha b^\beta$(avec $2<a<b$, donc $b\geq 5$):

D'où la relation

$$a^{\alpha-1}b^{\beta-1}(ab-a-b+1)=4\,\frac{(a^\alpha b^\beta-1)}{10}$$

ou

$$a^{\alpha-1}b^{\beta-1}[(10-4)\,ab-10a-10b+10]+4=0.$$

Or ceci est impossible; il suffit, pour le voir, de vérifier que l'on a

$6ab-10a-10b+10\geq 0$, ou $b\geq 5\dfrac{(a-1)}{3a-5}$ ou $\dfrac{5}{3}y\leq b$, en posant $y=\dfrac{a-1}{a-\frac{5}{3}}$.

Construisons l'hyperbole

$$y=\frac{a-1}{a-\frac{5}{3}}.$$

Pour $a\geq 2$ on a $y\leq 3$. Donc pour $a\geq 2$ on a bien $\dfrac{5}{3}y\leq 5\leq b$.—C.Q.F.D.

4° Soit $K=8$, alors p ne peut avoir plus de trois facteurs premiers:

A. Supposons d'abord qu'il en ait un seul d'où: $p=a^\alpha$. Alors on devrait avoir $a^{\alpha-1}(a-1)=\dfrac{8}{10}(a^\alpha-1)$ ou $a^{\alpha-1}(a-5)+4=0$, évidemment impossible dès que $a\geq 5$. Et pour $a=3$, il vient $-3^{\alpha-1}\times 2+4=0$, également impossible.

B. Supposons maintenant deux facteurs premiers: soit $p=a^{\alpha}b^{\beta}$. Alors

$$\phi(p)=8\frac{p-1}{10}$$

s'écrit

$$a^{\alpha-1}b^{\beta-1}(a-1)(b-1)=\frac{4}{5}(a^{\alpha}b^{\beta}-1) \text{ ou } a^{\alpha-1}b^{\beta-1}(ab-5a-5b+5)+4=0.$$

Or cette relation est impossible, si

(I) $$ab-5a-5b+5\geq 0 \text{ ou } b\geq 5\frac{a-1}{a-5}.$$

Construisons l'hyperbole $y=\dfrac{a-1}{a-5}$ et supposons d'bord $a\geq 7$:

Pour $a=7$, on a $y=3$, et pour $a\geq 7$, on a $y\leq 3$. Donc pour $a\geq 7$, on a $5y\leq 15$. Ainsi, dès que $a\geq 7$, la relation est bien impossible si $b\geq 17$.

Supposons maintenant $a\geq 7$ et $b<17$, donc $b=11$ ou 13.

Or pour $a=11$, on a $y=\dfrac{10}{6}=\dfrac{5}{3}$; donc pour $a\geq 11$, $y\leq\dfrac{5}{3}$ et $5y\leq 5\times\dfrac{5}{3}=8+\dfrac{1}{3}$ donc pour $a>7$, $5y$ est bien $\leq b$, si $b>8+\dfrac{1}{3}$ c'est-à-dire si $b\geq 11$.

Enfin si $a=7$, on a $y=3$, d'où $b-5y=b-15$. Si $b=11$ ou 13, $b-15=-4$ ou -2; donc pour qu'on ait $b-5y+\dfrac{4}{a^{\alpha-1}b^{\beta-1}(a-5)}=0$ ou $b-5y+\dfrac{2}{7^{\alpha-1}b^{\beta-1}}=0$, il faut que $\alpha=\beta=1$ et que $b=13$.

Donc nous avons *un cas d'exception:* $p=7\times 13=91$, pour lequel $\phi(p)$ est bien $=8\times\dfrac{p-1}{10}$, avec $\dfrac{p-1}{10}=9$ impair.

Enfin si $a<7$, a ne peut être égal à 5, sinon $p-1$ ne serait pas divisible par 10. Si $a=3$, $y=-1$, $b-5y=b+5$; *l'égalité à satisfaire devient*:

$$b+5+\frac{4}{3^{\alpha-1}b^{\beta-1}(3-5)}=0 \text{ ou } b+5=\frac{2}{3^{\alpha-1}b^{\beta-1}},$$

ce qui est impossible, même si $\alpha=\beta=1$.

C. Supposons enfin que p contienne *trois* facteurs premiers a, b, c; soit donc $p=a^{\alpha}b^{\beta}c^{\gamma}$, avec α, β, γ tous >0.

La relation $\phi(p)=8.\dfrac{p-1}{10}$, dont il s'agit de prouver l'impossibilité, s'écrit alors

$$a^{\alpha-1}b^{\beta-1}c^{\gamma-1}(a-1)\ (b-1)\ (c-1)=\frac{8}{10}(a^{\alpha}b^{\beta}c^{\gamma}-1),$$

ou

$$a^{\alpha-1}b^{\beta-1}c^{\gamma-1}[5(a-1)\ (b-1)\ (c-1)-4abc]+4=0.$$

Mais $(p-1)$ étant par hypothèse divisible par 10 les nombres a, b et c doivent être impairs; donc $a^{\alpha-1}b^{\beta-1}c^{\gamma-1}$ne peut diviser 4, et il faut qu'on ait $\alpha=\beta=\gamma=1$.

Il reste

$$5(a-1)\ (b-1)\ (c-1)-4abc+4=0,$$

ou

(1) $$[5(a-1)\ (b-1)-4ab]c-5(a-1)\ (b-1)+4=0.$$

Or un premier cas, pour lequel cette égalité est manifestement impossible est celui où le crochet multiplicateur de c est$\leqq 0$; on a alors $ab-5a-5b+5\leqq 0$

ou

(2) $$b(a-5)-5(a-1)\leqq 0.$$

Or ceci d'abord a lieu *quel que soit b pour $a\leqq 5$.* Mais a, b, c jouent jusqu'ici un rôle absolument symétrique. Donc l'impossibilité a lieu dès que l'un des nombres a, b, c est $\leqq 5$.

Supposons donc maintenant que les trois nombres impairs et inégaux a, b et c sont$\geqq 7$; soit donc $7\leqq a<b<c$ et par conséquent $7\leqq a$, $11\leqq b$, $13\leqq c$.

Reprenons la condition (2) ou $b\leqq 5\dfrac{a-1}{a-5}$ ou $b\leqq 5+5\dfrac{4}{a-5}$; quand $a\geqq 7$, elle est vérifiée pour $b\leqq 15$.

Donc supposons qu'on a $7\leqq a$, $17\leqq b$, $19\leqq c$.

La relation (1) sera encore évidemment impossible, quand on aura au contraire

$$[5(a-1)\ (b-1)-4ab]c-5(a-1)\ (b-1)\geqq 0,$$

ou

$$c\geqq 5\,\frac{ab-a-b+1}{ab-5a-5b+5}.$$

Or c étant supposé $\geqq 19$, cela aura lieu, si on a

$$19\geqq 5\,\frac{ab-a-b+1}{ab-5a-5b+5},\ \text{ou}\ b\geqq 45\times\frac{a-1}{7a-45}.$$

Comme b doit être $\geqq 17$, cela aura lieu aussi si on a

$$17\geqq 45\,\frac{a-1}{7a-45},\ \text{ou}\ a\geqq\frac{16\times 45}{7\times 17-45},\ \text{c'est-à-dire}\ a\geqq\frac{360}{37}=9,72;$$

ainsi, dont l'impossibilité est encore démontrée, dès que

$$b\geqq 17,\ c\geqq 19\ \text{et}\ a\geqq 11.$$

Reste le cas où a est <11 (toujours avec $b\geqq 17$, $c\geqq 19$). Comme a doit être $\geqq 7$, il faut alors que $a=7$. Dans ces conditions (1) devient

$$bc-15b-15c+17=0,$$

ou

(3) $$(c-1)\ (b-15)-14b+2=0.$$

Or cette égalité ne sera pas vérifiée si on a $(c-1)$ $(b-15) \geq 14b$, ou, $c-1$ étant $> b$, si $b-15 > 14$, c'est-à-dire si $b \geq 29$.

L'égalité (3) ne peut donc être vérifiée que si b satisfait à $17 \leq b \leq 23$, c'est-à-dire si $b=17$, 19 ou 23.

Enfin l'égalité (3) donne pour c:

$$c = \frac{15b-17}{b-15} = 15 + \frac{16 \times 13}{b-15}. \tag{4}$$

De cette façon on trouve, avec $a=7$, les couples de valeurs $b=17$ et $c=119$ (qui n'est pas premier, donc inacceptable), $b=23$ et $c=41$ (mais le nombre correspondant $p=7 \times 23 \times 41 = 6601$ n'est pas tel que $\frac{p-1}{10}$ soit impair et par conséquent est aussi à rejeter); enfin $b=19$ et $c=67$, qui conviennent.

En résumé le théorème est démontré dans le cas de $m=10$ (*les valeurs* $p=91$ *et* $p=7 \times 19 \times 67 = 8911$ *faisant seules exception*).—C.Q.F.D.

A NEW METHOD IN THE THEORY OF ALGEBRAIC NUMBERS

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The fundamental problem in the theory of algebraic numbers is the determination of the decomposition of integral algebraic numbers into their prime-ideal factors. In the theory of rational numbers this problem corresponds to the decomposition of the rational integers into prime factors.

This fundamental problem is equivalent to the following:

I. *In a given algebraic field $P(\theta)$, to find the decomposition of a rational prime into its prime-ideal factors.*

The algebraic number θ is defined by the irreducible equation

$$f(\theta) = \theta^n + a_1\theta^{n-1} + \ldots + a_n = 0, \tag{1}$$

where all coefficients $a_1, a_2, \ldots a_n$ are rational integers.

In the theory of algebraic fields, another important problem is the following:

II. *The determination of the discriminant of the field.*

The second problem is closely connected with the first.

The discriminant, d, of the field is defined as the discriminant of n numbers in the field forming a *basis* of the field. If D designates the discriminant of the equation (1), then

$$D = k^2 d, \tag{2}$$

where k is a rational integer called the *index* of the algebraic number θ.

Problem I was first treated by Dedekind, who solved the problem for the case that k is not divisible by a prime p, by means of his theorem:

When p is not a divisor of k, and if

$$f(x) \equiv \phi_1(x)^{e_1}\phi_2(x)^{e_2}\ldots\phi_s(x)^{e_s} (\text{mod. } p)$$

is the decomposition of $f(x)$ in prime-functions (mod. p), then

$$p = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\ldots\mathfrak{p}_s^{e_s},$$

where the prime-ideal $\mathfrak{p}_i$ is defined by $\mathfrak{p}_i = (p, \phi_i(\theta))$ and $N\mathfrak{p}_i = p^{m_i}$, m_i being the degree of $\phi_i(x)$.

Dedekind also gave a criterion to decide when p is a divisor of k.

It is now *a priori* possible that for a given prime, p, there might always be found such a number α of the field, that the index of α is not divisible by p; and, therefore, the theorem just mentioned would be applicable to the equation which α satisfies. But Dedekind also showed the existence of primes, which are common factors to all indices, k, in a field, so that his method did not in all cases solve problem I. Dedekind treated also problem II and proved

$$d = N\delta; \tag{3}$$

that is to say: *The discriminant of a field is always equal to the norm of an ideal,* δ, *called the "differente" of the field.*

For the composition of the differente δ Dedekind showed that when p is divisible by a power $\mathfrak{p}^e$ of a prime-ideal, $\mathfrak{p}$, then δ is divisible exactly by $\mathfrak{p}^{e-1}$ when e is not divisible by p, and by the power $\mathfrak{p}^e$ or a higher power of $\mathfrak{p}$, when e *is* divisible by p. The exact value of this exponent was not determined.

Kronecker introduced the *fundamental equation* of the field, and Hensel showed that by means of the decomposition modulo p of the fundamental equation problem I can be completely solved.

For problem II he did not by this method obtain more general results than those of Dedekind. To treat this problem Hensel introduced his theory of p-adic numbers, that is, a type of infinite series, which are in general divergent, but of which only the arithmetical nature of the coefficients is of importance. This method is of great theoretical interest, but practical applications are very difficult.

In this paper I shall give a brief exposition of a new method of treating algebraic problems, a method which has been very fertile in many researches, such as, for instance, the reducibility of equations. I shall show that problems I and II, mentioned above, can be completely solved. It will be seen that this method is particularly well adapted for dealing with numerical cases.

For problem I, I shall first show how the prime-ideal decomposition of a prime, p, can always be determined.

Let

$$f(x) \equiv \phi_1(x)^{e_1}\phi_2(x)^{e_2}\dots\phi_s(x)^{e_s} \pmod{p} \tag{4}$$

be the prime-function decomposition of $f(x)$ (mod. p). From (4) follows easily that there always exists an ideal decomposition for p,

$$p = \mathfrak{g}_1\,\mathfrak{g}_2\cdots\mathfrak{g}_s, \tag{5}$$

where the ideals $\mathfrak{g}_i$ are relative primes and $\mathfrak{g}_i = (p, \phi_i(\theta)^{e_i})$. To find the prime-ideal decomposition of p it is therefore sufficient to determine the prime-ideal decomposition of an ideal $\mathfrak{g} = (p, \phi(\theta)^e)$, where $\phi(x)$ is a prime-function divisor of $f(x)$ (mod. p).

Assume $f(x)$ written in the form

$$f(x) = \sum_{i=0}^{t} Q_i(x)\, p^{a_i}\, \phi(x)^i, \tag{6}$$

where $Q_i(x)$ are polynomials of degree less than m (m the degree of $\phi(x)$), and where the exponents a_i are so chosen that not all coefficients of $Q_i(x)$ are divisible by p. Then (6) may be called the *series* $(p, \phi_i(x))$ *of* $f(x)$.

The series $(p, \phi(x))$ of $f(x)$ is easily obtained by successive divisions by $\phi(x)$. The number t is the greatest integer contained in $\frac{n}{m}$; or, symbolically $t = \left[\frac{n}{m}\right]$. The series $(p, \phi(x))$ is uniquely determined.

Let us now represent the pairs of numbers $(t-i, a_i)$, in a system of rectangular coordinates, and for the points so obtained, construct the Newton polygon, starting from the origin. This polygon is of great importance in our investigation. Since $\phi(x)$ is a divisor of $f(x)$, modulo p, not all sides of this polygon will lie in the x-axis. Denote the sides not lying in the x-axis by

$$S_1, S_2, \ldots S_k.$$

Let us call the projections of these sides on the x- and y-axis,

$$l_1, l_2, \ldots l_k,$$

and

$$h_1, h_2, \ldots h_k, \text{ respectively.}$$

Since these numbers are always rational integers, we can put

$$l_i = e_i\lambda_i,$$

$$h_i = e_i\eta_i,$$

where e_i is the greatest common factor of l_i and h_i. Therefore λ_i is relatively prime to η_i. The quantities λ_i and η_i are represented geometrically as the projections of the part of a side S_i lying between two lattice-points.

It is then possible to show that an ideal $\mathfrak{a}$ has the ideal composition

$$\mathfrak{a} = \mathfrak{P}_1^{\lambda_1}\mathfrak{P}_2^{\lambda_2}\ldots\mathfrak{P}_k^{\lambda_k} \tag{7}$$

where all ideals $\mathfrak{P}_i$ are relatively prime, but generally not prime-ideals.

The prime-ideal decomposition of $\mathfrak{P}_i$ can, however, be derived by an examination of those terms $Q_i(x)p^{a_i}\phi(x)^i$ in the series $(p, \phi(x))$ of $f(x)$, which are represented by points lying on the corresponding side S_i. The sum of these terms will have the form

$$p^{\alpha}\phi(x)^{\beta}(Q_{i,0}(x)\phi(x)^{e_i\lambda_i} + Q_{i,1}(x)\phi(x)^{(e_i-1)\lambda_i}p^{\eta_i} + Q_{i,2}(x)\phi(x)^{(e_i-2)\lambda_i}p^{2\eta_i} + \ldots + Q_{i,e_i}(x)p^{e_i\eta_i}). \tag{8}$$

From (8) we derive the polynomial in x and y

$$F_i(x, y) = Q_{i,0}(x)y^{e_i} + Q_{i,1}(x)y^{e_i-1} + \ldots + Q_{i,e_i}(x),$$

which shall be called the *factor corresponding to the side* S_i of the polygon. If now

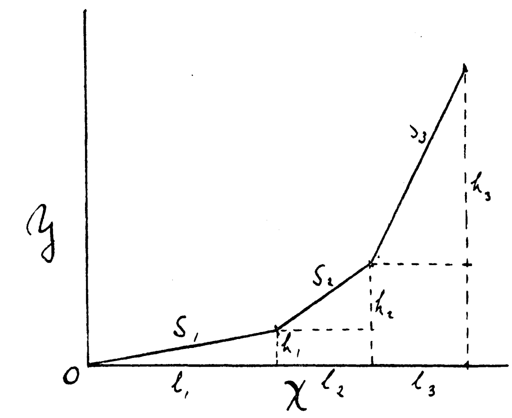
$$F_i(x, y) \equiv F_i^{(1)}(x, y)F_i^{(2)}(x, y)\ldots F_i^{(l_i)}(x, y) \pmod{p, \phi(x)} \tag{9}$$

is the decomposition of $F_i(x, y)$ into prime-functions for the double modulus $(p, \phi(x))$, then

$$\mathfrak{P}_i = \mathfrak{p}_i^{(1)} \mathfrak{p}_i^{(2)} \dots \mathfrak{p}_i^{(t_i)}$$

is the decomposition of an ideal $\mathfrak{P}_i$ in (7) into its prime-ideal factors. It is assumed that in (9) all prime-functions $F_i^{(j)}(x, y)$ are distinct.

We call the equation belonging to a primitive number of the field a *regular equation relative to* p, when, for every side of the polygon, the corresponding factor has no multiple prime-function divisors (mod. p, $\phi(x)$), in the decomposition (9), and when this is the case for all polygons belonging to the different prime-functions $\phi(x)$ dividing $f(x)$ modulo p.



It is now possible to show the existence of regular equations in all fields and relatively to every prime p. The prime-ideal decomposition of p can therefore be effected in the following way:

The equation $f(x) = 0$ is assumed to be chosen regular relative to p, and the decomposition of $f(x)$ into prime-functions may be given by (4). Then

$$p = \mathfrak{a}_1 \mathfrak{a}_2 \dots \mathfrak{a}_s,$$

where $\mathfrak{a}_i = (p, \phi(\theta)^{e_i})$. To find the decomposition of an ideal $\mathfrak{a} = (p, \phi(\theta)^e)$, we construct the polygon $(p, \phi(x))$ to $f(x)$ and determine the factors,

$$F_1(x, y),\ F_2(x, y) \dots F_k(x, y),$$

corresponding to the different sides. Therefore, when

$$F_i(x, y) \equiv F_i^{(1)}(x, y) F_i^{(2)}(x, y) \dots F_i^{(t_i)}(x, y) \ (\text{modd. } p, \phi(x))$$

is the decomposition of $F_i(x, y)$ (modd. p, $\phi(x)$) into prime-functions, we have

$$\mathfrak{a} = (\mathfrak{p}_1^{(1)} \mathfrak{p}_1^{(2)} \dots \mathfrak{p}_1^{(t_1)})^{\lambda_1} (\mathfrak{p}_2^{(1)} \dots \mathfrak{p}_2^{(t_2)})^{\lambda_2} \dots (\mathfrak{p}_k^{(1)} \dots \mathfrak{p}_k^{(t_k)})^{\lambda_k},$$

where $\mathfrak{p}_i^{(j)}$ are prime-ideals.

It can also be shown that the degree of a prime-ideal $\mathfrak{p}_i^{(j)}$ is equal to $\epsilon_i^{(j)} m$, where $\epsilon_i^{(j)}$ denotes the degree of $F_i^{(j)}(x, y)$ with respect to y; therefore

$$N\mathfrak{p}_i^{(j)} = p^{\epsilon_i^{(j)} m}.$$

The prime-ideals $\mathfrak{p}_i^{(j)}$ can also be easily determined as greatest common factors of certain numbers of the field.

By the same method we may also determine a fundamental system for the prime p, that is to say, n integral numbers

$$\omega_1, \omega_2, \ldots, \omega_n$$

of the field, such that for every integral number α we have

$$\alpha \equiv a_1\omega_1 + a_2\omega_2 + \ldots + a_n\omega_n \text{ (mod. } p),$$

where $a_1, a_2, \ldots a_n$ are integral numbers $0, 1, \ldots p-1$.

Problem II is closely connected with problem I, and its solution can easily be derived from the preceding results. When a fundamental system for a prime p is determined, the discriminant of this system will be exactly divisible by the same power of p as is the discriminant, d, of the field. From this remark we can derive the following theorem, which may be of interest:

Let L be the number of lattice points lying between the polygon $(p, \phi(x))$ of $f(x)$, not counting the points situated on the X-axis or on the ordinate of the end point of the polygon. When $f(x)$ is a regular equation relative to p, then the index k is exactly divisible by $p^{\Sigma mL}$, where the sum is taken over all polygons of $f(x)$ corresponding to the different prime-function divisors $\phi(x)$ of $f(x)$ (mod. p).

By the same method a formula was derived for the exact power of p dividing the discriminant D of the regular equation. By applying the relation (2), the complete composition of the discriminant, d, and of the "differente" δ can be found:

When the prime-ideal decomposition of p is

$$p = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\ldots\mathfrak{p}_s^{e_s},$$

then δ is exactly divisible by a prime-ideal $\mathfrak{p}_i$ to a power

$$\mathfrak{p}_i^{e_i-1+\rho_i} \quad (i=1, 2, \ldots s),$$

where $\rho_i=0$ if e_i is not divisible by p and $\rho_i \geqq 1$ if e_i is divisible by p.

By means of (3) we then find that the discriminant of the field is divisible by

$$p^{\sum\limits_{i=1}^{s}(e_i-1+\rho_i)f_i} = p^{n+\sum\limits_{i=1}^{s}(\rho_i-1)f_i}$$

The advantage of this method lies in the great simplicity of the determination of the numbers ρ_i. For these numbers there exists an upper limit first surmised by Dedekind and later proved by Hensel. This limit can be easily derived by the method of this paper. When e_i is exactly divisible by p^t, then

$$1 \leqq \rho_i \leqq te_i. \tag{10}$$

We can also find all the numbers ρ_i possible in accordance with (10).

The first part of the preceding results are published in the Acta Mathematica, vol. 44 and 46. The new results will be published in a later volume of the same journal.

SUR UNE MÉTHODE DE CALCUL DES IDÉAUX D'UN CORPS DU SECOND DEGRÉ

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I

Étude des nombres obtenus en donnant à x les valeurs entières 0, ±1, ±2, ... *dans le polynome à coefficients entiers*

$$f(x)=ax^2+bx+c.$$

1. Nous désignerons par D et nous appellerons déterminant du polynome le nombre

$$D=b^2-4ac.$$

Lorsqu'on établit la théorie des fonctions homogènes du second degré à deux variables on reconnait l'importance de D. Nous retrouverons son rôle dans cette étude; nous distinguerons deux cas:

$$D<0,\ D>0.$$

Nous laisserons de côté le cas de $D=0$, car nous serions ramenés à étudier le carré d'un polynome du premier degré.

2. Nous formerons le tableau des $f(0)$, $f(\pm 1)$, $f(\pm 2)$...décomposés en facteurs premiers.

Remarquons d'abord que le tableau des nombres donnés par

(1) $$ax^2+bx+c$$

est contenu dans le tableau des nombres donnés par l'une des deux formes:

$$x^2+x+r \text{ ou } x^2+r$$

suivant que b est pair ou impair.

En effet, si nous multiplions par a nous obtenons

$$a^2x^2+abx+ac\equiv(ax+b')^2+ac-b'^2$$

si b est pair et égal à $2b'$, et

$$a^2x^2+abx+ac\equiv(ax+b')\ (ax+b'+1)+ac-b'(b'+1)$$

si b est impair et égal à $2b'+1$.

En posant $ax+b'=X$ on a soit:

(2) $$X^2+r,$$

soit:

(3) $$X^2+X+r,$$

et nous voyons que les nombres du tableau ax^2+bx+c sont les nombres pris de a en a dans les polynomes (2) ou (3) suivant les cas.

J'exprimerai ce fait en disant qu'au point de vue arithmétique les polynomes (2) et (3) contiennent soit l'un, soit l'autre le polynome (1).

Remarquons que dans le premier cas $D=-4r$, dans le second $D=-4r+1$

3. Soit à étudier le polynome

$$x^2+\Delta;$$

nous distinguerons plusieurs cas:

$$(\alpha) \qquad \Delta=4p-1,$$

$$(\beta) \qquad \Delta=2p,$$

$$(\gamma) \qquad \Delta=4p+1.$$

(α) Soit $\Delta=4p-1$ c'est-à-dire impair.

Si pour $x=x_0<a$ on a $x_0{}^2+\Delta\equiv 0$ (mod. a), $a>2$ et premier, la seconde racine de la congruence (je parle des racines inférieures à a) sera de parité contraire à x_0, autrement dit: des deux racines de la congruence l'une est paire, l'autre est impaire, il suffira donc, pour trouver tous les diviseurs premiers impairs possibles, de donner à X dans $X^2+\Delta$ soit les valeurs paires, soit les valeurs impaires. Choisissons les valeurs impaires, posons $X=2x+1$ nous aurons:

$$4x^2+4x+1+4p-1\equiv 4\ (x^2+x+p),$$

c'est-à-dire qu'au point de vue des facteurs qui composent $X^2+\Delta$, $(\Delta=4p-1)$ lorsqu'on fait $X=(0,\ \pm 1,\ \pm 2,\dots)$, nous aurons le facteur 4, et les facteurs du tableau provenant de

$$f(x)=x^2+x+p,$$

et, comme $f(-1-x)=f(x)$, avec $x=0,\ 1,\ 2,\dots$

(β) Si Δ est de la forme $2p$, p impair, nous donnerons à X dans $X^2+\Delta$, les valeurs paires et nous étudierons $2x^2+p$ pour $x=0,\ 1,\ 2.\ \dots$

Si p est pair et égal à $2\Delta'$ en posant $X=2x$ on est ramené à $x^2+\Delta'$.

(γ) Si Δ est de la forme $4p+1$, nous donnerons à X dans $X^2+\Delta$ des valeurs impaires $2x+1$, nous aurons le facteur 2, et les facteurs provenant du tableau:

$$2x^2+2x+p+1$$

où l'on donne à x les valeurs 0, 1, 2, . . .

4. Soit maintenant le polynome

$$f(x)=x^2+x+p, \quad p>0,$$

et supposons que $f(x_0)$ ne soit pas premier, il admet un diviseur premier

$$a<\sqrt{f(x_0)},$$

$$f(x_0)\equiv 0 \pmod{a}.$$

Si $x_0>a$ nous aurions $x_0=aq+x_0'$ avec $x_0'<a$. Supposons donc $x_0<a$, sinon nous aurions considéré $f(x_0')$.

La congruence $f(x)\equiv 0 \pmod{a}$ admet comme racine inférieure à a, la racine $a-1-x_0$. Ecartons d'abord le cas $a-1-x_0=x_0$ et cherchons la condition pour que

$$a-1-x_0<x_0. \tag{4}$$

Le nombre a est inférieur ou égal à $\sqrt{x_0^2+x_0+p}$. Si donc la condition:

$$\sqrt{x_0^2+x_0+p}-1-x_0<x_0 \tag{5}$$

est satisfaite, la condition (4) l'est aussi. La condition (5) peut s'écrire

$$x_0^2+x_0+p<(2x_0+1)^2,$$

c'est-à-dire

$$3x_0^2+3x_0-p+1>0,$$

ou encore, puisque nous ne considérons que les valeurs positives de x ,

$$x_0>\frac{-3+\sqrt{12p-3}}{6}.$$

Les conditions qui précèdent nous permettent, en désignant par

$$x=E\left(\frac{-3+\sqrt{12p-3}}{6}\right)$$

le p!us grand entier contenu dans

$$\frac{-3+\sqrt{12p-3}}{6},$$

d'énoncer le théorème suivant:

Théorème I —Soit un polynome

$$x+x+p,$$

p positif, et soit:

$$x_1=E\left(\frac{-3+\sqrt{12p-3}}{6}\right);$$

pour toute valeur de $x>x_1$, ou bien $f(x_0)$ sera un nombre premier, ou bien il contiendra un facteur prem'e · a qu aura déja paru dans une décomposition de $f(x)$ en nombres premiers pour une valeur de $x<x_0$.

Remarque I.— Nous avions écarté hypothèse $a-1-x_0=x_0$. Dans ce cas, $x_0=\frac{a-1}{2}$,

$$f(x_0)=\frac{a^2-1}{4}+p=\frac{4p-1+a^2}{4}.$$

Puisque a divise $f(x_0)$ il divise $4p-1$ c es -à-dire Δ ce qu'on aurait pu prévoir, car dans ce cas la congruence a une racine double

Donc $f(x_0)=\frac{a(a+b)}{4}$, si $\Delta=4p-1=ab$ On voit en passant que $\frac{a+b}{4}$ et ses facteurs premiers font partie du tableau des diviseurs que nous formons.

Si $\Delta=4p-1$ l'un des facteurs a est de la forme $4p'+1$, l'autre de la forme $4q'-1$ et

$$\frac{a+b}{4}=p'+q'$$

est toujours entier.

Donc si b, comme a, est premier, l'hypothèse $\Delta=ab$ entraine la présence dans notre tableau de a, de b, de $\frac{a+b}{4}$ et de ses facteurs premiers; ceci résultait d'ailleurs de l'identité

$$\left(\frac{a-b}{2}\right)^2+\Delta=\left(\frac{a+b}{2}\right)^2.$$

Si Δ est décomposable en 3 facteurs a, b, c, il sera nécessairement de la forme

$$(4p'-1)\ (4q'+1)\ (4r'+1),$$

et nous trouverons dans notre tableau de facteurs premiers les diviseurs premiers de

$$p'+4q'r'+q'+r',$$
$$r'+4p'q'+p'-q',$$
$$q'+4q'r'+p'-r',$$

et ainsi de suite.

5. Supposons que $f(x_0)$ contienne le facteur premier a, x_0 étant supérieur à x_1; alors a aura déjà été obtenu antérieurement: on aura

$$f(x_0)=ab, \quad b=\frac{f(x_0)}{a}$$

avec l'hypothèse: $a<\sqrt{f(x_0)}$, $b>\sqrt{f(x_0)}$.

Si b n'est pas premier et admet un facteur premier c, c est plus grand que x_0, sans quoi il aurait déjà été trouvé. Supposons $c>x_0$; nous pouvons supposer $c<\sqrt{b}$ et par suite plus petit que $f(x_0)$, et nous pourrons répéter un raisonnement qui précède.

En somme nous avons le droit d'énoncer le théorème suivant.

Théorème II.—Pour $x_0 > x_1$, x_1 étant défini comme nous l'avons fait précédemment, $f(x_0)$ admet au plus un facteur premier nouveau [c'est-à-dire un seul facteur premier n'appartenant à aucun des nombres $f(x_i)$, $x_i < x_0$] et ce facteur premier nouveau est $c < \sqrt{f(x_0)}$.

Exemple 1. Appliquons ce qui précède à un polynome connu d'Euler,

$$x^2+x+41.$$

Ici

$$x_1 = E\left(\frac{-3+\sqrt{12\times 41-3}}{6}\right) = 3,$$

$$f(0)=41, f(1)=43, f(2)=47, f(3)=53.$$

Ces quatre nombres sont premiers pour $x > 3$. Ou bien $f(x)$ sera premier ou il admettra comme facteur premier un nombre obtenu précédemment; donc pour $f(x) < 41^2$, $f(x)$ est premier, c'est-à-dire que les 39 premiers nombres de la suite sont premiers.

J'en ai donné une raison élémentaire dans une communication à la Société Mathématique de France en 1911.

Les théorèmes que nous venons de démontrer pour x^2+x+p peuvent être démontrés de même pour

$$2x^2+2x+p$$

où

$$D=4(1-2p),$$

le nombre limite que je désignerai ici par x_2 étant déterminé par

$$2x^2+2x+p < (2x+1)^2,\ 2x^2+2x+1-p > 0,$$

$$x_2 = E\left(\frac{-1+\sqrt{2p-1}}{2}\right).$$

Exemple 2. Soit $2x^2+2x+19$,

$$D=-4\times 37,$$

$$x_2 = E\left(\frac{-1+\sqrt{37}}{2}\right) = 2,$$

$$f(0)=19, f(1)=23, f(2)=31.$$

Tous les nombres de la suite seront premiers tant que $2x^2+2x+19$ ne sera pas divisible par 19, 23, 31, . . . ; le premier nombre qui n'est pas premier est

$$2.18^2+2.18+19=19\times 37.$$

Ces théorèmes s'appliquent encore à

$$3x^2+3x+p,\quad D=-12p+9,$$

avec pour nombre limite x_3 déterminé par

$$3x^2+3x+p < (2x+1)^2,\ x^2+x+p+1 > 0,$$

$$x_3 = E\left(\frac{-1+\sqrt{4p-3}}{2}\right).$$

Exemple 3. Soit $3x^2+3x+11$ qui provient de X^2+X+31 dans lequel je pose $X=3x+1$; il vient

$$3(3x^2+3x+11),$$

$$x_3=E\left(\frac{-1+\sqrt{41}}{2}\right)=2.$$

$$f(0)=11,\ f(1)=17,\ f(2)=29$$

sont premiers; les autres, jusqu' à $f(10)$, seront premiers aussi,

$$f(10)=11\times 41.$$

Pour $4x^2+4x+p$, notre raisonnement ne vaut plus;

$$4x^2+4x+p<4x^2+4x+1$$

donnerait

$$p<1.$$

Enfin, pour x^2+p, on doit avoir

$$\sqrt{x_0^2+p}-x_0<x_0,$$

c'est-à-dire que le nombre limite ici est

$$x_4>E\left(\sqrt{\frac{p}{3}}\right).$$

Pour $2x^2+p$ le nombre limite est

$$x_5=E\left(\sqrt{\frac{p}{2}}\right).$$

Pour $3x^2+p$,

$$x_6=E(\sqrt{p}).$$

Exemple 4. Soit $2x^2+29$, polynome connu,

$$x_5=E\left(\sqrt{\frac{29}{2}}\right)=3,$$

$$f(0)=29,\ f(1)=31,\ f(2)=37,\ f(3)=43.$$

Nous aurons donc une suite de 28 nombres premiers.

6. Considérons le cas de $D<0$. Ce cas se subdivise encore en trois autres:

(α) $\Delta=4p+1$,
(β) $\Delta=2p$,
(γ) $\Delta=4p+3$.

Le polynome $x^2+\Delta$ se raménera comme précédemment aux formes

(α) x^2+x-p,
(β) $2x^2-p$,
(γ) $2x^2+2x-p$.

La démonstration du théorème sur le nombre limite x_1 prend ici une forme un peu différente, parce que tout d'abord les nombres obtenus sont négatifs;

au point de vue de la divisibilité par des nombres premiers, on peut les remplacer par leurs valeurs arithmétiques, de sorte que nous écrirons pour le premier

$$\sqrt{p-x_0-x_0^2}-1-x_0<x_0,\quad 0<5x_0^2+5x_0-p+1,$$

et la limite

$$x_1'=E\left(\frac{-5+\sqrt{20p+5}}{10}\right).$$

Si x_2' désigne l'entier contenu dans la racine de

$$x^2+x-p=0,$$

$$x_2'=E\left(\frac{-1+\sqrt{4p-1}}{2}\right).$$

Le théorème I est valable pour toutes les valeurs de x:

$$x_1'<x\leqq x_2'.$$

Que se passera-t-il ensuite pour $x_0>x_2'$? Ici x_0^2+x-p est positif et en reprenant le raisonnement du début nous devrons avoir:

$$x_0^2+x_0-p<4x_0^2+4x_0+1,$$

condition toujours satisfaite. Donc le théorème s'applique pour toutes les valeurs de x supérieures à x_1'.

Il existe de même des limites pour:

$$2x^2-p,\ 2x^2+2x-p,$$

$$3x^2-p,\ 3x^2+3x-p.$$

Le lecteur voudra bien établir lui-même ces limites.

Ici nous pourrons considérer aussi

$$4x^2+4x-p.$$

Nous devrons avoir tout d'abord:

$$\sqrt{p-4x_0-4x_0^2}<2x_0+1,\ 8x_0^2+8x_0-p+1>0,$$

$$x_0>\frac{-4+\sqrt{8p+8}}{8},$$

la limite

$$x_4'=E\left(\frac{-2+\sqrt{2(p+1)}}{4}\right).$$

Quant à la condition

$$4x_0^2+4x_0-p<4x_0^2+4x_0+1,$$

elle est toujours satisfaite.

Le raisonnement ne s'applique plus pour $5x^2+5x-p$.

Exemple 1. Soit x^2+x-73.

On a

$$x_1'=E\left(\frac{-1+\sqrt{1465}}{10}\right)=3,$$

$$f(0)=-73, f(1)=-71, f(2)=-67, f(3)=-61;$$

$f(4), f(5), f(6), f(7)$ seront premiers; en effet, on trouve pour leurs valeurs -53, -43, -31, -17;

$$f(8)=-1, f(9)=+17.$$

Les nombres que nous retrouverons ensuite seront premiers jusqu'à

$$f(17+7)=f(24)=17\times 31=f(31-1-6).$$

Exemple 2. Soit $2x^2-199$.

Un calcul simple montre que le nombre limite est 5.

$$f(0)=-199, f(1)=-197, f(2)=-191, f(3)=-181, f(4)=-167, f(5)=-149.$$

Tous les nombres qui suivent jusqu'à $f(10)$ sont premiers.

$$f(9)=-37, f(11)=43.$$

Le premier nombre non premier sera:

$$f(37-9)=f(28)=37^2 \text{ et le second } f(43-11)=f(32)=43^2.$$

Exemple 3. De même $3x^2+3x-107$ donne comme nombre limite $x_1'=3$:

$$f(0)=-107, f(1)=-101, f(2)=-89, f(3)=-71.$$

Les nombres suivants jusqu' à $f(5)=-17, f(6)=+19$, seront premiers. On en déduit que les nombres $f(17-5-1), f(19-6-1)$ ne sont pas premiers.

On trouve:

$$f(11)=17^2, f(12)=19^2.$$

Les nombres suivants jusqu'à $f(17+5)=f(22)=17\times 83$ sont premiers.

On trouve un résultat semblable pour

$$3x^2+3x-199.$$

Exemple 4. Soit $4x^2+4x-67$.

Le nombre limite est 2;

$$f(0)=-67, f(1)=-59, f(2)=-43.$$

Les nombres suivants sont premiers:

$$f(3)=-19, f(4)=13, f(5)=53, f(6)=101, f(7)=157, f(8)=f(13-4-1)=13\times 17.$$

Les nombres suivants seront premiers jusqu' à

$$f(13+4)=f(17).$$

II

Des cribles

Les théorèmes démontrés au sujet des polynomes de la forme x^2+x+p, $2x^2+p, \ldots$ nous donneront des cribles qui permettront de trouver assez rapidement de grands nombres premiers.

Prenons par exemple $4x^2+4x-457$. La limite x_1 est ici

$$E\left(\frac{(-1+\sqrt{457}}{2}\right)=11.$$

Formons donc:

$$f(0)=-457, f(1)=-449, f(2)=-433, f(3)=-409,$$
$$f(4)=-377=-13\times 29, f(5)=-337, f(6)=-289=-17^2,$$
$$f(7)=-233, f(8)=-169=-13^2, f(9)=-97,$$
$$f(10)=-17, f(11)=+71.$$

A partir de maintenant, les nombres rencontrés, ou seront premiers, ou seront divisibles par un nombre premier déjà trouvé; $f(x_0)$ sera divisible par 13 pour $x_0=13k+4$ ou $13k+8$, par 17 pour $x_0=17k+6$ ou $17k+10$, par 29 pour $x_0=29k+4$ ou $29k+24$.

Ces remarques nous permettent de faire un crible analogue à celui d'Erathostène.

Supposons que l'on forme $f(x_0)$, $x_0=0, 1, 2, \ldots, N$.

Nous barrerons d'abord les nombres de la forme:

$$457k,\ 457k-1,\ 449k+2,\ 449k-3,\ \ldots$$
$$13k+4,\ 13k+8,\ 17k+6,\ 17k+10,\ \ldots$$

Pour ces valeurs-là $f(x_0)$ n'est pas premier, nous formerons néanmoins $f(x_0)$ dont nous connaîtrons un diviseur premier ou peut-être plusieurs; si ce nombre nous donne un nombre premier nouveau p, nous trouverons les nombres de la forme $lp+x_0$ et nous recommencerons.

Nous finirons par avoir toutes les valeurs de $x<N$ pour lesquelles $f(x)$ est un nombre premier.

Je n'insiste pas sur ce procédé de calcul très simple que le lecteur développera facilement. Je veux montrer plutôt les formes algébriques des diviseurs que l'on obtient.

D'abord, rappelons que tous les diviseurs du tableau relatif à $f(x)$ appartiennent à une forme de déterminant D homogène et du second degré. Ce qui va suivre nous permettra de conclure qu' un nombre quelconque représentable par une forme

$$n=ax^2+bxy+cy^2$$

se présentera dans le tableau des diviseurs de $f(x)$. On pourra de plus prévoir comment et pour quelle valeur de x ce diviseur n se présentera dans notre tableau.

Je vais esquisser sur un exemple de quelle façon harmonieuse se répartissent les diviseurs.

Pour ne pas trop allonger ma communication, je ne ferai pas tous les calculs se rattachant à cette question. J'espère m'expliquer suffisamment pour que le lecteur puisse de lui-même compléter ce chapitre.

Prenons un exemple: $f(x) \equiv x^2+x+41$.

Supposons que ${x_0}^2+x_0+41=N$; $f(Nk+x_0)$, $f(Nk-1-x_0)$ seront divisibles par N:

$$f(x_0+Nk)=f(x_0)+kNf'_{x_0}+\frac{k^2}{1.2}N^2f''_{x_0} \doteq N(1+kf'_{x_0}+k^2N).$$

Nous pouvons choisir k arbitrairement et aussi x_0. Que nous fassions varier k ou que nous fassions varier x_0, nous aurons toujours un diviseur de notre tableau. Donc le polynome que je désigne par

$$f_k^1(x)=1+kf'(x)+\frac{k^2}{1}f(x)=(kx+1)^2+k(kx+1)+41k^2$$

admettra comme déterminant -163, qu'on le considère comme un polynome en k ou comme un polynome en x. De plus, nous voyons qu'il est de la forme:

$$x^2+xy+41y^2.$$

On voit de même que:

$$f(kN-1-x_0)=N[(kx_0-1)^2+k(kx_0-1)+41k^2].$$

Je désignerai par f_k^{-1} le polynome $(kx_0-1)^2+k(kx_0-1)+41k^2$.

Les nombrespremiers de la suite X^2+X+41 sont tout d'abord ceux qui correspondent aux valeurs de X de la forme:

$$X=k(x^2+x+41)+x,$$
$$X=k(x^2+x+41)-1-x.$$

Nous considérerons ensuite ceux qui correspondent à:

$$X_1=lf_k^1(x)+k(x^2+x+41)+x,$$
$$X_2=lf_k^1(x)-1-k(x^2+x+41)-x,$$
$$X_3=lf_k^{-1}+k(x^2+x+41)-1-x,$$
$$X_4=lf_k^{-1}-1-k(x^2+x+41)+x+1,$$
$$f(X_1)=f_k^1(x)f_{k,l}^{1,1}(x),$$
$$f(X_2)=f_k^1(x)f_{k,l}^{1,-1}(x),$$
$$f(X_3)=f_k^{-1}(x)f_{k,l}^{-1,1}(x),$$
$$f(X_4)=f_k^{-1}(x)f_{k,l}^{-1,-1}(x).$$

Telles sont les notations que nous emploierons: $f_{k,l}^{1,1}$, par exemple, n'est autre que

$$[(lk+1)x+l]^2+(lk+1)[(lk+1)x+l]+41(lk+1)^2.$$

Dans chacun de ces polynomes du second degré par rapport à k, l, x, je puis prendre pour variable soit k, soit l, soit x. Dans chaque cas le déterminant sera -163.

On peut continuer ainsi aussi longtemps que l'on voudra et former un polynome du second degré par rapport à autant de variables que l'on voudra, le déterminant du polynome du second degré obtenu en considérant une seule lettre comme variable et toutes les autres comme données sera toujours -163.

Remarquons que $f_{l,k}^{1,1}$ n'est pas égal à $f_{k,l}^{1,1}$; $f_{l,k}^{1,1}$ est égal à

$$[(lk+1)x+k]^2+(lk+1)\ (lx+1+k)+41(lk+1)^2.$$

Remarque.—Faisons $k=1$;

$$f_1^1(x)=f(x+1),\ f_1^{-1}\ (x)=f(x-1),$$

c'est-à-dire que dans la suite X^2+X+41 les premiers nombres non premiers sont:

$$41^2,\ 41\times 43,\ 43\times 47,\ 47\times 53,\ \dots$$

Faisons $k=2$;

$$f_2^1(x)=4x^2+163,\ f_2^{-1}(x)=4(x-1)^2+163.$$

Pour $k=3$

$$f_3^1(x)=9x^2+15x+373,\ f_3^{-1}(x)=9x^2+3x+367.$$

Les nombres représentés par ces polynomes, inférieurs à 41^2, sont premiers.

Puis à leur tour $f_2^1(x)$, $f_3^1(x)$, ..., peuvent ne pas être premiers et nous pourrons chercher à les décomposer en facteurs premiers: $f_1^1(x)$ admettra d'abord des diviseurs de la forme y^2+y+41 et de la forme $4x^2+163$.

Pour quelles valeurs de x a-t-on:

$$4x^2+163\equiv 0 \mod (y^2+y+41).$$

Il suffit de remarquer que

$$-163=(2y+1)^2-4(y^2+y+41)$$

pour voir que

$$4x^2+163\equiv 0 \mod (y^2+y+41)$$

est équivalent à

$$4x^2\equiv(2y+1)^2 \mod (y^2+y+41).$$

Les solutions sont donc:

$$x=\frac{k(y^2+y+41)\pm(2y+1)}{2};$$

pour $k=1$

$$x=\frac{y^2-y+40}{2},\qquad x=\frac{y^2+3y+42}{2}.$$

Cela nous montre que tout d'abord

$$f_2^1(x)=4x^2+163$$

est un nombre premier pour $x=0, 1, 2, \dots, 19$, ce que nous savions déjà. On a $f(20)=41\times 43$, $f(21)=41\times 47$ et ainsi de suite. Puis en faisant $k=2$ on trouve le quotient:

$$f_{2,2}^{1,1}=(4x+1)^2+4\times 163=(4x-1)^2+4(4x-1)+4^2\times 41=f_4^{-1}(x).$$

Il est toujours de la forme $x^2+xy+41y^2$. Je n'insiste pas: en continuant on verrait que tous les diviseurs sont de cette forme et on verrait aussi pour quelles valeurs de X un nombre de a forme

$$x^2+xy+41y^2$$

entre comme diviseur dans X^2+X+41.

On voit qu'il suffit de mettre y sous la forme $(ln\pm1)k$, $(ln\pm2)k$.

J'ai choisi le déterminant -163 qui n'admet qu'une classe de formes et l'on voit qu'il est facile de trouver la place d'un diviseur d'un diviseur dans le tableau, c'est-à-dire de trouver pour quelles valeurs de x

$$f(x)\equiv0 \bmod X^2+XY+41Y^2,$$

en même temps que l'on saura alors trouver les autres facteurs.

Si l'on avait pris un déterminant quelconque D qui admette plusieurs classes de formes, on verrait que tous les diviseurs peuvent s'écrire

$$aX^2+bXY+cY^2$$

avec $4ac-b^2=D$, et réciproquement tout nombre

$$n=aX^2+bXY+cY^2$$

se trouve dans le tableau, car il existe une forme adjointe telle que

$$(aX^2+bXY+cY^2)\ (a'X'^2+b'X'Y'+c'Y'^2)=\xi^2+\xi\eta+p\eta^2$$

et le calcul précédent laisse voir que ce nombre se trouve dans le tableau des diviseurs.

Tout ce que je viens de dire s'applique à un déterminant quelconque positif ou négatif.

J'ai abrégé pour arriver plus vite à un chapitre que je crois plus curieux.

III

1. Je voudrais montrer maintenant que l'étude du premier chapitre nous permet de trouver rapidement les idéaux d'un corps du second degré.

Considérons un corps du second degré et supposons par exemple que le déterminant de ce corps soit $D=-4p+1$; alors une forme principale du corps sera

$$x^2+xy+py^2.$$

Soit alors q un nombre premier qui peut se mettre sous cette forme:

$$q=X^2+XY+pY^2.$$

Si l'on désigne par ω et ω' les racines de

$$x^2+x+p=0,$$

on pourra écrire:

$$q=(X-Y\omega)\ (X-Y\omega').$$

$X-Y\omega$, $X-Y\omega'$ sont des entiers du corps $K(\sqrt{-4p+1})$ et nous dirons que q se décompose dans le corps en un produit de deux nombres entiers algébriques.

Si q n'est pas représentable par la forme principale, il n'est pas le produit de deux nombres entiers algebriques; nous le considérerons comme le produit de deux nombres idéaux conjugués

$$q=\mathfrak{q}\times\mathfrak{q}'.$$

$\mathfrak{q}$ et $\mathfrak{q}'$ sont dits deux idéaux premiers dont q est la norme.

Nous savons que le nombre des classes d'idéaux d'un corps est limité et que dans chaque classe il y a un représentant dont la norme est inférieure à $\sqrt{|D|}$ (Théorème de Minkowski).

Ceci posé nous avons démontré que pour $x>x_1$, x, étant le nombre limite relatif au polynome $f(x)$, nous avons au plus un diviseur nouveau. Si donc nous formons les $f(x)$ pour $x=1, 2, \ldots, x_1$ et si nous décomposons ces nombres en facteurs premiers nous trouverons parmi ces nombres des nombres qui se décomposent en deux nombres premiers du corps et qui appartiennent à la classe principale et peut-être des nombres qui ne se décomposent pas: nous déterminerons les classes d'idéaux auxquels ils appartiennent.

Supposons que nous ayons

$$f(x_0)=abcd,$$

où $x_0>x_1$ nombre limite et où d est un diviseur nouveau, a, b, c étant des diviseurs déjà obtenus et pour lesquels nous avons déterminé la classe d'idéaux auxquels ils appartiennent. On a $(x_0-\omega)\ (x_0-\omega')=\mathfrak{a}\mathfrak{a}'.\mathfrak{b}\mathfrak{b}'.\mathfrak{c}\mathfrak{c}'.\mathfrak{d}\mathfrak{d}'$, $(\mathfrak{a}\mathfrak{b}\mathfrak{c})\mathfrak{d}=x_0-\omega$.
La classe de l'idéal $\mathfrak{J}=(\mathfrak{a}\mathfrak{b}\mathfrak{c})$ est déjà connue et comme on a $\mathfrak{J}\mathfrak{d}\sim 1$, la classe de $\mathfrak{d}$ est connue: c'est la classe réciproque de $\mathfrak{J}$.

Remarque 1.—Nous savons que $d>\sqrt{x_0^2+x_0+p}$; nous serons donc sûr d'avoir un représentant de chaque classe lorsque nous aurons poussé la décomposition jusqu' à

$$\sqrt{x_0^2+x_0+p} \ > \ \sqrt{|D|}.$$

Remarque 2.—Un raisonnement analogue donnerait des résultats identiques pour la classe principale quelle que soit la forme de D.

Exemple 1. Cette méthode s'applique d'une façon particulièrement heureuse pour les corps qui n'ont qu'une classe d'idéaux. Il en est pour lesquels on le reconnait immédiatement Soient

$$x^2+x+11,\ K\sqrt{-43};\ x^2+x+17,\ K\sqrt{-67};\ x^2+x+41,\ K\sqrt{-163};\ldots.$$

Chacun de ces polynomes donne tout d'abord une suite de nombres premiers qui se décomposent dans le corps considéré en deux nombres premiers algébriques Puis, lorsqu'on aura un diviseur premier nouveau, il sera la norme d'un nombre appartenant à la classe réciproque de la classe principale, il appartiendra à la classe principale.

Donc les corps $K(\sqrt{-43}$, $K(\sqrt{-67})$, $K(-\sqrt{-163})$ n'ont qu'une seule classe d'idéaux: la classe principale.

Exemple 2. Considérons un autre cas simple $K(\sqrt{-58})$ auquel correspond

$$f(x)=x^2+58.$$

Le nombre 2 qui figure comme facteur ne se décompose pas:

$$2=\mathfrak{p}\mathfrak{p}',\ \mathfrak{p}'=\mathfrak{p},\ 2=\mathfrak{p}^2.$$

Si l'on donne à x des valeurs paires et des valeurs impaires, x^2+58 nous donne:

$$f_1(x)=(2x+\omega)\ (2x+\omega')=2(2x^2+29),$$
$$f_2(x)=(2x+1+\omega)\ (2x+1+\omega')=4x^2+4x+59.$$

Tous les nombres premiers de la forme:

$$f_1(x)=2x^2+29, f_1(0)=29, f_1(1)=31, f_1(2)=37\ldots$$

appartiennent à la classe réciproque de $\mathfrak{p}$, c'est-à-dire à la classe de $\mathfrak{p}$. Tous les nombres $f_2(0)=59, f_2(1)=67, f_3(1)=83, \ldots$, jusqu' à un certain rang appartiennent à la classe principale. On voit donc facilement que $K\sqrt{-58}$ n'a que deux classes d'idéaux: la classe principale et la classe $\mathfrak{p}=(2, \sqrt{-58})$.

Remarque. Une petite propriété curieuse de ce polynome connu de Bradlow me parait digne d'être signalée. Les 28 premiers nombres donnés par $2x^2+29$ sont premiers. Or, on a

$$f_1(x)+f_1(x+1)-1=f_2(x).$$

Or les 14 premiers nombres de la suite $f_2(x)$ sont premiers; les nombres 29, 31, 37, ... ont donc jusqu' à un certain rang la propriété suivante: la somme de deux nombres consécutifs diminuée d'une unité donne un nombre premier.

Exemple 3. $K(\sqrt{-91})$, $91=7\times13$.

Nous considérerons $f(x)=x^2+x+23$.

$$f(0)=23, f(1)=5^2, f(2)=29, f(3)=5\times7, f(4)=43,$$
$$f(5)=53, f(6)=5\times13, f(7)=79, f(8)=95=5\times19,\ \ldots$$

5, 7, 13 ne se décomposent pas, 7 est un diviseur du discriminant, donc $7=\mathfrak{p}_7^2$.

On a

$$5=\mathfrak{p}_5\mathfrak{p}_5'$$

mais, comme $f(3)=7\times5$, $\mathfrak{p}_5$ appartient à la classe réciproque de $\mathfrak{p}_7$ c'est-à-dire à $\mathfrak{p}_7$. $f(6)=65=5\times13$; $\mathfrak{p}_{13}$ appartient à la classe réciproque de $\mathfrak{p}_5$ et par suite à $\mathfrak{p}_7$.

Le corps $K(\sqrt{-91})$ n'a donc que deux classes: (1) et $(7, \sqrt{-91})$.

Exemple 4. Choisissons enfin un exemple moins simple:

$$K(\sqrt{-83}), f(x)=x^2+x+22,$$
$$f(0=22, f(1)=2^3\times3, f(2)=2^2\times7, f(3)=2\times17,$$
$$f(4)=2.3.7, f(5)=2^2\times13, f(6)=2^6,\ \ldots$$

2, 3, 7, 13, 17 ne se décomposent pas; nous avons donc:

$$2=\mathfrak{p}_2\mathfrak{p}_2'=(2,\ \omega)\ (2,\ \omega'),$$
$$3=\mathfrak{p}_3\mathfrak{p}_3'=(37+\omega)\ (37+\omega'),$$
$$7=\mathfrak{p}_7\mathfrak{p}_7'=(7,\ 2+\omega)\ (7,\ 2+\omega'),$$
$$f(6)=2^6=(6+\omega)\ (6+\omega')\sim\mathfrak{p}_2^6\mathfrak{p}_2'^6,\ \mathfrak{p}_2'^6\sim 1.$$

On a donc, si $\mathfrak{p}_2=\mathfrak{J}$, $\mathfrak{p}_2'=\mathfrak{J}^5$, d'après $f(1)$,

$$\mathfrak{p}_2^3\mathfrak{p}_3\sim 1,\ \mathfrak{p}_2^2\mathfrak{p}_3=\mathfrak{J}^6$$

donc

$$\mathfrak{p}_3=\mathfrak{J}^4,\ \mathfrak{p}_3'=\mathfrak{J}^2;$$

d'après $f(2)$

$$\mathfrak{p}_2^2\mathfrak{p}_7^2\sim 1$$

donc

$$\mathfrak{p}_7=\mathfrak{J}^4,\ \mathfrak{p}_7'=\mathfrak{J}^2.$$

Le corps a donc six classes d'idéaux:

$$(2,\ \omega)\ =\mathfrak{J},\ (7,\ 2+\omega)\ =\mathfrak{J}^2,\ (3,\ 1+\omega)\ =\mathfrak{J}^3,$$
$$(2,\ \omega')=\mathfrak{J}^5,\ (7,\ 2+\omega')=\mathfrak{J}^4,\ (3,\ 1+\omega')=\mathfrak{J}^3,$$
$$\mathfrak{J}^6\sim 1.$$

2. Appliquons cette méthode aux corps pour lesquels $D>0$. Là encore, lorsque le corps n'a qu'une classe, le résultat apparaît immédiatement.

Exemple 1. $K(\sqrt{437})$, $437=19\times 23$.

Nous considérerons $f(x)=x^2+x-109$.

Tous les nombres jusqu'à $f(13)$ sont premiers. On trouve de plus que $f(9)=-19$, $f(11)=+23$.

Nous avons une première série de nombres premiers qui se décomposent en un produit de nombres premiers algébriques; les suivants se décomposeront également.

Le corps n'a qu'une classe d'idéaux.

Exemple 2. $K(\sqrt{89})$.

Nous considérerons

$$f(x)=x^2+x-22.$$

$f(0)=-22, f(1)=-20, f(2)=-16, f(3)=-10, f(4)=-2, f(5)=+8, f(6)=+20.$

En désignant la norme par le symbole N nous avons:

$$N(4-\omega)=(4-\omega)\ (4-\omega')=2,$$
$$N(3-\omega)\ (4-\omega)=20,$$
$$N(34-8\omega)=20,$$
$$N(17-4\omega)=5;$$

2 appartient à la classe principale, 5 aussi; le corps n'a qu'une classe d'idéaux. *Profitons de notre tableau pour calculer une unité du corps.*

$$N(4-\omega)\ (5-\omega)=-16,$$
$$N(42-10\omega)=-16,$$
$$N(21-5\omega)=-4,$$
$$N(4-\omega)^2=N(38-9\omega)=+4,$$
$$N(21-5\omega)\ (38-9\omega)=-16,$$
$$N(1788-424\omega)=-16,$$
$$N(447-106\omega)=-1.$$

Le nombre $\epsilon=447-106\omega$ est alors une unité du corps.

Exemple 3. $K(\sqrt{398})$.

Il se traite comme le cas $K(\sqrt{-58})$. On trouve deux classes: la classe principale et la classe $\mathfrak{p}$ où $\mathfrak{p}^2=2$.

Les polynomes $f_1(x)=2x^2-199, f_2(x)=4x^2+4x-397$, ont des propriétés analogues à celles de $f_1(x)=2x^2+29$, $f_2(x)=4x^2+4x+59$.

Exemple 4. $K(\sqrt{1637})$; 1637 est premier.

Nous considérerons $f(x)=x^2+x-409$. Les nombres 1637 et 409 sont premiers. Nous formons $f(0)$ jusqu' à $f(20)$ et nous trouvons:

$$\mathrm{T}\left|\begin{array}{l}-407=-11\times37,\ -403=-13\times31,\ -397,\ -389,\ -379,\ -367,\\ -353,\ -337,\ -319=-11\times29, -299=-13\times23,\ -277,\ -253=-11\times23,\\ -227,\ -199,\ -169=-13^2,\ -137,\ -103,\ -67,\ -29,\ +11,\end{array}\right.$$

$$(20-\omega)\ (20-\omega')=11.$$

Le nombre 11 appartient à la classe principale, par suite 23, puisque $f(10)=-13\times23$, par suite 13, par suite tous les nombres premiers de ce tableau, ainsi que tous ceux que l'on trouvera par la suite. Le corps n'a donc qu'une classe d'idéaux

Remarque.—Le tableau T permet de décomposer rapidement les nombres premiers rationnels en idéaux premiers, ou encore de trouver les valeurs de x et y qui permettent de représenter 13, 23,. . par la forme $x^2+xy-409y^2$.

En effet:

$$N[(20-\omega)\ (12-\omega)]=-11^2\times23,$$
$$N[649-33\omega]=-11^2\times23,$$
$$N(59-3\omega)=-23, -23=59^2+59.3-409\times3^2.$$

On trouve de même:

$$N(79-4\omega)=13,$$
$$N(38-7\omega)=-31,$$
$$N(39-2\omega)=-37.$$

Cherchons enfin une unité du corps:

$$N(79-4\omega)^2=13^2,\ N(15-\omega)=-13^2,$$
$$N(12785-648\omega)\ (15-\omega)=-13^4,$$
$$N(456.807-23153\omega)=-13^4,$$
$$N[13^2(2703-137\omega)]=-13^4,$$
$$N(2703-137\omega)=-1.$$

A FOUNDATION FOR THE THEORY OF IDEALS

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I

The theory developed in this paper was presented in outline at the Australian meeting of the British Association for the Advancement of Science in 1914. During the two following years details were filled in and modifications introduced here and there. The text had been typed and most of the formulae transcribed from the written manuscript to the typewritten copy when, early in 1917, the work was definitely laid aside for activities which seemed nearer the needs of the time. Throughout the intervening years up to the presentation of the paper to the Mathematical Congress the author failed to find the small amount of time which would have sufficed to complete the transcription of the formulae and make some changes in their lettering. These final touches have only been given to the copy at the last moment before handing it over to the printer and doubtless further changes in the terminology could be made with advantage*.

The object of the paper is to lay a foundation for the theory of the ideals in particular and for the theory of the algebraic numbers in general on lines parallel to those on which the writer has developed the theory of the algebraic functions of one variable. With this object in view, it will be convenient to make use of Hensel's conception of the rational p-adic numbers. Any rational number is said to be divisible or not divisible by a prime p according as the numerator of the number in its reduced form is or is not divisible by p.

Relatively to a prime p any rational number, positive or negative, is characterized by, or can be uniquely represented by a series of the form

$$(1)\qquad c_{-k}p^{-k}+c_{-k+1}p^{-k+1}+\ldots+c_{-1}p^{-1}+c_0+c_1p+\ldots+c_rp^r+\ldots$$

where the coefficients c_s are all integral and included among the numbers $0, 1, \ldots, p-1$. The series of the above type constitute the rational p-adic numbers in their reduced form. These numbers may also be represented by series in powers of p with integral coefficients not all of which are included among the p integers just mentioned, in which case the numbers are said to be unreduced in form. While a rational p-adic number corresponds to each

*The word *foundation* has been substituted for the word *basis* in the title of the paper as presented both at the Australian meeting of the B.A.A.S. and at the Mathematical Congress of 1924, because of the technical sense in which the latter word is otherwise used by the author in the presentation of his theory. An account of the theory under the title "Division in relation to the algebraic numbers" was given in an address delivered to Section III of the Royal Society of Canada at its meeting in Ottawa in May, 1921.

rational number, it does not hold conversely that a rational number corresponds to each rational p-adic number. The lowest exponent in the series (1) for which the corresponding coefficient is different from 0 is called the order number of the p-adic number represented by the series. The operations of addition, subtraction, multiplication and division have been extended to the p-adic numbers by Hensel who has also enlarged the conception of the p-adic numbers so as to include certain series proceeding according to powers of an irrational element and involving irrational coefficients. With the aid of these generalized p-adic numbers he splits the left-hand side of an algebraic equation into linear p-adic factors. This, however, is superfluous for the purposes of this paper and for the development of the theory of the ideals in particular and we shall here make no use of these generalized p-adic numbers.

We shall have to do with polynomials in one unknown, or in an algebraic irrationality, with coefficients which are rational p-adic numbers, it may be ordinary rational numbers. The methods and terminology made use of will be analogous to those employed by the writer in connection with the algebraic functions. The p-adic number (1) is said to be integral with regard to the prime p if it involves no term with negative exponent. Otherwise it is said to be fractional with regard to p. The coefficient c_{-1} of p^{-1} we shall refer to as the *residue* of the number relative to the prime p,—it being understood, of course, that the number is expressed in its reduced p-adic form. The aggregate of the terms involving negative exponents we shall call the *principal part* of the number relative to the prime p. We shall have occasion to speak of *orders of coincidence* or *order numbers* of a rational function of an algebraic irrationality relative to a prime p and we shall find it convenient later on to define the term *adjoint* with reference to a prime.

Let us first consider a polynomial

$$f(x)=x^n-a_{n-1}x^{n-1}+\ldots+(-1)^n a_0 \tag{2}$$

in which the coefficients are ordinary rational numbers. This polynomial may be written as a product of linear factors

$$f(x)=(x-\epsilon_1)\ldots(x-\epsilon_n). \tag{3}$$

The coefficients in (2) are integral symmetric functions of the numbers $\epsilon_1, \ldots, \epsilon_n$.

Take $R(x)$ a polynomial in x with rational numerical coefficients and construct the product

$$F(X)=(X-R(\epsilon_1))\ldots(X-R(\epsilon_n)). \tag{4}$$

This may be written as a polynomial

$$F(X)=X^n-A_{n-1}X^{n-1}+\ldots+(-1)^n A_0 \tag{5}$$

where the coefficients are integral symmetric functions of the numbers $\epsilon_1, \ldots, \epsilon_n$. These coefficients are therefore also integral rational functions of $a_{n-1}, \ldots, a_0$. The polynomial $F(X)$ has then a definite relation to the polynomial $f(x)$ which relation is determined by the polynomial $R(x)$. We may suppose $a_{n-1}, \ldots, a_0$ to be arbitrary. The coefficients in $F(X)$ are definite integral rational functions of $a_{n-1}, \ldots, a_0$ depending for their forms solely on the form of the polynomial $R(x)$.

We might refer to $F(X)$ as the polynomial derived from the polynomial $f(x)$ through the polynomial $R(x)$. More briefly we shall speak of $F(X)$ as the *transform* of $f(x)$ through $R(x)$. If the coefficients in $R(x)$ as well as the coefficients in $f(x)$, other than the first, be arbitrary, it is evident that the coefficients in $F(X)$ will be integral rational functions of these two sets of arbitrary quantities with integral rational numerical coefficients.

In defining the term transform here we have assumed that the coefficient of the highest power of x in $f(x)$ is 1. We may if we will employ the term also where this is not the case meaning by the transform of $f(x)$ however always the same thing as the transform of the quotient of $f(x)$ by the coefficient of its highest term.

It is evident that the transforms of $f(x)$ through the polynomials $R(x)$ and $R_1(x)$ will be precisely the same when we have

(6) $$R(x) \equiv R_1(x) \pmod{f(x)}.$$

That is to say the coefficients in the transform of $f(x)$ through $R_1(x)$ will have precisely the same forms in the coefficients of $f(x)$ and $R(x)$ as the coefficients in the transform of $f(x)$ through $R(x)$ itself and that in the case where the coefficients of $f(x)$ and $R(x)$ are both regarded as arbitrary. Whatever the degree of $R(x)$ may be there will always be one and only one polynomial $R_1(x)$ of degree $n-1$ such that the congruence (6) holds. We may, if we will, suppose $R(x)$ itself to be this polynomial and write

(7) $$R(x) = \alpha_{n-1}x^{n-1} + \ldots + \alpha_0.$$

If the polynomial

(8) $$\bar{F}(X) = X^n - \bar{A}_{n-1}X^{n-1} + \ldots + (-1)^n\bar{A}_0$$

is derived from the polynomial $f(x)$ through the polynomial $\bar{R}(x)$ it is evident that in the polynomial derived from the polynomial $f(x)$ through the polynomial $R(x)\bar{R}(x)$ the last term will be $(-1)^nA_0\bar{A}_0$. We shall find it convenient to call the last term of a transform its norm term. The norm term of the transform of $f(x)$ through a product $R(x)\bar{R}(x)$ is then, to its sign, equal to the product of the norm terms of the transform of $f(x)$ through the polynomials $R(x)$ and $\bar{R}(x)$ respectively.

Let us now suppose $f(x)$ to be represented as the product of a number of factors not necessarily linear

(9) $$f(x) = f_1(x) \ldots f_r(x)$$

where

(10) $$f_s(x) = x^{n_s} - a^{(s)}_{n_s-1}x^{n_s-1} + \ldots + (-1)^{n_s}a^{(s)}_0, \quad (s = 1, 2, \ldots, r).$$

From $f_s(x)$ through $R(x)$ we derive a polynomial

(11) $$F_s(X) = X^{n_s} - A^{(s)}_{n_s-1}X^{n_s-1} + \ldots + (-1)^{n_s}A^{(s)}_0, \quad (s = 1, 2, \ldots, r).$$

Here the coefficients $A^{(s)}_{n_s-1}, \ldots, A^{(s)}_0$ are integral rational functions of the coefficients $a^{(s)}_{n_s-1}, \ldots, a^{(s)}_0$ in $f_s(x)$ whose forms depend solely on the form of $R(x)$.

Furthermore it is plain that the polynomial which is derived from $f(x)$ through the polynomial $R(x)$ is represented by the product

$$F(X)=F_1(X)\dots F_r(X). \tag{12}$$

Where then $f(x)$ represents the product of r polynomials $f_1(x), \dots, f_r(x)$ involving $n_1+\dots+n_r=n$ arbitrary coefficients and where $F_1(X), \dots, F_r(X)$ are the r polynomials derived from $f_1(x), \dots, f_r(x)$ respectively through $R(x)$ the product of the r derived polynomials is identical with the polynomial derived from $f(x)$ through the same polynomial $R(x)$. Otherwise stated, the transform of a product of a number of polynomial factors through a polynomial $R(x)$ is identical with the product of the transforms of the several factors through the same polynomial $R(x)$.

The transform of $f(x)$ through $R(x)$ is, as we have seen, a definite form in the coefficients of these polynomials, supposed to be arbitrary ordinary numbers. This same form we shall now employ to define the transform of $f(x)$ through $R(x)$ when in these polynomials we replace the coefficients, heretofore regarded as arbitrary ordinary numbers, by arbitrary p-adic numbers. Here too, it is evident that the transforms of $f(x)$ through $R(x)$ and $R_1(x)$ are the same when these polynomials satisfy the congruence (6). A polynomial with p-adic coefficients we shall briefly refer to as a p-adic polynomial, a polynomial with integral p-adic coefficients as an integral p-adic polynomial. It is plain that the identity which has just been stated holds also in the case where the polynomials in question are p-adic polynomials. That is to say, the transform of a product of a number of p-adic polynomials through a p-adic polynomial $R(x)$ is identical with the product of the transforms of the several factors through the same polynomial $R(x)$. There is no implication in what precedes that a p-adic polynomial can be represented as a product of linear factors or that it possesses anything in the nature of a root.

Where $F(X)$ and $\overline{F}(X)$ as in (5) and (8) are the transforms of $f(x)$ through polynomials $R(x)$ and $\overline{R}(x)$ respectively, it still holds good, in the case where the polynomials in question are all p-adic, that in the transform of $f(x)$ through the product $R(x)\overline{R}(x)$ the last term is $(-1)^n A_0\overline{A}_0$.

The coefficients of X^{n-1} in $F(X)$ and $\overline{F}(X)$ have the same linear form in terms of the coefficients of $R(x)$ and $\overline{R}(x)$ respectively. It is then immediately apparent that the coefficient of X^{n-1}in the transform of $f(x)$ through the polynomial $R(x)+R(x)$ is equal to the sum of the coefficients of x^{n-1} in $F(X)$ and $\overline{F}(X)$.

Suppose $F(X)$ and $\overline{F}(X)$ to be transforms of $f(x)$ through the polynomials $R(x)$ and $R(x)+p^kP(x)$ respectively where $R(x)$ and $P(x)$ are any two specific p-adic polynomials. The order number of any coefficient in $\overline{F}(X)$ will then evidently be the same as the order number of the corresponding coefficient in $F(X)$, when this order number is finite, if only the exponent k is chosen great enough. This follows immediately from the fact that the coefficients in $F(X)$ are definite integral polynomial forms in terms of the coefficients of $f(x)$ and $R(x)$ while the coefficients in $\overline{F}(X)$ are the same integral polynomial forms in terms of the coefficients of $f(x)$ and $R(x)+p^kP(x)$. For example, if in the coefficients of the p-adic polynomial $R(x)$ we discard terms of sufficiently high power in p we obtain a polynomial $\overline{R}(x)$ with ordinary rational coefficients through which $f(x)$ goes over into a polynomial $\overline{F}(X)$ whose coefficients have the same order numbers as the corresponding coefficients in $F(X)$ so long at least as these order numbers are finite.

Suppose, for a moment, the coefficients of the polynomial $f(x)$ to be p-adically integral and suppose $f(x)\equiv\overline{f}(x)$ $(mod\ p)$. Suppose also that the coefficients of the polynomial $R(x)$ are p-adically integral. The transforms of $f(x)$ and $\overline{f}(x)$ through $R(x)$ we shall designate by $F(X)$ and $\overline{F}(X)$ respectively. The corresponding coefficients in these transforms have the same forms in the coefficients of $f(x)$ and $R(x)$ and of $\overline{f}(x)$ and $R(x)$ respectively and because of the integral character of these forms we have $F(X)\equiv\overline{F}(X)$ $(mod\ p)$. This congruence evidently also holds good when, instead of taking specific p-adically integral numbers for the coefficients of $R(x)$, we regard these coefficients as arbitrary.

It is evident that a p-adic polynomial $f(x)$ is either divisible by an irreducible p-adic polynomial $f_1(x)$ or is relatively prime to it. It is readily seen that an irreducible p-adic polynomial which is a divisor of a product of p-adic polynomials must be a divisor of one of the factors. It follows that the representation of a p-adic polynomial as the product of irreducible p-adic polynomials is unique.*

Let us for a moment consider polynomials with ordinary coefficients. Suppose $\phi(x)$ and $\psi(x)$ to be two such polynomials

$$\phi(x)=b_m x^m+\ldots+b_0,\qquad \psi(x)=c_n x^n+\ldots+c_0. \tag{13}$$

The necessary and sufficient condition that these polynomials may have a common factor is constituted by the existence of polynomials $\phi_1(x)$ and $\psi_1(x)$ of degrees $m-1$ and $n-1$ respectively such that we have identically

$$\phi(x)\psi_1(x)+\psi(x)\phi_1(x)=0. \tag{14}$$

The existence of this identity, on the assumption that the polynomials $\phi_1(x)$ and $\psi_1(x)$ are not themselves identically 0, imposes on the coefficients of $\phi(x)$ and $\psi(x)$ a condition which may be written in determinantal form and which we shall represent by the notation

$$D(b_m,\ \ldots,\ b_0;\ c_n,\ \ldots,\ c_0)=0. \tag{15}$$

The determinant on the left-hand side of this equation is homogeneous in the set of coefficients $b_m, \ldots, b_0$ and in the set of coefficients $c_n, \ldots, c_0$ its degrees in the two sets being n and m respectively. It may be noted that the term $b_m^n c_0^m$ presents itself.

The preceding remarks evidently hold as well in the case where the coefficients of our polynomials are p-adic numbers as in the case where they are ordinary numbers. The equation (15) then gives the necessary and sufficient condition in order that the polynomials $\phi(x)$ and $\psi(x)$ may have a common factor when the coefficients $b_m, \ldots, b_0; c_n, \ldots, c_0$ are p-adic numbers as well as

*See Hensel, *Theorie der algebraischen Zahlen*, p. 65.

when they are ordinary numbers. When the condition (15) does not hold it is evident that we can choose the polynomials $\phi_1(x)$ and $\psi_1(x)$ so that the expression on the left-hand side of (14) is identically equal to an arbitrarily assigned p-adic number.

Returning to the case in which the coefficients of the polynomials are ordinary numbers, the condition that the equations $\phi(x)=0$ and $\psi(x)=0$ may have a root in common and therewith that the polynomials $\phi(x)$ and $\psi(x)$ may have a common factor can be written

(16) $$\psi(e_1)\psi(e_2)\ldots\psi(e_m)=0,$$

where $e_1, \ldots, e_m$ are the roots of the equation $\phi(x)=0$. On expressing the symmetric functions of $e_1, \ldots, e_m$ in terms of the coefficients $b_m, \ldots, b_0$ we evidently have identically

(17) $$D(b_m, \ldots, b_0; c_n, \ldots, c_0)=b_m^n\psi(e_1)\ldots\psi(e_m).$$

Still considering polynomials with ordinary numerical coefficients, let us revert to $f(x)$ in (2) and to $F(X)$ the transform of $f(x)$ through $R(x)$. The condition that a polynomial $F_1(X)$ may have a factor in common with $F(X)$ may be expressed in terms of the coefficients of these polynomials under the form (15). This condition, by (17), is identical with the equation

(18) $$F_1\{R(\epsilon_1)\}\ldots F_1\{R(\epsilon_m)\}=0$$

when the symmetric functions of $R(\epsilon_1), \ldots, R(\epsilon_m)$ have been expresssed in term of the coefficients of $F(X)$.

The equation (18) however can be represented in terms of the coefficients of $F_1(X)$, $f(x)$ and $R(x)$ on expressing the symmetric functions of $\epsilon_1, \ldots, \epsilon_m$ in terms of the coefficients of $f(x)$. By equation (17) the equation (18) so represented expresses the condition that $f(x)$ and $F_1\{R(x)\}$ may have a factor in common, such condition being identical in form with that obtained under (15) in order that $f(x)$ and $F_1\{R(x)\}$ may have a factor in common. The condition given under (15) in order that $f(x)$ and $F_1\{R(x)\}$ may have a factor in common is then identical with the condition given under (15) that $F(X)$ and $F_1(X)$ should have a factor in common, the coefficients of $F(X)$ being supposed to be expressed in terms of the coefficients of $f(x)$ and $R(x)$.

In the case where $\phi(x)$ and $\psi(x)$ have rational p-adic coefficients, however, the condition that they may have a common factor, as given by the relation (15), has just the same form as when the polynomials have ordinary numerical coefficients. When therefore the coefficients of $f(x)$ and $R(x)$ and therewith those of $F(X)$ are rational p-adic numbers as also the coefficients of $F_1(X)$, the condition that $F_1(X)$ may have a factor in common with $F(X)$, as given by the relation (15), has precisely the same form in the coefficients of $f(x)$, $R(x)$ and $F_1(X)$ as the condition given by the relation (15) in order that $F_1\{R(x)\}$ may have a factor in common with $f(x)$. If then $F_1(X)$ has a factor in common with $F(X)$ it must be that $F_1\{R(x)\}$ has a factor in common with $f(x)$ and conversely, and this is true whether the coefficients of the polynomials in question are ordinary numbers or p-adic numbers.

Assume for the moment that $f(x)$ is an irreducible p-adic polynomial. The p-adic polynomial $F(X)$ derived from $f(x)$ through $R(x)$ we shall suppose to be represented as a product of a number of irreducible p-adic polynomials:

$$F(X)=F_1(X)F_2(X)\dots$$

The irreducible p-adic polynomials $F_1(X)$, $F_2(X)$, $\dots$, we can readily prove to be identical with one another. Since $F(X)$ is divisible by $F_1(X)$ we know in fact from what precedes that $F_1\{R(x)\}$ must have a factor in common with $f(x)$ and must therefore be divisible by this irreducible p-adic polynomial. Similarly we know that $F_2\{R(x)\}$ must be divisible by $f(x)$. If now the irreducible p-adic polynomials $F_1(X)$ and $F_2(X)$ were distinct from one another, we could find p-adic polynomials $G_1(X)$ and $G_2(X)$ such that we would have identically

$$G_2(X)F_1(X)+G_1(X)F_2(X)=C$$

where C is a p-adic number other than 0. From this would follow

$$G_2\{R(x)\}F_1\{R(x)\}+G_1\{R(x)\}F_2\{R(x)\}=C.$$

This latter identity, however, involves a contradiction, for $F_1\{R(x)\}$ and $F_2\{R(x)\}$, being divisible by $f(x)$, the left-hand side of the identity would be divisible by this polynomial whereas the right-hand side is not so divisible. It follows that $F_1(X)$ and $F_2(X)$ are not distinct from one another. We conclude that the irreducible factors of $F(X)$ must all be the same and we can therefore write

$$F(X)=\{F_1(X)\}^k. \tag{19}$$

When then $f(x)$ is an irreducible p-adic polynomial, the polynomial $F(X)$ derived from it through a p-adic polynomial $R(x)$ is a power of an irreducible p-adic polynomial; it may be that it is itself an irreducible p-adic polynomial.

Let us now suppose that $f(x)$ is a polynomial with ordinary rational coefficients which is irreducible in the domain of the ordinary rational numbers. It may, however, be reducible in the domain of the rational p-adic numbers. We shall suppose it to be represented as the product of r irreducible p-adic polynomials,

$$f(x)=f_1(x)\dots f_r(x). \tag{20}$$

The irreducible factors $f_1(x)$, $\dots$, $f_r(x)$ must all be distinct from one another, or otherwise the application to $f(x)$ and its derivative $f'(x)$ of the process for finding the greatest common divisor would give a common divisor and therefore also a common divisor with ordinary rational coefficients. This, however, does not accord with our assumption that $f(x)$ is irreducible in the domain of the ordinary rational numbers. From the polynomial $f(x)$ expressed as a product of irreducible p-adic factors let us derive the polynomial $F(X)$ through the polynomial $R(x)$. The coefficients in the polynomial $R(x)$ we shall assume to be ordinary rational numbers. The coefficients in the polynomial $F(X)$ will therefore also be ordinary rational numbers. $F(X)$ may be represented as the product of r p-adic factors,

$$F(X)=F_1(X)\dots F_r(X). \tag{21}$$

Each p-adic factor $F_s(X)$ which here appears is derived through the polynomial $R(x)$ from a corresponding factor $f_s(x)$ of $f(x)$ and each such factor $F_s(X)$ is either itself an irreducible p-adic polynomial or a power of an irreducible p-adic polynomial. If one of the factors $F_s(X)$ then is not an irreducible p-adic polynomial $F(X)$ presents a multiple p-adic factor. In this case the application to $F(X)$ and its derivative $F'(X)$ of the process for finding the greatest common divisor would furnish a common divisor and therefore a common divisor with ordinary rational coefficients. The polynomial $F(X)$ derived from the irreducible* polynomial $f(x)$ through the polynomial $R(x)$ would then not be irreducible and would therefore be a power of an irreducible polynomial in X with ordinary rational coefficients.

Through the polynomial $R(x)$ the polynomial $f(x)$ goes over into $F(X)$ which is itself an irreducible polynomial or in any case a power of an irreducible polynomial. Suppose, what is usually the case, that $F(X)$ is an irreducible polynomial. Let ϵ designate a root of the polynomial $f(x)$. We know that $\bar{\epsilon}=R(\epsilon)$ is a root of the polynomial $F(X)$. We know, further, that we can write $\epsilon=\bar{R}(\bar{\epsilon})$ where $\bar{R}(\bar{\epsilon})$ is a polynomial in $\bar{\epsilon}$ with rational coefficients and of degree $n-1$. Evidently $\epsilon=\bar{R}\{R(\epsilon)\}$ and we therefore have $\bar{R}\{R(x)\}\equiv x \pmod{f(x)}$. We shall find it convenient on occasion to write $\epsilon=R^{-1}(\bar{\epsilon})$. We then have $R^{-1}\{R(x)\}\equiv x(\text{mod } f(x))$. The polynomial $F(X)$ goes over into the polynomial $f(x)$ through the polynomial $R^{-1}(X)$. Through $R^{-1}(X)$ we shall show that each p-adic factor $F_s(X)$ of $F(X)$ goes over into the p-adic factor $f_s(x)$ of $f(x)$ whence it came through $R(x)$.

For the moment let $R(x)$ and $\bar{R}(x)$ be any polynomials with ordinary rational coefficients. Suppose that $f(x)$ goes over into $F(X)$ through $R(x)$ and $F(X)$ into $\bar{F}(\bar{X})$ through $\bar{R}(X)$. Evidently $f(x)$ goes over into $\bar{F}(\bar{X})$ through $\bar{R}\{R(x)\}$. This holds where the coefficients of $f(x)$, $R(x)$ and $\bar{R}(x)$ are any ordinary numbers. This statement then constitutes an identity in the coefficients of these three polynomials. This identity would still hold good if the coefficients in $f(x)$, $R(x)$ and $\bar{R}(x)$ were arbitrary p-adic numbers. If then $f_s(x)$ goes over into $F_s(X)$ through $R(x)$ and $F_s(X)$ into $\bar{F}_s(\bar{X})$ through $\bar{R}(X)$ we see that $f_s(x)$ goes over into $\bar{F}_s(\bar{X})$ through $\bar{R}\{R(x)\}$, the coefficients of the several polynomials here in question being p-adic numbers.

Returning now to the case in which the coefficients in $f(x)$, $R(x)$ and $\bar{R}(x)$ are ordinary rational numbers and taking for $\bar{R}(x)$ in particular the polynomial $R^{-1}(x)$ already defined, we see where $f_s(x)$, a p-adic factor of $f(x)$, goes over into $F_s(X)$ through $R(x)$ and $F_s(X)$ into $\bar{F}_s(\bar{X})$ through $R^{-1}(X)$ that $f_s(x)$ goes over into $\bar{F}_s(\bar{X})$ through $R^{-1}\{R(x)\}\equiv x \pmod{f(x)}$. We see then that $\bar{F}_s(\bar{X})$ coincides with $f_s(x)$ and that therefore $F_s(X)$ goes over into $f_s(x)$ through $R^{-1}(X)$ when $f(x)$ goes over into $F(X)$ and therewith $f_s(x)$ into $F_s(X)$ through $R(x)$.

Suppose the coefficients in $f(x)$, $R(x)$ and $\bar{R}(x)$ to be arbitrary rational numbers, with the exception of the coefficient of x^n in $f(x)$, which has the value 1. As before we shall indicate the roots of $f(x)$ by $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. The polynomial $f(x)$ goes over into the polynomials $F(X)$, $\bar{F}(X)$ and $g(X)$ through the polynomials $R(x)$, $\bar{R}(x)$ and $R(x)+\bar{R}(x)$ respectively. We have the following factorizations:

$$F(X)=\prod_{s=1}^{n}(X-R(\epsilon_s)),\ \bar{F}(X)=\prod_{s=1}^{n}(X-\bar{R}(\epsilon_s)),\ g(X)=\prod_{s=1}^{n}(X-R(\epsilon_s)-\bar{R}(\epsilon_s)).$$

*The words reducible and irreducible will be understood to have reference to the domain of the ordinary rational numbers where the context does not indicate otherwise.

The coefficients of $F(X)$ are rational and integral in terms of the coefficients of $f(x)$ and $R(x)$, the coefficients of $\overline{F}(X)$ rational and integral in terms of the coefficients of $f(x)$ and $\overline{R}(x)$, the coefficients of $g(X)$ rational and integral in terms of the coefficients of $f(x)$, $R(x)$ and $\overline{R}(x)$.

Let us construct the polynomial

$$G(X)=\prod_{\substack{s=1,\ldots,n\\ t=1,\ldots,n}}\left\{X-R(\epsilon_s)-\overline{R}(\epsilon_t)\right\}.$$

This polynomial is of degree n^2 and its coefficients are integral and symmetrical in terms of the roots of each of the polynomials $F(X)$ and $\overline{F}(X)$. The coefficients of $G(X)$ are therefore rational and integral in terms of the coefficients of $F(X)$ and $\overline{F}(X)$. In consequence of this, or also because of the form of $G(X)$ as a product above, the coefficients of $G(X)$ are rational and integral in terms of the coefficients of $f(x)$, $R(x)$ and $\overline{R}(x)$. Also from the form of $G(X)$ as a product we see that each of its coefficients excepting the first is of dimension 1 at least in the coefficients of $F(X)$ and $\overline{F}(X)$ other than those of X^n.

From the product-forms for $G(X)$ and $g(X)$ we see that the latter polynomial is a factor of the former. We can then write

$$G(X)=g(X)h(X) \tag{22}$$

where evidently the coefficients in $h(X)$ as well as those in $g(X)$ and $G(X)$ are rational and integral in terms of the coefficients of $f(x)$, $R(x)$ and $\overline{R}(x)$. It is to be borne in mind that the coefficients in $G(X)$ are also rational and integral expressions in the coefficients of $F(X)$ and $\overline{F}(X)$.

The equation (22) may be regarded as an identity in the variable X and the coefficients of $f(x)$, $R(x)$ and $\overline{R}(x)$ regarded as arbitrary rational parameters. The identity continues to hold good when we replace the arbitrary rational parameters by arbitrary p-adic numbers. When then in $f(x)$, $R(x)$ and $\overline{R}(x)$ we replace the arbitrary rational coefficients by arbitrary p-adic numbers the p-adic polynomial $f(x)$ goes over into the p-adic polynomials $F(X)$, $\overline{F}(X)$ and $g(X)$ through the p-adic polynomials $R(x)$, $\overline{R}(x)$ and $R(x)+\overline{R}(x)$ respectively. In the identity (22) of the same form as before in the coefficients of $f(x)$, $R(x)$ and $\overline{R}(x)$ it is evident that $G(X)$ is of the same form as before in the coefficients of $F(X)$ and $\overline{F}(X)$, these coefficients now being p-adic. The coefficients of $G(X)$ are then polynomials in the coefficients of $F(X)$ and $\overline{F}(X)$ with integral rational numerical coefficients. If then $F(X)$ and $\overline{F}(X)$ are p-adically integral it follows that $G(X)$ is also p-adically integral. As a consequence* the factors $g(X)$ and $h(X)$ of $G(X)$ must be p-adically integral. Here, however, $g(X)$ is the polynomial into which $f(x)$ goes over through the polynomial $R(x)+\overline{R}(x)$. If then through the two p-adic polynomials $R(x)$ and $\overline{R}(x)$ a p-adic polynomial $f(x)$ goes over into polynomials $F(X)$ and $\overline{F}(X)$ which are p-adically integral it also goes over into an integral p-adic polynomial through the polynomial sum $R(x)+\overline{R}(x)$.

If the coefficients in the polynomials $F(X)$ and $\overline{F}(X)$ are all divisible by p with the exception only of the coefficient of the highest power of X in each of

*See Hensel, *Theorie der algebraischen Zahlen*, pp. 64-65.

them the coefficients in the polynomial $G(X)$ with the exception of that of the highest power of X will all be divisible by p. This follows from the fact that the coefficients of $G(X)$ with the exception of that of the highest power of X are of dimension 1 at least in the coefficients of $F(X)$ and $\overline{F}(X)$ other than those of X^n. In $g(X)$ then which is a factor of $G(X)$ the coefficients will evidently all be divisible by p with the exception of the coefficient of the highest power of X. Where then the coefficients in the transforms of a p-adic polynomial $f(x)$ through the p-adic polynomials $R(x)$ and $\overline{R}(x)$ respectively are all divisible by p with the exception only of the coefficient of the highest power of the variable in each of them the like is true also of the transform of $f(x)$ through the polynomial $R(x)+\overline{R}(x)$. This holds in particular in the case where the coefficients of any or all of the polynomials in question are ordinary rational numbers.

II

Let us consider a polynomial

$$(1) \qquad g(x)=g_m x^m+g_{m-1}x^{m-1}+\ldots+g_0$$

in which the coefficients $g_m, \ldots, g_0$ are integral p-adic numbers. We shall suppose that g_0 is divisible by p and that one at least of the inner coefficients is not divisible by p. We shall then prove that the polynomial $g(x)$ is reducible in the field of the p-adic numbers. Supposing g_k to be the last of the coefficients in $g(x)$ which is not divisible by p we shall prove that $g(x)$ has a p-adic polynomial factor of degree k in x. We shall in fact prove that we can effect a modular factorization of $g(x)$ in the form

$$(2) \quad g(x)\equiv(x^k+q_{k-1}x^{k-1}+\ldots+q_0)(Q_{m-k}x^{m-k}+Q_{m-k-1}x^{m-k-1}+\ldots+Q_0)(\text{mod}\, p^\lambda),$$

where λ may be chosen as great as we will, the coefficients $q_{k-1}, \ldots, q_0$; $Q_{m-k-1}, \ldots, Q_0$ being integral p-adic polynomials, the coefficients $q_{k-1}, \ldots, q_0$ in particular, being all $\equiv 0 (\text{mod}\, p)$. From the outset we assume $Q_{m-k}=g_m$.

The congruence (2) is equivalent to the two sets of congruences

$$g_t\equiv \sum_{s=0}^{t} q_{t-s}Q_s \ (\text{mod}\, p^\lambda), \ (t=0, 1, \ldots, k-1),$$

$$g_t\equiv \sum_{s=0}^{m-t} q_{k-s}Q_{t-k+s} \ (\text{mod}\, p^\lambda), \ (t=k, k+1, \ldots, m-1).$$

These two sets of congruences may also be written in the form:

$$(3) \qquad q_tQ_0\equiv g_t-\sum_{s=1}^{t} q_{t-s}Q_s, \ (\text{mod p}^\lambda), \ (t=0, 1, \ldots, k-1)$$

$$(4) \qquad Q_{t-k}\equiv g_t-\sum_{s=1}^{m-t} q_{k-s}Q_{t-k+s} \ (\text{mod}\, p^\lambda), \ (t=k, k+1, \ldots, m-1).$$

Taking $\lambda=1$ the congruences (3) are identically satisfied when we take 0 for the first term in each of the p-adic series $q_0, q_1, \ldots, q_{k-1}$. The congruences (4) evidently determine the first term in Q_{t-k} to be the same as the first term in g_t for $t=k, k+1, \ldots, m-1$. In particular the first term in Q_0, being $\equiv g_k \ (\text{mod}\, p)$, is different from 0. Let us now suppose the first terms and the coefficients of $p, p^2, \ldots, p^{\lambda-1}$ in $q_0, \ldots, q_{k-1}$ and $Q_{m-k-1}, \ldots, Q_0$ to have been so determined that the congruences (3) and (4) hold. We can then determine successively the coefficients of p^λ in $q_0, \ldots, q_{k-1}$ from the congruences (3) in terms of coefficients already determined in the q's and Q's so that in this set of congruences we can replace the modulus p^λ by the modulus $p^{\lambda+1}$. Thereafter we can determine in succession the coefficients of p^λ in $Q_{m-k-1}, \ldots, Q_0$ from the congruences (4) in terms of coefficients already determined in the q's and Q's so that in this set of congruences we can replace the modulus p^λ by the modulus $p^{\lambda+1}$. The principle of induction then shows us that the first terms and the coefficients of $p, p^2, \ldots, p^{\lambda-1}$ in $q_0, \ldots, q_{k-1}$; $Q_0, \ldots, Q_{m-k-1}$ may be so chosen that the modular factorization indicated in (2) holds for a value of λ as great as we will. It may be noted too that the first terms in $Q_0, \ldots, Q_{m-k-1}$ and the successive coefficients in these series and in the p-adic series $q_0, \ldots, q_{k-1}$ are uniquely determined as soon as we have taken 0 for the first term in each of these last k series. Also there is here no question of convergence, so that we can write

$$g(x) = (x^k + q_{k-1}x^{k-1} + \ldots + q_0)(Q_{m-k}x^{m-k} + Q_{m-k-1}x^{m-k-1} + \ldots + Q_0), \tag{5}$$

where $q_0, \ldots, q_{k-1}$; $Q_0, \ldots, Q_{m-k}$ are perfectly defined integral p-adic numbers the first term in each of the numbers $q_0, \ldots, q_{k-1}$ having the value 0.

If then in an integral p-adic polynomial $g(x)$ the end term g_0 is divisible by p while the inner coefficients are not all so divisible the polynomial is reducible in the p-adic field. It is immediately apparent that an integral p-adic polynomial must be reducible if either of the end coefficients is divisible by p while an inner coefficient is not divisible by p*.

As a corollary of the theorem just stated it follows that an irreducible p-adic polynomial with integral end coefficients cannot have a fractional inner coefficient. Nor can a p-adic polynomial with integral end coefficients and a fractional inner coefficient be a power of an irreducible p-adic polynomial. In particular if an irreducible p-adic polynomial $f_s(x)$, through a p-adic polynomial $R(x)$, goes over into $F_s(X)$ it follows that $F_s(X)$ must be p-adically integral if its norm term is p-adically integral. If the norm term of $F_s(X)$ is divisible by p its inner coefficients must evidently also all be divisible by p.

In the case where the coefficient of x^k in $g\ (x)$ is not divisible by p and where

$$g(x) \equiv x^{k'} g_1(x) \pmod{p} \tag{6}$$

we have proved that the polynomial $g(x)$ can be split into two p-adic factors which are $\equiv x^k$ and $\equiv g(x) \pmod{p}$ respectively. More generally where $g(x)$ is not divisible by p and where

$$g(x) \equiv g_1(x) g_2(x) \pmod{p}, \tag{7}$$

the factors $g_1(x)$ and $g_2(x)$ being relatively prime (mod p), the polynomial $g(x)$ splits up into two p-adic factors which are $\equiv g_1(x)$ and $\equiv g_2(x) \pmod{p}$ respectively. To prove this it suffices to consider the case in which 1 is the coefficient

*See Hensel, *Theorie der algebraischen Zahlen*, p. 74.

of the highest power of x in each of the polynomials $g(x)$, $g_1(x)$ and $g_2(x)$. We know that we can find polynomials $h_1(x)$ and $h_2(x)$ and a rational number c so that we have the identity

$$h_2(x)g_1(x)+h_1(x)g_2(x)=c. \tag{8}$$

It is evident that in this identity the polynomials involved and the constant c may all be supposed to be p-adically integral. We may also assume that the degrees of $h_1(x)$ and $h_2(x)$ are less than those of $g_1(x)$ and $g_2(x)$ respectively and we shall suppose, what is readily shown to be permissible, that the polynomials $h_1(x)$ and $h_2(x)$ and the constant c are none of them divisible by p. It is in fact evident from the identity (8) that if either of the polynomials $h_1(x)$ or $h_2(x)$ is divisible by p then are all three elements $h_1(x)$, $h_2(x)$ and c so divisible. Also c alone of the three cannot be divisible by p since this would be in contravention of the fact that the factorization of a polynomial, with rational integral coefficients, into prime factors (mod p) is unique.

The transform of the polynomial $g_1(x)$ through the polynomial $g_2(x)h_1(x)$ is the same as its transform through the polynomial $g_1(x)h_2(x)+g_2(x)h_1(x)$, that is to say, through the constant c. The transform in question is then $(x-c)^n$ in which the norm term is $(-1)^n c^n$. The product of the norm terms in the transforms of $g_1(x)$ through $g_2(x)$ and $h_1(x)$ respectively is therefore c^n. These norm terms then are neither of them divisible by p for both of them are p-adically integral and their product c^n is not divisible by p.

The transform of $g(x)$ through $g_2(x)$ is $\equiv$ (mod p) to the transform of the product $g_1(x)g_2(x)$ and is therefore $\equiv$ (mod p) to the product of the transforms of $g_1(x)$ and $g_2(x)$ through $g_2(x)$. The norm term in the transform of $g_1(x)$ through $g_2(x)$ is, however, as we have just seen, not divisible by p and the transform of $g_2(x)$ through $g_2(x)$ is evidently X^k where k is the degree of $g_2(x)$. The transform $G(X)$ of $g(x)$ through $g_2(x)$ will then satisfy a congruence of the form

$$G(X)\equiv X^k G_1(X) \pmod{p}, \tag{9}$$

where $G_1(X)$ is the transform of $g_1(x)$ through $g_2(x)$. Since the norm term in $G_1(X)$ is not divisible by p we see from this congruence that $G(X)$ must split up into two p-adic factors which are $\equiv$(mod p) to X^k and $G_1(X)$ respectively. The transform $G(X)$ of $g(x)$ through $g_2(x)$ is then neither an irreducible p-adic polynomial nor yet the power of an irreducible p-adic polynomial. It follows therefore that $g(x)$ is not p-adically irreducible. So long then as we can split $g(x)$ into factors (mod p) which are relatively prime to each other (mod p) we can factor $g(x)$ p-adically. If then $g(x)$ is p-adically irreducible it follows that it must be $\equiv$(mod p) to the product of a p-adic constant and a power of a polynomial which is irreducible (mod p). Otherwise said, if $g(x)$ is p-adically irreducible it must be $\equiv$(mod p) to the product of a p-adic constant and a power of a prime function (mod p). It is then immediately evident from the congruence (7) that $g(x)$ must split into two p-adic factors which are $\equiv$(mod p) to $g_1(x)$ and $g_2(x)$ respectively since these polynomials are relatively prime to each other (mod p).

If in the transform of an irreducible p-adic polynomial the norm term is divisible by p we have seen that the inner coefficients must all be divisible by p.

Let us suppose that the norm terms in the transforms of an irreducible p-adic polynomial $f(x)$ through the polynomials $R(x)$ and $\overline{R}(x)$ are both divisible by p. The inner coefficients in these transforms are therefore also divisible by p. From Section I it follows that the norm term and the inner coefficients in the transform of $f(x)$ through the polynomial $R(x)+\overline{R}(x)$ are all divisible by p. If then the norm terms are divisible by p in the transforms of the irreducible p-adic polynomial $f(x)$ through two p-adic polynomials $R(x)$ and $\overline{R}(x)$ the norm term is divisible by p in the transform of $f(x)$ through the polynomial $R(x)+\overline{R}(x)$.

Where $f(x)$ is an irreducible integral p-adic polynomial in which the coefficient of the highest power of x is 1 we see from the second paragraph preceding that we can write

$$f(x)\equiv\{\gamma(x)^k\} \pmod p \tag{10}$$

where $\gamma(x)$ is a prime function (mod p). The transform of $f(x)$ through the polynomial $\{\gamma(x)\}^k$ is then $\equiv X^n \pmod p$ for X^n is the transform of $f(x)$ through $f(x)$. The norm term in the transform of $f(x)$ through $\{\gamma(x)\}^k$ is then divisible by p. The norm term in the transform of $f(x)$ through $\gamma(x)$ must therefore be divisible by p. Where the norm term in the transform of $f(x)$ through a p-adic polynomial is divisible by p we shall show that $\gamma(x)$ must be a factor (mod p) of the polynomial in question.

Let $u(x)$ be an integral p-adic polynomial of the lowest possible degree which is not divisible by p and which is such that the norm term in the transform of $f(x)$ through $u(x)$ is divisible by p. We may evidently assume that the coefficient of the highest power of x in $u(x)$ is 1. Suppose $v(x)$ to be an integral p-adic polynomial which is not divisible by p and which transforms $f(x)$ into a polynomial whose norm term is divisible by p. We can write $v(x)=\phi(x)u(x)+u_1(x)$ where $u_1(x)$ is of degree lower than $u(x)$. Here $v(x)$ and $\phi(x)u(x)$ transform $f(x)$ into polynomials whose norm terms are divisible by p. It follows that the norm term in the transform of $f(x)$ through $v(x)-\phi(x)\,u(x)=u_1(x)$ must be divisible by p. Since $u_1(x)$ is of degree less than $u(x)$, however, this implies that $u_1(x)$ is divisible by p. Consequently we have $v(x)\equiv\phi(x)u(x) \pmod p$. The polynomial $v(x)$ then has $u(x)$ as factor (mod p). In particular $\gamma(x)$ must have $u(x)$ as factor (mod p). Since, however, $\gamma(x)$ is a prime function (mod p) it follows that $u(x)$ must identify itself with $\gamma(x) \pmod p$. Any p-adically integral polynomial $v(x)$ then which transforms $f(x)$ into a polynomial whose norm term is divisible by p must have the prime function $\gamma(x)$ as a factor (mod p).

III

Let us now consider the polynomial $f(x)$ with ordinary rational coefficients represented in formulae (2) and (3) Section I. We shall suppose $f(x)$ to be irreducible in the domain of the ordinary rational numbers. It may, however, be reducible in the domain of the p-adic numbers. Its representation as the product of irreducible p-adic factors we shall suppose to be given by formula (9), Section I, the factors $f_1(x), \ldots, f_r(x)$ now being regarded as irreducible p-adic polynomials. After the analogy of formula (10), Section II, we shall then have

$f_s(x) \equiv \{\gamma_s(x)\}^{k_s} (\mathrm{mod}\ p)$ and we can evidently write

$$(2) \qquad f(x) \equiv \prod_{s=1}^{r} \{\gamma_s(x)\}^{k_s} \ (\mathrm{mod}\ p)$$

where the polynomials $\gamma_s(x)$ are prime polynomials (mod p).

Let ϵ be a root of the polynomial $f(x)$. The algebraic corpus determined by ϵ we shall designate by the notation $C(\epsilon)$. The sum a_{n-1} of the n conjugate roots of $f(x)$ is called the trace of ϵ in the corpus. The product a_0 of the n roots is called the norm of ϵ. From formulae (9) and (10), Section I, we see that the trace of ϵ is represented by the sum

$$(3) \qquad a_{n-1} = a^{(1)}_{n_1-1} + a^{(2)}_{n_2-1} + \ldots + a^{(r)}_{n_r-1}$$

while its norm is represented by the product

$$(4) \qquad a_0 = a_0^{(1)}\, a_0^{(2)} \ldots a_0^{(r)}.$$

The elements $a^{(1)}_{n_1-1}, \ldots, a^{(r)}_{n_r-1}$ of the sum we call the p-adic *partial* traces of ϵ relative to the respective factors $f_1(x), \ldots, f_r(x)$ of $f(x)$ and the factors $a_0^{(1)}, a_0^{(2)}, \ldots, a_0^{(r)}$ of the product we call the p-adic *partial* norms of ϵ relative to these factors of $f(x)$.

Any rational function $R(\epsilon)$ of ϵ satisfies an equation $F(X)=0$ which may or may not be reducible. Whether this equation be reducible or irreducible the sum A_{n-1} of its n roots is called the trace of $R(\epsilon)$ in the corpus $C(\epsilon)$. The product A_0 of the n roots is called the norm of $R(\epsilon)$. Supposing $F(X)$ to be represented in formula (12), Section I, as the product of its p-adic factors $F_s(X)$ which correspond to the irreducible p-adic factors $f_s(x)$ of $f(x)$ we see that the trace of $R(\epsilon)$ is represented by the sum

$$(5) \qquad A_{n-1} = A^{(1)}_{n_1-1} + \ldots + A^{(r)}_{n_r-1}$$

while its norm is represented by the product

$$(6) \qquad A_0 = A_0^{(1)} A_0^{(2)} \ldots A_0^{(r)}.$$

The elements $A^{(1)}_{n_1-1}, \ldots, A^{(r)}_{n_r-1}$ of the sum are called the p-adic partial traces of $R(\epsilon)$ corresponding to the respective factors $F_1(X), \ldots, F_r(X)$ of $F(X)$ while the factors $A_0^{(1)}, \ldots, A_0^{(r)}$ of the product are called the p-adic partial norms of $R(\epsilon)$ corresponding to these factors. The coefficients $-a^{(s)}_{n_s-1}$ and $-A^{(s)}_{n_s-1}$ in the polynomials $f_s(x)$ and $F_s(X)$ respectively we refer to as the trace coefficients in these polynomials.

The order numbers of the p-adic partial traces of a number of the corpus we call the p-adic *partial* trace orders of the number. The order numbers of the p-adic partial norms $a_0^{(1)}, a_0^{(2)}, \ldots, a_0^{(r)}$ divided by the degrees of the p-adic polynomials $f_1(x), f_2(x), \ldots, f_r(x)$, that is divided by the integers $n_1, n_2, \ldots, n_r$, respectively, we shall call the *order numbers* of ϵ relative to the respective factors $f_1(x), f_2(x), \ldots, f_r(x)$ of $f(x)$. The order numbers of the p-adic partial norms $A_0^{(1)}, A_0^{(2)}, \ldots, A_0^{(r)}$ of $R(\epsilon)$ divided by $n_1, n_2, \ldots, n_r$ respectively, we shall call the *order numbers* of $R(\epsilon)$ relative to the p-adic factors $F_1(X), F_2(X), \ldots, F_r(X)$

of $F(X)$ or also the order numbers of $R(\epsilon)$ corresponding to the factors $f_1(x), f_2(x), \ldots, f_r(x)$ of $f(x)$.

It is readily seen that the order number of $R(\epsilon)$ relative to the factor $F_s(X)$ cannot be greater than the order number of the p-adic partial trace $A_{n_s-1}^{(s)}$. Otherwise the order number of $A_0^{(s)}$ would be greater than $n_s k_s$ where k_s indicates the order number of $A_{n_s-1}^{(s)}$. This, however, would imply that in $\overline{F}_s(X) = p^{-n_s k_s} F_s\ (p^k X)$ the norm term is divisible by p. The coefficient of X^{n_s} in $\overline{F}_s(X)$ is at the same time 1 and the coefficient of X^{n_s-1} has 0 for its order number. As a consequence $\overline{F}_s(X)$ and therefore also $F_s(X)$ could not be a power of an irreducible p-adic polynomial. $F_s(X)$, however, must be such a power since by hypothesis $f_s(x)$ is an irreducible p-adic polynomial. It follows that the order number of $R(\epsilon)$ corresponding to an irreducible p-adic factor $f_s(x)$ of $f(x)$ cannot be greater than the p-adic partial trace order of $R(\epsilon)$ corresponding to this factor. In particular if the order number of $R(\epsilon)$ corresponding to the factor $f_s(x)$ is positive so also is the p-adic partial trace order of $R(\epsilon)$ corresponding to this factor positive.

The p-adic order numbers of a number $R(\epsilon)$ of the corpus $C(\epsilon)$ have no more relation to the p-adic factors of one equation defining a primitive number of the corpus, but not possessing $R(\epsilon)$ as a root, than they have to the p-adic factors of any other irreducible equation satisfied by a primitive number of the corpus. It is convenient, however, to study the properties of the numbers of the corpus represented as rational functions of some one primitive number of the corpus. In terms of such primitive number we know that any number of the corpus can be represented as a polynomial of degree $n-1$ with rational coefficients. Let us select any irreducible equation $f(x)=0$ which is satisfied by a primitive number of the corpus. Suppose ϵ to be a root of the equation which defines the corpus. The numbers of the corpus we represent as polynomials in ϵ of degree $n-1$ with rational coefficients.

The irreducible p-adic factors of $f(x)$ selected in any arbitrary order we designate by the notation $f_1(x), f_2(x), \ldots, f_r(x)$ and, with explicit reference to the specific root ϵ of $f(x)$ we call these the factors of $f(x)$ of the 1st, 2nd, ..., rth *cycles* respectively. The order numbers of ϵ corresponding to these factors we refer to as the order numbers of ϵ for the 1st, 2nd, ..., rth cycles. When the polynomial $f(x)$ goes over into the polynomial $F(X)$ through the polynomial $R(x)$, the irreducible p-adic factors $f_1(x), f_2(x), \ldots, f_r(x)$ of $f(x)$ at the same time go over into the p-adic factors $F_1(X), F_2(X), \ldots, F_r(X)$ of $F(X)$. The order numbers of $R(\epsilon)$ corresponding to these latter factors we refer to as the order numbers of $R(\epsilon)$ for the 1st, 2nd, ..., rth cycles respectively. The order numbers of $R(\epsilon)$ for the several cycles we shall also on occasion refer to as the *orders of coincidence* of $R(\epsilon)$ corresponding to these cycles, recalling by this terminology the analogy existing between the theory here given and the theory of the algebraic functions.

In the case where roots of the equation $f(x)=0$ besides ϵ belong to the corpus $C(\epsilon)$ any such root can be represented as a polynomial $\phi(\epsilon)$ in ϵ of degree $n-1$ with rational coefficients. In this case the polynomial $f(x)$ goes over into $f(X)$ through the polynomial $\phi(x)$. It may be, however, that a p-adic factor $f_s(x)$ of $f(x)$ does not at the same time go over into $f_s(X)$ but into some other p-adic factor

$f_t(X)$. In such case the order of coincidence of $\phi(\epsilon)$ with the sth cycle would be the same as the order of coincidence of ϵ with the tth cycle. While, then, the p-adic orders of coincidence of the numbers ϵ and $\phi(\epsilon)$ with the r cycles are in the aggregate the same, the orders of coincidence of the two numbers with a specific cycle may be different.

Let ϵ_1 and ϵ_2 be two numbers of the corpus $C(\epsilon)$ of which ϵ_1 is primitive. We can write $\epsilon_1 = R(\epsilon)$, $\epsilon_2 = \overline{R}(\epsilon)$. The polynomial $f(x) = f_1(x)f_2(x)\ldots f_r(x)$ goes over into the polynomial $F(X) = F_1(X)F_2(X)\ldots F_r(X)$ through the polynomial $R(x)$, the p-adic factor $f_s(x)$ at the same time going over into the p-adic factor $F_s(X)$. Also the polynomial $f(x)$ goes over into the polynomial $\overline{F}(X) = \overline{F}_1(X)\overline{F}_2(X)\ldots\overline{F}_r(X)$ through the polynomial $\overline{R}(x)$, the p-adic factor $f_s(x)$ at the same time going over into the p-adic factor $\overline{F}_s(X)$. Here $\epsilon = R^{-1}(\epsilon_1)$, $\epsilon_2 = \overline{R}\{R^{-1}(\epsilon_1)\}$ and $F(X)$ goes over into $\overline{F}(\overline{X})$ through the polynomial $\overline{R}\{R^{-1}(x)\}$. The factor $F_s(X)$ however, goes over into the factor $f_s(x)$ through the polynomial $R^{-1}(X)$ and $f_s(x)$ goes over into $\overline{F}_s(X)$ through the polynomial $\overline{R}(x)$, so that $F_s(X)$ evidently goes over into $\overline{F}_s(\overline{X})$ through the polynomial $\overline{R}R^{-1}(X)$.

The orders of coincidence for ϵ_1 and ϵ_2 furnished by the factors $F_s(X)$ and $\overline{F}_s(X)$ respectively and corresponding to the order of coincidence of ϵ furnished by the factor $f_s(x)$ of $f(x)$ when the numbers of the corpus are expressed in terms of ϵ, are then corresponding orders of coincidence when ϵ_2 and the other numbers of the corpus are expressed in terms of ϵ_1. The correspondence here in question is therefore independent of the primitive number ϵ in terms of which we express the numbers of the corpus. The r orders of coincidence of any one number of the corpus then correspond in a perfectly definite order to the r orders of coincidence of any other number of the corpus and each number of a pair of corresponding orders of coincidence corresponds to the same order of coincidence of any third number of the corpus. We see then that we are justified in separating the order numbers of all numbers of the corpus into r classes or *cycles* the order numbers in any one cycle corresponding to one another and not depending in any manner on the form in which the numbers of the corpus may be represented. We may refer to the cycles as the 1st, 2nd, . . . , rth but the arrangement of the cycles to which we attach this terminology is an arbitrary one.

From Section I we see that any p-adic partial norm of a product of two numbers of the corpus is obtained on multiplying the corresponding p-adic partial norms of the numbers. It follows that the p-adic order numbers of a product of two or more numbers of the corpus are obtained on adding the corresponding order numbers of the factors.

It is readily seen that the reciprocal of a number of the corpus has as its p-adic partial norms the reciprocals of the corresponding p-adic partial norms of the number. The p-adic order numbers of the reciprocal of a number of the corpus are therefore the negatives of the p-adic order numbers of the number. As a consequence the p-adic order numbers of a quotient of two numbers of the corpus are obtained on subtracting from the p-adic order numbers of the numerator the corresponding p-adic order numbers of the denominator.

The order of coincidence of the sum of two numbers of the corpus with a given cycle is equal to the smaller of the orders of coincidence of the two numbers with the cycle in the case where these orders of coincidence are unequal and where the orders of coincidence are equal the order of coincidence of the sum of the two numbers with the cycle in question is equal to or greater than their common order of coincidence. To prove this let us first consider two numbers of the corpus $\epsilon_1 = R(\epsilon)$ and $\epsilon_2 = \overline{R}(\epsilon)$ whose orders of coincidence with the sth cycle are zero or positive. Such numbers we say are integral with regard to the sth cycle. Suppose $f(x)$ to go through $R(x)$ over into $F(X)$, the irreducible factor $f_s(x)$ at the same time going over into $F_s(X)$ and suppose $f(x)$ to go through $\overline{R}(x)$ over into $\overline{F}(X)$, the factor $f_s(x)$ at the same time going over into $\overline{F}_s(X)$. Since by hypothesis the numbers $R(\epsilon)$ and $\overline{R}(\epsilon)$ are integral with regard to the sth cycle, the norm terms in the polynomials $F_s(X)$ and $\overline{F}_s(X)$ are integral with regard to p and therewith these polynomials themselves are integral p-adic polynomials. It then follows from the next to last paragraph of Section I that the p-adic polynomial $f_s(x)$ goes over into an integral p-adic polynomial through $R(x) + \overline{R}(x)$. The number $R(\epsilon) + \overline{R}(\epsilon)$ is therefore integral with regard to the sth cycle. If, then ϵ_1 and ϵ_2 are two numbers of the corpus which are integral with regard to the sth cycle, their sum is also integral with regard to this cycle.

Let us now suppose ϵ_1 and ϵ_2 to be any two numbers of the corpus such that the order of coincidence of ϵ_1 with the sth cycle is not greater than the order of coincidence of ϵ_2 with this cycle. The product $\epsilon_1^{-1}\epsilon_2$ is then integral with regard to the sth cycle. The number 1 is integral with regard to all cycles and the sum $1 + \epsilon_1^{-1}\epsilon_2$ is therefore integral with regard to the sth cycle. The product $\epsilon_1(1 + \epsilon_1^{-1}\epsilon_2) = \epsilon_1 + \epsilon_2$ consequently has an order of coincidence with the sth cycle which is equal to or greater than the order of coincidence of ϵ_1 with this cycle. Suppose now that the order of coincidence of ϵ_1 is actually less than the order of coincidence of ϵ_2 with the sth cycle. The order of coincidence of $\epsilon_1^{-1}\epsilon_2$ with the sth cycle is then positive. The order of coincidence of $\epsilon' = 1 + \epsilon_1^{-1}\epsilon_2$ with the sth cycle is, however, not positive for, if it were positive, the order of coincidence of $\epsilon' - \epsilon_1^{-1}\epsilon_2 = 1$ with the cycle would be at least equal to the smaller of the orders of coincidence of the two numbers ϵ' and $\epsilon_1^{-1}\epsilon_2$ with the cycle and therefore positive, which it is not for 0 is the order of coincidence of 1 with any cycle. The order of coincidence of $\epsilon' = 1 + \epsilon_1^{-1}\epsilon_2$ with the sth cycle must then be 0. The order of coincidence of $\epsilon_1\epsilon' = \epsilon_1 + \epsilon_2$ with the sth cycle is therefore precisely equal to the order of coincidence of ϵ_1 with this cycle when the order of coincidence of ϵ_1 with the sth cycle is less than the order of coincidence of ϵ_2 with this cycle.

It will be convenient for us on occasion to speak of the p-adic order numbers of a polynomial or number $R(\epsilon)$ in which the coefficients are p-adic numbers other than rational numbers. The p-adic order numbers will in this case be defined with reference to the partial norm terms in the transform of $f(x)$ through $R(x)$ in precisely the same way as if the coefficients in $R(x)$ were rational numbers. In this case, too, it is evident that the p-adic order numbers corresponding to the sth cycle are the same for all numbers which are $\equiv R(\epsilon) \pmod{f_s(\epsilon)}$. Also the p-adic order number of a product of several numbers for a given cycle is equal to the sum of the p-adic order numbers of the separate factors corresponding to this cycle. Furthermore, the sth p-adic order number of a sum of two p-adic polynomials in ϵ is equal to the smaller of the sth p-adic order numbers of the two polynomials in the case where these order numbers are unequal, and where they are equal the sth p-adic order number of the sum of the two polynomials is equal to or greater than their common sth p-adic order number. To see this it is only necessary to note that the deletion of higher powers of p in the p-adic coefficients of the two polynomials and therewith in the coefficients of their sum, alters none of the p-adic order numbers here in question and reduces the polynomials to numbers of the corpus $C(\epsilon)$ for which numbers the theorem has already been proved.

IV

The degrees $n_1, n_2, \ldots, n_r$ of the p-adically irreducible factors $f_1(x), f_2(x), \ldots, f_r(x)$ of $f(x)$ we also refer to as the *degrees* of the 1st, 2nd, $\ldots$, rth p-adic cycles of the corpus $C(\epsilon)$. The orders of coincidence of numbers of the corpus $C(\epsilon)$ with the sth cycle are integral multiples of $1/n_s$. Among these orders of coincidence there will be a least positive one. Suppose this in its reduced form to be k/ν_s. Here of course ν_s is a factor of n_s. Suppose ϵ_1 to be a number of the corpus possessing this least positive order of coincidence. Since k and ν_s are relatively prime we can determine integers λ and ρ such that $k\lambda-\nu_s\rho=1$. The order of coincidence of ϵ_1^λ with the sth cycle is then $k\lambda/\nu_s=\rho+1/\nu_s$. The order of coincidence of $p^{-\rho}\epsilon_1^\lambda$ with this cycle is consequently $1/\nu_s$. If then k/ν_s is the smallest positive order of coincidence which a number of the corpus can have with the sth p-adic cycle, we see that we must have $k=1$. It follows that the order of coincidence of any number of the corpus with the sth cycle must be an integral multiple of $1/\nu_s$. That any integral multiple of $1/\nu_s$ may actually present itself as the order of coincidence of a number of the corpus with the sth cycle is self-evident. For example, λ/ν_s is the order of coincidence of ϵ_1^λ with the sth cycle. The number ν_s we call the order of the sth cycle. When $\nu_s=n_s$ the order of the sth cycle is equal to its degree. In any case the order of a cycle is a factor of its degree. The integer n_s/ν_s we call the grade of the sth cycle. The degree of a cycle is then equal to the product of its order and its grade.

We have just considered the order of coincidence of a number of the corpus with a single one of the p-adic cycles without reference to what its orders of coincidence with the remaining $r-1$ cycles may be and we have seen that its order of coincidence with the cycle in question may be any integral multiple of $1/\nu_s$ where ν_s is the order of the cycle. We shall furthermore show that any arbitrarily assigned integral multiples $\lambda_1/\nu_1, \ldots, \lambda_r/\nu_r$ of $1/\nu_1, \ldots, 1/\nu_r$ may actually present themselves simultaneously as the orders of coincidence of a number of the corpus with the 1st, 2nd, $\ldots$, rth cycles respectively and at the same time we shall show that any set of finite p-adic partial trace orders $j_1, j_2, \ldots, j_r$ may be associated with the set of orders of coincidence in question if only we have severally, for $s=1, 2, \ldots, r$ the number j_s as sth p-adic partial trace order compatible with λ_s/ν_s as order of coincidence with the sth cycle, without taking account of what the partial trace orders and orders of coincidence corresponding to the other $r-1$ cycles may happen to be.

To prove the propositions here in question we shall begin by representing any p-adic polynomial $g(x)$ of degree $n-1$ in the form

$$g(x)=g_1(x)f_2(x)\ldots f_r(x)+g_2(x)f_1(x)f_3(x)\ldots f_r(x)+\ldots+g_r(x)f_1(x)\ldots f_{r-1}(x) \tag{1}$$

where $g_1(x), \ldots, g_r(x)$ are p-adic polynomials of degrees $n_1-1, \ldots, n_r-1$ respectively. To obtain this representation of $g(x)$ we first determine $g_1(x)$ and $G_2(x)$ of degrees n_1-1 and $n_2+\ldots+n_r-1$ respectively so that

$$g_1(x)f_2(x)\ldots f_r(x)+G_2(x)f_1(x)=g(x).$$

Thereafter we determine $g_2(x)$ and $G_3(x)$ so that

$$g_2(x)f_3(x)\ldots f_r(x)+G_3(x)f_2(x)=G_2(x)$$

$g_3(x)$ and $G_4(x)$ so that

$$g_3(x)f_4(x)\ldots f_r(x)+G_4(x)f_3(x)=G_3(x)$$

and so on until finally we determine $g_{r-1}(x)$ and $G_r(x)$ so that

$$g_{r-1}(x)f_r(x)+G_r(x)f_{r-1}(x)=G_{r-1}(x).$$

Combining the above identities so as to eliminate $G_2(x), \ldots, G_{r-1}(x)$ and replacing $G_r(x)$ by the notation $g_r(x)$ we arrive at the representation for $g(x)$ given in (1). Here $g_1(x), \ldots, g_r(x)$ are determined as p-adic polynomials in x of degrees $n_1-1, \ldots, n_r-1$ respectively. That the representation of $g(x)$ in the form (1) is unique is apparent. For if there was another such representation

$$g(x)=\bar{g}_1(x)f_2(x)\ldots f_r(x)+\ldots\ldots+\bar{g}_r(x)f_1(x)\ldots f_{r-1}(x)$$

we should by subtraction obtain the identity

$$\sum_{s=1}^{r}\{g_s(x)-\bar{g}_s(x)\}f_1(x)\ldots f_{s-1}(x)f_{s+1}(x)\ldots f_r(x)=0.$$

From this, however, would follow the divisibility of $g_s(x)-\bar{g}_s(x)$ by $f_s(x)$ and we should consequently have identically

$$g_s(x)=\bar{g}_s(x), \quad (s=1, 2, \ldots, r).$$

Let us now select r numbers of the corpus, $g^{(1)}(\epsilon), g^{(2)}(\epsilon), \ldots, g^{(r)}(\epsilon)$ such that for $s=1, 2, \ldots, r$ the order of coincidence of $g^{(s)}(\epsilon)$ with the sth cycle is λ_s/ν_s while its sth p-adic partial trace order is j_s, its orders of coincidence with the other cycles and its other partial trace orders not being specified. Representing $g^{(s)}(x)$ in the form

$$g^{(s)}(x)=\sum_{t=1}^{r} g_t^{(s)}(x)f_1(x)\ldots f_{t-1}(x)f_{t+1}(x)\ \ldots\ f_r(x), \quad (s=1, 2, \ldots, r) \tag{2}$$

we see that $f_s(x)$ goes over into the same polynomial $F_s(X)$ through the polynomial $g^{(s)}(x)$ and through the polynomial $g_s^{(s)}(x)f_1(x)\ldots f_{s-1}(x)f_{s+1}(x)\ldots f_r(x)$ since this latter polynomial is the only element in the summation on the right-hand side of (2) which is not divisible by $f_s(x)$. Since λ_s/ν_s is the order of coincidence of $g^{(s)}(\epsilon)$ with the sth cycle the order number of the norm term in $F_s(X)$ is $n_s\lambda_s/\nu_s$. The order number of the trace term in $F_s(X)$ is by hypothesis j_s.

Construct the polynomial

$$G(x) = \sum_{s=1}^{r} g_s^{(s)}(x)f_1(x)\ldots f_{s-1}(x)f_{s+1}(x)\ldots f_r(x). \tag{3}$$

Through $G(x)$ the polynomial $f_s(x)$ goes over into the polynomial $F_s(X)$ since it goes over into this polynomial through the polynomial $g_s^{(s)}(x)f_1(x)\,.\,.f_{s-1}(x)f_{s+1}(x)\,.\,.f_r(x)$. Through $G(x)$ then the polynomials $f_1(x),\ldots,f_r(x)$ go over into the polynomials $F_1(X),\ \ldots,\ F_r(X)$ respectively and the polynomial $f(x)=f_1(x)\ldots f_r(x)$ at the same time goes over into the polynomial $F(X)=F_1(X)\ldots F_r(X)$ If the p-adic coefficients of $G(x)$ are at the same time rational numbers $G(\epsilon)$ is a number of the corpus and its orders of coincidence with the branches of the 1st, 2nd, ..., rth cycles are evidently $\lambda_1/\nu_1,\ \lambda_2/\nu_2,\ \ldots,\ \lambda_r/\nu_r$ since the order numbers of the norm terms in $F_1(X),\ F_2(X),\ \ldots,\ F_r(X)$ are $n_1\lambda_1/\nu_1,\ n_2\lambda_2/\nu_2,\ \ldots,\ n_r\lambda_r/\nu_r$ respectively. Also its p-adic partial trace orders are $j_1,\ j_2,\ \ldots,\ j_r$ respectively. If the coefficients of $G(x)$ are not all rational numbers we discard in its p-adic coefficients terms involving higher powers of p thus obtaining a polynomial $\overline{G}(x)$ whose coefficients are rational numbers. Assuming that the terms discarded are of sufficiently high power the polynomials $f_1(x),\ \ldots,\ f_r(x)$ will through the polynomial $\overline{G}(x)$ go over into polynomials $\overline{F}_1(X),\ \ldots,\ \overline{F}_r(X)$ whose norm terms have the same order numbers as the norm terms in the polynomials $F_1(X),\ \ldots,\ F_r(X)$ and whose trace terms also have the same order numbers as the trace terms in these polynomials. The number $\overline{G}(\epsilon)$ belongs to the corpus and has the orders of coincidence $\lambda_1/\nu_1,\ \lambda_2/\nu_2,\ \ldots,\ \lambda_r/\nu_r$ with the 1st, 2nd, ..., rth cycle respectively while its p-adic partial trace orders corresponding to these cycles are $j_1,\ j_2,\ \ldots,\ j_r$ respectively. Assigning then any arbitrary set of orders of coincidence $\lambda_1/\nu_1,\ \ldots,\ \lambda_r/\nu_r$ corresponding to the r p-adic cycles we have shown that there exist in the corpus numbers which actually possess this set of orders of coincidence. At the same time we have shown that among these numbers are ones which have any assigned set of finite partial trace orders $j_1,\ j_2,\ \ldots,\ j_r$, so selected that for $s=1,\ 2,\ \ldots,\ r$ separately the sth p-adic partial trace order j_s is compatible with λ_s/ν_s as order of coincidence of a number of the corpus with the sth cycle.

V

Where $f(x)$ is an irreducible polynomial in x of degree n with the roots $\epsilon_1,\ \epsilon_2,\ \ldots,\ \epsilon_n$ we know that any polynomial $R(x)$ of degree $n-1$ in x with rational coefficients can be represented in the form

$$R(x) = f(x)\sum_{s=1}^{n} R(\epsilon_s)/f'(\epsilon_s)(x-\epsilon_s). \tag{1}$$

From the form on the right-hand side we see that the coefficient of x^{n-1} in $R(x)$ is equal to the trace of the number $R(\epsilon)/f'(\epsilon)$ where ϵ is one of the roots $\epsilon_1,\ \epsilon_2,\ \ldots,\ \epsilon_n$. In a polynomial of degree $n-1$ in x we shall find it convenient to call the term involving x^{n-1} the *principal term* and its coefficient the *principal coefficient* of the polynomial. Where the principal coefficient is represented as a p-adic number in its reduced form the coefficient of p^{-1} in the p-adic series is called the *principal p-adic residue* of the polynomial.

The principal coefficient in $R(x)$ is the trace of a number $\rho(\epsilon)$ where $\rho(x)$ is a polynomial in x with rational coefficients which satisfies the congruence $\rho(x)f'(x)\equiv R(x)$ (mod $f(x)$). If $\rho(x)$ is assumed to be of degree $n-1$ it is completely determined by this congruence in terms of $R(x)$. Otherwise it is determined (mod $f(x)$). Conversely $R(x)$ is determined by the congruence in terms of $\rho(x)$. If now $f(x)$ through $\rho(x)$ goes over into the polynomial $F(X)$ the coefficient, with sign changed, of X^{n-1} in $F(X)$ is the trace of $\rho(\epsilon)$. From what has been said above then the coefficient, with sign changed, of X^{n-1} in $F(X)$ must be equal to the coefficient of x^{n-1} in the remainder obtained on dividing the product $\rho(x)f'(x)$ by $f(x)$. This holds for arbitrary rational values of the coefficients in $f(x)$ and $\rho(x)$. It is in fact the statement of the identity of two expressions each of which is integral in terms of the arbitrary coefficients in question. If then we replace these arbitrary rational coefficients by arbitrary p-adic numbers the identity still holds good. It follows that when a p-adic factor $f_s(x)$ of $f(x)$ through $\rho(x)$ goes over into a p-adic factor $F_s(X)$ the coefficient of X^{n_s-1} in $F_s(X)$, with sign changed, is identical with the coefficient of x^{n_s-1} in the remainder obtained on dividing the product $\rho(x)f_s'(x)$ by $f_s(x)$. In the case where the coefficients in $\rho(x)$ are ordinary rational numbers we see then that the sth partial p-adic trace of $\rho(\epsilon)$ is given by the coefficient of x^{n_s-1} in the remainder obtained on dividing $\rho(x)f_s'(x)$ by $f_s(x)$.

In formula (1), Section IV, we have given a representation for a p-adic polynomial $g(x)$ of degree $n-1$. A polynomial $R(x)$ of degree $n-1$ with rational coefficients can then be represented in the form

$$R(x)=\sum_{s=1}^{r}\rho_s(x)f_1(x)\ldots f_{s-1}(x)f_{s+1}(x)\ldots f_r(x) \tag{2}$$

where the polynomials $\rho_1(x), \rho_2(x), \ldots, \rho_r(x)$ are of degrees $n_1-1, n_2-1, \ldots, n_r-1$ respectively with rational p-adic coefficients. The principal coefficient of $R(x)$ we have seen to be the trace of $\rho(\epsilon)$ where $\rho(x)$ is a polynomial which satisfies the congruence

$$\rho(x)f'(x)\equiv R(x), \pmod{f(x)}. \tag{3}$$

This congruence may be written

$$\begin{aligned}\rho(x)\sum_{s=1}^{r}f_1(x)\ldots f_{s-1}(x)f_s'(x)f_{s+1}(x)\ldots f_r(x)\\ \equiv\sum_{s=1}^{r}\rho_s(x)f_1(x)\ldots f_{s-1}(x)f_{s+1}(x)\ldots f_r(x), \pmod{f(x)}\end{aligned} \tag{4}$$

and this is evidently equivalent to the r congruences

$$\rho(x)f_s'(x)\equiv\rho_s(x) \pmod{f_s(x)}, \quad (s=1, 2, \ldots, r). \tag{5}$$

From these congruences follows for $s=1, 2, \ldots, r$ that the coefficient of x^{n_s-1} in $\rho_s(x)$ is equal to the coefficient, with sign changed, of X^{n_s-1} in $F_s(X)$, the polynomial into which $f_s(x)$ goes over through $\rho(x)$. The coefficient of x^{n_s-1} in $\rho_s(x)$ is then equal to the sth p-adic partial trace of $\rho(\epsilon)$. In any polynomial $R(x)$ of degree $n-1$ in x with rational coefficients the principal coefficient is equal to the sum of the principal coefficients in the several elements of the summation in (2), and what we have here found is that each of these principal coefficients is equal to the corresponding p-adic partial trace of the number $\rho(\epsilon)$ where $\rho(x)$ is a polynomial which satisfies the congruence (3). The coefficient of x^{n_s-1} in $\rho_s(x)$, that is, the coefficient of x^{n-1} in the sth element of the summation in (2), we shall refer to as the sth partial principal coefficient of $R(x)$. The sth partial principal coefficient of $R(x)$ is then equal to the sth p-adic partial trace of $\rho(\epsilon)$.

Consider a succession m', m'', ..., of increasing multiples of $1/\nu_s$ as possible orders of coincidence of numbers of the corpus $C(\epsilon)$ with the sth cycle. Let ϵ_1, ϵ_2, ..., be numbers which possess these respective orders of coincidence with the sth cycle and which at the same time have the smallest possible sth partial trace orders k', k'', ... compatible with their orders of coincidence with the sth cycle. It is readily seen that the numbers k', k'', ... are arranged in increasing order of magnitude, that is to say that no number in the series is less than the one which precedes it. To see this we note first that the order of coincidence of the sum $\epsilon_1+\epsilon_2$ with the sth cycle is m'. Its sth p-adic trace order, in accord with our hypothesis, cannot therefore be less than k'. This sth p-adic trace order of the sum $\epsilon_1+\epsilon_2$ must, however, be equal to the lesser of the numbers k' and k'' if these are unequal. This is in contradiction with the statement of the sentence preceding if k'' is less than k'. It follows that k'' must be equal to or greater than k'. When then m, m', ... represent orders of coincidence corresponding to the sth cycle arranged in increasing order of magnitude the smallest possible corresponding sth partial trace orders k, k', ... also find themselves arranged in increasing order of magnitude. Furthermore, we have $m\leqq k$, $m'\leqq k'$, ... since, as has already been pointed out, the order of coincidence of any number of the corpus with a given cycle can never be greater than the partial trace order of the number corresponding to the same cycle.

It is evident that any integer, positive negative or 0 may be the sth p-adic partial trace order of a number of the corpus. For example, the sth p-adic partial trace of the number 1 is evidently n_s the degree of the sth cycle. Writing $n_s=p^{k_s}n_s'$ where p^{k_s} is the highest power pf p contained as factor in n_s, the sth p-adic partial trace order of 1 is k_s and the sth p-adic partial trace order of p^{-k_s} is evidently 0. The sth p-adic partial trace order of $p^{\lambda-k_s}$ is λ where λ is any integer positive or negative.

Any number of the corpus whose sth p-adic partial trace order is 0 must have with the sth cycle an order of coincidence which is 0 or negative. Indicate by $-m_s$ the greatest order of coincidence with the sth cycle which can be possessed by a number of the corpus whose sth p-adic partial trace order is 0. It is plain that 0 is at the same time the smallest sth p-adic partial trace order which is compatible with the sth p-adic order of coincidence $-m_s$. For, if a number ϵ should combine the sth p-adic order of coincidence $-m_s$ with the negative sth p-adic partial trace order $-k$ the number $p^k\epsilon$ would combine the sth p-adic order of coincidence $-m_s+k$ with the sth p-adic partial trace order 0, whereas $-m_s$ is the greatest sth p-adic order of coincidence which is compatible with the sth p-adic partial trace order 0. The number m_s is of course an integral multiple of $1/\nu_s$. Any number of the corpus whose sth p-adic order of coincidence is greater than $-m_s$ must evidently have an sth p-adic partial trace order which is positive. If the number ϵ has $-m_s$ as sth p-adic order of coincidence the sth p-adic order of coincidence of $p^{-1}\epsilon$ is $-m_s-1$ and the sth p-adic partial trace order of $p^{-1}\epsilon$ is obtained on subtracting 1 from the sth p-adic partial trace order of ϵ. It is then evident that the least sth p-adic partial trace order compatible with the sth p-adic order of coincidence $-m_s-1$ is -1 and that also $-m_s-1$ is the greatest sth p-adic order of coincidence compatible with the sth p-adic trace order -1.

Now the smallest sth p-adic partial trace order which is compatible with one of the sth p-adic orders of coincidence $-m_s-1/\nu_s, -m_s-2/\nu_s, \ldots, -m_s-(\nu_s-1)/\nu_s$ cannot be greater than the smallest sth p-adic partial trace order compatible with the sth p-adic order of coincidence $-m_s$ nor less than the smallest sth p-adic partial trace order compatible with the sth p-adic order of coincidence $-m_s-1$. The smallest sth p-adic partial trace order which is compatible with one of the sth p-adic orders of coincidence $-m_s-1/\nu_s, -m_s-2/\nu_s, \ldots, -m_s-(\nu_s-1)/\nu_s$ must consequently have the value 0 or the value -1. It cannot have the latter value, however, since $-m_s-1$ is the greatest sth p-adic order of coincidence compatible with the sth p-adic partial trace order -1. We conclude that 0 is the least possible sth p-adic partial trace order compatible with any one of the sth p-adic orders of coincidence

$$-m_s, \; -m_s-1/\nu_s, \; \ldots, \; -m_s-1+1/\nu_s. \tag{6}$$

The least sth p-adic partial trace order compatible with a given sth p-adic order of coincidence τ_s other than one of the ν_s numbers in (6) is evidently positive or negative according as τ_s is greater than the greatest or less than the least of these ν_s numbers. The least sth p-adic partial trace order in question is in fact given by t_s when τ_s is written in the form $t_s-m_s-\lambda_s/\nu_s$ where t_s is an integer and λ_s has one of the values $0, 1, \ldots, (\nu_s-1)$.

From what precedes we see that the sth p-adic partial trace of a number of the corpus must be p-adically integral if the sth p-adic order of coincidence of the number is $\geqq -m_s-1+1/\nu_s$ and that for any smaller sth p-adic order of coincidence there exist numbers of the corpus whose sth p-adic partial traces are not p-adically integral. Otherwise said the sth p-adic partial trace of the general number $\rho(\epsilon)$ of the corpus conditioned by the sth p-adic order of coincidence τ_s is or is not integral according as τ_s is not or is less than the number $-m_s-1+1/\nu_s$. We may also say that $\tau_s \geqq -m_s-1+1/\nu_s$ is the necessary and sufficient condition in order that the sth p-adic partial trace may be p-adically integral in the case of every number of the corpus whose sth p-adic order of coincidence is τ_s.

As a general rule the number m_s has the value 0. To see this we simply have to note that the sth p-adic partial trace of the number 1 is n_s, and the sth p-adic trace order of 1 is therefore 0 so long as n_s is not divisible by p. The sth p-adic order of coincidence of 1 is of course 0. So long then as n_s is not divisible by p the sth p-adic order of coincidence 0 is compatible with the sth p-adic partial trace order 0. We therefore have $m_s=0$ so long as n_s is not divisible by p

and 0 is in this case the least sth p-adic partial trace order compatible with any one of the sth p-adic orders of coincidence $0, \ -1/\nu_s, \ -2/\nu_s, \ \ldots, \ -1+1/\nu_s$.

VI

Suppose $R(\epsilon)$ to be any number of the corpus possessing a certain set of orders of coincidence $\tau_1, \tau_2, \ldots, \tau_r$ with the r p-adic cycles. The orders of coincidence of the number $f'(\epsilon)$ in particular with these cycles we shall designate by the symbols $\mu_1, \mu_2, \ldots, \mu_r$. We say of a set of orders of coincidence $\tau_1, \tau_2, \ldots, \tau_r$ that it is adjoint relatively to the equation $f(x)=0$ when its members satisfy the inequalities

$$(1) \qquad \tau_s \geqq \mu_s - m_s - 1 + 1/\nu_s, \quad (s=1, 2, \ldots, r).$$

Also in this case we say that the number $R(\epsilon)$ is adjoint relatively to the equation $f(x)=0$ for the prime p.

Where $\rho(\epsilon)$ is a number of the corpus connected with $R(\epsilon)$ by the relation $\rho(\epsilon)f'(\epsilon)=R(\epsilon)$ the orders of coincidence $\sigma_1, \sigma_2, \ldots, \sigma_r$ of $\rho(\epsilon)$ are connected with the orders of coincidence $\tau_1, \tau_2, \ldots, \tau_r$ of $R(\epsilon)$ by the relations

$$(2) \qquad \sigma_s + \mu_s = \tau_s, \quad (s=1, 2, \ldots, r).$$

If $R(\epsilon)$ is p-adically adjoint relatively to the equation $f(x)=0$ it is readily seen that its principal coefficient must be p-adically integral. For the orders of coincidence of $\rho(\epsilon)$ will in this case satisfy the inequalities

$$(3) \qquad \sigma_s \geqq -m_s - 1 + 1/\nu_s, \quad (s=1, 2, \ldots, r),$$

and the p-adic partial traces of $\rho(\epsilon)$ will therefore all be p-adically integral. This means, however, that the partial principal coefficients of $R(\lambda)$ as represented in the form (2), Section V, must all be integral. For the congruence (3), Section V, here holds good and the polynomials $\rho_s(x)$ are determined by the congruences (5), Section V. The total principal coefficient of $R(x)$ being equal to the sum of its partial principal coefficients must then also be p-adically integral. If, therefore, a number $R(\epsilon)$ is p-adically adjoint, its principal coefficient must be p-adically integral.

If on the other hand the orders of coincidence $\tau_1, \tau_2, \ldots, \tau_r$ are not all adjoint the principal coefficient in the general number $R(\epsilon)$ of the corpus possessing these orders of coincidence is not p-adically integral. To show this we note first that in this case there will be among the orders of coincidence $\sigma_1, \sigma_2, \ldots, \sigma_r$ of the number $\rho(\epsilon)$ ones which satisfy the inequality

$$\sigma_s < -m_s - 1 + 1/\nu_s.$$

With an order of coincidence σ_s which satisfies this inequality, however, a corresponding negative p-adic partial trace order is compatible. With the set of orders of coincidence $\sigma_1, \sigma_2, \ldots, \sigma_r$ then may be associated a set of p-adic partial trace orders including one or more which are negative. Select a number $\rho(\epsilon)$ of the corpus possessing the orders of coincidence $\sigma_1, \sigma_2, \ldots, \sigma_r$ connected with $\tau_1, \tau_2, \ldots, \tau_r$ by the relations (2), Section 6, the p-adic partial trace orders of $\rho(\epsilon)$ including among them one at least which is negative. Determine the number

$R(\epsilon)$ such that $\rho(\epsilon)f'(\epsilon)=R(\epsilon)$. The orders of coincidence of $R(\epsilon)$ are then $\tau_1, \tau_2, \ldots, \tau_r$. The polynomial $R(x)$ may then be represented in the form (2), Section V, where the p-adic polynomials $\rho_s(x)$ are determined by the congruences (5), Section V.

Since the p-adic partial traces of $\rho(\epsilon)$ are not all integral the same is true of the partial principal coefficients of $R(x)$ corresponding to the several elements of the summation in (2), Section V. Though the partial principal coefficients of $R(x)$ are not all p-adically integral it might happen through cancellation that the principal coefficient of $R(x)$ is p-adically integral. In such case we choose integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ which are not divisible by p and construct the p-adic polynomial

$$\overline{R}(x)=\sum_{s=1}^{r}\lambda_s\rho_s(x)f_1(x)\ldots f_{s-1}(x)f_{s+1}(x)\ldots f_r(x).$$

The partial principal coefficients of $\overline{R}(x)$ are not all p-adically integral and assuming the integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ to be properly chosen the principal coefficient of $\overline{R}(x)$ will evidently not be integral. Through the polynomial $R(x)$ the polynomial $f(x)=f_1(x)\ldots f_r(x)$ goes over into the polynomial $F(X)=F_1(X)\ldots F_r(X)$ the factor $f_s(x)$ going over into the factor $F_s(X)$. The factor $f_s(x)$ then also goes over into the factor $F_s(X)$ through the polynomial $\rho_s(x)f_1(x)\ldots f_{s-1}(x)f_{s+1}(x)\ldots f_r(x)$. Through the polynomial $\overline{R}(x)$ the polynomial $f(x)$ goes over into a polynomial $\overline{F}(X)=\overline{F}_1(X)\ldots\overline{F}_r(X)$ the factor $f_s(x)$ going over into the factor $\overline{F}_s(X)$. The factor $f_s(x)$ evidently also goes over into the factor $\overline{F}_s(X)$ through the polynomial $\lambda_s\rho_s(x)f_1(x)\ldots f_{s-1}(x)f_{s+1}(x)\ldots f_r(x)$. We see then that the partial trace term in $\overline{F}_s(X)$ is obtained on multiplying the partial trace term in $F_s(X)$ by λ_s while the partial norm term in $\overline{F}_s(X)$ is obtained on multiplying the partial norm term in $F_s(X)$ by $\lambda_s^{n_s}$. The order numbers of the partial trace terms in $F_s(X)$ and $\overline{F}_s(X)$ are then the same and the order numbers of their norm terms are also the same. This holds for $s=1, 2, \ldots, r$.

If therefore the coefficients of $\overline{R}(x)$ happen to be rational numbers the number $\overline{R}(\epsilon)$ is a number of the corpus in which the principal coefficient is not p-adically integral, the orders of coincidence and the partial trace orders of $\overline{R}(\epsilon)$ being at the same time the same as those of the number $R(\epsilon)$. If, however, the rational p-adic coefficients of $\overline{R}(x)$ are not rational numbers, on discarding terms of higher power in p in these coefficients we obtain a polynomial $\overline{\overline{R}}(x)$ in which the coefficients are rational numbers, the principal coefficient being p-adically fractional. At the same time if through $\overline{\overline{R}}(x)$ the polynomial $f_s(x)$ goes over into $\overline{\overline{F}}_s(X)$ the order number of the trace term in $\overline{\overline{F}}_s(X)$ will be the same as that of the trace term in $\overline{F}_s(X)$ and therefore the same as the order number of the trace term in $F_s(X)$ and the order number of the norm term in $\overline{\overline{F}}_s(X)$ will be the same as that of the norm term in $\overline{F}_s(X)$ and therefore the same as the order number of the norm term in $F_s(X)$ and this holds for $s=1, 2, \ldots, r$, it being assumed that the terms discarded are of sufficiently high power in p. The numbers $R(\epsilon)$ and $\overline{\overline{R}}(\epsilon)$ then have the same set of p-adic orders of coincidence and the same set of p-adic partial trace orders. Also the principal coefficient of $\overline{\overline{R}}(\epsilon)$ is p-adically fractional.

When then $\tau_1, \tau_2, \ldots, \tau_r$ constitute a set of p-adic orders of coincidence which is not adjoint there exist in the corpus numbers possessing such set of orders of coincidence in which the principal coefficient is p-adically fractional. The argument shows too that any set of finite trace orders compatible with the orders of coincidence in question is also at the same time compatible with the p-adically fractional character of the principal coefficient.

The principal result reached in the preceding argument may be stated in the form of a theorem thus: Adjointness relative to the equation $f(x)=0$ on the part of a set of p-adic orders of coincidence $\tau_1, \tau_2, \ldots, \tau_r$ constitutes the necessary and sufficient condition in order that every number of the corpus possessing this set of orders of coincidence may have a p-adically integral principal coefficient when represented as a polynomial of degree $n-1$ in ϵ.

We may readily obtain corresponding results with regard to the p-adic character of coefficients other than the principal coefficient in the polynomial expressions in terms of ϵ of the numbers of the corpus conditioned by a given set of p-adic orders of coincidence. The expression for a number $R(\epsilon)$ can be written in the form

$$(4) \qquad R(\epsilon)=\epsilon^{q+1}(a_{n-1}\epsilon^{n-q-2}+a_{n-2}\epsilon^{n-q-3}+\ldots+a_{q+1})+a_q\epsilon^q+\ldots+a_0.$$

The condition $f(\epsilon)=0$ can evidently be written in the form

$$(5) \quad \epsilon^{n-q-1}-a_{n-1}\epsilon^{n-q-2}+\ldots\mp a_{q+2}\epsilon\pm a_{q+1}=\epsilon^{-q-1}(\pm a_q\epsilon^q\mp a_{q-1}\epsilon^{q-1}\pm\ldots-(-1)^n a_0).$$

Writing

$$(6) \qquad g_q(\epsilon)=\epsilon^{n-q-1}-a_{n-1}\epsilon^{n-q-2}+\ldots\pm a_{q+2}\epsilon\pm a_{q+1}$$

and employing alternately for $g_q(\epsilon)$ the equivalent forms given in (5) we have

(7)

$$\begin{aligned} g_q(\epsilon)R(\epsilon)=&(\pm a_q\epsilon^q\mp a_{q-1}\epsilon^{q-1}\pm\ldots-(-1)^n a_0)(a_{n-1}\epsilon^{n-q-2}+a_{n-2}\epsilon^{n-q-3}+\ldots+a_{q+1}) \\ &+(\epsilon^{n-q-1}-a_{n-1}\epsilon^{n-q-2}+\ldots\mp a_{q+2}\epsilon\pm a_{q+1})\,(a_q\epsilon^q+\ldots+a_0). \end{aligned}$$

The expression on the right-hand side of (7) gives the product $g_q(\epsilon)R(\epsilon)$ in its reduced form and the coefficient of ϵ^{n-1} in this expression is a_q the coefficient of ϵ^q in $R(\epsilon)$. The coefficient of ϵ^q in $R(\epsilon)$ then coincides with the principal coefficient of the number $g_q(\epsilon)R(\epsilon)$ and is therefore equal to the trace of $g_q(\epsilon)R(\epsilon)/f'(\epsilon)$.

The p-adic orders of coincidence of $R(\epsilon)$ we shall designate by $\tau_1, \tau_2, \ldots, \tau_r$ and those of $g_q(\epsilon)$ by $\delta_1^{(q)}, \delta_2^{(q)}, \ldots, \delta_r^{(q)}$. The p-adic orders of coincidence of the product $g_q(\epsilon)R(\epsilon)$ will then be $\tau_1+\delta_1^{(q)}, \ldots, \tau_r+\delta_r^{(q)}$. The necessary and sufficient condition that the coefficient of ϵ^q shall be p-adically integral in all numbers $R(\epsilon)$ of the corpus which possess the orders of coincidence $\tau_1, \ldots, \tau_r$ coincides with the necessary and sufficient condition that the principal coefficient shall be p-adically integral in all numbers $g_q(\epsilon)R(\epsilon)$ of the corpus which possess the p-adic orders of coincidence $\tau_1+\delta_1^{(q)}, \ldots, \tau_r+\delta_r^{(q)}$. Adjointness on the part of these orders of coincidence is therefore evidently the necessary and sufficient condition in order that the principal coefficient may be p-adically integral in every number $g_q(\epsilon)R(\epsilon)$ which possesses these orders of coincidence. Adjointness on the part of the orders of coincidence $\tau_1+\delta_1^{(q)}, \ldots, \tau_r+\delta_r^{(q)}$ is then the necessary and sufficient condition in order that the coefficient of ϵ^q may be p-adically integral in the reduced forms of all numbers $R(\epsilon)$ of the corpus which possess the p-adic orders of coincidence $\tau_1, \ldots, \tau_r$.

The necessary and sufficient conditions that the coefficients of ϵ^q be p-adically integral in all numbers of the corpus, represented in their reduced forms, which possess the orders of coincidence $\tau_1, \ldots, \tau_r$ are then given by the inequalities

$$\tau_s \geqq \mu_s - \delta_s^{(q)} - m_s - 1 + 1/\nu_s, \ (s=1, 2, \ldots, r). \tag{8}$$

The statement just made holds for $q=0, 1, \ldots, n-1$. For $q=n-1$ we have $\delta_s^{n-1}=0$; $s=1, 2, \ldots, r$.

Among the numbers of the corpus represented as reduced polynomials in ϵ and possessing a given set of p-adic orders of coincidence $\tau_1, \ldots, \tau_r$ there is a lowest possible order number for the coefficient of ϵ^q. This lowest possible order number is evidently $-\iota_q$ where ι_q is the smallest integer compatible with the system of inequalities

$$\tau_s + \iota_q \geqq \mu_s - \delta_s^{(q)} - m_s - 1 + 1/\nu_s, \ (s=1, 2, \ldots, r). \tag{9}$$

Here ι_q may happen to be positive, negative or 0. In particular, the smallest order number of the principal coefficient compatible with the orders of coincidence in question is $-\iota_{n-1}$ where ι_{n-1} is the smallest integer compatible with the system of inequalities

$$\tau_s + \iota_{n-1} \geqq \mu_s - m_s - 1 + 1/\nu_s, \ (s=1, 2, \ldots, r). \tag{10}$$

If the coefficients in the equation $f(x)=0$ are p-adically integral, it is evident that all the coefficients must be p-adically integral in a number $R(\epsilon)$ whose p-adic orders of coincidence are adjoint, for in the case where the equation is p-adically integral there are no negative numbers included in the sets $\delta_1^{(q)}, \ldots, \delta_r^{(q)}$; $q=0, 1, \ldots, n-1$. Also if the coefficients in a number $R(\epsilon)$ are all p-adically integral, the p-adic orders of coincidence of the number are $\geqq 0$.

We might here derive certain results which we shall have occasion to utilize a little later on. We shall assume, for the time being, that the equation $f(x)=0$ is p-adically integral. Consider a set of finite order numbers $\tau_1, \ldots, \tau_r$ corresponding to the prime p. A specific polynomial in ϵ with p-adic coefficients may, or may not, be conditioned by this set of order numbers. In any case there exists a power p^β with the least integral exponent, positive, zero or negative, such that the product of the polynomial by this power is conditioned by the set of order numbers here in question. This least exponent we call the *index* of the polynomial for the set of order numbers $\tau_1, \ldots, \tau_r$.

A polynomial in ϵ with integral p-adic coefficients we call *primitive* when the coefficient of the highest power of ϵ is not divisible by p. Still having reference to the order numbers $\tau_1, \ldots, \tau_r$ we shall designate by $\beta_{n-1}, \ldots, \beta_0$ the smallest possible indices which can belong to primitive polynomials of degrees $n-1, \ldots, 0$ respectively. It is evident that we have $\beta_{n-1} \gtreqless \beta_{n-2} \gtreqless \ldots \gtreqless \beta_0$.

Construct a set of n polynomials

$$p^{\beta_s} Q_s(\epsilon), \ (s=0, 1, \ldots, n-1), \tag{11}$$

where by $Q_s(\epsilon)$ we designate a primitive polynomial of degree s with β_s as index.

It is readily seen that the general p-adic polynomial in ϵ which is conditioned by the set of order numbers $\tau_1, \ldots, \tau_r$ can be represented in the form

$$\sum_{s=0}^{n-1} ((p))_s p^{\beta_s} Q_s(\epsilon) \tag{12}$$

where $((p))_s$ designates an arbitrary integral p-adic number, the suffix s being employed to indicate that the arbitrary coefficients in the several p-adic series are independent of one another. To see this it suffices to prove that the index of a non-primitive polynomial of degree $s=0, 1, \ldots, n-1$ whose coefficients are not all divisible by p cannot be less than β_s.

It is evident, on the face of it, that the index of a polynomial in which some of the coefficients are p-adically non-integral is greater than the index of certain polynomials of the same degree whose coefficients are p-ically integral and not all divisible by p. Again, any non-primitive integral polynomial of degree s will be congruent (mod p) to a primitive polynomial of lower degree t and can, therefore, on division by $Q_t(\epsilon)$ evidently be represented in the form $pQ_t(\epsilon)r(\epsilon)+q_t(\epsilon)$ where $r(\epsilon)$ and $q_t(\epsilon)$ are integral p-adic polynomials, the latter being primitive and of degree t. Here the index of $pQ_t(\epsilon)r(\epsilon)$ is less than β_t while the index of the primitive polynomial $q_t(\epsilon)$ cannot be less than β_t. The index of the sum of these two polynomials, and therewith the index of the non-primitive polynomial of degree s here in question, must, therefore be $\geqq \beta_t$, and by consequence $\geqq \beta_s$. The index of any p-adically integral non-primitive polynomial of degree s, whose coefficients are not all divisible by p, is therefore $\geqq \beta_s$.

The n polynomials (11) are said to constitute a basis for the representation of the p-adic polynomials in ϵ which are conditioned by the set of order numbers $\tau_1, \ldots, \tau_r$. In each of the polynomials $Q_s(\epsilon)$ we may evidently suppose the coefficient of the highest power of ϵ to be 1. We may plainly also assume that the coefficients of the several powers of ϵ are rational numbers which are integral relatively to the prime p. We may even assume that these coefficients are polynomials in p, that is to say, that they are integers. In the case where the coefficients here in question are rational numbers, the n polynomials (11) are said to constitute a basis for the representation of the numbers of the corpus $C(\epsilon)$, which are conditioned by the set of order numbers $\tau_1, \ldots, \tau_r$. Any number of the corpus can be represented in the form

$$\sum_{s=0}^{n-1} c_s p^{\beta_s} Q_s(\epsilon) \tag{13}$$

where the coefficients c_s are rational numbers, these coefficients being p-adically integral in the case of numbers of the corpus which are conditioned by the set of p-adic order numbers $\tau_1, \ldots, \tau_r$.

Consider the general number of the form

$$R(\epsilon) = \sum_{s=1}^{n} \sum_{t=-i+1}^{i} a_{t-1,\, s-1} p^{t-1} \epsilon^{s-1} \tag{14}$$

conditioned by the orders of coincidence $\tau_1, \ldots, \tau_r$ where the coefficients $a_{t-1,\, s-1}$ are supposed to have values included among the numbers $0, 1, \ldots, p-1$ and where i is assumed to be at least so great that the numbers $\beta_{n-1}, \ldots, \beta_0$ are included in the range $-i$ to i. We then have $\beta_{n-1} \geqq -i$, $\beta_0 \leqq i$. The number of the coefficients $\alpha_{t-1,\,s-1}$ which are arbitrary is

$$d = ni - \sum_{s=0}^{n-1} \beta_s. \tag{15}$$

To see this it is only necessary to compare the form (14) with the form (13) on writing

$$R(\epsilon) \equiv \sum_{s=0}^{n-1} c_s p^{\beta_s} Q_s(\epsilon) \pmod{p^i}, \tag{16}$$

when it is evident, from the right hand side of the congruence, that in the p-adic multipliers of the powers ϵ^{n-1}, ϵ^{n-2}, ... considered in this order the powers of p which have independently arbitrary coefficients number $i-\beta_{n-1}$, $i-\beta_{n-2}$, ... respectively. It is here to be understood that the multipliers c_s are represented as polynomials in p with arbitrary coefficients, that is to say, with coefficients which are arbitrarily selected from among the numbers 0, 1, ..., $p-1$. If we assign specific values to the d arbitrary coefficients in the p-adic multipliers of the powers of ϵ in $R(\epsilon)$ we evidently determine the coefficients of powers of p which are lower than p^i in the multipliers $c_s\, p^{\beta_s}$, on the right hand side of (16) and therewith determine $R(\epsilon)$ (mod p^i) and consequently determine $R(\epsilon)$ completely. When we assign specific values to the d arbitrary coefficients in $R(\epsilon)$ therefore the remaining coefficients are determined.

VII

We say of two sets of orders of coincidence $\tau_1, \ldots, \tau_r$ and $\bar{\tau}_1, \ldots, \bar{\tau}_r$ corresponding to a prime p that they are complementary adjoint to each other, if they satisfy the inequalities

$$\tau_s + \bar{\tau}_s \geqq \mu_s - m_s - 1 + 1/\nu_s, \quad (s = 1, 2, \ldots, r). \tag{1}$$

When they satisfy the inequalities

$$\tau_s + \bar{\tau}_s \geqq i + \mu_s - m_s - 1 + 1/\nu_s, \quad (s = 1, 2, \ldots, r), \tag{2}$$

where i is an integer, they are said to be complementary adjoint to the order i.

If the orders of coincidence relative to the prime p of two numbers of the corpus $C(\epsilon)$ are complementary adjoint, the principal coefficient in their product is evidently integral relatively to p. Conversely, if the principal coefficient is integral relatively to the prime p in the product $\bar{R}(\epsilon)R(\epsilon)$ where $\bar{R}(\epsilon)$ is the general number of the corpus possessing the p-adic orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ the p-adic orders of councidence of $R(\epsilon)$ must be complementary adjoint to these orders of coincidence. For suppose we had a specific number $R(\epsilon)$ whose p-adic orders of coincidence $\tau_1, \ldots, \tau_r$ do not constitute a set which is complementary adjoint to the set $\bar{\tau}_1, \ldots, \bar{\tau}_r$. The set of p-adic orders of coincidence $\tau_1 + \bar{\tau}_1, \ldots, \tau_r + \bar{\tau}_r$ is then not adjoint. We can consequently find a number $R_1(\epsilon)$ which possesses precisely this non-adjoint set of orders of coincidence, and in which the principal coefficient is not p-adically integral. The number $R_1(\epsilon)/R(\epsilon)$ possesses the set of p-adic orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ and its product by $R(\epsilon)$ is the number $R_1(\epsilon)$ whose principal coefficient is not p-adically integral.

Where then $R(\epsilon)$ is a number whose set of p-adic orders of coincidence is not complementary adjoint to the set $\bar{\tau}_1, \ldots, \bar{\tau}_r$ the principal coefficient in the product $\overline{R}(\epsilon)R(\epsilon)$ is not p-adically integral where $\overline{R}(\epsilon)$ represents the general number of the corpus possessing the set of p-adic orders of coincidence $\tau_1, \ldots, \bar{\tau}_r$. We conclude, therefore, that in order that the set of p-adic orders of coincidence of a number $R(\epsilon)$ may be complementary adjoint to a given set of p-adic orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ it is necessary and it suffices that the principal coefficient in the product $\overline{R}(\epsilon)R(\epsilon)$ be p-adically integral where $\overline{R}(\epsilon)$ is the general number of the corpus which possesses the set of p-adic orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$.

In the theorem just proved $\overline{R}(\epsilon)$ is supposed to include just those numbers of the corpus which actually possess $\bar{\tau}_1, \ldots, \bar{\tau}_r$ as their p-adic orders of coincidence. No new condition, however, is imposed on $R(\epsilon)$ by supposing in the theorem $\overline{R}(\epsilon)$ to include all those numbers of the corpus none of whose p-adic orders of coincidence falls short of the corresponding order of coincidence in the set $\bar{\tau}_1, \ldots, \bar{\tau}_r$. For p-adic integral character on the part of the principal coefficient in the product $\overline{R}(\epsilon)R(\epsilon)$ where $\overline{R}(\epsilon)$ covers the narrower range of numbers contemplated in the statement of the theorem above, already suffices to ensure that the p-adic orders of coincidence of $R(\epsilon)$ are complementary adjoint to the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$, and therefore also to the orders of coincidence of all the numbers included under $\overline{R}(\epsilon)$ in its wider range, and this brings with it p-adic integral character on the part of the principal coefficient in the product $\overline{R}(\epsilon)R(\epsilon)$ where $\overline{R}(\epsilon)$ covers the wider range of numbers.

Another form of statement of our theorem which we shall find convenient in application is the following:

The necessary and sufficient conditions in order that $R(\epsilon)$ may have orders of coincidence which are complementary adjoint to the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$, are given by the vanishing of the principal residue* in the product $\overline{R}(\epsilon)R(\epsilon)$ where $\overline{R}(\epsilon)$ is the general number of the corpus conditioned by the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$.

To recognize our theorem under this form of statement it is only necessary to note that if $d_{-k}p^{-k}$ appears as lowest term in the principal coefficient of the product $\overline{R}(\epsilon)R(\epsilon)$, we have d_{-k} as principal residue in the product $p^{k-1}\overline{R}(\epsilon)R(\epsilon)$ and $p^{k-1}\overline{R}(\epsilon)$ is evidently included under the general number $\overline{R}(\epsilon)$ conditioned by the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$. Vanishing of the principal residue in the product $\overline{R}(\epsilon)R(\epsilon)$, where $\overline{R}(\epsilon)$ is the general number of the corpus conditioned by the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$, then brings with it the vanishing of the fractional part of the principal coefficient in the product.

Let us consider any number of the corpus

$$R(\epsilon) = \alpha_{n-1}\epsilon^{n-1} + \ldots + \alpha_0. \tag{3}$$

The coefficients $\alpha_{n-1}, \ldots, \alpha_0$ are rational numbers as yet undetermined. We shall suppose them to be represented as p-adic series. The coefficients $\alpha_{n-1}, \ldots, \alpha_0$ regarded as p-adic series are then subject only to the condition that they

*By the *principal residue* relative to a prime p of a polynomial $R(\epsilon)$ is to be understood the coefficient of p^{-1} in the reduced p-adic representation of the principal coefficient in $R(\epsilon)$.

represent rational numbers. Among these series we shall confine ourselves to those in which no powers of p lower than p^{-i} present themselves. Here i may be any assigned integer. In each one of the p-adic series $\alpha_{n-1}, \ldots, \alpha_0$ we may assume, to begin with, that any finite number of the coefficients are arbitrary.

If now we would require $R(\epsilon)$ to have a certain set of p-adic orders of coincidence $\tau_1, \ldots, \tau_r$ the conditions thereby imposed on the coefficients in the p-adic series $\alpha_{n-1}, \ldots, \alpha_0$ would be obtained on forcing the principal coefficient in the product $\overline{R}(\epsilon)R(\epsilon)$ to be integral with regard to p where $\overline{R}(\epsilon)$ is the general number of the corpus conditioned by the set of orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ which is complementary to the set $\tau_1, \ldots, \tau_r$. The number of the coefficients in the p-adic series $\alpha_{n-1}, \ldots, \alpha_0$ affected by the assigned set of orders of coincidence $\tau_1, \ldots, \tau_r$ is evidently finite. The conditions on these coefficients will all be expressed by the aid of congruences with p and powers of p as moduli, the congruences being linear in the coefficients in question. To see this, we simply have to take a succession of specific numbers $\overline{R}_1(\epsilon), \overline{R}_2(\epsilon), \ldots$ included under $\overline{R}(\epsilon)$ so that on forcing the principal coefficients to be integral in the products

$$\overline{R}_1(\epsilon)R(\epsilon),\ \overline{R}_2(\epsilon)R(\epsilon), \ldots\ldots$$

we successively impose further conditions on the coefficients in the p-adic series $\alpha_{n-1}, \ldots, \alpha_0$. That these conditions are in the nature of congruences which are linear in the coefficients in question, with powers of p as moduli, is self-evident. In order that these conditions may include all those implied in forcing the principal coefficient to be integral in the product $\overline{R}(\epsilon)R(\epsilon)$ it is necessary to take only a finite number of numbers $\overline{R}_1(\epsilon), \overline{R}_2(\epsilon), \ldots$, for the totality of different congruence conditions with powers of p as moduli, which a finite number of coefficients in the p-adic series $\alpha_{n-1}, \ldots, \alpha_0$ can support is finite. The requirement, then, that $R(\epsilon)$ should have a set of p-adic orders of coincidence, none of which fall short of the corresponding number in the set $\tau_1, \ldots, \tau_r$, imposes on the coefficients in the p-adic series $\alpha_{n-1}, \ldots, \alpha_0$ a finite number of conditions, all of which are expressible as congruences with powers of p as moduli.

VIII

As in the latter part of Section VI, so here again we shall find it convenient to assume that the equation $f(x)=0$ is integral relatively to the prime p. In formula (13), Section VI, we have given a form of representation for the general number of the corpus conditioned by the orders of coincidence $\tau_1, \ldots, \tau_r$. For the general number of the corpus conditioned by the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ we shall have the representation

$$\overline{R}(\epsilon) = \sum_{s=0}^{n-1} \bar{c}_s p^{\bar{\beta}_s} \overline{Q}_s(\epsilon) \tag{1}$$

where the polynomials $\overline{Q}_s(\epsilon)$ are all primitive, each one being of the degree indicated by its suffix and possessing the minimum index relative to the set of p-adic order numbers $\bar{\tau}_1, \ldots, \bar{\tau}_r$ for a primitive polynomial of such degree. The multipliers $\bar{c}_s$ are arbitrary rational numbers which are p-adically integral and each of the exponents $\bar{\beta}_s$ is the minimum index for a p-adically primitive polynomial of the degree indicated by its suffix. Here, too, we may take $\bar{Q}_0(\epsilon)=1$.

The general number of the corpus conditioned by the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ and represented in the form (1) can also be written in the more detailed form

$$\bar{R}(\epsilon)=\sum_{s=0}^{n-1}\sum_{t=\bar{\beta}_s}^{\infty}\bar{c}_{s,t}p^t\bar{Q}_s(\epsilon), \tag{2}$$

where the coefficients $\bar{c}_{s,t}$ are arbitrary, that is, where they may have any of the values 0, 1, ..., $p-1$, subject however to the condition that the p-adic series

$$\sum_{t=\bar{\beta}_s}^{\infty}\bar{c}_{s,t}p^t, \quad (s=0, 1, \ldots, n-1), \tag{3}$$

must represent rational numbers.

Consider the product

$$\bar{R}(\epsilon)\sum_{s=0}^{n-1}\sum_{t=-i}^{i-1}\alpha_{-t-1,\,n-s-1}p^{-t-1}\epsilon^{n-s-1}, \tag{4}$$

where the $2ni$ coefficients $\alpha_{-t-1,\,n-s-1}$ in the second factor are in the first place supposed to be arbitrary. On subjecting these coefficients to the conditions involved in equating to 0 the principal residue relative to p in the product we impose on the second factor the orders of coincidence $\tau_1, \ldots, \tau_r$. Now the product of the second factor in (4) by p^i or any higher power of p is evidently integral in form and the conditions imposed on the coefficients $\alpha_{-t-1,\,n-s-1}$ are all obtained when we equate to 0 the principal residue in the product

$$\sum_{s=0}^{n-1}\sum_{t=\bar{\beta}_s}^{i-1}\bar{c}_{s,t}\,p^t\bar{Q}_s(\epsilon)\sum_{s=0}^{n-1}\sum_{t=-i}^{i-1}\alpha_{-t-1,\,n-s-1}\,p^{-t-1}\epsilon^{n-s-1} \tag{5}$$

where the coefficients $\bar{c}_{st}$, are arbitrary. The number of these arbitrary coefficients is

$$\bar{d}=ni-\sum_{s=0}^{n-1}\bar{\beta}_s. \tag{6}$$

The conditions imposed on the second factor in the product (5) by the orders of coincidence $\tau_1, \ldots, \tau_r$ are then obtained on equating to 0 the principal residue in this product. This is, however, equivalent to equating to 0 the principal residues in the $\bar{d}$ products

$$p^\tau\bar{Q}_\sigma(\epsilon)\sum_{s=0}^{n-1}\sum_{t=-i}^{i-1}\alpha_{-t-1,\,n-s-1}p^{-t-1}\epsilon^{n-s-1}, \quad (\tau=\bar{\beta}_\sigma, \ldots, i-1;\ \sigma=0,1,\ldots,n-1), \tag{7}$$

as is readily seen. For the conditions obtained on equating to 0 the principal residues in the products (7) are evidently included under the conditions obtained on equating to 0 the principal residue in the product (5), and on equating to 0 the principal residues in the products (7) we shall show that the principal coefficients in these products must, as a consequence, be integral, and that therefore also the principal coefficients in the product (5) must be integral.

From among the products (7) select for consideration the $i-\overline{k}_\sigma$ products corresponding to a given value σ. Assign to τ in succession the values $i-1, \ldots, \overline{k}_\sigma$. In the product

$$p^{i-1}\overline{Q}_\sigma(\epsilon)\sum_{s=0}^{n-1}\sum_{t=-i}^{i-1}\alpha_{-t-1,\,n-s-1}p^{-t-1}\epsilon^{n-s-1} \tag{8}$$

evidently no power of p lower than p^{-1} presents itself. Equate to 0 the principal residue in this product. This is equivalent to satisfying a congruence of the form

$$\alpha_{-i,\,n-\sigma-1}+A_{-i,\,n-\sigma-1}\equiv 0 \pmod{p} \tag{9}$$

where $A_{-i,\,n-\sigma-1}$ is an expression linear in coefficients $\alpha_{-t-1,\,n-s-1}$ in which the second suffix $n-s-1$ is greater than $n-\sigma-1$. The product

$$p^{i-2}\overline{Q}_\sigma(\epsilon)\sum_{s=0}^{n-1}\sum_{t=-i}^{i-1}\alpha_{-t-1,\,n-s-1}p^{-t-1}\epsilon^{n-s-1} \tag{10}$$

will evidently now present no power of p lower than p^{-1}. Equate to 0 the principal residue in this product. This is equivalent to satisfying a congruence of the form

$$\alpha_{-i+1,\,n-\sigma-1}+A_{-i+1,\,n-\sigma-1}\equiv 0 \pmod{p} \tag{11}$$

where $A_{-i+1,\,n-\sigma-1}$ is obtained on adding to $p^{-1}(\alpha_{-i,\,n-\sigma-1}+A_{-i,\,n-\sigma-1})$ an expression which is linear in coefficients $\alpha_{-t-1,\,n-s-1}$ in which the second suffix is greater than $n-\sigma-1$. So proceeding, on equating to 0 in succession the principal residues in the $i-\overline{\beta}_\sigma$ products of the system (7) which correspond to the value σ we obtain the $i-\overline{\beta}_\sigma$ congruences

$$\alpha_{-\tau,\,n-\sigma-1}+A_{-\tau,\,n-\sigma-1}\equiv 0 \pmod{p},\ (\tau=i,\ i-1,\ \ldots,\ \overline{\beta}_\sigma+1), \tag{12}$$

where $A_{-\tau,\,n-\sigma-1}$ is obtained on adding to $p^{-1}(\alpha_{-\tau-1,\,n-\sigma-1}+A_{-\tau-1,\,n-\sigma-1})$ an expression linear in coefficients $\alpha_{-t-1,\,n-s-1}$ in which the second suffix is greater than $n-\sigma-1$. When the congruences (12) are satisfied, it is evident that the principal coefficients are integral in the $i-\overline{\beta}_\sigma$ products (7) which correspond to the value σ. We see, then, that the principal coefficients in all the products (7) must be integral when we equate to 0 the principal residues in all of these products. When we equate to 0 the principal residues in the products (7), then the principal coefficient in the product (5) must be integral, and we impose on the second factor of this product the orders of coincidence $\tau_1, \ldots, \tau_r$. The conditions imposed on the second factor in the product (5) by the orders of coincidence $\tau_1, \ldots, \tau_r$ are then furnished by the n sets of congruences of the type (12) constructed successively for the values $\sigma=0, 1, \ldots, n-1$.

The congruences (12) make the coefficients $\alpha_{-i,\,n-\sigma-1}, \ldots, \alpha_{-\overline{\beta}_\sigma-1,\,n-\sigma-1}$ ultimately depend for their values on coefficients $\alpha_{-t-1,\,n-s-1}$ in which the second suffix $n-s-1$ is greater than $n-\sigma-1$. For $\sigma=0$ the set of congruences (12) becomes

$$\alpha_{-\tau,\,n-1}\equiv 0 \pmod{p},\ (\tau=i,\ i-1,\ \ldots,\ \overline{\beta}_0+1), \tag{13}$$

and we therefore have $\alpha_{-\tau,\,n-1}=0$; $\tau=i,\ i-1,\ \ldots,\ \overline{\beta}_0$, since the coefficients

$a_{-t-1,\,n-s-1}$ are supposed to have their values chosen from among the numbers $0, 1, \ldots, p-1$. For $\sigma=1$ the set of congruences (12) becomes

$$a_{-\tau,\,n-2}+A_{-\tau,\,n-2}\equiv 0 \pmod{p},\ (\tau=i,\ _{i-1},\ \ldots,\ \bar{\beta}_1+1),$$

and through these congruences the coefficients $a_{-1,\,n-2}, \ldots, a_{-\bar{\beta}_1-1,\,n-2}$ ultimately depend for their values on coefficients $a_{-t-1,\,n-1}$ with second suffix $n-1$ among which $i-\bar{\beta}_0$ are 0 because of the congruences (13). Through the set of congruences (12) corresponding to the value $\sigma=2$ the coefficients $a_{-\tau,\,n-3}$ depend for their values on the values of coefficients $a_{-t-1,\,n-1}$ and $a_{-t-1,\,n-2}$ in which the second suffix has the value $n-1$ or $n-2$.

Through the sets of congruences (12) corresponding to the values $\sigma \doteq 0, 1, \ldots, n-1$ then we see that the $\bar{d}$ coefficients

$$a_{-\tau,\,n-\sigma-1},\ (\tau=i,\ i-1,\ \ldots,\ \bar{\beta}_\sigma+1;\ \sigma=0,\ 1,\ \ldots,\ n-1), \tag{14}$$

are successively determined in terms of the remaining $2ni-\bar{d}$ coefficients, which coefficients remain arbitrary. The set of orders of coincidence $\tau_1, \ldots, \tau_r$ then imposes on the coefficients $a_{-t-1,\,n-s-1}$ in the second factor of the product (4) just $\bar{d}$ independent congruence conditions (mod p), leaving $2ni-\bar{d}$ of these coefficients arbitrary.

Instead of representing $\bar{R}(\epsilon)$ in the form (2) suppose we represent it in the form

$$\bar{R}(\epsilon)=\sum_{s=0}^{n-1}\sum_{t=-i}^{\infty} a_{t,\,s}\,p^t\,\epsilon^s \tag{15}$$

where the values of the coefficients $a_{t,\,s}$ are selected from among the numbers $0, 1, \ldots, p-1$; these coefficients being otherwise subject only to the conditions implied in the assumption that the p-adic series

$$\sum_{t=-i}^{\infty} a_{t,\,s}\,p^t,\ (s=0,\ 1,\ \ldots,\ n-1), \tag{16}$$

represent rational numbers, and to the conditions imposed on them when the number $\bar{R}(\epsilon)$ is conditioned by the set of orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$.

In the form of representation of $\bar{R}(\epsilon)$ given in (1) the coefficient of ϵ^{n-1} is $\bar{c}_{n-1}\,p^{\bar{\beta}_{n-1}}$ which may represent any rational number whose order number is $\gtreqqless\bar{\beta}_{n-1}$. In (15) therefore, the coefficients $a_{t,\,n-1}$ for $t\gtreqqless\bar{\beta}_{n-1}$ are arbitrary, subject, of course, to the condition that the series (16) corresponding to the value $s=n-1$ represents a rational number. Whatever the choice of $\bar{c}_{n-1}\,p^{\bar{\beta}_{n-1}}$ may be we can by choice of $\bar{c}_{n-2}\,p^{\bar{\beta}_{n-2}}$ assign arbitrary values to the coefficients $a_{t,\,n-2}$ in (15) for $t\gtreqqless\bar{\beta}_{n-2}$. Whatever our choice of $\bar{c}_{n-1}\,p^{\bar{\beta}_{n-1}}$ and $\bar{c}_{n-2}\,p^{\bar{\beta}_{n-2}}$ may be, we can thereafter by choice of $\bar{c}_{n-3}\,p^{\bar{\beta}_{n-3}}$ assign arbitrary values to the coefficients $a_{t,\,n-3}$ in (15) for $t\gtreqqless\bar{\beta}_{n-3}$. So proceeding, we can evidently assign arbitrary values to the coefficients

$$a_{t,\,s},\ t\gtreqqless\bar{\beta}_s,\ (s=0,\ 1,\ \ldots,\ n-1), \tag{17}$$

in (15), subject only to the condition that the series (16) represent rational numbers. Furthermore, when these coefficients are determined the numbers $\bar{c}_{n-1}\,p^{\bar{\beta}_{n-1}}, \ldots, \bar{c}_0 p^{\bar{\beta}_0}$ are completely determined, and therewith the number $\bar{R}(\epsilon)$ as represented in (15). The number $\bar{R}(\epsilon)$ as represented in (15) and supposed to be conditioned by the set of orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ is then completely determined, when values are assigned to the system of numbers $a_{t,s}$ in (17). In the general number of the form

$$\sum_{s=0}^{n-1} \sum_{t=-i}^{i-1} a_{t,s}\,p^t \epsilon^s \tag{18}$$

conditioned by the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ therefore the coefficients

$$a_{t,s}, \ (t=\bar{\beta}_s, \ldots, i-1;\ s=0, 1, \ldots, n-1), \tag{19}$$

are arbitrary, the remaining coefficients being determined when values are assigned to these $\bar{d}=ni-\sum_{s=0}^{n-1}\bar{\beta}_s$ coefficients.

When arbitrary values are assigned to the $2ni$ coefficients $a_{-t-1,\,n-s-1}$ in the second factor of the product (4) this factor has precisely the same form as the double summation in formula (14), Section VI, when its coefficients $a_{t,s}$ are arbitrary. We have seen, however, that the set of orders of coincidence $\tau_1, \ldots, \tau_r$ imposes on the coefficients in the general expression of the form of the second factor in (4) precisely $\bar{d}$ independent congruence conditions (mod p), leaving therewith $2ni-\bar{d}$ of these coefficients arbitrary. On the coefficients in the form (18), therefore, the set of orders of coincidence $\tau_1, \ldots, \tau_r$ would impose $\bar{d}$ independent congruence conditions (mod p), leaving $2ni-\bar{d}$ arbitrary coefficients. Just as in the form (18) conditioned by the set of orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ there are $\bar{d}=ni-\sum_{s=0}^{n=1}\bar{\beta}_s$ arbitrary coefficients, so in the form (18) conditioned by the set of orders of coincidence $\tau_1, \ldots, \tau_r$ there must be $d=ni-\sum_{s=0}^{n-1}\beta_s$ arbitrary coefficients. For the number of these coefficients, however, we have obtained the expression $2ni-\bar{d}$. We, therefore, have

$$d+\bar{d}=2ni. \tag{20}$$

On substituting for d and $\bar{d}$ in this relation, the expressions $ni-\sum_{s=0}^{n-1}\beta_s$ and $ni-\sum_{s=0}^{n-1}\bar{\beta}_s$ we obtain

$$\sum_{s=0}^{n-1}\beta_s+\sum_{s=0}^{n-1}\bar{\beta}_s=0. \tag{21}$$

As $\bar{d}$ and d are the numbers of the independent congruence conditions (mod p), imposed on the coefficients of the general form (18) by the respective sets of orders of coincidence $\tau_1, \ldots, \tau_r$ and $\bar{\tau}_1, \ldots, \bar{\tau}_r$ so also will $\bar{d}$ and d evidently be the numbers of the independent congruence conditions (mod p) imposed by these respective sets of orders of coincidence on the coefficients $a_{t,s}$ in the general form on the right hand side of (15) when we suppose these coefficients, to commence with, to be subject only to the conditions implied in the assumption that the p-adic series in (16) represent rational numbers. Here, as throughout, it is of course assumed that the integer i has been taken sufficiently great. For the truth of the statement that the number of the independent congruence conditions (mod p) imposed on the coefficients in the general form on the right hand side of (15) by the set of orders of coincidence $\tau_1, \ldots, \tau_r$ is given by $\bar{d} = ni - \sum_{s=0}^{n-1} \bar{\beta}_s$ it evidently suffices that we have $-i \overline{\gtrless} \beta_{n-1}$. For among the $\bar{d} = ni - \sum_{s=0}^{n-1} \bar{\beta}_s$ conditions here in question will evidently be included the $n(i-i')$ conditions

$$(22) \qquad \alpha_{t,s} = 0, \ (t = -i, \ -i+1, \ \ldots, \ -i'-1; s = 0, 1, \ldots, n-1),$$

when we have $-i \overline{\gtrless} -i' \overline{\gtrless} \beta_{n-1}$.

For a representation of the form given in (15) we shall also find it convenient to employ the notation $p^{-i}((p,\epsilon))$. The notation $((p, \epsilon))$ then designates, for our purposes, a polynomial of degree $n-1$ in ϵ with coefficients which may be any rational numbers subject to the condition that they are integral relatively to the prime p.

IX

Suppose $\tau_1, \ldots, \tau_r$ and $\sigma_1, \ldots, \sigma_r$ to be any two sets of orders of coincidence corresponding to the prime p. By $R_\tau(\epsilon)$ and $R_\sigma(\epsilon)$ respectively, we shall designate the general numbers of the corpus conditioned by these sets of orders of coincidence. We wish to show that the number of the independent congruence conditions (mod p) imposed on the coefficients in $R_\tau(\epsilon)$ by the addition of $1/\nu_s$ to its order of coincidence τ_s with the sth cycle is precisely the same as the number of the independent congruence conditions (mod p) imposed on the coefficients in $R_\sigma(\epsilon)$ by the addition of $1/\nu_s$ to its order of coincidence σ_s with the sth cycle. Select any specific number $\rho_{\tau-\sigma}(\epsilon)$ of the corpus which possesses precisely the set of orders of coincidence $\tau_1-\sigma_1, \ldots, \tau_r-\sigma_r$ corresponding to the prime p. The general number of the corpus conditioned by the set of orders of coincidence $\tau_1, \ldots, \tau_r$ is then representable in the two forms $R_\tau(\epsilon)$ and $\rho_{\tau-\sigma}(\epsilon)R_\sigma(\epsilon)$. Identifying $R_\tau(\epsilon)$ with the reduced form of $\rho_{\tau-\sigma}(\epsilon)R_\sigma(\epsilon)$ we obtain a linear connection between the coefficients of $R_\tau(\epsilon)$ and the coefficients of $R_\sigma(\epsilon)$. If the addition of $1/\nu_s$ to the order of coincidence of $R_\tau(\epsilon)$ with the sth cycle subjects its coefficients to e_s independent congruence conditions (mod p), none of these congruence conditions can become identical on substituting in them for the coefficients of $R_\tau(\epsilon)$ their equivalents in terms of the coefficients of $R_\sigma(\epsilon)$. In adding $1/\nu_s$ to the order of coincidence τ_s of $R_\tau(\epsilon) = \rho_{\tau-\sigma}(\epsilon)R_\sigma(\epsilon)$ with the sth cycle we impose on the coefficients of $R_\sigma(\epsilon)$ not less than e_s independent congruence conditions (mod p). Furthermore, these conditions are all necessary if $\rho_{\tau-\sigma}(\epsilon)R_\sigma(\epsilon)$ is to have the order of coincidence τ_s+1/ν_s with the sth cycle, and they are therefore all necessary if $R_\sigma(\epsilon)$ is to have the order of coincidence σ_s+1/ν_s with the sth cycle. The number of independent congruence conditions (mod p) imposed on the coefficients of $R_\sigma(\epsilon)$ by the addition of $1/\nu_s$ to its order of coincidence with the sth cycle is therefore not less than the number of such conditions imposed on the coefficients of $R_\tau(\epsilon)$ by the addition of $1/\nu_s$ to its order of coincidence with the sth cycle. In like manner the number of independent congruence conditions (mod p) imposed on the coefficients of $R_\tau(\epsilon)$ by the addition of $1/\nu_s$ to its order of coincidence with the sth cycle is not less than the number of conditions imposed on the coefficients of $R_\sigma(\epsilon)$ by the addition of $1/\nu_s$ to its order of coincidence with the sth cycle. The addition of $1/\nu_s$ to the order of coincidence of the general number $R_\sigma(\epsilon)$ with the sth cycle then imposes precisely the same number of congruence conditions on its coefficients as are imposed on the coefficients of the general number $R_\tau(\epsilon)$ by the addition of $1/\nu_s$ to its order of coincidence with the sth cycle. The number of extra congruence conditions imposed on the coefficients of the general number conditioned by a set of orders of coincidence $\tau_1, \ldots, \tau_r$ on adding $1/\nu_s$ to its order of coincidence with the sth cycle is then a fixed number e_s independent of what the set of orders of coincidence $\tau_1, \ldots, \tau_r$ may be. It is then evident that if we add 1 to the order of coincidence τ_s of the general number $R_\tau(\epsilon)$ conditioned by the set of orders of coincidence $\tau_1, \ldots, \tau_r$ we impose on its coefficients precisely $\nu_s e_s$ independent congruence conditions (mod p).

X

The p-adic factor $f_s(x)$ of $f(x)$ is of degree n_s in x. Formally dividing $R_\tau(\epsilon)$ regarded as a polynomial of degree $n-1$ in ϵ by $f_s(\epsilon)$ we may represent the result by the congruence

$$R_\tau(\epsilon) \equiv \rho(\epsilon) \pmod{f_s(\epsilon)}, \tag{1}$$

where $\rho(\epsilon)$ is a polynomial in ϵ of degree n_s-1 with p-adic coefficients. The coefficients in $\rho(\epsilon)$ are linearly connected with the coefficients in $R_\tau(\epsilon)$. If we add 1 to the order of coincidence of $R_\tau(\epsilon)$ with the sth cycle, we impose certain congruence conditions (mod p) on the coefficients. We also at the same time impose some number n_s' of independent congruence conditions on the coefficients in $\rho(\epsilon)$. Supposing these latter coefficients to be expressed linearly in terms of the coefficients in $R_\tau(\epsilon)$ the number of independent congruence conditions imposed at the same time on the coefficients of $R_\tau(\epsilon)$ will be $\gtreqless n_s'$. Now the order of coincidence for the sth cycle corresponding to the form $p\rho(\epsilon)$ is evidently τ_s+1. Also the general form $p\rho(\epsilon)$ evidently lacks precisely n_s arbitrary coefficients which present themselves in the general form $\rho(\epsilon)$. We, therefore, have $n_s' \lesseqgtr n_s$ and consequently the number of the independent congruence conditions (mod p) imposed on the coefficients of $R_\tau(\epsilon)$ by the addition of 1 to its order of coincidence with the sth cycle is $n_s'' \lesseqgtr n_s$. The number of independent conditions imposed on the coefficients of $R_\tau(\epsilon)$ on adding 1 to each of its orders of coincidence $\tau_1, \ldots, \tau_r$ is, therefore, $n_1''+\ldots+n_r''$. The number of these conditions, however, is evidently n since the general number of the corpus conditioned by the orders of coincidence $\tau_1+1, \ldots, \tau_r+1$ is plainly $pR_\tau(\epsilon)$. We consequently have $n_1''+\ldots+n_r''=n$ and since we also have $n_1+\ldots+n_r=n$ and $n_s''\lesseqgtr n_s$ it must be that n_s is the number of independent congruence conditions (mod p) imposed on the coefficients in $R_\tau(\epsilon)$ by the addition of 1 to its order of coincidence with the sth cycle. This, however, we have already seen to be equal to $\nu_s e_s$. We, therefore, have $e_s = n_s/\nu_s$ for the number of the independent congruence conditions (mod p) imposed on the coefficients of the general number conditioned by any set of orders of coincidence $\tau_1, \ldots, \tau_r$, when we add $1/\nu_s$ to its order of coincidence with the sth cycle. The addition of l_s, a multiple of $1/\nu_s$ to the order of coincidence with the sth cycle of the general number of the corpus conditioned by a given set of orders of coincidence will then impose on its coefficients $n_s l_s$ further congruence conditions (mod p).

We shall say that a set of orders of coincidence $\sigma_1, \ldots, \sigma_r$ is on a *lower level* than a set of orders of coincidence $\tau_1, \ldots, \tau_r$ when the two sets do not coincide, and when their elements satisfy the inequalities $\sigma_s \overline{\overline{<}} \tau_s$; $s=1, 2, \ldots, r$. In raising the level of the general number conditioned by the set of orders of coincidence $\sigma_1, \ldots, \sigma_r$ to the level furnished by the set of orders of coincidence $\tau_1, \ldots, \tau_r$ the number of the independent conditions (mod p) imposed on its coefficients is evidently given by the sum

$$\sum_{s=1}^{r}(\tau_s-\sigma_s)n_s. \tag{2}$$

Let us now suppose the set of orders of coincidence $\sigma_1, \ldots, \sigma_r$ to be on a lower level than either of the complementary adjoint sets $\tau_1, \ldots, \tau_r$ or $\bar{\tau}_1, \ldots, \bar{\tau}_r$. Consider the general number of the corpus representable in the form

$$p^{-i}((p, \epsilon)) \tag{3}$$

where ϵ is assumed to be taken sufficiently great and designate by δ the number of the independent congruence conditions (mod p) imposed on the coefficients of this form by the set of orders of coincidence, $\sigma_1, \ldots, \sigma_r$. In accord with the results obtained in this section and in the preceding section, then, we would have for the numbers of the conditions imposed on the general form (3) by the sets of orders of coincidence $\tau_1, \ldots, \tau_r$ and $\bar{\tau}_1, \ldots, \bar{\tau}_r$ respectively, the expressions

$$\begin{aligned} \bar{d} &= \delta+\sum_{s=1}^{r}(\tau_s-\sigma_s)n_s,\\ d &= \delta+\sum_{s=1}^{r}(\bar{\tau}_s-\sigma_s)n_s. \end{aligned} \tag{4}$$

From these formulae, by subtraction, we obtain

$$d-\bar{d} = \sum_{s=1}^{r}(\bar{\tau}_s-\tau_s)n_s. \tag{5}$$

Combining this formula with formula (20), Section VIII, we derive

$$\begin{aligned} d &= ni+\tfrac{1}{2}\sum_{s=1}^{r}(\bar{\tau}_s-\tau_s)n_s,\\ \bar{d} &= ni+\tfrac{1}{2}\sum_{s=1}^{r}(\tau_s-\bar{\tau}_s)n_s. \end{aligned} \tag{6}$$

In formula (1), Section VII, the sign of equality gives the relations which here hold between the order numbers τ_s and $\bar{\tau}_s$. From these relations in combination with formulae (6) we deduce

$$\begin{aligned} \bar{d} &= ni+\sum_{s=1}^{r}\tau_s n_s-\tfrac{1}{2}\sum_{s=1}^{r}(\mu_s-m_s-1+1/\nu_s)n_s,\\ d &= ni+\sum_{s=1}^{r}\bar{\tau}_s n_s-\tfrac{1}{2}\sum_{s=1}^{r}(\mu_s-m_s-1+1/\nu_s)n_s, \end{aligned} \tag{7}$$

for the numbers of the independent congruence conditions (mod p) which are imposed on the general number of the form (3) by the sets of orders of coincidence $\tau_1, \ldots, \tau_r$ and $\bar{\tau}_1, \ldots, \bar{\tau}_r$, respectively.

From formula (6), Section VIII, combined with formulae (6) and (7) just preceding we obtain

$$(8)\qquad \begin{aligned} \sum_{t=0}^{n-1} \bar{\beta}_t &= -\sum_{s=1}^{r} \tau_s n_s + \tfrac{1}{2}\sum_{s=1}^{r} (\mu_s - m_s - 1 + 1/\nu_s) n_s = \tfrac{1}{2}\sum_{s=1}^{r} (\bar{\tau}_s - \tau_s) n_s, \\ \sum_{t=0}^{n-1} \beta_t &= -\sum_{s=1}^{r} \bar{\tau}_s n_s + \tfrac{1}{2}\sum_{s=1}^{r} (\mu_s - m_s - 1 + 1/\nu_s) n_s = \tfrac{1}{2}\sum_{s=1}^{r} (\tau_s - \bar{\tau}_s) n_s. \end{aligned}$$

With the aid of the relations connecting the order numbers τ_s and $\bar{\tau}_s$ these results may also be written in the form

$$(9)\qquad \begin{aligned} \sum_{t=0}^{n-1} \bar{\beta}_t &= \sum_{s=1}^{r} \bar{\tau}_s n_s - \tfrac{1}{2}\sum_{s=1}^{r} (\mu_s - m_s - 1 + 1/\nu_s) n_s, \\ \sum_{t=0}^{n-1} \beta_t &= \sum_{s=1}^{r} \tau_s n_s - \tfrac{1}{2}\sum_{s=1}^{r} (\mu_s - m_s - 1 + 1/\nu_s) n_s. \end{aligned}$$

From the last formula we may note that an addition of $1/\nu_s$ to the order of coincidence τ_s adds $n_s/\nu_s = e_s$ to the sum of the numbers β_t. The addition of any one of the numbers $1/\nu_s, 2/\nu_s, \ldots, 1-1/\nu_s, 1$ to the order of coincidence τ_s can, however, in no case increase any single one of the numbers β_t by more than 1. This evidently follows from the definition of the numbers β_t. The addition of $1/\nu_s$ to the order of coincidence τ_s then adds 1 to each of e_s of the numbers β_t and the addition of 1 to the order of coincidence τ_s adds 1 to each of $\nu_s e_s = n_s$ of the numbers β_t.

We shall here find it convenient to employ the form of representation for a p-adic polynomial given in formula (1), Section IV. Where $g(\epsilon)$ is a p-adic polynomial of degree $n-1$ in ϵ we shall write

$$(10)\qquad g(\epsilon) = \sum_{s=1}^{r} g_s(\epsilon) f_1(\epsilon) \ldots f_{s-1}(\epsilon) f_{s+1}(\epsilon) \ldots f_r(\epsilon).$$

The order of coincidence of $g(\epsilon)$ with the sth cycle is evidently the same as the order of coincidence of the product $g_s(\epsilon)f_1(\epsilon)\ldots f_{s-1}(\epsilon)f_{s+1}(\epsilon)\ldots f_r(\epsilon)$ with this cycle. The degree of $g_s(\epsilon)$ in ϵ is $\leqq n_s - 1$. If $g(\epsilon)$ is conditioned by the orders of coincidence $\tau_1, \ldots, \tau_r$ the product $g_s(\epsilon)f_1(\epsilon)\ldots f_{s-1}(\epsilon)f_{s+1}(\epsilon)\ldots f_r(\epsilon)$ is conditioned by the order of coincidence τ_s with the sth cycle. The number $g_s(\epsilon)$ is then conditioned by the order of coincidence τ_s' with the cycle in question where $\tau_s - \tau_s'$ is the order of coincidence of the product $f_1(\epsilon)\ldots f_{s-1}(\epsilon)f_{s+1}(\epsilon)\ldots f_r(\epsilon)$ with this cycle.

In (12), Section VI the general p-adic polynomial conditioned by the order numbers $\tau_1, \ldots, \tau_r$ is represented linearly in terms of the n elements given in formula (11), Section VI. In like manner, the general p-adic polynomial of degree $n_s - 1$ in ϵ which is conditioned by the order number τ_s' corresponding to the sth cycle can be represented in the form

$$(11)\qquad g_s(\epsilon) = \sum_{t=0}^{n-1} ((p))_t p^{\lambda_t} q_t(\epsilon)$$

where the polynomials $q_t(\epsilon)$ are primitive of degree indicated by the suffix and of minimum index for primitive polynomials of such degree. The index of a polynomial $q_t(\epsilon)$ will here be λ_t and the coefficient of ϵ^t in this polynomial we may suppose to be 1. We may, if we will, suppose the coefficients in the polynomials $q_t(\epsilon)$ to be actual integers. The forms of the polynomials $q_t(\epsilon)$ are evidently independent of the value of the order number τ_t'.

If we increase the order number of $g_s(\epsilon)$ for the sth cycle from τ_s' to $\tau_s'+1/\nu_s$ we impose e_s independent congruence conditions (mod p) on the coefficients of the polynomial as represented in (11). We know, however, that in increasing τ_s' to $\tau_s'+1/\nu_s$ we add 1 to each of e_s of the exponents λ_t. Otherwise said, in adding $1/\nu_s$ to the order of coincidence of $g_s(\epsilon)$ with the sth cycle we equate to 0 the first term in each of e_s of the p-adic series $((p))_t$. This evidently implies that the e_s corresponding elements $p^{\lambda_t}q_t(\epsilon)$ have precisely the order of coincidence τ_s' while each of the remaining n_s-e_s elements has an order of coincidence which exceeds τ_s'.

The general p-adic polynomial of degree n_s-1 in ϵ which is conditioned by the order number $\tau_s'+1/\nu_s$ corresponding to the sth cycle, we can represent in the form

$$h_s(\epsilon)=\sum_{t=0}^{n_s-1}((p))_t p^{\lambda_t'}q_t(\epsilon) \tag{12}$$

where the exponents λ_t' are the same as the corresponding exponents λ_t with the exception of e_s among them, say λ_α', λ_β', $\ldots$, λ_γ', which are obtained on adding 1 to each of the corresponding exponents λ_α, λ_β, $\ldots$, λ_γ. The e_s elements corresponding to the suffixes α, β, $\ldots$, γ have precisely the order number $\tau_s'+1$. These elements are evidently unaffected when we impose on $h_s(\epsilon)$ the order of coincidence $\tau_s'+2/\nu_s$. This increases by $1/\nu_s$ the order of coincidence which defines the general polynomial $h_s(\epsilon)$ and equates to 0 the first term in each of e_s of the p-adic series $((p))_t$. Also these p-adic series correspond to suffixes other than α, β, $\ldots$, γ. This means that among the elements in the summation on the right-hand side of (11) there are e_s which have precisely the order of coincidence $\tau_s'+1/\nu_s$ as well as e_s which have precisely the order of coincidence τ_s'. At the same time it appears that the remaining n_s-2e_s elements have orders of coincidence which are greater than $\tau_s'+1/\nu_s$. Continuation of the above reasoning evidently leads to the conclusion that the n_s elements in the summation on the right-hand side of (11) consist of ν_s sets of e_s elements each, the elements of these respective sets having precisely the orders of coincidence τ_s', $\tau_s'+1/\nu_s$, $\ldots$, $\tau_s'+1-1/\nu_s$ so long as the first terms in the p-adic series $((p))_t$ are all different from 0.

The numbers included under $g_s(\epsilon)$ as represented in (11) we can separate into p^{e_s} classes as follows: Assign independently to the first term in each one of the p-adic series $((p))_\alpha$, $((p))_\beta$, $\ldots$, $((p))_\gamma$ any one of the p values 0, 1, 2, $\ldots$, $p-1$. We thus obtain, all told, p^{e_s} combinations of simultaneous values for the first terms in these e_s series. The numbers under $g_s(\epsilon)$ corresponding to each combination constitute a class. The numbers of $p^{e_s}-1$ of the classes have precisely τ_s' as order of coincidence with the sth cycle. The numbers of the remaining class have $\tau_s'+1/\nu_s$ at least as order of coincidence with the sth cycle. The difference of two numbers of different classes will always have τ_s' precisely as order of coincidence with the cycle in question. The difference of two numbers of the same class will have τ'_s+1/ν_s at least as its order of coincidence with the cycle.

Corresponding to the classification of the numbers $g_s(\epsilon)$ just given we have a classification of the numbers $g_s(\epsilon)f_1(\epsilon)\ldots f_{s-1}(\epsilon)f_{s+1}(\epsilon)\ldots f_r(\epsilon)$ and therewith of the numbers $g(\epsilon)$ represented in (10). The numbers $g(\epsilon)$ conditioned by the order of coincidence τ_s for the sth cycle may be said to group themselves into p^{e_s} classes with regard to the order of coincidence τ_s+1/ν_s. The numbers of $p^{e_s}-1$ of these classes have precisely τ_s as order of coincidence with the sth cycle. The numbers of the remaining class have orders of coincidence with this cycle which are not less than τ_s+1/ν_s. The difference of two numbers of different classes will always have precisely τ_s as order of coincidence with the sth cycle. The difference of two numbers of the same class will always have τ_s+1/ν_s at least as its order of coincidence with the sth cycle.

Returning to the numbers $g_s(\epsilon)$ in (10), we can separate these numbers into p^{2e_s} classes with reference to the order of coincidence $\tau_s'+1/\nu_s$, as follows: Assign the values 0, 1, ..., $p-1$ independently, not only to the first terms in the series $((p))_\alpha$, $((p))_\beta$, ..., $((p))_\gamma$ which are attached to elements $p^{\lambda_t}q_t(\epsilon)$ in (11), which possess precisely τ_s' as order of coincidence with the sth cycle, but also to the first terms in the e_s series $((p))_{\alpha'}$, $((p))_{\beta'}$, ..., $((p))_{\gamma'}$, which are attached to elements which have $\tau_s'+1/\nu_s$ as order of coincidence with the cycle in question. The numbers $g_s(\epsilon)$ corresponding to each of the p^{2e_s} combinations of first terms so obtained we regard as a class with reference to the order of coincidence $\tau_s'+2/\nu_s$. The difference of two numbers of the same class will have $\tau_s'+2/\nu_s$ at least as order of coincidence with the sth cycle. The difference of two numbers of different classes will have an order of coincidence which is less than $\tau_s'+2/\nu_s$. The numbers of one of the p^{2e_s} classes will have orders of coincidence which are equal to or greater than $\tau_s'+2/\nu_s$. The numbers of $p^{e_s}-1$ classes will have $\tau_s'+1/\nu_s$ precisely as order of coincidence and the numbers of $p^{2e_s}-p^{e_s}$ classes will have τ_s' precisely as order of coincidence.

The numbers $g_s(\epsilon)f_1(\epsilon)\ldots f_{s-1}(\epsilon)f_{s+1}(\epsilon)\ldots f_r(\epsilon)$ and therewith the numbers $g(\epsilon)$ conditioned by the order of coincidence τ_s with the sth cycle and represented as in (10) we may then separate into p^{2e_s} classes with reference to the order of coincidence τ_s+2/ν_s. The difference of two numbers of the same class will have τ_s+2/ν_s at least as order of coincidence with the sth cycle. The difference of two numbers of different classes will have an order of coincidence which is less than τ_s+2/ν_s. The numbers of $p^{2e_s}-1$ classes will have orders of coincidence which are less than τ_s+2/ν_s. The numbers of $p^{e_s}-1$ of these classes will have τ_s+1/ν_s precisely as order of coincidence and the numbers of $p^{2e_s}-p^{e_s}$ of them will have τ_s precisely as order of coincidence. The numbers of one class will have orders of coincidence which are equal to or greater than τ_s+2/ν_s.

The above considerations can be extended, and we readily see that the numbers $g(\epsilon)$ conditioned by the order of coincidence τ_s can be separated into p^{ke_s} classes with regard to the order of coincidence τ_s+k/ν_s. The numbers of one of these classes will have orders of coincidence which are equal to or greater than τ_s+k/ν_s. The numbers of the remaining $p^{ke_s}-1$ classes will have orders of coincidence which are less than τ_s+k/ν_s. Of these classes $p^{ke_s}-p^{(k-1)e_s}, p^{(k-1)e_s}-p^{(k-2)e_s}, \ldots, p^{e_s}-1$ respectively are made up of numbers which have precisely the orders of coincidence $\tau_s, \tau_s+1/\nu_s, \ldots, \tau_s+(k-1)/\nu_s$ with the sth cycle. The difference of two numbers of the same class will have τ_s+k/ν_s at least as order of coincidence. The difference of two numbers belonging to different classes will have an order of coincidence which is less than τ_s+k/ν_s.

Where, in what precedes, we have $k>\nu_s$, our considerations with regard to the series $(p)_t$ in (11) will have to take account of terms besides the first in these p-adic series. This, however, introduces no additional difficulties.

Where we have $\sigma_s>\tau_s$ the number of classes of the numbers $g(\epsilon)$, conditioned by the order of coincidence τ_s, with reference to the order of coincidence σ_s is, in accord with what we have seen in the above, $p^{(\sigma_s-\tau_s)n_s}$.

In our above classification of the numbers $g(\epsilon)$, represented as in (32) the aggregate of numbers has been conditioned with reference to one cycle only, corresponding to a single prime.

We might further limit the numbers $g(\epsilon)$ under consideration without detriment to the classification arrived at in our preceding argument. We might, for example, limit ourselves to the consideration of numbers conditioned by a certain set of orders of coincidence corresponding to the several cycles associated with the prime p. Or we might also impose on the numbers $g(\epsilon)$ orders of coincidence corresponding to cycles associated with other primes, or we might limit ourselves to the consideration of numbers which are conditioned by a system of orders of coincidence corresponding to all cycles associated with all primes.

Suppose we consider the numbers $g(\epsilon)$ conditioned by a set of orders of coincidence $\tau_1, \tau_2, \ldots, \tau_r$ corresponding to the cycles associated with the prime p. We can effect a classification of these numbers with reference to the set of orders of coincidence $\sigma_1, \sigma_2, \ldots, \sigma_r$ where we have $\sigma_1 \gtreqless \tau_1, \ldots, \sigma_r \gtreqless \tau_r$, as follows: The elements $g_s(\epsilon)f_1(\epsilon)\ldots f_{s-1}(\epsilon)f_{s+1}(\epsilon)\ldots f_r(\epsilon)$ in (10) we separate into $p^{(\sigma_s-\tau_s)n_s}$ classes with reference to the order of coincidence σ_s corresponding to the sth cycle. It is plain, then, that we can separate the numbers $g(\epsilon)$ conditioned by the orders of coincidence $\tau_1, \ldots, \tau_r$ corresponding to the sth cycle into $p^{\Sigma(\sigma_s-\tau_s)n_s}$ classes. In one only of these classes will the numbers be conditioned simultaneously by the orders of coincidence $\sigma_1, \ldots, \sigma_r$. The difference of two numbers in the same class will always be conditioned by this set of orders of coincidence. The difference of two numbers in different classes will never be conditioned by this set of orders of coincidence.

Suppose we assign a system of orders of coincidence corresponding to all primes, all but a finite number of these orders of coincidence having the value 0. The number of cycles corresponding to a prime p_l we shall designate by r_l; the orders of coincidence assigned to these cycles we shall designate by $\tau_1^{(l)}, \ldots, \tau_{r_l}^{(l)}$. The complete system of orders of coincidence $\tau_1^{(l)}, \ldots, \tau_{r_l}^{(l)}$ for all primes p_l we shall call a basis of order numbers or of orders of coincidence for the building of numbers $g(\epsilon)$. Such basis we shall designate by the notation (τ). We shall say of a basis (τ) that it is on a lower level than a basis (σ) if no order number furnished by the former basis is greater than the corresponding order number furnished by the latter basis, and if at the same time some one at least of the order numbers furnished by (τ) is actually less than the corresponding order number furnished by (σ).

Employing a suffix or index l to designate numbers which have reference to the prime p_l and assuming that the basis (τ) is on a lower level than the basis (σ), we can separate the numbers $g(\epsilon)$ on the basis (τ) into a number of classes, such that the difference of any two numbers of the same class is on the basis (σ) whereas the difference of two numbers belonging to different classes is not on the basis (σ). One class only contains numbers which have orders of coincidence which place them on the basis (σ) and all the numbers of this class are on the basis (σ). The number of the classes in question will be given by the product

$$\prod_l p_l^{\sum_{s=1}^{r_l}(\sigma_s^{(l)}-\tau_s^{(l)})n_s^{(l)}} \tag{13}$$

To show this we suppose $\rho_1^{(l)}(\epsilon)$, $\rho_2^{(l)}(\epsilon)$, ..., to constitute a complete set of representatives of the classes of the numbers $g(\epsilon)$ on (τ) with reference to the orders of coincidence $\sigma_1^{(l)}$, ..., $\sigma_{r_l}^{(l)}$ corresponding to the single prime p_l only. Choose A_l an integer relatively prime to p_l but containing sufficiently high powers of all the other primes p_1, p_2, ... here in question, so that A_l has for all cycles other than those corresponding to the prime p_l order numbers at least as great as those furnished by the basis (σ). The numbers $A_l\rho_1^{(l)}(\epsilon)$, $A_l\rho_2^{(l)}(\epsilon)$, ... evidently constitute a complete set of representatives of the classes of the numbers $g(\epsilon)$ on (τ) with reference to the orders of coincidence $\sigma_1^{(l)}$, ..., $\sigma_{r_l}^{(l)}$ corresponding to the single prime p_l. Now for each prime p_l here in question, construct such a system of representatives

$$A_l\rho_1^{(l)}(\epsilon),\ A_l\rho_2^{(l)}(\epsilon),\ \ldots \tag{14}$$

Take all possible sums made up by adding representatives, one of which corresponds to each prime p_l here in question. The total number of sums so obtained is given by the product (13). Every number $g(\epsilon)$ on the basis (τ) is evidently congruent to one of these sums with reference to the basis (σ). The several sums are incongruent to one another with reference to the basis (σ). These sums, then, plainly constitute a complete system of representatives of the classes into which the numbers $g(\epsilon)$ on (τ) fall with reference to the basis (σ).

In what precedes we have supposed (τ) to be on a lower level than (σ). Suppose now that for the sth cycle corresponding to the prime p we have $\tau_s > \sigma_s$. Going back to the consideration of the numbers $g(\epsilon)$ on (τ) with reference only to the sth cycle corresponding to the prime p we see that these numbers constitute only one class with regard to this cycle, for they are all included under the numbers conditioned by the order of coincidence σ_s for this cycle. It is then plain that the number of classes into which the numbers $g(\epsilon)$ on (τ) group themselves so far as the single prime p is concerned, is given by

$$p^{\sum_{s=1}^{r_l} (\sigma_s - \tau_s)' n_s} \tag{15}$$

where the accent $'$ signifies that only those terms are to be included in the summation for which $\sigma_s - \tau_s$ is positive. If the basis (τ) is not on a lower level than (σ) we then readily see that the number of classes into which the numbers $g(\epsilon)$ on (τ) separate themselves with reference to the basis (σ), is given by the product

$$\prod_l p_l^{\sum_{s=1}^{r_l} (\sigma_s^{(l)} - \tau_s^{(l)})' n_s^{(l)}} \tag{16}$$

where the product is extended to all the primes p_l in question, the accent $'$ signifying that any difference $\sigma_s^{(l)} - \tau_s^{(l)}$ which is negative is to be replaced by 0 in the summation.

If we employ the notation (τ, σ) to designate the number of classes into which the numbers $g(\epsilon)$ on (τ) separate themselves with reference to the basis (σ) we evidently have

$$\frac{(\tau, \sigma)}{(\sigma, \tau)} = \prod_l p_l^{\sum_{s=1}^{r_l} (\sigma_s^{(l)} - \tau_s^{(l)}) n_s^{(l)}} \tag{17}$$

The product

$$\prod_l p_l^{\sum_{s=1}^{r_l} \tau_s^{(l)} n_s^{(l)}} \tag{18}$$

we call the norm of the basis (τ) and designate it by the notation $N(\tau)$. Employing this notation formula (17) takes the form

$$\frac{(\tau, \sigma)}{(\sigma, \tau)} = \frac{N(\sigma)}{N(\tau)}. \tag{19}$$

In the case where (τ) is on a lower level than (σ) we have $(\sigma, \tau) = 1$ and the last formula becomes

$$(\tau, \sigma) = \frac{N(\sigma)}{N(\tau)}. \tag{20}$$

When the orders of coincidence furnished by a basis (τ) all have the value 0 we call this the 0-basis. Let us consider all the numbers $g(\epsilon)$ of the corpus on the 0-basis, with reference to a basis which furnishes the order number $1/\nu$ corresponding to one of the cycles associated with the prime p, the other order numbers furnished by it all having the value 0. The grade of the particular cycle just referred to we shall indicate by e. The numbers $g(\epsilon)$ here in question are the integral algebraic numbers in the corpus. These numbers, in accord with what we have seen in the preceding, will be separated into p^e classes with reference to the single coincidence here in question, corresponding to the cycle of grade e. The numbers in $p^e - 1$ of these classes will have with this cycle precisely 0 as order of coincidence. The orders of coincidence with this cycle of the numbers in the remaining class will all be equal to or greater than $1/\nu$. The difference of two numbers belonging to different classes will here always have 0 as its order of coincidence with the special cycle in question. The order of coincidence with this cycle of the difference of two numbers which belong to the same class will always be equal to or greater than $1/\nu$.

We shall suppose $\omega_1, \omega_2, \ldots$ to be a complete system of representatives of the p^e-1 classes whose numbers possess the order of coincidence 0 with the special cycle. Let ω be any number belonging to any one of these p^e-1 classes. The numbers

$$\omega\omega_1, \omega\omega_2, \ldots$$

will then evidently also constitute a complete system of representatives of the p^e-1 classes since each of them possesses 0 as its order of coincidence with the special cycle, and since 0 is evidently also the order of coincidence with this cycle of the difference of any two of them. We may then write

$$\omega\omega_1 \equiv \omega_\alpha,\ \omega\omega_2 \equiv \omega_\beta,\ \ldots,\ (\text{mod } (1/\nu)),$$

where $\omega_\alpha, \omega_\beta, \ldots$, in some order, coincide with $\omega_1, \omega_2, \ldots$. We, therefore, have

$$\omega_1\omega_2 \ldots \omega^{p^e-1} \equiv \omega_1\omega_2 \ldots (\text{mod } (1/\nu)),$$

and consequently

$$\omega^{p^e-1} \equiv 1 \ (\text{mod } (1/\nu)). \tag{21}$$

This congruence is satisfied in particular when we replace ω by any one of the p^e-1 numbers $\omega_1, \omega_2, \ldots$ which are incongruent to one another mod $(1/\nu)$. It is evident that a congruence equation mod $(1/\nu)$ cannot have a larger number of roots, which are incongruent mod $(1/\nu)$, than is given by the degree of the equation. It is evident that we may here write

$$x^{p^e-1}-1 \equiv (x-\omega_1)\ (x-\omega_2) \ldots (\text{mod } (1/\nu)).$$

It is also evident that among the numbers $\omega_1, \omega_2, \ldots$ there are s which will satisfy any congruence

$$x^s-1 \equiv 0 \ (\text{mod } (1/\nu)),$$

where s is a factor of p^e-1. Where ω is one of the numbers $\omega_1, \omega_2, \ldots$ then it does not always happen that its lowest power which is congruent to 1 mod $(1/\nu)$ has p^e-1 as exponent, though it is evident that such lowest exponent must be a factor of p^e-1. It is readily shown, however, by a well-known method in the theory of numbers, that among the p^e-1 numbers $\omega_1, \omega_2, \ldots$ there are just $\phi(p^e-1)$ whose lowest powers which are congruent to 1 mod $(1/\nu)$ have p^e-1 as exponent. Suppose ω to be one of these $\phi(p^e-1)$ numbers. The p^e-1 powers

$$\omega, \omega^2, \ldots, \omega^{p^e-1} \tag{22}$$

are all incongruent to one another mod $(1/\nu)$. These powers are then representatives of the p^e-1 classes of numbers which are not congruent to 0 mod $(1/\nu)$. The number ω satisfies the congruence (21) and this is evidently not the congruence of lowest degree which is satisfied by ω. Suppose e' to be the lowest degree of a congruence mod $(1/\nu)$ which is satisfied by ω. This congruence can evidently be written in the form

$$g(x)=x^{e'}+g_{e'-1}x^{e'-1}+\ldots+g_0\equiv 0 \pmod{(1/\nu)},$$

where the values of the coefficients $g_{e'-1}, \ldots, g_0$ are included among the integers $0, 1, 2, \ldots, p-1$. The polynomial $g(x)$ is a prime function (mod p), since otherwise we could split $g(\omega)$ into two factors (mod p), and one of these factors would be $\equiv 0$ mod $(1/\nu)$ contrary to our hypothesis that e' is the lowest degree of a congruence mod $(1/\nu)$ which can be satisfied by ω. Dividing each of the powers of ω in (22) by $g(\omega)$ we find that each of these powers is congruent mod $(1/\nu)$ to a polynomial of the form

$$h_{e'-1}\omega^{e'-1}+\ldots+h_0 \tag{23}$$

where the coefficients $h_{e'-1}, \ldots, h_0$ have for values ones among the numbers $0, 1, 2, \ldots, p-1$. Each one of the p^e-1 classes of numbers represented by the powers of ω in (22), therefore, has a representative which is included under the form (23). Also the class which is $\equiv 0$ mod $(1/\nu)$ is represented by this form when each of the coefficients $h_{e'-1}, \ldots, h_0$ has the value 0. This class we shall call the 0-class.

It is evident that no two of the $p^{e'}$ numbers under the form (23) here considered can represent the same class of numbers mod $(1/\nu)$. Each of the $p^{e'}$ numbers under the form (23) then represents a different one of the p^e classes mod $(1/\nu)$, and each of these p^e classes is represented by one of the $p^{e'}$ numbers under the form (23). As a consequence, we have $p^{e'}=p^e$ and, therefore, $e'=e$. The congruence mod $(1/\nu)$ of lowest degree which is satisfied by ω then has the form

$$g(x)=x^e+g_{e-1}x^{e-1}+\ldots+g_0\equiv 0 \pmod{(1/\nu)}, \tag{24}$$

where $g(x)$ is a prime function mod p. Not only is $g(x)$ irreducible mod p. It is evidently irreducible mod $(1/\nu)$, since otherwise a polynomial in ω of degree less than e would be $\equiv 0$ mod $(1/\nu)$.

If ω does not belong mod $(1/\nu)$ to the exponent p^e-1 it can readily be shown by well-known methods in the theory of numbers to belong to an exponent p^h-1 where h is a factor of e. It can also be shown that h is the lowest degree of a congruence mod $(1/\nu)$, which can be satisfied by ω. The polynomial on the left-hand side of this congruence is evidently also a prime polynomial mod p, and such prime polynomial must plainly be a factor mod p of the left-hand side of any congruence mod $(1/\nu)$ which is satisfied by ω. The congruence mod $(1/\nu)$ here in question is readily shown to possess the set of h roots

$$\omega, \omega^p, \omega^{p^2}, \ldots, \omega^{p^{h-1}}. \tag{25}$$

This notation for the roots of the congruence may be employed whichever one of the h roots is represented by ω.

Suppose ω to be a number belonging to any one of the p^e classes mod $(1/\nu)$ while ω' is any number belonging to the 0-class. If $\gamma(x)$ is the prime polynomial mod p such that we have $\gamma(\omega)\equiv 0$ mod $(1/\nu)$ we evidently also have $\gamma(\omega+\omega')\equiv\gamma(\omega)\equiv 0$ mod $(1/\nu)$. Every number of the class represented by ω then satisfies the same congruence $\gamma(x) \equiv 0 \bmod (1/\nu)$ where $\gamma(x)$ is a prime polynomial mod p. The degree of $\gamma(x)$, as we have seen, is either e or a factor of e.

Suppose now that our preceding remarks have reference to the sth cycle. We shall find it convenient to designate by a suffix s symbols corresponding to this cycle. The grade and order of this cycle we shall then indicate by e_s and ν_s respectively. Any integral number ω of the corpus $C(\epsilon)$ then satisfies a congruence $\gamma_s(x) \equiv 0 \bmod (1/\nu_s)$ where $\gamma_s(x)$ is a prime polynomial mod p. The degree of this congruence is an integer h_s which is a factor of e_s and the congruence has h_s roots mod $(1/\nu_s)$ which can be represented, after the analogy of (25). The polynomials $\gamma_s(x)$ here in question are evidently identical with the polynomials designated by the same notation in formula 2, Section III, for these are also prime polynomials mod $(1/\nu_s)$, and in their case also we evidently have $\gamma_s(\epsilon) \equiv 0 \bmod (1/\nu_s)$.

In formula (2), Section III, we have $k_s h_s = n_s = \nu_s e_s$ and as h_s is a factor of e_s we see that k_s is a multiple of ν_s. When $h_s = e_s$ then also $k_s = \nu_s$. From the formula just referred to, we also see that the trace coefficient in $f_s(x)$ is $\equiv \bmod p$ to an integer which has ν_s as a factor. The partial trace of ϵ corresponding to the sth cycle is then divisible by p if ν_s contains p as factor. Conversely, we can show if ν_s is not divisible by p that there exist integral numbers of the corpus whose sth p-adic partial traces are not divisible by p. To this end consider an integral number ω of the corpus which satisfies a congruence $\gamma_s(x) \equiv 0 \bmod (1/\nu_s)$ where $\gamma_s(x)$ is a prime polynomial of degree e_s. For convenience, we shall for the moment represent the e_s roots of this congruence by the notation $\omega_1, \omega_2, \ldots, \omega_{e_s}$. These roots can be represented in terms of one of them after the analogy of (25), and the trace coefficient in $\gamma_s(x)$ is evidently $\equiv \bmod (1/\nu_s)$ to the negative of their sum.

Suppose now, if possible, that we have simultaneously

$$(26) \qquad \sum_{i=1}^{e_s} \omega_i^t \equiv 0 \bmod (1/\nu_s), \ (t=0, 1, \ldots, e_s-1),$$

and consequently

$$| \omega_i^t | \equiv 0 \ (\text{mod } (1/\nu_s)).$$

Squaring the determinant on the left-hand side of this congruence, we immediately deduce

$$\prod_{i=1}^{e_s} g'(\omega_i) \equiv 0 \ (\text{mod } (1/\nu_s)).$$

For some one ω of the numbers $\omega_1, \ldots, \omega_{e_s}$ we must then have

$$g'(\omega) \equiv 0 \ (\text{mod } (1/\nu_s)).$$

Since, however, no primitive polynomial mod p of degree lower than $g(\omega)$ is $\equiv 0 \bmod (1/\nu_s)$, this would mean that the coefficients in $g'(\omega)$ are all divisible by p. From this would follow that $g(x)$ is a polynomial in x^p and we could, therefore, write $g(x) = G(x^p)$. We would then have $g(\omega) \equiv G(\omega^p) \equiv 0 \bmod (1/\nu_s)$. We have, however, also $g(\omega^p) \equiv 0 \bmod (1/\nu_s)$. The congruence $g(x) \equiv 0 \bmod (1/\nu_s)$ would then have the root ω^p in common with the congruence of lower degree

$G(x) \equiv 0 \bmod (1/\nu_s)$. This is, however, not in accord with the fact that $g(x)$ is a prime polynomial of the lowest possible degree which can have one of the numbers $\omega_1, \ldots, \omega_{e_s}$ as a root mod $(1/\nu_s)$. It follows, therefore, that the e_s congruences in (26) are not simultaneously compatible.

For some one at least of the values $t=0, 1, \ldots, e_s-1$ we then have

$$\sum_{i=1}^{e_s} \omega_i^t \not\equiv 0 \ (\bmod\ (1/\nu_s)).$$

For the value of t in question we have in the transform of $\gamma_s(x)$ through x^t a trace coefficient which is $\not\equiv 0 \bmod (1/\nu_s)$, and therefore $\not\equiv 0 \bmod p$. In the transform of $\{\gamma_s(x)\}^{\nu_s}$ through x^t then the trace coefficient is divisible by p only in the case where ν_s is divisible by p. We see, therefore, that the sth p-adic partial traces of all the integral numbers of the corpus $C(\epsilon)$ are divisible by p when, and only when, ν_s the order of the sth p-adic cycle is divisible by p.

Going back now to the numbers m_s introduced in Section V, we see that these numbers have values other than 0 when and only when the orders ν_s of the corresponding cycles are divisible by p.

XI

Where we have to deal with several primes simultaneously, we shall distinguish them by suffixes as has already been done in the Section preceding. A polynomial of degree $n-1$ in ϵ with coefficients which are integral rational numbers we shall designate by the notation $((\epsilon))$. Consider a set of primes p_l with which we associate a system of sets of orders of coincidence $\tau_1^{(l)}, \tau_2^{(l)}, \ldots, \tau_{r_l}^{(l)}$. The number of the cycles corresponding to the prime p_l we designate by r_l; the degrees of these cycles we shall indicate by $n_1^{(l)}, n_2^{(l)}, \ldots, n_{r_l}^{(l)}$ respectively, and their orders by $\nu_1^{(l)}, \nu_2^{(l)}, \ldots, \nu_{r_l}^{(l)}$. Their respective grades are then given by $e_s^{(l)} = n_s^{(l)}/\nu_s^{(l)}$, $(s=1, 2, \ldots, r_l)$.

The numbers of the conditions imposed on the coefficients in the general form

$$p_1^{-i_1} p_2^{-i_2} \ldots ((\epsilon)) \tag{1}$$

by the several sets of orders of coincidence $\tau_1^{(l)}, \tau_2^{(l)}, \ldots, \tau_{r_l}^{(l)}$ are evidently independent of one another. If, for example, we multiply the number (1) by sufficiently high powers of all the primes involved save p_1 the conditions relative to p_1 required by the set of orders of coincidence $\tau_1^{(1)}, \tau_2^{(1)}, \ldots, \tau_{r_1}^{(1)}$ have not been diminished in number though the conditions relative to the other primes required by the corresponding sets of orders of coincidence are already fulfilled.

Conceive of a system of sets of orders of coincidence $\tau_1^{(l)}, \tau_2^{(l)}, \ldots, \tau_{r_l}^{(l)}$ corresponding to all primes p_l, the numbers $\tau_s^{(l)}$ being all 0 save in the case of a finite number of sets. Such a system we called a basis of orders of coincidence in Section X where we designated it by the notation (τ). The basis made up of the complementary adjoint sets of orders of coincidence we shall designate by the notation $(\bar{\tau})$. The bases (τ) and $(\bar{\tau})$ we shall call complementary adjoint bases.

Because of the independence of the conditions imposed on the general number of the form (1) by the several sets of orders of coincidence which go to constitute the basis (τ), we see by formulae (6) and (7), Section X, that the number of independent congruence conditions, relative to the several moduli $p_1, p_2, \ldots$, imposed on the general number of the form (1) by the basis (τ), is

$$(2)\quad n\sum_l i_l+\tfrac{1}{2}\sum_l \sum_{s=1}^{r_l}(\tau_s^{(l)}-\bar{\tau}_s^{(l)})n_s^{(l)}=n\sum_l i_l+\sum_l \sum_{s=1}^{r_l}\tau_s^{(l)}n_s^{(l)}-\tfrac{1}{2}\sum_l \sum_{s=1}^{r_l}(\mu_s^{(l)}-m_s^{(l)}-1+1/\nu_s^{(l)})n_s^{(l)},$$

the numbers $-i_1, -i_2, \ldots$ being assumed to be taken sufficiently small algebraically. It evidently suffices if for each prime p_l the number $-i_l$ is not greater than the least order number relative to p_l which can actually be possessed by the principal coefficient in a number $R(\epsilon)$ conditioned by the set of orders of coincidence $\tau_1^{(l)}, \ldots, \tau_{r_l}^{(l)}$.

In selecting the n numbers in formula (11), Section VI, to serve as a basis for the representation of the numbers of the corpus, conditioned by the set of orders of coincidence $\tau_1, \ldots, \tau_r$ relatively to the prime p it sufficed to determine them to a high enough power of p as modulus. If the orders of coincidence all have the value 0 the exponents $\beta_{n-1}, \ldots, \beta_0$, save in the case of a finite number of primes p, will all have the value 0. Apart from the finite number of exceptions, just referred to, then we are at liberty to select as basis any n polynomials $R_{n-1}(\epsilon), \ldots, R_0(\epsilon)$ of degrees $n-1, \ldots, 0$ respectively whose coefficients are rational numbers which are integral relatively to the primes p in question, the coefficient of the highest power of ϵ in a polynomial being in no case divisible by p.

It is then evident that we can select n numbers

$$(3)\qquad \prod_l p_l^{\beta_{n-1}^{(l)}}.R_{n-1}(\epsilon),\ \prod_l p_l^{\beta_{n-2}^{(l)}}.R_{n-2}(\epsilon),\ \ldots,\ \prod_l p_l^{\beta_0^{(l)}}.R_0(\epsilon)$$

which will simultaneously serve as basis for the sets of orders of coincidence $\tau_1^{(l)}, \ldots, \tau_{r_l}^{(l)}$ corresponding to the several primes p_l, which go to make up the basis of orders of coincidence (τ). The coefficients of the polynomials $R_{n-1}(\epsilon), \ldots, R_0(\epsilon)$ are here supposed to be integral relatively to each of the primes p_l. They must, therefore, all be integers positive or negative. Also the coefficient of the highest power of ϵ in each of the polynomials is assumed to be divisible by none of the primes p. Such coefficient must, therefore, have the value 1. For the rest the coefficients of the polynomials in question are conditioned with reference to a finite number of primes only. Any number on the basis (τ) can then be represented in the form

$$(4)\qquad \sum_{t=0}^{n-1} C_t \prod_l p_l^{\beta_t^{(l)}}.R_t(\epsilon)$$

where the coefficients C_t are rational numbers which are integral relatively to all primes p_l and which must therefore be integral rational numbers. Conversely, all numbers of the form (4) in which the coefficients C_t are integral rational numbers are evidently on the basis (τ).

For brevity we shall write

$$(5)\qquad \prod_l p_l^{\beta_t^{(l)}}=\Pi_t,\ (t=0, 1, \ldots, n-1).$$

The discriminant of the basis

$$\Pi_t . R_t(\epsilon),\ (t=0,\ 1,\ \ldots,\ n-1), \tag{6}$$

is given as the square of a determinant in the form

$$|\Pi_t . R_t(\epsilon_s)|^2 = \prod_{t=0}^{n-1} \Pi_t^2 . |R_t(\epsilon_s)|^2,\ (s=1,\ 2,\ \ldots,\ n), \tag{7}$$

where $\epsilon_1, \ldots, \epsilon_n$ are the roots of the equation $f(x)=0$.

We evidently have

$$|R_t(\epsilon_s)|^2 = |\epsilon_s^t|^2 = (-1)^{\frac{1}{2}n(n-1)} f'(\epsilon_1) \ldots f'(\epsilon_n) = (-1)^{\frac{1}{2}n(n-1)} Nf'(\epsilon) \tag{8}$$

where ϵ stands for one of the numbers $\epsilon_1, \ldots, \epsilon_n$ and the notation $Nf'(\epsilon)$ indicates the norm of $f'(\epsilon)$. From the designation at the beginning of Section VI of the orders of coincidence of $f'(\epsilon)$ with the cycles corresponding to the prime p by the symbols $\mu_1, \ldots, \mu_r$ we evidently have

$$Nf'(\epsilon) = \pm \prod_l p_l^{\sum_{s=1}^{r_l} \mu_s^{(l)} n_s^{(l)}} \tag{9}$$

Also

$$\prod_{t=0}^{n-1} \Pi_t^2 = \prod_l p_l^{2(\beta_{n-1}^{(l)} + \ldots + \beta_0^{(l)})}.$$

and from formula (9), Section X,

$$2 \sum_{t=0}^{n-1} \beta_t^{(l)} = 2 \sum_{s=1}^{r_l} \tau_s^{(l)} n_s^{(l)} - \sum_{s=1}^{r_l} (\mu_s^{(l)} - m_s^{(l)} - 1 + 1/\nu_s^{(l)}) n_s^{(l)}.$$

We therefore have

$$\prod_{t=0}^{n-1} \Pi_t^2 = \prod_l p_l^{2 \sum_{s=1}^{r_l} \tau_s^{(l)} n_s^{(l)} - \sum_{s=1}^{r_l} (\mu_s^{(l)} - m_s^{(l)} - 1 + 1/\nu_s^{(l)}) n_s^{(l)}}. \tag{10}$$

For the discriminant of the basis (6) we deduce

$$(-1)^{\frac{1}{2}n(n-1)} Nf'(\epsilon) \prod_{t=0}^{n-1} \Pi_t^2 = \pm \prod_l p_l^{2 \sum_{s=1}^{r_l} \tau_s^{(l)} n_s^{(l)} + \sum_{s=1}^{r_l} (m_s^{(l)} + 1 - 1/\nu_s^{(l)}) n_s^{(l)}} \tag{11}$$

This expression for the discriminant of the set of numbers (6) we shall also call the discriminant of the basis (τ) designating it at the same time by the notation $\Delta(\tau)$.

If in (11) we assign the value 0 to every order of coincidence $\tau_s^{(l)}$ we obtain for the discriminant of the 0-basis the expression

$$D = \pm \prod_l p_l^{\sum_{s=1}^{r_l} (m_s^{(l)} + 1 - 1/\nu_s^{(l)}) n_s^{(l)}} \tag{12}$$

D we call the fundamental number of the corpus.

It is evident on the face of it that the numbers built on a basis (τ) constitute an ideal according to Kummer's definition of an ideal. It is readily shown that these ideals include among them all the ideals in the generally accepted sense, and that the numbers built on a basis $(1/\nu)$ made up of a single coincidence corresponding to a given cycle constitutes a prime ideal in Kummer's sense. To see this we note first, where α is any number of an integral ideal $\mathfrak{a}$, in the ordinary sense, that $N(\alpha)$ is divisible by $N(\mathfrak{a})$. Now $N(\alpha)$ is not divisible by a prime p for every number α of the ideal unless every number of the ideal has a positive order of coincidence with one and the same cycle corresponding to the prime in question.

To prove this we assume $N(\mathfrak{a})$ to be divisible by the prime p and consider any number α of the ideal $\mathfrak{a}$. This number must have positive orders of coincidence with one or more cycles corresponding to the prime p. Any number of the ideal must have a positive order of coincidence with one at least of the set of one or more cycles here in question, since otherwise addition of the number to α would give us a number of the ideal whose orders of coincidence with the cycles corresponding to the prime p would all be 0 and $N(\alpha)$ would consequently not have p as a factor. If there exists a number of the ideal, which does not have a positive order of coincidence, with each one of the set of cycles referred to above, the addition of this number to α would give a number β of the ideal which has positive orders of coincidence with each one of a second and more restricted set of cycles included under the first set. If there exists a number of the ideal, which does not possess positive orders of coincidence with each of the cycles of this second set, addition of this number to β would give us a number of the ideal, having positive orders of coincidence with each one of a third set of cycles still more restricted than the second set and included under it. Continuation of this process must, after a finite number of sets, bring us to a set of one or more cycles corresponding to the prime p with each of which every number of the ideal $\mathfrak{a}$ has a positive order of coincidence. If then $N(\mathfrak{a})$ is divisible by p every number of the ideal $\mathfrak{a}$ has at least a positive order of coincidence with some one and the same cycle corresponding to p. In particular, every number of a prime ideal, as defined in the ordinary theory, must have a positive order of coincidence with some one cycle, and must therefore be included under the numbers of an ideal built on a basis $(1/\nu)$ consisting of a single coincidence corresponding to a prime p. Such prime ideal is then divisible by the ideal built on the basis $(1/\nu)$ and must therefore coincide with this ideal. The prime ideals of the ordinary theory, therefore, coincide with the ideals which we have defined by a single coincidence with some cycle corresponding to a prime p.

Every integral ideal in the ordinary sense can be represented as a product of prime ideals in the form

$$\mathfrak{p}_1^{t_1}\mathfrak{p}_2^{t_2}\dots\mathfrak{p}_w^{t_w}. \tag{13}$$

Here we may suppose each prime ideal to be defined by a single coincidence corresponding to some one cycle. The prime ideal $\mathfrak{p}_s$ we shall say corresponds to the sth cycle and has as order of coincidence $1/\nu_s$ with this cycle.

Consider the ideal built on a basis (τ) where the orders of coincidence furnished by the basis correspond to the w cycles to which the prime factors of the product (13) correspond, the order of coincidence furnished for the sth cycle being t_s/ν_s. It is readily seen that the ideal built on the basis (τ) coincides with the ideal represented by the product (13). For, as we know, the ideal (τ) can be represented as the product of powers of prime ideals, and in that product the prime ideal $\mathfrak{p}_s$ cannot appear to a lower power than $\mathfrak{p}_s^{t_s}$ since in such case the ideal represented by the product would evidently not be included under the ideal (τ). Also $\mathfrak{p}_s$ cannot appear in the product to a higher power than $\mathfrak{p}_s^{t_s}$ since in that case the ideal (τ) would plainly not be included under the ideal represented by the product. It follows, therefore, that the ideal (τ) is one and the same with the ideal represented by the product (13). The integral ideals of the ordinary theory then coincide with the ideals built on the integral bases (τ).

From the above one sees that any integral ideal is built on a basis (τ) in which the orders of coincidence are those which are common to all the numbers of the ideal. Taking any number α of the ideal built on (τ) we can evidently select another number β of the ideal which has in common with α only those orders of coincidence which are furnished by the basis (τ). Designating by $\mathfrak{v}$ the ideal made up of all the integral numbers of the corpus, the G.C.D. of the *principal* ideals $\mathfrak{v}\alpha$ and $\mathfrak{v}\beta$ is, in the notation of the ordinary theory, indicated by the sum $\mathfrak{v}\alpha+\mathfrak{v}\beta$. This G.C.D. is an ideal and the orders of coincidence common to all the numbers of this ideal evidently coincide with those furnished by the basis (τ). The ideal built on the basis (τ) is then identical with the ideal $\mathfrak{v}\alpha+\mathfrak{v}\beta$.

Fractional, as well as integral ideals, may be conceived as built on a basis (τ) and may be represented in the form (13), where a number of the exponents t_s are negative. In this case, too, the order of coincidence furnished by the basis (τ) for the sth cycle is t_s/ν_s. Suppose the exponents $t_1, t_2, \ldots, t_\lambda$ to be negative, and to have the values $-q_1, -q_2, \ldots, -q_\lambda$. The integral ideals $\mathfrak{p}_1^{q_1}, \ldots, \mathfrak{p}_\lambda^{q_\lambda}$ and $\mathfrak{p}_{\lambda+1}^{t_{\lambda+1}} \ldots, \mathfrak{p}_w^{t_w}$ are built on bases which we shall indicate by $(\tau)_1$ and $(\tau)_2$ respectively. The ideal (13) consists of all those numbers of the corpus whose product by the ideal $\mathfrak{p}_1^{q_1} \ldots \mathfrak{p}_\lambda^{q_\lambda}$ are included under the ideal $\mathfrak{p}_{\lambda+1}^{t_{\lambda+1}} \ldots \mathfrak{p}_w^{t_w}$. The ideal built on the basis (τ) evidently consists of all those numbers of the corpus whose products by the numbers of the ideal built on the basis $(\tau)_1$ are included under the numbers of the ideal built on the basis $(\tau)_2$. The ideal (13) then evidently coincides with the ideal built on the basis (τ).

XII

In Section V we have seen that the principal coefficient in a number $R(\epsilon)$ is the trace $T\{R(\epsilon)/f'(\epsilon)\}$. If $R(\epsilon)$ is adjoint relatively to a prime p, the trace here in question is integral relatively to this prime. If the product $f'(\epsilon)R(\epsilon)$ is adjoint relatively to p, the trace $T\{R(\epsilon)\}$ is integral relatively to the prime. If then the orders of coincidence of $R(\epsilon)$ relative to p are equal to or greater than the numbers

$$(1) \qquad -m_s-1+1/\nu_s, \ (s=1, 2, \ldots, r),$$

for the respective cycles corresponding to the prime the trace $\{TR(\epsilon)\}$ is integral relatively to p.

When the elements of the two bases (τ) and $(\bar{\tau})$ are connected by the relations

$$(2) \qquad \tau_s^{(l)}+\bar{\tau}_s^{(l)}=\mathfrak{m}_s^{(l)}-m_s^{(l)}-1+1/\nu_s^{(l)}, \ (s=1,\ 2,\ \ldots,\ r_l),$$

where the numbers $\mathfrak{m}_s^{(l)}$ represent the orders of coincidence of an actually existing number $\rho(\epsilon)$, we say that the bases (τ) and $(\bar{\tau})$ are complementary with regard to the level furnished by the number $\rho(\epsilon)$. In particular, when we have

$$\tau_s^{(l)}+\bar{\tau}_s^{(l)}=\mu_s^{(l)}-m_s^{(l)}-1+1/\nu_s^{(l)}, \ (s=1,\ 2,\ \ldots,r_l),$$

the bases are complementary with regard to the level furnished by the number $f'(\epsilon)$. When we have

$$(3) \qquad \tau_s^{(l)}+\bar{\tau}_s^{(l)}=-m_s^{(l)}-1+1/\nu_s^{(l)}, \ (s=1,\ 2,\ \ldots,\ r_l),$$

the bases are complementary with regard to the level furnished by the constant 1. In this case, we shall also say that the bases are complementary with regard to the 0-level. Furthermore, with reference to a single prime p_l we shall say that the sets of orders of coincidence $\tau_1^{(l)}, \ldots, \tau_{r_l}^{(l)}$ and $\bar{\tau}_1^{(l)}, \ldots, \bar{\tau}_{r_l}^{(l)}$ are complementary with regard to the 0-level, when they satisfy the conditions (3).

When, with reference to bases or sets of order numbers, we employ the term complementary without further qualifying words, we shall mean complementary with regard to the 0-level. The word complementary so employed one may note bears no reference to the representation of the numbers of the corpus in terms of one primitive number rather than another.

If the orders of coincidence of two numbers $R(\epsilon)$ and $\overline{R}(\epsilon)$ relative to the prime p are complementary with regard to the 0-level the trace of their product $T\{R(\epsilon)\overline{R}(\epsilon)\}$ is evidently integral relatively to p. Also if $R(\epsilon)$ represents the general number of the corpus $C(\epsilon)$ which is conditioned by a set of orders of coincidence $\tau_1, \ldots, \tau_r$ for a specific prime p the necessary and sufficient condition in order that the orders of coincidence $\bar{\tau}_1, \ldots, \bar{\tau}_r$ of a number $\overline{R}(\epsilon)$ may be complementary to the orders of coincidence $\tau_1, \ldots, \tau_r$ is that the trace of the product $R(\epsilon)\overline{R}(\epsilon)$ be integral relatively to p. For the trace of the product $R(\epsilon)\overline{R}(\epsilon)$ is the principal coefficient in the product $f'(\epsilon)R(\epsilon)\overline{R}(\epsilon)$ and the requirement that this principal coefficient be integral relatively to the prime p, in the case where $R(\epsilon)$ has the general character here in question, imposes, as we know, on $\overline{R}(\epsilon)$ the same conditions as the requirement that the principal residue relative to p in the product be 0.

The ideals $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ built on the bases (τ) and $(\bar{\tau})$ which are complementary with regard to the 0-level, we shall call complementary ideals. It is readily shown that this definition is in accord with the definition given in the ordinary theory of the ideals. For both definitions involve the integral character of the trace $T(\alpha\bar{\alpha})$ where α and $\bar{\alpha}$ are any arbitrary numbers belonging to the ideals $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ respectively, and, where the ideal $\mathfrak{a}$ is given, the integral character of this trace completely determines the ideal $\bar{\mathfrak{a}}$ as built on the basis $(\bar{\tau})$ as we know, from the theory developed in the present paper. It is evident that the product $\mathfrak{a}\ \bar{\mathfrak{a}}$ in the sense of the ordinary theory, of two complementary ideals, is the ideal $\bar{\mathfrak{o}}$ which is complementary to the ideal $\mathfrak{o}$ built on the 0-basis, that is to say, to the ideal made up of all the integral numbers of the corpus.

Where $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ are the bases of complementary ideals $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ we have from formula (11) Section XI for the product of the discriminants of the ideals

$$\Delta(\alpha_1, \alpha_2, \ldots, \alpha_n)\Delta(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n) = \Delta(\tau)\Delta(\bar{\tau}) = 1. \tag{4}$$

The discriminant of the ideal $\bar{\mathfrak{d}}$ is D^{-1} where D is the fundamental number of the corpus.

The modulus $\bar{\mathfrak{a}}$ which is complementary to a modulus $\mathfrak{a}$ of order n in the corpus $C(\epsilon)$ is made up of all those numbers $\bar{\alpha}$ in the corpus for which we have $T(\alpha\bar{\alpha})$ integral where α represents an arbitrary number of the modulus $\mathfrak{a}$. This is an immediate consequence of the definition ordinarily given for the complementary moduli. If then we represent the numbers of the corpus in terms of the number ϵ we see in the product $f'(\epsilon)R(\epsilon)\overline{R}(\epsilon)$ where $R(\epsilon)$ and $\overline{R}(\epsilon)$ represent the general numbers of the moduli $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ respectively that the principal coefficient is integral, for this coefficient, as we know, is $T\{R(\epsilon)\overline{R}(\epsilon)\}$. Also evidently we can define the form of the general number $\overline{R}(\epsilon)$ of the modulus $\bar{\mathfrak{a}}$ by the condition that in the product $f'(\epsilon)R(\epsilon)\overline{R}(\epsilon)$ the principal residues relative to the various primes should be 0 where $R(\epsilon)$ represents the general number of the modulus $\mathfrak{a}$ and where $\overline{R}(\epsilon)$, to begin with, is of any form which is sufficiently general to include the complementary modulus $\bar{\mathfrak{a}}$. We might extend the meaning of the expression, complementary adjoint, and say of a modulus $\bar{\mathfrak{a}}$ that it is complementary adjoint to a modulus $\mathfrak{a}$ relatively to a prime p and with reference to the equation $f(x) = 0$ when in the product $R(\epsilon)\overline{R}(\epsilon)$ of the general numbers of the two moduli the principal residue relative to p is identically 0. This is equivalent to saying that the trace $T\{R(\epsilon)\overline{R}(\epsilon)/f'(\epsilon)\}$ is identically integral.

In what precedes we have developed the conception of the p-adic order numbers of an algebraic number and by their aid we have arrived at a clear-cut definition of the ideals which should serve as a satisfactory basis on which to build up the theory of these moduli. We have also obtained results that illustrate the use of the conception, which conception, for the rest, should prove an effective aid in handling the algebraic numbers in general for, as is evident, an algebraic number regarded as a member of a corpus is determined to a unit factor by its p-adic order numbers.

ON THE ANALOGUE OF BERNOULLIAN NUMBERS IN QUADRATIC FIELDS

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Let m denote a positive integer without square factors and let us consider the quadratic number-field $k(\sqrt{-m})$, which we suppose to have only one class of ideals. This happens when $m=1, 2, 3, 7, 11, 19, 43, 67, 163$, no further case of this kind being known. Putting

$$\theta=\sqrt{-m}, \text{ when } m=1 \text{ or } 2,$$

$$\theta=\frac{1+\sqrt{-m}}{2}, \text{ when } m=3, 7, 11, 19, 43, 67, 163,$$

let us consider the Weierstrassian function $\wp(u)$ with the periods 2ω, $2\omega\theta$, where

$$\omega=\int_{e_1}^{\infty}\frac{dx}{\sqrt{4x^3-g_2x-g_3}}$$

and e_1 denotes one of the roots of the equation

$$4x^3-g_2x-g_3=0,$$

the invariants g_2, g_3 being, as is indicated in the table below, suitably chosen rational numbers. From the expansion of $\wp(u)$ in power series

$$\wp(u)=\frac{1}{u^2}+\sum_{n=2}^{\infty}\frac{2^{2n}E_n}{2n}\frac{u^{2n-2}}{(2n-2)!}$$

we infer that the E_n are rational numbers (we have $E_n=0$ unless $n\equiv 0 \pmod 2$ in the case $m=1$, or $n\equiv 0 \pmod 3$ in the case $m=3$), which may be considered as analogues of Bernoullian numbers. Not only can they be expressed by means of the sums

$$\sum\frac{1}{(a+b\theta)^{2n}} \quad (a \text{ and } b \text{ integers})$$

extending to all the integers $a+b\theta$, zero excluded, of the quadratic field considered, namely

$$E_n=\frac{2n!}{(4\omega)^{2n}}\sum\frac{1}{(a+b\theta)^{2n}},$$

but their arithmetical properties also present the most striking analogies with those of Bernoullian numbers, Staudt's theorem included. Hurwitz was the first who studied the arithmetical properties of the numbers E_n in the case $m=1$ in a well known and very remarkable paper*. Cases $m=3$ and $m=2$ have also been considered by K. Matter† and E. Dintzl‡ respectively, but Matter's result is incomplete and his deduction is erroneous, while Dintzl has stated his result without sufficient proof.

Further investigation and development of Hurwitz's original method has enabled us to obtain and rigorously prove the following results analogous to those given by Staudt's theorem concerning the ordinary Bernoullian numbers. First, suppose $m=2$, 3 or 7; then every number E_n, provided $n\equiv 0$ (mod. 3) in the case $m=3$, can be resolved into a sum of partial fractions as follows:

$$E_n = C_n + \frac{q}{2^k} + \sum \frac{(\epsilon . 2a)^{\frac{2n}{p-1}}}{p}. \tag{1}$$

Here C_n denotes the integral part of E_n and the sum is extended to all the odd prime numbers p capable of being represented by the form: $p=a^2+mb^2$ and such that $2n$ is divisible by $p-1$. The sign of a in (1) is determined by the condition

$$a\equiv 1 \text{ (mod. 4) in the case } m=2,$$

$$\left(\frac{a}{m}\right)=1 \text{ in the case } m=3 \text{ or } 7.$$

Moreover

$$\epsilon=\begin{cases}(-1)^{\frac{a^2-1}{8}+\sigma\frac{b+\sigma}{2}+\frac{b(b+1)}{2}}, & \sigma\equiv b \text{ (mod. 2) when } m=2,\\ (-1)^{\frac{p-1}{2}} & \text{when } m=3,\\ 1 & \text{when } m=7,\end{cases}$$

and

$$\frac{q}{2^k}=\begin{cases}0 & \text{when } m=2,\\ \dfrac{(-1)^n}{4} & \text{when } m=3,\\ \dfrac{1}{2} & \text{when } m=7.\end{cases}$$

It must be borne in mind that n is supposed to be divisible by 3 in the case $m=3$, for otherwise E_n is $=0$.

*A. Hurwitz, *Ueber die Entwickelungskoefficienten der lemniskatischen Funktionen.* Mathematische Annalen, Bd. 51.

†K. Matter, *Die den Bernoullischen Zahlen analogen Zahlen im Körper der dritten Einheitswurzeln.* Vierteljahresschrift Natur. Gesell. Zürich, 45 Jahrg. 1900.

‡E. Dintzl, *Uber die Zahlen im Körper K* ($\sqrt{-2}$), *welche den Bernoullischen Zahlen analog sind.* Wiener Berichte, 1909.

In the cases $m=11, 19, 43, 67, 163$ the resolution of E_n into a sum of partial fractions is somewhat simpler, namely

$$E_n=C_n+\sum\frac{(\epsilon a)^{\frac{2n}{p-1}}}{p}.$$

Here again C_n is the integral part of E_n and p denotes all the odd primes capable of being represented by the form $\alpha^2+\alpha\beta+\dfrac{1+m}{4}\beta^2$ and satisfying the congruence $2n\equiv 0 \pmod{p-1}$. As to the number a it is completely determined by the equation

$$4p=a^2+mb^2$$

and the condition

$$\left(\frac{a}{m}\right)=1.$$

Moreover

$$\epsilon=(-1)^{\frac{p+1}{2}}.$$

Now let us consider the numbers E_n' and E_n'' defined respectively by the equations

$$\sum\frac{1}{(a+b\theta)^{2n}}=\frac{(4\omega)^{2n}}{2n!}E_n' \qquad b\equiv 0 \pmod{2},$$

$$\sum\frac{1}{(a+b\theta)^{2n}}=\frac{(4\omega)^{2n}}{2n!}E_n'' \qquad b\equiv 1 \pmod{2},$$

the corresponding sums being extended to all the integers $a+b\theta$ of the given field with either an *even* or an *odd* b respectively. Evidently we have

$$E_n'+E_n''=E_n$$

When $m=3$ or 7 E_n' and E_n'' are rational numbers and

$$E_n'=C_n'+\frac{\beta}{2^k}+\sum\frac{(\epsilon\,.\,2a)^{\frac{2n}{p-1}}}{p} \qquad (n\equiv 0 \pmod 3 \text{ if } m=3),$$

$$E_n''=C_n''+\frac{\beta}{2^k},$$

where C_n', C_n'' are integers and

$$\frac{\beta}{2^k}=\begin{cases}\dfrac{(-1)^n}{8}, & \text{when } m=3,\\[2mm] \dfrac{1}{4}, & \text{when } m=7,\end{cases}$$

all the other notations being the same as above.

When $m=11, 19, 43, 67, 163$ E_n' and E_n'' are no longer rational. They can however be expressed as follows

$$E_n' = \frac{2^{2n}+1}{2^{n+1}} E_n - \frac{nH_n}{2^{4n}},$$

$$E_n'' = \frac{2^{2n}-1}{2^{2n+1}} E_n + \frac{nH_n}{2^{4n}},$$

H_n being an algebraic integer. It is remarkable that the fractional parts of E_n' and E_n'' are rational; in fact we have

$$E_n' = C_n' + \frac{q}{2} + \sum \frac{(\epsilon a)^{\overline{\frac{2n}{p-1}}}}{p},$$

$$E_n'' = C_n'' + \frac{q}{2},$$

where $q=0$ or 1 and C_n', C_n'' are algebraic integers; ϵ, a, p having the same meaning as above.

Table of the numbers E_n

Field $K(\sqrt{-2})$: $g_2=\frac{40}{3}$, $g_3=\frac{224}{27}$,

$E_2=\frac{1}{3}$	$=-1+\frac{2^2}{3}$,	$3=1^2+2.1^2$,
$E_3=\frac{2}{3}$	$=-2+\frac{2^3}{3}$,	
$E_4=\frac{2.5}{3}$	$=-2+\frac{2^4}{3}$,	
$E_5=\frac{2^2.5^2.7}{3.11}$	$=10+\frac{2^5}{3}+\frac{6}{11}$,	$11=(-3)^2+2.1^2$,
$E_6=\frac{2^2.5^2.7}{3}$	$=212+\frac{2^6}{3}$,	
$E_7=\frac{2^3.5^2.7^2}{3}$	$=3224+\frac{2^7}{3}$,	
$E_8=\frac{2^3.5^4.7^2.13}{3.17}$	$=62366+\frac{2^8}{3}-\frac{6}{17}$,	$17=(-3)^2+2.2^2$,
$E_9=\frac{2^4.5^3.7^2.13.67}{3.19}$	$=1497338+\frac{2^9}{3}+\frac{2}{19}$,	$19=1^2+2.3^2$,
$E_{10}=\frac{2^4.5^4.7^2.13.233}{3.11}$	$=C_{10}+\frac{2^{10}}{3}+\frac{6^2}{11}$,	

$$E_{11} = \frac{2^5.5^5.7^3.11.13}{3} = C_{11} + \frac{2^{11}}{3},$$

$$E_{12} = \frac{2^5.5^3.7^3.13.11971}{3} = C_{12} + \frac{2^{12}}{3}.$$

Field $K(\sqrt{-3})$: $g_2 = 0,\ g_3 = 4,$

$E_n = 0$, unless n is divisible by 3.

$$E_3 = \frac{3^2}{4.7} = -\frac{1}{4} + \frac{4}{7}, \qquad 7 = (-2)^2 + 3.1^2,$$

$$E_6 = \frac{3^5.5^2}{4.7.13} = 14 + \frac{1}{4} + \frac{4^2}{7} + \frac{2}{13}, \qquad 13 = 1^2 + 3.2^2,$$

$$E_9 = \frac{3^8.5^3.11}{4.7.19} = 16949 - \frac{1}{4} + \frac{4^3}{7} - \frac{8}{19}, \qquad 19 = 4^2 + 3.1^2,$$

$$E_{12} = \frac{3^{11}.5^3.11^2.17}{4.7.13} = 125134365 + \frac{1}{4} + \frac{4^4}{7} + \frac{2^2}{13},$$

$$E_{15} = \frac{3^{14}.5^6.11^2.17.23}{4.7.31} = 4073427533687 - \frac{1}{4} + \frac{4^5}{7} + \frac{4}{31},$$

$$31 = (-2)^2 + 3.3^2,$$

$$E_{18} = \frac{3^{17}.5^7.11^3.17^2.23.29.43}{4.7.13.19.37} = C_{18} + \frac{1}{4} + \frac{4^6}{7} + \frac{2^3}{13} + \frac{8^2}{19} - \frac{10}{37},$$

$$37 = (-5)^2 + 3.2^2,$$

$$E_{21} = \frac{3^{20}.5^8.11^3.17^2.23.29.431}{4.7.43} = C_{21} - \frac{1}{4} + \frac{4^7}{7} - \frac{8}{43}, \qquad 43 = 4^2 + 3.3^2,$$

$$E_{24} = \frac{3^{23}.5^8.11^4.17^2.23^2.29.41.313}{4.7.13} = C_{24} + \frac{1}{4} + \frac{4^8}{7} + \frac{2^4}{13},$$

$$E_{27} = \frac{3^{26}.5^{10}.11^4.17^3.23^2.29.41.47.1201}{4.7.19} = C_{27} - \frac{1}{4} + \frac{4^9}{7} - \frac{8^3}{19},$$

$$E_{30} = \frac{3^{29}.5^{13}.11^5.17^3.23^2.29^2.41.47.53.74743}{4.7.13.31.61} = C_{36} + \frac{1}{4} + \frac{4^{10}}{7} + \frac{2^5}{13} + \frac{4^2}{31} + \frac{14}{61},$$

$$61 = 7^2 + 3.2^2,$$

and so on.

Field $K(\sqrt{-7})$: $g_2 = 140,\ g_3 = 392,$

$$E_2 = \frac{7}{2}, \qquad = 3 + \frac{1}{2},$$

$$E_3 = \frac{3^2.7}{2} = 31 + \frac{1}{2},$$

$$E_4 = \frac{3.5.7^2}{2} = 367 + \frac{1}{2},$$

$$E_5 = \frac{3^3.5^2.7^3}{2.11} = 10523 + \frac{1}{2} + \frac{4}{11}, \qquad 11 = 2^2 + 7.1^2,$$

$$E_6 = \frac{3^4.5^2.7^3}{2} = 347287 + \frac{1}{2},$$

$$E_7 = \frac{3^4.5^2.7^5}{2} = 17017087 + \frac{1}{2},$$

$$E_8 = \frac{3^4.5^3.7^5.13}{2} = 1106110687 + \frac{1}{2},$$

$$E_9 = \frac{3^8.5^3.7^5.13}{2} = C_9 + \frac{1}{2},$$

$$E_{10} = \frac{3^6.5^4.7^6.13.17^2}{2.11} = C_{10} + \frac{1}{2} + \frac{4^2}{11},$$

$$E_{11} = \frac{3^7.5^4.7^7.11.13.17.19}{2.23} = C_{11} + \frac{1}{2} + \frac{8}{23}, \qquad 23 = 4^2 + 7.1^2,$$

$$E_{12} = \frac{3^7.5^3.7^7.13.17.19.353}{2} = C_{12} + \frac{1}{2},$$

$$E_{13} = \frac{3^{10}.5^5.7^8.13^2.17.19}{2} = C_{13} + \frac{1}{2},$$

$$E_{14} = \frac{3^9.5^5.7^{10}.13^2.17.19.359}{2.29} = C_{14} + \frac{1}{2} + \frac{2}{29}, \qquad 29 = 1^2 + 7.2^2,$$

and so on.

Field $K(\sqrt{-11})$: $\quad g_2 = \frac{2^7.11}{3}, \quad g_3 = \frac{2^6.7.11^2}{3^3},$

$$E_2 = \frac{2^4.11}{3.5} = 12 + \frac{1}{3} - \frac{3}{5}, \qquad 4.3 = 1^2 + 11.1^2, \quad 4.5 = 3^2 + 11.1^2,$$

$$E_3 = \frac{2^2.11^2}{3} = 161 + \frac{1}{3},$$

$$E_4 = \frac{2^9.11^2}{3.5} = C_4 + \frac{1}{3} + \frac{3^2}{5},$$

$$E_5 = \frac{2^7.5.7.11^2}{3} = C_5 + \frac{1}{3},$$

$$E_6 = \frac{2^6.7.11^3.283}{3.5} \qquad = C_6 + \frac{1}{3} - \frac{3^3}{5},$$

$$E_7 = \frac{2^{12}.7^2.11^4}{3} \qquad = C_7 + \frac{1}{3},$$

$$E_8 = \frac{2^{14}.7^2.11^5.13}{3.5} \qquad = C_8 + \frac{1}{3} + \frac{3^4}{5},$$

$$E_9 = \frac{2^{10}.7^2.11^5.13.467}{3} \qquad = C_9 + \frac{1}{3},$$

$$E_{10} = \frac{2^{16}.11^5.13.17.19061}{3.5} \qquad = C_{10} + \frac{1}{3} - \frac{3^5}{5},$$

$$E_{11} = \frac{2^{12}.7^2.11^7.13.17.19.2744}{3.23} \qquad = C_{11} + \frac{1}{3} + \frac{9}{23}, \qquad 4.23 = 9^2 + 11.1^2,$$

and so on.

Field $K(\sqrt{-19})$: $\qquad g_2 = 2^7.19,\ g_3 = 2^6.19^2,$

$$E_2 = \frac{2^4.19}{5} \qquad = 61 - \frac{1}{5}, \qquad 4.5 = 1^2 + 19^2,$$

$$E_3 = \frac{2^2.3^2.19^2}{7} \qquad = C_3 - \frac{3}{7}, \qquad 4.7 = (-3)^2 + 19.1^2,$$

$$E_4 = \frac{2^9.3.19^2}{5} \qquad = C_4 + \frac{1}{5},$$

$$E_5 = \frac{2^7.3^3.5.19^3}{11} \qquad = C_5 + \frac{5}{11}, \qquad 4.11 = 5^2 + 19.1^2,$$

$$E_6 = \frac{2^6.3^4.19^3.1513}{5.7} \qquad = C_6 - \frac{1}{5} + \frac{3^2}{7},$$

$$E_7 = 2^{12}.3^4.7.19^4$$

$$E_8 = \frac{2^{14}.3^4.19^4.38623}{5.17} \qquad = C_8 + \frac{1}{5} - \frac{7}{17}, \qquad 4.17 = 7^2 + 19.1^2,$$

$$E_9 = \frac{2^{10}.3^8.13.19^4.16001}{7} \qquad = C_9 - \frac{3^3}{7},$$

$$E_{10} = \frac{2^{16}.3^6.19^5.4973449}{5.11} \qquad = C_{10} - \frac{1}{5} + \frac{5^2}{11},$$

$$E_{11} = \frac{2^{15}.3^7.11.19^6.3316183}{23} \qquad = C_{11} + \frac{4}{23}, \qquad 4.23 = 4^2 + 19.2^2,$$

and so on.

Field $K(\sqrt{-43})$: $g_2=2^8.5.43,\ g_3=2^6.3.7.43^2$,

$E_2 = 2^5.43$,

$E_3 = 3^3.2^2.43^2$,

$E_4 = 2^{11}.3.5.43^2$,

$$E_5 = \frac{2^8.3.{}^45^2.7.43^3}{11} = C_5+\frac{1}{11}, \qquad 4.11=1^2+43.1^2,$$

$$E_6 = \frac{2^6.3^4.5^2.7.43^3.3209}{13} = C_6+\frac{3}{13}, \qquad 4.13=(-3)^2+43.1^2,$$

$E_7 = 2^{14}.3^5.5^2.7^2.43^4$

$$E_8 = \frac{2^{15}.3^4.5^3.7^2.43^4.6299}{17} = C_8+\frac{5}{17}, \qquad 4.17=(-5)^3+43.1^2,$$

$E_9 = 2^{10}.3^9.5^3.7^2.43^5.1787$

$$E_{10} = \frac{2^{18}.3^6.5^4.7^2.43^5.810893}{11} = C_{10}+\frac{1}{11},$$

$$E_{11} = \frac{2^{16}.3^8.5^4.7^2.11.19.43^6.171079}{23} = C_{11}-\frac{7}{23}, \qquad 4.23=(-7)^2+43.1^2,$$

and so on.

Field $K(\sqrt{-67})$: $g_2=2^7.5.11.67,\ g_3=2^6.7.67^2.31$,

$E_2 = 2^4.11.67$,

$E_3 = 2^2.3^2.31.67^2$,

$E_4 = 2^9.3.5.11^2.67^2$,

$E_5 = 2^7.3^3.5^2.7.31.67^3$,

$E_6 = 2^6.3^4.5^2.7^2.67^3.5867$,

$E_7 = 2^{12}.3^4.5^2.7^2.11^2.31.67^4$,

$$E_8 = \frac{2^{14}.3^4.5^4.7^2.11.13.67^4.16123}{17} = C_8-\frac{1}{17}, \qquad 4.17=1^2+67.1^2,$$

$$E_9 = \frac{2^{10}.3^7.5^3.7^2.11.13.31.67^5.118509}{19} = C_9-\frac{3}{19}, \qquad 4.19=(-3)^2+67.1^2,$$

$E_{10} = 2^{16}.3^6.5^4.7^4.11.13.67^5.211793$

$$E_{11} = \frac{2^{15}.3^7.5^5.7^3.11^2.13.31.67^6.1385107}{23} = C_{11}-\frac{5}{23}, \qquad 4.23=(-5)^2+67.1^2,$$

and so on.

Field $K(\sqrt{-163})$, $g_2 = 2^8.5.23.29.163,$

$g_3 = 2^6.7.11.19.127.163^2,$

$E_2 = 2^5.23.29.163,$

$E_3 = 2^2.3^2.11.19.127.163^2,$

$E_4 = 2^{11}.3.5.23^2.29^2.163^2,$

$E_5 = 2^8.3^3.5^2.7.19.23.29.127.163^3,$

$E_6 = 2^6.3^4.5^3.7.163^3.14649862729,$

$E_7 = 2^{14}.3^4.5^2.7^2.11.19.23^2.29^2.127.163^4,$

$E_8 = 2^{15}.3^4.5^3.7^2.11.13.23.29.163^4.768889061,$

$E_9 = 2^{10}.3^8.5^4.7^2.11.13.127.163^5.155003745377,$

$E_{10} = 2^{18}.3^8.5^4.7^2.11.13.17.23^2.29^2.163^5.8998348729.$

Integers up to E_{19} inclusive increase extremely rapidly.

GENERAL CLASS NUMBER RELATIONS WHOSE DEGENERATES INVOLVE INDEFINITE FORMS

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1. Let each of $f(x)$, $g(x)$ take a single definite value for each positive, zero or negative integer value x, and satisfy respectively

$$f(-x)=f(x),\ g(-x)=-g(x),\ g(0)=0.$$

Beyond these conditions $f(x)$, $g(x)$ are arbitrary in the widest sense.

If for integers a_i, b_i $(i=0, 1, \ldots)$,

$$a_o+\sum_{i=1} a_i \cos b_i t=0$$

is an identity in t, then

$$a_o f(0)+\sum_{i=1} a_i f(b_i)=0.$$

There is a corresponding theorem for sines and $g(x)$. Both are included as very special cases in a theorem established elsewhere.*

The passage from circular to arbitrary odd or even functions is the simplest example of what I have called paraphrase.

To obtain paraphrases involving F, F_1, where as customary $F(n)$ is the number of odd, $F_1(n)$ the number of even classes of binary quadratic forms of negative determinant $-n$, we proceed as above indicated from those identities between circular functions which are derived by equating like powers of q in identities between theta quotients and products of which at least one contains F or F_1 in its Fourier expansion.

As a necessary preliminary to paraphrase the theta expansions are reduced to the form

$$A_o+\sum_{s=1} A_s q^s,$$

in which $A_n (n=0, 1, \ldots)$ is a function, in general involving circular functions, of the divisors of the exponent n. We shall say that such an expansion is in divisor form.

The resulting class number paraphrases have been called singly infinite†,

*Trans-Amer. Math. Soc., vol. 21, p. 1.

†Quarterly Journal of Mathematics, January 1923. There are several obvious misprints in the paper, particularly in transpositions of upper summation limits.

for evidently any such paraphrase by special choices of f or g, as for instance

$$f(x)=(-1)^x,\ f(x)=1^x,\ f(x)=x^{2m},\ g(x)=x^{2m-1},$$

gives rise to an infinity of different class number relations free from arbitrary functions.

In each of the singly infinite class number paraphrases cited there occur also the integers u, v, y, z representing a fixed integer k in an associated quaternary quadratic form,

$$k=auv+byz,$$

where a, b are constant integers.

The restriction that the associated form be of the second degree is removed in a forthcoming paper*.

Finally, by paraphrasing identities between products of more than two divisor expansions, we can increase the number of indeterminates in the associated forms at will. The scope of this powerful method of paraphrase is limited only by our ability to obtain the divisor expansions of theta quotients and products of increasing complexity.

I have reduced all of the theta expansions of Hermite, Kronecker, Biehler, Appell, Petr, Humbert and several others to divisor form, and for some time have been accumulating divisor expansions of many other theta products and quotients; a first selection of about 200 will appear in the *Messenger of Mathematics*. This set is meant in references later to the list of expansions.

By means of the list it is possible to extend the known class number relations indefinitely in many directions. Each of the associated quadratic forms in the class number paraphrases which I have so far published contains an even number of indeterminates. It is equally easy, however, to derive such paraphrases in which the number of indeterminates in the associated quadratic form is odd. It is also easy to obtain class number paraphrases in which several associated forms, not necessarily quadratic, are involved.

To illustrate some of the foregoing remarks it will suffice here to exhibit a class number paraphrase in which the associated quadratic form is ternary, and to show how the list of expansions is to be used for the derivation of further relations involving ternary quadratic forms. This particular application of the method is of peculiar interest because, for certain special choices of f, the class number paraphrases degenerate to class number relations in which the associated quadratic forms are binary and indefinite, that is, of positive determinants. One special choice of f effecting this is $f=\phi$, where

$$\phi(x)=1 \text{ if } |x|=c,\ \ \phi(x)=0 \text{ if } |x|\neq c,$$

and c is a constant integer.

We recall that class number relations involving an indefinite binary quadratic form were first obtained from theta expansions in 1901 by Petr, and independently in 1907 by Humbert†.

*Annali di Matematiche, 1924.

†Cf. Dickson's *History of the Theory of Numbers*, vol. 3, chap. 6.

Let us start from one of the simplest of Petr's and Humbert's functions,

$$\Psi(x) \equiv \theta_2\theta_3 \frac{\theta_1^2(x)\,\theta_2(x)\,\theta_3(x)}{\theta_0^2(x)},$$

and paraphrase the identity

$$(1) \qquad \Psi(x) \times \theta_0 = \theta_1(x) \times \theta'_1 \frac{\theta_1(x)\,\theta_2(x)\,\theta_3(x)}{\theta_0^2(x)}.$$

2. The divisor form of the expansion of $\Psi(x)$ is

$$2\Sigma q^n[\Sigma' F(8n-b^2)\cos bx - 2\Sigma'(2t-\tau)\cos(2t+\tau)x],$$

Σ referring to all integers $n>0$, the first Σ' to all odd integers $b \lessgtr 0$ such that $8n-b^2>0$, the second to all pairs (t,τ) of integers >0 such that τ is odd, $n=t\tau$, $\tau<\sqrt{2n}$. The divisor expansion of the second factor on the right of (1) is

$$-2\Sigma q^{\frac{\beta}{4}}\left[\Sigma'(-1|\delta)(\delta-d)\sin\left(\frac{\delta+d}{2}\right)x\right],$$

in which Σ refers to all positive integers $\beta\equiv 3 \bmod 4$, Σ' to all pairs (d,δ) of integers >0 such that $\beta=d\delta$, $d<\sqrt{\beta}$. By the process outlined in §1, it is immediately seen from these expansions that the paraphrase of (1) is

$$(2) \qquad \Sigma(-1|\delta_1\mu_2)(\delta_1-d_1) f\left(\frac{\delta_1+d_1}{2}+\mu_2\right)$$

$$= \Sigma i^{\lambda_2}[F(8n-2\lambda_2^2-b^2) f(b) - 2(2t_1-\tau_1) f(2t_1+\tau_1)],$$

in which $(m|n)$ is the Legendre-Jacobi symbol, $i=\sqrt{-1}$, and the summations refer, for k a fixed integer >0, to all integers d_1, δ_1, μ_2, t_1, τ_1, λ_2, b such that

$$(3) \qquad 4k = d_1\delta_1 + \mu_2^2 = 4t_1\tau_1 + \lambda_2^2,$$

subject to the restrictions

$$(4) \qquad d_1\delta_1 \equiv 3 \bmod 4, \quad d_1,\ \delta_1>0,\ d_1<\delta_1;$$

$$\mu_2 \lessgtr 0,\ \text{odd}; \qquad \lambda_2 \gtreqless 0\ \text{even}; \qquad \tau_1\ \text{odd},\ <2t_1;$$

$$b \lessgtr 0,\ \text{odd}; \qquad 8n-2\lambda_2^2-b^2>0.$$

The quadratic form associated with (2) is therefore of the type y^2+uv.

3. To obtain from (2) a degenerate involving an indefinite binary quadratic form, we take $f=\phi$, where

$$\phi(x)=1 \text{ if } |x|=1, \quad \phi(x)=0 \text{ if } |x| \neq 1.$$

The only arguments giving non-vanishing f's are those for which

$$\frac{\delta_1+d_1}{2}+\mu_2=\pm 1, \quad b=\pm 1;$$

so that (3) reduces in this case to

$$(5) \qquad 4n=\left(\frac{\delta_1+d_1}{2}\pm 1\right)^2+\mu_2^2,$$

subject to the conditions (4). Hence for s, t integers we have, since $d_1\delta_1\equiv 3 \bmod 4$, $d_1<\delta_1$,

$$(6) \qquad \frac{\delta_1+d_1}{2}=2s, \quad \frac{\delta_1-d_1}{2}=2t-1, \quad s>0, \quad t>0.$$

It follows then that in this special degenerate case, (3), (4) are equivalent to

$$(7) \qquad 4n=4s_1^2-(2t_1-1)^2+(2s_1-1)^2, \; s_1,\, t_1>0, \; s_1\geqq t_1,$$

$$(8) \qquad 4n=4s_2^2-(2t_2-1)^2+(2s_2+1)^2, \; s_2,\, t_2>0, \; s_2\geqq t_2,$$

with $b\lesseqgtr 0$ odd, s_1, t_1, s_2, t_2 integers. Hence (2) degenerates to

$$(9) \qquad -2\Sigma[(-1)^{t_1}(2t_1-1)-(-1)^{t_2}(2t_2-1)=\Sigma(-1)^a F(8n-1-8a^2)$$

the Σ on the left referring to all t_1, t_2 defined by (7), (8), that on the right to all integers $a\gtreqless 0$ for which $8n-1-8a^2>0$. But it is easily seen that (7), (8) are equivalent to

$$(10) \qquad 8n-1=x^2-2y^2, \; x,\; y\geqq 0, \; x>2y,$$

where x, y are integers, necessarily odd, and that (9) becomes the class number relation of Humbert

$$(11) \qquad \Sigma(-1)^a F(8n-1-8a^2)=\Sigma y(-1)^{\frac{x+y}{2}},$$

which therefore appears as a degenerate of (2).

4. Without taking space to write out the resulting paraphrases and their degenerates involving indefinite binary quadratic forms, we shall indicate a few of the numerous identities suggested by the list of expansions which lead to such results. The fertility of the list is best suggested by giving a selection of those identities which are based upon the single function Ψ already used as an illustration. The divisor expansions of all the quotients separated by the sign $\times$ are given in the list. The six specimens which follow by no means exhaust the simplest possibilities. We have

$$(12) \qquad \Psi(x)\times\theta_3=\theta_2(x)\times\theta_2\theta_3^2\,\frac{\theta_1^2(x)\theta_3(x)}{\theta_0^2(x)},$$

$$(13) \qquad \Psi(x)\times\theta_2=\theta_3(x)\times\theta_2^2\theta_3\,\frac{\theta_1^2(x)\theta_2(x)}{\theta_0^2(x)},$$

for which the second factor on the right in each case involves class number functions;

$$(14) \qquad \Psi(x)\times\theta_0(q^2)=\theta_1(2x,\, q^2)\times\theta_1'\theta_3\,\frac{\theta_1(x)\theta_3(x)}{\theta_0^2(x)},$$

$$\Psi(x) \times \theta_2(i\sqrt{q}) = \theta_1(x, i\sqrt{q}) \times \theta_1'\theta_2 \frac{\theta_1(x)\theta_2(x)}{\theta_0^2(x)}, \tag{15}$$

$$\Psi(x) \times \theta_1'(\sqrt{q}) = \theta_1(x, \sqrt{q}) \times \theta_1'^2 \frac{\theta_1(x)\theta_2(x)\theta_3(x)}{\theta_0^3(x)}, \tag{16}$$

$$\Psi(x) \times \theta_2(\sqrt{q}) = \theta_2(x, \sqrt{q}) \times \left[\theta_2\theta_3 \frac{\theta_1(x)}{\theta_0(x)}\right]^2, \tag{17}$$

in all four of which the only function involving class numbers is $\Psi(x)$.

As an example of the numerous Ψ identities whose paraphrases involve class numbers and an associate quaternary quadratic form leading to degenerates concerning only definite forms, we may mention one of the simplest,

$$\Psi(x) = \theta_2 \frac{\theta_1(x)\theta_2(x)}{\theta_0(x)} \times \theta_3 \frac{\theta_1(x)\theta_3(x)}{\theta_0(x)}. \tag{18}$$

By inspection of the exponents of q occurring in the list of expansions it is easy to foresee the type of the associated quadratic forms, either for the paraphrases or for their degenerates, without calculation.

Conversely, given a specific quadratic form, it is possible to decide from examination of the list whether or not the form can be an associate in a class number paraphrase or in any of its degenerates.

ON SOME NEW CLASS-NUMBER RELATIONS

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After the first examples of class-number relations had been discovered by Kronecker in 1857, several mathematicians pursued investigations on the same subject along two different lines. The first of these, inaugurated by Kronecker himself, is based on the theory of transformations of elliptic modular functions and seems to afford the clearest insight into the relations in question. It has been highly developed and brought to a certain conclusion in memorable papers by Gierster and especially by Hurwitz. The second method has been indicated by Hermite. Its foundations are much more elementary since it operates only upon the trigonometrical developments of certain combinations of the theta-functions. Hermite's method has been developed and applied to the deduction of a great number of new formulae by several authors among whom we cite P. Nazimoff, K. Petr and particularly G. Humbert, whose remarkable paper*, on account of its completeness and many new results obtained must be regarded as fundamental.

But there exists a third method of investigating the class-number relations, which hitherto has attracted but little attention, although its outlines have been sketched by Liouville. Liouville'smethod is based upon certain general formulae, due to Liouville himself, and which, although closely related to the trigonometrical developments of Hermite, can be obtained by means of the simplest algebraical considerations. "Mes formules," says Liouville, "se rattachent aussi à la théorie des fonctions elliptiques, mais elles contiennent plutôt cette théorie qu'elles n'en dépendent. Je les obtiens toutes au moyen de l'algèbre la plus simple." It will be no wonder then, if Liouville's method, when sufficiently developed, may prove to be as powerful as that of Hermite, presenting the indisputable advantage over it of being completely elementary. It seems, from this point of view, that the whole of Liouville's arithmetical methods would be worth studying more closely. Stieltjes seems to be the only mathematician who occupied himself systematically with Liouville's methods and he advanced so far by their aid as to be able to obtain many peculiar class-number relations

*G. Humbert. *Formules relatives aux nombres de classes des formes quadratiques binaires et positives.* Jour. de Math., 6[e] Série, t. III, 1907.

which have been proved long afterwards by Petr and Humbert by means of the elliptic functions. Careful examination of Liouville's methods leads to the conclusion that they may be completed and developed so as to constitute a simple and uniform way of obtaining all the known class-number relations, those of Hurwitz perhaps excluded, and many new ones in addition. But as the whole development would require more time than we have at our disposition, we shall only point out some of the new results which we have arrived at and which do not seem to be easily obtainable otherwise.

Let $G^*(n)$ denote the number of all the classes of binary quadratic forms $ax^2+2bxy+cy^2$ of a given negative determinant $b^2-ac=-n$, and $F^*(n)$ the number of those classes which are capable of representing *odd* numbers. We introduce two new arithmetical functions, $G(n)$ and $F(n)$, which are defined as follows:

$$\left.\begin{array}{l} G(n)=G^*(n)-\tfrac{1}{2}, \text{ when } n=s^2, \\ G(n)=G^*(n)-\tfrac{2}{3}, \text{ when } n=3s^2, \end{array}\right\} s \text{ an integer,}$$

$$G(n)=G^*(n) \text{ otherwise, with the exception}$$

$$G(0)=-\tfrac{1}{12};$$

and

$$F(n)=F^*(n)-\tfrac{1}{2}, \quad \text{when } n=s^2, \quad s \text{ odd,}$$

$$F(n)=F^*(n) \quad \text{otherwise, with the exception}$$

$$F(0)=0.$$

We have then the following first group of relations which are partly known:

$$(1)\quad 2\Sigma(-1)^h F(n-2h^2)=(-1)^{\frac{(n-1)(n-2)}{2}} \sum_{n=x^2-2y^2,\ x\pm 2y>0} x \;+\; 2(-1)^{\frac{(n-1)(n-2)}{2}}\{s\}_{n=2s^2,\ s>0},$$

$$(2)\quad 2\sum_{h=0,\ \pm 1,\ \pm 2,\dots} F(n-2h^2) = 2\sum_{n=d\delta,\ \delta\ \text{odd}} \left(\frac{2}{\delta}\right) d - \sum_{n=x^2-2y^2,\ x\pm 2y>0} x \;-2\{s\},\ n=2s^2,\ s>0.$$

$$(3)\quad 8\sum_{i=1,3,5,\dots}(-1)^{\frac{i^2-1}{8}} F\left(\frac{m-i^2}{2}\right)=2\sum_{m=x^2-8y^2,\ x>3y>0}\left(\frac{2}{x}\right)y \;+\; \sum_{m=s^2+8t^2,\ s>0}\left(\frac{-2}{s}\right)s \;-\left\{\left(\frac{2}{s}\right)s\right\},\ m=s^2,\ s>0;$$

$$m\equiv 1 \pmod 8.$$

$$(4)\quad \Sigma(-1)^{\frac{i^2-1}{8}} F\left(\frac{m-i^2}{2}\right)=(-1)^{\frac{m-7}{8}} \sum_{m=8x^2-y^2,\ 2x>y>0} x \;;\quad m\equiv 7 \pmod 8.$$

The notations in these formulae explain themselves except that a symbol such as $\{s\}$, $n=2s^2$, $s>0$, denotes a quantity equal to zero, save when $n=2s^2$, $s>0$, in which case it is equal to s. The relations (1) and (2) are those given by Stieltjes; (4) has been given by Humbert; (3) seems to be new.

Next we have the following formulae:

$$(5)\quad 2\sum_{h=0,\pm1,\pm2,\dots} F(n-3h^2)=2^{\alpha-1}\left[3^{\beta+1}-(-1)^{\alpha+\beta}\left(\frac{m}{3}\right)\right]S-\sum_{2n=x^2-3y^2,\ x\pm3y>0} x\quad -\{3s\},\ n=3s^2,\ s>0.$$

$$S=\sum\left(\frac{3}{\delta}\right)d,\ m=d\delta,\ n=2^{\alpha}\,3^{\beta}\,m,\ m \text{ odd and not divisible by } 3.$$

$$(6)\quad \sum_{h=0,\pm1,\pm2,\dots}(-1)^h F(2n-3h^2)=(-1)^{n-1}\sum_{n=x^2-3y^2,\ x\pm3y>0} x\quad -\{3s\},\ n=6s^2,\ s>0.$$

$$(7)\quad 2\sum_{h=0,\pm1,\pm2,\dots}(-1)^h F(m-3h^2)=\sum_{2m=x^2-3y^2,\ x\pm3y>0}(-1)^{\frac{x+y}{2}}y\quad +\sum_{4m=i^2+3j^2,\ i,j\text{ odd},>0}(-1)^{\frac{j-1}{2}}j+\{s\}\,;\ m=3s^2, m\text{ odd}.$$

$$(8)\quad 6\sum_{h=0,\pm1,\pm2,\dots}(-1)^h F\left(\frac{m-h^2}{3}\right)=-\sum_{2m=3x^2-y^2,\ x\pm y>0}(-1)^{\frac{x+y}{2}}y-\sum_{4m=i^2+3j^2,\ i,j\text{ odd},>0}(-1)^{\frac{i-1}{2}}i\ +\{s\},\ m=s^2, s>0, m\text{ odd}.$$

$$(9)\quad \sum_{h=0,\pm1,\pm2,\dots}(-1)^h F\left(\frac{2n-h^2}{3}\right)=(-1)^{n-1}\sum_{n=3x^2-y^2,\ x\pm y>0} x\quad -\left\{\frac{s}{2}\right\},\ 2n=s^2,\ s>0.$$

Some particular cases of (6) and (7) have been given by Professor Mordell. The following results do not seem to have been noticed:

$$(10)\quad 8\sum_{h=0,\pm1,\pm2,\dots}F(m-5h^2)=-4\sum_{\substack{x+y\equiv0\\4m=5x^2-y^2,\ 5x>3y>0}} y\ -2\sum_{x+y\equiv2 \pmod 4} y+N(m=x^2+y^2+z^2+5t^2)-\{10s\}, m=5s^2, s>0;\ m\text{ odd}.$$

$$(11)\quad \sum_{h=0,\pm1,\pm2,\dots}F(2m-5h^2)=\quad -2\sum_{2n=5x^2-y^2,\ 5x>3y>0} y\quad +N(8m=i^2+j^2+k^2+5l^2),\ m\text{ odd}.\quad (i,j,k,l\text{ odd},>0)$$

$$(12)\quad 2\sum_{h=0,\pm1,\pm2,\dots}F(4m-5h^2)=\quad -4\sum_{4m=5x^2-y^2,\ 5x>3y>0} y\quad +N(m=x^2+y^2+z^2+5t^2)-\{10s\}, m=5s^2, s>0,\ m\text{ odd}.$$

$$(13)\quad \sum_{h=0,\pm1,\pm2,\dots}F(8n-5h^2)=-4\sum_{2n=5x^2-y^2,\ 5x>3y>0} y\quad +4N(8n=i^2+j^2+k^2+5l^2)-\{10s\},\ n=10s^2,\ s>0.\quad (i,j,k,l\text{ odd},>0)$$

$$(14)\quad \sum_{h\equiv\pm\sqrt{m}\pmod 5}F\left(\frac{m-h^2}{5}\right)\ =\ -2\sum_{m=i^2-5j^2,\ i>3j>0} j\quad +4N(4m=i^2+5j^2+5k^2+5l^2)-\left\{\frac{s}{2}\right\},\ m=s^2,\ s>0.\quad (i,j,k,l\text{ odd},>0)\quad m\ even$$

$$(15)\quad \sum_{h\equiv\pm\sqrt{m}\pmod 5}F\left(\frac{m-h^2}{5}\right)\ =\ -\tfrac{1}{2}\sum_{\substack{i+j\equiv0\\4m=i^2-5j^2,\ i>3j>0}} j\quad -\tfrac{1}{4}\sum_{i+j\equiv2\pmod 4} j\quad +\tfrac{1}{8}N(m=x^2+5y^2+5z^2+5t^2)-\left\{\frac{s}{4}\right\}, m=s^2,\ s>0,\ m\ odd.$$

$$(16)\quad 6\sum_{h=0,\pm1,\pm2,\dots}(-1)^h G\left(n-\frac{3h^2-h}{2}\right)=\sum_{24n+1=3(2\alpha+1)^2-2(3\beta+1)^2,\ 2\alpha+1\pm(3\beta+1)>0}(-1)^{\alpha-1}(3\beta+1)\quad -\left\{\left(\frac{3}{s}\right)\frac{s}{2}\right\}, 24n+1=s^2, s>0.$$

Here $N(m=x^2+y^2+z^2+5t^2)$ etc., indicate the number of solutions of the equation $m=x^2+y^2+z^2+5t^2$, etc. Their explicit values have been given by Liouville. The purely arithmetical deduction of all these formulae requires additional developments and is therefore reserved for another occasion.

THE EXPANSION OF DETERMINANTS AND PERMANENTS IN TERMS OF SYMMETRIC FUNCTIONS

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Introduction

In this communication reference is made to two papers* in which I have shown that when the constituents of a determinant are expressed in an umbral notation which necessitates the use of one alphabet of letters only, the determinant is a symmetric function of such letters. Ordinary determinants which are denoted by a square array in two dimensions are of Class II and each constituent in the array is expressed by two letters of the Greek alphabet, one letter being placed over the other. When the determinant is of Class σ a constituent of a σ-dimensional array is expressed by σ letters of the Greek alphabet placed vertically over one another. Such a constituent is said to be an umbra of Class σ while the letters of the Greek alphabet may be also termed the umbral letters. A Permanent, which only differs from a Determinant in having all of the signs positive on expansion, is more convenient than the Determinant in the present discussion because the signs of the various terms can be excluded from consideration.

I demonstrated also that when $\sigma=2$ the expansion of the Permanent is expressible as the sum of p_n symmetric functions where p_n denotes the number of the partitions of n and n denotes the number of different Greek letters. When $n=3$, for example, the Permanent is briefly denoted by the notation

$$\overset{+\quad +}{\begin{pmatrix} \alpha\beta\gamma \\ \alpha\beta\gamma \end{pmatrix}}$$

and the corresponding determinant by

$$\begin{pmatrix} \alpha\beta\gamma \\ \alpha\beta\gamma \end{pmatrix}$$

and we have the expansion

*P. A. MacMahon: *Researches in the Theory of Determinants*, Trans. Camb. Phil. Soc., Vol. XXIII, No. V, pp. 89-135, 1924.

P. A. MacMahon: *The Symmetrical Functions of which the General Determinant is a particular case*, Proc. Camb. Phil. Soc., Vol. XXII, Part 5, 1925.

$$\left(\overset{+}{\alpha}\beta\overset{+}{\gamma} \atop \alpha\beta\gamma\right) = \begin{vmatrix} \alpha & \alpha & \alpha \\ \alpha & \beta & \gamma \\ \\ \beta & \beta & \beta \\ \alpha & \beta & \gamma \\ \\ \gamma & \gamma & \gamma \\ \alpha & \beta & \gamma \end{vmatrix}^{+} = \sum{}^{\alpha\beta\gamma}_{\alpha\beta\gamma} + \sum{}^{\alpha\beta\gamma}_{\alpha\gamma\beta} + \sum{}^{\alpha\beta\gamma}_{\beta\gamma\alpha}$$

It will be observed that the notation

$$\left(\overset{+}{\alpha}\beta\overset{+}{\gamma} \atop \alpha\beta\gamma\right)$$

for the Permanent of Class II in three letters indicates that, the first row of letters being retained unpermuted, the second row is to be given in succession all permutations and the 3! (n!) products thus produced are to be added together to obtain the expansion.

Note that when $\sigma=1$, we have for n letters the simple algebraic product

$$\alpha\beta\gamma\ldots\nu.$$

The object of the present paper is the enumeration of the symmetric functions in terms of which the expansion may be exhibited when $\sigma>2$, and n is any integer.

1. When the Permanent

$$\left(\overset{+}{\alpha}\beta\overset{+}{\gamma} \atop \alpha\beta\gamma\right)$$

of Class II is examined it is found that the expansion is separable into three types of terms in agreement with the types of the circular substitutions which, operating upon the letters in the first row, produce the permutations.

Thus the permutation $\alpha\beta\gamma$ is produced by the substitution $(\alpha)(\beta)(\gamma)$ of type $(1)^3$,

"	"	$\alpha\gamma\beta$	"	"	"	$(\alpha)(\beta\gamma)$	of type (1) (2),
"	"	$\gamma\beta\alpha$	"	"	"	$(\beta)(\gamma\alpha)$	
"	"	$\beta\alpha\gamma$	"	"	"	$(\gamma)(\alpha\beta)$	
"	"	$\beta\gamma\alpha$	"	"	"	$(\alpha\beta\gamma)$	of type (3).
"	"	$\gamma\alpha\beta$	"	"	"	$(\alpha\gamma\beta)$	

We may denote the symmetric function expansion by

$$\sum\binom{\alpha\beta\gamma}{(\alpha)(\beta)(\gamma)} + \sum\binom{\alpha\beta\gamma}{(\alpha)(\beta\gamma)} + \sum\binom{\alpha\beta\gamma}{(\beta\gamma\alpha)}$$

or by

$$\sum\binom{\alpha\beta\gamma}{(1)^3} + \sum\binom{\alpha\beta\gamma}{(1)(2)} + \sum\binom{\alpha\beta\gamma}{(3)}.$$

In general we write, when $\sigma=2$,

$$\left(\overset{+}{\alpha}\beta\gamma\ldots\overset{+}{\nu} \atop \alpha\beta\gamma\ldots\nu\right) = \sum\binom{\alpha\beta\gamma\ldots\nu}{(1)^{t_1}(2)^{t_2}(3)^{t_3}\ldots},$$

the summation being for every partition

$$(1^{l_1}2^{l_2}3^{l_3}\ldots)$$

of the number n.

2. The above has been set forth merely to introduce the notation which has been found to be suitable in the generalizations to which we now proceed.

For the cubic Permanent, $\sigma=3$, we have, in the brief notation, three rows of umbral lettters

$$\overset{+\qquad\quad +}{\begin{pmatrix} \alpha\beta\gamma\ldots\nu \\ \alpha\beta\gamma\ldots\nu \\ \alpha\beta\gamma\ldots\nu \end{pmatrix}}.$$

The expansion is obtained by writing the second, and also the third, row in each of the $n!$ permutations so as to obtain $(n!)^2$ products of the n^3 umbrae

$$\begin{matrix} \alpha & \alpha & \alpha \\ \alpha & \alpha & \beta\ldots \\ \alpha, & \beta, & \alpha, \end{matrix}$$

n together.

The n^3 umbrae present themselves in n groups, commencing with the letters $\alpha, \beta, \ldots \nu$ respectixely.

Each product involves an umbral factor from each of these groups.

If we take together (in respect of each row) all those permutations which are derived from substitutions of the same circular type we find

$$p_n^2$$

types of products of umbrae, and each of these types is expressible as the sum of one or more symmetric functions. Hence we find that the Permanent is expressible on expansion as a sum of p_n^2 *groups* of symmetric functions.

In general, for the Permanent of Class σ in n umbral letters, the expansion is obtained by writing the 2nd, 3rd, $\ldots$, σth rows of letters, each of them in $n!$ permutations, so as to obtain $(n!)^{\sigma-1}$ products of the n^σ umbrae, n together. We then find, corresponding to the types of circular substitutions, $p_n^{\sigma-1}$ types of products of umbrae and show that the Permanent can be expanded into a sum of $p_n^{\sigma-1}$ groups of symmetric functions.

2.1 To elucidate the matter take the simplest cubic Permanent

$$\overset{+\;\;+}{\begin{pmatrix} \alpha\beta \\ \alpha\beta \\ \alpha\beta \end{pmatrix}} \quad \sigma=3,\ n=2.$$

The expansion is

$$\sum\begin{pmatrix} \alpha\beta \\ (\alpha)(\beta) \\ (\alpha)(\beta) \end{pmatrix}+\sum\begin{pmatrix} \alpha\beta \\ (\alpha)(\beta) \\ (\alpha\beta) \end{pmatrix}+\sum\begin{pmatrix} \alpha\beta \\ (\alpha\beta) \\ (\alpha)(\beta) \end{pmatrix}+\sum\begin{pmatrix} \alpha\beta \\ (\alpha\beta) \\ (\alpha\beta) \end{pmatrix}$$

$$\begin{matrix} & \alpha\beta & & \alpha\beta & & \alpha\beta & & \alpha\beta \\ = & \alpha\beta & + & \alpha\beta & + & \beta\alpha & + & \beta\alpha \\ & \alpha\beta & & \beta\alpha & & \alpha\beta & & \beta\alpha \end{matrix}$$

Here $p_2^2=4$, and each of the four groups contains one symmetric function and each symmetric function one term.

3. In the cubic Permanent in three umbral letters

$$\overset{+\quad +}{\begin{pmatrix} \alpha\beta\gamma \\ \alpha\beta\gamma \\ \alpha\beta\gamma \end{pmatrix}}$$

we have, in the first place, the Tableau of substitutions

$$\begin{array}{ccc} \alpha\beta\gamma & \alpha\beta\gamma & \alpha\beta\gamma \\ (1)^3 & (1)^3 & (1)^3 \\ (1)^3 & (1)(2) & (3) \end{array}$$

$$\begin{array}{ccc} \alpha\beta\gamma & \alpha\beta\gamma & \alpha\beta\gamma \\ (1)(2) & (1)(2) & (1)(2) \\ (1)^3 & (1)(2) & (3) \end{array}$$

$$\begin{array}{ccc} \alpha\beta\gamma & \alpha\beta\gamma & \alpha\beta\gamma \\ (3) & (3) & (3) \\ (1)^3 & (1)(2) & (3) \end{array}$$

The numbers of permutations indicated by the row representations

$$(1)^3, \quad (1)(2), \quad (3)$$

are given by the coefficients on the right hand side of the symmetric function formula

$$3!\, h_3 = s_1^3 + 3s_1s_2 + 2s_3$$

and are thus 1, 3, 2 respectively.

3.1 Hence, by multiplication, the numbers of terms in the constituents of the above Tableau are respectively

$$\begin{array}{c} 1,\ 3,\ 2 \\ 3,\ 9,\ 6 \\ 2,\ 6,\ 4 \end{array}$$

3.2 The number of terms in any single symmetric function, that we have under consideration, must be a divisor of 3!. Thence it is clear that neither of the integers 9, 4 which present themselves in the last numerical Tableau can arise from a single symmetric function. The remaining seven of the groups do, as a matter of fact, arise from a single symmetric function. Thus the number of symmetric functions must be at least eleven. The exact number is in fact eleven as shown by the complete expansion which follows:

$$\overset{+\qquad +}{\begin{pmatrix} \alpha\beta\gamma \\ \alpha\beta\gamma \\ \alpha\beta\gamma \end{pmatrix}}$$

$$= \sum\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\beta\gamma\\ \alpha\beta\gamma\end{smallmatrix} + \sum\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\beta\gamma\\ \alpha\beta\gamma\end{smallmatrix} + \sum\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\beta\gamma\\ \beta\gamma\alpha\end{smallmatrix}$$

$$+ \sum\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\\ \beta\alpha\gamma\end{smallmatrix} + \sum\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\\ \alpha\gamma\beta\end{smallmatrix} + \sum\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\\ \beta\alpha\gamma\end{smallmatrix} + \sum\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\\ \beta\gamma\alpha\end{smallmatrix}$$

$$+ \sum\begin{smallmatrix}\alpha\beta\gamma\\ \beta\gamma\alpha\\ \alpha\beta\gamma\end{smallmatrix} + \sum\begin{smallmatrix}\alpha\beta\gamma\\ \beta\gamma\alpha\\ \alpha\gamma\beta\end{smallmatrix} + \sum\begin{smallmatrix}\alpha\beta\gamma\\ \beta\gamma\alpha\\ \beta\gamma\alpha\end{smallmatrix} + \sum\begin{smallmatrix}\alpha\beta\gamma\\ \beta\gamma\alpha\\ \gamma\alpha\beta\end{smallmatrix}$$

exhibiting 11 symmetric functions in $p_3^2(=9)$ groups.

3.3 The numbers of terms in the several symmetric functions are shown in the Tableau

1	3	2
3	3, 6	6
2	6	2, 2

and the sum of these numbers is $(3!)^2$.

The 36 terms have all been set out in R. F. Scott's Treatise on Determinants. In the Tableau of the expansion (above) there is quasi-symmetry about the principal diagonal. The representative products of the symmetric functions beneath the diagonal are derivable from the corresponding ones above by interchange of the second and third rows. Moreover in the principal diagonal the symmetric fractions are unaltered by an interchange of the second and third rows, although the representative products *may* be changed.
Thus

$$\sum\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\gamma\beta\\ \beta\alpha\gamma\end{smallmatrix} \text{ becomes } \sum\begin{smallmatrix}\alpha\beta\gamma\\ \beta\alpha\gamma\\ \alpha\gamma\beta\end{smallmatrix}.$$

The representative products are changed but the functions are identical.

4. We pass to the Cubic Permanent in four umbral letters

$$\overset{+\qquad +}{\begin{pmatrix} \alpha\beta\gamma\delta \\ \alpha\beta\gamma\delta \\ \alpha\beta\gamma\delta \end{pmatrix}}.$$

The expansion into $p_4^2(=25)$ groups of symmetric functions is indicated by the Tableau, in which only the circular substitutions which appertain to the second and third rows are given:

$(1)^4$ $(1)^4$	$(1)^4$ $(1)^2(2)$	$(1)^4$ $(2)^2$	$(1)^4$ $(1)(3)$	$(1)^4$ (4)
$(1)^2(2)$ $(1)^4$	$(1)^2(2)$ $(1)^2(2)$	$(1)^2(2)$ $(2)^2$	$(1)^2(2)$ $(1)(3)$	$(1)^2(2)$ (4)

$(2)^2$ $(1)^4$	$(2)^2$ $(1)^2(2)$	$(2)^2$ $(2)^2$	$(2)^2$ $(1)(3)$	$(2)^2$ (4)
$(1)(3)$ $(1)^4$	$(1)(3)$ $(1)^2(2)$	$(1)(3)$ $(2)^2$	$(1)(3)$ $(1)(3)$	$(1)(3)$ (4)
(4) $(1)^4$	(4) $(1)^2(2)$	(4) $(2)^2$	(4) $(1)(3)$	(4) (4)

4.1 From the formula

$$4!\, h_4 = s_1^4 + 6s_1^2s_2 + 3s_2^2 + 8s_1s_3 + 6s_4$$

we derive the numerical Tableau

1	6	3	8	6
6	36	18	48	36
3	18	9	24	18
8	48	24	64	48
6	36	18	48	36

which exhibits the numbers of terms in the several groups. Of these numbers some are and some are not divisors of 4!. Those which are divisors are, as a matter of fact, connected with a single symmetric function but I am not prepared to say that this is always the case.

Those which are not divisors are necessarily connected with more than one symmetric function. It is only necessary to examine the groups which are connected with such numbers which lie either in the principal diagonal or *above* it.

4.2 The expansion is represented in the Tableau which follows:

$$\overset{+\qquad +}{\begin{pmatrix} \alpha\beta\gamma\delta \\ \alpha\beta\gamma\delta \\ \alpha\beta\gamma\delta \end{pmatrix}}$$

is equal to the som of the functions in the Tableau.

$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\gamma\delta\\ \alpha\beta\gamma\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\gamma\delta\\ \beta\gamma\delta\alpha\end{smallmatrix}$
$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\\ \alpha\beta\gamma\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\\ \alpha\beta\delta\gamma\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\\ \alpha\delta\gamma\beta\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\\ \beta\alpha\gamma\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\\ \beta\alpha\delta\gamma\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\beta\delta\\ \beta\alpha\delta\gamma\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\\ \alpha\gamma\delta\beta\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\\ \beta\gamma\alpha\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\beta\delta\gamma\\ \beta\gamma\delta\alpha\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\beta\delta\\ \beta\delta\alpha\gamma\end{smallmatrix}$
$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\\ \alpha\beta\gamma\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\\ \alpha\beta\delta\gamma\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\\ \alpha\gamma\beta\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\\ \beta\alpha\delta\gamma\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\\ \gamma\delta\alpha\beta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\\ \alpha\gamma\delta\beta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\\ \beta\gamma\delta\alpha\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\alpha\delta\gamma\\ \gamma\delta\beta\alpha\end{smallmatrix}$
$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \alpha\beta\gamma\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \alpha\beta\delta\gamma\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\gamma\alpha\delta\\ \alpha\beta\delta\gamma\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \beta\alpha\delta\gamma\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \alpha\gamma\delta\beta,\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \alpha\delta\beta\gamma\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \beta\delta\gamma\alpha,\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \beta\gamma\alpha\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \beta\gamma\delta\alpha\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \alpha\gamma\delta\beta\\ \beta\delta\alpha\gamma\end{smallmatrix}$
$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\gamma\delta\alpha\\ \alpha\beta\gamma\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\gamma\delta\alpha\\ \alpha\beta\delta\gamma\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\delta\alpha\gamma\\ \alpha\gamma\beta\delta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\gamma\delta\alpha\\ \beta\alpha\delta\gamma\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\gamma\delta\alpha\\ \gamma\delta\alpha\beta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\gamma\delta\alpha\\ \alpha\gamma\delta\beta\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\delta\alpha\gamma\\ \alpha\gamma\delta\beta\end{smallmatrix}$	$\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\gamma\delta\alpha\\ \beta\gamma\delta\alpha\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\delta\alpha\gamma\\ \gamma\alpha\delta\beta\end{smallmatrix}$ $\sum\begin{smallmatrix}\alpha\beta\gamma\delta\\ \beta\gamma\delta\alpha\\ \gamma\alpha\delta\beta\end{smallmatrix}$

exhibiting $25(=p_4^2)$ groups of symmetric functions.

4.3 The subjoined Tableau shows the number of terms in each symmetric function:

1	6	3	8	6
6	6 24 6	6 12	24 24	24 12
3	6 12	3 6	24	12 6
8	24 24	24	8 8 24 24	24 24
6	24 12	12 6	24 24	6 24 6

the sum of the numbers being $(4!)^2$.

4.4 Finally the next Tableau shows the number of symmetric functions in each group:

1	1	1	1	1
1	3	2	2	2
1	2	2	1	2
1	2	1	4	2
1	2	2	2	3

in all 43 symmetric functions in the expansion of the cubic Permanent (or Determinant) which involves four umbral letters.

5. I pass on to consider the enumeration in general. Let

$$N_{\sigma,n}$$

denote the number of symmetric functions which present themselves in the expansion of the Permanent of Class σ in n letters.

So far we know that

$$N_{\sigma,1}=1, \quad N_{1,n}=1, \quad N_{2,n}=p_n=\text{number of partitions of } n.$$

5.1 We can readily find the value of $N_{\sigma,2}$.

The leading umbral factor of the representative product of a symmetric function in two umbral letters, must be headed by the letter α and there is no other condition. The remaining $\sigma-1$ letters in the umbra may, each, be either α or β. They are thus $2^{\sigma-1}$ in number. The remaining umbral factor is derived by the substitution $(\alpha\beta)$ from the leading factor. Each group involves but one symmetric function and each symmetric function but one term. Hence

$$N_{\sigma,2}=2^{\sigma-1}.$$

5.2 When $n=3$, a leading factor of the representative umbral product of a symmetric function is such that its position in the alphabetical order, in which the whole of the umbrae may be arranged, cannot be improved by any substitution that may be impressed upon the umbral letters. The leading letter of the factor must be α and the letter β must occur before the letter γ. There are no other conditions. It is readily seen that the number of different leading umbral factors is

$$\tfrac{1}{2}(3^{\sigma-1}+1).$$

Thus for $\sigma=3$, these are

$$\begin{matrix} \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \beta & \beta & \beta \\ \alpha, & \beta, & \alpha, & \beta, & \gamma \end{matrix}$$

Any symmetric function of the umbral letters α, β, γ must give rise to 1, 2, 3 or 6 terms. Let A_k denote the number of symmetric functions, enumerated by $N_{\sigma,3}$, which give rise to k terms. Since the whole number of terms in the expansion of

$$\begin{pmatrix} \overset{+}{\alpha}\overset{+}{\beta}\gamma \\ \alpha\beta\gamma \\ \alpha\beta\gamma \\ \cdot\,\cdot \\ \cdot\,\cdot \end{pmatrix}$$

is $(3!)^{\sigma-1}$ it follows that

$$A_1+2A_2+3A_3+6A_6=6^{\sigma-1}.$$

Now $A_1=1$. Further the leading factor of a representative product which is enumerated by A_2 may be any such factor with the exception of that one in which *only* the letter α occurs, because any factor

$$\begin{matrix}\alpha\\ \alpha\\ \beta\\ \gamma\\ \alpha\\ \beta\\ \cdot\cdot\\ \cdot\cdot\end{matrix}$$

gives rise, by circular procession, to the product

$$\begin{matrix}\alpha\beta\gamma\\ \alpha\beta\gamma\\ \beta\gamma\alpha\\ \gamma\alpha\beta\\ \alpha\beta\gamma\\ \beta\gamma\alpha\\ \cdots\\ \cdots\end{matrix}$$

which has only two values, the second value appearing by the substitution $(\beta\gamma)$. Hence

$$A_2=\tfrac{1}{2}(3^{\sigma-1}-1).$$

The representative products which appertain to A_3, must be of form

$$\begin{matrix}\alpha\beta\gamma\\ \alpha\beta\gamma\\ \alpha\gamma\beta\\ \alpha\gamma\beta\\ \alpha\beta\gamma\\ \alpha\gamma\beta\\ \cdots\\ \cdots\end{matrix}$$

where the second factor is headed by β and the remaining letters of the factor may, each, be either β or γ—but subject to the condition that the letter γ occurs at least once. The third factor is obtained from the second by the substitution $(\beta\gamma)$.

The number of such products is $2^{\sigma-1}-1$. Hence

$$A_3=2^{\sigma-1}-1.$$

Substituting in the relation above

$$1+3^{\sigma-1}-1+3(2^{\sigma-1}-1)+6A_6=6^{\sigma-1}$$

whence

$$A_6=\tfrac{1}{2}(2^{\sigma-1}-1)\ (3^{\sigma-2}-1)$$

and

$$A_1+A_2+A_3+A_6=6^{\sigma-2}+3^{\sigma-2}+2^{\sigma-2}.$$

Finally

$$N_{\sigma,3}=6^{\sigma-2}+3^{\sigma-2}+2^{\sigma-2},$$

verifying the values 1, 3, 11 which have been already found for $N_{1,3}$, $N_{2,3}$, and $N_{3,3}$.

Further

$$N_{4,3}=6^2+3^2+2^2=49,$$

a number which the writer has also verified.

It would be possible, but laborious, to similarly evaluate $N_{\sigma,4}$.

6. I propose, however, on this occasion, to suggest a formula for $N_{\sigma,n}$ which has the appearance of truth. I recall the symmetric function identity

$$3!\,h_3=s_1{}^3+3s_1s_2+2s_3$$

and the observation, already made, that the coefficients on the right hand side, *viz.*:

$$1,\ 3,\ 2$$

are divisors of 3! The divisors, conjugate to these,

$$6,\ 2,\ 3$$

are integers which appear exclusively in the value of $N_{\sigma,3}$:

$$N_{\sigma,3}=6^{\sigma-2}+3^{\sigma-2}+2^{\sigma-2}.$$

6.1 We must now see if the value of $N_{\sigma,2}$ can be similarly obtained from

$$2!\,h_2=s_1{}^2+s_2$$

We find the divisors 1, 1 of 2! and the conjugate divisors 2, 2.

Moreover

$$N_{\sigma,2}=2^{\sigma-1}=2^{\sigma-2}+2^{\sigma-2},$$

a result which is singularly suggestive.

6.2 It is worth while also to remark that for the case $N_{\sigma,1}$

$$1!\,h_1=s_1.$$

The divisor 1 of 1! presents itself and the conjugate divisor is also unity. We are therefore in agreement with

$$N_{\sigma,1}=1^{\sigma-2}.$$

6.3 A further crucial test is supplied by the result $N_{3,4}=43$.

We have the formula

$$4!\,h_4=s_1^4+6s_1^2s_2+3s_2^2+8s_1s_3+6s_4$$

and are presented with the divisors

$$1,\ 6,\ 3,\ 8,\ 6$$

of 4! and the conjugate divisors

$$24,\ 4,\ 8,\ 3,\ 4.$$

The identity

$$24+8+4+4+3=43$$

is in agreement with

$$N_{\sigma,4}=24^{\sigma-2}+8^{\sigma-2}+4^{\sigma-2}+4^{\sigma-2}+3^{\sigma-2}$$

because we have found above

$$N_{3.4}=43.$$

6.4 I propose, for my present purpose, to call the numbers which thus present themselves, viz., 1; 2, 2; 6, 3, 2; 24, 8, 4, 4, 3; etc., Permanent Numbers of the various orders.

They may be easily calculated from the formula

$$n!\, h_n = \sum \frac{n!}{1^{k_1}.\, 2^{k_2}.\, 3^{k_3} \ldots k_1!\, k_2!\, k_3! \ldots} s_1^{k_1} s_2^{k_2} s_3^{k_3} \ldots$$

where

$$\sum_t t k_t = n$$

and the summation is for all partitions

$$(1^{k_1}\, 2^{k_2}\, 3^{k_3} \ldots)$$

of n; k_1, k_2, $k_3 \ldots$ being repetitional numbers. The Permanent Numbers clearly have the form

$$1^{k_1}.\, 2^{k_2}.\, 3^{k_3} \ldots k_1!\, k_2!\, k_3! \ldots$$

and one is derived from every partition of the number n. There are p_n Permanent Numbers of Order n.

We may also write

$$h_n = \sum \frac{s_1^{k_1} s_2^{k_2} s_3^{k_3} \ldots}{B_{(1^{k_1}\, 2^{k_2}\, 3^{k_3} \ldots)}}$$

and then $B_{(1^{k_1} 2^{k_2} 3^{k_3} \ldots)}$ is the Permanent Number derived from the partition $(1^{k_1}\, 2^{k_2}\, 3^{k_3} \ldots)$. The theorem is then, presumably

$$N_{\sigma, n} = \Sigma (1^{k_1}.\, 2^{k_2}.\, 3^{k_3} \ldots k_1!\, k_2!\, k_3! \ldots)^{\sigma-2} = \Sigma B^{\sigma-2}_{(1^{k_1}\, 2^{k_2}\, 3^{k_3} \ldots)}$$

the summation being for every partition

$$(1^{k_1}\, 2^{k_2}\, 3^{k_3} \ldots)$$

of the number n.

6.5 The Permanent Numbers for the first ten orders are:

Order 1
1

Order 2
2, 2

Order 3
6, 3, 2

Order 4
24, 8, 4, 4, 3

Order 5
120, 12, 8, 6, 6, 5, 4

Order 6
720, 48, 48, 18, 18, 16, 8, 8, 6, 6, 5

Order 7

5040, 240, 72, 48, 48, 24, 24, 18, 12, 12, 10, 10, 8, 7, 6

Order 8

40320, 1440, 384, 360, 192, 96, 96, 36, 36, 36, 32, 32,
30, 24, 16, 15, 12, 12, 12, 10, 8, 7

Order 9

362880, 10080, 2160, 960, 480, 384, 288, 162, 144, 144, 120, 48, 48,
40, 36, 36, 36, 32, 32, 24, 24, 20, 20, 18, 15, 14, 14, 12, 9, 8

Order 10

3628800, 80640, 15120, 5760, 3840, 2880, 1152, 768, 720, 600, 432, 192, 192, 162,
144, 144, 144, 144, 72, 72, 72, 64, 64, 64, 60, 50, 48, 42, 40, 30, 30, 24, 24, 24,
21, 20, 18, 16, 16, 14, 10, 9

6.6 For cubic Permanents, the enumeration is:

no. of letters	1	2	3	4	5	6	7	8	9	10.
no. of symmetric functions	1	4	11	43	161	901	5579	43206	378288	3742738.

The ratios of these numbers to $n!$ are

1 2 1.83 1.79 1.34 1.11 1.09 1.07 1.004 1.003,

so that

$$N_{3,n}/n!$$

appears to approach unity as $n \to \infty$.

6.7 The presumed formula has been established

(i) when $n=1, 2, 3$ and σ has all values,

(ii) when $\sigma=1$, 2, and n has all values,

(iii) when $\sigma=3$, $n=4$.

The confirmation or denial of the theorem adumbrated in this communication appears to be a *desideratum.*

THEOREMS OF FINITENESS IN FORMAL CONCOMITANT THEORY, MODULO P*

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PART 1. MODULAR INVARIANT ELEMENTS

I. *Introduction.* This paper, a study in the algorisms of invariant theory, has for one of its objects that of giving proof of the finiteness of the system of formal modular concomitants; that is, the invariants and covariants of an arbitrary binary quantic f of order m under the total linear homogeneous group G modulo p, an odd prime. An auxiliary theorem employed relates to finite systems for a single modular transformation τ. Part 1 of the paper treats the extension to modular fields of the method of invariant elements which I have used to advantage in previous papers on algebraic and differential invariants and which here furnishes one of the simplest methods possible for the proposed finiteness theorem.

II. *Covariants of modular transformations.* The coefficients of the transformation,

$$\tau : x_1 = \alpha_1 x_1' + \alpha_2 x_2',\ x_2 = \beta_0 x_1' + \beta_1 x_2',$$

are regarded as variable parameters, in that they represent the sets of residues modulo p for which τ ranges over the $(p^2-p)\ (p^2-1)$ transformations of G. A set of generators of G is $\tau_1,\ \tau_2,\ \tau_3$, as follows:

$$(1) \qquad \tau_1 : x_1 = x_1' + t x_2',\ x_2 = x_2';\ \tau_2 : x_1 = x_1',\ x_2 = \lambda x_2';\ \tau_3 : x_1 = x_2',\ x_2 = -x_1'.$$

It is well known that the invariancy of a rational integral function, in the rational domain $R(1)$, of the coefficients of

$$f = (a_0,\ \ldots,\ a_m)(x_1,\ x_2)^m,$$

when the latter is transformed by τ_3, makes the function an invariant (mod. p) of the permutation $s = \prod_{i=0}^{m} (a_{m-i},\ (-1)^m a_i)$, that invariancy under τ_2 represents the fact of isobarism of the function after the weights of all terms are reduced modulo $p-1$, and that the invariants under τ_1 are the formal modular seminvari-

*The initial research in the subject, aside from the brief paper of 1903 by A. Hurwitz (Archiv. der Math. u. Phys., 5), is by Dickson, *The Madison Colloquium Lectures* (1913).

ants* of f. Here the coefficients $a_0, \ldots, a_m$ are assumed to be arbitrary variables and they are not preceded in f by binomial coefficients.

If a binary quantic in $R(1)$ of degree-order (i, ω) in the coefficients and variables of f is invariantive with respect to transformation of f by τ_1, τ_2 and τ_3 (and τ) it is a formal concomitant modulo p.

The salient method of this paper consists in the development, first, of a complete system of concomitants (mod. p), of τ whose parametric coefficients are restricted to represent, always, integers of some defined set of residues. These invariants belong to a domain which includes α_1, α_2, β_0, β_1 among its defining elements and may be represented by $R(1, \tau, r_{\pm 1})$ where $r_{\pm 1}$ are yet to be defined. By the choice of polynomials in the forms of the above complete system, under definite methods of choice, we obtain concomitants which simplify, by the fact that all expressions in α_1, α_2, β_0, β_1 factor out and cancel, into the usual domain for covariants, $R(1, 0, 0)$. As a result we are able to study the question of finiteness of the concomitant system of f of the latter domain as an instance under Hilbert's lemma (cf., Part 2).

The following quadratic form is a covariant of τ, where the parameters α_1, α_2, β_0, β_1 represent the sets of residues for which $D = \alpha_1\beta_1 - \alpha_2\beta_0 \not\equiv 0$ (mod. p):

(2) $$\Gamma = \beta_0 x_1^2 + (\beta_1 - \alpha_1)x_1x_2 - \alpha_2 x_2^2.$$

The invariant relation is

(3) $$\Gamma' \equiv D^{-1}\Gamma (\text{mod. } p).$$

This form remains invariant by its linear factors being invariantive, separately. Thus, if

(4) $$\Gamma \equiv \beta_0(x_1 - r_{+1}x_2)\ (x_1 - r_{-1}x_2)\ (\text{mod. } p),$$

we find, after writing

(5) $$f_{+1} = x_1 - r_{+1}x_2, \quad f_{-1} = x_1 - r_{-1}x_2,$$
$$f'_{\pm 1} \equiv \rho_{\pm 1}^{-1} f_{\pm 1} (\text{mod. } p),$$

where ρ_{+1}, ρ_{-1} are factors of D in $R(1, \tau, r_{\pm 1})$, viz.,

(6) $$2\rho_{\pm 1} \equiv \alpha_1 + \beta_1 \pm (r_{-1} - r_{+1})\beta_0 \ (\text{mod. } p),$$
$$\rho^2_{\pm 1} - (\alpha_1 + \beta_1)\rho_{\pm 1} + D \equiv 0 \ (\text{mod. } p);\ \rho_{+1}\rho_{-1} \equiv D \ (\text{mod. } p).$$

Formal reductions give, readily,

(7) $$\beta_0 r_{+1} r_{-1} = -\alpha_2, \quad \beta_0(r_{+1} + r_{-1}) = \alpha_1 - \beta_1,$$
$$\beta_0^2(r_{+1} - r_{-1})^2 = (\alpha_1 + \beta_1)^2 - 4D (= \Delta;\ \textit{Def.}).$$

*Dickson, *loc. cit.*, Lecture III, p. 40.

III. *Use of the GF* $[p^2]$. If the congruence obtained by writing Γ in non-homogeneous form is irreducible (mod. p):

$$F(r) = \beta_0 r^2 + (\beta_1 - \alpha_1) r - \alpha_2 \equiv 0 \text{ (mod. } p), \tag{8}$$

then p, $F(r)$ define the $GF\,[p^2]$, the roots of (8) being $\eta \pm x\zeta = r_{\pm 1}$, where η, ζ are integers and x is a galoisian imaginary. Thus, whereas the coefficients in τ have only to satisfy Fermat's theorem for real residues of p, the parameters* r_{+1}, r_{-1}, which are dependent upon α_1, α_2, β_0, β_1, may represent either real residues or distinct conjugate galoisian imaginaries. Their type as numbers is determined by the congruence,

$$(\rho_{+1}^{p} - \rho_{+1})(\rho_{+1}^{p+1} - D) \equiv 0 \text{(mod. } p), \tag{9}$$

assuming that $\delta = r_{-1} - r_{+1} \not\equiv 0$ (mod. p). If $\delta \equiv 0$ the roots $r_{\pm 1}$ are real and congruent. To prove the statement concerning (9) let the first factor be zero. Then $\delta^p - \delta \equiv 0$ (mod. p) and δ, and therefore $r_{\pm 1}$, is a real residue of p. If the second factor of (9) vanishes, $\delta^p + \delta \equiv 0$ (mod. p) and hence $\delta(\not\equiv 0)$, r_{+1}, r_{-1} are galoisian imaginaries. Conversely, according as r_{+1}, r_{-1} are real or imaginary $(\delta \not\equiv 0)$, the first or the second congruence in δ will hold and the first or the second respective factors of (9) vanish. This separation into cases on the basis of real or imaginary roots $r_{\pm 1}$ is fundamental.

IV. *The case* $\delta \equiv 0$, $\Delta \equiv 0$ *(mod. p)*. If $\beta_0 \equiv 0$ (mod. p), τ takes the form of the parametric transformation whose characteristic invariants are the isobaric seminvariants and $F(r) \equiv 0$ has only one root.

If $\Delta \equiv 0$ (mod. p) then $F \equiv 0$ again has only one root, r_{+1} being congruent to r_{-1}. The one covariant (5), viz., $y_1 = x_1 - r x_2$, is then real. If we use y_1 instead of x_1, τ is reduced (by a linear transformation upon x_1) to the form,

$$y_1 = \alpha_1' y_1', \; x_2 = \beta_0' y_1' + \beta_1' x_2', \tag{10}$$

$$\alpha_1' = \tfrac{1}{2}(\alpha_1 + \beta_1) \not\equiv 0 \text{(mod. } p), \; \beta_1' = \beta_0 r + \beta_1.$$

Since the characteristic invariants of (10) are the isobaric anti-seminvariants we obtain the following:

Lemma. With the condition (I): $\beta_0 \Delta \equiv 0$ *(mod. p), assumed as a hypothesis, the invariant problem of* τ *is determined as that of the modular seminvariants and the symmetrically related anti-seminvariants.*

Since we are concerned with full invariants and covariants, a problem which is different from that of the seminvariant systems, in modular theory, we shall assume (II): $\beta_0 \Delta \not\equiv 0$ (mod. p). It is not inferred that invariancy under condition (II) is a necessary and sufficient condition that a polynomial in $a_0, \ldots, a_m, x_1, x_2$ should be a formal modular concomitant, but we confine attention to case (II) up to the last paragraph of this paper where it is shown that systems in $R(1, 0, 0)$, which we construct under condition (II), also possess the seminvariant property†.

*The group G can be generated by transformations for which $\Delta \not\equiv 0$.

†The term parameter is used for a letter that can assume each value of a set. Only such relations are derived for a parameter as hold for all of the numbers of the set.

V. *Invariant elements.* A binary quantic can be factored, (modulo p), into linear factors:

$$(11)\qquad f=(a_0,\ \dots,\ a_m)(x_1,x_2)^m=\prod_{i=1}^{m}(a_1^{(i)}x_1+a_2^{(i)}x_2)=\prod_{i=1}^{m}a_x^{(i)}=\prod_{i=1}^{m}b_x^{(i)},$$

and if we substitute, for x_1, x_2, from the inverse of (5), viz.,

$$(12)\qquad x_1=(r_{-1}f_{+1}-r_{+1}f_{-1})/\delta,\quad x_2=(f_{+1}-f_{-1})/\delta,$$

we obtain, for f, an expansion in terms of f_{+1}, f_{-1} as arguments. The coefficient of $f_{+1}^{m-i}f_{-1}^{i}$ is a homogeneous form of the ith elementary symmetric function*:

$$(13)\quad \phi_{m-2i}=(-1)^i\Sigma\prod_{j=1}^{i}(a_1^{(j)}r_{+1}+a_2^{(j)})\prod_{k=i+1}^{m}(a_1^{(k)}r_{-1}+a_2^{(k)})/\delta^m,\ (i=0,\ \dots,\ m),$$

and the forms ϕ_{m-2i} are formal invariants, mod. p, in $R(1,\ \tau,\ r_{\pm1})$. The invariant relations are

$$(14)\qquad \phi'_{m-2i}\equiv\rho_{+1}^{m-2i}D^i\phi_{m-2i}\ (\text{mod. } p).$$

The forms ϕ_{m-2i} are linear in $a_0,\ \dots,\ a_m$ and linearly independent. We refer to them as the *invariant elements.*

VI. *Invariant systems which belong to the domain* $R(1,\ \tau,\ 0)$. The forms ϕ_{m-2i}, $(-1)^m\phi_{-m+2i}$, are conjugates in that they are interchanged by the transposition $s=(r_{+1}r_{-1})$. The notation is such, therefore, that conjugate symbols ϕ are interchanged by a change of sign. By (7), (8) we obtain

$$(15)\qquad \phi_{m-2i}=P_i+Q_ir_{+1}+R_ir_{-1},\ (-1)^m\phi_{-(m-2i)}=P_i+Q_ir_{-1}+R_ir_{+1},$$

where P_i, Q_i, R_i do not contain $r_{\pm1}$ but are rational in the coefficients of τ. Their domain may be represented by $R(1,\ \tau,\ 0)$.

The set ϕ_{m-2i}, $f_{\pm1}$ form a complete system of concomitants, modulo p, of f, under τ, in $R(1,\ \tau,\ r_{\pm1})$†, the transformation τ being subject only to the condition (II), $(D\not\equiv0)$. In order to construct systems in $R(1,\ \tau,\ 0)$ we consider the products,

$$(16)\qquad \psi_{+1}=\phi_m^{x_0}\phi_{m-2}^{x_1}\dots\phi_{-m}^{x_m}f_{+1}^{\sigma_1}f_{-1}^{\sigma_2},\ \ \pm\psi_{-1}=\phi_{-m}^{x_0}\phi_{-m+2}^{x_1}\dots\phi_m^{x_m}f_{-1}^{\sigma_1}f_{+1}^{\sigma_2}.$$

These are also conjugates and reducible to expressions similar to those in (15). Hence the binomials,

$$(17)\quad P_{+1}=\psi_{+1}+(-1)^{em}\psi_{-1},\ P_{-1}=(\psi_{+1}-(-1)^{em}\psi_{-1})/\delta,\ (e=x_0+\dots+x_m),$$

belong to $R(1,\ \tau,\ 0)$. From (14),

$$(18)\qquad \psi'_{+1}=\rho_{+1}^{a}D^b\psi_{+1},\ \psi'_{-1}=\rho_{+1}^{-a}D^c\psi_{-1},$$

where,

$$(19)\qquad a=\sum_{i=0}^{m}(m-2i)x_i-\sigma_1+\sigma_2,\ b=\sum_{i=0}^{m}ix_i-\sigma_2,\ c=\sum_{i=0}^{m}(m-i)x_i-\sigma_1.$$

*Trans. Amer. Math. Soc., vol. 21 (1920), p. 287; vol. 18 (1917), p. 451.

†Trans. Amer. Math. Soc., vol. 21 (1920), p. 288.

The following lemma may be proved by methods closely analogous to those of the well-known proof of the corresponding theorem for algebraical* concomitants:

Lemma. If a rational, integral formal concomitant, I, of f, modulo p, belongs to the domain $R(1, \tau, 0)$, the multiplier M in its invariant relation, $I' \equiv MI$ (mod. p), is congruent to a power of D.

We may prove, next, two theorems upon concomitants of τ.

Theorem. A necessary and sufficient condition in order that the forms $P_{\pm 1}$ (cf. (17)) *should be invariantive under the arbitrary transformation τ, subject to condition* (II), *is that a certain one of two congruences in x_i, σ_1, σ_2, should be satisfied,* viz.,

$$\text{(20)} \quad \begin{aligned} &\text{(i) } a = \sum_{i=0}^{m}(m-2i)x_i - \sigma_1 + \sigma_2 \equiv 0 (\text{mod. } p-1), \\ &\text{(ii) } a \equiv 0 \ (\text{mod. } p+1); \end{aligned}$$

in the respective cases of real and imaginary roots $r_{\pm 1}$.

The only relations to be employed in the verification of $P'_{\pm 1} = D^a P_{\pm 1}$, assumed as the invariant relation for $P_{\pm 1}$, in accordance with the above lemma, are Fermat's theorem for the parameters α_1, α_2, β_0, β_1, the condition (II) and the relation expressive of the fact that $r_{\pm 1}$ are real; or that they are imaginary, viz., the following congruences, respectively:

$$\text{(21)} \quad \Delta^{\frac{1}{2}(p+1)} - \Delta \equiv 0 \ (\text{mod. } p), \ \Delta^{\frac{1}{2}(p+1)} + \Delta \equiv 0 \ (\text{mod. } p).$$

The invariant problem would be altered if any other relation should be assumed. A consequence of this is the following:

Lemma. The parameters involved satisfy no relation $\rho^{\beta}_{+1} \equiv D^{\gamma}$ except for exponents β which are divisible by $p+1$.

In fact, if we assume this congruence, there follows,

$$\text{(22)} \quad \rho^{p\beta}_{+1} \equiv (D^{\gamma})^p \equiv \rho^{\beta}_{+1} (\text{mod. } p),$$

and we should have

$$\text{(23)} \quad \rho^{\beta}_{+1}(\rho^{(p-1)\beta}_{+1} - 1) \equiv 0 \ (\text{mod. } p).$$

Hence, if (9) is satisfied by its second factor being zero, $\beta = (p+1)d$.

If (9) is satisfied by the vanishing of its first factor, there are only p distinct powers of ρ_{+1}. They are ρ^{σ}_{+1}, $(\sigma = 0, \ldots, p-1)$.

But, in

$$\text{(24)} \quad (2\rho_{+1})^{\sigma} = (x+y)^{\sigma}, \quad x = \alpha_1 + \beta_1, \ y = \delta\beta_0,$$

there are odd powers of y whose numerical multipliers in the form of binomial coefficients $\binom{\sigma}{i}$ are not congruent to zero. Indeed all of the numbers,

*Grace and Young, *Algebra of Invariants*, 1 edition, p. 21.

$$(25)\qquad \binom{p-i}{j},\ (i=1,\ \ldots,\ p-1;\ j=0,\ \ldots,\ p-i),$$

are incongruent to zero. Hence ρ_{+1}^{σ} belongs to the domain $R(1, \tau, r_{\pm 1})$ and could not be congruent to a power of D by means of the relations which define the set τ, (cf. (21)).

The proof of our theorem is now as follows. We have, by hypothesis,

$$(26)\qquad P'_{+1} = D^{a}P_{+1},\ P'_{-1} = D^{a'}P_{-1},$$

hence, from (18),

$$(27)\qquad \rho_{+1}^{a}D^{b} \equiv \rho_{+1}^{-a}D^{c} = D^{a}\ (\text{mod. } p).$$

In view of the last lemma, if $r_{\pm 1}$ are real parameters, we can have only $a\equiv 0$ (mod $p-1$), $(\rho_{+1}^{a}\equiv 1)$. If $r_{\pm 1}$ are galoisian imaginaries all cases of (27) give $a\equiv 0$ (mod. $p+1$), $(\rho_{+1}^{a}=D^{v})$.

Theorem. A fundamental system of concomitants of the binary quantic f under τ, *in* $R(1,\tau,0)$, *when* $r_{\pm 1}$ *are real parameters for which condition* (II) *holds true, is given by the totality of irreducible solutions in positive integers of the diophantine equations,*

$$(28)\qquad a=0,\ \pm a=p-1,\ \pm a=2(p-1),\ \ldots,\ \pm a=\lambda(p-1),\ (\lambda\leqq \tfrac{1}{4}(m+1)^2),$$

it being, therefore, finite.

A concomitant, $I(a_0,\ \ldots,\ a_m;\ x_1,\ x_2;\ \alpha_1,\ \alpha_2,\ \beta_0,\ \beta_1)$, satisfying the hypotheses of the theorem and expressed, by substitution from (12) and the inverse of the linear equations, (in $a_0,\ \ldots,\ a_m$),

$$(29)\qquad \phi_{m-2i} = \phi_{m-2i}(a_0,\ \ldots,\ a_m),\ (i=0,\ \ldots,\ m),$$

uniquely in terms of invariant elements, consists of terms ψ (cf. (16)) in combinations $F_{\pm 1}$, as below, satisfying the same exponential rules (20) as are satisfied by the terms of $P_{\pm 1}$:

$$(30)\qquad F_{+1} = g_{+1}\psi_{+1} + (-1)^{em}g_{-1}\psi_{-1},\ F_{-1} = (g_{+1}\psi_{+1} - (-1)^{em}g_{-1}\psi_{-1})/\delta.$$

The quantities $g_{\pm 1}$ are conjugate expressions of the domain $R(1,\ \tau,\ r_{\pm 1})$.
From (17),

$$(31)\qquad \psi_{+1} = \tfrac{1}{2}(P_{+1}+\delta P_{-1}),\ \psi_{-1} = \tfrac{1}{2}(P_{+1}-\delta P_{-1})(-1)^{em}.$$

Hence we obtain

$$(32)\qquad \begin{aligned} F_{+1} &= \tfrac{1}{2}[(g_{+1}+g_{-1})P_{+1} + \delta(g_{+1}-g_{-1})P_{-1}],\\ F_{-1} &= \tfrac{1}{2}[\delta^{-1}(g_{+1}-g_{-1})P_{+1} + (g_{+1}+g_{-1})P_{-1}],\end{aligned}$$

and since the parentheses (with their multipliers δ^{ν}; $\nu=0,\ \pm 1$), and also $P_{\pm 1}$, belong to the domain $R(1,\ \tau,\ 0)$, it is shown clearly by (32) how reduction from $R(1,\ \tau,\ r_{\pm 1})$ to $R(1,\ \tau,\ 0)$ occurs, in I, in general. The concomitant I is a linear combination of forms $P_{\pm 1}$ with coefficients which are rational in the defining elements of the domain $R(1,\ \tau,\ 0)$.

The concomitant I may, without loss of generality, be assumed to be homogeneous; its properties of isobarism are to be noticed later (cf. § VIII).

Now, since the forms $\psi_{\pm 1}$ have been placed, by formulae (17), in one to one correspondence with the pair $P_{\pm 1}$ and I is linear, within $R(1, \tau, 0)$, in the forms $P_{\pm 1}$, the problem of the complete system, thus reduced to that of determining the irreducible forms $P_{\pm 1}$, is reduced to the problem of finding all irreducible products $\psi_{\pm 1}$. For if a solution, $x_0, \ldots, x_m, \sigma_1, \sigma_2$, of an equation (28) is reducible and ψ_{+1} is the product which is furnished by this solution, we have

$$\psi_{+1} = \psi'_{+1}\psi''_{+1},\ \psi_{-1} = \psi'_{-1}\psi''_{-1}, \tag{33}$$

and each factor ψ on the right is furnished by some solution of an equation (28). The forms $P_{\pm 1}$, which correspond to $\psi_{\pm 1}$ through the equations (17), are then reducible also. In fact, if

$$\begin{aligned} P'_{+1} &= \psi'_{+1} + (-1)^{e'm}\psi'_{-1}, \quad P'_{-1} = (\psi'_{+1} - (-1)^{e'm}\psi'_{-1})/\delta, \\ P''_{+1} &= \psi''_{+1} + (-1)^{e''m}\psi''_{-1},\ P''_{-1} = (\psi''_{+1} - (-1)^{e''m}\psi''_{-1})/\delta, \end{aligned} \tag{34}$$

we find,

$$\begin{aligned} P_{+1} &= \psi_{+1} + (-1)^{em}\psi_{-1} \quad = \tfrac{1}{2}(P'_{+1}P''_{+1} + \delta^2 P'_{-1}P''_{-1}), \\ P_{-1} &= (\psi_{+1} - (-1)^{em}\psi_{-1})/\delta = \tfrac{1}{2}(P'_{+1}P''_{-1} + P'_{-1}P''_{+1}), \end{aligned} \tag{35}$$

which proves $P_{\pm 1}$ to be, under the premises, reducible in the domain $R(1, \tau, 0)$.

A lemma on monomials, which is a special case of the well-known theorem of Hilbert on systems of polynomials* (cf. (51), § IX of the present paper), may now be quoted.

Lemma. If a product of positive integral powers of n letters,

$$y_1^{k_1} y_2^{k_2} \ldots y_n^{k_n}, \tag{36}$$

be formed in such a way that the exponents satisfy certain prescribed conditions, then, although the number of products satisfying the conditions may be infinite, yet a finite number of them can be chosen so that every other is divisible by one at least of this finite number.

In the present case the letters y_i are the forms $\phi_{m-2i}, f_{\pm 1}$ and the products $\psi_{\pm 1}$ form the infinite system which satisfies conditions (viz., (28)) as required by the lemma. All forms $\psi_{\pm 1}$ are divisible by and hence reducible in terms of one or more of the finite number given by the set of irreducible solutions $x_0, \ldots, x_m, \sigma_1, \sigma_2$ obtained by combining the sets given by the different equations (28). The formulae of reduction are those of (33). Corresponding to an irreducible solution $x_0, \ldots, x_m, \sigma_1, \sigma_2$, we have an irreducible pair $\psi_{\pm 1}$ and, from (17), an irreducible pair $P_{\pm 1}$ of the domain $R(1, \tau, 0)$.

Note that if a letter x_i of a solution is $\geqq p-1$, then, that solution is reducible† in terms of solutions of equations previous in the sequence (28) to the equation

*Grace and Young, *Algebra of Invariants* (1903), p. 178.

†The case where ψ_{+1} is one of the absolute concomitants $\phi_{m-2i}^{p-1}, f_{+1}^{p-1}$, is an exception.

containing the aforesaid x_i. An exponent in the expression for ψ_{+1} will be as great as $p-1$ if a exceeds the number obtained by making all exponents $p-2$ in positive terms of a and $x_j=0$ in negative terms, that is, if

$$a \geqq \tfrac{1}{4}(m+1)^2(p-2). \tag{37}$$

This proves the theorem and, by analogous reasoning, we deduce the following:

Corollary. A finite fundamental system of concomitants of the binary quantic f under τ, in $R(1, \tau, 0)$, when the expressions $r_{\pm 1}$ are parametric galoisian imaginaries for which condition (II) *holds true, is given by the totality of irreducible solutions, in positive integers, of the diophantine equations,*

$$a=0,\ \pm a=p+1,\ \pm a=2(p+1),\ \ldots,\ \pm a=\lambda(p+1). \tag{38}$$

VII. *Examples of concomitants as functions of invariant elements.* We obtain by (12) and the method of (29) the following representations of known formal modular concomitants of the quadratic, $(p=3)$:

$$f=a_0x_1^2+2a_1x_1x_2+a_2x_2^2,$$

$$\begin{aligned} f_4 &= a_0x_1^4+a_1(x_1^3x_2+x_1x_2^3)+a_2x_2^4 \\ &= \{(r^3_{-1}-r_{+1})[\phi_2f^4_{+1}+\phi_0f^3_{+1}f_{-1}]-(r^3_{+1}-r_{-1})[\phi_0f_{+1}f^3_{-1}+\phi_{-2}f^4_{-1}] \\ &\quad +(r^3_{-1}-r_{-1})[\phi_0f^4_{+1}+\phi_{-2}f^3_{+1}f_{-1}]-(r^3_{+1}-r_{+1})[\phi_2f_{+1}f^3_{-1}+\phi_0f^4_{-1}]\} \div \delta^3, \end{aligned} \tag{39}$$

$$\begin{aligned} q &= a_0^2a_2+a_0a_2^2+a_0a_1^2+a_1^2a_2-a_0^3-a_2^3 \\ &= \{(r_{+1}-r_{-1})[(r^3_{+1}-r_{-1})\phi_2\phi_0^2-(r^3_{-1}-r_{+1})\phi_2\phi^2_{-2}+(r^3_{+1}-r_{+1})\phi_2^2\phi_0 \\ &\quad +(r^3_{+1}-r_{-1})\phi_2^2\phi_{-2}-(r^3_{-1}-r_{-1})\phi_0\phi^2_{-2}-(r^3_{-1}-r_{+1})\phi_0^2\phi_{-2}]-[(r^3_{+1}-r_{+1})^2+1]\phi_2^3 \\ &\quad -[(r^3_{-1}-r_{-1})^2+1]\phi^3_{-2}+[(r^3_{+1}-r_{+1})\,(r^3_{-1}-r_{-1})-(r_{+1}-r_{-1})^2+1]\phi_0^3\} \div \delta^4. \end{aligned} \tag{40}$$

When the roots $r_{\pm 1}$ are real these expressions reduce, modulo 3, to those following:

$$\begin{aligned} f_4 &\equiv \phi_2f^4_{+1}+\phi_0f^3_{+1}f_{-1}+\phi_0f_{+1}f^3_{-1}+\phi_{-2}f^4_{-1}, \\ q &\equiv \phi_2\phi_0^2+\phi_2\phi^2_{-2}+\phi_2^2\phi_{-2}+\phi_0^2\phi_{-2}-\phi_2^3-\phi^3_{-2}, \end{aligned} \tag{41}$$

these being the same functions, respectively, of ϕ_2, ϕ_0, ϕ_{-2}, f_{+1}, f_{-1}, that f_4, q are of a_0, a_1, a_2, x_1, x_2; but, when $r_{\pm 1}$ are imaginary, we find

$$\begin{aligned} f_4 &\equiv \phi_0f^4_{+1}+\phi_{-2}f^3_{+1}f_{-1}+\phi_2f_{+1}f^3_{-1}+\phi_0f^4_{-1}, \\ q &\equiv \phi_2^2\phi_0+\phi_0\phi^2_{-2}+\phi_2^3-\phi_0^3+\phi^3_{-2}+\delta^{-2}(\phi_2^3-\phi_0^3+\phi^3_{-2}). \end{aligned} \tag{42}$$

Part 2. The Finiteness of Formal Modular Systems

VIII. *General properties. Terminology.* A simple method of proof of the finiteness of the system of formal concomitants, modulo p, of f is obtained by connecting the theory of the previous paragraphs with Hilbert's theorem on systems of polynomials. We first state certain fundamental facts concerning the representation of invariants and covariants in the form which is illustrated by (39), (40).

Functionality. If $r_{\pm 1}$ are real parameters the expression for a formal modular concomitant I in terms of invariant elements is the same function of $\phi_{m-2i}, f_{+1}, f_{-1}$ respectively, that I is of a_i, x_1, x_2. For then the coefficients in (12) are real residues. Hence, if the invariant relation for I, under transformation of f by τ, is

(43) $$I(a'_0, \ldots, a'_m; x'_1, x'_2) = D^r I(a_0, \ldots, a_m; x_1, x_2),$$

we have

(44) $$I(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1}) \equiv (-\delta)^{-r} I(a_0, \ldots, a_m; x_1, x_2) \pmod{p},$$

which is the property stated.

If $r_{\pm 1}$ are imaginary I is not necessarily the same function, F, of $\phi_{m-2i}, f_{+1}. f_{-1}$ as of a_i, x_1, x_2 (cf, (42)). We do not in this case determine explicitly the general quantic F as a function of invariant elements although its form is not wholly undefined. Since, when $r_{\pm 1}$ are imaginary parameters, $r^p_{\pm 1} \equiv r_{\pm 1} \pmod{p}$, and the roots of $f_{\pm 1}$ are cogredient to the variables,

(45) $$\Omega = f^p_{+1} \frac{\partial}{\partial f_{-1}} + f^p_{-1} \frac{\partial}{\partial f_{+1}},$$

is an invariantive operator. From the expression for a quadratic in terms of invariant elements,

$$f = \phi_2 f^2{}_{+1} + 2\phi_0 f_{+1} f_{-1} + \phi_{-2} f^2{}_{-1},$$

we then obtain

(46) $$\tfrac{1}{2}\Omega f = \phi_0 f^{p+1}_{+1} + \phi_{-2} f^p_{+1} f_{-1} + \phi_2 f_{+1} f^p_{-1} + \phi_0 f^{p+1}_{-1},$$

a covariant which reduces to f_4 when $p=3$.

The expression for the general F in terms of $\phi_{m-2i}, f_{\pm 1}$ is unique.

Term-wise invariance. The various terms $\psi_{\pm 1}$ (cf. (16)) of I (or F) are invariantive and the multipliers in their invariant relations are all congruent to D^α, modulo p, where α is the index of I (or F). When $r_{\pm 1}$ are real the formula for α is, from (19), (27),

(47) $$w - \omega = \alpha + \xi(p-1),$$

where w is the weight, ω the order and α the index of a term of I. Here the weights of $\phi_{m-2i}, f_{+1}, f_{-1}$, respectively, are defined to be i, 1, 0. If $r_{\pm 1}$ are imaginary the analogous relation is, for terms of F,

(48) $$w - \omega + v = \alpha + \eta(p-1),$$

where $a = (p+1)v$.

Isobarism. Since, in the relation (47) for terms of $I(\phi_m, \ldots, \phi_{-m}, f_{+1}, f_{-1})$, ω and α are constant, w can vary only so that its various values differ by multiples of $p-1$, that is, I is isobaric modulo $p-1$. The relation (48) contains three variables w, v, η and does not give an analogous conclusion for the weights of terms of the quantic $F(\phi_m, \ldots, \phi_{-m}, f_{+1}, f_{-1})$. We can write the expression of (20),ii, however, in the form,

$$md-2w+\omega=v(p+1), \tag{49}$$

where d is the degree (in $a_0, \ldots, a_m$), ω the order and w the weight of a term of F. Hence twice the weights of the various terms of F are integers which differ at most by multiples of $p+1$. This property of F will be referred to as *even isobarism* modulo $p+1$. In the example (46) v is also even, and the form Ωf is actually isobaric modulo $p+1$.

Terminology relating to number fields. Let an arbitrary formal modular concomitant I be expressed, by the substitutions inverse to (29), and those of (12), in the form, $I=C$, of which (39) and (40) are illustrations. The form C is thus obtained from I by regarding $r_{\pm 1}$ to be algebraic indeterminate facients*. Also, if the reverse transformations are made, C returns to the quantic I from which it arose without our changing $r_{\pm 1}$ from the status of algebraic indeterminates, provided integral numbers occurring in the developments are reduced modulo p. We shall say that such reductions are made *algebraically with reference to* $r_{\pm 1}$.

Subsequently the hypothesis is made that $r_{\pm 1}$ become real *modular parameters* and C reduces (mod. p) to $I(\phi_m, \ldots, \phi_{-m};\ f_{+1}, f_{-1})$, or, if $r_{\pm 1}$ are assigned to be galoisian imaginaries the reduction is to $F(\phi_m, \ldots, \phi_{-m};\ f_{+1}, f_{-1})$ (modd. p, $F(r)$) (cf. (8)). In either of these latter two cases $r_{\pm 1}$ are subject to the condition (II) of the Lemma in paragraph IV.

If, at the time when we make the hypothesis that $r_{\pm 1}$ shall be modular parameters we should apply (7), (8) to express C in terms of †$\alpha_1, \alpha_2, \beta_0, \beta_1$, (and r_{+1}, r_{-1} in the first degree, only), then, the two forms $I(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1})$, $F(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1})$ would be obtained by substituting in C from the following well-known relations, respectively:

$$\Delta^{\frac{1}{2}(p+1)}-\Delta\equiv 0,\ \Delta^{\frac{1}{2}(p+1)}+\Delta\equiv 0 \pmod{p}. \tag{50}$$

These two congruences, although involving $\alpha_1, \alpha_2, \beta_0, \beta_1$, only, distinguish the cases of real and imaginary parameters $r_{\pm 1}$.

IX. *Complete systems which belong to the domain* $R(1, 0, 0)$. Some of the notations of the preceding paragraph are adhered to in this. The relation $I=C$, of which (39) is an example, subsisting as a result of transformation of an arbitrary formal modular concomitant $I(a_0, \ldots, a_m; x_1, x_2)$ of the domain $R(1, 0, 0)$, the letters $r_{\pm 1}$ being algebraic indeterminates, is said to determine the *system of quantics* C. The operation of making the letters $r_{\pm 1}$, in C, assume real modular parametric values subject to (II), § IV, is said to define the *system of quantics* $I(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1})$, whereas that of substituting galoisian imaginaries $r_{\pm 1}$ for the facients $r_{\pm 1}$ in C is said to define the *system of quantics* $F(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1})$.

It is desired to formulate a method of construction of the infinitude of

*Cayley employed this term for the variables in a binary form. The cases are similar.

†The parameters α_1, β_1 enter these quantics only in the combination $\gamma_1=\beta_1-\alpha_1$.

quantics C as polynomials, in the letters $\phi_{m-2i}, f_{+1}, f_{-1}$, forming a system S in the sense of Hilbert's theorem*, viz.,

If an infinite system S of homogeneous polynomials in n letters satisfies any law sufficiently explicit to locate an arbitrarily chosen polynomial within or without the system, then, there exists a set of these forms, $F_1, \ldots, F_s$, finite in number, each within S and such that any form F' belonging to S is expressible in the form,

$$F' = N_1F_1 + N_2F_2 + \ldots + N_sF_s, \tag{51}$$

where $N_1, \ldots, N_s$ are also homogeneous polynomials in the same n variables but do not necessarily belong to S.

We define the system S in the present instance by means of three hypotheses, each hypothesis being accompanied by a set of explanatory conclusions.

Hypothesis (1). Construct a polynomial Φ in the letters $\phi_{m-2i}, f_{\pm 1}$, with coefficients which are integers, homogeneous in the set ϕ_{m-2i} and also in the set f_{+1}, f_{-1} and possessing the properties,

(*a*) The letters $r_{\pm 1}$ (in $\phi_{m-2i}, f_{\pm 1}$) are real modular parameters which satisfy condition (II) of paragraph (IV).

(*b*) The exponents of the terms of Φ satisfy the congruence (20), i.

(*c*) The quantic Φ is isobaric modulo $p-1$.

(*d*) When the explicit formulas (cf. (5), (13)) are substituted for $\phi_{m-2i}, f_{\pm 1}$, the form Φ reduces, by the cancellation† of all parameters $\alpha_1, \alpha_2, \beta_0, \beta_1, r_{\pm 1}$, to a polynomial, $\Phi_1(a_0, \ldots, a_m; x_1, x_2)$, in the coefficients and variables of f, which belongs to the domain $R(1, 0, 0)$.

Explanatory conclusions relating to Hypothesis (1). We observe as follows:

(α) Forms Φ exist to satisfy hypothesis (1) for every quantic f. In fact every form f possesses a formal modular concomitant Γ (mod. p), and we may employ the method of direct transformation of Γ, as was done in the case of example (39), and obtain $\Gamma = C$. Then, assigning $r_{\pm 1}$ to be real, C is congruent to a form Φ of hypothesis (1).

(β) The polynomial Φ is term-wise invariant. This follows directly from the value of b in (19) and the property of isobarism modulo $p-1$.

(γ) The quantic Φ_1 is a concomitant mod. p of f under transformations τ for which $r_{\pm 1}$ are real parameters satisfying (II) of paragraph (IV).

(δ) The quantic Φ_1 may or may not be invariantive under τ if $r_{\pm 1}$ are galoisian imaginaries.

In proof of (γ) let f be transformed into $f' = (a_0', \ldots, a_m')(x_1', x_2')^m$, by the transformation τ described, and expand f' in the arguments,

$$f'_{\pm 1} = x_1' - r_{\pm 1}x_2', \tag{52}$$

with the result,

$$f' = \sum_{i=0}^{m} \phi'_{m-2i} f'^{m-i}_{+1} f'^{i}_{-1}.$$

*Mathematische Annalen, vol. 36 (1890), p. 473.

†Compare (Illustration) Trans. Amer. Math. Soc., 20 (1919), p. 208.

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We have, from (d),

$$\Phi_1(a_0, \ldots, a_m; x_1, x_2) \equiv \Phi(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1}).$$

Hence,

$$\Phi_1(a'_0, \ldots, a'_m; x'_1, x'_2) \equiv \Phi(\phi'_m, \ldots, \phi'_{-m}; f'_{+1}, f'_{-1}).$$

Since the polynomial Φ possesses the property of term-wise invariance,

$$\Phi(\phi'_m, \ldots, \phi'_{-m}; f'_{+1}, f'_{-1}) = D^\alpha \Phi(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1}),$$

and, consequently,

$$\Phi_1(a'_0, \ldots, a'_m; x'_1, x'_2) \equiv D^\alpha \Phi_1(a_0, \ldots, a_m; x_1, x_2). \tag{53}$$

Hypothesis (2). Transform $\Phi_1(a_0, \ldots, a_m; x_1, x_2)$ by the substitutions, algebraic with reference to $r_{\pm 1}$, obtained from (12) and the inverse of (29) (cf. (39)). The result is,

(a) A polynomial Φ_2, (later to be identified as a quantic of the system C), which reduces from the form $\Phi_2(\phi_m, \ldots; f_{+1}, f_{-1})$, to the form $\Phi_1(a_0, \ldots, a_m; x_1, x_2)$, algebraically with reference to $r_{\pm 1}$, when the explicit values of ϕ_{m-2i}, $f_{\pm 1}$ are substituted.

Explanatory conclusions relating to Hypothesis (2).

(α) If the facients $r_{\pm 1}$ in Φ_2 are replaced by real modular parameters $r_{\pm 1}$ which satisfy the condition (II) of paragraph (IV), we have,

$$\Phi_2(\phi_m, \ldots) \equiv \Phi(\phi_m, \ldots), \quad \text{(cf. Hyp. (1))}. \tag{54}$$

(β) If the facients $r_{\pm 1}$ in Φ_2 are replaced by galoisian imaginaries $r_{\pm 1}$ which satisfy the aforesaid condition (II), we have,

$$\Phi_2(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1}) \equiv \Psi(\phi_m, \ldots, \phi_{-m}; f_{+1}, f_{-1}), \quad \text{(cf. (42))}. \tag{55}$$

Note that Ψ is usually a different function of its arguments $\phi_m, \ldots, \phi_{-m}, f_{+1}, f_{-1}$ from what Φ_1 is of $a_0, \ldots, a_m, x_1, x_2$.

Hypothesis (3). Assume that $\Phi(\phi_m, \ldots, f_{+1}, f_{-1})$ is such a function that $\Psi(\phi_m, \ldots; f_{+1}, f_{-1})$ has the properties,

(a) The quantic Ψ is term-wise invariant, the exponents of the various terms of Ψ satisfying the congruence (20), ii.

(b) The quantic Ψ is evenly isobaric, *i.e.*, the equations (48) and (49) are assumed to be satisfied by its terms.

It is clear that the quantic $\Phi_1(a_0, \ldots, a_m; x_1, x_2)$, defined by these hypotheses, is a formal modular concomitant of f, under τ, in $R(1, 0, 0)$, with the condition (II) of §IV satisfied. Moreover, theoretically, all formal modular concomitants can be constructed as forms $\Phi_1(a_0, \ldots; x_1, x_2)$ by the methods of the hypotheses.

The following identifications with the functions alluded to at the beginning of the present paragraph IX may be mentioned, to wit; the function Φ_1 is an example of I, while Φ_2 is a quantic C and Ψ is a quantic F.

Theorem. Every concomitant of the infinitude of formal concomitants modulo $p > 2$ of f, of the domain R (1, 0, 0), is expressible as a linear combination

(51) *of forms of a definite set from the infinitude, with coefficients* N_j *which are quantics of the domain* $R(1, \tau, 0)$ *or of* $R(1, 0, 0)$, *invariantive under the transformation* τ *of* (II), §IV.

In proof, the hypotheses (1), (2), (3) define a Hilbert system, S, which consists of the totality of polynomials Φ_2 in the letters $\phi_{m-2i}, f_{\pm 1}$, $(i=0, \ldots, m)$. Hence there exists, within S, a finite set of quantics Φ_2, as $\Phi_2^{(1)}, \ldots, \Phi_2^{(s)}$, such that any polynomial Φ_2 of S is expressible in the form,

$$\Phi_2 \equiv N_1\Phi_2^{(1)} + \ldots + N_s\Phi_2^{(s)} \pmod{p}. \tag{56}$$

The congruence is formal, and the reduction of one side of (56) to the other, algebraic with reference to $r_{\pm 1}$. Since Φ_2 represents a formal modular concomitant of the domain $R(1, 0, 0)$ (Hyp. (2) (a)), the quantics N_j are polynomials, in the letters $\phi_{m-2i}, f_{+1}, f_{-1}$, which reduce to $R(1, \tau, 0)$, if not to $R(1, 0, 0)$, when these letters are written explicitly.

To prove that N_j is invariantive let the facients $r_{\pm 1}$be replaced by real modular parameters $r_{\pm 1}$. Then the forms Φ_2, $\Phi_2^{(t)}$, reduce to forms Φ of Hyp. (2), (a) and Hyp. (3), each function Φ_2, $\Phi_2^{(t)}$ being then the same function of $\phi_{m-2i}, f_{\pm 1}$ as of a_i, x_1, x_2, respectively. Since (56) is merely a certain grouping of the terms, $\phi_m^{x_0} \ldots \phi_{-m}^{x_m} f_{+1}^{\sigma_1} f_{-1}^{\sigma_2}$, of Φ_2(cf. (33)), it follows that N_j now satisfies Hyp. (1), (β) and is invariantive (when $r_{\pm 1}$ are real).

Let $r_{\pm 1}$ in (56) be replaced by parametric galoisian imaginaries. Then the set Φ_2, $\Phi_2^{(t)}$ reduce to forms Ψ of Hyp. (2), (β) and Hyp. (3). Hence the quantics N_j then have, also, the properties Hyp. (3), (a), (b) and are invariantive by (a). This proves the theorem.

Theorem. The quantics N_j *are forms of the domain* $R(1, 0, 0)$ *or can be so chosen.*

Assume $r_{\pm 1}$ to be real. Then N_j is a polynomial in $\phi_{m-2i}, f_{\pm 1}$, and we can show that the coefficients in this polynomial are integers. Any simultaneous concomitant of f and a quadratic, $g = b_0x_1^2 + b_1x_1x_2 + b_2x_2^2$, may be expressed in terms of invariant elements, $r_{\pm 1}$ being real parameters, by the substitution,

$$\begin{pmatrix} a_0, & \ldots, & a_m, & b_0, & \ldots, & b_2, & x_1, & x_2, \\ \phi_m, & \ldots, & \phi_{-m}, & \psi_2, & \ldots, & \psi_{-2}, & f_{+1}, & f_{-1} \end{pmatrix}, \tag{57}$$

where ψ_2, ψ_0, ψ_{-2} are the invariant elements of g. But the quantic N_j is a simultaneous concomitant of f and the quadratic form,

$$\Gamma = \beta_0 x_1^2 + (\beta_1 - \alpha_1)x_1x_2 - \alpha_2 x_2^2,$$

and the expression for the latter in terms of invariant elements is $\Gamma = f_{+1}f_{-1}$. Hence, if we write,

$$N_j = Y(a_0, \ldots, a_m, x_1, x_2, \beta_0, \gamma_1, \alpha_2),$$

we find (cf. (44))

$$Y(a_0, \ldots, a_m, x_1, x_2, \beta_0, \gamma_1, \alpha_2) = Y(\phi_m, \ldots, \phi_{-m}, f_{+1}, f_{-1}, 0, 1, 0). \tag{58}$$

The coefficients of the polynomial on the right are integers.

The relation (56) is now merely two forms in which a polynomial Φ_2, in $\phi_{m-2i}, f_{+1}, f_{-1}$, may be arranged, such that Φ_2, $\Phi_2^{(t)}$are concomitants in $R(1, 0, 0)$. In all parts of this relation let us make the replacements,

$$s = (\phi_m a_0)(\phi_{m-2}\, a_1)\dots(f_{+1}x_1)(f_{-1}x_2),$$

and designate the result as (56_1). Then, (cf. (44)), Φ_2, $\Phi_2^{(t)}$ assume their explicit forms as concomitants in $R(1, 0, 0)$. Also N_j becomes the polynomial in a_i, x_1, x_2 to which it would reduce by simplifying it, as an expression in multinomials $\phi_{m-2i}, f_{\pm 1}$, by multiplication. For suppose that this were not the case. Then this reduction of N_j is, necessarily, to a function of a_i, x_1, x_2 and β_0, γ_1, α_2, also. Then the substitution inverse to s, performed upon (56) (reduced)*, would give a form of (56) which contradicts (58) in that the coefficients in N_j would be expressions in the parameters α_1, α_2, β_0, β_1 and not integers.

Since N_j is now the same function of a_i, x_1, x_2 as of $\phi_{m-2i}, f_{\pm 1}$ its expression in terms of a_i, x_1, x_2 is invariantive with respect to all transformations ($r_{\pm 1}$ real) (II), §IV, of f, and isobaric modulo $p-1$, (Hyp. 1, (c), (β)). It is symmetric (or alternating) under the substitution $\sigma = (a_0 a_m)(a_1 a_{m-1})\dots(x_1 x_2)$, (cf. (30)); $g_{\pm 1}$ are integers in the present case.

Lemma. The polynomial N_j, in a_i, x_1, x_2, is seminvariantive.

Assume

$$\beta_1 = 0,\ \alpha_2 = 1, \text{ with } \beta_0 \neq 0,$$

such that

$$\Delta^{\frac{1}{2}(p-1)} = (4\beta_0 + \alpha_1^2)^{\frac{1}{2}(p-1)} \equiv 1 \pmod{p}. \tag{59}$$

Then the following are transformations (II), §IV, with $r_{\pm 1}$ real:

$$\tau' : x_1 = \alpha_1 x_1' + \alpha_2 x_2',\ x_2 = \beta_0 x_1';\quad \tau_3' : x_1 = x_2',\ x_2 = x_1',$$

and N_j is invariantive under

$$\tau'\tau'_3 : x_1 = x_1' + \alpha_1 x_2',\ x_2 = \beta_0 x_2'. \tag{60}$$

Hence, since N_j is isobaric modulo $p-1$, it is a seminvariant. Since there is but one number $r < 3$ such that $r^{\frac{1}{2}(3-1)} \equiv 1 \pmod{3}$, the case $p = 3$ forms an exception to the method (59). Here, however, the substitutions,

$$T : x_1 = 2x_1',\ x_2 = x_1' + x_2';\quad T_1 : x_1 = x_1',\ x_2 = x_1' + 2x_2',$$

which also belong to (II), §IV with $r_{\pm 1}$ real, with the property of symmetry of N_j, make the latter an anti-seminvariant. It is therefore a seminvariant due to its symmetry with reference to σ.

It follows that N_j is, in $R(1, 0, 0)$, a formal concomitant under the total group $G_{(p^2-p)(p^2-1)}$,the properties of symmetry, isobarism and seminvariance being sufficient conditions. By the same reasoning $\Phi_2^{(t)}$ are concomitants of G, (cf. §II).

We now transform (56_1) by the transformations of Hyp. (2). The result is a representation of Φ_2 which is essentially (56) but concerning which we have

*Every relation (56) has a form in terms of ϕ_{m-2i}, $f_{\pm 1}$ and a reduced form in terms of a_0, a_1, ...

the new information that the functions N_j simplify, algebraically as to the facients $r_{\pm 1}$, into formal modular concomitants (under G) in $R(1, 0, 0)$. Thus these forms N_j are *quantics* C, also.

Hence, if Hilbert's theorem be applied to the concomitants N_j, regarding the latter as Φ_2 quantics, the typical concomitant Φ_2 of S is ultimately expressed as a finite polynomial in the finite set $\Phi_2^{(1)}, \ldots, \Phi_2^{(s)}$. Thus,

Theorem. The system of all formal concomitants modulo $p > 2$ of a binary quantic f of order m is finite.

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FORMAL MODULAR INVARIANTS OF FORMS IN q VARIABLES*

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Suppose a form of order m in q variables

$$f(x_1, x_2, \ldots, x_q) = \sum a_{i_1, i_2, \ldots, i_q} x_1^{i_1} x_2^{i_2} \ldots x_q^{i_q},$$

where the summation is over all non-negative integral values of $i_1, i_2, \ldots, i_q$ such that $i_1+i_2+\ldots+i_q=m$, and where the x's are independent variables, subjected to a linear homogeneous substitution T:

$$\begin{aligned} x_1 &= t_{11}X_1+t_{12}X_2+\ldots+t_{1q}X_q, \\ x_2 &= t_{21}X_1+t_{22}X_2+\ldots+t_{2q}X_q, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_q &= t_{q1}X_1+t_{q2}X_2+\ldots+t_{qq}X_q. \end{aligned}$$

In the substitution T, the t_{ij} are elements of the Galois field $GF[p^n]$ and M, the determinant of the transformation, is a non-zero mark of $GF[p^n]$.

Let

$$F(X_1, X_2, \ldots, X_q) = \sum A_{i_1, i_2, \ldots, i_q} X_1^{i_1} X_2^{i_2} \ldots X_q^{i}$$

be the form into which the substitution T transforms the form f. A polynomial P in the coefficients of f is defined as a *formal modular invariant* of f under all non-singular linear homogeneous substitutions T over the $GF[p^n]$, if

$$P(A_{m00\ldots0}, \ldots, A_{00\ldots0m}) = M^{\lambda} P(a_{m00\ldots0}, \ldots, a_{00\ldots m}).$$

The exponent λ is called the index of the invariant P. If $\lambda=0$, P is an *absolute* formal modular invariant of f. We notice that each $A_{ij\ldots l}$ is a linear polynomial in the a's with coefficients in $GF[p^n]$.

HOMOGENEITY OF INVARIANTS. *Every formal modular invariant of a q-ary form is the sum of homogeneous parts, each of which is a formal modular invariant.*

Proof: Consider an invariant i, where

$$i = j_1+j_2+\ldots+j_r,$$

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each j being homogeneous in the a's and no two j's having the same degree. We are to prove that each j is an invariant. Since i and j_k are polynomials in the a's we may take I and J_k to be the same polynomials in the corresponding A's and

$$I = J_1 + J_2 + \ldots + J_r = M^\lambda i = M^\lambda (j_1 + j_2 + \ldots + j_r).$$

Since each A is linear in the a's we see from the fact that all the j's have different degrees that

$$J_k = M^\lambda j_k, \qquad (k = 1, 2, \ldots, r).$$

Not only is each j an invariant, but it is an invariant whose index is the same as the index of i.

WEIGHT: *Weight* is defined with respect to any particular variable x_r, which thus plays a rôle distinct from that of the other variables. We may arrange f in powers of x_r:

$$f = b_0 + b_1 x_r + \ldots + b_j x_r\ ,$$

where $b_0, b_1, \ldots, b_j$ are polynomials in all the x's except x_r. Each a occurs in one and only one of $b_0, b_1, \ldots, b_j$. To each a in b_0 we ascribe the weight 0, to each a in b_1 the weight 1, etc. If the weight of $a_{ij \ldots l}$ is s, the weight of $a^u_{ij \ldots l}$ is su, and the weight of a product is the sum of the weights of the factors. It is readily seen that if weight is determined with respect to x_s the weight of $a_{i_1, i_2, \ldots, i_s, \ldots, i_q}$ is i_s.

THEOREM: *If weights be assigned to the coefficients in any form, with respect to any variable, the weights of any two terms will be congruent to each other, mod* $(p^n - 1)$. *In virtue of this theorem a formal modular invariant is said to be modularly isobaric, mod* $(p^n - 1)$.

Example: The determinant

$$\begin{vmatrix} a^{p^{3n}} & b^{p^{3n}} & c^{p^{3n}} \\ a^{p^{2n}} & b^{p^{2n}} & c^{p^{2n}} \\ a & b & c \end{vmatrix}$$

is a formal modular invariant of the ternary linear form

$$f = ax_1 + bx_2 + cx_3.$$

If the weight be determined with respect to x_1, the weight of a is 1 and b and c are each of weight 0; if x_2 is the special variable, b is of weight 1, while a and c are each of weight 0; if x_3 is the special variable, the weight of c is 1 and a and b each have the weight 0. The truth of the theorem is readily verified in this case. Indeed, to give an illustration; if weight is determined with respect to x_1, the weight of the term $a^{p^{3n}} b^{p^{2n}} c$ is p^{3n}, and the weight of the term $ab^{p^{2n}} c^{p^{3n}}$ is 1; furthermore,

$$p^{3n} \equiv 1 \bmod (p^n - 1).$$

Proof of theorem: Apply to f the transformation

$$x_i = X_i, \qquad (i = 1, 2, \ldots, r-1, r+1, \ldots, q),$$

$x_r = \alpha X_r$, where α is a primitive root in the field.

This multiplies each a by α^w where w is the weight of that a. Consequently, each term of the invariant is multiplied by α^w where w is the weight of the term. But since α is the determinant of the transformation

$$I' = \alpha^\lambda I$$

and therefore

$$\alpha^\lambda = \alpha^{w_1} = \alpha^{w_2} = \ldots = \alpha^{w_l}.$$

If $w_1, w_2, \ldots, w_l$ are the weights we have $w_1 \equiv w_2 \equiv w_3 \equiv \ldots \equiv w_l \equiv \lambda \mod (p^n - 1)$.

THEOREM: *If the invariant I of a form f in q variables of order π is of degree i and of weight $\equiv w \mod (p^n-1)$, then $i\pi - qw \equiv 0 \mod (p^n-1)$; and this is true no difference what variable plays the special rôle with respect to weight.*

Example: In the example of an invariant of the ternary linear form above, $i = p^{3n} + p^{2n} + 1$, $\pi = 1$, $q = 3$, $w = 1$, and the relation $i\pi - qw \equiv 0 \mod (p^n - 1)$ is verified, since

$$p^{3n} + p^{2n} + 1 \equiv 3 \mod (p^n - 1).$$

Proof of above theorem: Apply the special transformation

$$x_i = \alpha \xi_i, \ (i = 1, 2, \ldots, q),$$

of determinant $M = \alpha^q$, in which α is a primitive root in the $GF[p^n]$; f is transformed into a form F, whose coefficients are the products of those of f by α^π. Hence in the polynomial P in the coefficients of F, the term corresponding to any term in the same polynomial in the coefficients of f is the product of that term by $\alpha^{\pi i}$; this latter is equal in the field to $M^w = \alpha^{qw}$. Consequently

$$\pi i - qw \equiv 0 \mod (p^n - 1).$$

ANNIHILATORS: Olive C. Hazlett has made a careful study* of the annihilators of modular invariants and covariants of binary forms. The particular result of hers which will be useful to us is the following: Every formal modular invariant of a system of forms

$$f_i = a_{0i} x^q + a_{1i} x^{q-1} y + a_{2i} x^{q-2} y^2 + \ldots + a_{qi} y^q, \quad (i = 1, 2, \ldots, r),$$

is annihilated by

$$\delta_k = \sum_{l=o}^{\infty} \frac{1}{[k + l(p-1)]!} \sum_{i=1}^{q} \Omega_i^{k+l(p-1)}, \quad (k = 1, p, p^2, \ldots, p^{n-1}),$$

where

$$\Omega_i = q a_{0i} \frac{\partial}{\partial a_{1i}} + (q-1) a_{1i} \frac{\partial}{\partial a_{2i}} + (q-2) a_{2i} \frac{\partial}{\partial a_{3i}} + \ldots + a_{q-1 i} \frac{\partial}{\partial a_{qi}}.$$

In other words if a polynomial P is an invariant of the system f_i, it satisfies the following n relations:

$$\delta_k P \equiv 0 \ (\text{mod } p), \quad (k = 1, p, \ldots, p^{n-1}).$$

*Annals of Mathematics, Second Series, vol. 23 (1921-22), p. 198.

In order to see the nature of the corresponding annihilators of formal modular invariants of forms in more than two variables, let us first examine the case in which there are three variables, x, y, z. We may arrange the form in descending powers of one variable, z say:

$$f(x, y, z) = a_m z^m + (a_{m-1}x + b_{m-1}y)z^{m-1} + \dots + a_0 x^m + b_0 x^{m-1} y + \dots + k_0 y^m.$$

An invariant of $f(x, y, z)$ under all non-singular linear transformations is, of course, invariant under all transformations

$$x' = t_{11}x + t_{12}y, \quad y' = t_{21}x + t_{22}y, \quad z' = z,$$

where

$$\begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix} \neq 0.$$

Hence an invariant may be written in the form

$$I_0 a_m^l + I_1 a_m^{l-1} + \dots + I_l,$$

where $I_0, I_1, \dots, I_l$ are invariants of the system of forms

$$\begin{gathered} a_{m-1}x + b_{m-1}y, \\ a_{m-2}x^2 + b_{m-2}xy + c_{m-2}y^2, \\ \dots\dots\dots\dots\dots\dots \\ a_0 x^m + b_0 x^{m-1} y + \dots\dots + k_0 y^m. \end{gathered}$$

Hence by Miss Hazlett's result we have a set of annihilators of any invariant of the ternary form.

In like manner if we have an invariant of a form of q variables we can write it in the form

$$I_0 J_l + I_1 J_{l-1} + \dots + I_l,$$

where J_l contains all the terms of degree l in the coefficients of terms of f which do not involve x and y, J_{l-1} contains all terms of degree $l-1$ in the same coefficients, etc., while the I's contain only coefficients of terms which contain x and y. We have then in the same way a set of annihilators of invariants of forms in q variables. We may, of course, choose any two variables to play the rôle played by x and y in the above discussion.

As a simple example, consider the linear form

$$f = a_1 x_1 + a_2 x_2 + \dots + a_{q-1} x_{q-1} + a_q x_q$$

and let x_{q-1}, x_q take the place of x and y, respectively. Annihilators of the invariants of

$$a_{q-1} x_{q-1} + a_q x_q$$

will be annihilators of the invariants of f. These annihilators are

$$\Omega + \frac{\Omega^{p^n}}{(p^n)!} + \ldots,$$

$$\frac{\Omega^p}{p!} + \frac{\Omega^{p+p^n-1}}{(p+p^n-1)!} + \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\Omega^{p^{n-1}}}{(p^{n-1})!} + \frac{\Omega^{p^{n-1}+p^n-1}}{(p^{n-1}+p^n-1)!} + \ldots,$$

where

$$\Omega = a_{q-1}\frac{\partial}{\partial a_q} \quad \text{or} \quad a_q \frac{\partial}{\partial a_{q-1}}.$$

As we may choose two variables from q in $\frac{1}{2}q(q-1)$ ways and from each choice obtain $2n$ annihilators, we have $nq(q-1)$ annihilators of the invariants of a q-ary form under linear substitutions over the $GF[p^n]$.

The general theorems which have just been proved have a close resemblance to general theorems in the theory of algebraic invariants. When we come to the study of invariants of given forms, we shall be impressed more with the contrast between the two theories than with their similarity. A single linear form has no algebraic invariants, but does possess formal modular invariants; we have given above an example of an invariant of the ternary linear form. By a fundamental system of invariants of a form we mean a finite number s of invariants of this form, which have the property that every invariant of the form is a polynomial (with coefficients in the field) in these s invariants. In the following pages we shall show that a q-ary linear form possesses a fundamental system of invariants and that this system has exactly q members.

The problem of finding a fundamental system of invariants of the linear form in q variables under the group G of all non-singular linear homogeneous transformations with coefficients in the $GF[p^n]$ is formally equivalent to the problem of finding a fundamental system of invariants of G. This is easily seen in the following manner. Given a linear q-ary form

$$f = c_1x_1 + c_2x_2 + \ldots + c_qx_q$$

in which $c_1, c_2, \ldots, c_q$ are independent variables, we shall apply to it the non-singular transformation T:

$$x_i = a_{i1}X_1 + a_{i2}X_2 + \ldots + a_{iq}X_q, \quad (i = 1, 2, \ldots, q),$$

where the a's are elements of the $GF[p^n]$.
The resulting form is

$$F = C_1X_1 + C_2X_2 + \ldots + C_qX_q,$$

where

$$C_1 = a_{11}c_1 + a_{21}c_2 + \ldots + a_{q1}c_q,$$
$$C_2 = a_{12}c_1 + a_{22}c_2 + \ldots + a_{q2}c_q,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$C_q = a_{1q}c_1 + a_{2q}c_2 + \ldots + a_{qq}c_q.$$

Comparing T with the above transformation, we see that there is a one-to-one reciprocal correspondence between the formal modular invariants of the linear form f under all non-singular linear homogeneous substitutions over the $GF[p^n]$ and the invariants of all non-singular linear homogeneous substitutions over the $GF[p^n]$. Taking the latter point of view, L. E. Dickson solved the problem for the binary group, and also for the general group in papers which appeared in the Transactions of the American Mathematical Society in 1911. The present solution of the problem is a consequence of the former point of view; it gives an interesting example of the application of the general theorems proved above and contrasts in many ways with Professor Dickson's solution.

FUNDAMENTAL SYSTEM OF INVARIANTS OF THE BINARY LINEAR FORM

THEOREM: *The expressions*

$$L_2 = a\Pi(b+at) = ab^{p^n} - a^{p^n}b,\quad Q_{21} = \frac{ab^{p^{2n}} - a^{p^{2n}}b}{L_2} = b^{p^n(p^n-1)} + \dots$$

(the product in the former taken with t ranging over all marks of the $GF[p^n]$), form a fundamental system of formal modular invariants of the linear form

$$f = ax + by$$

under all non-singular homogeneous linear substitutions over the $GF[p^n]$.

Proof: An invariant I is divisible by b or it is not. If it is divisible by b it is divisible by L_2 and $= L_2^r J$ where J is not divisible by b. If J is not simply a mark of the field it remains to prove that J, an invariant not divisible by b, is a polynomial in L_2 and Q_{21}. Since J is not divisible by b it is not divisible by a and therefore contains a term in b alone; so that, after being divided by a suitable mark of the field, it may be written in the form

$$J = b^s + \dots + ka^s.$$

If we give to b the weight 1 and to a the weight 0, we infer that $s \equiv 0 \bmod (p^n - 1)$, for the term ka^s has the weight 0 and by a previous theorem s, the weight of the term b^s, must be congruent to the weight of ka^s.

The weight s is also divisible by p^n, for J is annihilated by

$$\Omega + \frac{\Omega^{p^n}}{p^n!} + \dots,$$

$$\frac{\Omega^p}{p!} + \frac{\Omega^{p+p^n-1}}{(p+p^n-1)!} + \dots,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{\Omega^{p^{n-1}}}{(p^{n-1})!} + \frac{\Omega^{p^{n-1}+p^n-1}}{(p^{n-1}+p^n-1)!} + \dots\dots,$$

where $\Omega = a\dfrac{\partial}{\partial b}$.

Let us operate with the above annihilators on

$$J=b^s+\lambda_{p^n-1}\,a^{p^n-1}\,b^{s-p^n+1}+\lambda_{2(p^n-1)}\,a^{2(p^n-1)}\,b^{s-2p^n+2}+\ldots+ka^s,$$

remembering that s is a multiple of p^n-1 and that k is not zero. We obtain

$$\frac{s(s-1)\ldots(s-q+1)}{q!}\,a^qb^{s-q}+\text{terms containing lower powers of } b,$$

where q takes the values $1, p, p^2, \ldots, p^{n-1}$. Since the result of the operation is to be zero in the field and s is a rational integer,

$$\frac{s(s-1)\ldots(s-q+1)}{q!}\equiv 0 \pmod p, \quad (q=1, p, p^2, \ldots, p^{n-1}).$$

From this it follows that s is divisible by p^n. Consequently

$$J=b^{\lambda p^n(p^n-1)}+\ldots+ka^{\lambda p^n(p^n-1)}$$

where λ is some integer.

Now

$$Q_{21}=b^{p^n(p^n-1)}+\ldots.$$

Subtract from J the power of Q_{21} which has the same leading term as J. If the difference is not zero it will contain the factor a; hence it will contain the factor b and may be treated as I. By induction, L_2 and Q_{21} form a fundamental system of invariants of $ax+by$.

Corollary: The expressions

$$\begin{vmatrix} x^{p^n} & y^{p^n} \\ x & y \end{vmatrix}, \quad \frac{\begin{vmatrix} x^{p^{2n}} & y^{p^{2n}} \\ x^{p^n} & y^{p^n} \end{vmatrix}}{\begin{vmatrix} x^{p^n} & y^{p^n} \\ x & y \end{vmatrix}}$$

are a fundamental system of invariants of the linear group

$$x=\alpha X+\beta Y, \quad y=\gamma X+\delta Y, \quad (\alpha\delta-\beta\gamma\neq 0),$$

where $\alpha, \beta, \gamma, \delta$ *are marks of* $GF[p^n]$.

INVARIANTS OF THE TERNARY LINEAR FORM. In order to make clear the methods which we shall use in finding a fundamental system of invariants of the linear form in q variables we shall now treat the case of three variables. For convenience we shall arrange the terms of an invariant of

$$f=a_1x_1+a_2x_2+\ldots+a_qx_q$$

in the following manner. Of two terms, each of which contains a_q we shall write first that which involves the higher power of a_q unless they involve it to the same power; in the latter case the one (if either) which involves a_{q-1} to the higher power; if both involve both a_{q-1} and a_q to the same power, that shall be written first which involves a_{q-2} to the higher power, unless both involve a_{q-2} to the same power. As the terms are different, by continuing in this way we can ultimately decide which shall be written first and thus give to the terms of the invariant a definite order.

All the terms involving the highest power of a_q which occurs will be written first in a definite order, next will come all those containing a_q to the next lower power, etc. If an invariant I be written in descending powers of a_q, viz.,

$$I=A_0a_q^m+A_1a_q^{m-1}+\dots+A_m,$$

$A_0a_q^m$ is called the leading term and, as proved above, $A_0, A_1, \dots, A_m$ are invariants of

$$\phi=a_1x_1+a_2x_2+\dots+a_{q-1}x_{q-1}.$$

We have remarked in the preceding that

$$L_2=\begin{vmatrix} a & b \\ a^{p^n} & b^{p^n} \end{vmatrix}$$

is the product of a by the product of all non-proportional linear expressions of the form $b+at$ obtained by letting t range over all marks of the $GF\ [p^n]$.

Likewise

$$L_q=\begin{vmatrix} a_1^{p^{(q-1)n}} & a_2^{p^{(q-1)n}} & a_3^{p^{(q-1)n}} & \dots a_q^{p^{(q-1)n}} \\ a_1^{p^{(q-2)n}} & a_2^{p^{(q-2)n}} & a_3^{p^{(q-2)n}} & \dots a_q^{p^{(q-2)n}} \\ \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots a_q \end{vmatrix}$$

is the product of all the linear expressions

$$l_1a_1+l_2a_2+\dots+l_qa_q,$$

in which $l_1, l_2, \dots, l_q$ are marks not all zero of the $GF\ [p^n]$ and such that, of the coefficients l_i not zero, the one with the smallest subscript is unity:

$$P=\prod_{k=1}^{q}\prod_{l}(a_1+l_{k+1}a_{k+1}+\dots+l_qa_q)$$

where the inner product extends over the $p^{n(q-k)}$ sets $l_{k+1}, \dots, l_q$ of $q-k$ marks of the field. Hence the term

$$\prod_{k=1}^{q} a_k^{p^{n(q-k)}}=a_1^{p^{n(q-1)}}\dots a_{q-1}^{p^n}a_q$$

occurs once and only once in the expansion of P and has the coefficient unity. This term is the product of the elements in the main diagonal of the determinant L_q. Since L_q is invariant under the group of homogeneous linear transformations over $GF\ [p^n]$ and has the factor a_1, it follows that L_q is identical with the product P*.

*This theorem is due to E. H. Moore, Bull. of the Amer. Math. Soc., Vol. 2 (1896), p. 189. The proof given here is an almost verbatim reproduction of that of L. E. Dickson, Trans. Amer. Math. Soc., Vol. 12 (1911), p. 76.

Similarly,

$$[l_1, l_2, \dots, l_q] = \begin{vmatrix} a_1^{l_1 n} & a_2^{l_1 n} \dots\dots a_q^{l_1 n} \\ a_1^{l_2 n} & a_2^{l_2 n} \dots\dots a_q^{l_2 n} \\ \dots & \dots \\ a_1^{l_q n} & a_2^{l_q n} \dots\dots a_q^{l_q n} \end{vmatrix}$$

has the factor $L_q = [m-1, m-2, \dots, 1, 0]$. To $[m, m-1, \dots, s+1, s-1, \dots, 1, 0]L_m$ we give with Dickson the notation Q_{ms}.

We shall now prove the THEOREM: *Any three invariants whose leading terms are $L_2 c^{p^{2n}}$, $Q_{21} c^{p^{2n}(p^n-1)}$, $c^{p^{2n}(p^n-1)}$ form a fundamental system of invariants of $f = ax + by + cz$.*

Corollary I: Every invariant of the ternary form $ax+by+cz$ is a polynomial in

$$L_3 = \begin{vmatrix} a^{p^{2n}} & b^{p^{2n}} & c^{p^{2n}} \\ a^{p^n} & b^{p^n} & c^{p^n} \\ a & b & c \end{vmatrix} = L_2 c^{p^{2n}} + \dots,$$

$$Q_{31} = \frac{1}{L_3} \begin{vmatrix} a^{p^{3n}} & b^{p^{3n}} & c^{p^{3n}} \\ a^{p^{2n}} & b^{p^{2n}} & c^{p^{2n}} \\ a & b & c \end{vmatrix} = Q_{21} c^{p^{2n}(p^n-1)} + \dots,$$

$$Q_{32} = \frac{1}{L_3} \begin{vmatrix} a^{p^{3n}} & b^{p^{3n}} & c^{p^{3n}} \\ a^{p^n} & b^{p^n} & c^{p^n} \\ a & b & c \end{vmatrix} = c^{p^{2n}(p^n-1)} + \dots .$$

Corollary II: Every invariant of the ternary linear group is a polynomial in

$$\lambda_3 = \begin{vmatrix} x_1^{p^{2n}} & x_2^{p^{2n}} & x_3^{p^{2n}} \\ x_1^{p^n} & x_2^{p^n} & x_3^{p^n} \\ x_1 & x_2 & x_3 \end{vmatrix},$$

$$q_{31} = \frac{1}{\lambda_3} \begin{vmatrix} x_1^{p^{3n}} & x_2^{p^{3n}} & x_3^{p^{3n}} \\ x_1^{p^{2n}} & x_2^{p^{2n}} & x_3^{p^{2n}} \\ x_1 & x_2 & x_3 \end{vmatrix},$$

$$q_{32} = \frac{1}{\lambda_3} \begin{vmatrix} x_1^{p^{3n}} & x_2^{p^{3n}} & x_3^{p^{3n}} \\ x_1^{p^n} & x_2^{p^n} & x_3^{p^n} \\ x_1 & x_2 & x_3 \end{vmatrix}.$$

Proof of above theorem: An invariant I of the ternary form is or is not divisible by c. If it is, $I = L_3^n J$ where J is not divisible by c and hence not divisible by a. Consequently, J, if it is not simply a mark of the field, contains either the term $k_1 c^\gamma$ or the term $k_2 b^\beta c^\alpha$, where k_1 and k_2 are constants. Under the first supposition, after division by k_1, we have an invariant

$$J = c^\gamma + \lambda_1 c^{\gamma-1} + \dots + \lambda_{\gamma-1} c + J_0,$$

where J_0 is not zero and does not contain c. Since the weight of J_0 with respect to c is zero, γ the weight of c^γ is divisible by p^n-1. As $\lambda_1, \lambda_2, \ldots, \lambda_{\gamma-1}$, J_0 are invariants of the binary linear form, any term in any one of them which contains b only will contain it to a power which is a multiple of $p^n(p^n-1)$ and there is in J_0 a term in b alone, since J contains a term in c alone.

Now let us use the annihilators

$$\Omega + \frac{\Omega^{p^n}}{(p^n)!} + \ldots,$$

$$\Omega^p + \frac{\Omega^{p+p^n-1}}{(p+p^n-1)!} + \ldots + \frac{\Omega^{p^{n+1}}}{(p^{n+1})!} + \ldots,$$

$$\Omega^{p^2} + \frac{\Omega^{p^2+p^n-1}}{(p^2+p^n-1)!} + \ldots + \frac{\Omega^{p^{n+2}}}{(p^{n+2})!} + \ldots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\Omega^{p^{n-1}}}{(p^{n-1})!} + \frac{\Omega^{p^{n-1}+p^n-1}}{(p^{n-1}+p^n-1)!} + \ldots + \frac{\Omega^{p^{2n-1}}}{(p^{2n-1})!} + \ldots.$$

When we operate with these annihilators on c^γ we obtain terms which are numerical multiples of $bc^{\gamma-1}$, $b^p c^{\gamma-p}$, $b^{p^2} c^{\gamma-p^2}, \ldots, b^{p^{2n-1}} c^{\gamma-p^{2n-1}}$ and no one of these can cancel with any other or with any other term arising in the operations unless $p=2$, $n=1$, for $p^{rn-1}<p^n(p^n-1)$. Note that $p^{n+i}(i=1, 2, \ldots, n-1)$ occurs in the i-th annihilator above, since $p^i+p^i(p^n-1)=p^{n+i}$. The coefficients of these terms must then be $\equiv 0 \pmod p$, whence

$$\binom{\gamma}{s} \equiv 0 \pmod p, \quad (s=1, p, p^2, \ldots, p^{2n-1})$$

and from this it follows that

$$\gamma \equiv 0 \pmod{p^{2n}}.$$

Therefore every invariant J has the same leading term as a power of Q_{32}. Subtracting this power, we have left an invariant, no term of which is free of both a and b. If this invariant is divisible by c we treat it as we did the original I and we shall ultimately arrive at an invariant K which is neither divisible by c nor contains a term in c alone. If K is not simply a mark of the field, let

$$K = \ldots + (b^\beta + \ldots)c^\gamma + \ldots + K_0,$$

the terms of K being arranged in order in the manner described above. A term $b^\beta c^\gamma$ occurs in K, for since K is not divisible by c there is a term involving a and b only; it must involve both, for if there were a term in a alone or in b alone there would be a term in c alone. Of all terms in K which involve both b and c but not a, let $b^\beta c^\gamma$ be the one in which the exponent of c is the greatest. Every term in K to the left of $b^\beta c^\gamma$ contains a. If we use the same annihilators as above we shall find that the terms arising from operating on $b^\beta c^\gamma$ can be crossed out only by terms arising from operation on terms to the right of $b^\beta c^\gamma$ and also containing b and c only. But $\beta = kp^n(p^n-1)$ and if $b^{\beta'} c^{\gamma'}$ is the term in

K nearest to $b^{\beta}c^{\gamma}$, $\beta'=k'p^n(p^n-1)$ where $k'<k$. Hence as above, γ is divisible by $p^{2n}(p^n-1)$.

If in K, we write c for a, a for b, and b for c, it will be unchanged, since the determinant of the substitution is unity. The terms of K which do not contain a become the terms which do not contain c; in particular, $b^{\beta}c^{\gamma}$ becomes $a^{\beta}b^{\gamma}$ and is the term in K_0 in which b has the highest degree. Since K_0 is an invariant of the linear form in two variables, it is a polynomial in L_2 and Q_{21}. Hence $\gamma \geqq p^n\beta$. Since $\beta=l(p^{2n}-p^n)$, $\gamma \geqq lp^{2n}(p^n-1)$. But we have just proved that γ is divisible by $p^{2n}(p^n-1)$. Hence if we have two invariants whose leading terms are $c^{p^{2n}(p^n-1)}$ and $Q_{21}c^{p^{2n}(p^n-1)}$ we can find a multiple of powers of these which has as its leading term $b^{\beta}c^{\gamma}$, and after subtracting this we can treat any other term, which contains b and c but not a, in like manner, until we finally obtain either zero or an invariant divisible by a which we may treat in the same way as the original I. Thus the theorem is proved.

THE CASE OF q VARIABLES

THEOREM. *Any q invariants of the linear form in q variables*

$$f=a_0x_0+a_1x_1+\ldots+a_qx_q$$

whose leading terms are

$$a_0a_1^{p^n}a_2^{p^{2n}}a_3^{p^{3n}}\ldots a_q^{p^{qn}},\; a_1^{p^nt}a_2^{p^{2n}t}a_3^{p^{3n}t}\ldots a_q^{p^{qn}t},\;\ldots\ldots\ldots\ldots,\; a_q^{p^{qn}t},$$

where $t=p^n-1$, *are a fundamental system of invariants of f under all non-singular linear homogeneous transformations over the Galois Field $GF[p^n]$.*

Proof: We shall assume as proved the corresponding theorem for $q-1$ variables and prove our theorem by mathematical induction. As in the proofs for the special cases $q=2$, $q=3$, we need consider only an invariant J not simply a mark of the field and not divisible by a_q. If J contains a term containing a_q only, after division by a properly chosen mark of the field, let

$$J=a_q^{\gamma}+\ldots+J_0=\sum_{i=o}^{\gamma}\lambda_i a_q^{\gamma-i},$$

where $\lambda_0=1$, $\lambda_{\gamma}=J_0$ and where J_0 does not contain a_q and is not zero. Since J is modularly isobaric, γ is divisible by p^n-1. As each λ_i is an invariant of the binary linear form, any term in any λ_i which contains a_{q-1} only will contain it to a degree which is a multiple of $p(p^n-1)$. Now using the annihilators

$$\Omega+\frac{\Omega^{p^n}}{(p^n)!}+\ldots,$$

$$\Omega^{p}+\frac{\Omega^{p+p^n-1}}{(p+p^n-1)!}+\ldots,$$

$$\Omega^{p^2}+\frac{\Omega^{p^2+p^n-1}}{(p^2+p^n-1)!}+\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\Omega^{p^{n-1}}}{(p^{n-1})!}+\frac{\Omega^{p^{n-1}+p^n-1}}{(p^{n-1}+p^n-1)!}+\ldots,$$

where $\Omega=a_{q-1}\frac{\partial}{\partial a_q}$, we see that it is impossible then for the annihilation to take place unless the coefficients in the terms derived from

$$\frac{\Omega^i}{i!}a_q^\gamma, \quad [i=1, p, p^2, \ldots, p^{n-1}, \ldots, <p^{n(q-1)}(p^n-1)],$$

are all $\equiv 0 \pmod p$. Now this list in square brackets contains all powers of p up to and including the $(qn-1)$th, for

$$p^{qn-1}<p^{(q-1)n}(p^n-1),$$

unless $p=2, n=1$. Also, since $p^i+p^i\,(p^{(q-1)n}-1)=p^{(q-1)n+i}$,

$$\Omega^{p^{(q-1)n+i}}/(p^{(q-1)n+i})!$$

occurs in the i-th annihilator above. Since no power of a_{q-1} not a power of $a_{q-1}^{p^{(q-1)n}(p^n-1)}$ can occur in J_0, γ must have the factor $p^{(q-1)n}(p^n-1)$. We then have

$$\gamma\equiv 0,\ \frac{\gamma(\gamma-1)\ldots(\gamma-p+1)}{p!}\equiv 0,\ \ldots,\ \frac{\gamma(\gamma-1)\ldots(\gamma-p^{qn-1}+1)}{(p^{qn-1})!}\equiv 0 \pmod p,$$

whence $\gamma\equiv 0 \pmod{p^{qn}}$. Hence

$$J=a_q^{kp^{qn}(p^n-1)}+\ldots+J_0.$$

This expression has the same leading term as the k-th power of any invariant whose leading term is $a^{p^{qn}(p^n-1)}$. Subtracting this power we have left an invariant no term of which contains a_q only. If the resulting invariant is divisible by a_q it may be treated as was the original invariant, and we shall ultimately arrive at an invariant K which is neither divisible by a_q nor contains a term in a_q only. If K is not simply a mark of the field, let $K=\ldots+(a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_{q-1}^{\alpha_{q-1}}+\ldots)a_q^{\alpha_q}+\ldots$ As K is not divisible by a_q there is in K a term free of a_q and therefore a term free of a_0 and in the above we suppose that of all such terms $a_1^{\alpha_1}\ldots a_q^{\alpha_q}$ is the leading one. As above, using annihilators in which $\Omega=a_{q-1}\frac{\partial}{\partial a_q}$, we have $a_q\equiv 0 \pmod{p^{qn}(p^n-1)}$.

By using the transformation

$$a_0=a_q,\quad a_1=a_0,\quad a_2=a_1,\quad a_3=a_2,\ \ldots\ldots,\ a_q=a_{q-1},$$

the term becomes $a_0^{\alpha_1}a_1^{\alpha_2}\ldots a_{q-1}^{\alpha_q}$. Furthermore by the same reasoning as in the case $q=2$, we have $\alpha_q\geqq p^n\alpha_{q-1}$.

Hence if we have any q invariants, whose leading terms are

$$a_0\ \ a_1{}^{p^n}\,a_2{}^{p^{2n}}\,a_3{}^{p^{3n}}\ldots a_q{}^{p^{qn}},\ a_1{}^{p^{nt}}\,a_2{}^{p^{2nt}}\,a_3{}^{p^{3nt}}\ldots a_q{}^{p^{qnt}},\ldots\ldots\ldots\ldots,\ a^{p^{qnt}},$$

where $t=p^n-1$, we can find an invariant, which is a polynomial in these, having the same leading term as K. The theorem follows by induction.

Corollary I. Since L_q, Q_{q1}, Q_{q2}, $\ldots$, Q_{qq-1} *have exactly the leading terms of the* q *invariants in the above theorem, they form a fundamental system of formal modular invariants of the linear form in* q *variables.*

Corollary II. Every invariant of the q-ary linear group is a polynomial in λ_q, q_{q1}, q_{q2}, $\ldots$, q_{qq-1}, *these invariants being those obtained from* L_q, Q_{q1}, $\ldots$, Q_{qq-1}, *by replacing* a_1, a_2, $\ldots$, a_q *by* x_1, x_2, $\ldots$, x_q *respectively.*

A NEW THEORY OF LINEAR TRANSFORMATIONS AND PAIRS OF BILINEAR FORMS

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It is customary to develop the theory of pairs of bilinear forms having the matrices M and N, and by considering the special case in which N is the identity (or unit) matrix I to deduce the theory of the canonical form of a linear transformation T. We here proceed in reverse order and first develop independently a simple theory of linear transformation and later deduce the theory of equivalence of pairs of matrices and hence of pairs of bilinear forms. We avoid the introduction of irrationalities and employ only rational processes, so that our theory holds for any given field (or domain of rationality). We obtain a simple interpretation of invariant factors.

Moreover, we avoid the consideration of matrices whose elements are any polynomials in a variable λ as well as elementary transformations of matrices.

Start with any given linear transformation

$$T: \qquad \xi_i' = \alpha_{i1}\xi_1 + \ldots + \alpha_{in}\xi_n, \qquad (i=1, \ldots, n),$$

whose determinant $|\alpha_{ij}|$ may or may not be zero. We shall say that T replaces ξ_i by ξ_i'. Let x_1 be any linear homogeneous function of $\xi_1, \ldots, \xi_n$ and let T replace it by x'_1. If x'_1 is not the product of x_1 by a constant, write x_2 for x'_1 and consider similarly the function x'_2 by which T replaces x_2. If x'_2 is not a linear combination of x_1 and x_2, we write x_3 for x'_2. In this manner, we obtain a chain of linearly independent linear functions $x_1, x_2, \ldots, x_a$ of the ξ_i such that T implies

$$(1) \qquad x_1' = x_2,\ x_2' = x_3,\ \ldots,\ x'_{a-1} = x_a,\ x'_a = \text{lin. func. } x_1, \ldots, x_a.$$

Select x_1 so that it is the leader of such a chain of maximal length a. If $n=a$, (1) is the desired canonical form of T.

If $n>a$, let η_1 be any linear homogeneous function of the ξ_i which is linearly independent of $x_1, \ldots, x_a$. As before, T implies

$$\eta_1' = \eta_2,\ \eta'_2 = \eta_3,\ \ldots,\ \eta'_{b-1} = \eta_b,\ \eta_b' = \text{lin. func. } x_1, \ldots, x_a, \eta_1, \ldots, \eta_b.$$

It is possible to choose a linear function ρ of $x_1, \ldots, x_a$ so that, if T replaces ρ by ρ', ρ' by ρ'', etc., and if we introduce the new variables

$$y_1 = \eta_1 + \rho,\ y_2 = \eta_2 + \rho',\ \ldots,\ y_b = \eta_b + \rho^{(b-1)},$$

then T implies

(2) $$y'_1=y_2,\ y'_2=y_3,\ \ldots,\ y'_{b-1}=y_b,\ y'_b=\text{lin. func. } y_1,\ \ldots,\ y_b,$$

where the last function involves no one of $x_1, \ldots, x_a$. Let η_1 and hence y_1 be chosen so that b is the maximal length of a chain whose leader y_1 is linearly independent of $x_1, \ldots, x_a$. If $n=a+b$, (1) and (2) together give the desired canonical form of T.

But if $n>a+b$, there exists a third chain $z_1, \ldots, z_c$ such that

(3) $$z'_1=z_2,\ z'_2=z_3,\ \ldots,\ z'_{c-1}=z_c,\ z'_c=\text{lin. func. } z_1,\ \ldots,\ z_c,$$

such that c is the maximal length of a chain whose leader z_1 is linearly independent of $x_1, \ldots, x_a, y_1, \ldots, y_b$.

This process proves that T has a rational canonical form* C composed of partial transformations (1), (2), (3), etc., and such that $a, b, c, \ldots$ have the maximal properties described.

If we subtract λ from each diagonal element of the matrix A of a linear transformation T, we obtain the characteristic matrix (or λ-matrix) of A or T. The determinant of this λ-matrix is called the characteristic determinant of A or T.

Let $\alpha_1, \ldots, \alpha_k$ be the characteristic determinants of the partial transformations on the variables of the first, $\ldots$, kth chains of a canonical transformation C. Then α_i is divisible by α_{i+1} for every i. Moreover, the g.c.d. of all $(n-i)$-rowed determinants of the λ-matrix M of C is the product $\alpha_{i+1}\alpha_{i+2}\ldots\alpha_k$ if $i<k$, but is unity if $i\geqq k$.

By choice of the signs in

$$\alpha_1=\pm I_n,\ \alpha_2=\pm I_{n-1},\ \ldots,\ \alpha_k=\pm I_{n-k+1},$$

we obtain polynomials $I_n, \ldots$ in λ in which the coefficient of the highest power of λ is unity. Write $I_{n-k}=1, \ldots, I_1=1$. Then I_j is the well-known jth invariant factor of M. By the above results, I_j divides I_{j+1}, and

(4) $$I_j=\frac{G_j}{G_{j-1}}, \qquad (j=1, \ldots, n),$$

where G_j is the g.c.d. of all j-rowed determinants of M chosen so that the coefficient of the highest power of λ in G_j is unity, while $G_o=1$. By (4), the G's uniquely determine the I's, and conversely.

It is readily proved that G_j does not change when we introduce new variables into a linear transformation T. Hence T has the same invariant factors I_j as its canonical form C. The invariant factors other than unity of any linear transformation T are therefore (apart from signs) the characteristic determinants of the partial transformations of the canonical form C of T.

Two linear transformations (or matrices) S and T are called *similar* if there exists a matrix B whose determinant is not zero such that $BSB^{-1}=T$.

*It is easy to deduce the classical canonical form and the supplement relating to the irrationalities appearing in the new variables.

Then T may be derived from S by the introduction of new variables defined by a linear transformation of matrix B. Hence two linear transformations (or matrices) are similar if and only if their λ-matrices have the same invariant factors.

Two pairs of n-rowed square matrices M, N and R, S (or bilinear forms with those matrices) are called equivalent if and only if there exist n-rowed matrices P and Q whose determinants are not zero such that

$$PMQ=R, \qquad PNQ=S. \tag{5}$$

This is true if and only if $M-\lambda N$ and $R-\lambda S$ have the same invariant factors, provided the determinants of N and S are not zero. First, if (5) hold, then $MN^{-1}=J$ is similar to $RS^{-1}=K$ since

$$PJP^{-1}=PMQQ^{-1}N^{-1}P^{-1}=RS^{-1}=K,$$

so that the λ-matrices of J and K have the same invariant factors, nd the samea is true of

$$(J-\lambda I)N=M-\lambda N, \ (K-\lambda I)S=R-\lambda S.$$

Conversely, if the latter have the same invariant factors, J and K are similar, so that there exists a matrix P whose determinant is not zero such that $PJP^{-1}=K$. Then

$$P(J-\lambda I)P^{-1}=K-\lambda I, \ P(MN^{-1}-\lambda I)P^{-1}=RS^{-1}-\lambda I,$$

$$P(M-\lambda N)N^{-1}P^{-1}=(R-\lambda S)S^{-1}.$$

Writing Q for $N^{-1}P^{-1}S$, we get $P(M-\lambda N)Q=R-\lambda S$ or (5).

All of the preceding results were obtained by rational processes and hence hold for any field.

Next consider a pair of bilinear forms ϕ and ψ in the variables $x_1, \ldots, x_r$, $y_1, \ldots, y_s$ for the singular case in which either $r\neq s$ or else $r=s$ and the determinant of $u\phi+v\psi$ is zero identically in u, v. The theory is due to Kronecker (Sitzungsber. Akad. Berlin, 1890, 1225-37), who introduced irrationalities in order to employ the canonical form of Weierstrass for a pair of bilinear forms in the non-singular case. For the latter we may however employ the rational canonical form which follows from the foregoing theory. Hence we may replace Kronecker's theory by a purely rational one, valid for any field.

The new theory of which this is a bare outline will appear in a book on modern theories of algebra to be published by Sanborn & Co. of Chicago.

COMMUTATIVE CONJUGATE CYCLES IN SUBGROUPS OF THE HOLOMORPH OF AN ABELIAN GROUP

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If K represents any regular substitution group the holomorph H of K may be defined as the substitution group composed of all the substitutions on the letters of K which transform K into itself, and the group of isomorphisms of K may be defined as the subgroup formed by all the substitutions of H which omit a given letter. In the present article it will be assumed that K is abelian, and all the subgroups of H which involve K and have the property that all their conjugate cycles are commutative will be determined. For the sake of clearness we shall first consider the case when K is cyclic and has an order which is of the form p^m, p being a prime number.

§1. Holomorph of a Cyclic Group

If K is cyclic and if s_1 and s_2 represent any two non-identity substitutions of K then s_1 transforms the cycles of s_2 according to a cycle of order d where d is the quotient obtained by dividing the order of s_1 by the highest common factor of the orders of s_1 and s_2. In particular, a necessary and sufficient condition that s_1 transforms every cycle of s_2 into itself is that the order of s_1 divides that of s_2. The truth of these statements results directly from the well known fact that the only substitutions on the letters of a cycle which are commutative with this cycle are the powers of this cycle.

Suppose that the order of K is p^m. If i represents an operator in the group of isomorphisms of K such that i is not commutative with the operators of order p contained in K then i has p^m conjugates under K and i omits just one of the letters of K. Moreover, no cycle of i is transformed under H into any power of itself besides its first power. These statements result almost directly from the fact that the characteristic subgroup of order $p^m(p-1)$ which is generated by the operators of order $p-1$ contained in H, when $p>2$, is of class p^m-1 and involves only regular substitutions. Hence no cycle appears in more than one substitution of this subgroup, and no cycle of this subgroup except those of K is transformed into any except its first power under H. In fact, if such a cycle c were transformed under H into its αth power, where $\alpha \not\equiv 1$ mod the order of c, then c and c^α would appear in the same co-set of H with respect to K. By multiplying the substitutions of this co-set and of the one which involves the αth power of the substitution in which c appears by the inverse of the latter substitution we would obtain a product of class greater than 0 and less than p^m-1. As this is impossible, since this product would appear in the given characteristic subgroup, our theorem is eatablished for this subgroup.

Every substitution of H which is not commutative with the substitutions of order p contained in K involves at least one cycle containing a number of letters exceeding unity and dividing $p-1$. When this substitution is raised to some power of p the resulting substitution will be regular and of degree p^m-1. Hence it results that the co-set to which this substitution belongs cannot contain two cycles whose orders divide $p-1$ and which are powers of each other, since such cycles cannot appear in a co-set belonging to the characteristic subgroup noted above. The conjugates of a cycle whose order divides $p-1$ under a subgroup of H which involves K must involve at least two cycles which have a common letter, since the number of letters in these conjugates execeds p^m. As these two cycles cannot be powers of each other they are non-commutative. This completes a proof of the following theorem: *Every subgroup of* H *which includes* K *and involves cycles which contain a number of letters which exceeds one and divides* $p-1$ *involves some conjugate cycles that are not commutative.*

The preceding theorem evidently has meaning only when p is an odd prime number. When $p=2$ the corresponding theorem is that every subgroup of H which involves K and also substitutions which are not commutative with the substitutions of order 4 contained in K involves some conjugate cycles which are not commutative. A proof of this theorem results almost directly from the fact that the dihedral subgroup of H which involves K contains a subgroup of order 8 which has two octic groups for transitive constituents whenever $m>2$. As these octic groups involve conjugate cycles of order 2 which are non-commutative the dihedral subgroup of H must have this property. Since the group of isomorphisms of K involves a cyclic subgroup of order 2^{m-2} which omits four letters of K, and transforms into themselves the substitutions of order 4 contained in K, it results that every co-set of H, which contains substitutions which are not commutative with the substitutions of order 4 found in K, contains four substitutions involving as constituents substitutions of order 2 which with two constituents of order 4 in K constitute two simply isomorphic octic groups. Hence it results that *every subgroup of* H *which involves* K *but no non-commutative conjugate cycles when* $p=2$ *must contain a central involving the substitutions of order* 4 *contained in* K.

In the preceding paragraph we made use incidentally of the dihedral group whose order is divisible by 4. It is well known that when such a group is constructed by extending a regular cyclic group whose order k exceeds 2 by means of substitutions on the letters of this regular group half of the remaining substitutions are of degree k while the other half are of degree $k-2$. The cycles of the substitution of order 2 found in the given regular group are all found either in these substitutions of degree k or in these substitutions of degree $k-2$ as $k/2$ is odd or even. In particular, when k is a power of 2 these cycles appear always among the substitutions of degree $k-2$. In every case one of these non-invariant substitutions of order 2 contains at most one of the cycles in question; so that one of these cycles appears either in every one of the non-invariant substitutions of degree k or in every one of the substitutions of degree $k-2$.

From what precedes it results directly that every subgroup of H which involves K and has the property that every pair of conjugate cycles contained therein is commutative must involve the same number of substitutions of order p^m in every co-set with respect to K. In what follows it will be assumed that such a subgroup is non-regular and hence its order must exceed p^m. It is easy to see that the order of the group of isomorphisms of K under this subgroup cannot exceed $p^{\frac{m}{2}}$ since the order of the commutator subgroup of this subgroup must be equal to the order of this group of isomorphisms. In fact, each operator of the group of isomorphisms under consideration gives rise to a commutator whose order is equal to the order of this operator. These commutators must clearly be powers of the substitutions of order p^m which give rise to them. All groups of inner isomorphisms of the subgroups in question clearly have two and only two invariants. These invariants are equal to each other.

Having proved that the order of the largest subgroup of H which contains K and involves only commutative conjugate cycles cannot exceed $p^{\frac{3m}{2}}$, it remains to prove that its order has exactly this value when m is even while it has the value $p^{\frac{3m-1}{2}}$ when m is odd. To prove this theorem it may first be noted that if $c_1, c_2, \ldots, c_\lambda$ represent a set of cycles such that each one except the last is transformed into the one following it by a substitution s of order p^m contained in K, while the last is transformed into the first by the same substitution, and if the order of these cycles does not exceed $p^{\frac{m}{2}}$ then will $c_1.c_2^2. \ldots c_\lambda^\lambda$ be a substitution which omits at least p letters of K and transforms s into itself multiplied by $c_1.c_2. \ldots c_\lambda$. That is, $c_1.c_2^2. \ldots c_\lambda^\lambda$ is a substitution in the group of isomorphisms of K, and all the substitutions in this group of isomorphisms whose orders are powers of p and not greater than $p^{\frac{m}{2}}$ can be found in this manner.

From what precedes it results that all the cycles in H whose orders are powers of p and do not exceed $p^{\frac{m}{2}}$ are found in K and hence all the conjugates of these cycles under H are commutative. It remains only to prove that the conjugates of the cycles of the substitutions whose orders exceed $p^{\frac{m}{2}}$ in the group generated by K and the substitutions of H whose orders divide $p^{\frac{m}{2}}$ are also commutative. To prove this fact it may first be noted that the commutators of this group are found in its central and that these commutators do not interchange any of the cycles of the substitutions in question. Hence these commutators must be composed of constituents which are powers of the cycles in question. This completes a proof of the following theorem: *A necessary and sufficient condition that a subgroup of the holomorph of a cyclic group of order* p^m, *p being an odd prime number, which includes this cyclic group, involves only commutative conjugate cycles, is that this subgroup can be constructed by extending this cyclic group by means of substitutions whose orders divide* $p^{\frac{m}{2}}$. *When $p=2$ then these extending substitutions must be commutative with the substitutions of order* 4 *contained in K.*

Some of the preceding statements may become clearer by noting that whenever substitutions of different orders appear in a co-set of H with respect to K then all the substitutions in this co-set except those of the lowest order are regular and of degree p^m. This results directly from the facts that the order of every such substitution is a power of p and that a power of such substitution must equal a substitution of K which is not the identity. In particular, all the substitutions of H which are of degree p^m are regular substitutions. The only other substitutions of H which are regular on the letters which they actually involve are those whose orders divide $p-1$ and the p^2-p substitutions of order p and degree p^m-p^{m-1}. The latter substitutions are of lower degree than any others, besides the identity, contained in H.

When K is a cyclic group of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_\lambda^{\alpha_\lambda}$, where $p_1, p_2, \ldots, p_\lambda$ are distinct prime numbers which divide its order, and if c_1 is any cycle of K whose order is not divisible by a higher power of p than its $\frac{1}{2}\alpha_s$ power, where $s=1, 2, \ldots, \lambda$, and if $c_1, c_2, \ldots, c_\gamma$ are the successive transforms of c_1 under a generator of K, then will the substitution $c_1.c_2^2. \ldots c_\gamma^\gamma$ transform this generator into itself multiplied by $c_1.c_2. \ldots c_\gamma$. Hence it results that all the substitutions whose orders do not exceed the largest possible value of the order of c, subject to the conditions noted above, in the subgroup composed of all the substitutions of H which leave invariant all the substitutions of prime order in K, as well as those of order 4, if the order of K is divisible by 4, are composed of cycles found in K. Hence it results, just as in the case when the order of K is a power of a single prime number, that in the group generated by K and all the substitutions of the form $c_1.c_2^2. \ldots c_\gamma^\gamma$, subject to the conditions noted above, has the property that all its conjugate cycles are commutative. No other co-sets of H with respect to K can appear in such a group. Hence the following theorem: *The largest subgroup of the holomorph of a cyclic group K which includes K and has the property that every two of its conjugate cycles are commutative is of order $p_1^{s_1}p_2^{s_2} \ldots p_\lambda^{s_\lambda}$, where the order of K is $p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_\lambda^{\alpha_\lambda}$ and s_β is the largest integer which does not exceed $3\,^\alpha\beta/2$, $\beta=1, 2, \ldots, \lambda$. When the order of K is divisible by 4 its substitutions of this order are in the central of this largest subgroup.* Moreover, every subgroup of H, which involves K and has the property that all of its conjugate cycles are commutative, is contained in the largest subgroup noted in this theorem. It is well known that H is the direct product of the subgroups $H_1, H_2, \ldots, H_\lambda$ which are simply isomorphic, in order, with the holomorphs of the Sylow subgroups of K. The subgroups $H_1, H_2, \ldots, H_\lambda$ have the same transitive constituents as the corresponding Sylow subgroups of K. These transitive constituents, therefore, come under the developments given above. In particular, the class of H is $k(p_\alpha-1)/p_\alpha$, where k is the order of K and p_α is the smallest prime number which divides this order, whenever p_α is odd or k is divisible by p_α^2. When k is the double of an odd number, p_α is the smallest odd prime which divides k.

§2. Holomorph of the General Abelian Group

It may first be noted that if a substitution s of the holomorph of the regular abelian group K does not transform into itself every subgroup of K then s and K generate a group involving conjugate cycles which are not commutative. This follows directly from the fact that if s transforms a cyclic group of K into a different cyclic group these two groups must involve two cycles of the same order having a letter in common but not all their letters in common. Two such cycles could not be commutative. As all the cycles in the same substitution are conjugate under K two such cycles would have to be conjugate under the group generated by K and s, and hence these conjugate cycles could not be commutative. It may be added that it is sometimes possible to extend a regular *non-abelian* group by means of a substitution on its own letters which does not transform into itself every subgroup of this regular group so that all the conjugate cycles in the extended group are commutative. For such a non-abelian regular group we may take the group of order 16 in which the generators of the cyclic subgroups of order 8 are transformed into their fifth powers. Hence the theorem noted at the beginning of this paragraph does not apply to the holomorphs of non-abelian groups.

For the sake of simplicity we shall consider first the case when K is a prime power abelian group and hence K may be supposed to represent a regular group of order p^m. From the preceding paragraph it results that every subgroup of the holomorph H of K, which includes K and in which every pair of conjugate cycles is composed of commutative cycles, is found in that part of H which corresponds to the invariant operators in the group of isomorphisms of K. If p^α is the largest invariant of K the number of these invariant operators is known to be $p^{\alpha-1}(p-1)$. To the operators whose orders divide $p-1$ in this group of isomorphisms there correspond regular substitutions of H, and all of these regular substitutions which are not found in K are of degree p^m-1. Hence all the cycles which appear in such a co-set are distinct and no two such cycles are powers of each other. In particular, such a co-set could not appear in a subgroup of H which has the property that all the conjugate cycles are commutative. Since some power of every invariant operator of the group of isomorphisms of K whose order is not of power of p is one of these $p-1$ operators we may confine our attention to the subgroup of order $p^{\alpha-1}$ in the given group of invariant operators when $p>2$.

In fact, since every substitution of the subgroup composed of the invariant substitutions in the group of isomorphisms of K transforms into itself at least one regular cyclic subgroup of K of every order it follows from the preceding section that we may confine our attention to the subgroup whose order does not exceed $p^{\frac{\alpha-1}{2}}$ in the given subgroup composed of invariant substitutions when $p>2$. For similar reasons, when $\alpha>2$, we may confine our attention to the corresponding subgroup of order $2^{\frac{\alpha-1}{2}}$ when $p=2$. It remains therefore only to prove that when K is extended by such substitutions in its group of isomorphisms the resulting group has the property that all conjugate cycles are commutative.

To prove this theorem it may be first noted how all such substitutions may be found. An important feature to bear in mind is that all the cycles of such substitutions appear in K. To prove this fact, we note first that every independent generating substitution of K whose order is p^λ transforms according to a cyclic group of order p^λ the transitive constituents of the group generated by any other independent generating substitution of K. It is possible to construct according to the method explained in the preceding section a substitution involving only cycles of K which transforms one of the independent generators of K into any power corresponding to an operator whose order divides $p^{\frac{a-1}{2}}$ in its group of isomorphisms such that this substitution is commutative with each of the other generating substitutions of K. The product of substitutions thus selected for the various independent generators of K separately, such that these substitutions transform every independent generator into the same power, will be a substitution which transforms every substitution of K into the same power and involves only cycles of K. The fact that when K is extended by means of such a substitution all the conjugate cycles are commutative results just as in the preceding section.

It may be noted that the lowest degree of an invariant substitution besides the identity in the group of isomorphisms of K is equal to $p^m - p^n/p^\gamma$ where γ is equal to the number of the independent generators of highest order contained in K. Since the holomorph of a general abelian group is the direct product of the holomorphs of its Sylow subgroups the preceding considerations give rise to the following general theorem: *If K represents an abelian group of order $p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_\lambda^{\alpha_\lambda}$, where $p_1, p_2, \ldots, p_\lambda$ represent distinct prime numbers, and if H is a subgroup of the holomorph of K which includes K and has the property that all its conjugate cycles are commutative then H may be obtained by extending K by means of substitutions which involve only cycles of K and is a subgroup of a group whose order is the order of K multiplied by*

$$p^{\frac{1}{2}(\beta_i - 1)}, \quad (i=1, 2, \ldots, \lambda),$$

where $p_i^{\beta_i}$ *is the largest invariant in the Sylow subgroup of order* $p_i^{\alpha_i}$ *and* $[\frac{1}{2}(\beta_i - 1)]$ *represents the largest integer which does not exceed* $\frac{1}{2}(\beta_i - 1)$. Moreover, when the order of K is even and the largest invariant in its Sylow subgroup of even order does not exceed 4 then $p_i^{\beta i}$ is to represent the largest invariant in the Sylow subgroups of odd order only.

In all the cases noted above the subgroups of H which involve K and whose conjugate cycles are commutative were obtained by extending K by means of substitutions in its group of isomorphisms which involve only cycles found in K. This necessary condition can easily be proved to be also sufficient when K is cyclic and has an order which is a power of a prime. It was noted above that this power cannot be equal to 4. In all other cases a substitution of the group of isomorphisms of K whose order is a larger power of p than $p^{\frac{m}{2}}$, p^m being the order of K, must involve at least one cycle which does not appear in K, since such a substitution could not be commutative with the substitutions of K of the same order as this substitution. When $p > 2$, it was noted above that all the remaining substitutions of the group of isomorphisms of K which do not appear in the subgroups under consideration involve at least one cycle whose order is prime to p. When $p=2$, these remaining substitutions involve cycles of order 2 which do not appear in K. Hence the following theorem has been established: *When K is a cyclic prime power regular group whose order is not* 4 *then a necessary and sufficient condition that a subgroup of the holomorph of K which involves K contains only commutative conjugate cycles is that this subgroup can be constructed by extending K by means of substitutions in its group of isomorphisms involving only cycles of K.*

If K is any regular group whose holomorph is H then any substitution of H which involves at least one cycle of K is commutative with the substitution of K which contains this cycle. When a substitution s in the group of isomorphisms of K involves only cycles of K it must therefore be commutative with all the substitutions of K which involve these cycles. These substitutions generate a subgroup of K whose transitive constituents are regular, and hence s must be composed of constituents which are substitutions of the conjoints of the transitive constituents. In particular, when K is abelian all the substitutions of K with which s is commutative generate a subgroup of K, and s omits all the letters of one and only one of the transitive constituents of this subgroup. In fact, if a substitution transforms any regular group into itself and omits at least one of the letters of this regular group then the number of its letters which it omits is exactly equal to the number of the substitutions with which it is commutative. Hence the following theorem: *If s represents a substitution in the group of isomorphisms of a regular abelian group K and if s involves only cycles of K then K involves a subgroup K_1 such that s omits all the letters of a transitive constituent of K_1 but no other letter, and s is composed of substitutions found in the other transitive constituents of K_1.*

From this theorem it results directly that whenever a regular abelian group K is extended by means of a substitution s which involves only cycles of K then this extended group can be constructed by establishing a certain isomorphism between the transitive constituents of a subgroup of K such that the group thus obtained is invariant under the remaining substitutions of K. The substitution s together with those of K give rise to a number of commutators which is equal to the index of K_1 under K, and each of these commutators involves one and only one constituent involving as many letters as are omitted by s. The entire substitution s can readily be constructed from any one of these constituents, since each of them is transformed into the letters of every other under K. In the particular case when K is cyclic the construction of s from such a constituent is especially simple as was noted in the preceding section.

DIFFERENTIAL COMBINANTS AND ASSOCIATED PARAMETERS

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PART 1. BINARY DIFFERENTIAL PARAMETERS

I. *Introduction.* Simple statements of the algorisms of the symbolic theory of differential parameters, which was originated by H. Maschke*, are followed in the first part of this paper by a treatment of binary associated forms, a theory not previously developed for differential quantics. The new theory of combinants outlined in Part 1 for binary forms and extended to n variables in Part 2 has noteworthy practical and theoretical implications among which is the translation principle which establishes correspondence between combinant parameters of sets of n-ary forms and differential parameters of space of $n+1$ dimensions.

Transform a differential quantic,

$$(1)\qquad F=a_0(x_1, x_2)dx_1{}^m+\binom{m}{1}a_1(x_1, x_2)dx_1{}^{m-1}dx_2+\ldots+a_m(x_1, x_2)dx_2{}^m,$$

by arbitrary functional substitutions,

$$(2)\qquad T: x_i=x_i\,(y_1, y_2),\ dx_i=\frac{\partial x_i}{\partial y_1}dy_1+\frac{\partial x_i}{\partial y_2}dy_2,\ (i=1, 2).$$

This gives rise to invariants ϕ of both the absolute and the relative type: $\phi'=\phi$ or $\phi'=R\phi$, primes indicating functions of y_1, y_2. A desirable explicit notation is

$$(3)\qquad \phi(a_0,\ \ldots,\ a_m;\frac{\partial^r a_i}{\partial x_1{}^{r-s}\partial x_2{}^s};\frac{\partial u}{\partial x_1};\ldots;\ dx_1,\ dx_2),$$

and it indicates a function of the functions $a_i(x_1, x_2)$ and their x_1, x_2 derivatives together with derivatives of extraneous arbitrary functions, $(u'=u)$, the whole being a polynomial in the differentials dx_1, dx_2.

If λ is any symbolical or actual function of x_1, x_2 we use the following abbreviations†:

$$(4)\qquad d\lambda=\frac{\partial\lambda}{\partial x_1}dx_1+\frac{\partial\lambda}{\partial x_2}dx_2=\lambda_1dx_1+\lambda_2dx_2=\lambda_{dx}.$$

*Ricci and Levi-Civita, Mathematische Annalen, vol. 54 (1901), p. 125; Maschke, Trans. Amer. Math. Soc., vol. 1 (1900), p. 197; vol. 4 (1903), p. 445.

†Thus subscripts of symbols shall indicate partial derivatives.

Accordingly either F or ϕ may be represented symbolically; thus,

$$F=f_{dx}^m=g_{dx}^m,\ \phi=\alpha_{dx}^w=\beta_{dx}^w,\ \text{(Maschke)}. \tag{5}$$

II. *The invariant relation. Complementing factors.* The transformed of F by the relations (2) is

$$F'=\left[\left(f_1\frac{\partial x_1}{\partial y_1}+f_2\frac{\partial x_2}{\partial y_1}\right)dy_1+\left(f_1\frac{\partial x_1}{\partial y_2}+f_2\frac{\partial x_2}{\partial y_2}\right)dy_2\right]^m \tag{6}$$

hence,

$$a_i'=\left(f_1\frac{\partial x_1}{\partial y_1}+f_2\frac{\partial x_2}{\partial y_1}\right)^{m-i}\left(f_1\frac{\partial x_1}{\partial y_2}+f_2\frac{\partial x_2}{\partial y_2}\right)^i,\ (i=0,\ \ldots,\ m), \tag{7}$$

expressions which are linear in $a_0, \ldots, a_m$; $(f_1^{m-j}f_2^j=a_j)$. If we differentiate a_i' successively, using the operators,

$$\frac{\partial}{\partial y_1}=\frac{\partial x_1}{\partial y_1}\frac{\partial}{\partial x_1}+\frac{\partial x_2}{\partial y_1}\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial y_2}=\frac{\partial x_1}{\partial y_2}\frac{\partial}{\partial x_1}+\frac{\partial x_2}{\partial y_2}\frac{\partial}{\partial x_2}, \tag{8}$$

we obtain for each derivative, $\partial^r a_i'/\partial y_1^{r-s}\,\partial y_2^s$, a linear expression in the derivatives of orders from zero to the rth of $a_0, \ldots, a_m$. The explicit verification of the identity $\phi'=R\phi$ is made by the substitution of these linear expressions, the relations obtained by operating with (8) upon the equation $u'=u$, and the inverse of relations (2), in ϕ'. It follows that if ϕ is of the rational, integral type, no generality is lost by the assumption that it is homogeneous in dx_1, dx_2 and in the set consisting of $a_0, \ldots, a_m$ and their x_1, x_2 derivatives up to the order r of ϕ. A parameter which is non-homogeneous in the *orders* of derivatives may be reducible into invariantive constituents which are homogeneous as to these orders. An example is Beltrami's second differential parameter ($m=2$, cf. (5)),

$$\Delta_2 u=c(f,c(f,u))=c(f,c)(f,u)+c^2\begin{vmatrix} f_1 & f_2 \\ (f,u_1) & (f,u_2)\end{vmatrix}+c^2\begin{vmatrix} f_1 & f_2 \\ (f_1,u) & (f_2,u)\end{vmatrix}, \tag{9}$$

(f, u) indicating the jacobian of f and u.

In $\Delta_2 u$ and in the following relations expressive of the cogrediency of the operators $c\partial/\partial x_2$, $-c\partial/\partial x_1$, as applied to absolute invariants, with the differentials dx_1, dx_2, the expression c is the reciprocal of the $m(m-1)$-th root of the discriminant of F:

$$c\frac{\partial}{\partial x_2}=c'\frac{\partial x_1}{\partial y_1}\frac{\partial}{\partial y_2}+\frac{\partial x_1}{\partial y_2}\left(-c'\frac{\partial}{\partial y_1}\right),\ -c\frac{\partial}{\partial x_1}=c'\frac{\partial x_2}{\partial y_1}\frac{\partial}{\partial y_2}+\frac{\partial x_2}{\partial y_2}\left(-c'\frac{\partial}{\partial y_1}\right). \tag{10}$$

If ϕ is relative and an invariantive operational relation, $\Delta'=t\Delta$, involving first x_1, x_2 derivatives, is applied to $\phi'=R\phi$, we have

$$\Delta'\phi'=tR\Delta\phi+t\phi\Delta R, \tag{11}$$

and, since R is usually a function of x_1, x_2, this is not a relation of invariancy. But, if ϕ is first complemented by an invariantive multiplier μ which renders it absolute, $\Delta\mu\phi$ will be a parameter. Thus when $R=D^k$, $(D=(x_1,\ x_2))$, we may complement with c^k. Then,

$$\Delta' c'^k\phi'=t\Delta c^k\phi, \tag{12}$$

and $\Delta c^k\phi$ is a differential parameter. The following are some of the invariantive operators which are used to generate invariants from invariants:

The jacobian of two absolute parameters,

$$(13)\qquad (\phi', \psi') = D(\phi, \psi);$$

The total differential,

$$(14)\qquad \frac{\partial\phi'}{\partial y_1}dy_1 + \frac{\partial\phi'}{\partial y_2}dy_2 = \frac{\partial\phi}{\partial x_1}dx_1 + \frac{\partial\phi}{\partial x_2}dx_2;$$

The operator obtained by substituting for dx_1, dx_2 in any covariantive absolute parameter the cogredient elements,

$$(14_1)\qquad c\frac{\partial}{\partial x_2},\ -c\frac{\partial}{\partial x_1};$$

Polar operators,

$$(15)\qquad dw_1\frac{\partial}{\partial(dx_1)} + dw_2\frac{\partial}{\partial(dx_2)},$$

dw_1, dw_2 being cogredient to dx_1, dx_2.

III. *Typical representation**. If P, Q are two absolute parameters which are functionally independent and free from dx_1, dx_2, the differentials,

$$(16)\qquad dP = P_{dx},\ dQ = Q_{dx},$$

are linearly independent absolute parameters. They are linear in dx_1, dx_2 and if we transform F by the inverse of (16) we obtain

$$(17)\qquad F = [(f, Q)dP + (P, f)dQ]^m \div (P, Q)^m$$
$$= \left[A_0(dP)^m + \binom{m}{1}A_1(dP)^{m-1}(dQ) + \ldots + A_m(dQ)^m\right] \div (P, Q)^m.$$

This is the abstract form of a typical representation of F. The coefficients A_i are relative parameters, in fact, $A_i' = D^m A_i$, where

$$(18)\qquad A_i = (f, Q)^{m-i}(P, f)^i,\ (i = 0, \ldots, m).$$

Since the transformations employed to produce (17) are arbitrary we have a theorem: *Every parameter of F which is rational and integral is a polynomial in dP, dQ, c, in parameters of the form $B_i = A_i/(P, Q)^m$ and their derivatives with regard to P and Q, together with derivatives of arbitrary functions ($u' = u$). These arguments form an associated system.*

Indeed if we solve $P = P(x_1, x_2)$, $Q = Q(x_1, x_2)$ for x_1, x_2 the present theory becomes the regular theory with y_1, y_2 replaced by P, Q, (17) being the transformed form (cf. (7)), Moreover the derivatives of B_i with respect to P, Q are invariants; in fact, from $A_i' = D^m A_i$, we have $B_i' = B_i$ and since $P' = P$, $Q' = Q$, there follows,

*Hermite, Journal für Math., vol. 52 (1856), p. 1. Sylvester, Cambridge and Dublin Math. Journal, vol. 7 (1852), p. 52.

$$\frac{\partial B_i'}{\partial P'} = \frac{\partial B_i}{\partial P}, \quad \frac{\partial B_i'}{\partial Q'} = \frac{\partial B_i}{\partial Q}, \tag{19}$$

which are the invariant properties required. Thus all of the associated forms are parameters.

Among the simplest associated systems is that obtained by employing the method (15) and combining F with u, v, *i.e.*, $u'_{dy} = u_{dx}$, $v'_{dy} = v_{dx}$. We obtain

$$P = c^m(f, u)^m, \quad Q = c^m(f, u)^{m-r}(f, v)^r, \tag{20}$$

in which r may have any value from 1 to $m-1$. The linear covariants P_{dx}, Q_{dx} are then (cf. (4)),

$$\begin{aligned} dP = {} & mc^{m-1}(f, u)^{m-1}\{[c_1(f, u) + c(f_1, u) + c(f, u_1)]dx_1 \\ & + [c_2(f, u) + c(f, u_2) + c(f_2, u)]dx_2\}, \quad [(f_1, u) = f_{11}u_2 - f_{12}u_1], \end{aligned} \tag{21}$$

$$\begin{aligned} dQ = {} & c^{m-1}(f, u)^{m-r-1}(f, v)^{r-1}\{[mc_1(f, u)(f, v) + (m-r)c(f, v)((f_1, u) + (f, u_1)) \\ & + rc(f, u)((f_1, v) + (f, v_1))]dx_1 + [mc_2(f, u)(f, v) + (m-r)c(f, v) \\ & \times((f, u_2) + (f_2, u)) + rc(f, u)((f, v_2) + (f_2, v))]dx_2\}. \end{aligned}$$

The explicit form of (P, Q) is, $(\gamma = c^m)$,

$$\begin{aligned} & \gamma(f, u)^{m-1}(g, u)^{m-r-1}(g, v)^{r-1}\{(m-r)(f, u)(g, v)(\gamma, (g, u)) \\ & + r(f, u)(g, u)(\gamma, (g, v)) + m(g, u)(g, v)((f, u), \gamma) \\ & + m(m-r)\gamma(g, v)((f, u), (g, u)) + mr\gamma(g, u)((f, u), (g, v))\}. \end{aligned} \tag{22}$$

When $m = 2$ we might use, instead of dQ, the simplest linear covariant whose coefficients are linear expressions in Christoffel's triple index symbols of the first kind, viz.,

$$(f, c)f_{dx} + c\left[\left(\begin{bmatrix} 21 \\ 1 \end{bmatrix} - \begin{bmatrix} 11 \\ 2 \end{bmatrix}\right)dx_1 + \left(\begin{bmatrix} 22 \\ 1 \end{bmatrix} - \begin{bmatrix} 12 \\ 2 \end{bmatrix}\right)dx_2\right]. \tag{23}$$

IV. *Combinants**. The definition of a binary differential combinant is the following: Suppose that $f = \alpha^m_{dx}$, $g = \beta^m_{dx}$, $\ldots$, $l = \mu^m_{dx}$, is a set of n differential forms of order m $(n > 1)$, and that ϕ is any simultaneous differential parameter of the set (S) which satisfies, also, the added relation of invariancy,

$$\phi'' = (\xi\eta\zeta\ldots)^k\phi, \tag{24}$$

when the forms of S are themselves transformed linearly according to the substitutions,

$$\begin{aligned} f'' &= \xi_1 f + \eta_1 g + \ldots + \sigma_1 l, \\ g'' &= \xi_2 f + \eta_2 g + \ldots + \sigma_2 l, \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ l'' &= \xi_n f + \eta_n g + \ldots + \sigma_n l, \end{aligned} \tag{25}$$

where ξ_i, η_i, $\ldots$, are constants. Then ϕ is a combinant of S.

*Sylvester, Cambridge and Dublin Math. Journal, vol. 8 (1853), p. 256. White, Amer. Jour. Math., vol. 17 (1895), p. 235.

The jacobian of two forms, *i.e.*,

$$(26)\qquad (f,\ g)=(\alpha_{dx},\ \beta_{dx})\alpha_{dx}^{m-1}\beta_{dx}^{m-1}=\begin{vmatrix}\alpha_{dx}^{m-1}(\alpha_1)_{dx} & \alpha_{dx}^{m-1}(\alpha_2)_{dx}\\ \beta_{dx}^{m-1}(\beta_1)_{dx} & \beta_{dx}^{m-1}(\beta_2)_{dx}\end{vmatrix}$$
$$=\alpha_{dx}^{m-1}\beta_{dx}^{m-1}[(\alpha_1,\ \beta_1)dx_1^2+((\alpha_1,\ \beta_2)+(\alpha_2,\ \beta_1))dx_1dx_2+(\alpha_2,\ \beta_2)dx_2^2],$$

if complemented by the reciprocal of the m^2-th root of the algebraic combinant resultant of f and g is an absolutely invariantive differential combinant of the first order of f and g.

It is linear in the determinants,

$$(27)\qquad (a_i:b_j)=\frac{\partial a_i}{\partial x_1}\frac{\partial b_j}{dx_2}-\frac{\partial b_i}{\partial x_1}\frac{\partial a_j}{\partial x_2}\equiv a_{ix_1}b_{jx_2}-a_{jx_2}b_{ix_1}.$$

The first transvectant, $(f,\ g)^1$, which is the algebraic jacobian, is a combinant of zero order in the differential theory, linear in the determinants,

$$(28)\qquad (a_ib_j)=a_ib_j-b_ia_j.$$

Theorem. If a differential parameter ϕ of order r of S is absolute as a result of being complemented by c, any algebraic combinant invariant, a necessary and sufficient condition in order that ϕ should be a combinant is that it should satisfy an independent set of the system of linear partial differential operators,

$$(29)\qquad \left(a\frac{\partial}{\partial b}\right),\ \left(b\frac{\partial}{\partial a}\right),\ \left(a\frac{\partial}{\partial c}\right),\ \ldots;$$

$$\left(a\frac{\partial}{\partial b}\right)=\sum_{i=0}^{m}a_i\frac{\partial}{\partial b_i}+\sum_{i=0}^{m}a_{ix_1}\frac{\partial}{\partial b_{ix_1}}+\ldots+\sum_{i=0}^{m}a_{ix_1{}^{r-s}x_2{}^s}\frac{\partial}{\partial b_{ix_1{}^{r-s}x_2{}^s}}.$$

Partial differentiations of a_i and b_i are indicated by subscripts; $(s=0,\ \ldots,\ r)$.

The relations (25) are equivalent to linear substitutions upon the functional coefficients of the quantics, viz.,

$$(30)\qquad \begin{aligned} a_i''&=\xi_1a_i+\eta_1b_i+\ldots+\sigma_1l_i,\\ b_i''&=\xi_2a_i+\eta_2b_i+\ldots+\sigma_2l_i,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ l_i''&=\xi_na_i+\eta_nb_i+\ldots+\sigma_nl_i.\end{aligned}$$

Hence the derivatives of these functions are transformed linearly,

$$(31)\qquad \begin{aligned} a''_{ix_1{}^{u-v}x_2{}^v}&=\xi_1a_{ix_1{}^{u-v}x_2{}^v}+\eta_1b_{ix_1{}^{u-v}x_2{}^v}+\ldots+\sigma_1l_{ix_1{}^{u-v}x_2{}^v},\\ b''_{jx_1{}^{u-v}x_2{}^v}&=\xi_2a_{jx_1{}^{u-v}x_2{}^v}+\eta_2b_{jx_1{}^{u-v}x_2{}^v}+\ldots+\sigma_2l_{jx_1{}^{u-v}x_2{}^v},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

In proof of the necessity, the relation (24) holds under the special case $\xi_1=\eta_2=\ldots=\sigma_n=1$, $\xi_2=\delta$ (an infinitesimal), with all other constants zero. Then $(\xi\eta\ldots\sigma)=1$. Let ϕ be represented in the form (3), and, by Taylor's theorem,

$$(32)\qquad \phi(a_0, \ldots, b_i+\delta a_i, \ldots, b_{jx_1}{}^{u-v}{}_{x_2}{}^{v}+\delta a_{jx_1}{}^{u-v}{}_{x_2}{}^{v}, \ldots) = \phi(a_0, \ldots, b_i, \ldots, b_{jx_1}{}^{u-v}{}_{x_2}{}^{v}, \ldots)+\delta\left(a\frac{\partial}{\partial b}\right)\phi(a_0, \ldots, b_i, \ldots),$$

from which we have $\left(a\dfrac{\partial}{\partial b}\right)\phi=0$. In the same way ϕ satisfies the remaining operators (29).

Reasoning conversely, assume $\left(b\dfrac{\partial}{\partial a}\right)\phi=0$, then, since ϕ is absolute, $\left(b''\dfrac{\partial}{\partial a''}\right)\phi''=0$, *i.e.*,

$$(33)\qquad \sum_{i=0}^{m}\sum_{v=0}^{u}\sum_{u=0}^{r}(\xi_2 a_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\eta_2 b_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\ldots)\frac{\partial\phi''}{\partial(\xi_1 a_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\eta_1 b_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\ldots)}$$
$$=\sum_{i=0}^{m}\sum_{v=0}^{u}\sum_{u=0}^{r}\Bigg[\xi_2\frac{\partial\phi''}{\partial(\xi_1 a_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\eta_1 b_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\ldots)}\frac{\partial(\xi_1 a_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\ldots)}{\partial\xi_1}$$
$$+\eta_2\frac{\partial\phi''}{\partial(\xi_1 a_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\eta_1 b_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\ldots)}\frac{\partial(\xi_1 a_{ix_1}{}^{u-v}{}_{x_2}{}^{v}+\ldots)}{\partial\eta_1}+\ldots\Bigg]=0.$$

Hence,

$$(34)\qquad \delta_{21}\phi''=\left(\xi_2\frac{\partial}{\partial\xi_1}+\eta_2\frac{\partial}{\partial\eta_1}+\ldots+\sigma_2\frac{\partial}{\partial\sigma_1}\right)\phi''=0.$$

It is clear that we can obtain in this way n^2 equations;

$$(35)\qquad \delta_{\tau\nu}\phi''\begin{cases}=0, & (\nu\neq\tau),\\ =i_\nu\phi'', & (\nu=\tau),\end{cases}$$

where i_ν is the order of ϕ'' in the parameters $\xi_\nu, \eta_\nu, \ldots, \sigma_\nu$. Referring to small variations of $\xi_\nu, \ldots, \sigma_\nu$ we combine*

$$(36)\qquad d\phi''=\frac{\partial\phi''}{\partial\xi_\nu}d\xi_\nu+\frac{\partial\phi''}{\partial\eta_\nu}d\eta_\nu+\ldots+\frac{\partial\phi''}{\partial\sigma_\nu}d\sigma_\nu,$$

with the subset,

$$(37)\qquad \begin{aligned}&\xi_\nu\frac{\partial\phi''}{\partial\xi_\nu}+\eta_\nu\frac{\partial\phi''}{\partial\eta_\nu}+\ldots+\sigma_\nu\frac{\partial\phi''}{\partial\sigma_\nu}=i_\nu\phi'',\\ &\xi_\tau\frac{\partial\phi''}{\partial\xi_\nu}+\eta_\tau\frac{\partial\phi''}{\partial\eta_\nu}+\ldots+\sigma_\tau\frac{\partial\phi''}{\partial\sigma_\nu}=0,\ (\tau\neq\nu),\end{aligned}$$

solving the latter for the partial derivatives by Cramer's rule.

Thus,

$$(38)\qquad \frac{d\phi''}{\phi''}=i_\nu\frac{d\Delta}{\Delta},\ \big(\Delta=(\xi\eta\ldots\sigma)\big).$$

After integrating we determine the constant of integration by means of the identical transformation as a case of (25), and obtain, finally,

*Gordan, Mathematische Annalen, vol. 5 (1872), p. 114.

(39) $$\phi'' = \Delta^{i_\nu}\,\phi, \quad (i_1 = i_2 = \ldots = i_n).$$

Hence the conditions (29) are also sufficient and ϕ is a combinant.

V. *Generation of combinants.* Consider the set of linear forms,

(40) $$\xi a_{ix_1^{u-v}x_2^v} + \eta b_{ix_1^{u-v}x_2^v} + \zeta c_{ix_1^{u-v}x_2^v} + \ldots + \sigma l_{ix_1^{u-v}x_2^v},$$
$$(v = 0, \ldots, u;\quad u = 0, \ldots, r;\quad i = 0, \ldots, m).$$

If we transform this system by the substitutions,

(41) $$\begin{aligned} \xi &= \xi_1\xi'' + \xi_2\eta'' + \ldots + \xi_n\sigma'', \\ \eta &= \eta_1\xi'' + \eta_2\eta'' + \ldots + \eta_n\sigma'', \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \sigma &= \sigma_1\xi'' + \sigma_2\eta'' + \ldots + \sigma_n\sigma'', \end{aligned}$$

the transformed forms are those given by the relations (30), (31). It follows that combinants, in their relations with the scheme of transformations (25), are simultaneous invariants of the set (40). Hence,

Theorem. Combinants are, if rational and integral except for a power of c, homogeneous polynomials in determinants of the matrix,

(42) $$\| a_{ix_1^{u-v}x_2^v},\ b_{ix_1^{u-v}x_2^v},\ \ldots,\ l_{ix_1^{u-v}x_2^v} \|,$$
$$(v = 0, \ldots, u;\quad u = 0, \ldots, r;\quad i = 0, \ldots, m),$$

of order n.

The general formula for these determinants is,

(43) $$\begin{vmatrix} a_{i_1x_1^{r_1-s_1}x_2^{s_1}}, & b_{i_1x_1^{r_1-s_1}x_2^{s_1}}, & \ldots, & l_{i_1x_1^{r_1-s_1}x_2^{s_1}} \\ a_{i_2x_1^{r_2-s_2}x_2^{s_2}}, & b_{i_2x_1^{r_2-s_2}x_2^{s_2}}, & \ldots, & l_{i_2x_1^{r_2-s_2}x_2^{s_2}} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i_nx_1^{r_n-s_n}x_2^{s_n}}, & b_{i_nx_1^{r_n-s_n}x_2^{s_n}}, & \ldots, & l_{i_nx_1^{r_n-s_n}x_2^{s_n}} \end{vmatrix}.$$

It does not represent hessians or wronskians or any determinants which do not have the same derivative in each element of a given row.

Theorem. The four differential processes (13), (14), (14_1), (15), *produce combinants from combinants.*

Referring to any polynomial which is rational and integral in determinants (43) and in c as a combinant gradient it is clear that the x_1 or x_2 partial derivative of a gradient is a gradient by the rule for differentiating a determinant. In order, therefore, that a derivative operator Δ should generate a combinant from a combinant it is necessary and sufficient that the operator itself should satisfy an invariant relation,

$$\Delta' = D^k\Delta.$$

The operators (13), (14), (14_1), (15) satisfy this condition. These processes can all be iterated but prior to each operation of finding the jacobian (13) of two absolute combinants we must complement by c.

VI. *The case of two quantics.* Any algebraic odd transvectant of two differential forms f, g of order m is a differential combinant of zero order. After complementing these by proper powers of c we can apply the four processes (13), ..., (15) and generate any number of combinant parameters of any order. We can also start a series from

$$\phi = c(f, g) = c(\alpha_{dx}, \beta_{dx})\alpha_{dx}^{m-1}\beta_{dx}^{m-1}, \text{ (cf. (26))}.$$

An arbitrary function $(u' = u)$, also, may be used as the second form ψ in (13). Thus we obtain

$$c(c(f, g), u) = (c, u)\phi + c^2((\alpha_{dx}, \beta_{dx}), u)\alpha_{dx}^{m-1}\beta_{dx}^{m-1} + (m-1)c^2\alpha_{dx}^{m-2}\beta_{dx}^{m-2}(\alpha_{dx}, \beta_{dx})[(\alpha_{dx}, u)\beta_{dx} + (\beta_{dx}, u)\alpha_{dx}].$$

Most of the forms in the above sequences are covariantive, involving dx_1, dx_2. The algebraic $2m$th transvectant of (f, g) with itself is a combinant invariant of order one and partial degrees $i_1 = i_2 = 2$. If $m = 2$ this combinant is, (cf. (27)),

$$\Lambda = (a_0 : b_0)(a_2 : b_2) - (a_0 : b_1)(a_1 : b_2) - (a_0 : b_1)(a_2 : b_1)$$
$$- (a_1 : b_0)(a_1 : b_2) - (a_1 : b_0)(a_2 : b_1) + \frac{1}{12}(a_0 : b_2)^2$$
$$+ \frac{4}{3}(a_1 : b_1)^2 + \frac{1}{12}(a_2 : b_0)^2 + \frac{2}{3}(a_0 : b_2)(a_1 : b_1)$$
$$+ \frac{1}{6}(a_0 : b_2)(a_2 : b_0) + \frac{2}{3}(a_1 : b_1)(a_2 : b_0).$$

Part 2. Forms in N Variables

VII. *Resultants.* Assume a set of p differential quantics of order m in n variables with coefficients which are functions of $x_1, \ldots, x_n$, each affected by a numerical multiplier which is a multinomial coefficient,

$$(44) \qquad f^{(r)} = \sum \frac{m!}{m_1! \ldots m_n!} a^{(r)}_{m_1 \ldots m_n} dx_1^{m_1} \ldots dx_n^{m_n} = a_{dx}^{(r)m} = b_{dx}^{(r)m},$$
$$(r = 1, \ldots, p;\ m_1 + \ldots + m_n = m),$$

where $p \geqq n$ and

$$(45) \qquad a_{dx}^{(r)} = a_1^{(r)}dx_1 + a_2^{(r)}dx_2 + \ldots + a_n^{(r)}dx_n = \frac{\partial a^{(r)}}{\partial x_1}dx_1 + \ldots + \frac{\partial a^{(r)}}{\partial x_n}dx_n.$$

A proof of the existence of complementing factors for differential combinants of the set may be based upon a formula for n-ary resultants, due to Poisson.

The forms $f^{(r)}$ can be written in the notation $f^{(r)}(z_1, \ldots, z_{n-1})$, $(r = 1, \ldots, p)$, where $z_i = dx_i/dx_n$. The system of algebraic equations,

$$(46) \qquad f^{(1)}(z_1, \ldots, z_{n-1}) = 0, \ldots, f^{(n-1)}(z_1, \ldots, z_{n-1}) = 0,$$

possesses $k = m^{n-1}$ roots, $(z_1^{(t)}, \ldots, z_{n-1}^{(t)})$, $(t = 1, \ldots, k)$, and a function whose vanishing is a necessary and sufficient condition for a common root of

$$f^{(1)}=0, \ldots, f^{(n)}=0,$$

is, therefore*,

$$R = \rho_0^{\mu} \prod_{t=1}^{k} f^{(n)} (z_1^{(t)}, \ldots, z_{n-1}^{(t)}).$$

The form R is a polynomial in the coefficients of the n forms $f^{(1)}, \ldots, f^{(n)}$. If we operate by any operator $\left(a^{(r)} \frac{\partial}{\partial a^{(n)}}\right)$, $(r=1, \ldots, n-1)$, (cf. (29)), there results,

$$\left(a^{(r)} \frac{\partial}{\partial a^{(n)}}\right) R = \rho_0^{\mu} \Sigma f^{(r)}(z_1^{(1)}, \ldots, z_{n-1}^{(1)}) \prod_{t=2}^{k} f^{(n)}(z_1^{(t)}, \ldots, z_{n-1}^{(t)}). \tag{47}$$

This vanishes since any $f^{(r)}$, $(r=1, \ldots, n-1)$, is satisfied by the roots $(z_1^{(t)}, \ldots, z_{n-1}^{(t)})$, $(t=1, \ldots, k)$. Similarly, for any r, s, $r \neq s$,

$$\left(a^{(r)} \frac{\partial}{\partial a^{(s)}}\right) R=0,$$

hence R is an algebraic combinant of the set (44), $(p=n)$. A power of R, C, may be employed as a complementing factor of combinants of this set.

VIII. *Combinants of forms in n variables.* A combinant ϕ is any simultaneous parameter of the set $f^{(r)}$ which remains invariant, save for a multiplier which is a power of $D'=(\lambda\mu \ldots \sigma)$, when the forms are transformed by the substitutions,

$$f^{(r)\prime\prime}=\lambda_r f^{(1)}+\mu_r f^{(2)}+\ldots+\sigma_r f^{(p)}, \; (r=1, \ldots, p). \tag{48}$$

The transformations induced upon the coefficients and their derivatives by (48) are,

$$\begin{aligned}
a_{i..l}^{(r)\prime\prime} &= \lambda_r a_{i..l}^{(1)} + \mu_r a_{i..l}^{(2)} + \ldots + \sigma_r a_{i..l}^{(p)}, \\
a_{i..lx_1}^{(r)\prime\prime} &= \lambda_r a_{i..lx_1}^{(1)} + \mu_r a_{i..lx_1}^{(2)} + \ldots + \sigma_r a_{i..lx_1}^{(p)}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{i..lx_1^{s_1}..x_n^{s_n}}^{(r)\prime\prime} &= \lambda_r a_{i..lx_1^{s_1}..x_n^{s_n}}^{(1)} + \mu_r a_{i..lx_1^{s_1}..x_n^{s_n}}^{(2)} + \ldots + \sigma_r a_{i..lx_1^{s_1}..x_n^{s_n}}^{(p)},
\end{aligned} \tag{49}$$

$(s_1 + \ldots + s_n = t$, a constant; $r = 1, \ldots, p$; $t=0, \ldots, \tau$, the order of ϕ; $i+\ldots+l=m)$. An example of ϕ is the jacobian of $f^{(1)}, \ldots, f^{(n)}$, viz.,

$$\phi^{(1)}=(a_{dx}^{(1)}, a_{dx}^{(2)}, \ldots, a_{dx}^{(n)}) a_{dx}^{(1)m-1} a_{dx}^{(2)m-1} \cdots a_{dx}^{(n)m-1} = \Xi_{dx}^{mn}. \tag{50}$$

The coefficients of $\phi^{(1)}$ are linear polynomials in the

$$N=\binom{m+n-1}{m}^n$$

determinants,

*Netto, *Algebra*, vol. 2, p. 81.

$$(51)\qquad \begin{vmatrix} a^{(1)}_{i_1\cdot\cdot l_1x_1} & a^{(2)}_{i_1\cdot\cdot l_1x_1} & \cdots\cdots & a^{(n)}_{i_1\cdot\cdot l_1x_1} \\ a^{(1)}_{i_2\cdot\cdot l_2x_2} & a^{(2)}_{i_2\cdot\cdot l_2x_2} & \cdots & a^{(n)}_{i_2\cdot\cdot l_2x_2} \\ \cdots & \cdots & \cdots & \cdots \\ a^{(1)}_{i_n\cdot\cdot l_nx_n} & a^{(2)}_{i_n\cdot\cdot l_nx_n} & \cdots & a^{(n)}_{i_n\cdot\cdot l_nx_n} \end{vmatrix}, \begin{pmatrix} i_a+\ldots+l_a=m, \\ i_a,\ldots,l_a=0,\ldots,m. \end{pmatrix},$$

in particular the leading coefficient Ξ_1^{mn} is the determinant given by (51) when, for all values of α, $i_\alpha=m$, $j_\alpha=\ldots=l_\alpha=0$.

Now if the reasoning of paragraph V be applied to the present more general case we readily deduce the following:

Theorem. Combinants are homogeneous functions of determinants of order p of the matrix, (cf. (57)),

$$(52)\qquad \left\| a^{(1)}_{i\ldots lx_1^{s_1}\ldots x_n^{s_n}},\ldots\ldots, a^{(p)}_{i\ldots lx_1^{s_1}\ldots x_n^{s_n}} \right\|, \begin{pmatrix} s_1+\ldots+s_n=t; \\ t=0,\ \ldots\ \tau, \\ i+j+\ldots+l=m. \end{pmatrix}.$$

A necessary and sufficient condition for combinant concomitance is the satisfaction of a complete independent set of the linear operators,

$$(53)\qquad \sum\left(a^{(r)}_{i\ldots lx_1^{s_1}\ldots x_n^{s_n}} \frac{\partial}{\partial a^{(s)}_{i\ldots lx_1^{s_1}\ldots x_n^{s_n}}} \right), \ (r\neq s),$$

where the summation extends over all subscript derivatives represented in the matrix (52).

Among the operators, in addition to jacobians, which produce combinants from combinants should be mentioned the total differential,

$$dy_1\partial/\partial y_1+\ldots+dy_n\partial/\partial y_n=dx_1\partial/\partial x_1+\ldots+dx_n\partial/\partial x_n.$$

Conversely some combinants are integral invariants,

$$\int_{(y)}\phi'=\int_{(x)}\phi,$$

in which the integrand ϕ is also a combinant. Take

$$\delta=\zeta^{(1)}\frac{\partial}{\partial x_1}+\ldots+\zeta^{(n)}\frac{\partial}{\partial x_n},$$

where $\zeta^{(1)},\ldots,\zeta^{(n)}$ are any elements cogredient to $dx_1,\ldots,dx_n$, then $\delta\phi$ is a combinant. The iterations of δ, the first being

$$\delta^2=(\delta)^2+\sum_{i=1}^{n}\zeta^{(i)}\theta_i,$$

in which $(\delta)^2$ is the formal square of δ and $\theta_i=\zeta_i^{(1)}\partial/\partial x_1+\ldots+\zeta_i^{(n)}\partial/\partial x_n$, are abstractly analogous to Christoffel's covariant derivatives.

IX. *Translation principle**. One interpretation of a set of $n+1$ differential forms in $dx_1, \ldots, dx_n$,

$$(54) \qquad d\xi_i = d\eta_i^m = g^{(i)}_{m0\ldots0}\, dx_1{}^m + m g^{(i)}_{m-110\ldots0} dx_1{}^{m-1} dx_2 + \ldots + m(m-1) g^{(i)}_{m-2110\ldots0} \times dx_1{}^{m-2} dx_2 dx_3 + \ldots, \quad (i=1, \ldots, n+1),$$

when $m=2$, is that they represent the $n+1$ expressions for the squared n dimensional interval ds^2 corresponding to $n+1$ choices of a coordinate system. I shall retain the word "interval" as a convenient term for $d\xi_i = d\eta_i^m$ for any value of m.

Correspondence between any combinant parameter in n variables representing an invariant property of n dimensional space and a certain parameter in $n+1$ variables, expressive of an invariantive principle in space of $n+1$ dimensions, may be established by combining with the set of intervals $d\xi_i$ of (54) a system of n intervals in the form of n of the $n+1$ linear forms of a contravariant vector,

$$(55) \qquad ds_e^m = \frac{\partial v^{(e)}}{\partial \xi_1} d\xi_1 + \frac{\partial v^{(e)}}{\partial \xi_2} d\xi_2 + \ldots + \frac{\partial v^{(e)}}{\partial \xi_{n+1}} d\xi_{n+1}, \; (e=1, \ldots, n).$$

We obtain, thus, n quantics in the n differentials $dx_1, \ldots, dx_n$, whose coefficients are bilinear forms in the functions $g^{(i)}_{rs}\ldots$ on the one hand and in the coefficients $\partial v^{(e)}/\partial \xi_i = v_i^{(e)}$ on the other, viz.,

$$(56) \qquad \begin{aligned} ds_e^m = {} & (v_1{}^{(e)} g^{(1)}_{m0\ldots0} + v_2{}^{(e)} g^{(2)}_{m0\ldots0} + \ldots + v^{(e)}_{n+1} g^{(n+1)}_{m0\ldots0}) dx_1{}^m \\ & + m(v_1{}^{(e)} g^{(1)}_{m-110\ldots0} + v_2{}^{(e)} g^{(2)}_{m-110\ldots0} + \ldots + v^{(e)}_{n+1} g^{(n+1)}_{m-110\ldots0}) dx_1{}^{m-1} dx_2 \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + m(m-1)\, (v_1{}^{(e)} g^{(1)}_{m-2110\ldots0} + v_2{}^{(e)} g^{(2)}_{m-2110\ldots0} + \ldots + v^{(e)}_{n+1} g^{(n+1)}_{m-2110\ldots0}) \\ & \times dx_1{}^{m-2} dx_2 dx_3 + \ldots \end{aligned}$$

Let

$$ds_e^m = G^{(e)}_{m0\ldots0} dx_1{}^m + m G^{(e)}_{m-110\ldots0} dx_1{}^{m-1} dx_2 + \ldots + m(m-1) G^{(e)}_{m-2110\ldots0} \times dx_1{}^{m-2} dx_2 dx_3 + \ldots,$$

then the typical determinant of which a combinant parameter I of the set $ds_1{}^m, \ldots, ds_n^m$, (generally of any set $f^{(r)}$, $(r=1, \ldots, n)$), is a function, is,

$$(57) \qquad G = \begin{vmatrix} G^{(1)}_{i_1\ldots l_1 x_1}{}^{s_{11}}\ldots{}_{x_n}{}^{s_{n1}}, & \ldots\ldots, & G^{(n)}_{i_1\ldots l_1 x_1}{}^{s_{11}}\ldots{}_{x_n}{}^{s_{n1}} \\ G^{(1)}_{i_2\ldots l_2 x_1}{}^{s_{12}}\ldots{}_{x_n}{}^{s_{n2}}, & \ldots\ldots, & G^{(n)}_{i_2\ldots l_2 x_1}{}^{s_{12}}\ldots{}_{x_n}{}^{s_{n2}} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ G^{(1)}_{i_n\ldots l_n x_1}{}^{s_{1n}}\ldots{}_{x_n}{}^{s_{nn}}, & \ldots, & G^{(n)}_{i_n\ldots l_n x_1}{}^{s_{1n}}\ldots{}_{x_n}{}^{s_{nn}} \end{vmatrix},$$

$$(s_{1k} + \ldots + s_{nk} = t_k; \; t_k = 0, \ldots, \tau; \; i_k + j_k + \ldots + l_k = m).$$

We assume that $d/dx_1, \ldots, d/dx_n$ are permutable with $v_i^{(e)}$. If the combinant I is of zero order, *i.e.*, algebraic, (permitted to contain, however, derivatives of

*W. F. Meyer, *Apolarität und Rationale Curven* (1883), p. 18.

any order of arbitrary functions $(u'=u)$, no derivatives of functions $g^{(e)}_{ik..lk}$ occur in G and the vector ds^m_e remains arbitrary. If the order of I is >0 in derivatives of coefficients of the set $f^{(r)}$, $(r=1, \ldots, n)$, the coefficients of the $d\xi_i$ in ds^m_e must be arbitrary constants. Then $v^{(e)}$ is a linear form in $\xi_1, \ldots, \xi_{n+1}$.

By a known theorem on determinants of matrices G can be expanded into the formula,

$$G=\sum_{a=1}^{n+1}(v^{(1)}, v^{(2)}, \ldots, v^{(n)})_a\,(g_{(1)}, g_{(2)}, \ldots, g_{(n)})_a, \tag{58}$$

in which a subscript (k) of g represents the actual subscript,

$$i_k j_k \ldots l_k x_1^{s_{1k}} \ldots x_n^{s_{nk}},$$

$(v^{(1)}, \ldots, v^{(n)})_a$ is the determinant of order n formed by deleting the a-th column of the matrix,

$$\left\|\begin{matrix} v_1^{(1)} & v_2^{(1)} & \ldots & v_{n+1}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ v_1^{(n)} & v_2^{(n)} & \ldots & v_{n+1}^{(n)} \end{matrix}\right\|, \tag{59}$$

and $(g_{(1)}, \ldots, g_{(n)})_a$ is the determinant obtained by deleting the a-th column of the matrix,

$$\left\|\begin{matrix} g^{(1)}_{(1)} & g^{(2)}_{(1)} & \ldots & g^{(n+1)}_{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ g^{(1)}_{(n)} & g^{(2)}_{(n)} & \ldots & g^{(n+1)}_{(n)} \end{matrix}\right\|. \tag{60}$$

If we wished to preserve the numerical equality of the expressions to be derived next, with those from which they are derived, the next step in this development would be to solve $ds^m_e=0$, $(e=1, \ldots, n)$, and we should find,

$$d\xi_1 : d\xi_2 : \ldots : d\xi_{n+1} = (v^{(1)}, \ldots, v^{(n)})_1 : (v^{(1)}, \ldots, v^{(n)})_2 : \ldots : (v^{(1)}, \ldots, v^{(n)})_{n+1}. \tag{61}$$

To preserve the quality of invariance it is only necessary to observe that the sets,

$$\begin{gathered} d\xi_1,\ d\xi_2,\ \ldots,\ d\xi_{n+1}\ , \\ (v^{(1)}, \ldots, v^{(n)})_1,\ (v^{(1)}, \ldots, v^{(n)})_2,\ \ldots,\ (v^{(1)}, \ldots, v^{(n)})_{n+1}\ , \end{gathered} \tag{62}$$

are cogredient to each other. We thus preserve invariancy and, by (61), numerical equivalence also, except for a factor of proportionality, when we write,

$$G=\sum_{a=1}^{n+1}(g_{(1)}, g_{(2)}, \ldots, g_{(n)})_a\, d\xi_a = \left|\begin{matrix} d\xi_1 & d\xi_2 & \ldots & d\xi_{n+1} \\ g^{(1)}_{(1)} & g^{(2)}_{(1)} & \ldots & g^{(n+1)}_{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ g^{(1)}_{(n)} & g^{(2)}_{(n)} & \ldots & g^{(n+1)}_{(n)} \end{matrix}\right|. \tag{63}$$

Theorem. If a combinant I of the set $f^{(r)}$, $(r=1, \ldots, n)$, remains equal to zero, then the set of intervals $(d\xi_1, \ldots, d\xi_{n+1})$, is constrained to satisfy an equation $J=0$, *where J is the covariantive parameter obtained by substituting $(n+1)$th order determinants G (cf. (63)) for the corresponding nth order determinants of the matrix* (52), $n=p$, *of which I is a function. The correspondence of I with J is independent of the measure-code* of the vector ds_e^m.*

For example, if, to obtain a simpler notation in the binary case, we write $g_{jk}^{(i)}=a_i^k$, the combinant Λ of paragraph VI gives, by the translation principle, the quadratic parameter,

$$\begin{aligned}\Lambda_{d\xi}^2 = &|d\xi_1 a_{2x_1}^0 a_{3x_2}^0||d\xi_1 a_{2x_1}^2 a_{3x_2}^2| - |d\xi_1 a_{2x_1}^0 a_{3x_2}^1||d\xi_1 a_{2x_1}^1 a_{3x_2}^2| - |d\xi_1 a_{2x_1}^0 a_{3x_2}^1||d\xi_1 a_{2x_1}^2 a_{3x_2}^1| \\ &- |d\xi_1 a_{2x_1}^1 a_{3x_2}^0||d\xi_1 a_{2x_1}^1 a_{3x_2}^2| - |d\xi_1 a_{2x_1}^1 a_{3x_2}^0||d\xi_1 a_{2x_1}^2 a_{3x_2}^1| + \tfrac{1}{12}|d\xi_1 a_{2x_1}^0 a_{3x_2}^2|^2 \\ &+ \tfrac{4}{3}|d\xi_1 a_{2x_1}^1 a_{3x_2}^1|^2 + \tfrac{1}{12}|d\xi_1 a_{2x_1}^2 a_{3x_2}^0|^2 + \tfrac{2}{3}|d\xi_1 a_{2x_1}^0 a_{3x_2}^2||d\xi_1 a_{2x_1}^1 a_{3x_2}^1| \\ &+ \tfrac{1}{6}|d\xi_1 a_{2x_1}^0 a_{3x_2}^2||d\xi_1 a_{2x_1}^2 a_{3x_2}^0| + \tfrac{2}{3}|d\xi_1 a_{2x_1}^1 a_{3x_2}^1||d\xi_1 a_{2x_1}^2 a_{3x_2}^0|.\end{aligned}$$

The parameters obtained by this principle involve two sets of variables, $x_1,\ldots,x_n$ and $\eta_1,\ldots,\eta_{n+1}$. Also we note that a solution of the set of equations (54) contains at least n arbitrary functions,

$$u_j(x_1,\ldots,x_n,\eta_j)=0,\ (j=1,\ldots,n): \tag{64}$$

hence, for the variables $x_1,\ldots,x_n$ in these parameters, may be substituted as many arbitrary functions of $\eta_1,\ldots,\eta_n$.

The question of complete systems of combinant parameters of a given order, which is not considered in this article, is a problem of interest. Complete systems of differential parameters of quadratic forms have been investigated to some extent†.

*Eddington, *Theory of Relativity* (1923), p. 48.

†Ricci and Levi-Civita, *loc. cit.*; Wright, Amer. Jour. Math., vol. 27 (1905), p. 323.

ON THE SOLUTION OF QUINTIC EQUATIONS

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1. Introduction. Equations of the second, third, and fourth degrees, as is well known, may be solved with a finite number of radicals. If, however, the equation is of higher degree than the fourth, this is, as a rule, no longer possible. Equations of the fifth or higher degrees must consequently be solved by other irrationalities than radicals, and it is solutions of this kind that will be the subject of this paper. We will consequently not occupy ourselves with the many methods, graphic as well as analytic, (for instance developments in series according to powers of the coefficients) for numerical calculation of the roots.

When seeking for other irrationalities than radicals for the solution of algebraic equations, it is natural to try if it is not possible, by means of a suitable specialization of certain transcendental functions, previously known and investigated, to obtain the desired algebraic irrationalities.

The first step in this direction was taken by Hermite* in 1858. We know that when in the cubic equation

$$x^3-3x+2a=0$$

the parameter a is expressed in terms of an auxiliary variable α on putting

$$a=\sin\alpha,$$

the three roots are given as functions of α by the expressions

$$2\sin\frac{\alpha}{3},\quad 2\sin\frac{\alpha+2\pi}{3},\quad 2\sin\frac{\alpha+4\pi}{3}.$$

Hermite considered the quintic equation of the form

(1) $$x^5-x-a=0$$

and showed that the parameter a as well as the five roots x might be expressed in terms of an auxiliary variable by certain series well known in the theory of elliptic functions. The method is thus analogous to that indicated for the solution of the foregoing cubic equation.

The result of Hermite's elegant analysis is as follows:
Let K and K' be the periods of an elliptic integral corresponding to a modulus $\boldsymbol{k}$, which is a solution of the biquadratic equation

$$k^4+A^2k^3+2k^2-A^2k+1=0,\qquad a=\frac{2}{\sqrt[4]{5^5}}A.$$

*Comptes Rendus Acad. Sciences, Paris, t. 46 (1858).

Then the new auxiliary variable used by Hermite is

$$\omega = i\frac{K'}{K}$$

If, as is customary, we put

$$q = e^{-\pi\frac{K'}{K}} = e^{i\pi\omega},$$

the fourth root of the modulus may be expressed in terms of ω as follows:

$$\sqrt[4]{k} = \phi(\omega) = \sqrt{2}\,\sqrt[8]{q}\,\frac{\Sigma q^{2m^2+m}}{\Sigma q^{m^2}}, \quad \sqrt[4]{k'} = \psi(\omega); \quad k^2 + k'^2 = 1.$$

If now we put

$$u = \phi(\omega), \qquad v = \phi(5\omega)$$

there will exist the following algebraic relation between v and u, the so-called *modular equation*,

$$u^6 + v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0.$$

Galois had already observed that the modular equation might be reduced to an equation of the fifth degree. Hermite succeeded in working out this reduction, by which the modular equation was transformed into a quintic equation of the form (1) where the parameter a, expressed in terms of ω, is given by

$$A = \frac{1 + \phi^8(\omega)}{\phi^2(\omega)\psi^4(\omega)} = \frac{1 + k^2}{\sqrt{k}\,k'}.$$

If we write

$$\Phi(\omega) = \left[\phi(5\omega) - \phi\left(\frac{\omega}{5}\right)\right]\left[\phi\left(\frac{\omega+16}{5}\right) - \phi\left(\frac{\omega+2.16}{5}\right)\right]\left[\phi\left(\frac{\omega+2.16}{5}\right) - \phi\left(\frac{\omega+3.16}{5}\right)\right]$$

then the five roots of the quintic are given by

$$x_i = \frac{1}{\sqrt[4]{2^4 5^3}}\,\frac{\Phi(\omega + (i-1).16)}{\phi(\omega)\psi^4(\omega)}, \quad (i = 1, 2, 3, 4, 5).$$

For the cubic equation the auxiliary quantity α may be easily eliminated, the roots x then being directly expressed in terms of the parameter a; *the analogous transformation is not possible in the above solution of the quintic.* Thus x is not found directly by means of a, but it is possible to calculate for any value of ω the corresponding value of a and the roots x. If, however, a be given and we wish to find the corresponding value of ω, we have to do with a transcendental problem.

The methods worked out by Kronecker, Brioschi and Klein* employ the theory of elliptic functions and have in common with Hermite's method the two essential features that they express a certain parameter as well as the roots by an auxiliary variable and that they make use of the modular equation or of the closely allied Jacobian equation. They undertake either to reduce the modular equation of the sixth degree to a quintic of the form in question or to form a

*See for instance: F. Klein, *Vorlesungen über das Ikosaeder*, Leipzig, 1884.

resolvent equation of the sixth degree which coincides with the modular equation or with a Jacobian equation. It is the merit of Gordan and Klein to have connected the solution of the quintic equation with the theory of the icosahedron. The equation of the icosahedron [of the sixtieth degree] is intimately connected with the solution of the modular equation, since, apart from certain numerical irrationalities, it has the same group as the modular equation. In all these solutions the roots are not *directly* expressed in terms of the parameter since we cannot eliminate the auxiliary variable. Schwarz* has given a solution of the icosahedron equation where the roots are given directly in terms of the parameter by means of a relation between two Riemannian P functions, which is equivalent to the use of Gauss' hypergeometric series.

There exist also solutions of the quintic equation by means of certain integrals by Heymann† and Mellin.‡

2. THE TRINOMIAL EQUATION. The general quintic equation may, by elementary operations [solution of equations of a lower degree than the fifth] be transformed into

$$x^5 = gx^m + \beta$$

where m is one of the numbers 1, 2, 3, 4. All these equations are included under the form

$$x^n = gx^m + \beta \tag{2}$$

where n and m are positive whole numbers, prime to each other, and where $n > m$. We can show§ that if the roots of equation (2) are considered as functions of the variable

$$\zeta = (-1)^{n-m} \frac{n^n}{m^m (n-m)^{n-m}} \frac{\beta^{n-m}}{g^n},$$

they are particular integrals of a higher hypergeometric differential equation of order $n-1$,

$$\zeta^{n-2}(\zeta-1)\frac{d^{n-1}\Lambda}{d\zeta^{n-1}} + \zeta^{n-3}(A_1\zeta - B_1)\frac{d^{n-2}\Lambda}{d\zeta^{n-2}} + \ldots + (A_{n-2}\zeta - B_{n-2})\frac{d\Lambda}{d\zeta} + C\Lambda = 0,$$

where the quantities A,B,C are constants. We can obtain the group of this differential equation and consequently find how the roots vary and interchange around the critical points just as completely as in the case of the binomial equation. So far as I know they are the first groups found where $n-1>3$. These differential equations were introduced into analysis by Thomae|| and have been the subject of important investigations by Goursat¶. The type of functions

*H. Weber: *Lehrbuch der Algebra*. Bd. 2 (1896), p. 432.

†W. Heymann: *Studien über die Transformation und Integration der Differential-und Differenzengleichungen* (Leipzig, 1891).

‡Hj. Mellin: Annales Academiæ Scientiarum Fennicae, Ser. A, t. VII, No. 7 and No. 8.

§Comptes Rendus Acad. Sciences, Paris, t. 171, p. 778, p. 1047, t. 177, p. 23; see also t. 171, p. 1370, t. 172, p. 1155.

||Mathematische Annalen, Bd. 2, 1870, p. 427.

¶Annales de l'École Normale Supérieure, t. 12, 1883, 2e série, p. 272.

(higher hypergeometric functions) which solve these differential equations is in Goursat's notation,

$$F(\zeta)=F\begin{pmatrix} a_1, a_2, \dots, a_{n-2}, a_{n-1} \\ b_1, b_2, \dots, b_{n-2}, \zeta \end{pmatrix} = \sum_0^\infty c_s \zeta^s,$$

where

$$c_0=1,\ c_s=\frac{(a_1, s).(a_2, s)\ \dots\ (a_{n-2}, s)\ (a_{n-1}, s)}{(1, s).(b_1, s)\ \dots\ (b_{n-3}, s)\ (b_{n-2}, s)},$$

and where the symbol (λ, μ) means

$$(\lambda, \mu)=\lambda(\lambda+1)(\lambda+2)\ \dots\ (\lambda+\mu-1).$$

The constants A_i, B_i, C in the differential equation are determined by the quantities a_i and b_i.

The ratio

$$\frac{c_{s+1}}{c_s}=\frac{(s+a_1)\ (s+a_2)\dots(s+a_{n-1})}{(s+1)\ (s+b_1)\dots(s+b_{n-2})}$$

is always a definite rational function of s with numerator and denominator of degree $n-1$. Goursat generalized these functions and called all functions which can be developed in series possessing the property that the ratio of the two coefficients c_{s+1} and c_s is a rational function of s with numerator and denominator of a *fixed degree independent of s*, hypergeometric functions. These functions form a clearly defined class which has been investigated.

Let i_0 and i_1 be the two whole numbers ($i_0 \gtreqless n$, $i_1<m$) given by

$$(i_0-1)m+1=i_1 n,$$

and let

$$a_i=\frac{i-1}{n}-\frac{1}{n(n-m)},\quad (i=1, 2, \dots, i_0-1),$$

$$a_i=\frac{i}{n}-\frac{1}{n(n-m)},\quad (i=i_0, i_0+1, \dots, n-1),$$

$$b_i=\frac{i}{m}-\frac{1}{m(n-m)},\quad (i=1, 2, \dots, i_1-1),$$

$$b_i=\frac{i+1}{m}-\frac{1}{m(n-m)},\quad (i=i_1, i_1+1, \dots, m-1),$$

$$b_i=\frac{i-m+1}{n-m},\quad (i=m, m+1, \dots, n-2),$$

$$F_0(\zeta)=F\begin{pmatrix} a_1, a_2, \dots, a_{n-2}, a_{n-1} \\ b_1, b_2, \dots, b_{n-2}, \zeta \end{pmatrix},$$

$$F_{n-i-1}(\zeta)=F\begin{pmatrix} a_1+1-b_i, a_2+1-b_i, \dots, a_{n-2}+1-b_i, a_{n-1}+1-b_i \\ 2-b_i, b_1+1-b_i, \dots, b_{n-2}+1-b_i, \zeta \end{pmatrix},$$

$$\nu=e^{\frac{2\pi i}{n-m}},\qquad \delta=e^{\frac{2\pi i}{m}}.$$

Let us suppose $g=1$.

Then we have, if $|\zeta|<1$ or $\zeta=1$, the $n-m$ roots given by

$$(3)\qquad x_i=F_0(\zeta)+\sum_{\kappa=1}^{n-m-1}\theta_\kappa \nu^{i(1-\kappa m)}\,\zeta^{1-b_\kappa+m-1}\,F_\kappa(\zeta),\qquad (i=1,2,\ldots,n-m),$$

where the coefficients θ_κ are numerical quantities. The m roots $x_{n-m+1},\ldots,x_n$, are, (when $m<n-1$), given by

$$(4)\qquad x_{n-m+i}=-\frac{n-m}{m}\theta_{\kappa_0}\zeta^{1-b_{\kappa_0}+m-1}F_{\kappa_0}(\zeta)+\sum_{s=1}^{m-1}\Delta_{m-s}\,\delta^{i(1-sn)}\,\zeta^{1-b_s}F_{n-s-1}(\zeta),$$

$$(i=1,2,\ldots,m),$$

where the symbols Δ_{m-s} are numerical coefficients and where

$$m\kappa_0-1\equiv 0 \bmod (n-m).$$

When $|\zeta|>1$ we have the n roots

$$(5)\qquad y_i=\beta^{\frac{1}{n}}\sum_{s=1}^{n-m}c_s\,\epsilon^{i(1+(s-1)m)}\,\zeta^{-a_s}\,\psi_{s-1}\left(\frac{1}{\zeta}\right),\qquad (i=1,2,\ldots,n),$$

where $\epsilon=e^{\frac{2\pi i}{n}}$, where the coefficients c_s are numerical constants, and where

$$\psi_{s-1}\left(\frac{1}{\zeta}\right)=F\left(\begin{matrix}a_s,\ a_s+1-b_1,\ \ldots,\ a_s+1-b_{n-3},\ a_s+1-b_{n-2}\\ a_s+1-a_1,\ a_s+1-a_2,\ \ldots,\ a_s+1-a_{n-1},\ \dfrac{1}{\zeta}\end{matrix}\right).$$

When the n roots of the trinomial equation are considered as functions of β, *the critical points*, besides $\beta=\beta_0=0$, (when $m>1$), are given by those values of β which make $\zeta=1$. We have then the following $n-m+1$ critical points,

$$\beta_0=0,\ \ \beta_1=-c\nu^m,\ \ \beta_2=-c\nu^{2m},\ \ldots\ldots\ldots,\beta_{n-m}=-c$$

where

$$c=\frac{n-m}{m}\left(\frac{m}{n}\right)^{\frac{n}{n-m}}>0.$$

Let us consider for the moment such values of β as make $|\zeta|<1$. In order to fix the notation of the roots (3) and (4) we choose for β a definite value, *for example, a negative* value so that $(-\beta)^{\frac{1}{m}}$ is positive. If from each of the critical points we draw cuts which do not intersect each other, and which are not crossed when β varies in a continuous manner, then the notation is fixed for any value of β. We can then prove *the following law for the permutation of the roots:*

$$x_1\xrightarrow{\beta_1}x_n,\ \ x_2\xrightarrow{\beta_2}x_n,\ \ldots,\ \ x_{n-m}\xrightarrow{\beta_{n-m}}x_n,$$

$$x_{n-m+i}\underset{+}{\xrightarrow{\beta_0}}x_{n-m+i+1},\qquad (i=1,2,\ldots,m);$$

whereas

$$x_i \xrightarrow{\beta_h} x_i, \qquad (i \lessgtr h,\ i \text{ and } h \gtreqless n-m),$$

$$x_i \xrightarrow{\beta_0} x_i, \qquad (i=1,\ 2,\ \ldots,\ n-m).$$

By the symbol $x_1 \xrightarrow{\beta_1} x_n$, for instance, we denote that x_1 is changed into x_n when β describes a small closed contour about β_1. By the symbol $\xrightarrow[+]{\beta_0}$ we denote that β describes a small closed contour in a positive direction around $\beta=\beta_0=0$.

There is thus found the *group of the higher hypergeometric differential equation of order $n-1$ having $x_1,\ x_2,\ \ldots,\ x_n$ as particular integrals, when these roots are regarded as functions of ζ; for when ζ describes a closed contour around $\zeta=1$ we can arrange it so that β will describe a closed contour around $\beta=-c$.*

3. We shall now apply this theory to the quintic equation where we consider the cases $m=1,\ 2,\ 3$; to each value of m we have a corresponding differential equation of order 4 and a particular group. We will first examine the case $m=1$, for which the equation is

$$x^5=gx+\beta.$$

Here we have

$$\zeta=\frac{5^5}{4^4}\,\frac{\beta^4}{g^5}.$$

If we have $|\zeta|<1$ and if we simplify matters by taking $g=1$, we have the five roots given by

$$x_1= \quad iF_0(\zeta)+\frac{1}{4}\beta F_1(\zeta)+\frac{5}{2^5}i\beta^2F_2(\zeta)-\frac{5}{2^5}\beta^3F_3(\zeta),$$

$$x_2=-\ F_0(\zeta)+\frac{1}{4}\beta F_1(\zeta)+\frac{5}{2^5}\ \beta^2F_2(\zeta)+\frac{5}{2^5}\beta^3F_3(\zeta),$$

$$x_3=-iF_0(\zeta)+\frac{1}{4}\beta F_1(\zeta)-\frac{5}{2^5}i\beta^2F_2(\zeta)-\frac{5}{2^5}\beta^3F_3(\zeta),$$

$$x_4= \quad F_0(\zeta)+\frac{1}{4}\beta F_1(\zeta)-\frac{5}{2^5}\beta^2F_2(\zeta)+\frac{5}{2^5}\beta^3F_3(\zeta),$$

$$x_5=-\beta F_1(\zeta),$$

where $i=\sqrt{-1}$, and $F_0,\ F_1,\ F_2,\ F_3$ are the following hypergeometric functions of order 4,

$$F_0(\zeta)=F\begin{pmatrix}-\dfrac{1}{20}, & \dfrac{3}{20}, & \dfrac{7}{20}, & \dfrac{11}{20} \\ \dfrac{1}{4}, & \dfrac{2}{4}, & \dfrac{3}{4}, & \zeta\end{pmatrix},\quad F_1(\zeta)=F\begin{pmatrix}\dfrac{1}{5}, & \dfrac{2}{5}, & \dfrac{3}{5}, & \dfrac{4}{5} \\ \dfrac{2}{4}, & \dfrac{3}{4}, & \dfrac{5}{4}, & \zeta\end{pmatrix},$$

$$F_2(\zeta)=F\begin{pmatrix} \frac{9}{20}, & \frac{13}{20}, & \frac{17}{20}, & \frac{21}{20} \\ \frac{3}{4}, & \frac{5}{4}, & \frac{6}{4}, & \zeta \end{pmatrix}, \quad F_3(\zeta)=F\begin{pmatrix} \frac{7}{10}, & \frac{9}{10}, & \frac{11}{10}, & \frac{13}{10} \\ \frac{5}{4}, & \frac{6}{4}, & \frac{7}{4}, & \zeta \end{pmatrix}.$$

If $|\zeta|>1$, the five roots are given by

$$y_1=\epsilon\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right)+\frac{\epsilon^2}{5}\beta^{-\frac{3}{5}}\psi_1\left(\frac{1}{\zeta}\right)-\frac{\epsilon^3}{25}\beta^{-\frac{7}{5}}\psi_2\left(\frac{1}{\zeta}\right)+\frac{\epsilon^4}{125}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

$$y_2=\epsilon^2\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right)+\frac{\epsilon^4}{5}\beta^{-\frac{3}{5}}\psi_1\left(\frac{1}{\zeta}\right)-\frac{\epsilon}{25}\beta^{-\frac{7}{5}}\psi_2\left(\frac{1}{\zeta}\right)+\frac{\epsilon^3}{125}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

$$y_3=\epsilon^3\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right)+\frac{\epsilon}{5}\beta^{-\frac{3}{5}}\psi_1\left(\frac{1}{\zeta}\right)-\frac{\epsilon^4}{25}\beta^{-\frac{7}{5}}\psi_2\left(\frac{1}{\zeta}\right)+\frac{\epsilon^2}{125}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

$$y_4=\epsilon^4\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right)+\frac{\epsilon^3}{5}\beta^{-\frac{3}{5}}\psi_1\left(\frac{1}{\zeta}\right)-\frac{\epsilon^2}{25}\beta^{-\frac{7}{5}}\psi_2\left(\frac{1}{\zeta}\right)+\frac{\epsilon}{125}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

$$y_5=\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right)+\frac{1}{5}\beta^{-\frac{3}{5}}\psi_1\left(\frac{1}{\zeta}\right)-\frac{1}{25}\beta^{-\frac{7}{5}}\psi_2\left(\frac{1}{\zeta}\right)+\frac{1}{125}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

where ϵ is a primitive root of $\epsilon^5=1$ and where

$$\psi_0\left(\frac{1}{\zeta}\right)=F\begin{pmatrix} -\frac{1}{20}, & \frac{7}{20}, & \frac{9}{20}, & \frac{1}{5} \\ \frac{4}{5}, & \frac{3}{5}, & \frac{2}{5}, & \frac{1}{\zeta} \end{pmatrix}, \psi_1\left(\frac{1}{\zeta}\right)=F\begin{pmatrix} \frac{3}{20}, & \frac{9}{10}, & \frac{13}{20}, & \frac{2}{5} \\ \frac{6}{5}, & \frac{4}{5}, & \frac{3}{5}, & \frac{1}{\zeta} \end{pmatrix},$$

$$\psi_2\left(\frac{1}{\zeta}\right)=F\begin{pmatrix} \frac{7}{20}, & \frac{11}{20}, & \frac{17}{20}, & \frac{3}{5} \\ \frac{7}{5}, & \frac{6}{5}, & \frac{4}{5}, & \frac{1}{\zeta} \end{pmatrix}, \psi_3\left(\frac{1}{\zeta}\right)=F\begin{pmatrix} \frac{11}{20}, & \frac{13}{10}, & \frac{21}{20}, & \frac{4}{5} \\ \frac{6}{5}, & \frac{7}{5}, & \frac{6}{5}, & \frac{1}{\zeta} \end{pmatrix}.$$

By these two sets of formulae the roots are given *directly* for any set of values of β. We also know the variation of the roots when β describes an arbitrary curve in its plane. The critical points are here

$$\beta_1=-ci,\ \beta_2=c,\ \beta_3=ci,\ \beta_4=-c,$$

where

$$c=4\left(\frac{1}{5}\right)^{\frac{5}{4}}>0.$$

Then we have

$$x_1\xrightarrow{\beta_1}x_5,\ x_2\xrightarrow{\beta_2}x_5,\ x_3\xrightarrow{\beta_3}x_5,\ x_4\xrightarrow{\beta_4}x_5.$$

Thus we have found the group of the linear differential equation of order 4. We can also find into which of the roots y one of the roots x may be changed when we vary β from a point where $|\zeta|<1$ to a point where $|\zeta|>1$.

4. Let us now consider the quintic equation in the form

$$x^5 = gx^2 + \beta.$$

We have

$$\zeta = \frac{5^5}{2^2 3^3} \frac{\beta^3}{g^5}.$$

If we suppose $g=1$ and $|\zeta|<1$, we have the five roots given by

$$x_1 = \nu F_0(\zeta) + \frac{\nu^2}{3}\beta F_1(\zeta) - \frac{1}{3}\beta^2 F_2(\zeta),$$

$$x_2 = \nu^2 F_0(\zeta) + \frac{\nu}{3}\beta F_1(\zeta) - \frac{1}{3}\beta^2 F_2(\zeta),$$

$$x_3 = F_0(\zeta) + \frac{1}{3}\beta F_1(\zeta) - \frac{1}{3}\beta^2 F_2(\zeta),$$

$$x_4 = \frac{1}{2}\beta^2 F_2(\zeta) - \sqrt{-\beta}F_3(\zeta),$$

$$x_5 = \frac{1}{2}\beta^2 F_2(\zeta) + \sqrt{-\beta}F_3(\zeta),$$

where ν is a primitive root of the equation $\nu^3=1$, and where

$$F_0(\zeta) = F\begin{pmatrix} -\frac{1}{15}, & \frac{2}{15}, & \frac{8}{15}, & \frac{11}{15} \\ \frac{5}{6}, & \frac{1}{3}, & \frac{2}{3}, & \zeta \end{pmatrix}, \quad F_1(\zeta) = F\begin{pmatrix} \frac{4}{15}, & \frac{7}{15}, & \frac{13}{15}, & \frac{16}{15} \\ \frac{4}{3}, & \frac{7}{6}, & \frac{2}{3}, & \zeta \end{pmatrix},$$

$$F_2(\zeta) = F\begin{pmatrix} \frac{3}{5}, & \frac{4}{5}, & \frac{6}{5}, & \frac{7}{5} \\ \frac{5}{3}, & \frac{3}{2}, & \frac{4}{3}, & \zeta \end{pmatrix}, \quad F_3(\zeta) = F\begin{pmatrix} \frac{1}{10}, & \frac{3}{10}, & \frac{7}{10}, & \frac{9}{10} \\ \frac{7}{6}, & \frac{1}{2}, & \frac{5}{6}, & \zeta \end{pmatrix}.$$

If, however, $|\zeta|>1$, the roots are given by

$$y_1 = \epsilon\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right) + \frac{\epsilon^3}{5}\beta^{-\frac{2}{5}}\psi_1\left(\frac{1}{\zeta}\right) - \frac{\epsilon^2}{125}\beta^{-\frac{8}{5}}\psi_2\left(\frac{1}{\zeta}\right) + \frac{\epsilon^4}{625}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

$$y_2 = \epsilon^2\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right) + \frac{\epsilon}{5}\beta^{-\frac{2}{5}}\psi_1\left(\frac{1}{\zeta}\right) - \frac{\epsilon^4}{125}\beta^{-\frac{8}{5}}\psi_2\left(\frac{1}{\zeta}\right) + \frac{\epsilon^3}{625}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

$$y_3 = \epsilon^3\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right) + \frac{\epsilon^4}{5}\beta^{-\frac{2}{5}}\psi_1\left(\frac{1}{\zeta}\right) - \frac{\epsilon}{125}\beta^{-\frac{8}{5}}\psi_2\left(\frac{1}{\zeta}\right) + \frac{\epsilon^2}{625}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

$$y_4 = \epsilon^4\beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right) + \frac{\epsilon^2}{5}\beta^{-\frac{2}{5}}\psi_1\left(\frac{1}{\zeta}\right) - \frac{\epsilon^3}{125}\beta^{-\frac{8}{5}}\psi_2\left(\frac{1}{\zeta}\right) + \frac{\epsilon}{625}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

$$y_5 = \beta^{\frac{1}{5}}\psi_0\left(\frac{1}{\zeta}\right) + \frac{1}{5}\beta^{-\frac{2}{5}}\psi_1\left(\frac{1}{\zeta}\right) - \frac{1}{125}\beta^{-\frac{8}{5}}\psi_2\left(\frac{1}{\zeta}\right) + \frac{1}{625}\beta^{-\frac{11}{5}}\psi_3\left(\frac{1}{\zeta}\right),$$

where ϵ is a primitive root of the equation $\epsilon^5=1$ and where

$$\psi_0\left(\frac{1}{\zeta}\right)=F\begin{pmatrix}-\frac{1}{15}, & \frac{1}{10}, & \frac{3}{5}, & \frac{4}{15} \\ \frac{4}{5}, & \frac{2}{5}, & \frac{1}{5}, & \frac{1}{\zeta}\end{pmatrix}, \psi_1\left(\frac{1}{\zeta}\right)=F\begin{pmatrix}\frac{2}{15}, & \frac{3}{10}, & \frac{4}{5}, & \frac{7}{15} \\ \frac{6}{5}, & \frac{3}{5}, & \frac{2}{5}, & \frac{1}{\zeta}\end{pmatrix},$$

$$\psi_2\left(\frac{1}{\zeta}\right)=F\begin{pmatrix}\frac{8}{15}, & \frac{7}{10}, & \frac{6}{5}, & \frac{13}{15} \\ \frac{8}{5}, & \frac{7}{5}, & \frac{4}{5}, & \frac{1}{\zeta}\end{pmatrix}, \psi_3\left(\frac{1}{\zeta}\right)=F\begin{pmatrix}\frac{11}{15}, & \frac{9}{10}, & \frac{7}{5}, & \frac{16}{15} \\ \frac{9}{5}, & \frac{8}{5}, & \frac{6}{5}, & \frac{1}{\zeta}\end{pmatrix}.$$

The critical points in this case are

$$\beta_0=0,\ \beta_1=-c\nu^2,\ \beta_2=-c\nu,\ \beta_3=-c,$$

where

$$c=3\left(\frac{2}{5}\right)^{\frac{5}{3}}>0.$$

The permutation scheme is now

$$x_1\xrightarrow{\beta_1}x_5,\quad x_2\xrightarrow{\beta_2}x_5,\quad x_3\xrightarrow{\beta_3}x_5,\quad x_4\underset{+}{\xrightarrow{\beta_0}}x_5.$$

Thus we have found another group of a higher hypergeometric differential equation of order 4.

5. Consider the equation of form

$$x^5=gx^3+\beta.$$

We have

$$\zeta=\frac{5^5}{2^2 3^3}\frac{\beta^2}{g^5}.$$

If we put $g=1$ we obtain for the five roots,

$$x_1=-F_0(\zeta)+\frac{1}{2}\beta F_1(\zeta),$$

$$x_2=\quad F_0(\zeta)+\frac{1}{2}\beta F_1(\zeta),$$

$$x_3=-\frac{1}{3}\beta F_1(\zeta)+\frac{\delta}{2}(-\beta)^{\frac{1}{3}}F_2(\zeta)+\frac{3}{4}\delta^2(-\beta)^{\frac{5}{3}}F_3(\zeta),$$

$$x_4=-\frac{1}{3}\beta F_1(\zeta)+\frac{\delta^2}{2}(-\beta)^{\frac{1}{3}}F_2(\zeta)+\frac{3}{4}\delta(-\beta)^{\frac{5}{3}}F_3(\zeta),$$

$$x_5=-\frac{1}{3}\beta F_1(\zeta)+\frac{1}{2}(-\beta)^{\frac{1}{3}}F_2(\zeta)+\frac{3}{4}(-\beta)^{\frac{5}{3}}F_3(\zeta),$$

where δ is a primitive root of the equation $\delta^3=1$, and where

$$F_0(\zeta)=F\begin{pmatrix} -\frac{1}{10}, & \frac{1}{10}, & \frac{3}{10}, & \frac{7}{10} \\ \frac{1}{6}, & \frac{5}{6}, & \frac{1}{2}, & \zeta \end{pmatrix}, \quad F_1(\zeta)=F\begin{pmatrix} \frac{2}{5}, & \frac{3}{5}, & \frac{4}{5}, & \frac{6}{5} \\ \frac{3}{2}, & \frac{2}{3}, & \frac{4}{3}, & \zeta \end{pmatrix},$$

$$F_2(\zeta)=F\begin{pmatrix} \frac{1}{15}, & \frac{4}{15}, & \frac{7}{15}, & \frac{14}{15} \\ \frac{7}{6}, & \frac{1}{3}, & \frac{2}{3}, & \zeta \end{pmatrix}, \quad F_3(\zeta)=F\begin{pmatrix} \frac{11}{15}, & \frac{14}{15}, & \frac{17}{15}, & \frac{23}{15} \\ \frac{11}{6}, & \frac{5}{3}, & \frac{4}{3}, & \zeta \end{pmatrix}.$$

The critical points are

$$\beta_0=0,\ \beta_1=+c,\ \beta_2=-c,$$

where

$$c=\frac{2}{3}\left(\frac{3}{5}\right)^{\frac{5}{2}}>0,$$

and the scheme of permutation is

$$x_1\overset{\beta_1}{\longrightarrow}x_5,\quad x_2\overset{\beta_2}{\longrightarrow}x_5,\quad x_3\underset{+}{\overset{\beta_0}{\longrightarrow}}x_4,\quad x_4\underset{+}{\overset{\beta_0}{\longrightarrow}}x_5.$$

6 We know that according to the general theory of implicit functions there exist developments of the roots in powers of the coefficients provided certain conditions are fulfilled. Consequently we can obtain developments of one of the forms

$$(6) \qquad x=\Sigma A_k\beta^k \text{ or } x=\Sigma B_k g^k,$$

where the coefficients A depend upon g and the coefficients B upon β. We have seen that if the roots are considered as functions of the variable ζ they are given by sums of generalized hypergeometric functions of ζ of order $n-1$. It is possible, of course, to use a definition of a hypergeometric function other than the one we have used above.

Thus, in their interesting memoirs, Haley*, Heyman*, Mellin* and others call the series (6) a hypergeometric function of g provided β equals unity. But this definition implies that the ratio of two consecutive coefficients

$$\frac{B_{k+1}}{B_k}$$

is not a rational function of rank k of the coefficients of fixed degree independent of the rank k. In order to make this point clear let us consider the cubic equation

$$x^3=gx+\beta.$$

If we put $\beta=1$, the roots are hypergeometric functions of g according to the definition used by the above named authors. On the other hand my method

*Heymann and Mellin, *loc. cit.* Haley: Quarterly Journal of Mathematics, Vol. V. See also my note: Comptes Rendus Acad. Sciences, Paris, t. 177, p. 23, séance du 2 juillet, 1923.

gives the roots expressed as a sum of Gaussian hypergeometric functions of the new variable

$$\zeta = \frac{27}{4}\frac{\beta^2}{g^3}$$

as follows,

$$x_i = \sqrt{g}\left[(-1)^{3i}F\left(-\frac{1}{6},\frac{1}{6},\frac{1}{2},\zeta\right)+\frac{1}{3}\sqrt{\frac{\zeta}{3}}\,F\left(\frac{1}{3},\frac{2}{3},\frac{3}{2},\zeta\right)\right],\ (i=1,\ 2),$$

$$x_3 = -\frac{2}{3}\sqrt{\frac{g\zeta}{3}}\,F\left(\frac{1}{3},\frac{2}{3},\frac{3}{2};\zeta\right),$$

provided $|\zeta|<1$ or $\zeta=1$; and the analogous formula holds when $|\zeta|>1$. Hence it is apparent that my investigations and those referred to above belong to different fields.

The developments here given for trinomial equations can in large measure be extended so as to hold for the general algebraic equation if generalized hypergeometric functions of several variables are introduced.*

*See my notes: Comptes Rendus Acad. Sciences, Paris t. 171, 1920, p. 1370, t. 172, 1921, pp. 309 and 1155. Congrès des Mathématiques à Helsingfors, 1922, p. 174.

NUMBER OF DIMENSIONS OF AN ABSTRACT SET

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Preliminary Remarks*

(*a*) The purpose of this paper is to compile in a summarized and systematized form, definitions and theorems which have been given by the author in seven different memoirs or which have not yet been published.

(*b*) As a historical outline of the evolution of the notion of number of dimensions will be published elsewhere [7] by the author, he will here dispense with a survey of this notion and with descriptions of other interesting definitions which have been proposed.

Introduction

1. In a note published in the Comptes Rendus de l'Académie des Sciences de Paris, t. 148, 1909, p. 1152, the author gave a definition of the number of dimensions applicable to the very general case where the space (L) considered is one of the spaces where the convergence of sequences of points has been defined.

It has since occurred to him, that the hypothesis of the existence of a definition of convergence of sequences is not essential for the purpose. What is wanted is a definition of the continuity of transformations, which in its turn can be defined when the operation of derivation of sets has been defined.

We have the following definitions: A *topological space*† is a system (P, K) consisting of a range P, that is, a well determined aggregate of elements or points and a relation K which will derive from any set F of points of the space a set F' (eventually a void one), called the *derived set* of F, of which each element is said to be an *accumulation-point* of F—which point, of course, belongs to the space. A one-one transformation of a set F of points of a topological space S into a set G of points of a topological space T—which need not be distinct from S—is said to be *continuous* if every point a of F which is an accumulation-point of an

*For references in the text the reader will consult the appended bibliography in which each memoir is designated by a number.

†We think that a topological space should be any space to which topological considerations apply. It seems that this may be actually expressed as above. This should make obsolete the expression "topologischer Raum" used by Professor Hausdorff for a space that I should prefer to describe as Hausdorff's space.

arbitrary subset f of F is transformed into a point b of G which is an accumulation-point of the set g into which f has been transformed*.

Two sets F, G are said to be *homeomorphic* when their points may be put into correspondence by a one-one transformation which is bicontinuous, that is, continuous from F to G and from G to F.

2. *Remark.* It is to be noticed in that definition F and G play the roles of topological spaces, as no accumulation-point of F (or G) or of a subset of F (or G) is considered unless it belongs to F (or G).

To make this clearer, it might be better to say then that the two topological sets F and G are *intrinsically* homeomorphic. Since it is often important to bear in mind and to bring into the notion of homeomorphism the relation between a set and the topological space in which it is imbedded, it might be interesting to define homeomorphism of two sets F and G *relatively* to two topological spaces S and T.

Such a homeomorphism should be obtained when there is a homeomorphism between two closed supersets M and N, containing respectively F and G and contained respectively in S and T.

In the sequel we will leave untouched this more precise kind of homeomorphism and deal exclusively with intrinsic homeomorphisms.

3. *Definition of numbers (or types) of dimensions.* Let F, G be two topological sets. F will be said to have a number (or type) of dimensions smaller than or equal to G, if F is homeomorphic with a part of G. This shall be denoted by

$$dF \leq dG.$$

Two sets H, K will be said to have the same type (or number) of dimensions if at the same time

$$dH \leq dK; \; dK \leq dH.$$

If $dH \leq dK$ but no part of H is homeomorphic with K, then H will be said to have a smaller number (or type) of dimensions than $K: dH < dK$.

Given two sets L, M, four cases present themselves:

$$dL < dM; \; dL > dM; \; dL = dM; \; dL \text{ and } dM \text{ are not comparable.}$$

It should be noted that if L and M are homeomorphic, $dL = dM$, but that the converse is not true [2, p. 147]. This is an outstanding difference with the theory of cardinal numbers which otherwise runs along parallel lines. It has been proved recently by S. Banach (Fundamenta Math., t. 6, 1924, p. 239) that if $dL = dM$, then either L is homeomorphic with M or $L = L + L_2$, $M = M_1 + M_2$ so that L_1, L_2 are respectively homeomorphic with M_1, M_2.

It is obvious that if $dH \leq dK$ the cardinal number of H is at most equal to that of K; in particular, all sets of which the number of dimensions is smaller than that of one countable set are countable sets.

*To be quite correct a second alternative should be allowed: that the set g reduces itself to the element b. This will become impossible when the transformation is a one to one correspondence.

4. *General Properties of the number of dimensions.* It is obvious [2, p. 147] that if $dL \leq dM \leq dN$, then $dL \leq dN$; that if $dL < dM \leq dN$, then $dL < dN$; that if $dL = dM = dN$ then $dL = dN$.

It is obvious that if L is a part of M, $dL \leq dM$.

The above definition of the number of dimensions is well adapted to the topological classification of sets. We will show this by studying first the numbers of dimensions of sets of points on a line, on a plane, . . . Later on we will examine the case of functional spaces.

5. *Spaces with a finite number of dimensions.* Let us call R_n a topological space where each po nt is determined by an ordered sequence of n real variables. Assume also that a point X of R_n is an accumulation-point of a subset F of R_n when it is the limiting element of an infinite sequence $X^{(1)}, X^{(2)}, \ldots$ of distinct points of F. Then it is obvious that R_n is homeomorphic with a part of R_{n+p}. Thus

(1) $$dR_1 \leq dR_2 \leq \ldots \leq dR_n \leq \ldots$$

It is also easy to see [2, p. 165] that if two spaces R_n, R_{n+p} have the same number of dimensions, all numbers of the sequence (1) are equal from a certain rank on and increase before attaining that rank. As Lüroth [IV] proved that R_2 is not homeomorphic with a part of R_1, then $dR_1 < dR_2$ and there remains a choice between two cases: either the numbers of the sequence (1) are all dis-t'nct or they are distinct up to a certain rank and all equal after this rank.

Now it seems strange that there should be one such privileged rank; so that the simple notion above defined of number of dimensions leads one to believe that

(2) $$dR_1 < dR_2 < \ldots < dR_n < \ldots$$

These considerations were presented in 1910 [2, p. 165]; since that date, in 1911 has appeared an elaborate proof by Brouwer [III] which substantiates the statement which we had put forward as a guess. Having proved the inequalities (2), there is no reason why we should not replace briefly each quantity dR_n by its index n. So that we will say that R_n is a space of which the type of dimensions may be denoted by n, or more briefly that *R_n is an n-dimensional space.* Let us now study first the types of dimensions ≤ 1.

6. *Numbers of dimensions of linear sets.* If $dG \leq dR_1$, G is homeomorphic with a set F of points of R_1, that is with a linear set F. So that when dealing with sets of which the types of dimensions are ≤ 1, we may replace them by linear sets.

It is easy to prove [2, p. 152] that for a linear set *to have a number of dimensions equal to unity*, it is necessary and sufficient that this set have at least *one interior point*, that is, *contain a whole* (great or small) *interval.*

Consider now the linear sets which have a number of dimensions smaller than unity. They are the sets complementary to sets which are everywhere *dense* on the whole straight line. It is interesting to note that *there is a number of dimensions which is the greatest of all those smaller than unity* [2, p. 154]. It may be described as the *number of dimensions of the set H_1 constituted of all* *irrational numbers* [2, p. 154]. As the *number of dimensions of* H_1 *is equal to that of every linear perfect everywhere discontinuous set* [7], it results that the most general linear set which has the greatest number of dimensions smaller than unity is *a set* which is nowhere dense and which contains *a perfect subset* [7].

I have proved that *there exists an infinite number of types of dimensions smaller than unity.* I stated that *this infinite number is non-countable* (2, p. 149). The proof was not correct. But Paul Mahlo [II] has given a formal proof and the statement itself remains true.

Among these numbers of dimensions, it is interesting to note that *there is one type of dimensions which is the greatest of all types of dimensions of countable linear sets* [2, p. 151]. It can be described as *the type of dimensions of the set* C_1 *of all rational numbers* [2, p. 151[. M. W. Sierpinski has proved [I; 6] that two countable linear sets, each dense in itself, are homeomorphic. It follows that the most general linear set of which *the type of dimensions* is equal to that of C_1 *is any linear set which is countable and contains a subset dense in itself**.

7. *Numbers of dimensions between* 1 *and* 2. If a number of dimensions is 2, it may be considered as the number of dimensions of a plane set E. If this number is not smaller than 1, there is a part of E which is homeomorphic with a straight line. This part will then be a continuous curve.

So that *every number of dimensions which is* ≥ 1 *and* ≤ 2 *denotes the number of dimensions of a plane set* E *of which a part at least is a continuous curve.*

8. *Finite number of dimensions.* Similar results may be obtained for every number of dimensions which is finite, that is, which relates to a set homeomorphic with a part of the n-dimensional space R_n, for a sufficiently large integral value of n.

It is interesting to note that *every countable set of points of any space* R_n *(with a finite integral number* n *of dimensions) is homeomorphic with a linear countable set* [2, p. 151]. As a consequence, *the number of dimensions of every countable set of points of any space* R_n *(with an integral number* n *of dimensions) is at most equal to the number of dimensions of the set* C_1 *of all rational numbers.*

In particular it is possible to prove that C_1 *is homeomorphic with the set* C_n *of all points of the* n*-dimensional space* R_n *which have only rational coordinates.*

These results are easily proved on making use of a more precise proposition since it concerns non-countable sets.

The number of dimensions of the set H_n *of all points of the* n*-dimensional space* R_n *which have only irrational coordinates is smaller than unity.*

More precisely: *whatever may be the integral number* n, *the sets* H_n *and* H_1 *are homeomorphic* [2, p. 154]. This furnishes a very simple proof of a theorem given by Sierpinski [I, see also 6]: *If* M *and* N *are each a countable, dense in itself, set of points of the respective finite-dimensional spaces* R_m *and* R_n *(distinct or not), they are homeomorphic.* For, a convenient translation will make them respectively parts of H_m and H_n (both homeomorphic with H_1), so that M and N are

*Such a set is in Denjoy's terminology any linear countable set which is not "clairsemé".

homeomorphic with two countable, dense in themselves, linear sets. The proof of Sierpinski's theorem is much easier when $m=n=1$.

The previous theorem concerns countable sets. For the non-countable sets a theorem somewhat analogous has been proved by L. Antoine [VIII]. *If P and Q are two perfect discontinuous* sets of points of the respective finite-dimensional spaces R_m and R_n (distinct or not), they are homeomorphic.*

Thus all the numbers of dimensions, from unity (included) to n (included), may be obtained by adding points of R_n to the set H_n which is, however, non-countable and everywhere dense in R_n.

Among the numbers of dimensions which are smaller than n, there is one which is the greatest, namely the number of dimensions of the set Δ_n of points of which at least one coordinate is irrational [2, p. 168]. *Furthermore, the necessary and sufficient condition that the number of dimensions of a set E of points of the n-dimensional space R_n be equal to n, is that E contain an interior point* [III].

Infinite number of Dimensions

Introduction

9. *Properties of topological spaces.* We shall now consider sets contained in spaces more general than the n-dimensional spaces R_n. These spaces may have some of the properties of R_n, and may have not all of them. It will then be useful to give names to some of these properties. As long as we are dealing with R_n these names are not needed, since R_n or even the sets in R_n possess all these properties. This is the reason why they may appear cumbersome to those who have not dealt frequently with functional fields.

A set E is *separable* when there exists a countable set N extracted from E, such that every point of E belongs either to N or to the derived set N' of N.

A set E is *topologically homogeneous* when for every pair A, B of its points there exists a one-one and bicontinuous transformation of E into itself which brings A into B. We recall also that a closed set is a *continuum* when it contains more than one point and cannot be divided into two disconnected closed sets.

We shall now define several categories of spaces more or less analogous to the euclidean space.

10. *Vectorial Spaces.* Combining and modifying two definitions, arrived at independently by S. Banach [V] and N. Wiener [VI], I propose to define a topologically vectorial space in the following way:

11. *Vector field.* A vector field is a system $(\sigma, +, \cdot, ||)$ determined by a set σ of elements which will be called vectors, and three operations (denoted by $+, \cdot, ||$) which will be interrelated with σ as follows:

Let ξ, η, ζ be any three vectors of σ and a, b any two real numbers. Then

1. $\xi+\eta$ is a well determined vector of σ.
2. $\xi+\eta=\eta+\xi$.
3. $(\xi+\eta)+\zeta=\xi+(\eta+\zeta)$.

*Discontinuous here means containing no continuous subset.

4. If $\xi+\eta=\xi+\zeta$, then $\eta=\zeta$.
5. There exists in σ a vector which may be denoted by O such that $\xi+O=\xi$.

6. $a.\xi$ is a well determined vector of σ.
7. If $a\neq 0$ and $a.\xi=a.\eta$ then $\xi=\eta$.
8. If $\xi\neq O$ and $a.\xi=b.\xi$ then $a=b$.
9. $a.(\xi+\eta)=a.\xi+a.\eta$.
10. $(a+b).\xi=a.\xi+b.\xi$.
11. $1.\xi=\xi$.
12. $(ab).\xi=a.(b.\xi)$.

13. $|\xi|$ is a well determined number ≥ 0.
14. $|\xi|=0$ is equivalent to $\xi=O$.
15. $|a.\xi|=|a||\xi|$.
16. $|\xi+\eta|\leq|\xi|+|\eta|$.

(Banach adds the condition (denoted by him I_7) that $a.\xi=O$ is equivalent to: either $a=0$ or $\xi=O$. But this condition is a consequence of the postulates which he calls II_2 and II_3, which are here 14 and 15. Other reductions of the number of the above conditions have been lately pointed out by T. H. Hildebrandt and P. Flamant (Ann. Soc. Polonaise Math., 1926).

12. *Vectorial space.* A set P of elements or points will be called a vectorial space if one vector field σ may be associated with P so as to fulfil the following conditions: To express these conditions call an ordered pair of points of P, the sequence of two points of P, associated in a certain order.

I. To each ordered pair A, B of points of the space corresponds a well determined vector ξ of σ. And this correspondence is expressed as $\overline{AB}=\xi$.

II. Being given a vector ξ of σ and a point A of the space, there exists a unique point B of the space such that $\overline{AB}=\xi$.

III. For every point A of the space $\overline{AA}=O$.

IV. For every three points A, B, C of the space

$$\overline{AB}+\overline{BC}=\overline{AC}.$$

Until now, nothing has been said about infinitesimal properties.

A topological space will be called a *topological vectorial space* when

V. $$\lim_{n\to\infty}|\overline{AA_n}|=0$$

shall be the necessary and sufficient condition for every infinite sequence $A_1, A_2, \ldots$ of points of the class to converge to an arbitrarily chosen point A of the class.

Banach assumes that the operation $|\overline{AB}|$ which furnishes what we may call the absolute value of the vector $\overline{AB}$ gives rise to a generalization of Cauchy's convergence criterion. We will consider this as a further property of a topological vectorial space which may or may not be fulfilled.

As illustrations of the above definition, we may quote the N-dimensional space R_n as a topological vectorial space. In this space, Cauchy's theorem can be generalized by taking $\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}$ for $|\overline{AB}|$, where the x's and the y's are the coordinates of A and B.

13. *Spaces where the limit is defined in terms of "distance."* Without being vectorial a topological space may retain some of the above mentioned properties.

A topological space will be called a *(D)-space** when the presupposed definition of accumulation-points in this topological space can be expressed in terms of *distance*. That is:

1. To every pair of points A, B of the space, there corresponds a number≥ 0 called the "distance" of A, B, denoted by (A, B) and such that

$$(A, B) = (B, A) \geq 0.$$

2. (A, B) is zero when and only when A, B are not distinct points.
3. For every three points A, B, C

$$(A, B) \leq (A, C) + (C, B).$$

4. Lim $(A, A_n) = 0$ is the necessary and sufficient condition that the sequence $A_1, A_2, \ldots\ldots$ converges to A.

It is obvious that every topological vectorial space is a class (D). For a "distance" can be defined as $(A, B) = |\overline{AB}|$.

A (D)-space will be said to be *complete* when it is possible to define in this class its distance in such a way that it admits of a generalization of Cauchy's convergence criterion, without changing the derived sets.

It is easy to prove [7] that a topological vectorial space is a (D)-space which is a homogeneous continuum†. We will find an example of a topological vectorial space which is not separable §9. There are also very probably topological spaces which are not complete §9.

14. *Compact sets.* In order to generalize the notion of bounded linear sets in terms independent of the notion of distance, we define as compact any set of which every infinite subset has a derived set which is not void.

It may or may not happen that a topological space consists of the points of a countable sequence of compact sets. We shall give examples of both cases. Thus the following theorem [3]‡ will be useful:

If a class (D) may be divided into a countable sequence of compact sets, it is separable.

15. *Finite dimensional spaces.* It should be interesting to find topological properties common to all sets which have a finite number of dimensions. By such sets we mean all those which have a number of dimensions at most equal to an integral number.

*I had formerly introduced this notion in my Theses (1906) under the name of space E which I now use when condition 3° is not imposed. Professor Hausdorff called in 1914, metric space, what I call space (D).

†It should be interesting to investigate what are the conditions (in terms of distance or in terms of derivation of sets) for a homogeneous continuum which is a (D)-space to be a topological vectorial space.

‡The hypothesis made in this proof that the class is complete is not necessary; the assumption that the class is perfect may be dropped when the meaning of "separable set" is the one adopted here.

If $dE \leq dR_n$, that is if E is homeomorphic with a set r_n of points of R_n, it is obvious *that* E like r_n *is separable.*

E like r_n can be divided into a countable sequence of compact disconnected sets [3, p. 15; 7].

Finally if a topological space S has a finite number of dimensions, it is obvious that by putting it into correspondence with a set of points of one of the R_n's, it is possible to define on S a distance, namely the distance of the corresponding points on R_n.

Thus we have proved that *every topological space which has a finite number of dimensions is a separable* (D)*-space* [4, p. 161].

On the other hand, it need be neither complete nor compact, nor homogeneous, nor need it be a continuum.

Also a separable (D)-space need not have a finite number of dimensions as will become obvious from what follows.

16. *Number of dimensions of a countable set.* We have seen that if a *countable* set E belongs to a space R_n of which the number of dimensions is finite, then E is homeomorphic with a linear point set. This proposition does not hold true for a countable set of points taken in an arbitrary topological space. For instance, if in this space some sets constituted of finite numbers of distinct points are closed and have at least one element of accumulation, it is obvious that such a space cannot be homeomorphic with a linear set (which should be finite and should have accumulation points)*.

It is then so much the more interesting to note that *when it belongs to a separable* (D)*-space, a countable set is always homeomorphic with a linear set.* This may also be stated as follows: among the numbers of dimensions of all the countable sets of points taken each in an arbitrary separable (D)-space, there is one which is the greatest and it is equal to the number of dimensions of the linear set C_1—of which the points are all those with rational values.

An interesting, though indirect proof [4] consists of two steps which have themselves intrinsic values.

First it may be proved that every separable (D)-space is homeomorphic with one of the sets of points of a very simple separable (D)-space which will be called E_ω. It suffices, for obtaining this result, to use exactly the same proof which was given in 1910 by the author for a similar but different purpose [3, p. 12].

The space E_ω consists of all points X which are each determined by an infinite (ordered) sequence of coordinates $x_1, x_2, \ldots, x_n$. In this space a sequence of points $X^{(1)}, X^{(2)}, \ldots$ is said to converge to a point X, when the coordinates of $X^{(p)}$ converge respectively and *independently* to the coordinates of X when $p \rightarrow \infty$.

I have proved in my thesis that this space E_ω is a separable complete

*A linear set is here a set of points of R_1, that is, a set of points of a straight line where the limit is defined as usual in terms of the euclidean distance.

(D)-space where the distance of two points X, Y, may *for instance* be defined as*

$$(X, Y) = \frac{|x_1 - y_1|}{1 + |x_1 - y_1|} + \dots + \frac{1}{n!}\, \frac{|x_n - y_n|}{1 + |x_n - y_n|} + \dots .$$

It is also easy to prove that this space is a homogeneous continuum.

Let us then take arbitrarily a separable (D)-space. By definition there is a countable set N of points of this space $A_0, A_1, A_2, \dots$ such that the space is the sum $N+N'$. Assign now to any point A of the given space, the point X of E_ω of which the coordinates are

(3) $x_1 = (A_1, A) - (A_1, A_0)$; $x_2 = (A_2, A) - (A_2, A_0)$; $\dots$; $x_n = (A_n, A) - (A_n, A_0)$; $\dots$.

When A describes the given space, X describes a part F of E_ω. The given space is homeomorphic with F, as is shown by the equality (which is easy to prove [2, p. 161]),

$$(A, B) = \text{upper bound of } (x_n - y_n)$$

when n varies, where $y_1, y_2, \dots$ are the coordinates of the point Y of E_ω corresponding to the B of the given space.

It is always possible to assume in any (D)-space that the distance is bounded, by replacing, if necessary, (A,B) by $\dfrac{(A, B)}{1+(A, B)}$. If such a distance has been chosen, the set F will be compact.

Finally: *any given separable (D)-space is homeomorphic with a compact set of points of the above defined space E_ω.*

As E_ω is itself a separable (D)-space, it results that *among the numbers of dimensions of all separable (D)-spaces, the greatest is the number of dimensions of the space E_ω.*

Now, we have seen that H_n is homeomorphic with H_1 whatever may be the integer n. It is interesting to note that *the linear set H_1 of all irrational numbers is also homeomorphic with the set H_ω of all points of the space E_ω of each of which the infinite sequence of coordinates consists of irrational numbers* [5; 7].

On the contrary, *the set C_ω of all the points of the space E_ω, of which all the coordinates are rational, is not homeomorphic to C_1.* For C_1 is countable and C_ω is not. But *C_ω, like C_n, is homeomorphic with a linear set.* For, a translation such that $x_n' = x_n + \sqrt{2}$, $n = 1, 2, \dots$ makes C_ω homeomorphic with a part of H_ω.

The proof which we alluded to is now as follows: given a countable set F of points in a separable (D)-space, and knowing that this space is homeomorphic with a part of E_ω, we see that F is homeomorphic with a countable set G of points of E_ω. A convenient translation will carry G over a part of H_ω, and,

*The distance might also be taken equal to the lower bound of

$$(X, Y)_n + \frac{1}{n}$$

when n varies, $(X, Y)_n$ being a distance for the points $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ in R_n, a distance chosen arbitrarily but not decreasing when n increases, for instance

$$(X, Y)_n = |x_1 - y_1| + \dots + |x_n - y_n|.$$

as H_ω is homeomorphic with H_1, we see that F is homeomorphic with a part of H_1, that is with a linear set.

The theorem due to Sierpinski [1] and given above may be extended to more general spaces than the finite dimensional space.

Let M, N be two countable sets of points of two respective separable (D)-spaces. We have seen that they are homeomorphic with two countable linear sets, M_1 and N_1. If M and N are dense in themselves, so are M_1 and N_1 which consequently are homeomorphic. Thus: *two countable, dense in themselves, sets of points of two respective (distinct or not) separable (D)-spaces are homomorphic* [4]. Furthermore *each such set is topologically homogeneous*, as has been proved by Sierpinski [1] in the case of linear sets.

17. *∞-dimensional sets and spaces.* The number of dimensions of a set will be naturally said to be infinite when it is greater than any integer. In other words an ∞-dimensional set is a set which, for every integer n, contains a set which is homeomorphic with the n-dimensional space n. As illustrations we shall quote the following very simple ∞-dimensional spaces.

1. Let R be a space of which each point X is defined by an ordered infinite sequence of real numbers $x_1, x_2, \ldots, 0, 0, \ldots$ which are equal to zero from a certain rank (varying in general with the point X). In this space a sequence of points $X^{(1)}, X^{(2)}, \ldots$ will be said to converge to a point X if the coordinates of $X^{(p)}$ converge respectively to those of X when p tends to infinity.

Such a space is obviously an ∞-dimensional space. It is also a (D)-space which is separable and can be divided into a countable sequence of compact sets. But it is not complete [7].

2. Let P be the polynomial space, that is, the space of which each element Q is a polynomial. We assume furthermore that an infinite sequence $Q_1, Q_2, \ldots$ converges to P when these polynomials (of which the degrees may differ) converge uniformly to Q in every finite interval.

This polynomial space is also ∞-dimensional. It is also a (D)-space which is separable and can be divided into a countable sequence of compact sets. But it is not complete [7].

It might be interesting to find out whether the two spaces R and P which are very similar are homeomorphic, as also on the other hand whether they have the same number of dimensions as those which we shall study now and which have at any rate at least the same number of dimensions. The property that R and P may be divided into countable sequences of compact sets does not belong to the following spaces and these spaces are complete, whereas P and R are not. This suggests that P and R have a smaller number of dimensions.

18. *List of many important topological spaces which have the same (infinite) number of dimensions.* We have seen above that every separable (D)-space is homeomorphic with a (compact) part of the space E_ω. A similar proposition had been proved by Urysohn [VII] except that E_ω was replaced by the Ω or Hilbert space. Let us recall that this space consists of all the points X determined each by an infinite sequence of real numbers $x_1, x_2, \ldots$ such that Σx_n^2 con-

verges, and that in this space the limit of a sequence of points is defined in terms of distance, the distance between the point X and the point $Y(y_1, y_2, \ldots)$ being

$$(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots\ldots}$$

Now I proved that both E_ω and Ω are separable (D)-spaces, so that by the two theorems quoted above we get $d\Omega \leq dE_\omega$; $dE_\omega \leq d\Omega$ and finally $dE_\omega = d\Omega$. Obviously E_ω and Ω have an infinite number of dimensions, so that we now have two ∞-dimensional spaces of great importance in Analysis which have the same number of dimensions.

It is interesting to note that a similar proof will extend the same result to many other important classes [5; 7]. We then get the following final result:

Among the most important fields considered in Analysis or Geometry there are a number which have the same (infinite) number of dimensions. A list (far from exhaustive) of these spaces which have the same (infinite) number of dimensions is given below.

Space I: Elements: integral functions of a complex variable z. Limit of a sequence $f_1, f_2, \ldots, f_n, \ldots$: an element f such that $f_n(z) - f(z)$ converges uniformly to zero in every bounded region of the complex plane.

Space C_n: Elements: functions of a real variable x, which in a fixed interval have n continuous derivatives. Limit of a sequence $f_1, f_2, \ldots$ an element f, such that $f_p(x) - f(x), f_p'(x) - f'(x), \ldots, f_p^{(n)}(x) - f^{(n)}(x)$ converge uniformly to zero in the given interval.

Space C: Elements: continuous functions of a real variable in a fixed interval. Limit: $f_1(x), \ldots, f_p(x), \ldots$ have $f(x)$ as a limit when $f_p(x) - f(x)$ converges uniformly to zero in the given interval.

Space Ω_1: Elements: functions of which the square is Lebesgue-integrable in a fixed interval (a, b). Limit in terms of a distance which for f and ϕ is: $\sqrt{\int_a^b [f(x) - \phi(x)]^2 dx}$.

Space $\mathfrak{M}$: Elements: functions which in a fixed interval (a, b) are Lebesgue-measurable. Limit: $f_1, f_2, \ldots, f_n, \ldots$ tend to f when for every $\epsilon > 0$ and $\eta > 0$, there is an integer p such that the measure of the set of points, where $|f(x) - f_n(x)| > \epsilon$, is smaller than η for $n > p$.

Space Γ: Elements: oriented Jordan curves. Limit: a sequence of elements $\gamma_1, \gamma_2, \ldots$ converge to γ when there is for every n a homeomorphism between γ_n and γ such that the distance between two corresponding points tends uniformly to zero.

Space Σ: Similar space of Jordan surfaces instead of Jordan curves.

Space $\mathfrak{M}_1$: Elements: all measurable sets of points on (for instance) a straight segment. Limit: a sequence of elements $E_1, E_2, \ldots E_n, \ldots$ converges to E when the measure of the set common to E and E_n converges to zero.

Space E_ω: See §16.

Space Ω (or Hilbert space). See above.

Space A: Elements: absolutely converging series. Limit: is expressed in terms of a certain distance, the distance between the two series U, V of which the elements are $u_1, u_2, \dots$ and $v_1, v_2, \dots$ being

$$(U, V) = |u_1 - v_1| + |u_2 - v_2| + \dots$$

Space S: Elements: converging series of real numbers. Limit: in terms of distance. With the previous notation:

$$(U, V) = \text{upper bound of } |(u_1 + \dots + u_n) - (v_1 + \dots + v_n)|.$$

19. *Properties of these spaces.* It is interesting to note that each of the spaces

$$I, C_n, C, \Omega_1, \mathfrak{M}, \Gamma, \Sigma,$$

forms a separable homogeneous continuum which is a complete (D)-space [7]. Furthermore, unlike R_n, P and R, none of them can be separated into a countable sequence of compact sets [3; 7].

Some of the spaces such as R_n, C_n, C, Ω_1, Γ, Σ, Ω, A, S are obviously topologically vectorial spaces. Others, P, R, I, E_ω, might be considered as *locally* vectorial spaces by the use of a proper definition of the meaning of "locally vectorial space".

An interesting question to solve would be which ones of these spaces are homeomorphic. (It is obvious for instance by the Fatou-Riesz-Fischer theorem that Ω and Ω_1 are homeomorphic and even with invariance of the distance).

20. *A greater infinite number of dimensions.* A slight change in the definition of the space E_ω, will now make it a more complex space which has a greater number of dimensions than all the preceding ones.

This new space which will be called D_ω is made up of all the points X, each of which is determined by a bounded infinite ordered sequence of real coordinates $x_1, x_2, \dots, x_n, \dots$ and in this space a sequence of points $X^{(1)}, X^{(2)}, \dots$ is said to converge to a point X when the coordinates of $X^{(p)}$ converge uniformly to the corresponding coordinates of X. It can easily be proved [7] that this space D_ω *is a complete topological vectorial space* which is therefore a *homogeneous continuum.* But this space *is not separable* and therefore no separable space may have a number of dimensions $\geqq dD_\omega$.

As a matter of fact, it has been proved that *every separable (D)-space has a smaller number of dimensions than D_ω* [2, p. 161]. That is, every separable (D)-space is homeomorphic with a part of the space D_ω. But the proof, by means of the correspondence stated in (3), §16, gives something more. Not only is it possible to show that this correspondence is homeomorphic, but the formula (4) §16 shows that *this correspondence* also leaves *distances invariant.* An important consequence is that a *complete* separable (D)-space corresponds in D_ω to a separable *closed* set.

We have noticed that every countable set of points of a separable (D)-space is homeomorphic with a linear set. It would be interesting to ascertain whether this proposition holds true for every countable set of points of D_ω.

21. *Still greater numbers of dimensions.* We have seen that there is a rather simple space, the space D_ω which has a greater number of dimensions than all the separable (D)-spaces. This space D_ω has the same cardinal number as the continuum. This shows that its number of dimensions may be surpassed by many others. For example, it is smaller than the number of dimensions of the space U. U will here denote a space consisting of all the functions of a real variable x $(o \leq x \leq 1)$ with the usual definition of uniform limit of a sequence of functions [7].

22. *Addition of dimensions.* My first definition of the number of dimensions included a definition of the addition of such numbers [2, p. 147]. This definition of addition I still consider as useful; but I do not now regard it as essential; and to show this I have delayed its introduction in this paper up to the present paragraph.

Let us first define for any two topological spaces G, H their topological product GH. It will be the set consisting of all the pairs AB of elements drawn one, A, from G, one, B, from H. An element AB of GH shall be considered as an accumulation element of a subset E of GH, E consisting of pairs CD, when either A is an accumulation-point of the set of the elements C, and B is either an element of the set K of the elements D or belongs to K' , or conversely.

Then by definition $dG+dH=dGH$. It is easy to see that

$$dG+dH \geq dG \text{ and } dG+dH \geq dH.$$

But it may happen [2, p. 155] that

$$dG+dH=dG.$$

It is also easily seen that

$$(dG+dH)+dK=dG+(dH+dK).$$

23. *Integral numbers of dimensions.* Now it is also easy to see that R_{n+p} is homeomorphic with R_nR_p so that

$$dR_{n+p}=dR_n+dR_p, \text{ when } n,\ p \text{ are integers.}$$

This supplies a new motive for denoting more briefly dR_n by n.

Furthermore the notion of topological product is useful to build up new numbers of dimensions and, for instance, to show that there is an infinite number of distinct numbers of dimensions between two consecutive integers.

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L'EXPRESSION LA PLUS GÉNÉRALE DE LA «DISTANCE» SUR UNE DROITE

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L'utilité de la conception des espaces à une infinité de coordonnées peut être manifestée dans la solution suivante d'un problème dont l'énoncé ne fait intervenir que l'espace à une dimension. L'introduction d'un espace supérieur n'est certainement pas nécessaire pour résoudre le problème, mais elle met sur le chemin de la solution et permet de décrire cette solution sous une forme géométrique plus suggestive.

Le problème est le suivant: attacher à tout couple de points x, x' d'une droite indéfinie Δ, un nombre (x, x') (qu'on pourra appeler *distance généralisée* de x et de x'), nombre satisfaisant aux conditions suivantes:

1° on a $(x, x')=(x', x)\geq 0$;

2° (x, x') est positif si $x\neq x'$, nul si $x=x'$;

3° quels que soient x, x', x'' on a

$$(x', x'')\leq(x, x')+(x, x'');$$

4° la condition nécessaire et suffisante pour que $x_1, x_2, \ldots, x_n, \ldots$, convergent vers x, est que la suite de nombres $(x, x_1), (x, x_2), \ldots, (x, x_n), \ldots$ converge vers zéro.

Une solution évidente de ce problème s'obtient en prenant

$$(x, x')=|x-x'|\times\text{constante}.$$

On obtiendrait une solution plus générale en prenant pour (x, x'), la distance géométrique des deux points correspondants à x, x' dans une transformation biunivoque et bicontinue de la droite Δ en elle même.

Mais ce ne serait pas encore la solution la plus générale. On sera mis sur le chemin de celle ci, si l'on note encore la solution particulière consistant à prendre pour (x, x') la distance euclidienne entre les deux points X, X' correspondants à (x, x') dans une transformation biunivoque et bicontinue de la droite Δ en une courbe de Jordan (extrémités, s'il y en a, exclues) sans points multiples. En effet, une telle expression de (x, x') convient quel que soit le nombre de dimensions de l'espace où est tracée la courbe de Jordan. On est donc conduit à essayer si l'expression obtenue quand le *nombre de dimensions devient infini* est une solution et la plus générale.

Comme dans ce cas la notion de suite convergente peut être conçue de plusieurs manières, il faut choisir. Un théorème que nous avons précédemment obtenu sur les ensembles dits séparables nous a fait préférer la suivante. *Nous appellerons espace* D_ω un espace dont chaque point X est défini par une suite infinie de nombres réels X_1, X_2, ... qui seront les coordonnées de X et où une suite $X^{(1)}, \ldots, X_n^{(n)}, \ldots$ est dite converger vers X lorsque les coordonnées de $X^{(n)}$ convergent uniformément vers les coordonnées de même rang de X.

On peut définir dans cet espace, une «distance» satisfaisant aux conditions 1°, 2°, 3°, 4° en désignant par (X, X') la borne supérieure des valeurs absolues $|X_n - X'_n|$ des coordonnées de même rang de X et de X'*.

Ceci étant, la *solution générale du problème* est la suivante: *on considère dans l'espace à une infinité de coordonnées* D_ω *une courbe de Jordan (extrémités, s'il y en a, exclues) sans points multiples.* Par définition même cette courbe de Jordan correspond dans une transformation biunivoque et bicontinue convenable à la droite Δ; *on prendra pour distance généralisée* (x, x') *de deux points de la droite* Δ *la distance définie plus haut des deux points correspondants de la courbe de Jordan.*

La même méthode serait sans doute employée avec succès si Δ, au lieu d'être une droite était un plan, l'espace à trois dimensions, etc.

Nota. Les résultats énoncés ci-dessus ont été établis dans un mémoire paru sous le même titre dans *l'American Journal of Mathematics* (vol. 47, 1925, p. 10). Feu Urysohn a ensuite montré qu'on pouvait, dans ce qui précède, remplacer l'espace D_ω par l'espace séparable qu'il a appelé l'espace métrique universel (Bull. Sc. Math., t. LI, 1927, p. 1-38).

*Pour que la distance soit finie quand les coordonnées sont finies, on conviendra de n'admettre un point X dans $D\omega$, que si les coordonnées de X sont bornées dans leur ensemble.

SUR UNE REPRÉSENTATION PARAMÉTRIQUE INTRINSÈQUE DE LA COURBE CONTINUE LA PLUS GÉNÉRALE*

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Utilité d'une représentation intrinsèque. On sait qu'on peut obtenir une infinité de représentations paramétriques d'une même courbe continue en effectuant dans l'une de ces représentations des changements de variable. Si $a(t)$, $b(t)$, $c(t)$ sont des fonctions uniformément continues dans l'intervalle (0, 1) et non à la fois constantes dans un même sous-intervalle, il suffit de faire le changement de variable $t=\lambda(u)$ où $\lambda(u)$ est uniformément continue et croissante de 0 à 1 pour obtenir la représentation paramétrique la plus générale de la courbe

$$x=a(t), \quad y=b(t), \quad z=c(t), \quad 0\leqslant t\leqslant 1. \tag{1}$$

Mais il est évident que la plus ou moins grande simplicité de la courbe ne se réflète pas toujours exactement dans chacune de ses représentations paramétriques. La courbe pourra avoir, en tous points, une tangente, une courbure, . . . sans que les fonctions $a(t)$, $b(t)$, $(c(t)$ fussent même dérivables.

Il y aurait donc intérêt à posséder pour chaque courbe une représentation paramétrique dont les particularités soient commandées par celles de la courbe indépendamment des singularités introduites par un changement de variable plus ou moins compliqué. En particulier, la variable employée devrait être, pour un point donné de la courbe indépendante de la position dans l'espace de la courbe considérée comme rigide. Ce serait *une représentation intrinsèque.*

Lorsque la courbe est rectifiable, une représentation paramétrique intrinsèque tout indiquée s'obtient en prenant l'arc pour variable. Mais certaines questions d'analyse et d'Analysis situs font appel aux courbes continues les plus générales, pour lesquelles le choix de l'arc pour paramètre n'est plus possible. Il n'est donc peut-être pas sans intérêt de déterminer une réprésentation paramétrique intrinsèque de la courbe continue la plus générale.

Exemple d'une représentation intrinsèque. Soit C une courbe continue représentée par exemple sous la forme (1). Il s'agit de déterminer un changement de variable $t=h(\sigma)$ qui transforme la représentation paramétrique (1) dans une représentation intrinsèque

$$x=\phi(\sigma), \quad y=\psi(\sigma), \quad z=\theta(\sigma), \quad 0\leqslant\sigma\leqslant 1.$$

*Le mémoire dont cette communication est le résumé sans démonstration a paru depuis le Congrès dans le Jour. de Math., *t.* IV, Fasc. III, 1925, p. 281-297.

Nous opèrerons de la façon suivante. Appelons $Os(MM')$ *l'oscillation de l'arc* MM' *de la* courbe C, c'est-à-dire la longueur de la plus grande corde de cet arc. Il existe sur la courbe au moins un point que nous pouvons désigner par $M_{\frac{1}{2}}$ tel qu'en désignant par M_0, M_1 les extrémités de C on ait

$$Os(M_0M_{\frac{1}{2}}) = Os(M_{\frac{1}{2}}M_1).$$

De même, il existe au moins un point $M_{\frac{1}{4}}$ et un point $M_{\frac{3}{4}}$ tels que

$$Os(M_0M_{\frac{1}{4}}) = Os(M_{\frac{1}{4}}M_{\frac{1}{2}}),\ Os(M_{\frac{1}{2}}M_{\frac{3}{4}}) = Os(M_{\frac{3}{4}}M_1), \text{ etc.}$$

Nous définirons d'une façon générale M_r, r étant une fraction (entre 0 et 1) dont le dénominateur est une puissance de 2,

$$r = \frac{K}{2^n},$$

et nous adopterons pour paramètre un nombre σ qui prend la valeur r au point M_r de C. On démontre facilement que cela est possible (la démonstration sera publiée ailleurs). On voit que la représentation obtenue ne sera seule de son espèce que si les points M_r sont bien déterminés pour chaque valeur de r de la forme indiquée. Il est manifeste que cela n'a pas lieu nécessairement pour toute courbe continue. Inversement, il n'y a pas toujours indétermination. Par exemple, la représentation ainsi obtenue pour un segment de droite ou un arc de cercle est unique et n'est autre que celle qui consiste à prendre σ proportionnel à la longueur.

Propriétés de la représentation intrinsèque qui vient d'être définie. 1° On dit qu'une suite de courbes continues C_n *converge* vers une courbe continue C si l'on peut établir entre les points P_n de C_n et les points P de C une homéomorphie H_n telle que la distance PP_n de deux points correspondants converge uniformément vers zéro. Si l'on considère une représentation paramétrique *quelconque* de C_n, que nous pouvons écrire sous la forme condensée

$$P_n = F_n(t), \quad 0 \leqslant t \leqslant 1,$$

en général $F_n(t)$ ne convergera pas uniformément. Mais si l'on considère une représentation intrinsèque de C_n de l'espèce indiquée plus haut

$$P_n = \Phi_n(\sigma), \quad 0 \leqslant \sigma \leqslant 1,$$

I. On peut extraire de la suite des $\Phi_n(\sigma)$ au moins une suite uniformément convergente.

II. Toute suite uniformément convergente extraite de la suite des Φ_n a pour limite une fonction réalisant une représentation intrinsèque de C de l'espèce indiquée plus haut.

2° *Ensembles compacts de courbes continues.* On appelle ainsi un ensemble E de courbes continues tel que chacun de ses sous-ensembles ait un dérivé non vide. Considérons un ensemble de représentations paramétriques quelconques

$$P = F(t), \quad 0 \leqslant t \leqslant 1,$$

une pour chaque courbe C d'un ensemble E de courbes. D'après un théorème connu d'Arzela, si les fonctions $\Phi(\sigma)$ sont bornées et également continues dans leur ensemble, l'ensemble E est compact.

Mais la réciproque n'est pas exacte. Si l'ensemble E est compact, les fonctions $F(t)$ seront nécessairement bornées dans leur ensemble, *mais elles pourront ne pas être également continues.* Tout ce qu'on peut dire c'est que si E est compact, l'un au moins des systèmes de représentations paramétriques des courbes de E doit être formé de fonctions également continues. Pour vérifier si E est compact, il faudrait théoriquement essayer au hasard tous les systèmes possibles de représentations paramétriques jusqu'à ce qu'on en aie trouvé un formé de fonctions également continues ou jusqu'à ce qu'on ait vérifié qu'il n'en existe aucun. L'emploi de la représentation paramétrique intrinsèque évite cette difficulté, car: *un système de représentations paramétriques intrinsèques (de l'espèce indiquée plus haut) des courbes d'un ensemble E est nécessairement formé de fonctions également continues si l'ensemble E est compact.*

Propriétés infinitésimales d'une représentation paramétrique. Il est évident qu'une courbe peut avoir une tangente, un cercle osculateur, sans qu'il en résulte nécessairement pour une représentation paramétrique arbitraire de la courbe que les coordonnées ont des dérivées premières, secondes. . . .

D'autre part, si on s'adresse à une représentation paramétrique intrinsèque, il est évident que les propriétés infinitésimales de la courbe doivent se refléter en quelque manière dans les propriétés infinitésimales des fonctions qui réalisent cette représentation. Il serait intéressant d'étudier cette réaction.

Nous avons pu démontrer qu'en employant la représentation intrinsèque proposée, on est assuré que, dans les cas où elle est formée de fonctions dérivables (en au moins un point) *les dérivées des coordonnées ne peuvent être nulles en même temps.* Par suite, la tangente existe et on en connait immédiatement les paramètres directeurs. Dans le cas pourtant plus particulier où les courbes sont rectifiables, le même résultat ne serait pas assuré si l'on adoptait comme paramètre intrinsèque l'arc de la courbe.

Il serait également intéressant d'essayer d'utiliser notre représentation intrinsèque pour résoudre la question suivante; une courbe de Jordan ayant une tangente en chaque point, est-il possible d'en trouver une représentation paramétrique formée de fonctions partout dérivables? Pour montrer que cette question mérite d'être posée, nous avons construit un exemple d'une courbe rectifiable ayant une tangente partout et dont la représentation paramétrique en fonction de l'arc n'est pas dérivable partout.

Pour résoudre la question posée, et des questions analogues, il y a lieu d'observer que si l'on fait disparaitre l'indétermination éventuelle de la représentation intrinsèque définie ci-dessus, la réaction des propriétés infinitésimales de la courbe sur celles de ses fonctions représentatives sera sans doute plus marquée.

Il n'est donc pas sans intérêt de montrer comment on pourrait réaliser l'unicité de la représentation intrinsèque. Il suffit de préciser le choix du point $M_{\frac{1}{2}}$ et d'opérer ensuite de même pour les points $M_{\frac{1}{4}}$, $M_{\frac{3}{4}}$. . . .

Or, s'il y a plus d'un point M tel que la fonction

$$Os(M_0M) - Os(MM_1)$$

s'annule, il y en a évidemment une infinité formant un arc fermé $N_1'N_1''$ intérieur à l'arc M_0M_1.

Un choix simple consisterait à prendre N_1' pour $M_{\frac{1}{2}}$; mais on romprait ainsi la symétrie du rôle joué par M_0 et M_1. On évitera cet inconvénient de la façon suivante:

Opérons sur $N_1'N_1''$ comme sur M_0M_1. Ou bien, il y aura un seul point M tel que $Os(N_1'M) = Os(MN_1'')$ et on prendra ce point M pour $M_{\frac{1}{2}}$. Ou bien ces points M formeront un arc fermé $N_2'N_2''$ intérieur à $N_1'N_1''$, etc. Ou bien alors on arrivera au bout d'un certain nombre d'opérations à un seul point qu'on prendra pour $M_{\frac{1}{2}}$ ou bien les arcs successifs $N_1'N_1''$, $N_2'N_2''$, . . . auront un point limite qu'on prendra pour $M_{\frac{1}{2}}$.

Note supplémentaire. Dans mon mémoire cité plus haut, j'avais posé (en note p. 295) la question de savoir si on ne pourrait assurer la dérivabilité des coordonnées d'une courbe continue en supposant la continuité de la tangente. M. Valiron a répondu par l'affirmative dans les Nouv. Ann. Math. 1926. Il a même prouvé que cela suffit pour assurer aussi la continuité des dérivées des coordonnées. En outre, la courbe est alors rectifiable et l'arc peut être pris pour paramètre réalisant ces conditions.

LES ENSEMBLES BIEN DÉFINIS, NON MESURABLES B

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Il y a vingt ans, au moment où M. Lebesgue écrivait son important Mémoire sur les fonctions représentables analytiquement, on pouvait dire qu'on connaissait toutes les opérations qui permettaient, dans l'état de la science à ce moment, de définir effectivement les ensembles d'une part et les fonctions d'autre part. C'étaient, pour les ensembles, les opérations d'addition et de soustraction effectuées un nombre fini ou une infinité dénombrable de fois à partir des intervalles et, pour les fonctions, l'opération de passage à la limite, effectuée un nombre fini ou une infinité dénombrable de fois à partir des polynomes.

Toutefois, dès qu'on eut constaté que tous les ensembles d'une part et les fonctions d'autre part qu'on savait définir effectivement pouvaient être obtenus à l'aide des opérations susdites, on fut en mesure de définir d'autres ensembles et d'autres fonctions qui ne rentraient pas dans la classe de ceux qui avaient été déjà étudiés et qu'on appelle aujourd'hui *mesurables B*. Naturellement, si l'on voulait obtenir un ensemble non mesurable B en utilisant seulement les opérations d'addition, de soustraction et de multiplication des ensembles à partir des intervalles, il faudrait appliquer ces opérations une infinité non dénombrable de fois. Ce fait a suggéré la pensée qu'il était impossible de définir effectivement un ensemble non mesurable B autrement qu'en utilisant une infinité non dénombrable de fois les opérations élémentaires. Or, en 1917, M. Souslin prouva qu'il existe des opérations qui doivent être regardées comme tout à fait élémentaires et qui, effectuées sur les ensembles mesurables B, conduisent à des ensembles qui ne sont pas mesurables B: une telle opération est par exemple la projection orthogonale d'un ensemble plan mesurable B, par exemple d'un F de première classe de M. Lebesgue. Ce fait ne présente d'ailleurs rien de paradoxal: il prouve seulement que l'opération qui consiste à prendre la projection d'un ensemble mesurable B ne se réduit pas, en cas général, à une infinité dénombrable d'additions de soustractions ou de multiplications des ensembles mesurables B.

Les opérations élémentaires qu'on a utilisé jusqu'à présent, ont-elles été énumérées et, effectuées sur les ensembles élémentaires, connait-on les classes d'ensembles auxquelles elles conduisent? Je pense que non et qu'il serait important de le faire. Il se peut toutefois qu'on puisse tout de suite nommer une nouvelle opération élémentaire, mais l'énumération de toutes celles qui sont utilisées dans l'état actuel de la science ne me semble pas dépourvue d'intérêt.

Je donnerai ici quelques exemples d'opérations élémentaires qui, effectuées sur les ensembles mesurables B, conduisent aux ensembles non mesurables B.

(1) Soit E un ensemble plan mesurable B, par exemple un F de classe 1. Opérons une translation rectiligne de cet ensemble: soit H le trait tracé par cet ensemble; H peut être un ensemble non mesurable B.

(2) Soit E un ensemble plan mesurable B; coupons-le par des droites P parallèles par exemple à l'axe des ordonnées et remplaçons tout produit PE par sa fermeture $\overline{PE}=PE+(PE)'$: l'ensemble ainsi obtenu peut être non mesurable B.

(3) Soit $f(x)$ une fonction continue d'une variable réelle, E un ensemble mesurable B: l'ensemble de toutes les valeurs que $f(x)$ prend sur E peut être non mesurable B.

(4) Soit $f(x, y)$ une fonction continue de deux variables réelles, définie pour $0 \leq x \leq 1$, $0 \leq y \leq 1$. Considérons seulement les valeurs de x pour lesquelles il existe une valeur unique de y, telle que $f(x, y)=0$ et soit H l'ensemble de tous les y correspondants. L'ensemble H peut être non mesurable B*.

(5) L'ensemble de toutes les valeurs qu'une fonction continue d'une variable réelle prend une infinité dénombrable de fois peut être non mesurable B, comme nous l'avons prouvé récemment avec M. Mazurkiewicz.†

(6) Soit $E_1, E_2, E_3, \ldots$ une suite infinie d'ensembles fermés, $A(E_1, E_2, E_3, \ldots)$ la somme de tous les produits

$$E_{n_1} E_{n_2} E_{n_3} \ldots,$$

étendus à toutes les suites descendentes d'ensembles, extraites de la suite $E_1, E_2, E_3, \ldots$, c'est-à-dire telles que

$$E_{n_1} \supset E_{n_2} \supset E_{n_3} \supset \ldots, \quad n_1 < n_2 < n_3 < \ldots.$$

L'ensemble $A(E_1, E_2, E_3, \ldots)$ peut être non mesurable B‡.

(7) Soit E un ensemble mesurable B, p un point donné, H l'ensemble des distances des points de E au point p. Si E est un ensemble linéaire, H est naturellement mesurable B; or, si E est un ensemble plan, H peut être non mesurable B. On en peut déduire que l'ensemble de toutes les distances de deux points d'un ensemble (plan) mesurable B peut être non mesurable B. (Remarquons que l'ensemble de toutes les distances de deux points d'un ensemble mesurable L peut être non mesurable L).

Il est remarquable que toutes les sept opérations citées, effectuées sur les ensembles d'une part, les fonctions d'autre part, mesurables B, conduisent, soit aux ensembles A de M. Souslin, soit aux complémentaires de ces ensembles. Nous voyons donc comment les ensembles A, dont la théorie est due à MM. Souslin et Lusin, interviennent d'une façon tout à fait naturelle dans la Théorie des Ensembles et dans l'Analyse et quel rôle important ils y jouent.

Il existe cependant des opérations qui peuvent être regardées comme élémentaires et dont la nature n'est pas encore étudiée. Je ne citerai ici qu'une

*Cet exemple montre comment des problèmes très simples et naturels concernant les fonctions continues conduisent à des ensembles très compliqués, même non mesurables B.

†Fundamenta Mathematicae, t. VI (1924).

‡Fundamenta Mathematicae, t. VI.

seule opération dont on ne sait pas si, effectuée sur les ensembles mesurables B, elle conduit toujours aux ensembles mesurables L.

Soit E un ensemble mesurable B et soit H l'ensemble de tous les points p de E pour lesquels il existe un segment pq de longueur 1 et tel que l'intérieur de pq ne contienne aucun point de E. Quelle est la nature de l'ensemble E? M. Urysohn a prouvé dans le cas particulier où E est fermé, que, H est un ensemble A (pas nécessairement mesurable B); or, est-il alors nécessairement de mesure superficielle nulle? demande M. Banach.

Ma conclusion est la suivante:

Les opérations qui nous permettent aujourd'hui de définir effectivement les ensembles à partir des ensembles effectivement définis n'ont pas été énumérées et ni les propriétés des classes d'ensembles auxquels elles conduisent, ni leurs rapports mutuels n'ont été étudiés d'une façon satisfaisante. Les résultats de MM. Baire, Borel, Lebesgue, Souslin et Lusin ont *amorcé* cette étude qui doit être poursuivie. Elle est indispensable si l'on veut aborder le problème important posé récemment par M. Borel: quels sont les nombres réels que nous savons définir?

ON A CERTAIN TYPE OF CONNECTED SET WHICH CUTS THE PLANE

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It has been shown* by J. R. Kline that a continuum† which contains more than one point, and which remains connected upon the removal of any connected subset, is a simple closed curve (Jordan curve)‡. It has been shown§ by C. Kuratowski that if no subcontinuum of a bounded continuum C cuts|| C, then C is a simple closed curve.

The purpose of the present paper is to investigate some of the properties of a plane connected set M which contains more than one point, and which remains connected upon the omission of any connected subset. It should be noted that neither of the words "bounded" and "closed" is used in this characterization of M. In fact, M may be unbounded, punctiforme¶, or both. In any case, it will be shown in §1 that if A and B are any two distinct points of M then M is the sum of two sets, M_1 and M_2, which are irreducibly connected** from A to B and such that $M_1-(A+B)$ and $M_2-(A+B)$ are mutually separated††. It will also be shown in §3 that a set M having these properties always cuts the plane.

*J. R. Kline, *Closed and connected sets which remain connected upon the removal of certain connected subsets*, Fundamenta Mathematicae, V (1924), pp. 3-10.

†A point set is said to be *closed* if it contains all its limit points. A point set is said to be *connected* if, no matter how it be divided into two subsets, at least one of these contains a limit point of the other. A *continuum* is a closed and connected point set.

‡If A and B are distinct points, an *arc* from A to B is a bounded continuum which contains A and B and which is disconnected by the omission of any one of its points which is distinct from A and B. A and B are called the *end-points* of the arc, and any other point of the arc is called an *inner* point. A *simple closed curve* is the sum of two arcs which have in common only their end-points. A point set M is *bounded* if there exists a circle K such that every point of M lies interior to K.

§C. Kuratowski, *Contribution à l'étude de continus de Jordan*, Fundamenta Mathematicae, V (1924) pp. 112-122.

||If C is a continuum and N is a subset of C, then N is said to *cut* C provided there exist at least two points A and B of the set $C-N$ which do not lie in a subcontinuum of $C-N$.

¶A set is said to be *punctiforme* if it contains no subcontinuum which contains more than one point.

**A set K is said to be *irreducibly connected* from A to B if K is a connected set containing A and B and no proper connected subset of K contains both A and B. (A set N is a *proper* subset of set K provided N is a subset of K and $K-N$ is not vacuous).

††Two point sets M and N are said to be *mutually exclusive* if they have no point in common. If M and N are mutually exclusive and neither contains a limit point of the other, then they are said to be *mutually separated.*

In §2 of this paper is a collection of theorems which I have found necessary to the demonstration of the theorem of §3.

§1. *Theorem* 1. *Let M be a connected point-set such that if N is any connected subset of M, $M-N$ is connected, and let A and B be any two distinct points of M. Then M is the sum of two sets, M_1 and M_2, which are irreducibly connected from A to B, and such that $M_1-(A+B)$ and $M_2-(A+B)$ are mutually separated sets.*

Proof. $M-A$ is connected by hypothesis. The set $M-(A+B)$ is not connected. For if $M-(A+B)$ were connected, then $M-[M-(A+B)]=A+B$ would be connected, which is absurd.

Since $M-(A+B)$ is not connected,

$$M-(A+B)=K+L,$$

where K and L are mutually separated sets.

As $M-A$ is a connected set which is disconnected by the omission of one point, B, it follows* that $K+B$ and $L+B$ are connected sets. Similarly, since $M-B$ is a connected set which is disconnected by the omission of a point A, $K+A$ and $L+B$ are connected sets. Then the sets $K+A+B$ and $L+A+B$ are connected as each is the sum of two connected sets having points in common.

Let

$$K+A+B=M_1, \quad L+A+B=M_2.$$

M_1 and M_2 are irreducibly connected from A to B. For suppose that M_1 is not irreducibly connected from A to B. Then there exists a subset, X, of K, such that M_1-X is connected and contains A and B. By hypothesis, the set $M-(M_1-X)$ must be connected. But

$$M-(M_1-X)=X+L$$

and X and L are mutually separated since K and L are mutually separated. Hence the supposition that M_1 is not irreducibly connected from A to B leads to a contradiction. Similarly, it can be shown that M_2 is irreducibly connected from A to B. Then M is the sum of two sets, M_1 and M_2, which are irreducibly connected from A to B, and such that $M_1-(A+B)$ $[=K]$ and $M_2-(A+B)$ $[=L]$ are mutually separated sets.

Theorem 2. *Let M be a connected set such that if N is any connected subset of M, $M-N$ is strongly connected*†. *Then M is a simple closed curve.*

Proof. By Theorem 1, if A and B are two distinct points of M,

$$M=M_1+M_2, \quad M_1=K+A+B, \quad M_2=L+A+B,$$

*Cf. B. Knaster and C. Kuratowski, *Sur les ensembles connexes,* Fundamenta Mathematicae, II (1921), pp. 206-255, Th. VI.

†A point set K is *strongly* connected if every two points of K lie in a subcontinuum of K.

where M_1 and M_2 are irreducibly connected from A to B and K and L are mutually separated.

K and L are connected*. Since

$$M-L=M_1,$$

M_1 is strongly connected, and A and B lie in a subcontinuum, N, of M_1. As every subcontinuum of a set irreducibly connected between two points is an arc†, N is an arc. Clearly M_1 and N are identical, as otherwise M_1 could not be irreducibly connected from A to B. Hence M_1 is an arc from A to B. Similarly, M_2 is an arc from A to B.

Since M is the sum of two arcs which have only their end-points in common, it is a simple closed curve.

§2. C. Carathéodory has made‡ an admirable analysis of the structure of the boundary of a simply connected domain. In all that follows, I shall assume familiarity with his work, as well as with the work of Miss Marie Torhorst in the same connection§.

Theorem A. *For every interval of prime ends‖ of a simply connected domain G there exist infinitely many distinct points of the boundary of G that are accessible for prime ends of that interval¶.*

Proof. Let E be a prime end of the interval, I, distinct** from the ends of the interval†† I. Let P be a principal point‡‡ of E. Then E can be defined by a chain of cross-cuts§§, q_1, q_2, q_3, ... which lie on concentric circles that converge to their common centre‖‖. There exists a positive integer, n, such that the prime ends corresponding to¶¶ q_i $(i \geqq n)$ are prime ends of I. For every value of i, let E_i be one of the prime ends corresponding to q_i, and P_i the end-point of q_i in E_i. The points P_n, P_{n+1}, P_{n+2}, ... are distinct points satisfying the theorem.

By an argument similar to that used above the following theorem can be established:

*Cf. B. Knaster and C. Kuratowski, *loc. cit.*, Corollaire XXIV.

†Cf. B. Knaster and C. Kuratowski, *loc. cit.*, Corollaire XXVIII.

‡C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete,* Mathematische Annalen 73 (1913), pp. 323-370.

§Marie Torhorst, *Über den Ränder der einfach zusammenhängender Gebiete,* Mathematische Zeitschrift, 6 (1921), pp. 44-65.

‖*Primenden.*

¶It should be noted that this is not the same as stating that every interval of prime ends contains infinitely many prime ends of the first or second kind. This theorem states that every interval of prime ends contains infinitely many prime ends of the first or second kinds whose accessible points are distinct.

**When two prime ends are not identical, I shall call them *distinct.*

††By the *ends of an interval of prime ends* I mean the prime end which precedes all others in the interval and the prime end which is preceded by all others in the interval.

‡‡*Hauptpunkt.*

§§*Querschnittskette.*

‖‖Cf. C. Carathéodory, *loc. cit.*, Satz. VIII.

¶¶Cf. M. Torhorst, *loc. cit.*, p. 49, §§ 26 and 27.

Theorem A′. *Every prime end, E, of a simply connected domain G is the limit of a convergent sequence* of prime ends of G of the first or second kind, whose accessible points are all distinct. If P is a principal point of E, then the prime ends of this sequence can be so chosen that their accessible points have P as a sequential limit point*†.

Definition. Let G be a simply connected domain, M a set of points of G, P a point of the boundary of G, and E a prime end of G that contains P. Then P is a *limit point of M for the prime end E* provided there exists a sequence of points of M, $P_1, P_2, P_3, \ldots$, which converges‡ to the prime end E and has P as a sequential limit point.

Theorem B. *Let M be a subset of a simply connected domain, G, and let P be a limit point of M on the boundary of G. Then there exists a prime end of G, E_P, which contains P, and such that P is a limit point of M for the prime end E_P.*

Proof. Since P is a limit point of M, there exists a sequence of points of M, $P_1, P_2, P_3, \ldots$, which has P as a sequential limit point.

Let G be mapped conformally by means of an analytic function, $f(z)$, on the interior of the unit-circle, $|z|<1$. For every value of n, let the image of P_n in $|z|<1$ be x_n. Then the set of points $x_1, x_2, x_3, \ldots$, has at least one limit point, x, on the unit circle $|z|=1$.

As x is a limit point of $x_1, x_2, x_3, \ldots$, there exists a subsequence of this sequence, $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$, which has x as a sequential limit point. Let E_P be that prime end of G which corresponds to x§. Then the sequence of points $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$, converges to the prime end E_P, and since P is a sequential limit point of this sequence it follows that P is a limit point of M for the prime end E_P.

Theorem C. *Let E be a prime end of a simply connected domain G, and P a point contained in E. Then the chain of crosscuts $q_1, q_2, q_3, \ldots$, which determines E can be so chosen that for any positive integer, n, there exists a circle, K, with centre at P, such that no point of q_n lies either interior to, or on, K.*

Proof. E can be defined by a chain of cross-cuts, $q_1, q_2, q_3, \ldots$, which lie on concentric circles that converge to their common centre. If P is the common centre of these circles, the circle, K, of which q_{n+1} is a portion, satisfies the theorem. If P is not the common centre of these circles, there exists a positive number, N, such that for $i>N$, the circle of which q_i is a portion neither encloses nor contains P. Let k be a positive integer greater than N. Then the chain of cross-cuts $q_k, q_{k+1}, q_{k+2}, \ldots$, and a circle K with centre P and radius less than $\delta(P, C_k)$‖ (where C_k is the circle of which q_k is a portion) satisfy the theorem.

Theorem D. *Let A, B, C, D be four distinct points of the boundary, β, of a simply connected domain G, accessible, respectively, for the prime ends E_A, E_B, E_C, E_D,*

*Cf. C. Carathéodory, *loc. cit.*, p. 351, footnote.

†A point P is a *sequential limit point* of a sequence of points $P_1, P_2, P_3, \ldots$, provided that if, K is any circle with centre at P there exists a positive integer n such that for $i>n$, P_i lies interior to K.

‡Cf. C. Carathéodory, *loc. cit.*, § 11, III.

§Cf. C. Carathéodory, *loc. cit.*, Satz XIII.

‖M and N being point-sets, $\delta(M, N)$ denotes the lower limit of all the distances xy where x is any point of M and y is any point of N.

where E_A and E_B separate E_C and E_D. Let AB and CD be arcs which lie, except for their end-points, A, B, and C, D, entirely in G, and such that the prime ends corresponding to AB are E_A and E_B, and the prime ends corresponding to CD are E_C and E_D. Let t be an arc with end-points A and B and containing no point of CD. Under these conditions, there exists a simple closed curve, J, which consists of a sub-arc of t and a sub-arc of AB, and such that C is interior to J and D exterior to J, or vice versa.

Proof. The arc CD separates G into two domains, G_1 and G_2*, such that E_A and E_B are contained in G_1 and G_2, respectively. Denote the boundary of G_1 by β_1 and the boundary of G_2 by β_2.

Let

$$AB \times G_1 + A = N_1, \quad AB \times G_2 + B = N_2.$$

On t, in the order from A to B, there exists a last point, x, of N_1. To show this, add to N_1 all its limit points, and call the resulting set $\overline{N}_1$†. Since the latter set is closed, there will exist on t, in the order from A to B, a last point of $\overline{N}_1$. Since the only limit points of N_1 that do not belong to N_1 lie on CD, this point must be a point of N_1, by virtue of the fact that t contains no points of CD.

After x, on t, in the order from x to B, let y be the first point of N_2. The existence of y can be established by an argument similar to that used to show the existence of x. The points x and y are distinct; for $x = A$ or x is a point of G_1, and therefore x is not a limit point of N_2, or a point of N_2.

That portion‡ of t from x to y is an arc t_1. That portion of AB from x to y is an arc t_2; t_1 and t_2 have only x and y in common, hence their sum is a simple closed curve J.

On t_1, in the order from x to y, let x' be the first point of β_1. On t_1, in the order from y to x, let y' be the first point of β_2; t_2, together with those portions of t_1 from x to x' and from y to y', forms an arc $x'y'$ which lies, except for x' and y', wholly in G. Let the prime ends of G corresponding to $x'y'$ be $E_{x'}$ and $E_{y'}$, where $x'y'$ converges to x' in $E_{x'}$ and to y' in $E_{y'}$.

$E_{x'}$ and $E_{y'}$ are separated by E_C and E_D, since $E_{x'}$ and $E_{y'}$ are contained in G_1 and G_2, respectively. Therefore there will exist, on CD, in the order from C to D, a first point, w, of $x'y'$, and in the order from D to C a first point z of $x'y'$. Between C and w, and between D and z on CD there can be no points of J, since $J - x'y'$ is a subset of t.

Hence if C and D are both interior (or exterior) to J, all points of the arcs Cw and Dz (subsets of CD), except w and z, are interior (or exterior) to J. No

*Cf. C. Carathéodory, *loc. cit.*, §§ 5-7.

†If M is any point-set, I shall hereafter in this paper use the symbol $\overline{M}$ to denote M together with its limit points.

‡If S is an ordered set of elements, and x and y are elements of S, that portion of S from x to y is the set of all elements $[e]$ such that $x = e$, or $y = e$, or e is between x and y.

point of that portion of t_2 from w to z is a limit point of the rest of J, except w and z, and all such points lie in G. Hence there exist, on Cw and Dz, points w' and z', respectively, which can be joined by an arc $w'z'$ that contains no point of $\beta+J$, and no point of Cw or Dz except w' and z'. The sum of the arcs Cw' (subset of CD), $w'z'$, and $z'D$ (subset of CD), is an arc which joins C and D, lies wholly in G, except for C and D, and converges to the prime ends E_C and E_D, but contains no point of the arc $x'y'$. This is impossible, since E_C and E_D are separated by $E_{x'}$ and $E_{y'}$. It follows that C and D cannot both be interior (or exterior) to J.

Corollary. If A and B separate C and D on a simple closed curve K, AB and CD are arcs joining A, B, and C, D, respectively, and lying, except for their end-points, interior to K, and t is an arc from A to B that contains no point of CD, then there exists a simple closed curve J consisting of a sub-arc of AB and a sub-arc of t, such that C is interior to J and D exterior to J, or vice versa.

The notion of *prime part* of a continuum has been recently introduced by Hans Hahn*. If P is a point of a continuum M, the prime part of M which contains P is the set of all points $[x]$ belonging to M such that for every positive number e there exists a finite set of irregular† points, $P_1, P_2, P_3, \ldots, P_n$, of M such that the distances $PP_1, P_1P_2, P_2P_3, \ldots, P_nx$ are all less than e. A bounded continuum M is a *simple continuous arc of prime parts* provided there exist two distinct prime parts of M, p and q, such that M is disconnected by the omission of any one of its prime parts which is distinct from p and q‡. The prime parts p and q are called the *extremities* of the arc.

Theorem E. *Let E be a prime end of a simply connected domain, G, and let P be any point of G. Then there exists a chain of cross-cuts, $q_1, q_2, q_3, \ldots$, defining E, and a simple continuous arc, t, of prime parts such that (i) the extremities of t are P and ω, the set of principal points of E, (ii) $t-\omega$ is a subset of G, and converges to E, (iii) every prime part of t, except ω, is a point, (iv) for every n, q_n and t have only one point in common.*

Proof. It is; of course, only necessary to show that this theorem is satisfied for some particular point of G and a particular choice of the cross-cuts defining E, since then it easily follows that the theorem is satisfied for all points of G.

Carathéodory shows§ the existence of a domain G', subset of G, and a prime end E' of G' which is contained in E and contains only principal points of E. He then shows|| that the image in G of that radius of the unit circle (upon the interior of which G' has been mapped conformally) which joins the centre to the image of E' is a curve c' which begins in a point P of G' and converges to E' and to E. The only points of the boundary of G that are limit points of c'

*Hans Hahn, *Über irreduzible Kontinua*, Wiener Berichte, 130 (1921), pp. 217-250.

†An *irregular* point of M is a point at which M is not connected im kleinen. Cf. H. Hahn, Wiener Berichte, 123 (1914) (2433).

‡For a more general definition of simple continuous arc of prime parts, see R. L. Moore, *Concerning the prime parts of certain continua which separate the plane*, Proc. Nat. Acad. Sci., 10 (1924), pp. 170-175.

§*loc. cit.*, p. 361, Satz XIX.

||*loc. cit.*, p. 362, § 43.

are the points of ω, and every one of these is a limit point of c'. Let $c'+\omega=c$. The curve c is a simple continuous arc of prime parts whose extremities are P and ω. If E is a prime end of the first or second kind, c is of course a simple continuous arc in the ordinary sense. If E is a prime end of the third or fourth kind, then ω is a prime part of c. This is easily shown as follows: ω being a continuum*, and c clearly connected im kleinen at all points except those belonging to ω, it is only necessary to show that c is not connected im kleinen at any point of ω. Suppose c were connected im kleinen at a point x of ω. Let y be a point of ω distinct from x. Let K_1 be a circle with centre x and radius less than $\delta(x, y)$. Then there exists a circle K_2 concentric with K_1 and such that if z is any point of c interior to K_2, z can be joined to x by an arc zx of c which lies wholly interior to K_1. Let z be a point of $c-\omega$, and let z' be the first point of ω on zx in the order from z to x. That portion of zx from z to z' is an arc zz' which lies, except for z', wholly in G. The prime end corresponding to zz' is E, since every subset of c which has a limit point on the boundary of G converges to E. Then z' is accessible for E and $\omega=z'$. Thus the supposition that c is not connected im kleinen at x leads to a contradiction.

The proof that every prime part of t except ω, is a point, and that t is a simple continuous arc of prime parts, is only dependent on the fact that the conformal mapping of G' upon the interior of the unit circle establishes a one-to-one continuous correspondence between the points of G' and the points interior to the unit circle.

It remains to show that the chain of cross-cuts defining E can be so chosen as to satisfy (iv). It will, however, be simpler to first choose the chain of cross-cuts defining E and then modify the simple continuous arc of prime parts c. Let x be any point of ω and let $q_1, q_2, q_3, \ldots$, be a chain of cross-cuts (defining E) which consist of portions of circles that converge to their common centre, x. With no loss of generality it can be assumed that P does not lie in g_1, where g_1 is the first domain of the chain of domains (Gebietskette) determined by the above chain of cross-cuts. On c, in the order from P to ω, let P_1 be the first point of q_1, and P_1' the last point of q_1; after P_1' let P_2 be the first point of q_2 and P_2' the last point of q_2; after P_2' let P_3 be the first point of q_3 and P_3' the last point of q_3; and so on. Let P_1'' be a point of the arc $P_1'P_2$ (subset of c) which is distinct from P_1' and P_2 and at a distance less than 1 from P_1'. Then P_1 can be joined to P_1'' by an arc P_1P_1'' which lies, except for P_1, wholly in g_1, contains no point of $P_1'P_2$ other than P_1'', and is such that every point of P_1P_1'' is at a distance less than 1 from some point of the arc P_1P_1' (subset of q_1). In general, if P_n'' is a point of the arc $P_n'P_{n+1}$ at a distance from P_n' less than $1/n$, there exists an arc P_nP_n'' joining P_n and P_n'' which lies, except for P_n, wholly in g_n (the n^{th} domain of the chain of domains determined by the chain of cross-cuts $q_1, q_2, q_3, \ldots$), contains no point of $P_n'P_{n+1}$ other than P_n'', and such that every point of P_nP_n'' is at a distance less than $1/n$ from some point of the arc P_nP_n' (subset of q_n). The sum of the arcs PP_1, P_1P_1'', $P_1''P_2$ (subset of $P_1'P_2$), $\ldots$, P_nP_n'', $P_n''P_{n+1}$ (subset of $P_n'P_{n+1}$), $\ldots$, together with the set ω, is an arc t

*Cf. C. Carathéodory, *loc. cit.*, Satz XX.

of prime parts. The conditions of the theorem are satisfied by t and the chain of cross-cuts $q_1, q_2, q_3, \ldots$.

Corollary. Let E_1 and E_2 be distinct prime ends of a simply connected domain, ω and ω' the sets of principal points of E_1 and E_2, respectively, and suppose that ω and ω' are mutually exclusive. Then there exist a simple continuous arc, t, of prime parts and chains of cross-cuts defining E_1 and E_2 such that (i) the extremities of t are ω and ω', (ii) $t-\omega-\omega'$ is a subset of G, (iii) every prime part of t, except ω and ω', is a point, (iv) $t-\omega-\omega'$ approximates ω in E_1 and ω' in E_2, (v) every cross-cut of the chains defining E_1 and E_2 has only one point in common with t.

Definition. A set having the properties of t as stated in this corollary will be called a *cross-cut of prime parts.* If either ω or ω' reduces to a single point, the set will still be called a cross-cut of prime parts. If both ω and ω' reduce to a single point, the set will be called, as usually, a cross-cut.

Theorem F. *If K is a cross-cut of prime parts of a simply connected domain G, then K divides G into two mutually exclusive domains, G_1 and G_2.*

Proof. Let G be mapped conformally by means of an analytic function $f(z)$ on the interior of the unit circle, $|z|<1$. The image of K, together with its limit points on the circle $|z|=1$, is an arc* which lies, except for its end-points, wholly interior to the unit circle $|z|=1$. This arc divides the interior of the unit circle into two mutually exclusive domains. The remainder of the proof should be obvious.

The following theorem can now be established by means of an argument similar to that used in proving Theorem D:

Theorem D′. *Let AB and CD be a cross-cut and a cross-cut of prime parts, respectively, of a simply connected domain, G, such that the prime ends E_A and E_B corresponding to AB are separated by the prime ends E_C and E_D corresponding to CD. Let the end-points of AB be A and B, and the extremities of CD be ω and ω', and suppose that A, B, ω and ω' are mutually exclusive point-sets. Let t be an arc with end-points A and B and containing no point of CD. Under these conditions there exists a simple closed curve J, which consists of a sub-arc of t and a sub-arc of AB, and is such that ω is interior to J and ω' exterior to J, or vice versa.*

Theorem G. *Let E_1 be a prime end of a simply connected domain G, A and B distinct points contained in E_1, and C a set irreducibly connected from A to B such that $C-A-B$ is a subset of G and A and B are limit points of $C-A-B$ only for the prime end E_1. Let K be a cross-cut of prime parts of G, such that the prime ends corresponding to K are E_1 and E_2; K satisfies the conditions siated in the corollary of Theorem E, and neither A nor B is a principal point of E_2. Then if G_1 and G_2 are the domains into which K separates G, A and B are not both limit points of $G_1\times(C-A-B)$, or of $G_2\times(C-A-B)$.*

Proof. Suppose both A and B are limit points of $G_2\times(C-A-B)$. Let x and y be distinct points of $C-A-B$, and let the order of x, y, A, and B on C be† A, x, y, B. Let those portions of C from A to x and from y to B

*Cf. C. Carathéodory, *loc. cit.*, Satz XIII.

†Cf. B. Knaster and C. Kuratowski, *loc. cit.*, Th. XXIII.

be denoted by C_1 and C_2, respectively. Then A is not a limit point of C_2 and B is not a limit point of C_1*. But A is a limit point of $C_1 \times G_2$ and B is a limit point of $C_2 \times G_2$.

Let $q_1, q_2, q_3, \ldots$, be the chain of cross-cuts defining E_1, and composed (see proof of Theorem E) of portions of circles that converge to their common centre, w, a point of ω (the set of principal points of E_1). It can be assumed that for every n, E_2 is not contained in g_n (where $g_1, g_2, g_3, \ldots$, is the chain of domains determined by the chain of cross-cuts defining E_1). It can also be assumed that q_n does not have A or B as an end-point.

A and B are secondary points (Nebenpunkte) of E_1. For, since A and B are limit points of $C-A-B$ only for the prime end E_1, A and B are limit points of C_1 and C_2, respectively, for the prime end E_1 (Theorem B); and were A, say, a principal point of E_1, then would A be a limit point of C_2†, which is impossible. As, in addition, neither A nor B is a principal point of E_2, A and B are not points of K.

Hence there exists a circle K_A, with centre A, such that if R_A is the circular domain‡ bounded by K_A,

$$\bar{R}_A \times (K + q_n + C_2 + B) = 0; \quad (n = 1, 2, 3, \ldots),$$

and a circle K_B, with centre B, such that if R_B is the circular domain bounded by K_B,

$$\bar{R}_B \times (K + q_n + C_1 + \bar{R}_A) = 0; \quad (n = 1, 2, 3, \ldots).$$

For every positive integer n, that portion of K from P_n, the point common to q_n and K, to ω separates g_n into two mutually exclusive domains, $g_n^{(1)}$ and $g_n^{(2)}$ (Theorem F), which are subsets of G_1 and G_2, respectively. Then A is a limit point of $C_1 \times g_n^{(2)}$ and B a limit point of $C_2 \times g_n^{(2)}$.

Accordingly there exists, common to $g_1^{(2)}$ and R_A, a point A_1 of C_1, and common to R_B and $g_1^{(2)}$ a point B_1 of C_2.

There exists a positive integer, i, such that neither g_i nor q_i contains A_1 or B_1. Common to $g_i^{(2)}$ and R_A there exists a point A_2 of C_1 and common to $g_i^{(2)}$ and R_B there exists a point B_2 of C_2.

Since

$$C_1(\bar{C}_2 + \bar{R}_B) = 0,$$

A_1 and A_2 lie in a connected subset of G, viz., $C_1 - A$, which contains no point of the closed set $\bar{C}_2 + \bar{R}_B$, and hence (as will be demonstrated in Theorem H) A_1 and A_2 are the end-points of an arc, a_1, which lies wholly in G and contains no point of the set $\bar{C}_2 + \bar{R}_B$. Similarly, B_1 and B_2 are the end-points of an arc, b_1, which lies wholly in G and contains no points of $\bar{C}_1 + \bar{R}_A + a_1$.

On a_1, in the order from A_1 to A_2, let A_1' be the first point of K_A, and A_2' the last point of K_A. Such points will exist since $\bar{R}_A$ and q_i have no points in common. A_1' is a point of G_2, but not of $g_i^{(2)}$, and A_2' is a point of both G_2

*Cf. B. Knaster and C. Kuratowski, *loc. cit*,. Th. XIX.

†Cf. C. Carathéodory, *loc. cit.*, § 43.

‡*i.e.*, the bounded domain whose boundary is K_A.

and $g_i^{(2)}$ Let H_1 be that maximal connected subset* of $K_A \times G$ determined by A_1', and H_2 that maximal connected subset of $K_A \times G$ determined by A_2'. H_2 and H_1 are subsets of $g_i^{(2)}$ and $G_2 - g_i^{(2)}$, respectively.

There exists a subset, t_1, of $a_1 + \overline{H}_1 + \overline{H}_2$ which has the following properties: (i) t_1 is a cross-cut of G, (ii) if $E_i^{(2)}$ is that prime end of G_2 which corresponds to q_i and is contained in G_2, the prime ends $E_A^{(1)}$ and $E_A^{(2)}$ corresponding to t_1 are separated by E_1 and $E_i^{(2)}$; furthermore, $E_A^{(1)}$ and $E_A^{(2)}$ are prime ends of G contained in G_2, and indeed, $E_A^{(2)}$ is contained in $g_i^{(2)}$, but $E_A^{(1)}$ is not contained in $g_i^{(2)}$.

In a similar manner, there can be shown to exist a cross-cut t_2 of G, subset of $a_2 + K_B$, which has the following properties: (i) the prime ends of G corresponding to t_2, $E_B^{(1)}$ and $E_B^{(2)}$, are separated by E_1 and $E_i^{(2)}$; (ii) $E_B^{(1)}$ and $E_B^{(2)}$ are contained in G_2, and $E_B^{(2)}$ is contained in $g_i^{(2)}$; (iii) t_1 and t_2 have no point in common. Property (iii) is a result of the fact that $a_1 + K_A$ and $a_2 + K_B$ have no points in common.

The prime ends E_1, E_2, $E_i^{(2)}$, $E_A^{(1)}$, $E_A^{(2)}$, $E_B^{(1)}$ and $E_B^{(2)}$ must occur in one of the following cyclical orders:

(1) $E_1, E_A^{(2)}, E_B^{(2)}, E_i^{(2)}, E_A^{(1)}, E_B^{(1)}, E_2, E_1,$

(2) $E_1, E_B^{(2)}, E_A^{(2)}, E_i^{(2)}, E_A^{(1)}, E_B^{(1)}, E_2, E_1,$

(3) $E_1, E_A^{(2)}, E_B^{(2)}, E_i^{(2)}, E_B^{(1)}, E_A^{(1)}, E_2, E_1,$

(4) $E_1, E_B^{(2)}, E_A^{(2)}, E_i^{(2)}, E_B^{(1)}, E_A^{(1)}, E_2, E_1.$

In orders (1) and (4) $E_A^{(1)}$ and $E_A^{(2)}$ are separated by $E_B^{(1)}$ and $E_B^{(2)}$; but t_1 and t_2 have no point in common and hence these orders are impossible.

Consider order (2). Let j be a positive integer such that g_j contains no point of t_2. Let D_1 and D_2 be the domains into which t_2 divides G, D_1 being that one of these domains that contains g_j. As t_1 contains points of H_1, and H_1 contains no points of t_2, H_1 is a subset of D_2. $C_1 - A$ contains a point of H_1, viz., A_1'. Also $C_1 - A$ contains a point, A_3, of g_j. Then $C_1 - A$ is a connected subset of G that contains a point of D_1 and a point of D_2, but no point of t_2. This is impossible. Hence order (2) is impossible.

In a similar manner it can be shown that order (3) is impossible.

Hence the supposition that A and B are both limit points of $G_2 \times (C - A - B)$ leads to a contradiction. In a similar manner it can be shown that the supposition that A and B are both limit points of $G_1 \times (C - A - B)$ leads to a contradiction.

Theorem H. *Let G be a bounded domain, K any closed set of points and N a connected subset of G which contains no points of K. Then every pair of distinct points of N are the end-points of an arc which lies in G and contains no point of K.*

Proof. Let β be the boundary of G; β lies interior to some simple closed curve J. Let I be the interior of J. Then $G + \beta$ is a subset of I†. The set

*If M is a point-set, and P a point of M, that maximal connected subset of M determined by P is the set of all points $[x]$ of M such that x and P lie in a connected subset of M.

†Cf. R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, 15 (1922), pp. 254-260, Lemma 2.

of points $J+I-G$ is a closed set L. The set of points which K has in common with $J+I$ is a closed set M. Then $L+M$ is a closed subset of $J+I$. As N has no points in common with $L+M$, it is clear that N is a connected subset of an open subset, $T[=J+I-(L+M)]$, of $J+I$, and hence every two points of N are the end-points of an arc of T*. As T is a subset of G the theorem is proved.

The following lemma was proved by B. Knaster and C. Kuratowski†:

Lemma 1. *M and N being two mutually separated point-sets, at least one of which is bounded, and A and B being two points of M and N, respectively, there exists a bounded continuum, K, which cuts‡ the plane between A and B and which contains no point of $M+N$.*

Theorem I. *If M and N are two mutually separated connected point-sets, atl east one of which is bounded, there exists a simply connected bounded domain, G, which contains M, but does not contain N, or vice versa.*

Proof. If A and B are points of M and N, respectively, then by the above lemma there exists a bounded continuum, K, which cuts the plane between A and B, and hence between M and N. As only one of the domains complementary to a bounded continuum is unbounded, one of the sets M, N, say M, lies in a bounded domain, G, complementary to K. That the boundary of G is connected has been proved by Brouwer§. That G is simply connected follows from a theorem proved by R. L. Moore||.

§3. *Theorem* 3. *In a plane, S, let C_1 and C_2 be point-sets which are irreducibly connected from A to B (points of S), and such that $C_1-(A+B)$ and $C_2-(A+B)$ are mutually separated. Then the set C_1+C_2 cuts the plane S.*

Proof. I. Let one of the sets C_1, C_2, be bounded.

Let

$$C_1-(A+B)=F_1,\ C_2-(A+B)=F_2.$$

That F_1 and F_2 are connected sets has been proved by Knaster and Kuratowski¶.

By Theorem I there exists a simply connected bounded domain, G, which contains, say, F_1, but contains no points of F_2, and whose boundary, β, is a continuum which cuts the plane between F_1 and F_2.

A and B are points of β, since A and B are both limit points of F_1 and F_2, and hence of point-sets interior and exterior to G. Then there exists a prime

*Cf. R. L. Moore, *Concerning continuous curves in the plane, loc. cit.*, Th. 1.

†*loc. cit.*, p. 233, th. 37. Although in the statement of their theorem, Messrs. Knaster and Kuratowski do not mention the boundedness of the continuum K, it clearly follows from their proof.

‡A point-set K *cuts* the plane between two points A and B that do not belong to K if there exists no subcontinuum of $S-K$ (where S is the set of all points in the plane) that contains A and B.

§L. E. J. Brouwer, *Beweis des Jordanschen Kurvensatzes*, Mathematische Annalen 69 (1910), pp. 169-175.

||*Concerning Continuous Curves in the Plane, loc. cit.*, Th. 2.

¶*loc. cit.*, Corollaire XXIV.

end*, E_A, which contains A and is such that A is a limit point of F_1 for this prime end (Theorem B). Similarly, there exists a prime end E_B which contains B, and is such that B is a limit point of F_1 for this prime end.

(*a*) Suppose E_A and E_B are distinct prime ends. Then they are the ends of two intervals of prime ends, I_1 and I_2, which have in common only E_A and E_B. By Theorem A, there exists in I_1 a prime end E_C which is distinct from E_A and E_B, and which contains a point C that is accessible for E_C and is distinct from A and B. In I_2 there exists a prime end E_D which is distinct from E_A, E_B, and E_C, and which contains a point D that is accessible for E_D and is distinct from A, B, and C. C and D are the end-points of an arc, a, which lies, except for C and D, wholly in G, and which converges to each of the prime ends E_C, E_D.

I shall show that M cuts the plane between C and D. For suppose there exists, in $S-M$, a continuum, K, which contains C and D. Then the set $a+K$ forms a continuum, N. A and B must lie in the same domain complementary to N, since they lie in a connected set, C_2, which contains no point of N.

E_C and E_D are the ends of two intervals of prime ends, J_1, of which E_A is a prime end, and J_2, of which E_B is a prime end. Let $q_1, q_2, q_3, \ldots,$ be a chain of cross-cuts defining E_A, such that for any positive integer i there exists a circle with centre at A which neither contains nor encloses any point of q_i. (Theorem C). Let $g_1, g_2, g_3, \ldots,$ be the corresponding chain of regions. Then there exists a positive integer, n, such that the prime ends corresponding to q_n are prime ends of J_1 distinct from E_C and E_D, and such that neither g_n nor q_n contains points of a.

Similarly, let $t_1, t_2, t_3, \ldots,$ be a chain of cross-cuts defining E_B, such that for every positive integer i there exists a circle with centre at B which neither contains nor encloses any point of t_i. Let $u_1, u_2, u_3, \ldots,$ be the corresponding chain of regions. Then there exists a positive integer m such that the prime ends corresponding to t_m are prime ends of J_2 distinct from E_C and E_D, and such that neither t_m nor u_m has points in common with $a+g_n+q_n$.

Let R_A be a circular domain with centre† A and of such a radius that $\bar{R}_A$ contains no point of the set $N+q_n+B$. Let R_B be a circular domain with centre B and of such a radius that $\bar{R}_B$ contains no point of the set $N+t_m+\bar{R}_A$.

Since A is a limit point of F_1 for the prime end E_A, there is a point of F_1, say A_1, which is common to g_n and R_A. Similarly, there is a point of F_1, say B_1, common to u_m and R_B. A_1 and B_1 are distinct points, since R_A and R_B are mutually exclusive. As F_1 is a connected subset of G which contains A_1 and B_1 and contains no point of K, there exists, in G, by virtue of Theorem H, an arc, c, which has A_1 and B_1 as its end-points, and contains no point of K.

Join A and A_1 by a straight line interval, s_1. Let A_2 be the last point of the set $c\times g_n$ on s_1 in the order from A_1 to A. That such a point exists is easily shown as follows:

Let

$$L=\bar{R}_A\times\bar{g}_n\times c.$$

*Hereafter, unless otherwise indicated, "prime end" will be understood to mean "prime end of G".

†By the *centre* and *radius* of a circular domain R are meant the centre and radius, respectively, of the circle which forms the boundary of R.

L is a closed set, since it is the set of points common to three closed sets. Also, L is non-vacuous, since A_1 is a point of L. Let A_2 be the last point of L on s_1 in the order from A_1 to A. A_2 is not a point of the boundary of g_n, since the only points of this boundary that c contains are points of q_n, and no points of q_n lie in R_A. Hence A_2 is a point of c in g_n. That it is the last point of $c \times g_n$ on s_1 in the order from A_1 to A is obvious.

After A_2, on s_1, in the order from A_1 to A, let A' be the first point of β.

That portion of c from B_1 to A_2, together with that portion of s_1 from A_2 to A', is an arc $A'B_1$, which lies, except for A', wholly in G. Let the prime end which corresponds to this arc be $E_{A'}$. $E_{A'}$ is contained in g_n and is therefore a prime end of the interval J_1, distinct from the ends of J_1.

Join B and B_1 by a straight line interval, s_2. Let B_2 be the last point of the set $A'B_1 \times g_n$ on s_2 in the order from B_1 to B. After B_2, on s_2, in the order from B_1 to B, let B' be the first point of β.

That portion of $A'B_1$ from A' to B_2, together with that portion of s_2 from B_2 to B', is an arc $A'B'$. Besides the prime end $E_{A'}$, there corresponds to this arc a prime end $E_{B'}$, containing B', and belonging to the interval J_2, and distinct from the ends of J_2.

A' and A are in the same domain complementary to N, since R_A contains no points of N. Similarly, B' and B are in the same domain complementary to N. Hence, as A and B lie in the same domain complementary to N, A' and B' lie in the same domain complementary to N.

Hence there exists an arc, t, which has A' and B' as end-points, and which has no points in common with N. *A fortiori*, t and a have no points in common. Hence by Theorem D there exists a simple closed curve, J, which consists entirely of points of t and $A'B'$, and is such that C lies interior to J and D exterior to J, or vice versa. But K has no points in common with t or $A'B'$, and therefore no points in common with J. This is absurd, since K contains both C and D. Thus the supposition that C and D lie in a subcontinuum of $S-M$ must be false and M cuts the plane between C and D.

(*b*) Suppose E_A and E_B are the same prime end E_{AB}. If either A or B is a limit point of F_1 for any prime end distinct from E_{AB}, a method of argument similar to that in (*a*) may be used. If A and B are limit points of F_1 only for the prime end E_{AB}, we may proceed as follows:

Let C be a principal point of E_{AB}. Then C is distinct from A and B, since A and B are secondary points of E_{AB}, as shown in the proof of Theorem G. Let D be a point of β which is not a principal point of E_{AB}, which is distinct from A and B, and which is accessible for a prime end E_D that is distinct from E_{AB}*.

By the corollary of Theorem E there exists a cross-cut of prime parts, a,

*The existence of such a point is easily shown as follows: As E_{AB} contains secondary points, let x be a secondary point of E_{AB} distinct from A and B. Let R be a circular domain with centre x and of such a radius that it contains no principal point of E_{AB}, and neither of the points A, B. R contains some point, y, of G. On a straight line interval joining x and y, let D be the first point of β in the order from y to x.

which has the following properties: (i) ω (the set of principal points of E_{AB}) and D are the extremities of a, (ii) the set $a-\omega-D$ approximates ω in E_{AB} and D in E_D. By Theorem F, a separates G into two mutually exclusive domains G_1 and G_2, and by Theorem G, A and B are not both limit points of one of the sets $F_1 \times G_1$, $F_1 \times G_2$. Hence A is a limit point of $F_1 \times G_1$, say, but not of $F_1 \times G_2$, and B is a limit point of $F_1 \times G_2$, but not of $F_1 \times G_1$.

Suppose that there exists a continuum, K, which contains C and D, but contains no points of M, and let $a+K=N$.

Let K_A be a circular domain with centre A and of such a radius that $\overline{K}_A$ contains no points of the set $N+B+F_1 \times G_2$. Let K_B be a circular domain with centre B and of such a radius that $\overline{K}_B$ contains no points of the set $N+\overline{K}_A+F_1 \times G_1$. Let A_1 and B_1 be points of F_1 in K_A and K_B, respectively. There exists, in G, by Theorem H, an arc c which has A_1 and B_1 as end-points and contains no point of K.

Join A and A_1 by a straight line segment interval, s_1. Let A_2 be the last point of $c \times G_1$ on s_1 in the order from A_1 to A, and after A_2 let A' be the first point of β. That portion of c from B_1 to A_2, together with that portion of s_1 from A_2 to A' is an arc $A'B_1$, which lies, except for A', wholly in G. The prime end, $E_{A'}$, which corresponds to this arc, is a prime end of G distinct from E_{AB} and E_D. Furthermore, $E_{A'}$ is a prime end of G_1, since A_1 is a point of G_1.

Join B and B' by a straight line s_2. Let B_2 be the last point of $A'B_1 \times G_2$ on s_2 in the order from B_2 to B, and after B_2 let B' be the first point of β. That portion of $A'B_1$ from A' to B_2, together with that portion of s_2 from B_2 to B', is an arc $A'B'$. Besides $E_{A'}$, there corresponds to this arc a prime end $E_{B'}$, distinct from E_{AB} and E_D and also a prime end of G_2.

As $E_{A'}$ and $E_{B'}$ are prime ends of G_1 and G_2, respectively, as well as prime ends of G, they are separated by E_{AB} and E_D.

A' and B' lie in the same domain complementary to N, and hence there exists an arc, t, which has A' and B' as end-points and contains no points of N.

(The remainder of the proof is similar to that of (a), except that Theorem D′ is used to obtain the contradiction instead of Theorem D.)

II. If neither of the sets C_1, C_2, is bounded, proceed as follows:

Let x be a point of C_1-A-B, and let R be a circular domain with centre x and containing no points of C_2. Let P be a point of R which does not belong to C_1, and let K_P be a circle with centre P and lying wholly in R. By an inversion of the plane, S, about K_P, C_2 goes into a set C_2', interior to K_P and therefore bounded. If M' denotes the set into which C_1+C_2 goes under this inversion, the proof of I may be used to show that M' cuts the plane between two points C' and D' of $S-M'$, neither of which is identical with P, and hence if the images under the inversion of C' and D' are C and D, respectively, M cuts the plane between C and D.

In selecting the points C' and D' the centre of K_P must be avoided. In case (a) of I this causes no difficulty if Theorem A is applied. In case (b) of I, if E_{AB} is a prime end of the fourth kind, C' may be selected as a principal point of ω distinct from P, the centre of K_P. However, if E_{AB} is of the second kind, P may be the principal point of E_{AB}. Then the proof of I (*b*) shows that M' cuts the plane between P and some point D'. Let R_P be a circular domain with centre P and of such a radius that R_P contains neither A' nor B', the images, respectively, of A and B. There exist points of β in R_P which are joined to P by continua of β that lie wholly in R_P and hence contain neither A' nor B', and therefore no point of M'*. Let C' be such a point. Then M' cuts the plane between C' and D'.

*Cf. Anna M. Mullikin, *Certain theorems relating to plane connected point-sets*, Trans. Amer. Math. Soc., 24 (1922), pp. 144-162.

ON A GENERALIZATION OF FABRY'S AND SZÁSZ'S THEOREMS CONCERNING THE SINGULARITIES OF POWER SERIES

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The analytic function defined by the power series

$$f(z)=\sum_{n=1}^{\infty} a_n z^n,\ a_n=|a_n|e^{i\phi_n},\quad \overline{\lim_{n\to\infty}}\ |a_n|^{\frac{1}{n}}=1, \tag{1}$$

has one or more singularities on the circle of convergence. The number, position and quality of these singular points are functions of the infinite number of variables $a_1, a_2, a_3, \ldots.$

It has been shown by the author in some recently published memoirs,* that all the investigations concerning this subject can be considerably simplified and at the same time generalized by means of the so-called L series of coefficients

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots.$$

chosen from the set $a_1, a_2, a_3, \ldots$ in such a manner that

$$\lim_{q\to\infty} |a_{n_q}|^{\frac{1}{n_q}}=1.$$

Clearly every set $a_1, a_2, \ldots$ given by (1) includes an unlimited number of L series.

In the memoirs just cited the author has generalized the well known theorems of Vivanti-Dienes, Fatou-Polya and of Hadamard concerning singularities on the circle of convergence. The first and a special case of the third generalizations are reprinted as the second and third auxiliary theorems in part I of this memoir. The present communication is a further addition to this general theory based upon the systematic use of L series.

It seems quite probable that the results obtained by this method are an individual property of power series and cannot be extended to more general Dirichlet's series.

*(*a*) Rendiconti dei Lincei. XXXII (5), 1° sem. (1923) *Sur les singularités der séries entières*, p. 26-29. *Nouveaux théorèmes sur les singularités des séries entières*, p. 83-85. *Sur les singularites des séries entières*, p. 528-531.

(*b*) Rozpravy ces. akademie Praha. XXXII, tr. II, c. 35 (1923). *O singularitách rady mocninné lezících na krunici konvergencni*, p. 1-15. See also the extract in French: Bulletin internat. de l'Académie des Sciences de Bohême (1923), *Sur les singularités des séries entières situées sur le cercle de convergence*, p. 1-3.

I. Auxiliary Theorems.

We use in the following some auxiliary theorems. The *first* is Hadamard's* multiplication theorem:

First Auxiliary Theorem: If two analytic functions are defined by the convergent power series

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

$$\psi(x) = b_0 + b_1 x + b_2 x^2 + \ldots,$$

the only singularities of the function

$$f(x) = a_0 b_0 + a_1 b_1 x + a_2 b_2 x^2 + \ldots$$

will be points whose affixes γ_{ij} are the product of affixes of the singular points α_i and β_j of the first two functions.

For our purpose the proof of a special form of this theorem due to Pringsheim† is quite sufficient.

The *second* and *third* auxiliary theorems‡ are as follows, the *third* being a generalization of the well-known Hadamard's theorem§, concerning power series with an unlimited number of zero coefficients:

Second Auxiliary Theorem: If corresponding to some L series $a_{n_1}, a_{n_2}, \ldots, a_{n_q}, \ldots$, chosen from the coefficients of (1), *a series of angles $\psi_1, \psi_2, \ldots, \psi_q, \ldots$ can be selected in such a manner, that*

$$(a) \qquad \lim_{q\to\infty} \left\{ \cos(\phi_{n_q} + \psi_q) \right\}^{\frac{1}{n_q}} = 1,$$

$$(b) \qquad \cos\{\phi_n + \psi_q\} \geqq 0,$$

for all indices n satisfying one of the inequalities

$$n_q(1-\mu) \leqq n \leqq n_q(1+\mu), \quad (q = 1, 2, 3, \ldots),$$

where μ is a positive constant independent of q, then the point $z=1$ is a singularity of the function (1).

Third Auxiliary Theorem: If an L series $a_{n_1}, a_{n_2}, \ldots, a_{n_q}, \ldots$, derived from the coefficients of (1) *exists, which has the property, that*

$$a_n = 0$$

for all indices n satisfying one of the inequalities

$$n_q(1-\mu) \leqq n \leqq n_q(1+\mu), \quad (q = 1, 2, 3, \ldots),$$

where μ is a positive constant independent of q, with the sole exception of the central coefficients a_{n_q}, then the circle of convergence is the natural boundary of the function (1).

*Acta Mathematica 22 (1898), p. 55.

†Münchener Berichte (1912), p. 11-92.

‡Consult for proof (*b*), p. 12 and 14, theorems (I) and (II) for $r_q = 0$ or Bulletin intern., p. 2-3 theorems (I) and (II).

§Jour. de Math. (4), vol. 8 (1892), p. 101-186.

It is obvious from this, that every coefficient of the L series used, *e.g*,, the coefficient $a_{n_{10}}$, as non-vanishing can be a member of only one group defined by

$$n_{10}(1-\mu) \leqq n \leqq n_{10}(1+\mu),$$

which fact involves the easily verified consequence

$$n_{q+1} \geqq n_q(1+\mu_1), \quad (q=1, 2, 3, \ldots),$$

where μ_1 is a positive constant dependent only on μ.

The *fourth* auxiliary theorem, which will be stated at the end of this section after we have proved a preliminary result, refers to a type of integral functions. Suppose that the positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_\mu, \ldots$ have the properties

$$\frac{\lambda_\nu}{\nu} \to \infty, \lambda_{\nu+1}-\lambda_\nu \geqq 1, \quad (\nu=1, 2, 3, \ldots).$$

Then the integral function

$$g(x)=\prod_{\nu=1}^{\infty}\left(1-\frac{x^2}{\lambda_\nu^2}\right)$$

is of the "Minimaltypus der Ordnung eins"* and therefore

$$|g(x)| < e^{\delta|x|},$$

where δ is some positive constant independent of x, if only $|x|$ is greater than a suitably chosen $R(\delta)$. The consequence of this is

$$\text{(2)} \qquad \overline{\lim_{|x|\to\infty}} |g(x)|^{\frac{1}{|x|}} \leqq 1.$$

If now a series of positive numbers

$$a_1 < a_2 < a_3 < \ldots a_n < \ldots$$

s given in such a way, that†

$$a_n \to \infty, |a_n - \lambda_\nu| \geqq \kappa, \quad (n, \nu=1, 2, 3, \ldots),$$

then we can prove the following inequality‡:

$$|g(a_n)|^{\frac{1}{a_n}} > e^{-\delta},$$

where δ is as small a positive number as desired, if only n is sufficiently great.

To prove this preliminary theorem we first observe that for some given a_m only one index n can verify the relation

$$\lambda_{n-1} < a_m < \lambda_n.$$

*Cf. Pringsheim, *loc. cit.*, p. 85, 87-88.

†κ is a positive constant independent of n and ν.

‡This inequality represents a remarkable property of the function $g(x)$. In the special case $\frac{a_m}{\lg m} \to \infty$ the inequality has been proved by Szász: see Mathematische Annalen 85 (1922) s. 99-110, *Ueber Singularitäten von Potenzreihen und Dirichletschen Reihen e.s.o.*

We have now

$$|g(a_m)|^{-1}=\prod_{\nu=1}^{n-1}\frac{1}{\dfrac{a_m^2}{\lambda_\nu^2}-1}\cdot\prod_{\nu=n}^{\infty}\frac{1}{1-\dfrac{a_m^2}{\lambda_\nu^2}}$$

$$\leqq\prod_{\nu=1}^{n-1}\frac{\lambda_\nu}{a_m-\lambda_\nu}\cdot\prod_{\nu=n}^{\infty}\left(1+\frac{a_m^2}{(\lambda_\nu+a_m)(\lambda_\nu-a_m)}\right)$$

$$\leqq\frac{a_m^{n-1}}{\kappa(\kappa+1)\ldots(\kappa+n-2)}\cdot\prod_{\nu=n}^{\infty}\left(1+\frac{a_m^2}{\lambda_\nu(\kappa+\nu-n)}\right).$$

To simplify the proof* further we introduce the series of auxiliary numbers

$$\eta_m=\operatorname{Max}\frac{\nu}{\lambda_\nu},\quad \nu\geqq m,\quad (m=1,\ 2,\ 3,\ \ldots.).$$

Since

$$\frac{\nu}{\lambda_\nu}\to 0,$$

it is obvious that

$$\eta_m\to 0 \text{ if } m\to\infty.$$

It follows that

$$\frac{\nu}{\lambda_\nu}\leqq\eta_n \text{ if } \nu\geqq n,$$

and therefore

$$\frac{1}{\lambda_\nu}\leqq\frac{\eta_n}{\nu},$$

which gives

$$\prod_{\nu=n}^{\infty}\left(1+\frac{a_m^2}{\lambda_\nu(\kappa+\nu-n)}\right)\leqq\prod_{\nu=n}^{\infty}\left(1+\frac{(a_m\sqrt{\eta_n})^2}{\nu(\nu+\kappa-n)}\right).$$

If now the number n is so great, that

$$n\kappa>1,\ n+\kappa-2\geqq 0,$$

then putting $\nu=n+\mu$ we find

$$(n+\mu)\ (\kappa+\mu)>(\mu+1)^2$$

and therefore

$$\prod_{\nu=n}^{\infty}\left(1+\frac{(a_m\sqrt{\eta_n})^2}{\nu(\kappa+\nu-n)}\right)<\prod_{\mu=1}^{\infty}\left(1+\frac{(a_m\sqrt{\eta_n})^2}{\mu^2}\right)=\frac{\sin\pi i a_m\sqrt{\eta_n}}{\pi i a_m\sqrt{\eta_n}}.$$

see Göttinger Nachrichten Math. Phys. Klasse 1921; *Neuer Beweis und Verallgemeinerungen des*
*This follows the lines of the proof of a similar theorem used by F. Carlson and E. Landau:
Fabryschen Lückensatzes.

Hence returning to the function $g(x)$, we have

$$|g(a_m)|^{-1} < \frac{a_m^{n-1}}{\kappa(\kappa+1)\ldots(\kappa+n-2)} \cdot \frac{\sin \pi i a_m \sqrt{\eta_n}}{\pi i a_m \sqrt{\eta_n}}$$

$$< \frac{a_m^{n-1}}{(n-1)!} \frac{(n-1)}{\kappa} e^{\pi a_m \sqrt{\eta_n}}$$

$$< \frac{n-1}{\kappa} \frac{(ea_m)^{n-1}}{(n-1)^{n-1}} \cdot e^{\pi a_m \sqrt{\eta_n}}$$

$$< e^{ea_m \left\{ \frac{\pi\sqrt{\eta_n}}{e} + \frac{n-1}{ea_m} lg \frac{ea_m}{n-1} + \frac{1}{ea_m} lg \frac{n-1}{\kappa} \right\}},$$

But the numbers n and m are conditioned by the inequalities

$$e\lambda_{n-1} < ea_m < e.\lambda_n,$$

$$e\frac{\lambda_{n-1}}{n-1} < \frac{ea_m}{n-1} < \frac{e.n}{n-1} \cdot \frac{\lambda_n}{n}.$$

Now if $m \to \infty$ (we recall that $a_m \to \infty$) then also $n \to \infty$ and therefore*

$$\frac{\lambda_{n-1}}{n-1} \to \infty, \frac{\lambda_n}{n} \to \infty, \frac{ea_m}{n-1} \to \infty, \frac{n-1}{e.a_m} lg \frac{ea_m}{n-1} \to 0, \ \eta_n \to 0, \ \frac{1}{ea_m} lg \frac{n-1}{\kappa} \to 0.$$

We arrive at the following result: if one chooses some small positive number δ, then an integer $m(\delta)$ can be found such that for each $m > m(\delta)$, the inequality

(3) $$|g(a_m)|^{\frac{1}{a_m}} > e^{-\delta}$$

holds. Hence

(4) $$\underline{lim}_{m \to \infty} |g(a_m)|^{\frac{1}{a_m}} \geqq 1.$$

But from (2) we have,

$$\overline{\lim_{m \to \infty}} |g(a_m)|^{\frac{1}{a_m}} \leqq 1.$$

and from the two inequalities we finally obtain the *fourth auxiliary theorem* expressed by the following formula:

Fourth Auxiliary Theorem:

$$\lim_{m \to \infty} |g(a_m)|^{\frac{1}{a_m}} = 1.$$

*As to the last of these relations we have:

$$\frac{lg(n-1)}{a_m} < \frac{lg(n-1)}{\lambda_{n-1}} = \frac{lg(n-1)}{n-1} \frac{n-1}{\lambda_{n-1}} \to 0.$$

II. The Generalized Theorem of Fabry.

E. Fabry has proved the following well known theorem*:

If in the power series

$$f(z) = \sum_{p=0}^{\infty} a_p z^{m_p}$$

the conditions

$$m_1 < m_2 < m_3 < \dots, \quad \lim_{p \to \infty} \frac{m_p}{p} = \infty,$$

are satisfied, then the circle of convergence is a natural boundary for $f(z)$.

Quite recently the validity of this theorem has been shown to hold for the general Dirichlet series

$$f(z) = \sum_{p=1}^{\infty} a_p \, e^{-\lambda_p s} \text{ if } \frac{\lambda_p}{p} \to \infty.$$

The author of this important generalization is O. Szász†, while a still more general form is due to F. Carlson and E. Landau‡.

In all these theorems the condition

$$\frac{\lambda_p}{p} \to \infty$$

must hold for *all* indices p, that is to say, for all members of the series.

The following pages contain a generalization of the theorem relating to power series *only*: there is, however, here no longer the necessity of imposing the condition $\frac{m_p}{p} \to \infty$ on all the members of the series.

Theorem A. Let $a_{n_1}, a_{n_2}, \dots, a_{n_q}, \dots$ be an L series derived from the coefficients of

$$f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

where

$$n_{q+1} \geqq n_q(1 + \mu_1)$$

and the positive number μ_1 is independent of q.

Let us arrange in an infinite series $a_{m_1}, a_{m_2}, \dots, a_{m_\nu}, \dots$, the non-vanishing

Annales de l'École Normale Supérieure (3) 13 (1896), p. 381-382, Acta Mathematica 22 (1899), p. 86. The theorem above is quoted in a form adapted to our notation.

†*loc. cit.* p. 106.

‡*loc. cit.*

coefficients of $f(z)$ which satisfy the conditions

$$n_q(1-\mu)\leqq n\leqq n_q(1+\mu),\ n\neq n_q,\quad (q=1,\,2,\,3,\,\ldots),$$

where μ is a suitably chosen positive constant, independent of q.*

If the condition

$$\frac{m_\nu}{\nu}\to\infty,$$

is satisfied, then the circle of convergence is a natural boundary for $f(z)$.

To prove this we first form the integral function

$$g(y)=\prod_{\nu=1}^{\infty}\left(1-\frac{y^2}{m_\nu^2}\right),$$

which is of the form indicated in the fourth auxiliary theorem.

Then the power series†

$$\sum_{\nu=1}^{\infty}g(\nu)z^\nu$$

defines an analytic function with the sole singular point $z=1$.

Next we form a new power series

$$F(z)=\sum_{\nu=1}^{\infty}g(\nu)a_\nu z^\nu,$$

all the singularities of which must (in accordance with the first auxiliary theorem) be such as are possible for the series

$$f(z)=\sum_{\nu=1}^{\infty}a_\nu z^\nu.$$

But of all the coefficients of $F(z)$ satisfying the inequality

$$n_q(1-\mu)\leqq n\leqq n_q(1+\mu)$$

only one, $a_{n_q}g(n_q)$, does not vanish. Furthermore

$$\lim|a_{n_q}g(n_q)|^{\frac{1}{n_q}}=1,$$

as a consequence of the fourth auxiliary theorem and the condition

$$\lim\,|a_{n_q}|^{\frac{1}{n_q}}=1.$$

Now it is obvious that the function $F(z)$ has all the qualities requisite for the application of the third auxiliary theorem. The circle of convergence is

*We choose the μ in such a manner that each a_{n_q} is a member of only one group defined by $n_q(1-\mu)\leqq n\leqq n_q(1+\mu)$. (Cf. the third auxiliary theorem).

†Consult for proof Pringsheim *loc. cit.*

therefore the natural boundary of $F(z)$ and since $f(z)$ has the same singularities, it is also the natural boundary of $f(z)$.

As an example we construct the following series:

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Put the coefficients with indices n satisfying one of the inequalities

$$10^q - 10^{q-3} \leqq n \leqq 10^q + 10^{q-3}, \quad (q = 3, 4, 5, \ldots),$$

equal to zero with the following exception:

$$a_{10^q} = 1$$

and for coefficients with prime indices

$$1 = a_3 = a_5 = a_7 = a_{11} = \ldots.$$

On all the remaining coefficients with indices n such that

$$10^q + 10^{q-3} < n < 10^{q+1} - 10^{q-2}, \quad (q = 3, 4, 5, \ldots),$$

no limitations need be imposed except naturally

$$\overline{\lim} |a_n|^{\frac{1}{n}} = 1.$$

The choice of these latter coefficients cannot affect the singular character of the natural boundary $|z| = 1$.

It is obvious that the original Fabry's theorem is a special form of (A).

The theorem (A) remains a "Lückensatz," which means that the vanishing of some coefficients is essential. More explicitly stated: The replacing of the zero coefficients by a series $a_{r_1}, a_{r_2}, \ldots, a_{r_\nu}, \ldots$ with the properties

$$|a_{r_\nu}| > 0, \quad \overline{\lim} |a_{r_\nu}|^{\frac{1}{r_\nu}} = 1, \quad (\nu = 1, 2, 3, \ldots),$$

cannot be done without invalidating our proof. This remark is not superfluous because in the third auxiliary theorem, which represents a generalization of Hadamard's "Lückensatz," a suitable replacing of zero coefficients is permissible (cf. (b) second auxiliary theorem).

III. The Generalization of Szász's Theorem.

Szász (*loc. cit.*) has recently published a theorem concerning general *Dirichlet* series, which, restricted to power series, may be stated as follows:

Consider the power series with real coefficients

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \overline{\lim} |a_n|^{\frac{1}{n}} = 1.$$

Let the series of coefficients $a_1, a_2, \ldots$ *have the following properties*

$$a_1 \geqq 0,\ a_2 \geqq 0,\ \ldots,\ a_{q_1} \geqq 0;\ a_{q_1+1} < 0,\ a_{q_1+2} \leqq 0,\ \ldots,\ a_{q_2-1} \leqq 0,$$

$$a_{q_2} \leqq 0;\ a_{q_2+1} > 0,\ a_{q_2+2} \geqq 0,\ \ldots.$$

If now

$$\lim_{\nu \to \infty} \frac{q_\nu}{\nu} = \infty,$$

then $z=1$ *is a singular point of* $f(z)$.

This is a generalization of a well known theorem of Vivanti*. We now propose to prove a further generalization:

Theorem B. Let $a_{n_1}, a_{n_2}, \ldots, a_{n_q}, \ldots$ *be some L series derived from the coefficients of*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \overline{\lim} |a_n|^{\frac{1}{n}} = 1,$$

and $\psi_1, \psi_2, \ldots, \psi_q, \ldots$ *a corresponding series of angles selected in such a manner that*†

1° $$\lim_{q \to \infty} \{\cos(\phi_{n_q} + \psi_q)\}^{\frac{1}{n_q}} = 1.$$

Let μ *be a positive constant independent of* q. *The first of the indices n defined by*

$$n_q(1-\mu) \leqq n \leqq n_q(1+\mu),$$

which satisfies the inequality

2° $$\cos(\phi_n + \psi_q) < 0,$$

we denote by n_{q1}; *the next index which satisfies the inequality*

3° $$\cos(\phi_n + \psi_q) > 0$$

we denote by $n_{q2}+1$, *so that*

$$\cos(\phi_{n_{q2}} + \psi_q) \leqq 0$$

and so forth.

For simplicity of presentation denote the series of positive integers

$$n_{1,1} < n_{1,2} < n_{1,3} < \ldots < n_{2,1} < n_{2,2} < \ldots < n_{q1} < n_{q2} < \ldots$$

by

$$n_{1,1} = p_1,\ n_{1,2} = p_2, \ldots$$

*Riv. di Matem. 3 (1893), p. 111-114.

†Obviously the equation 1° implies the condition $\cos(\phi_{n_q} + \psi_q) > 0$ for sufficiently large values of q.

If this series has the property

$$\lim_{\nu \to \infty} \frac{p_\nu}{\nu} = \infty,$$

then $z=1$ *is a singular point of* $f(z)$.

To prove this we start with the function

$$g(y) = \prod_{\nu=1}^{\infty} \left(1 - \frac{y^2}{p_\nu^2}\right)$$

and make use of the first auxiliary theorem.

Clearly $g(y)$ is negative for every value of y, which lies within one of the intervals

$$(p_1, p_2),\ (p_3, p_4),\ \ldots$$

and is positive for every other positive value of y; in particular it is positive and different from zero, for every value

$$y = n_1, n_2, \ldots, n_q, \ldots$$

because these numbers are all *outside* the intervals. Therefore the values of $g(y)$ attached to these arguments satisfy the equation

$$\lim |g(n_q)|^{\frac{1}{n_q}} = 1.$$

This follows from the fourth auxiliary theorem.

Consider now the power series

$$F(z) = \sum_{\nu=1}^{\infty} g(\nu) a_\nu z^\nu$$

which is derived by applying Hadamard's multiplicative process to the following two functions

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^\nu,$$

$$h(z) = \sum_{\nu=1}^{\infty} g(\nu) z^\nu.$$

The function $h(z)$ has the sole singular point $z=1$. Hence it follows from the first auxiliary theorem, that $F(z)$ has no singular points other than those which are possible for $f(z)$. The series $F(z)$, however, obviously satisfies the conditions (a) and (b) of the *second* auxiliary theorem, because the real part of $g(\nu)a_\nu e^{i\vartheta_\nu}$ cannot be negative for indices ν defined by

$$n_q(1-\mu) \leqq \nu \leqq n_q(1+\mu), \quad (q=1, 2, 3, \ldots).$$

Hence the function $F(z)$ and consequently also $f(z)$ has the singularity $z=1$.

Clearly the original Szász's theorem can be obtained as a specialization of B.

CORRESPONDANCE ENTRE LA FONCTION ET LA FRACTION DÉCIMALE

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1. L'on sait qu'une suite de chiffres, formant un nombre réel positif peut présenter autant de diversité et résumer autant de complications qu'une fonction analytique d'un nombre quelconque de variables. L'ensemble de telles fonctions a la puissance de l'ensemble des nombres réels positifs, et même, si l'on veut, des nombres compris entre 0 et 1.

Mais ce fait théorique n'est pas d'un grand intérêt tant qu'on ne définit pas d'une manière concrète la correspondance entre ces deux êtres mathématiques, la fonction et la fraction décimale. Par contre, si l'on arrivait à établir une correspondance définie, effective, entre les éléments déterminants de la fonction et la suite de chiffres composant le nombre décimal, il serait possible de ramener l'étude de certaines questions d'analyse à la considération de problèmes sur les fractions décimales, et inversement.

J'ai établi un tel mode de correspondance pour des classes étendues de fonctions, à l'aide de ma théorie des *spectres numériques** et des *transmutations fonctionnelles*. Dans ce travail, je voudrais résumer les résultats auxquels je suis parvenu à cet égard, en me bornant aux fonctions d'une seule variable.

2. *Correspondance entre les fractions décimales et les séries de puissances à coefficients entiers.*

J'appellerai, pour abréger le langage, *séries* (E) les séries de puissances

$$f(z) = M_0 + M_1 z + M_2 z^2 + \dots \qquad (1)$$

à coefficients M_k, nombres entiers et ayant un rayon de convergence non nul. Les fonctions d'une variable, susceptibles d'être représentées par de telles séries, seront désignées comme *fonctions* (E).

Les séries (E) s'introduisent dans un grand nombre de questions d'analyse et de théorie des nombres. Aussi ont-elles été l'objet d'importants travaux (MM. Borel, Polya, Fatou, Cahen, etc.).

Dans le cas où les M_k sont des entiers à partie réelle et à coefficient de i positifs, je désigne comme *spectre* de la fonction (1) une fraction décimale obtenue en écrivant bout à bout, à la suite les uns des autres, les nombres $M_0, M_1, M_2, \dots$

*dont les premiers éléments ont paru dans l'ouvrage: *Les spectres numériques* (Gauthier-Villars, Paris, 1919).

et en séparant M_0 des autres entiers de la suite par la virgule décimale, ces entiers pouvant être séparés entre eux par des zéros en nombre arbitraire.

Lorsque les parties réelles et les coefficients de i des M_k sont de signes quelconques, on remplacera les M_k par les $\theta_k M_k$, où θ_k est l'une des quatre quantités $1, -1, i, -i$, choisie de manière à ce que la partie réelle et le coefficient de i des $\theta_k M_k$ soient positifs. Dans le cas où les M_k sont tous réels et positifs, on prendra $\theta_k = 1$.

Une fonction (E), donnée, a une infinité de spectres différant entre eux par le nombre de zéros interposés entre les entiers correspondants M_k. L'un parmi ces spectres s'exprime directement, à l'aide de la fonction $f(z)$, par une intégrale définie rattachée à cette fonction, et dans certains cas même, sous forme explicite, en termes finis. La convergence de la série (1) au voisinage de $z=0$ assure l'existence d'un nombre positif fixe A tel que, pour toute valeur de n, on ait $\sqrt[n]{M_n} < A$; désignons par c un entier quelconque supérieur ou égal à $\log A$. Formons la fonction auxiliaire

(2) $$\chi(x) = \sum_{n=0}^{\infty} \theta_n q^{n^2} x^n, \qquad (q = 10^{-\frac{c}{2}}),$$

et l'intégrale définie

$$S = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{ti})\, \chi\left(\frac{qe^{-ti}}{\rho}\right) dt,$$

où

$$\rho = \text{const. arbitraire} < \frac{1}{A}.$$

On peut toujours déterminer l'entier c de manière que la valeur numérique de S représente un spectre de $f(z)$.

Dans le cas où M_n n'augmente pas indéfiniment avec n, en formant la fonction auxiliaire

(4) $$\psi(x) = \theta_0 M_0 + \theta_1 M_1 x + \theta_2 M_2 x^2 + \ldots$$

la valeur numérique S' de $\psi(10^{-h})$ où h est un nombre entier supérieur ou égal au nombre de chiffres des M_k, *représente également un spectre de $f(z)$.*

Le spectre étant donné par l'intégrale (3), le coefficient M_0 est donné par la partie entière de la valeur numérique S, et le coefficient M_k coïncide avec le groupe de chiffres significatifs de S commençant par la $\left[\frac{k(k-1)}{2} c + 1\right]$-ième et se terminant par la $\left[\frac{k(k+1)}{2} c\right]$-ième décimale de S; le n-ième chiffre de M_k coïncide avec la $\left[\frac{k(k+1)}{2} c - n + 1\right]$-ième décimale de S.

Lorsque le spectre est donné par la valeur numérique S' de $\psi(10^{-h})$, où $\psi(x)$ est la fonction (4), le coefficient M_k coïncide avec le groupe de chiffres significatifs de S' commençant par la décimale de rang $(k-1)h+1$ et se terminant par la décimale de rang kh; le chiffre decimale de rang n de M_k est formé par la décimale de rang kh.

La fonction $f(z)$, l'une quelconque parmi les fonctions (E), se trouve ainsi parfaitement déterminée par l'ensemble des données suivantes:

1° la suite θ_0, θ_1, θ_2 se rapportant aux signes des M_k;

2° l'entier positif c ou h qui lui correspond;

3° la fraction décimale S ou S' représentant le spectre de $f(z)$.

Ainsi, par exemple, la seule fonction (E) se corrpondant à

$$\theta_k=1,\ h=1,\ S'=3,\ 121212\ldots\ldots=\frac{103}{33}$$

est la fonction rationnelle

$$f(z)=\frac{3+z-z^2}{1+z^2}.$$

On a ainsi une correspondance effective entre les fractions décimales et les fonctions (E). Il est, par exemple, manifeste que la fraction décimale, spectre de $f(z)$, ne saurait se réduire à un nombre entier (réel ou imaginaire) sans que la fonction correspondante se réduise à une constante; elle ne saurait avoir un nombre limité de décimales que si la fonction se réduit à un polynome. Une fraction périodique (simple ou mixte) correspondant à une valeur finie et fixe de h et à $\theta_k=1$, définit une fonction rationnelle; si la fraction est un nombre algébrique, la fonction est elle-même algébrique, etc.

3. *Sur certaines transmutations fonctionnelles.*

On a défini une *transmutation fonctionnelle* lorsqu'on a indiqué une loi permettant de déduire de toute fonction ϕ une autre fonction Φ dont la forme dépend de celle de ϕ, et l'on dira que la transmutation, appliquée à la fonction ϕ, donne pour *transmuée* Φ.

La notion de transmutation est, pour les fonctions, exactement l'équivalent de ce qu'est celle de transformation ponctuelle pour les points de l'espace. On établit une analogie aussi complète que possible entre les deux conceptions en considérant chaque fonction comme correspondant à un point de *l'espace fonctionnel,* dans lequel une *catégorie* quelconque de fonctions représenterait un *champ fonctionnel.* Une transmutation peut n'avoir un sens et n'être définie que dans un champ fonctionnel déterminé, de même qu'en géométrie ordinaire une transformation ponctuelle peut n'être définie que pour les points d'une région déterminée de l'espace, d'une surface, d'une ligne.

Parmi les divers moyens d'établir effectivement une correspondance entre les fonctions (E) et les fonctions $f(z)$ d'une autre catégorie, l'un consisterait à appliquer à f une transmutation déterminée *telle que la transmuée soit une fonction* (E). Nous désignerons comme une *transmutation* $\Delta(f)$ *compatible avec* $f(z)$ *toute transmutation* qui, appliquée à f, remplit ces conditions.

Par exemple, la transmutation,

$$\Delta(f)=f(-i\log z)$$

est compatible avec la fonction

$$f(z)=\frac{e^{zi}}{(1-e^{zi})^2},$$

qu'elle transforme en série (E),

$$z-2z^2-3iz^3+4z^4+\ldots\ldots;$$

la transmutation

$$\Delta(f)=\int_o^\infty e^{-t}f(zt)dt$$

est compatible avec $f(z)=e^{mz}$ ($m=$nombre entier), qu'elle transforme en série (E),

$$1+mz+m^2z^2+m^3z^3+\ldots\ldots;$$

la transmutation

$$\Delta(f)=\alpha f(\beta z),$$

où α et β sont deux nombres entiers convenablement déterminés, est compatible avec toute fonction *algébrique* développable en série de puissances de z ayant pour coefficients des nombres *commensurables*, etc.

4. *Catégories spectrales de fonctions.*

Nous dirons que plusieurs fonctions $f_1(z)$, $f_2(z)$, $f_3(z)$, ... *appartiennent à une même catégorie spectrale* s'il existe pour chacune d'elles une région du plan des z dans laquelle ces fonctions admettent un même $\Delta(f)$, ne différant d'une fonction à une autre que par les valeurs numériques d'un certain nombre de paramètres qu'il contient. Une catégorie spectrale de fonctions se trouve ainsi déterminée par la forme de $\Delta(f)$ qui lui convient, comme l'est une catégorie de surfaces par la forme du ds^2 qui la caractérise.

Telles seraient, par exemple, les catégories suivantes:

1° fonctions développables en série de puissances à coefficients a_n, nombres entiers ou ayant un nombre fini de décimales; cette catégorie de fonctions admet

$$\Delta(f)=\alpha f(z), \quad (\alpha=\text{const.});$$

2° fonctions à a_n commensurables dont le dénominateur n'augmente pas plus vite que α^n ($\alpha=$constante fixe) et n'a qu'un nombre limité de facteurs premiers (catégorie embrassant toutes les fonctions algébriques à a_n commensurables); cette catégorie admet

$$\Delta(f)=\beta f(\gamma z), \quad (\beta \text{ et } \gamma \text{ constants});$$

3° fonctions à a_n *racine carrée* d'un nombre entier, admettant

$$\Delta(f)=\frac{1}{2\pi}\int_o^{2\pi} f(\rho e^{ti})f\left(\frac{ze^{-ti}}{\rho}\right)dt, \quad (\rho=\text{const.});$$

4° fonctions à a_n commensurables dont le nombre de chiffres de la partie non périodique et celui de la période restent finis, ou bien présentent une régularité donnée à l'avance, ou bien n'augmentent pas plus vite qu'une fonction donnée $\phi(n)$;

5° fonctions à a_n commensurables, satisfaisant à une équation différentielle algébrique d'un ordre fini;

6° fonctions à a_n tels que les produits $a_n \alpha_n$ soient des nombres entiers, les α_n étant des nombres variant avec n et tels que $\sqrt[n]{\alpha_n}$ reste fini pour n infini; etc.

5. *Correspondance entre les fractions décimales et les diverses catégories de fonctions.*

Une transmutation $\Delta(f)$, appliquée aux fonctions d'une catégorie (f), établit une correspondance entre les f et leurs transmuées (E) de manière que f étant connue, (E) sera déterminée. Or, dans la section 2, nous avons établi une correspondance entre les fractions décimales et la catégorie de fonctions (E). Nous avons ainsi une manière concrète *d'établir une correspondance entre les fonctions d'une catégorie spectrale et les fractions décimales.* La *correspondance* est définie à l'aide

1° d'un $\Delta(f)$ compatible avec la catégorie de fonctions considérée;

2° du spectre S ou S' de la transmuée (E) correspondant à la fonction considérée;

3° d'un ensemble de données qualitatives rattachées à la fonction.

Ainsi, parmi les fonctions (f) admettant

$$\Delta(f) = \int_o^\infty e^{-t} f(zt)dt,$$

la seule qui corresponde à

$$h=2, \quad \theta_k=1, \quad S'=0{,}636363\ldots = \frac{7}{11}$$

est la fonction

$$f(z)=63e^z-1;$$

parmi les fonctions (f) admettant

$$\Delta(f)=f''(z)-\lambda f(z), \qquad (\lambda=\text{const.}),$$

la seule correspondant à

$$h=3, \quad \theta_k=1, \quad S'=0{,}02731831831\ldots = \frac{9997}{333000}$$

est l'intégrale générale de l'équation linéaire du second ordre

$$\frac{d^2y}{dz^2}+\lambda y=\frac{27z+291z^2}{1-z}.$$

On arrive ainsi à déterminer des fonctions par des conditions très larges et dépendant, dans un autre mode de déterminations, d'un nombre *infini* de paramètres, à l'aide de données numériques en nombre *fini*, ce nombre restant immuable pour les fonctions d'une même catégorie.

C'est, par exemple, ainsi qu'une fonction $f(z)$ développable en série de puissances à coefficients a_n nombres entiers, réels et positifs, est complètement déterminée par un nombre entier positif (dépendant de la vitesse de croissance de son $\sqrt[n]{a_n}$ avec n) et une fraction décimale (son spectre).

Une fraction algébrique, développable en série de puissances à coefficients nombres commensurables, est complètement déterminée par un ensemble de données qualitatives (précisées par les θ_k), trois nombres entiers (dont deux sont fournis par le théorème d'Eisenstein et le troisième indique la vitesse de croissance de $\sqrt[n]{a_n}$ avec n) et une fraction décimale (spectre de sa transmuée).

Envisageons encore, à titre d'exemple, ce problème en apparence indéterminé: déterminer une courbe plane $y=f(z)$ dont la sous-tangente est développable en série de puissances de x à coefficients M_k nombres entiers positifs ne dépassant pas un nombre fixe donné M, connaissant la longueur de la sous-tangente pour une valeur particulière convenablement choisie de x. Pour peu que le problème soit parfaitement déterminé, il suffit de connaitre la longueur de la sous-tangente pour $x=10^{-h}$ où h est un nombre entier supérieur au nombre de chiffres de la partie entière de M. Dans le cas, par exemple, où les M_k sont inférieurs à 10 et où la sous-tangente pour $x=0,1$ est égale à la périphérie du cercle de rayon 1, la courbe cherchée est définie par l'équation

$$y=\frac{y_0}{\phi(10)}\,\phi(x),$$

où

$$\phi(x)=1+\frac{x}{6}-\frac{x^2}{72}-\frac{81}{432}x^3+\ldots,$$

le coefficient λ_n de x^n étant déterminé par la relation de récurrence

$$(n+1)M_0\lambda_{n+1}+nM_1\lambda_n+(n-1)M_2\lambda_{n-1}+\ldots+M_n\lambda_1=\lambda_n$$

avec $M_0=6$ et M_n égal à la n-ième décimale du nombre 2π.

On traiterait de la même manière les problèmes de ce genre dans lesquels des données analogues concerneraient le rayon de courbure en un point variable de la courbe, la longueur de son arc compté à partir d'un point fixe, etc.

6. *Cas général.*

La correspondance précédente entre les fonctions et les fractions décimales n'est pas bornée à des catégories spéciales de fonctions. J'ai établi à cet égard le théorème suivant:

A toute fonction analytique et pour un cercle donné C décrit autour d'un point ordinaire de la fonction, on peut faire correspondre l'ensemble d'un nombre décimal et d'un nombre entier positif qui, avec l'adjonction d'un ensemble de données qualitatives, contient tous les éléments pour la détermination de la fonction dans le cercle C, avec une approximation donnée à l'avance.

Divers éléments caractéristiques des fonctions analytiques se laissent également condenser en un seul nombre décimal et un ensemble de données qualitatives. Tel est, par exemple, le cas des *singularités* d'une fonction comprises dans un cercle ayant pour centre un point ordinaire pour la fonction ; ou bien le cas des *zéros* d'une fonction compris dans un cercle, etc. . . . Ainsi, étant donnée une fonction $f(z)$, on peut, d'après un théorème connu de M. Borel,* pour la détermination de ses singularités à l'intérieur d'un cercle quelconque décrit

*E. Borel: *Leçons sur les fonctions méromorphes*, 1903, p. 35-36.

autour d'un quelconque de ses points ordinaires, substituer à $f(z)$ une fonction (E). Celle-ci ayant un spectre, on a par là *une correspondance entre $f(z)$ et un nombre décimal contenant tous les éléments pour la détermination complète des singularités de $f(z)$ dans un cercle donné.*

Le même procédé appliqué à

$$\frac{1}{f(z)}, \quad f(z)-a, \quad f'(z), \text{ etc.}$$

établit une pareille correspondance entre $f(z)$ et un nombre décimal déterminant les valeurs pour lesquelles $f(z)$ s'annule, prend une valeur donnée a, atteint au maximum ou minimum, etc.

7. *Quelques applications.*

Le mode de correspondance ainsi établi entre la fonction et la fraction décimale conduit, entre autres applications, à établir des relations entre certains problèmes sur les fonctions et des problèmes de la théorie des nombres.

Ainsi, à une certaine fonction rationelle, facile à former, correspond une fraction décimale indiquant, à simple vue, de combien de manières un entier variable se laisse exprimer comme combinaison linéaire homogène d'une suite d'entiers positifs.

A la fonction définie par la série de Lambert correspond une fraction décimale indiquant, par la suite de ses décimales, le nombre de diviseurs d'un entier variable, ainsi que le nombre des nombres premiers inférieurs à un nombre donné.

Une fraction décimale, rattachée à une fonction déterminée et formée à l'aide de cette fonction, fournit par les groupes de ses décimales, soit directement, soit par un calcul connu, les valeurs numériques des coefficients du développement de la fonction, ou bien un coefficient voulu, ou même le chiffre d'un rang voulu d'un coefficient.

Etant donné un champ fonctionnel, la correspondance entre les fonctions qu'il embrasse et les fractions décimales qui s'y rattachent, fournit un procédé de *numérotage* de ces fonctions. Chaque fonction se trouve marquée par un nombre qui la détermine parfaitement dans le champ fonctionnel dont elle fait partie.

ON THE SUFFICIENT CONDITIONS FOR ANALYTICITY OF FUNCTIONS OF A COMPLEX VARIABLE

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§1 Generalizing a theorem of Pompéiu, Dr. Looman has given the following sufficient condition for the analyticity of a continuous function $f(z)$ of the complex variable $z=x+yi$ throughout a domain D.

Let z be a point of D, γ a square containing z, $\mu(\gamma)$ its area, $I(\gamma)$ the integral of $f(z)$ along the contour of γ, and let us consider the function

$$\sigma(z)=\limsup_{\mu(\gamma)\to 0}\frac{|I(\gamma)|}{\mu(\gamma)};$$

then $f(z)$ is analytic throughout D, if $\sigma(z)$ is everywhere finite and almost everywhere equal to zero.*

The following extension may be given to Looman's result:

If the function

$$\lambda(z)=\liminf_{\mu(\gamma)\to 0}\frac{|I(\gamma)|}{\mu(\gamma)}$$

is almost everywhere equal to zero and if the function $\sigma(z)$ is finite except at the points of an enumerable set in D, then the continuous function $f(z)$ is analytic throughout D.

§2 In our proof, we shall make use of the following conceptions:

A *portion* of a perfect point-set P is the set that P has in common with a rectangle $a<x<b$, $c<y<d$, provided this common part be non-vacuous.

A point-set S is *non-dense with regard to a perfect set P*, if every portion of P contains another portion of P having no point in common with S.

If an enumerable point-set S is the common part of open point-sets Ω_1, Ω_2, . . . , then S is non-dense with regard to any perfect set.

§3 Let us now consider with Dr. Looman the point-set P, each point z of which has the property that in any neighbourhood of z there exist squares γ with $I(\gamma)\neq 0$.

*Nieuw Archief voor Wiskunde, 2[e] reeks, XIV, 3, p. 234-239. "Almost everywhere" is used for the well known German term "fast überall".

P has no isolated points; for if a were an isolated point of P, we could find a neighbourhood Ω of a, having the property that each point of Ω different from a has a neighbourhood for which every square γ within it has its $I(\gamma)$ equal to zero; then $f(z)$, which was assumed continuous, would be analytic throughout Ω, and a would not belong to P. It is clear then that P is closed, and consequently P is either perfect or vacuous.

§4 For our theorem it is sufficient to prove that P is vacuous. Let us assume the contrary and consider the set Ω_n consisting of all open squares γ for which $|I(\gamma)|>n.\mu(\gamma)$. The common part S of $\Omega_1, \Omega_2, \ldots$ is evidently the set of points z where $\sigma(z)=\infty$, for (1) if $\sigma(z)=\infty$, z belongs to a square of Ω_n, for any value of n, and z lies in S; (2) if z lies in S it lies in squares $\gamma_1, \gamma_2, \ldots$ for which $|I(\gamma_n)| > n.\mu(\gamma_n)$. Assuming $|f(z)|<M$ throughout D (which is permitted) and calling a_n the side of γ_n we have $a_n < \dfrac{4M}{n} \rightarrow 0$, and hence $\sigma(z)=\infty$.

Since S is assumed enumerable, S is non-dense with regard to P (§2), and P contains a (non-vacuous) portion Q having no point in common with S.

Now it is easy to see that Q contains a (non-vacuous) portion in which $\sigma(z)$ is limited, for otherwise in any portion of Q this function would be unlimited and we could find a point z of Q where $\sigma(z)=\infty$, by the method of contracting squares γ_n, $I(\gamma_n)$ becoming infinite.

We have then proved the existence of a rectangle R having the following properties:

(1) in any point z of R, belonging to P, we have

$$|\sigma(z)|<\text{a fixed number } N,$$

and as $\sigma(z)=0$ for z not belonging to P, this inequality holds throughout R.

(2) R contains points of P in its interior.

§5 We next note that for any square γ lying within R the inequality

$$|I(\gamma)|<N.\mu(\gamma)$$

holds, for otherwise we could find a point z of R, where $|\sigma(z)|\geqq N$, by the method of quartering squares γ_n, $|I(\gamma_n)|\geqq N.\mu(\gamma_n)$.

§6 Now we consider an arbitrary square γ within R. From our supposition, $\lambda(z)=0$ almost everywhere, it follows that γ contains a set A of measure $=\mu(\gamma)$, each point z of which lies within a set of squares $\gamma_1, \gamma_2, \ldots$ with

$$\lim_{n\rightarrow\infty} \mu(\gamma_n)=0 \text{ and } \frac{|I(\gamma_n)|}{\mu(\gamma_n)}<\epsilon,$$

ϵ being an arbitrarily small positive number.

Making use of a metric theorem of Vitali*, we can find a finite number of

*C. Carathéodory, *Vorlesungen über reelle Funktionen*, p. 299.

such squares, interior to γ and exterior to one another, and which cover a part of A of measure $> (1-\epsilon).\mu(A) = (1-\epsilon).\mu(\gamma)$. The sum of the absolute values of the integrals I around these squares is less than $\epsilon.\mu(\gamma)$. The remaining part of γ has an area $< \epsilon\mu(\gamma)$, and if we divide it into squares and apply §5 we find that the integral of $f(z)$ around this remaining part has an absolute value less than $\epsilon.N.\mu(\gamma)$.

Hence $|I(\gamma)| < \epsilon.\mu(\gamma) + \epsilon.N.\mu(\gamma)$; since ϵ is arbitrarily small, we find $I(\gamma) = 0$; and since γ is an arbitrary square within R, we see that $f(z)$ is analytic throughout R, and therefore R will not contain points of P. This contradicts §4 and our proof is thus completed.

SUR LES VALEURS EXCEPTIONNELLES DES FONCTIONS MULTIFORMES

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Je me propose de donner la démonstration d'une proposition concernant le nombre des valeurs exceptionnelles d'une classe de fonctions transcendantes ayant un nombre fini de branches.

Je considère les fonctions définies par l'équation

$$u^\nu + g_1 u^{\nu-1} + g_2 u^{\nu-2} + \dots + g_\nu = 0, \tag{1}$$

$g_1, g_2, \dots, g_\nu$ étant des fonctions entières qui ne sont pas toutes des polynomes.

On sait que ces fonctions prennent dans le domaine de l'infini toutes les valeurs sauf $2\nu - 1$ au plus.

Lorsque les fonctions g_i sont linéairement distinctes, c'est-à-dire lorsqu'il n'y a aucune relation linéaire à coefficients constants de la forme

$$c_0 + c_1 g_1 + c_2 g_2 + \dots + c_\nu g_\nu = 0,$$

la limite supérieure $2\nu - 1$ peut être remplacé par ν.

C'est ce que nous allons démontrer. Posons

$$f(x, u) \equiv u^\nu + g_1 u^{\nu-1} + g_2 u^{\nu-2} + \dots + g_\nu.$$

Les valeurs exceptionnelles de la fonction définie par l'équation (1) sont les valeurs de u pour lesquelles $f(x, u)$ est une constante, ou un polynome, ou un polynome multiplié par une fonction entière dépourvue de racines. Comme les fonctions g_i sont linéairement distinctes, nous ne pouvons pas avoir de valeurs exceptionnelles du premier type.

Je vais montrer que le nombre des valeurs correspondant, soit à un polynome non constant, soit à un polynome multiplié par une exponentielle, ne dépasse pas ν.

En effet soient $\nu + 1$ telles valeurs $u_0, u_1, u_2, \dots, u_\nu$. Il y en aura deux au moins qui correspondent à un polynome multiplié par une exponentielle, car si pour ν valeurs $u_0, u_1, u_2, \dots, u_{\nu-1}$, $f(x, u)$ se réduisait à un polynome, en résolvant le système des ν équations:

$$f(x, u_0) = p_0(x),$$
$$f(x, u_1) = p_1(x),$$
$$\dots\dots\dots\dots$$
$$f(x, u_{\nu-1}) = p_{\nu-1}(x),$$

(ce qui est toujours possible) on définirait les fonctions $g_1, g_2, \dots, g_\nu$ comme des fonctions algébriques, et puisqu'elles sont uniformes elles seraient toutes des polynomes (et même des constantes), ce qui est contraire à l'hypothèse.

J'envisage donc les équations:

$$\begin{aligned} &u_0^\nu + g_1 u_0^{\nu-1} + g_2 u_0^{\nu-2} + \dots + g_\nu = p_0 e^{\phi_0}, \\ &u_1^\nu + g_1 u_1^{\nu-1} + g_2 u_1^{\nu-2} + \dots + g_\nu = p_1 e^{\phi_1}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &u_\nu^\nu + g_1 u_\nu^{\nu-1} + g_2 u_\nu^{\nu-2} + \dots + g_\nu = p_\nu e^{\phi_\nu}; \end{aligned}$$

parmi les $\phi_0, \phi_1, \dots, \phi_\nu$ il peut y avoir des constantes.
J'élimine $g_1, g_2, \dots, g_\nu$ (ce qui est toujours possible) et j'obtiens

$$\lambda = \lambda_0 p_0 e^{\phi_0} + \lambda_1 p_1 e^{\phi_1} + \dots + \lambda_\nu p_\nu e^{\phi_\nu}, \tag{2}$$

$\lambda, \lambda_0, \lambda_1, \dots, \lambda_\nu$ étant des constantes différentes de zéro;

$$\lambda = \begin{vmatrix} u_0^\nu & u_0^{\nu-1} \dots u_0 & 1 \\ u_1^\nu & u_1^{\nu-1} \dots u_1 & 1 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ u_\nu^\nu & u_\nu^{\nu-1} \dots u_\nu & 1 \end{vmatrix}, \quad \lambda_0 = \begin{vmatrix} u_1^{\nu-1} & u_1^{\nu-2} \dots u_1 & 1 \\ u_2^{\nu-1} & u_2^{\nu-2} \dots u_2 & 1 \\ \dots & \dots & \dots \\ u_\nu^{\nu-1} & u_\nu^{\nu-2} \dots u_\nu & 1 \end{vmatrix}, \dots$$

Soit μ le nombre des termes ($\mu \leq \nu - 1$) qui sont des constantes; comme nous pouvons supposer que ce sont les μ premiers on aura l'identité

$$-\lambda + \lambda_0 p_0 + \lambda_1 p_1 + \dots + \lambda_\mu p_\mu + \lambda_{\mu+1} p_{\mu+1} e^{\phi_{\mu+1}} + \dots + \lambda_\nu p_\nu e^{\phi_\nu} \equiv 0.$$

Je vais montrer que le terme non-exponentiel ne peut pas être identiquement nul. En effet, s'il en était ainsi, nous aurions entre les $g_1, g_2, \dots, g_\nu$, la relation suivante:

$$\begin{aligned} &-\lambda + \lambda_0 u_0^\nu + \lambda_1 u_1^\nu + \dots + \lambda_\mu u_\mu^\nu + g_1(\lambda_0 u_0^{\nu-1} + \lambda_1 u_1^{\nu-1} + \dots + \lambda_\mu u_\mu^{\nu-1}) \\ &+ g_2(\lambda_0 u_0^{\nu-2} + \lambda_1 u_1^{\nu-2} + \dots + \lambda_\mu u_\mu^{\nu-2}) + \dots + g_\nu(\lambda_0 + \lambda_1 + \dots + \lambda_\mu) \equiv 0, \end{aligned}$$

relation qui entraîne, puisque les g_i sont linéairement indépendants,

$$\begin{aligned} &\lambda_0 u_0^{\nu-1} + \lambda_1 u_1^{\nu-1} + \dots + \lambda_\mu u_\mu^{\nu-1} \equiv 0, \\ &\lambda_0 u_0^{\nu-2} + \lambda_1 u_1^{\nu-2} + \dots + \lambda_\mu u_\mu^{\nu-2} \equiv 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\lambda_0 u_0 \quad + \lambda_1 u_1 \quad + \dots + \lambda_\mu u_\mu \quad \equiv 0, \\ &\lambda_0 \quad + \lambda_1 \quad + \dots + \lambda_\mu \quad \equiv 0. \end{aligned}$$

Or les $\lambda_0, \lambda_1, \dots, \lambda_\mu$ ne sont pas nuls; par conséquent le déterminant

$$\lambda_\nu = \begin{vmatrix} 1 & u_0 & u_0^2 \dots\dots u_0^{\nu-1} \\ 1 & u_1 & u_1^2 \dots\dots u_1^{\nu-1} \\ \dots & \dots & \dots \\ 1 & u_\mu & u_\mu^2 \dots\dots u_\mu^{\nu-1} \\ \dots & \dots & \dots \\ 1 & u_{\nu-1} & u_{\nu-1}^2 \dots u_{\nu-1}^{\nu-1} \end{vmatrix} \qquad (\mu \leq \nu - 1)$$

doit être identiquement nul, ce qui est impossible car λ_ν est le déterminant de Van der Monde qui est différent de zéro.

Donc l'identité (2) est impossible, ce qui démontre notre proposition:

Théorème—*Une transcendante algébroïde à ν branches, définie par une équation de la forme*

$$u^\nu + g_1(x)u^{\nu-1} + g_2(x)u^{\nu-2} + \ldots + g_\nu(x) = 0,$$

$g_i(x)$ désignant des fonctions entières linéairement distinctes et dont une au moins n'est pas un polynome, prend dans le domaine de l'infini toutes les valeurs, sauf ν au plus.

Cette valeur maximum ν est atteinte; par exemple considérons les algébroïdes

$$u^\nu + e^x - 1 = 0.$$

Les $\nu^{i\grave{e}mes}$ racines de l'unité sont des valeurs exceptionnelles pour l'algébroïde.

SUR CERTAINES ÉQUATIONS FONCTIONNELLES

PAR M. JACQUES TOUCHARD,

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Je me propose d'indiquer certaines équations fonctionnelles, satisfaites par des fonctions algébriques et d'en déduire des relations entre les coefficients du développement de ces fonctions par la série de Lagrange.

I

Considérons, en premier lieu, l'équation

$$xz^2 - z + 1 = 0 \tag{1}$$

et supposons x réel et plus grand que $\frac{1}{4}$.

Soit z_1 la racine située dans le demi-plan au-dessus de l'axe réel et soit z_2 la racine imaginaire conjuguée. On a

$$z_1 = 1 + e^{2\phi i},$$

$$z_2 = 1 + e^{-2\phi i},$$

ϕ étant un angle compris entre 0 et $\frac{\pi}{2}$, qui est défini par l'égalité

$$4x \cos^2 \phi = 1.$$

Donnons à ϕ une autre valeur θ, également comprise entre 0 et $\frac{\pi}{2}$, ce qui revient à faire sur x une certaine substitution $[x, u(x)]$, définie par la formule

$$4u^2(x) \cos^2 \theta = 1.$$

Par cette substitution, les fonctions z_1 et z_2 deviennent respectivement

$$y_1 = z_1[u(x)] = 1 + e^{2i\theta},$$

$$y_2 = z_2[u(x)] = 1 + e^{-2i\theta},$$

qui sont racines de l'équation

$$u(x)y^2 - y + 1 = 0.$$

1° Je suppose d'abord $\theta = \frac{\pi}{2} - \phi$, d'où

$$y_1 = 2 - z_2,$$
$$y_2 = 2 - z_1,$$
$$u(x) = \frac{1}{4\sin^2\phi} = \frac{x}{4x-1}.$$

On a donc les équations fonctionnelles

$$(2) \qquad \begin{cases} z_1\left(\dfrac{x}{4x-1}\right) = 2 - z_2(x), \\ z_2\left(\dfrac{x}{4x-1}\right) = 2 - z_1(x). \end{cases}$$

Il est maintenant facile de voir que si x est imaginaire et si

$$\text{P.R. } x > \tfrac{1}{4}, \text{ on a aussi P.R. } \frac{x}{4x-1} > \tfrac{1}{4},$$

et que si

$$\text{P.R. } x < \tfrac{1}{4}, \text{ on a aussi P.R. } \frac{x}{4x-1} < \tfrac{1}{4}.$$

Comme, d'autre part, les racines z_1 et z_2 se permutent autour des points $x = \frac{1}{4}$ et $x = \infty$, il en résulte que la droite $x = \frac{1}{4}$ partage le plan en deux régions. Dans celle de droite, ont lieu les équations (2), tandis qu'on a, dans la région de gauche

$$(3) \qquad \begin{cases} z_1\left(\dfrac{x}{4x-1}\right) = 2 - z_1(x), \\ z_2\left(\dfrac{x}{4x-1}\right) = 2 - z_2(x). \end{cases}$$

2° Supposons maintenant x réel et compris entre $\frac{1}{4}$ et $\frac{1}{2}$, d'où

$$0 < \phi < \frac{\pi}{4},$$

et prenons

$$\theta = \frac{\pi}{4} + \phi.$$

Nous aurons

$$y_1 = z_1[u(x)] = 1 + e^{2i(\phi + \frac{\pi}{4})} = 1 - i + iz_1(x).$$

D'autre part,

$$u(x) = \frac{1}{4\cos^2\left(\dfrac{\pi}{4} + \phi\right)}$$

ou

$$(4) \qquad 2u(x) = \frac{1}{1 - \sin 2\phi}, \quad 2x = \frac{1}{1 + \cos 2\phi},$$

puis

$$\frac{2u(x)-1}{2u(x)} = \pm\frac{1}{2x}\sqrt{4x-1},$$

et comme, d'après (4), $u(x)$ est $>\frac{1}{2}$, c'est le signe $+$ qui convient. On a donc l'équation fonctionnelle

$$z_1\left[\frac{2x^2+x\sqrt{4x-1}}{(2x-1)^2}\right]=1-i+iz_1(x).$$

3° Cherchons, en supposant encore $0<\phi<\frac{\pi}{4}$, l'argument de la fonction

$$z_1(x^2)=1+e^{2i\theta}.$$

Nous aurons

$$\cos\theta=2\cos^2\phi.$$

Faisons maintenant $\theta'=\frac{\theta}{2}$, il en résulte pour x la substitution

$$u(x)=\frac{1}{4\cos^2\frac{\theta}{2}}=\frac{1}{2+4\cos^2\phi}=\frac{x}{1+2x}.$$

D'autre part

$$y_1=z_1\left(\frac{x}{1+2x}\right)=1+e^{2i\theta'}=1+e^{i\theta}=\frac{1+e^{2i\theta}}{2\cos\theta}.$$

On a donc les équations fonctionnelles

$$(5)\qquad \begin{cases} z_1\left(\dfrac{x}{1+2x}\right)=1+xz_1(x^2),\\ z_2\left(\dfrac{x}{1+2x}\right)=1+xz_2(x^2).\end{cases}$$

Les relations (5) sont caractéristiques de l'équation (1), tandis que les équations (3) ne le sont pas et peuvent se déduire de (5).

Ajoutant, en effet,

$$z\left(\frac{x}{1+2x}\right)=1+xz(x^2),$$

et

$$z\left(\frac{-x}{1-2x}\right)=1-xz(x^2);$$

on obtient

$$z\left(\frac{x}{1+2x}\right)+z\left(\frac{x}{2x-1}\right)=2,$$

et en posant $\frac{x}{1+2x}=y$ on retombe sur (3).

II

L'équation

$$xz^2+az+b=0$$

donne naissance aux équations fonctionnelles

(A) $$z\left(\frac{x}{\frac{4b}{a^2}x-1}\right)=-\frac{2b}{a}-z(x),$$

(B) $$z\left(\frac{\frac{a}{\sqrt{b}}x}{\frac{2\sqrt{b}}{a}x-1}\right)=-\frac{b}{a}-\frac{x\sqrt{b}}{a}z(x^2).$$

III

Soit en général $F(x, z)=0$ une équation algébrique de degré m en z. On peut effectuer sur x une substitution $[x, u(x)]$, $u(x)$ étant une fonction donnée, et en posant

$$y=z[u(x)],$$

l'équation précédente deviendra

$$\Phi(x, y)=0.$$

Si, entre $F=0$ et $\Phi=0$, on élimine x, l'équation résultante,

$$K(y, z)=0,$$

permettra de développer y en série de puissances de z. Ces puissances peuvent être réduites à un exposant inférieur à m et l'on obtient ainsi une série de relations fonctionnelles

$$z[u(x)]=a_1z^{m-1}+a_2z^{m-2}+\ldots+a_m,$$

dans lesquelles les fonctions a_i varient en général avec la racine z considérée.

Inversement, on peut se donner une transformation générale de Tchirnhausen

(6) $$y=a_1z^{m-1}+a_2z^{m-2}+\ldots+a_m,$$

$a_1, a_2, \ldots, a_m$ étant des fonctions connues de x, arbitrairement choisies.

Ayant formé l'équation en y

$$\Psi(x, y)=0,$$

éliminons y entre cette équation et

$$F(u, y)=0$$

où u désigne une nouvelle variable; nous obtiendrons un résultant

$$R(x, u)=0$$

dont les racines u définiront des substitutions $[x, u(x)]$, donnant lieu à des relations fonctionnelles telles que (6), ces substitutions variant en général avec la racine z considérée.

Si l'on convient au-contraire qu'une même substitution $[x, u(x)]$ doit donner lieu pour un certain groupe, et notamment pour un groupe circulaire de racines z de $F(x, z)=0$, à une même équation fonctionnelle, c'est-à-dire à une formule telle que (6), dans laquelle les fonctions a_i ne changent pas quand on permute entre elles deux racines quelconques du groupe, les fonctions a_i ne seront plus arbitraires. En particulier, si l'on veut que le groupe de racines en question contienne toutes les racines de $F(x, z)=0$, on formera les fonctions symétriques fondamentales des m fonctions y, données par la formule (6), et l'on obtiendra, si l'on considère $u(x)$ comme donnée, m équations de condition entre les fonctions a_i.

Si l'on désire de plus, dans le but de donner une forme simple à l'équation fonctionnelle (6), que les fonctions a_i soient en nombre $\mu < m$, les m équations de condition obtenues donneront, après élimination de $a_1, a_2, \ldots, a_\mu$, $m-\mu$ équations de condition entre les coefficients de $F(x, z)$.

Les considérations précédentes s'appliquent immédiatement à l'équation

$$xz^2-z+1=0.$$

La substitution $[x, u(x)]$ donne, en faisant $y=z[u(x)]$,

$$uy^2-y+1=0$$

et

$$y=az+b.$$

On a alors

$$\begin{cases} y_1+y_2=a(z_1+z_2)+2b, \\ y_1y_2=a^2z_1z_2+ab(z_1+z_2)+b^2, \end{cases}$$

ou bien

$$\begin{cases} \dfrac{x}{u} = a+2bx, \\ \dfrac{x}{u} = a^2+ab+b^2x. \end{cases}$$

On a donc l'équation de condition

$$a^2+ab+b^2x=a+2bx.$$

En d'autres termes, l'équation en u s'abaisse au premier degré et donne naissance à l'équation fonctionnelle

$$z\left(\frac{x}{a+2bx}\right)=az(x)+b.$$

Pour généraliser ce résultat, je considérerai l'équation

$$F(x, z)=xz^m+p_1z^{m-1}+p_2z^{m-2}+\ldots+p_m=0, \tag{7}$$

et je déterminerai les coefficients constants p_i, de telle sorte que cette équation n'ait, à distance finie, qu'un seul point critique d'ordre m.

L'élimination de x entre $F=0$ et $\frac{\partial F}{\partial z}=0$ donne immédiatement l'égalité

$$p_1 z^{m-1}+2p_2 z^{m-2}+3p_3 z^{m-3}+\dots+mp_m=0$$

et, en identifiant avec $p_1(z+\alpha)^{m-1}$, on obtient les relations

$$(8) \qquad \frac{\nu p_\nu}{p_1}=\binom{m-1}{\nu-1}\alpha^{\nu-1}, \qquad (\nu=2,\ 3,\ \dots, m),$$

qui lient les coefficients $p_3, p_4, \dots, p_m$ à p_1 et p_2 et qui définissent la valeur critique

$$z=-\frac{2}{m-1}\frac{p_2}{p_1}.$$

Par substitution dans (7) on obtient la valeur critique de x,

$$x=\frac{m-1}{2m}\frac{p_1^2}{p_2}.$$

Il est visible que, lorsque les conditions (8) sont satisfaites, toutes les racines de (7) ne forment qu'un seul système circulaire. On peut le vérifier et calculer la valeur approchée des m racines au voisinage du point critique, savoir

$$m \text{ pair}\dots\dots z=-\frac{2p_2}{(m-1)p_1}+\frac{2p_2}{(m-1)p_1}\left[1-\frac{2mp_2\,x}{(m-1)p_1^2}\right]^{\frac{1}{m}},$$

$$m \text{ impair}\dots z=-\frac{2p_2}{(m-1)p_1}+\frac{2p_2}{(m-1)p_1}\left[\frac{2mp_2\,x}{(m-1)p_1^2}-1\right]^{\frac{1}{m}}.$$

Maintenant, si l'on fait la substitution

$$\left[x,\ \frac{x}{\frac{2mp_2\,x}{(m-1)p_1^2}-1}\right],$$

un calcul direct conduit à l'équation fonctionnelle que je voulais établir:

$$z_i\left[\frac{x}{\frac{2mp_2\,x}{(m-1)p_1^2}-1}\right]=-\frac{2}{m-1}\frac{p_2}{p_1}-z_j(x),$$

dans laquelle les indices i et j peuvent prendre les valeurs $1, 2, \dots, m$, suivant la région du plan où se trouve le point x.

IV

Pour tirer le parti que j'ai annoncé des équations fonctionnelles établies au paragraphe II, il suffit de généraliser une formule d'Euler relative à la théorie des différences.

Soient $u_0, u_1, \ldots, u_n, \ldots$ une suite de nombres et $\Delta^n u_0$ la différence $n^{ième}$ du premier d'entre eux; soit encore

$$f(x) = \sum_0^\infty u_n x^n; \tag{9}$$

Euler a montré que

$$\frac{1}{1+x} f\left(\frac{x}{1+x}\right) = \sum_0^\infty \Delta^n u_0 x^n.$$

Plus généralement, désignons par $\Delta_m^n u_0$ la fonction linéaire définie par l'équation symbolique

$$\Delta_m^n u_0 = (u-m)^n,$$

où m est un paramètre arbitraire; on tire de (9)

$$u_n = \mathcal{E} \frac{f(z)}{[z^{n+1}]},$$

d'où

$$\Delta_m^n u_0 = \mathcal{E} f(x) \frac{(1-mz)^n}{z^{n+1}},$$

et

$$\sum_{n=0}^\infty \Delta_m^n u_0 x^n = \mathcal{E} \frac{f(z)}{[z-(1-mz)x]} = \mathcal{E} \frac{1}{1+mx} \frac{f(z)}{\left[z - \dfrac{x}{1+mx}\right]},$$

ou encore

$$\frac{1}{mx+1} f\left(\frac{x}{mx+1}\right) = \sum_{n=0}^\infty \Delta_m^n u_0 x^n.$$

Concevons que l'on ait développé par la série de Lagrange une racine de l'équation

$$xz^2 + az + b = 0$$

sous la forme

$$z(x) = \sum_0^\infty u_n x^n;$$

les équations (A) et (B) donneront lieu aux relations symboliques:

$$\left(u - \frac{4b}{a^2}\right)^n + \frac{4b}{a^2}\left(u - \frac{4b}{a^2}\right)^{n-1} = (-1)^{n-1} u_n,$$

$$\left(u - \frac{2b}{a^2}\right)^{2n} + \frac{2b}{a^2}\left(u - \frac{2b}{a^2}\right)^{2n-1} = 0,$$

$$\left(u - \frac{2b}{a^2}\right)^{2n+1} + \frac{2b}{a^2}\left(u - \frac{2b}{a^2}\right)^{2n} = -\left(\frac{b}{a^2}\right)^{n+1} u_n.$$

En particulier si l'on considère la suite

$$(v)\quad 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ldots\ldots,$$

$$v_{n-1}=\frac{(n+2)(n+3)\ldots 2n}{n!},$$

les nombres v_n sont les coefficients du développement de ζ^2, ζ étant la racine, régulière à l'origine, de

$$xz^2-z+1=0,$$

et l'on a:

$$\Delta_2^{2n+1}v_0=0,\qquad \Delta_2^{2n+2}v_0=v_n,\qquad (-1)^n\Delta_4^n v_0=v_n,$$

$$v_n=4^n v_0-\binom{n}{1}4^{n-1}v_1+\binom{n}{2}4^{n-2}v_2-\ldots\ldots,$$

$$\Delta^n v_0=2^n v_0-\binom{n}{1}2^{n-1}\Delta^1 v_0+\binom{n}{2}2^{n-2}\Delta^2 v_0-\ldots\ldots,$$

et

$$\frac{(n+2)\ (n+3)\ldots 2n}{n!}=2^{n-1}+\frac{1}{2}\binom{n-1}{2}2^{n-3}\frac{2}{1}+\frac{1}{3}\binom{n-1}{4}2^{n-5}\frac{4.3}{1.2}$$

$$+\frac{1}{4}\binom{n-1}{6}2^{n-7}\frac{6.5.4}{1.2.3}+\ldots\ldots.$$

SUR L'*INTÉGRATION LOGIQUE* DES ÉQUATIONS DIFFÉRENTIELLES: APPLICATIONS AUX ÉQUATIONS DE LA GÉOMÉTRIE ET DE LA MÉCANIQUE

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GÉNÉRALITÉS

Groupe de rationalité. Je rappellerai d'abord ce qu'il faut entendre par «*intégration logique*» d'une équation différentielle ordinaire:

$$\frac{d^n y}{dx^n} = f\left(x,\, y,\, \frac{dy}{dx},\, \ldots,\, \frac{d^{n-1}y}{dx^{n-1}}\right) \tag{1}$$

dont le second membre est une fonction de $(n+1)$ arguments appartenant à un certain domaine de rationalité (Δ) qui sera précisé plus tard.* L'équation (1) étant remplacée par

$$X(z) = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} y' + \ldots + \frac{\partial z}{\partial y^{(n-1)}} f = 0 \tag{2}$$

où $y' = \frac{dy}{dx}$, ..., qui en définit les intégrales premières, sa solution générale peut être donnée par les relations implicites,

$$z_i(x,\, y,\, y',\, \ldots,\, y^{(n-1)}) = C_i, \quad (i = 1,\, \ldots,\, n),$$

où les z_i sont un système fondamental de solutions de (2) et les C_i des constantes arbitraires. Or à l'égard d'une équation quelconque à $(n+1)$ variables $x,\ x_1, \ldots,\ x_n$

$$X(z) = \frac{\partial z}{\partial x} + A_1 \frac{\partial z}{\partial x_1} + \ldots + A_n \frac{\partial z}{\partial x_n} = 0 \tag{3}$$

dont les coefficients A_i sont des fonctions de $x,\ x_1,\ \ldots,\ x_n$ d'un domaine (Δ), deux circonstances peuvent se présenter:

1° Quel que soit le système fondamental particulier (z_i) choisi, il n'y a pas d'équations rationnelles en $z_i, \frac{\partial z_i}{\partial x_k}, \frac{\partial^2 z_i}{\partial x_h dx_k}, \ldots$ (les dérivées des $\frac{\partial z}{\partial x}$ sont donc

*Un exposé sommaire des points essentiels de la théorie, qui remonte à 1893, a été donné au Congrès International de Cambridge, 1912 (voir Proceedings, I, p. 438-497, II, p. 145-159); d'autres applications ont été indiquées au Congrès International de Strasbourg, 1920 (voir Comptes Rendus du Congrès, p. 356-380). Les références s'y rapportant seront indiquées simplement par les lettres C et S.

exclues) à coefficients rationnels dans (Δ), vérifiées par ce système. L'équation (3) est dite *générale*: les fonctions $z_1, \ldots, z_n$ sont définies par les relations: $X(z_i)=0$, à une transformation ponctuelle quelconque près. Ce sont des transcendantes, fonctions des $(n+1)$ arguments x, x_i, *attachées dans le domaine* (Δ) *au groupe ponctuel général* Γ_n.

2° Pour certains systèmes fondamentaux (z_i), il existe des relations de la forme indiquée plus haut. L'équation (3) est *spéciale*. On peut donner à ces relations—lorsqu'on veut définir les transcendantes (z_i) *les plus simples*—la forme:

$$(\Sigma) \qquad \Omega_i\left(z_1, \ldots, z_n, \frac{\partial z_1}{\partial x_1}, \ldots, \frac{\partial z_n}{\partial x_n}, \frac{\partial^2 z_1}{\partial x_1^2}, \ldots\right) = \alpha_i\,(x, x_1, \ldots, x_n), \quad (i=1, \ldots, k),$$

où les Ω_i sont des fonctions rationnelles de leurs arguments qui constituent *l'ensemble des invariants différentiels caractéristiques* d'un certain groupe continu (fini ou infini) de transformations en $z_1, \ldots, z_n$—groupe Γ—quand on regarde les z_i comme fonctions des x_k, *non transformés*.

Ces invariants Ω_i sont *rationnellemdent* distincts mais non fonctionnellement distincts: ils peuvent être liés par, des relations, nécessairement algébriques. Tout autre invariant de Γ, rationnel par rapport aux mêmes éléments, est fonction rationnelle des Ω_i. Le système (Σ) est *irréductible*—c'est-à-dire que toute relation de même nature, compatible avec les équations (Σ), donc vérifiée par *un* système fondamental au moins—est une conséquence nécessaire des équations (Σ). Il est aussi *primitif*, c'est-à-dire qu'aucune transformation: $z_i=\phi_i(Z_1, \ldots, Z_n)$, qui conserve aux premiers membres de (Σ) leur caractère rationnel, ne permet d'abaisser l'ordre différentiel minimum de (Σ) ou d'augmenter le nombre des équations d'un ordre donné ou même d'abaisser le degré d'une certaine résolvante algébrique qui définit les dérivées *principales* au moyen des dérivées *paramétriques*.

Dans ces conditions, le groupe Γ est dit *groupe de rationalité* de (3); les transcendantes $z_1, \ldots, z_n$ sont des fonctions des $(n+1)$ arguments x, x_i *attachées au groupe* Γ *dans le domaine* (Δ).

Elles sont définies par (Σ) à *des transformations* près de Γ, donc, en général inséparables.

On peut prendre pour Γ l'un des *types* de groupes ponctuels en $z_1, \ldots, z_n$ déterminés *a priori* par S. Lie. On peut d'ailleurs trouver ces types directement, par voie algébrique.

Enfin Γ peut-être remplacé par Γ', pourvu que la transformation (z, z') change tout invariant rationnel Ω_i en un invariant rationnel Ω'_i.

Faire l'*intégration logique* de (1) c'est *d'abord* déterminer le groupe de rationalité de l'équation (2) correspondante.

Pour cette détermination il y a lieu d'envisager les divers types de groupes possibles et d'établir pour chacun d'eux un ensemble d'invariants caractéristiques Ω_i. J'ai montré (C, p. 27-37) qu'on peut choisir les Ω_i de manière qu'ils subissent, pour toute transformation de Γ_n, une transformation projective—qu'on peut même ramener à la forme linéaire par *adjonction* d'un certain déterminant fonctionnel—et indiqué la formation du *système résolvant* qui définit les Ω_i en partant des invariants du groupe ponctuel général Γ_n, c'est-à dire des A_i. Si Γ est le groupe de rationalité de (2), le *système résolvant* pour les Ω_i possède une *solution rationnelle dans* (Δ) *et une seulement.*

Réduction du groupe de rationalité. Une fois fixé le groupe Γ de rationalité, sa *structure* permet de ramener la détermination des z_i à une succession *théorique* d'opérations, dans le cas où Γ possède des sous-groupes invariants. En fait, ces opérations, dont l'ensemble seul est déterminé, donnent une *décomposition théorique* du problème d'intégration de (Σ) qui est le cadre naturel de l'étude, au point de vue de la théorie des fonctions, des transformations subies par les fonctions z_i lorsque les variables complexes x, x_i décrivent respectivement dans leur plan des contours fermés. Ces transformations forment le *groupe de monodromie* de (Σ). On peut encore dire que les z_i s'obtiennent en partant des α_i choisis dans (Δ) par des *adjonctions successives* de transcendantes *attachées* à des groupes *simples* (qui ne possèdent pas de sous-groupe invariant) dans un domaine de rationalité convenable (D).

Il reste pour terminer à étudier par les méthodes de la théorie des fonctions, les systèmes auxiliaires successifs, à groupe simple, que l'on a dû former (Cf. plus loin, *Etude fonctionnelle*).

Remarques. I. Parmi les solutions z_i de (3), il peut y en avoir de rationnelles; on peut les englober dans la théorie générale et écrire tout de suite les relations $z_i=R_i(x, x_1, \ldots, x_n)$ $(i=1, \ldots, k)$ correspondantes [qui sont des équations de (Σ)] auquel cas les transformations de Γ seront de la forme

$$Z_i=z_i, \quad (i=1, \ldots, k); \; Z_{k+l}=\phi_l(z_1, \ldots, z_n), \quad (l=1, \ldots, n-k),$$

ou bien *adjoindre* les z_i, ce qui donne dans un champ algébrique (D), une équation

$$X(z)=\frac{\partial z}{\partial x}+A_1\frac{\partial z}{\partial x_1}+\ldots+A_{n-k}\frac{\partial z}{\partial x_{n-k}}=0, \tag{4}$$

à $(n-k+1)$ variables seulement, dépendant des paramètres $z_1, \ldots, z_k$. Le groupe Γ correspond au groupe de (4) quand les z_i $(i=1, \ldots, k)$ sont des paramètres arbitraires; si l'on fixe certaines de ces constantes, il peut y avoir réduction.

II. La distinction la plus importante relative aux groupes Γ est cette relative à la *primitivité*. Une équation (3) est *imprimitive* si certaines des solutions $z_1, \ldots, z_n$, par exemple les k premières seules, peuvent vérifier un système (Σ). Dans ce cas il existe un système de $(n-k)$ équations linéaires, (B) qu'on peut écrire:

$$\text{(B)} \qquad \begin{array}{c} \dfrac{\partial z}{\partial x_1}+B_{11}\dfrac{\partial z}{\partial x_{k+1}}+\ldots+B_{1,n-k}\dfrac{\partial z}{\partial x_n}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial z}{\partial x_k}+B_{k1}\dfrac{\partial z}{\partial x_{k+1}}+\ldots+B_{k,\,n-k}\dfrac{\partial z}{\partial x_n}=0, \end{array}$$

dont les coefficients sont rationnels—ou *algébriques* dans (Δ)—qui forment avec (3) un système jacobien dont les solutions sont $z_1, \ldots, z_k$. Les B_{ij} sont rationnels quand un seul groupe de solutions $z_1, \ldots, z_k$ peut être ainsi *isolé*, ils sont algébriques s'il y en a plusieurs. En commençant la recherche par celles des systèmes (B) à une équation, on voit aisément que les B_{ij} ne peuvent dépendre de constantes arbitraires que si les z sont en tout ou en partie, des transcendantes *attachées* à des groupes *linéaires*. Dans certains cas, l'essentiel de la théorie d'intégration logique peut se retrouver dans la théorie donnée par M. Émile Picard pour les équations différentielles linéaires. Les transcendantes $z_1, \ldots, z_n$ dépendant de $(n+1)$ arguments peuvent s'obtenir par des opérations successives portant sur des fonctions où un seul argument varie. Il en est également ainsi pour tout groupe de rationalité *fini*.

Quand l'équation (3) est *primitive*, $z_1, \ldots, z_n$ sont *inséparables*. La décomposition de Γ en une suite de groupes dont chacun est invariant maximum dans le précédent, donne la seule décomposition *théorique* du problème de l'intégration de (Σ).

Extension du domaine de rationalité. Le domaine (Δ) peut d'abord être le domaine *absolu*, ensemble des fonctions rationnelles à coefficients rationnels numériques (puis algébriques) de x, x_i; on peut l'étendre en y ajoutant des paramètres constants, puis des fonctions algébriques de x, x_i.

D'une manière générale on peut *adjoindre* à (Δ) tout élément u fonction de $x, x_1, \ldots, x_n$ satisfaisant à un système (S) de relations d'ordre quelconque:

$$(S) \qquad F_i\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x_1}, \ldots \middle| x, x_1, \ldots, x_n\right) = 0, \quad (i=1, \ldots, h),$$

dont les premiers membres sont des polynomes entiers en $u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x_1}, \ldots$ à coefficients rationnels dans (Δ), pourvu que ces relations, compatibles, déterminent *d'une manière unique* le calcul (addition, multiplication, dérivation) des fonctions rationnelles du nouveau domaine. Si le système (S) n'est pas *irréductible* (c'est-à-dire s'il existe des relations de même nature, compatibles avec (S) sans en être des conséquences nécessaires), il faudra donc ajouter à (S) des *inégalités* pour exclure ces relations. Je dis alors que u est *bien défini*. Par exemple, on peut adjoindre l'exponentielle Ce^x, bien définie par $\frac{\partial u}{\partial x} = u$, avec $u \neq 0$.

Les fonctions *arbitraires* de moins de $(n+1)$ arguments figurent parmi les fonctions bien définies.

Un système complet de $(n+1-k)$ équations linéaires par rapport aux dérivées $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}$ dont les coefficients appartiennent à (Δ) donne naturellement pour ses solutions *les plus simples* $z_1, \ldots, z_k$, des transcendantes *attachées* au groupe ponctuel général à k arguments, Γ_k, ou à l'un de ses sous-groupes *types* Γ. La théorie s'étend donc aux systèmes complets linéaires. Il suit de là qu'elle s'applique à la détermination des *intégrales complètes* des systèmes d'équations non linéaires à une inconnue (C, p. 54).

Les groupes de rationalité correspondants peuvent être pris parmi les types des groupes de *transformations de contact*, mais les sous-groupes de groupes semblables aux groupes de contact sont parfois plus commodes.

Étude fonctionnelle. Enfin, il est clair que l'on peut répéter les raisonnements qui conduisent aux conclusions précédentes *en se bornant au voisinage d'un point ou d'un domaine singulier* pour l'équation (3), pourvu que les coefficients se comportent *dans ce voisinage* comme des fonctions rationnelles de (Δ), c'est-à-dire soient méromorphes. On déduit de là une classification rationnelle des domaines singuliers de (3), basée sur l'existence d'un groupe Γ *au voisinage* du domaine singulier (C, p. 22). L'étude, au point de vue de la théorie des fonctions, des transcendantes $z_1, \ldots, z_n$ *attachées* à un groupe G, même simple, et en particulier au groupe général Γ_n dans le domaine (Δ), comportera donc la détermination des groupes Γ pour les divers domaines singuliers et celle des transformations subies par $z_1, \ldots, z_n$ quand on passe du voisinage d'un domaine singulier à celui d'un autre domaine singulier.

Observons, en passant, que la recherche, par le moyen d'équations différentielles rationnelles, de transcendantes *uniformes qui ne se ramènent pas à celles définies par des équations linéaires*, doit conduire nécessairement à des transcendantes *attachées à un groupe infini* dans le domaine absolu (Δ). Si ces transcendantes ne s'obtiennent pas par des quadratures successives, ce groupe infini est *primitif*. C'est essentiellement parce que le groupe de rationalité de l'équation de M. Painlevé: $y''=6y^2+\alpha x$ est le groupe *infini* à 2 arguments, *primitif* et même *simple*, donné par $\dfrac{\partial(\Phi, \Psi)}{\partial(\phi, \psi)}=1$ où ϕ, ψ sont les deux intégrales de

$$X(f)=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\,y'+\frac{\partial f}{\partial y'}\,(6y^2+\alpha x)=0$$

que les transcendantes y, y' sont *nouvelles*. Il en est de même, *a fortiori*, pour les autres équations de M. Painlevé. (Cf. Bulletin des Sciences Math., juillet 1915).

Équation du Premier Ordre

L'application de la théorie aux équations du premier ordre

$$\frac{dy}{dx}=A(x, y), \tag{1}$$

donne des résultats très simples.

L'équation

$$X(z)=\frac{\partial z}{\partial x}+A\frac{\partial z}{\partial y}=0, \tag{2}$$

lorsqu'elle est *spéciale*, peut posséder:

(α) une solution rationnelle dans (Δ) : $z=R(x, y)$,

(β) une solution z pour laquelle $\left(\dfrac{\partial z}{\partial y}\right)^n$, (où n est entier positif), est rationnel.

Si l'on a $\left(\frac{\partial z}{\partial y}\right)^n = K(x, y)$ la fonction K rationnelle dans (Δ) satisfait à la résolvante:

$$X(K) + nK\frac{\partial A}{\partial y} = 0,$$

qui n'a qu'une seule solution rationnelle.

La transcendante z est définie aux transformations près ($Z = \epsilon z + a$), $\epsilon^n = 1$, a est constant, qui forment le groupe Γ.

(γ) une solution z pour laquelle $\frac{\partial^2 z}{\partial y^2} : \frac{\partial z}{\partial y}$ est rationnel.

Si l'on pose $\frac{\partial^2 z}{\partial y^2} = J\frac{\partial z}{\partial y}$, J est la seule solution rationnelle de la résolvante:

$$X(J) + J\frac{\partial A}{\partial y} + \frac{\partial^2 A}{\partial y^2} = 0.$$

La transcendante z est *attachée* au groupe Γ linéaire ($Z = az + b$). Elle s'obtient par deux quadratures superposées, avec *adjonction* de l'exponentielle.

(δ) Une solution z pour laquelle l'invariant *projectif* de *Cayley-Schwarz*

$$I = \frac{\frac{\partial^3 z}{\partial y^3}}{\frac{\partial z}{\partial y}} - \frac{3}{2}\left(\frac{\frac{\partial^2 z}{\partial y^2}}{\frac{\partial z}{\partial y}}\right)^2$$

est rationnel dans (Δ)

La résolvante en I, avec une seule solution rationnelle, est:

$$X(I) + 2I\frac{\partial A}{\partial y} + \frac{\partial^3 A}{\partial y^3} = 0.$$

Dans ce cas z est défini aux transformations projectives près $\left(Z = \frac{az+b}{cz+d}\right)$ où a, b, c, d sont des constantes, qui forment Γ.

Le système différentiel qui définit z peut-être remplacé par un système de deux équations de Riccati définissant J, et des quadratures définissant z, mais cette procédure ne décompose pas le problème; le groupe projectif Γ est *simple*. On peut aussi remplacer le système de Riccati par un système linéaire à deux inconnues, dont le groupe est le groupe linéaire spécial.

Ainsi, en dehors de ces cas bien précis, l'équation est *générale*. On doit l'étudier par les méthodes de la théorie des fonctions (domaines singuliers et groupes Γ correspondants, etc.).

J'ai montré en détail (C. p. 6) quel intérêt il y a, lorsque l'équation (1) est spéciale, à définir implicitement y au moyen de x et de la constante z par la méthode précédente. Si l'on envisage la relation $y = f(x, x_0, y_0)$ qui donne la solution y se réduisant à y_0 pour $x=x_0$, y est une fonction de *trois arguments* et la solution *principale* en x_0 pour (2) est: $u=f(x_0, x, y)$. Or déjà dans le cas simple où $\left(\frac{\partial z}{\partial y}\right)^n$ est rationnel, u est défini par les équations

$$X(u)=0,\ K(x_0, u)\left(\frac{\partial u}{\partial y}\right)^n=K(x, y)$$

aux transformations (u, v) près déterminées par:

$$K(x_0, u)\left(\frac{\partial u}{\partial v}\right)^n=K(x_0, v)$$

qui dépendent de x_0 *et aussi de la forme de* K, et sont en général transcendantes.

Détermination du groupe de rationalité. La détermination effective, pour une équation entièrement déterminée dans un domaine (Δ), du groupe de rationalité Γ, se ramène essentiellement à celle des polynomes irréductibles, P, formés avec les arguments de (Δ), qui satisfont à une identité:

$$X(P)=\frac{\partial P}{\partial x}+A_1\frac{\partial P}{\partial x_1}+\ldots+A_n\frac{\partial P}{\partial x_n}=MP$$

où M est également un polynome de (Δ), c'est-à-dire à la recherche des équations: $P=0$, *invariantes* par l'opération $X(f)$. C'est là un problème difficile qui relève en général de recherches arithmétiques.

J'ai indiqué en particulier (C, p. 13) comment dans le cas le plus simple d'une équation

$$\text{(1)}\qquad \frac{dy}{dx}=\frac{\alpha(x, y)}{\beta(x, y)}$$

où α et β sont des polynomes en x et y, on peut—s'il n'existe que deux *valeurs remarquables* de z, pour lesquelles $P-zQ$, indécomposable en général, possède un facteur multiple—*reconnaître si l'équation admet une solution rationnelle:* $z=\frac{P}{Q}$. On y fait voir aussi que la méthode s'applique encore dans le cas de 3 et 4 valeurs remarquables, pour certaines valeurs particulières des exposants des facteurs multiples. Les recherches classiques de Darboux (Bulletin des Sciences Math. 1878) et celles de H. Poincaré et P. Painlevé (Cf. *Leçons de Stockholm*), où l'on se préoccupe de limiter le degré de P et Q par l'étude des intégrales de (1) au voisinage des points singuliers, montrent la difficulté de la question.

La *recherche du groupe de rationalité* d'une équation

$$\text{(1)}\qquad Xdy-Ydx=0$$

c'est-à-dire d'abord celle de toutes les intégrales particulières algébriques de l'équation ou de tous les polynomes irréductibles $f(x, y)$ qui satisfont à une identité

$$\Omega(f)=X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}=Mf,$$

où M est un polynome de degré $m-1$, quand X et Y sont de degré m, m'a conduit (Comptes Rendus Acad. Sciences, Paris, 27 janvier 1919) aux conclusions suivantes:

Supposons, avec H. Poincaré, que les points communs aux courbes $X=0$, $Y=0$ soient simples et à distance finie. Si l'on désigne par μ et μ' les racines de l'équation en M, au point singulier x, y:

$$S=\left(\frac{\partial X}{\partial x}-M\right)\left(\frac{\partial Y}{\partial y}-M\right)-\frac{\partial Y}{\partial x}\frac{\partial X}{\partial y}=0,$$

lorsque la courbe $f=0$ a en ce point un point multiple d'ordre n, le polynome M y prend une valeur $p\mu+p'\mu'$ où p et p' sont des entiers positifs tels que $p+p'=n$. On déduit de là, en éliminant les coefficients de M dans les équations:

$$M(\xi_i, \eta_i)=p_i\mu_i+p_i'\mu_i'$$

relatives aux m^2 points singuliers, $\frac{m(m-1)}{2}$ relations linéaires

$$(\Delta) \qquad \sum\theta_i(p_i\mu_i+p'_i\mu'_i)=0$$

où les θ_i sont des déterminants formés avec les coordonnées ξ_j, η_j des divers points singuliers. Ces relations, où les p_i, p_i' seuls sont inconnus, doivent donc admettre des solutions *entières*, *positives ou nulles*, pour ces inconnues.* (On voit pourquoi elles avaient échappé au géomètres faisant l'étude locale des points singuliers). La courbe M ne peut passer (sauf cas exceptionnels) que par $\frac{m(m+1)}{2}-1$ points singuliers: chaque courbe $f=0$ passe au moins par $\frac{m(m-1)}{2}+1$ de ces points, pour lesquels $p_i+p_i'>0$.

En outre, comme en tous les points singuliers qui ne sont pas des *nœuds* (points où $\frac{\mu'}{\mu}$ est un nombre rationnel *positif*), il passe au plus *deux* branches d'intégrales algébroïdes, si f est *irréductible*, on a en ces points: $p+p'\leq 2$.

D'autres conseq́uences des relations (Δ) résultent du fait que toutes leurs solutions entières et positives sont des combinaisons linéaires, à coefficients entiers positifs, de *k solutions fondamentales*: le polynome M le plus général possible est donc $\alpha_1M_1+\ldots+\alpha_kM_k$ où les α_i sont des entiers positifs quelconques. Lorsque f est un polynome irréductible, les α doivent être pris de façon que *pour tout point autre qu'un nœud la multiplicité soit au plus* 2. Il est clair que la limitation des α en résulte généralement. Or leur détermination donne, pour tous les points singuliers, les entiers p et p', donc aussi la multiplicité de f; on en déduit aisément le degré possible de f.

Quand l'intégrale de (1) est algébrique, on peut l'écrire $z=\frac{P}{Q}$ où P et Q sont irréductibles et ne s'annulent qu'aux nœuds. Soit M le polynome $\alpha_1M_1+\ldots+\alpha_kM_k$ correspondant; pour les valeurs remarquables z où $f=Qz-P$ passe par un

*D'autres relations, en nombre $(m+1)$, se déduisent de l'étude des points à l'infini des deux courbes: $X=0$, $Y=0$.

point singulier autre qu'un nœud, on aura en général un autre groupement $\beta_1 M_1 + \ldots + \beta_k M_k$ identique à M, donc au moins une identité:

$$\alpha_1 M_1 + \ldots + \alpha_k M_k = \beta_1 M_1 + \ldots + \beta_k M_k.$$

Enfin si (1) possède h solutions particulières algébriques, le système (Δ) admet des solutions dépendant de h entiers positifs arbitraires; on a donc $k \geq h$.

Exemples. La détermination du groupe de rationalité d'une équation donnée, rencontrée dans une question de Géométrie par exemple, n'exige pas toujours, heureusement, une discussion arithmétique. Si l'équation renferme des paramètres, en observant que poùr certaines valeurs des paramètres la difficulté de l'intégration s'abaisse et permet de trouver le groupe G, on peut affirmer que Γ contiendra G comme sous-groupe; si l'on a plusieurs groupes tels que G, Γ contiendra un ensemble de groupes semblables aux différents G, quand ces sous-groupes ne correspondent pas aux mêmes intégrales.

J'ai réussi de cette manière (*C*, p. 19) à trouver le groupe de rationalité de l'équation différentielle des lignes de courbure pour la *surface des ondes* de Fresnel, c'est-à-dire, en fait, à intégrer cette équation à 3 paramètres, à l'aide d'intégrales abéliennes. Cette question avait occupé divers géomètres, dont Cayley. Peu après, j'ai pu former le groupe de rationalité de l'équation différentielle des lignes *asymptotiques*, pour la *surface générale du troisième degré.* Cette équation s'intègre encore en égalant à une constante une intégrale de différentielle algébrique. Elle dépend de quatre paramètres et j'ai pu étudier en détail sa réduction dans le cas où la surface possède des singularités.

Un autre exemple précis, emprunté à la Mécanique, sera donné à la fin de ce travail: le groupe de rationalité du système qui définit le mouvement d'un solide pesant autour d'un point fixe.

Pour les équations de la Géométrie (j'appelle ainsi celles où figurent certains paramètres ou certaines fonctions d'un ou de plusieurs arguments, dont on ne connait pas la nature transcendante—qui sont donc *a priori* arbitraires), j'ai indiqué (C, II, p. 2) des méthodes *régulières* qui permettent d'utiliser des propriétés géométriques connues de familles de courbes ou de surfaces pour déterminer ces familles en partant de leurs équations différentielles. Ces méthodes ne sont d'ailleurs que l'application particulière d'une théorie qui conduit à *utiliser au mieux la connaissance fortuite de relations rationnelles entre les z_i, solutions d'une équation linéaire*

$$X(z) = 0, \tag{2}$$

à $(n+1)$ variables $x, x_1, \ldots, x_n$, leurs dérivées de tous ordres: $\frac{\partial z_i}{\partial x_k}$, $\frac{\partial^2 z_i}{\partial x_h \partial x_k}$, $\ldots$, et les variables, ces dernières ne figurant que par des fonctions du domaine (Δ), *pour l'intégration logique de l'equation* (2).

En fait, on déduit toujours de ces relations la connaissance sous forme de fonction de (Δ), des invariants rationnels caractéristiques d'un certain groupe G. Ce groupe G est nécessairement un groupe qui renferme Γ; il peut être le groupe de rationalité, Γ, lui-même.

Applications

On a rangé sous cette rubrique toute une série de recherches où l'on se propose de former *a priori*, parmi les équations de catégories données, où l'une des variables *au moins* figure rationnellement, tous les *types* d'équations qui peuvent s'intégrer logiquement par des moyens fixés d'avance, en d'autres termes qui posséderont un groupe de rationalité donné, Γ, dans le domaine (Δ) auquel appartiendront les coefficients de l'équation. Les plus intéressantes sont d'abord celles où, *en dernière analyse*, on sera ramené à une intégration d'une équation du premier ordre se faisant par quadratures. Des signes de quadrature suffisent en fait à expliciter les opérations nécessaires pour définir les coefficients de l'équation à former. J'ai montré, par des exemples, ce qu'il faudrait envisager comme *opération explicite* pour traiter des cas plus étendus (S, p. 356-380).

Dans la plupart de ces recherches on a été conduit à intégrer logiquement des équations différentielles, ou aux dérivées partielles, d'ordre élevé. On a pu le faire par une analyse intime de la question, en exprimant essentiellement que les transformations des éléments $z_1, z_2, \ldots, z_n$ d'un système fondamental appartiennent au groupe Γ de rationalité fixé d'avance. Il a été nécessaire d'introduire explicitement et de regarder comme variables, d'abord indépendantes, les éléments qui déterminent les *singularités* des coefficients ou des intégrales de l'équation à déterminer et aussi quand la question exige l'étude d'équations aux dérivées partielles, les *caractéristiques, au sens d'Ampère*, de ces équations aux dérivées partielles. Après avoir rappelé rapidement quelques-unes des recherches antérieures et leur intérêt géométrique, nous montrerons par de nouveaux exemples, la puissance de la méthode adoptée.

I. Le premier exemple est donné par les équations

$$(A) \qquad \frac{dy}{dx} = \frac{\alpha(y)}{\beta(y)}$$

où α et β sont deux polynomes en y de degré donné, dont les coefficients sont des fonctions de x à déterminer. Les transformations qui conservent la forme de l'équation permettent d'adopter un certain *type* quand le degré n de $\alpha(y)$ est fixé et j'ai pu former méthodiquement les équations $X(z) = \frac{\partial z}{\partial x} + \frac{\alpha(y)}{\beta(y)} \frac{\partial z}{\partial y} = 0$, dont le groupe de rationalité est l'un des trois groupes α, β, γ signalés plus haut, c'est-à-dire pour lesquelles z, $\left(\frac{\partial z}{\partial y}\right)^n$ ou $\frac{\partial^2 z}{\partial y^2} : \frac{\partial z}{\partial y}$ sont rationnels en y, les coefficients appartenant à un domaine (Δ) qui comprend les coefficients de $\alpha(y)$ et de $\beta(y)$ (S, p. 358). J'ai étudié en détail les équations $\frac{dy}{dx} = \frac{P_3(y)}{P_1(y)}$ sur une de leurs formes réduites qui se trouve être *l'équation de la Balistique extérieure;* la méthode est très longuement expliquée dans les Annales de l'École Normale Supérieure, 1920. On sait que ces équations sont, en particulier, celles qui permettent de trouver les *lignes géodésiques des surfaces spirales.*

La détermination des *lignes de longueur nulle des surfaces réglées* se ramène aisément aussi à l'intégration d'une équation de la forme générale, dont on peut indiquer tous les cas de réduction aux quadratures.

II. Un autre exemple—lié à la *déformation infiniment petite des surfaces minima*—est celui des cas de réduction aux quadratures de l'équation linéaire:

$$\frac{d^2y}{dx^2}=[\phi(x)+h]y$$

ou mieux de l'équation de Riccati:

$$\frac{d\rho}{dx}+\rho^2=\phi(x)+h$$

où $\phi(x)$ est à déterminer et h, paramètre, demeure arbitraire. Il nous a conduit à l'extension naturelle des recherches mémorables d'Hermite et de M. É. Picard sur l'équation de Lamé. A noter que, soit dans l'étude de cette dernière équation soit dans celle du groupe des équations: $\dfrac{dy}{dx}=\dfrac{P_3(y)}{P_1(y)}$, nous avons retrouvé pour des cas de réduction du groupe de rationalité (mais non à des groupes intégrables) les transcendantes de M. Painlevé.

III. J'ai montré sur des exemples (S, p. 365) le rôle que joue dans une équation du premier ordre, un paramètre qui demeure arbitraire et figure rationnellement, *en principe*, comment on peut établir les formes: $\dfrac{dy}{dx}=R(\phi)$ où R est le quotient de polynomes de degrés *donnés* en ϕ à coefficients dépendant de x, y, de manière que cette équation s'intègre par quadratures.

Dans ce dernier exemple, où il s'agit de déterminer des fonctions des deux variables x, y, interviennent, pour la première fois, des équations aux dérivées partielles à deux variables et à une fonction inconnue mais *d'ordre quelconque*. Le rôle des *variables caractéristiques d'Ampère* y apparait.

IV. Il se précise dans l'étude (S, p. 374) des équations du second ordre:

$$\frac{d^2y}{dx^2}=F(x, y)$$

où F est à déterminer de façon que cette équation possède une intégrale première rationnelle en $\dfrac{dy}{dx}$, auquel cas l'équation s'intègre complètement par quadratures.

Dans les pages suivantes, nous examinons spécialement l'équation des géodésiques—prise sous forme *type* avec des coordonnées symétriques—et aussi, sur des exemples particuliers, l'équation générale:

$$\frac{d^2y}{dx^2}=R\left(\frac{dy}{dx}\right)$$

où R est le quotient de deux polynomes en $\frac{dy}{dx}$, de degrés donnés, dont les coefficients seront des fonctions de x, y à déterminer.

Équation des lignes géodésiques

Je me propose la détermination des éléments linéaires $ds^2=4\lambda dudv$ pour lesquels l'équation différentielle des lignes géodésiques

$$v''=\frac{1}{\lambda}\frac{\partial\lambda}{\partial u}v'-\frac{1}{\lambda}\frac{\partial\lambda}{\partial v}v'^2 \tag{1}$$

possède une intégrale première *rationnelle* en v'.

On sait que l'intégration de (1) et celle de l'équation aux dérivées partielles:

$$F=\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}-\lambda=0 \tag{2}$$

sont des problèmes équivalents, la correspondance entre les deux étant définie par:

$$\frac{\partial z}{\partial u}=\sqrt{\lambda v'}, \qquad \frac{dz}{\partial v}=\sqrt{\frac{\lambda}{v'}}.$$

Jacobi a d'ailleurs observé que si l'on connait une solution z de (2) dépendant d'un paramètre ϕ,

$$\frac{\partial z}{\partial\phi}=\psi$$

donne l'intégrale générale de (1).

On n'avait pu traiter le problème (2) que dans les cas où il existe une intégrale $\phi(p, q)=\text{const.}$ de l'équation

$$(F, \phi)=\frac{\partial F}{\partial u}\frac{\partial\phi}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial\phi}{\partial u}+\frac{\partial F}{\partial v}\frac{\partial\phi}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial\phi}{\partial v}=0$$

où $p=\frac{\partial z}{\partial u}$, $q=\frac{\partial z}{\partial v}$, intégrale qu'on peut supposer homogène, *linéaire* en p, q (ds^2 de révolution, Bour), *quadratique* en p, q (ds^2 de Liouville et de Lie) ou *fractionnaire* et du premier degré en p, q (O. Bonnet). Tout un chapitre des *Leçons sur la théorie des surfaces* de Darboux (livre VI, Chap. IV) témoigne du vain effort des géomètres: Bour, O. Bonnet, Maurice Lévy, Darboux, pour étendre ces résultats.

Intégrales entières d'ordre impair. La fonction $\phi(p, q)$, étant supposée *homogène* en p, q, nous étudions d'abord le cas où elle est d'ordre impair: $m=2n+1$.

Pour simplifier l'écriture, considérons au lieu de v' la variable $w=\lambda v'$ (c'est-à-dire p^2); l'équation (1) donne alors l'équation aux dérivées partielles:

$$A(f)=\lambda\frac{\partial f}{\partial u}+w\frac{\partial f}{\partial v}+2w\frac{\partial\lambda}{\partial u}\frac{\partial f}{\partial w}=0. \tag{3}$$

Si l'on possède une solution: $\phi=\phi(w)$ une autre solution ψ sera donnée par la quadrature

$$d\psi=\frac{\lambda dv-wdu}{w^{3/2}\dfrac{\partial\phi}{\partial w}} \tag{4}$$

où w est défini par l'équation $\phi=\phi(w)$ au moyen de u, v et du paramètre ϕ.

Dans l'hypothèse où ϕ, intégrale de $(F, \phi)=0$, est un polynome entier en p, q d'ordre $2n+1$, il est possible (en changeant au besoin u, v en de nouvelles fonctions de ces arguments, c'est-à-dire en passant à une *équation type*) de lui donner la forme:

$$\phi=w^{-\frac{m}{2}}(w-\gamma_1)\ldots(w-\gamma_m). \tag{5}$$

Le système des relations que doivent vérifier les γ_i—qui exprime que les facteurs $(w-\gamma_i)$ sont invariants par l'opération $A(f)$—et l'inconnue λ, peut être remplacé par:

$$\lambda^m=-\gamma_1\ldots\gamma_m=\zeta_1\ldots\zeta_m \tag{6}$$

et

$$\lambda\frac{\partial\phi_i}{\partial u}+\zeta_i\frac{\partial\phi_i}{\partial v}=0, \qquad (i=1, \ldots, m), \tag{S}$$

où les ζ_i sont les racines de l'équation $\dfrac{\partial\phi}{\partial w}=0$, et les ϕ_i, les expressions: $\phi_i=\phi(\zeta_i)$. Ces ϕ_i sont, à un autre point de vue, les *variables caractéristiques d'Ampère* pour l'équation unique aux dérivées partielles d'ordre m à une fonction et à deux variables u, v que l'on peut former pour remplacer le système (S) (*équation formée déjà par Bour*): ce sont les combinaisons intégrables des divers systèmes de caractéristiques de Cauchy pour l'équation en question, *chaque système n'en admettant qu'une.*

L'emploi des variables indépendantes $\phi_1, \ldots, \phi_m$ définies aussi par les relations: $\lambda dv-\zeta_i du=B_i d\phi_i$, va nous conduire à la détermination des multiplicateurs B_i en $\phi_1, \ldots, \phi_m$.

Les γ_i et les ζ_i sont des fonctions algébriques de $\phi_1, \ldots, \phi_m$. En posant $\gamma_i=x_i^2$, on reconnait sans difficulté que l'on a:

$$dx_1^2+\ldots+dx_m^2=H_1^2d\phi_1^2+\ldots+H_m^2d\phi_m^2$$

ce qui permet de faire usage, pour la transformation des dérivées, des formules des systèmes orthogonaux à n variables. (Cf. Darboux, *Systèmes orthogonaux*, Livre I, Ch. VI).

On trouve ainsi que w défini par: $\phi=\phi(w)$ satisfait aux équations:

$$\frac{\partial w}{\partial\phi_i}=\frac{w}{\zeta_i\phi''(\zeta_i)(\zeta_i-w)} \tag{7}$$

où l'on a posé:

$$\phi''(\zeta_i) = \left(\frac{\partial^2\phi}{\partial w^2}\right)_{w=\zeta_i}.$$

Les γ_h satisfont aux mêmes équations. Il en est encore de même des ζ_h pour $(h \neq i)$. Enfin on a

$$\frac{\partial\phi''(\zeta_i)}{\partial\phi_h} = \frac{-2\phi''(\zeta_i)}{\phi''(\zeta_h)(\zeta_h-\zeta_i)^2} \tag{8}$$

et

$$\frac{1}{H_i^2} = -4\phi_i\zeta_i\phi''(\zeta_i). \tag{9}$$

Envisageons maintenant la différentielle exacte en u, v:

$$d\psi = \frac{\lambda dv - w du}{w^{3/2}\dfrac{\partial\phi}{\partial w}};$$

elle se trouve être rationnelle en w et peut être décomposée en fractions simples. Si l'on tient compte des relations $\lambda dv - \zeta_i du = B_i d\phi_i$, on peut lui donner la forme:

$$d\psi = \frac{A_1 d\phi_1}{w-\zeta_1} + \ldots + \frac{A_m d\phi_m}{w-\zeta_m} \tag{10}$$

où les A_i sont des fonctions inconnues de $\phi_1, \ldots, \phi_m$, liées simplement aux B_i.

Nous cherchons à déterminer les A_i de manière que $d\psi$ soit une différentielle exacte en $\phi_1, \ldots, \phi_m$, quelle que soit la fonction w donnée par $\phi=\phi(w)$. (Cela aura lieu en particulier pour $w=\gamma_i$). On trouve ainsi $A_i = \dfrac{\partial\theta}{\partial\phi_i}$ où θ est la solution générale du système d'équations de Laplace

$$\frac{\partial}{\partial\phi_i}\left(\frac{1}{\zeta_h}\frac{\partial\theta}{\partial\phi_h}\right) = \frac{\partial}{\partial\phi_h}\left(\frac{1}{\zeta_i}\frac{\partial\theta}{\partial\phi_i}\right) \tag{T}$$

solution qui dépend de m fonctions arbitraires d'un argument, qui est l'un des éléments $\phi_1, \ldots, \phi_m$.

En observant que w est une solution de (T) qui dépend du paramètre ϕ et désignant par $w_1, \ldots, w_m$ les solutions de $\phi=\phi(w)$ qui se réduisent à $\zeta_1, \ldots, \zeta_m$ pour $\phi=\phi_1, \ldots, \phi=\phi_m$, on peut écrire cette solution générale:

$$\theta = \int w_1 F_1(\phi)d\phi + \ldots + \int w_m F_m(\phi)d\phi,$$

où les $F_i(\phi)$ sont arbitraires, les intégrales étant prises entre limites constantes ou le long de contours fermés dans le plan ϕ.

Observons encore que λ considéré comme fonction de $\phi_1, \ldots, \phi_m$ satisfait aux relations:

$$\frac{\partial\lambda}{\partial\phi_i} = \frac{\lambda}{2\zeta_i^2\phi''(\zeta_i)}.$$

Il reste à établir entre les éléments $\phi_1, \ldots, \phi_m$ et les variables u, v les relations qu'exige l'identité en w:

$$d\psi = \frac{\lambda dv - w du}{w^{3/2}\dfrac{\partial\phi}{\partial w}} = \frac{A_1 d\phi_1}{w-\zeta_1} + \ldots + \frac{A_m d\phi_m}{w-\zeta_m} \tag{11}$$

Si l'on pose $(w-\zeta_1)\ldots(w-\zeta_m)=P(w)=w^m+p_1w^{m-1}+\ldots+p_m$, on déduit de cette identité:

$$(\lambda dv - wdu)w^n = \Sigma A_i d\phi_i[w^{m-1}+(p_1+\zeta_i)w^{m-2}+\ldots. \tag{12}$$
$$+(p_{m-1}+p_{m-2}\zeta_i+\ldots.+\zeta_i^{m-1})].$$

On conclut de là, d'abord, de proche en proche, les relations:

$$\text{(U)}\qquad \begin{aligned} &\Sigma A_i\zeta_i^k d\phi_i=0, \qquad (k=0,\ 1,\ \ldots,\ n-2),\\ &\Sigma A_i\zeta_i^{n-1}d\phi_i=-du.\end{aligned}$$

Les autres, qui se présentent sous une apparence plus compliquée, se ramènent aisément à la forme:

$$\text{(V)}\qquad \begin{aligned} &\Sigma A_i\left(\frac{1}{\zeta_i}\right)^k d\phi_i=0, \qquad (k=1,\ \ldots,\ n),\\ &\lambda^{m-1}\Sigma A_i\left(\frac{1}{\zeta_i}\right)^{n+1}d\phi_i=dv.\end{aligned}$$

Cette forme, symétrique du premier groupe, s'en déduit aussi en remplaçant u par v et $\sqrt{w}$ par $\dfrac{\lambda}{\sqrt{w}}$.

Ce système de m relations aux différentielles totales à $(m+2)$ variables, *possède m combinaisons intégrables** que que l'on peut construire méthodiquement: Il est commode pour les obtenir de calculer les dérivées en $\phi_1, \ldots, \phi_m$ des fonctions symétriques des γ_i, c'est-à-dire des coefficients $a_1, \ldots, a_m$ définis par

$$(w-\gamma_1)\ldots(w-\gamma_m)=w^m+a_1w^{m-1}+\ldots+a_m;$$

d'où l'on déduit aussi, simplement, les dérivées de $p_1, \ldots, p_m$ sans calculer les dérivées $\dfrac{\partial\zeta_i}{\partial\phi_i}$.

Nous avons ainsi, par exemple, les combinaisons intégrables des premiers membres:

*Ce sont évidemment des combinaisons linéaires des relations

$$d\psi_i=\sum\frac{A_k d\phi_k}{\gamma_i-\zeta_k},$$

où $d\psi_i=\dfrac{\lambda dv-\gamma_i du}{\gamma_i^{3/2}\left(\dfrac{\partial\phi}{\partial w}\right)_{w=\gamma_i}}$.

$$(\Sigma)\quad \begin{aligned}
&\sum \frac{\partial\theta}{\partial\phi_i}\, d\phi_i = d\theta,\\
&\sum \frac{\partial\theta}{\partial\phi_i}\left(a_1+\frac{m}{2}\zeta_i\right) d\phi_i = d\theta_1,\\
&\sum \frac{\partial\theta}{\partial\phi_i}\left[a_2-\frac{1}{m}\left(\frac{m}{2}-3\right)a_1^2+2a_1\zeta_i+\frac{m}{2}\zeta_i^2\right] d\phi_i = d\theta_2,\\
&\sum \frac{\partial\theta}{\partial\phi_i}\left\{a_3-\frac{2}{m}(m-4)\,a_1a_2-\frac{2}{3m}\left[\frac{2}{m}(m-2)(m-4)+\frac{(m-8))}{m}\right]a_1^3\right.\\
&\qquad\left.+\left[2a_2-\frac{(m-8)}{m}a_1^2\right]\zeta_i+3a_1\zeta_i^2+\frac{m}{2}\zeta_i^3\right\} d\phi_i = d\theta_3,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

où l'on voit apparaître, comme coefficients des $\frac{\partial\theta}{\partial\phi_i}$ des combinaisons linéaires à *coefficients constants* de monomes de *même poids* en $\zeta_1, \dots, \zeta_m$, tels que, pour l'expression $d\theta_k$, les éléments: a_k, a_1a_{k-1}, a_2a_{k-2}, $a_1^2a_{k-2}$, $\dots$, ou les produits par ζ_i^l des éléments de poids $(k-l)$.

Ces expressions, assez compliquées, pourraient se calculer directement. Leur formation se continue jusqu'à celle de $d\theta_n$. Il n'y intervient que les n relations du premier groupe (U). (On obtient ensuite toutes les autres par le passage de u à v et de $\sqrt{w}$ à $\frac{\lambda}{\sqrt{w}}$, en ayant égard à l'invariance de $d\psi$).

La première, évidente, donne

$$\sum \frac{\partial\theta}{\partial\phi_i}\frac{1}{\zeta_i}\, d\phi_i = d\theta_{-1}$$

et l'on a ensuite, par exemple:

$$\lambda^2\sum \frac{\partial\theta}{\partial\phi_i}\left(\frac{1}{\zeta_i}\right)^2 d\phi_i = d\theta_{-2}.$$

En se reportant aux équations (U), (V) on voit donc qu'elles entraînent les relations

$$(\Omega)\quad \begin{aligned}
&\theta=c,\quad \theta_1=c_1,\ \dots,\ \theta_{n-2}=c_{n-2},\quad \theta_{n-1}=u_0-u,\\
&\theta_{-1}=c_{-1},\ \dots,\ \theta_{-n}=c_{-n},\qquad \theta_{-(n+1)}=v-v_0.
\end{aligned}$$

où les c sont des constantes, relations qui sont en nombre m et déterminent $\phi_1, \dots, \phi_m$ en fonction de u, v pour toute solution θ de (T).

Si l'on prend pour θ la solution générale de (T), on a donc par la formule:

$$\lambda^m = -\gamma_1\dots\gamma_m = \zeta_1\dots\zeta_m$$

l'expression *la plus générale* de λ en u, v, pour laquelle l'équation: $pq=\lambda(u, v)$, admet une intégrale $\phi(p, q) = \phi$ entière en p, q, et d'ordre $m = 2n+1$; on a d'ailleurs par la quadrature de $d\psi$ l'intégrale générale de l'équation des géodésiques.

Par exemple, dans le cas d'une intégrale du troisième ordre, il y a trois variables caractéristiques ϕ_1, ϕ_2, ϕ_3 et les relations qui les définissent en u et v sont, sous forme intégrable:

$$(\Sigma) \qquad \begin{aligned} &\sum \frac{\partial\theta}{\partial\phi_i} d\phi_i = d'\theta = -du, \\ &\sum \frac{\partial\theta}{\partial\phi_i} \frac{1}{\zeta_i} d\phi_i = d'\theta_{-1} = 0, \\ \lambda^2 &\sum \frac{\partial\theta}{\partial\phi_i} \left(\frac{1}{\zeta_i}\right)^2 d\phi_i = d'\theta_{-2} = dv, \end{aligned}$$

c'est-à-dire que ϕ_1, ϕ_2, ϕ_3 sont définis en u, v par:

$$\theta + u = u_0, \qquad \theta_2 - v = -v_0, \qquad \theta_{-1} = c.$$

Intégrales entières d'ordre pair. Nous nous bornerons à signaler les modifications à apporter à la méthode précédente dont le principe subsiste.

L'intégrale d'ordre $2n$ peut recevoir la forme:

$$\phi = w^{-n}(w-\gamma_1)\dots(w-\gamma_{2n})$$

comme plus haut. Il existe encore $m = 2n$ variables caractéristiques $\phi_i = \phi(\zeta_i)$ où les ζ_i sont les racines de l'équation

$$\frac{1}{\phi}\frac{\partial\phi}{\partial w} = \frac{-n}{w} + \frac{1}{w-\gamma_1} + \dots + \frac{1}{w-\gamma_n} = 0.$$

En posant $\gamma_i = x_i^2$, on définit toujours un système orthogonal de l'espace euclidien à $2n$ dimensions $x_1, \dots, x_{2n}$; les formules (7), (8), (9), du paragraphe précédent subsistent.

Si l'on considère la différentielle $d\psi$, elle est ici

$$d\psi = \frac{\lambda dv - w du}{w^{\frac{3}{2}} \dfrac{\partial\phi}{\partial w}} = \frac{2}{n} \frac{(\lambda dv - w du) w^{n-\frac{1}{2}}}{(w-\zeta_1)\dots(w-\zeta_{2n})};$$

on la rend rationnelle en remplaçant w par ω^2 et l'on peut alors décomposer en fractions simples et tenir compte des relations: $\lambda dv - \zeta_i du = B_i d\phi_i$, ϕ_i ne changeant pas quand on remplace $\sqrt{\zeta_i}$ par $-\sqrt{\zeta_i}$. Ceci nous permet encore d'écrire:

$$\frac{(\lambda dv - w du) w^{n-1}\sqrt{w}}{(w-\zeta_1)\dots(w-\zeta_{2n})} = \sqrt{w}\left(\frac{A_1 d\phi_1}{w-\zeta_1} + \dots + \frac{A_{2n} d\phi_{2n}}{w-\zeta_{2n}}\right)$$

et l'on déterminera les A_i en $\phi_{1n}, \dots, \phi_{2n}$ par la condition que le second membre soit différentielle exacte pour toute fonction w de $\phi_1, \dots, \phi_{2n}$ définie par $\phi = \phi(w)$.

On obtient ainsi $A_i = \dfrac{\partial\theta}{\partial\phi_i}$ où θ satisfait à un système d'équations de Laplace, différent de celui obtenu pour $m=2n+1$:

$$(T_1) \qquad \frac{\partial}{\partial\phi_h}\left(\frac{\lambda}{\zeta_i}\frac{\partial\theta}{\partial\phi_i}\right)=\frac{\partial}{\partial\phi_i}\left(\frac{\lambda}{\zeta_h}\frac{\partial\theta}{\partial\phi_h}\right).$$

La solution générale de (T_1) se déduit alors de ce que $\sqrt{w}$ (et par suite $\sqrt{\gamma_i}$) est une solution particulière de ce système quel que soit ϕ; on peut l'écrire:

$$\theta=\int\sqrt{w_1}F_1(\phi)d\phi+\ldots+\int\sqrt{w_{2n}}F_{2n}(\phi)d\phi$$

où les F_I sont $2n$ fonctions arbitraires et les w_i les racines de $\phi=\phi(w)$ se réduisant à ζ_i pour $\phi=\phi_i$.

Il reste à établir entre les variables caractéristiques ϕ_i et les variables u, v, les relations qu'exige l'identité:

$$d\psi=\frac{(\lambda dv-wdu)w^{n-1}\sqrt{w}}{(w-\zeta_1)\ldots(w-\zeta_{2n})}=\sqrt{w}\left(\frac{A_1d\phi_1}{w-\zeta_1}+\ldots+\frac{A_{2n}d\phi_{2n}}{w-\zeta_{2n}}\right)$$

où $A_i=\dfrac{\partial\theta}{\partial\phi_i}$, qui a lieu quel que soit w donné par $\phi=\phi(w)$.

Nous savons *d'avance* que ces relations donneront $2n$ combinaisons intégrables qui sont, par exemple, celles qu'on obtient en remplaçant w par γ_i et regardant dans la première expression de $d\psi$, λ, γ_i, ζ_h comme dépendant de u et v seuls. Les expressions des λ, γ_i, ζ_h en u, v étant pour l'instant inconnues, on écrira les identités qui résultent de:

$$(\lambda dv-wdu)w^{n-1}=P(w)\left(\frac{A_1d\phi_1}{w-\zeta_1}+\ldots+\frac{{}_{2u}Ad\phi_{2n}}{w-\zeta_{2n}}\right)$$

où l'on a posé:

$$P(w)=(w-\zeta_1)\ldots(w-\zeta_{2n})=w^{2n}+p_1w^{2n-1}+\ldots+p_{2n}.$$

On trouve ainsi, de proche en proche, un premier groupe formé des équations:

$$(U) \qquad \begin{aligned}&\Sigma A_i\zeta_i^k d\phi_i=0, \qquad (k=0,\ 1,\ \ldots,\ n-2),\\ &\Sigma A_i\zeta_i^{n-1}d\phi_i=-du,\end{aligned}$$

puis, en faisant usage de la relation $P(\zeta_i)=0$ et des équations (U), on forme le second groupe

$$(V) \qquad \begin{aligned}&\sum A_i\left(\frac{1}{\zeta}\right)^k d\phi_i=0, \qquad (k=1,\ \ldots,\ n-1),\\ &\lambda^{2n-1}\sum A_i\left(\frac{1}{\zeta_i}\right)d\phi_i=dv.\end{aligned}$$

Ces groupes sont analogues à ceux que nous avons donné pour $m=2n+1$. Les combinaisons intégrables peuvent se former méthodiquement, en calculant d'abord les dérivées des coefficients $a_1, \ldots, a_{2n}$ du polynome $(w-\gamma_1)\ldots(w-\gamma_{2n})$ par rapport aux ϕ_i, ou celles des p_i qui s'en déduisent aisément.

On obtient ainsi, par exemple, pour le premier groupe (V):

$$(\Sigma_1)\quad \begin{aligned}
&\sum \frac{\partial\theta}{\partial\phi_i}\, d\phi_i = d\theta,\\
&\sum \frac{\partial\theta}{\partial\phi_i}\left(n\zeta_i + \frac{a_1}{2}\right) d\phi_i = d\theta_1,\\
&\sum \frac{\partial\theta}{\partial\phi_i}\left[n\zeta_i^2 + \frac{3}{2}a_1\zeta_i + \frac{a_2}{2} - \frac{(2n-5)}{8n}a_1^2\right] d\phi_i = d\theta_2,\\
&\sum \frac{\partial\theta}{\partial\phi_i}\left\{n\zeta_i^3 + \frac{5}{2}a_1\zeta_i^2 + \frac{3}{2}\left[a_2 - \frac{(2n-7)}{4n}a_1^2\right]\zeta_i\right.\\
&\qquad\left. + \frac{1}{2}\left[a_3 - \frac{(2n-7)}{4n}a_1a_2 + \frac{(2n-5)\,(2n-7)}{3.8n^2}a_1^3\right]\right\} d\phi_i = d\theta_3,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

où l'on voit apparaître, avec des coefficients constants que l'on calculera régulièrement, tous les termes de même poids par rapport à ζ_i et aux coefficients $a_1, \ldots, a_{2n}$, où a_k aura le poids k et ζ_i le poids 1.

Ces combinaisons intégrables ne sont pas les mêmes que pour $m=2n+1$; il ne faut pas s'en étonner puisque c'est $\sqrt{w}$ qui est solution du système (T_1).

Les combinaisons intégrables du second groupe peuvent se déduire des précédentes par l'échange de u en v, *mutatis mutandis.*

On a, par exemple:

$$\lambda \sum \frac{\partial\theta}{\partial\phi_i}\left(\frac{1}{\zeta_i}\right) d\phi_i = d\theta_{-1}$$

et

$$\lambda^3 \sum \frac{\partial\theta}{\partial\phi_i}\left[\left(\frac{1}{\zeta_i}\right)^2 + \left(\frac{1}{\zeta_i}\right)\frac{a_{2n-1}}{2n\,a_{1n}}\right] d\phi_i = d\theta_{-2}$$

pour les deux premières.

L'ensemble des équations:

$$(\Omega_1)\quad \begin{aligned}
&\theta = c, \quad \theta_1 = c_1, \ \ldots, \ \theta_{n-1} = u_0 - u,\\
&\theta_{-1} = c_{-1}, \ \ldots\ldots\ldots, \ \theta_{-n} = v - v_0
\end{aligned}$$

définit alors les ϕ au moyen de u et v. On en déduit λ en u, v et la question est résolue.

Par exemple dans le cas simple $n=2$ où il existe quatre caractéristiques, les combinaisons intégrables sont:

$$\sum \frac{\partial\theta}{\partial\phi_i}\, d\phi_i = d\theta = 0, \qquad \sum \frac{\partial\theta}{\partial\phi_i}\,\frac{\lambda}{\zeta_i}\, d\phi_i = d\theta_{-1} = 0,$$

$$\sum \frac{\partial\theta}{\partial\phi_i}\left(2\zeta_i + \frac{a_1}{2}\right) d\phi_i = d\theta_1 = -du, \qquad \sum \frac{\partial\theta}{\partial\phi_i}\cdot\frac{1}{\lambda}\left(\frac{a_4}{\zeta_i^2} + \frac{a_3}{4\zeta_i}\right) d\phi_i = d\theta_{-2} = dv,$$

l'équation aux ζ étant: $\zeta^4 + \frac{a_1}{2}\zeta^3 - \frac{a_3}{2}\zeta - a_4 = 0.$

Intégrales de la forme: $\phi=\sigma w^{m_0}(w-\gamma_1)^{m_1}\dots(w-\gamma_p)^{m_p}$.

L'étude des intégrales rationnelles en w de l'équation

$$A(f)=\lambda\frac{\partial f}{\partial u}+w\frac{\partial f}{\partial v}+2w\frac{\partial\lambda}{\partial u}\frac{\partial f}{\partial w}=0 \tag{3}$$

peut se faire en suivant la même méthode: les systèmes orthogonaux peuvent y intervenir encore. Il y a lieu seulement de faire quelques modifications lorsque u et v sont des variables caractéristiques.

Nous aborderons tout de suite le cas plus général où:

$$\phi=\sigma w^{m_0}(w-\gamma_1)^{m_1}\dots(w-\gamma_p)^{m_p}$$

est une intégrale de (3), les m_i étant des constantes quelconques. Ce cas devrait s'écrire suivant nos principes

$$\Phi=\log\phi=\log\sigma+m_0\log w+\sum m_i\log(w-\gamma_i)$$

c'est-à-dire que $\frac{\partial\Phi}{\partial w}$ est rationnel en w. Il ouvre donc la voie à une recherche plus étendue, celle de tous les cas où, pour une intégrale ϕ, $\frac{\partial\phi}{\partial w}$ est rationnel en w.

On trouve sans difficulté que les relations $w-\gamma_i=0$, sont invariantes par $A(f)$ et qu'aux racines ζ_i, en nombre p, correspondent des fonctions $\phi_i=\phi(\zeta_i)$, variables caractéristiques, intégrales des équations $\lambda\frac{\partial f}{\partial u}+\zeta_i\frac{\partial f}{\partial v}=0$.

Dans l'hypothèse où m_0 et $m_0+m_1+\dots+m_p$ sont différents de zéro, u et v ne sont pas caractéristiques et l'on peut prendre: $\sigma=1$ et $\lambda^{-2m_0}=\gamma_1^{m_1}\dots\gamma_p^{m_p}$.

On a les variables euclidiennes x_i du système orthogonal en posant $\gamma_i=\frac{x_i^2}{m_i}$. Toute fonction w définie par $\phi=\phi(w)$ satisfait encore aux équations:

$$\frac{\partial w}{\partial\phi_h}=\frac{w}{\zeta_h\phi''(\zeta_h)\,(\zeta_h-w)}$$

et l'on aura aussi

$$\frac{\partial\phi''(\zeta_i)}{\partial\phi_h}=\frac{-2\phi''(\zeta_i)}{\phi''(\zeta_h)\,(\zeta_h-\zeta_i)^2},$$

mais, bien entendu, les expressions des γ et des ζ en $\phi_1,\dots,\phi_p$ dépendent de $m_0, m_1, \dots, m_p$.

La différentielle

$$d\psi=\frac{\lambda du-wdv}{w^{3/2}\frac{\partial\phi}{\partial w}}$$

sera ici remplacée par $\phi d\psi$, ce qui revient à changer ϕ en Φ et l'on aura encore pour cette différentielle une expression rationnelle en $\sqrt{w}$:

$$\phi d\psi = \frac{(\lambda dv - wdu)(w-\gamma_1)\dots(w-\gamma_p)}{\sqrt{w}(w-\zeta_1)\dots(w-\zeta_p)}$$

qui, décomposée en fractions simples, donne simplement, eu égard aux valeurs des coefficients:

$$\phi d\psi = A\frac{\lambda dv}{\sqrt{w}} - \mu\sqrt{w}du + \sum \frac{\sqrt{w}A_i d\phi_i}{w-\zeta_i}$$

où $A = \dfrac{m_0+m_1+\dots+m_p}{m_0}$, et μ est une fonction déterminée de $\phi_1, \dots, \phi_p$:

$$\mu = -\frac{\Sigma m_i\gamma_i}{2(m_0+\dots+m_p)}.$$

Si l'on écrit que $\phi d\psi$ (où ϕ peut être regardé comme une constante) est une différentielle exacte en $\phi_1, \dots, \phi_p, u, v$, on trouve en dernière analyse que l'on peut poser:

$$A_i = \frac{\partial\theta}{\partial\phi_i}$$

avec, pour θ, l'expression:

$$\theta = A\lambda v - \mu u + \omega.$$

ω désignant la solution générale du système en $\phi_1, \dots, \phi_p$:

$$(\Omega) \qquad \frac{\partial}{\partial\phi_h}\left(\frac{\lambda}{\zeta_i}\frac{\partial\omega}{\partial\phi_i}\right) = \frac{\partial}{\partial\phi_i}\left(\frac{\lambda}{\zeta_h}\frac{\partial\omega}{\partial\phi_h}\right),$$

entièrement analogue à celui trouvé pour les intégrales d'ordre pair. La fonction λ est une solution de ce système et il en est de même de μ (donc aussi de θ). On a, en outre, par exemple:

$$\frac{\partial\lambda}{\partial\phi_i} = \frac{\lambda}{2\zeta_i^2\phi''(\zeta_i)}, \qquad \frac{\partial\mu}{\partial\phi_i} = \frac{1}{2\zeta_i\phi''(\zeta_i)}.$$

Toutes les fonctions $\sqrt{w}$ verifiant la relation $\phi = \phi(w)$ où ϕ est une constante satisfont à (Ω); on déduit encore de là, par quadratures, la solution générale de (Ω).

Il reste à établir entre les caractéristiques $\phi_1, \dots, \phi_p$ et les variables u, v, les relations (en nombre p) qui détermineront tout en u, v et qui résultent de l'identité en w:

$$\frac{(\lambda dv - wdu)\ (w-\gamma_1)\dots(w-\gamma_p)}{(w-\zeta_1)\dots(w-\zeta_p)} = \lambda A\,dv - \mu wdu + \sum\frac{wA_i d\phi_i}{w-\zeta_i}.$$

On y remplacera A_i par son expression

$$A_i = \frac{\partial\theta}{\partial\phi_i} = A\frac{\partial\lambda}{\partial\phi_i}w - \frac{\partial\mu}{\partial\phi_i}u + \frac{\lambda\omega}{\partial\phi_i}$$

et l'on obtiendra un système complètement intégrable, en $d\phi_1, \dots, d\phi_p, du, dv$ dépendant linéairement des $\dfrac{\partial\omega}{\partial\phi_i}$. Les combinaisons intégrables peuvent toujours être formées méthodiquement. Il sera commode pour le calcul de considérer les polynomes:

$$P(w) = (w - \zeta_1) \dots (w - \zeta_p) = w^p + P_1 w^{p-1} + \dots + P_p,$$

et

$$(w - \gamma_1) \dots (w - \gamma_p) = w^p + a_1 w^{p-1} + \dots + a_p$$

et de former les dérivées des P_i et des a_k, (liés simplement) en $\phi_1, \dots, \phi_p$.

En égalant à des constantes les intégrales du système, on a les p relations qui définissent $\phi_1, \dots, \phi_p$ en u, v. La fonction $\lambda(u, v)$ est alors connue; tout est fini.

On voit avec quelle simplicité se détermine la forme la plus générale d'élément linéaire: $ds^2 = 4\lambda(u, v) du dv$ pour laquelle le problème des lignes géodésiques admet une intégrale du type considéré et le rôle essentiel des caractéristiques d'Ampère. Il convient d'ajouter qu'aucune trace de ces caractéristiques n'apparaît dans les recherches antérieures sur ce sujet (O. Bonnet, Maurice Lévy, Laguerre) bornées à des cas *extrêmement* particuliers.

Toute solution particulière ω de (Ω) conduit à un élément linéaire; quand on ne veut pas l'élément *le plus général*, qui possède la propriété indiquée, la fonction $\lambda(u, v)$ peut donc être de nature plus simple.

Remarques sur l'équation des géodésiques. La puissance de la méthode adoptée est loin d'être bornée aux problèmes précédents. Elle s'appliquera par exemple à la recherche des éléments linéaires: $ds^2 = 4\lambda du dv$, pour lesquels l'équation des géodésiques admet *deux* intégrales premières rationnelles en $\frac{dv}{du}$ ou w, problème *beaucoup plus simple* que ceux que l'on vient de traiter. Nous indiquerons, à propos d'un exemple plus général, la voie à suivre pour le résoudre.

D'une manière générale le groupe de rationalité étant sous-groupe type de celui qui est défini par: $\frac{\partial(\Phi, \Psi)}{\partial(\phi, \psi)} = 1$, on voit que si l'équation est *imprimitive*—et s'il existe une seule équation: $\frac{\partial f}{\partial v} + \mu \frac{\partial f}{\partial w} = 0$, formant avec $A(f) = 0$ un système complet, auquel cas μ est *rationnel* en w—la solution ϕ peut être obtenue isolément par ces deux équations. On a donc nécessairement: $\Phi = F(\phi)$ et par suite $\Psi = \frac{\psi}{F'(\phi)} + F(\phi)$ pour les transformations du groupe de rationalité.

Il en résulte que si $F = \epsilon\phi + a$ avec $\epsilon^n = 1$, ou bien si $F = a\phi + b$, ϕ et ψ s'obtiendront par des quadratures. Nous pourrons déterminer $\lambda(u, v)$ *de la manière la plus générale* pour que ces circonstances se présentent: les dérivées de ϕ ou celles de $\log \phi$ sont alors rationnelles en w.

Si l'on suppose que la détermination de ϕ exige l'intégration d'un système de Riccati, il n'en sera plus de même, du moins avec l'emploi de signes de quadrature. *L'inversion* d'une équation de Riccati entre en jeu. Il restera encore à examiner le cas de deux familles d'intégrales définies séparément: μ est quadratique en w et les transformations du groupe de rationalité étant: $\Phi = F(\phi)$ $\Psi = G(\psi)$ on a nécessairement $F'(\phi) = a$, $G'(\psi) = \frac{1}{a}$, c'est-à-dire que les dérivées de log ϕ et de log ψ sont rationnelles en w. On pourra former tous les cas où il en est ainsi.

Le dernier cas—où il existe plus de deux familles d'intégrales qu'on peut isoler—conduit aussi à des transformations linéaires des intégrales qui s'obtiennent encore par des quadratures.

Lorsque l'équation est *primitive*, le seul cas de réduction donnerait l'équivalent d'une équation différentielle linéaire, *primitive aussi*. *L'inversion* d'une telle équation à coefficients rationnels peut être nécessaire pour obtenir $\lambda(u, v)$ de la manière la plus générale. Tout ceci sera développé ailleurs.

Ajoutons enfin que l'on peut traiter de la même manière toute équation du second ordre: $y'' = F(x, y, y')$ où F est *rationnel* en y' et qui provient d'un *problème de variations*. Jacobi a en effet montré que lorsque l'équation précédente se présente dans la recherche d'une valeur extrême de l'intégrale $\int f(x, y, y')dx$, l'expression $M = \frac{\partial^2 f}{\partial y'^2}$ est un dernier multiplicateur pour l'équation:

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y' + \frac{\partial \phi}{\partial y'} F = 0,$$

c'est-à-dire que le groupe de rationalité relatif aux solutions fondamentales de cette équation ϕ, ψ, est en général déterminé par:

$$\frac{\partial(\Phi, \Psi)}{\partial(\phi, \psi)} = 1.$$

Il suffira donc, par exemple, que les dérivées de f ou de $\log f$ soient rationnelles en y', pour que l'on puisse déterminer f de la manière la plus générale, dans une catégorie donnée, de façon à permettre par quadratures l'intégration de l'équation de Lagrange:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0.$$

ÉQUATIONS DU SECOND ORDRE: $y'' = R(y')$ OU R EST RATIONNEL EN LA DÉRIVÉE PREMIÈRE y'

Nous avons étudié jusqu'à présent la réduction de certaines équations du second ordre: $[y'' = F(x, y)$, équation des géodésiques] où l'expression de la dérivée seconde y'' est rationnelle en y' et où *de plus* la connaissance d'un multiplicateur réduit le groupe de rationalité au groupe primitif donné par: $\frac{\partial(\Phi, \Psi)}{\partial(\phi, \psi)} = 1$. On peut traiter sans beaucoup plus de difficulté la question tout à fait générale de la réduction aux quadratures de l'équation

$$y'' = R(y'), \tag{1}$$

où R est le quotient de deux polynomes en y' de *degrés donnés*, dont les coefficients sont des fonctions de x, y *à déterminer*.

Les cas les plus simples sont ceux où l'équation (1) possède une intégrale algébrique (par suite une intégrale rationnelle) en y', soit $\phi = R_1(y')$ et où l'équation $dy - y'dx = 0$ correspondante est réductible aux quadratures quel que soit ϕ. Cela exige que pour l'équation aux dérivées partielles: $X(\psi) = \frac{\partial \psi}{\partial x} + y' \frac{\partial \psi}{\partial y} = 0$, l'un des éléments: ψ, $K = \left(\frac{\partial \psi}{\partial y}\right)^n$ ou $J = \frac{\partial^2 \psi}{\partial y^2} : \frac{\partial \psi}{\partial y}$, soit rationnel en y'.

D'autres cas de réduction: celui où $I = \{\psi, y\}$ est rationnel en y', qui conduit pour J à un système de Riccati—et celui où l'on peut ajouter une relation $\frac{\partial \psi}{\partial \phi} - \rho \frac{\partial \psi}{\partial y} = 0$ où ρ est rationnel (ou algébrique à deux valeurs) en y', sont ég lement intéressants—mais on ne peut les traiter *en général* par le simple emploi de signes de quadrature. Une classe d'équations particulièrement importante est $y'' = P_3(y')$, où P est un polynome du troisième degré en y'; on sait qu'elle conserve sa forme par une transformation ponctuelle quelconque exécutée sur x et y; cela nous permettrait d'adopter un *type réduit* comme point de départ; nous l'étudierons ailleurs.

Nous nous bornerons ici, *à titre d'exemple*, à traiter deux cas simples. Il convient d'observer que, par des transformations n'exigeant que des quadratures, l'on passera toujours à des équations *types*, de manière à avoir des problèmes sans indétermination.

On supposera que l'équation

(1) $$y'' = R(y')$$

possède une intégrale première *entière* en y' et de degré n:

(2) $$\phi = a_0 y'^n + a_1 y'^{n-1} + \dots + a_n = \phi(y')$$

et l'on cherchera, en fait, les cas de réduction aux quadratures de l'équation (2).

Si l'on suppose dans l'identité en y'

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y' + \frac{\partial \phi}{\partial y'} R(y') = 0$$

que $\frac{\partial \phi}{\partial y'}$ et $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y'$ n'aient pas de diviseur commun en y' les deux termes de $R(y')$ se déduiront de la connaissance de $\phi(y')$ et leur degré en résultera. Si ces degrés sont donnés d'avance il y aura un plus grand commun diviseur, de degré donné, aux polynomes $\frac{\partial \phi}{\partial y'}$ et $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y'$. On peut donc envisager d'abord la réduction de (2) sans faire intervenir (1)—cette dernière n'influant que par la valeur réduite imposée au quotient $\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y'\right) : \frac{\partial \phi}{\partial y'}$ quand $\phi(y')$ est rationnel en y'.

I. *Intégrale Rationnelle en y', pour l'équation* (2).

(*a*) Supposons donc $\frac{\partial\phi}{\partial y'}$ et $\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}y'$ sans diviseur commun en y' et cherchons si (1) et par suite (2) peut posséder une autre intégrale rationnelle en y': $\psi=\psi(y')$. Pour fixer les idées on admettra ici qu'elle est *entière* et d'ordre m:

$$\psi=b_0y'^m+b_1y'^{m-1}+\ldots+b_m.$$

La condition nécessaire et suffisante est l'identité en y':

$$(\phi,\ \psi)=\frac{\partial\psi}{\partial y'}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}y'\right)-\frac{\partial\phi}{\partial y'}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}y'\right)=0$$

qui donne entre les a_i et les b_k un système (Σ) d'équations aux dérivées partielles en nombre $(n+m+1)$ seulement. Nous rendons ce système déterminé en remplaçant y par une nouvelle inconnue, donnée par une quadrature partielle, de manière à avoir $a_0=1$. Un changement de la variable x donne alors $b_0=x$ (ou exceptionnellement $b_0=1$).

On obtient immédiatement l'intégrale première: $mxa_1-n(y+b_1)=X(x)$, où $X(x)$ est essentiel.

Les autres intégrales du système (Σ) sont données comme suit:

Le polynome $\frac{\partial\psi}{\partial y'}$ doit être divisible par $\frac{\partial\phi}{\partial y'}$; en écrivant l'identité $\frac{\partial\psi}{\partial y'}=\Lambda(y')\frac{\partial\phi}{\partial y'}$, on a $(n-1)$ conditions algébriques entières en a_i, b_k.

La différentielle $d\psi$ peut s'écrire, en regardant y' comme donné en x, y par $\phi=\phi(y')$, sous la forme:

$$d\psi=\left[\frac{\partial\psi}{\partial y}-\Lambda(y')\frac{\partial\phi}{\partial y}\right](dy-y'dx)=K(dy-y'dx)$$

où K est en y' de degré $(m-1)$. Si l'on désigne par u_i l'une des $(m-1)$ racines de $K(y')=0$, l'équation $y'=u_i$ a pour intégrale, indifféremment $\psi(u_i)$ et $\phi_i=\phi(u_i)$. On exprime que u_i, inconnu, satisfait à $K(u_i)=0$ par l'ensemble d'équations

$$\psi(u_i)=F_i(\phi_i),\quad \phi_i=\phi(u_i),$$
$$\Lambda(u_i)=\frac{\partial F_i}{\partial\phi_i}.$$

Les variables auxiliaires ϕ_i sont les caractéristiques d'Ampère du système(Σ). Il s'y ajoute x.

On a ainsi $(m-1)$ relations nouvelles avec autant de fonctions arbitraires d'un argument.

Il ne manque plus qu'une intégrale. Elle s'écrit:

$$n(u_1+\ldots+u_{m-1})+ma_1=f(x).$$

On la trouve en exprimant l'intégrabilité de $d\psi$ quand les racines de $K(y')=0$ sont en évidence. Elle dépend, en fait, de a_1, b_1, a_2, b_2 et des équations qui lient ces éléments.

Il est d'ailleurs possible, en remplaçant y par $y+\mu(x)=Y$, d'annuler $f(x)$, c'est-à-dire de choisir l'équation *type* (2) de manière que: $n\Sigma u_i+ma_1=0$.

(*b*) Comment tout ceci se modifie-t-il quand $\dfrac{\partial\phi}{\partial y'}$ et $\dfrac{\partial\phi}{\partial x}+\dfrac{\partial\phi}{\partial y}y'$ ont un plus grand commun diviseur de degré donné, M?

Si l'on pose $M=(y'-\omega_1)\ldots(y'-\omega_k)$, pour tout ω_i, l'équation $y'-\omega_i=0$ a pour intégrale

$$\phi_i=\phi(\omega_i)=\text{const.}$$

mais cela ne nous conduit pas à la solution. (On pourrait cependant partir de l'expression explicite de $\dfrac{\partial\phi}{\partial y'}$, c'est-à-dire de MB pour présenter les résultats qui suivent). Nous observerons qu'en posant

$$\frac{\partial\phi}{\partial y'}=MB, \qquad \frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}y'=MA$$

où A et B sont premiers entre eux, on aura nécessairement pour l'intégrale ψ, entière en y',

$$\frac{\partial\psi}{\partial y'}=\Lambda(y').B \qquad \frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}y'=\Lambda(y')A$$

et

$$d\psi=\frac{\left[M\dfrac{\partial\psi}{\partial y}-\Lambda(y')\dfrac{\partial\phi}{\partial y}\right]}{M}(dy-y'dx)=K(dy-y'dx)$$

lorsque $\phi=\phi(y')$ définit y'.

On a donc entre $\dfrac{\partial\psi}{\partial y'}$ et $\dfrac{\partial\phi}{\partial y'}$ un plus grand commun diviseur $B(y')$ de degré $n-1-k$ et ceci entraîne entre les a_i et les b_k, $(n-1-k)$ relations algébriques entières.

Le multiplicateur $K(y')$ admet $(m+k-1)$ racines u_i, qui conduisent à autant de systèmes:

$$\psi(u_i)=F_i(\phi_i), \quad \Lambda(u_i)=\frac{\partial F_i}{\partial\phi_i}, \quad \phi_i=\phi(u_i),$$

et par suite à autant d'intégrales de (Σ) avec les fonctions arbitraires $F_i(\phi_i)$.

On a donc toujours ainsi $(m+n-2)$ intégrales, auxquelles il suffira d'ajouter: $mxa_1-n(y+b_1)=X(x)$ et $n(\Sigma u_i-\Sigma\omega_i)+ma_1=f(x)$ [où $f(x)$ peut d'ailleurs être annulé] pour obtenir un système de $(m+n)$ équations définissant les a_i et les b_k en x, y.

On observera que $B(y')$, plus grand commun diviseur de $\dfrac{\partial\psi}{\partial y'}$ et $\dfrac{\partial\phi}{\partial y'}$, a ses coefficients entiers en a_i, b_k et que par suite les coefficients de M, donc $\Sigma\omega_i$, sont rationnels en ces éléments.

On n'a traité ici que le cas général, où les racines de $K(y')=0$ sont distinctes; il n'y a aucune difficulté à étendre la solution au cas de racines multiples.

Remarque. Les deux exemples précédents donnent en principe la *détermination des transformations de contact du plan*

$$d\phi-\lambda d\psi=\rho(dy-y'dx)$$

pour lesquelles ϕ *et* ψ *sont rationnels en* y'. Cette détermination est faite en fixant arbitrairement les degrés des deux termes de $\phi(y')$ [dans le cas actuel, le degré n de $\phi(y')$].

II. *Multiplicateur* K, *rationnel en* y', *pour l'équation* (2).

Nous allons, *à titre d'exemple*, traiter le cas où, pour une équation du second ordre:

(1) $$y''=\lambda(y'-\mu),$$

linéaire en y', il existe une intégrale

(2) $$\phi=a_0y'^n+a_1y'^{n-1}+\ldots+a_n$$

et de telle sorte que l'équation (2) puisse s'intégrer par la quadrature:

(3) $$d\psi=K(dy-y'dx),$$

où K est rationnel en y'.

Le multiplicateur K est susceptible de diverses formes; nous ne discuterons pas leur ensemble, nous bornant à la plus simple: $K=P(y'):\dfrac{\partial\phi}{\partial y'}$ où le *polynome* $P(y')$ *n'a pas de racines multiples*.

Faisons l'hypothèse $a_0=1$, qui conduit à une équation (1) *type*.

L'identité en y':

(4) $$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}y'+\lambda(y'-\mu)\frac{\partial\phi}{\partial y'}=0$$

exprime essentiellement que pour toute racine ω_i de $\dfrac{\partial\phi}{\partial y'}=0$, l'équation $y'=\omega_i$ a pour intégrale $\phi_i=\phi(\omega_i)=\text{const.}$ Ces conditions étant vérifiées, la division de $\dfrac{\partial\phi}{\partial x}+\dfrac{\partial\phi}{\partial y}y'$ par $\dfrac{\partial\phi}{\partial y'}$ donne λ et μ sans ambiguité. On a donc ici le cas extrême où le plus grand commun diviseur de $\dfrac{\partial\phi}{\partial x}+\dfrac{\partial\phi}{\partial y}y'$ et $\dfrac{\partial\phi}{\partial y'}$ est d'ordre $(n-1)$ en y'. Supposons $P(y')=\rho(y'-u_1)\ldots(y'-u_p)$, la résolvante en K (condition d'intégrabilité de $d\psi$) nous donne:

$$\frac{1}{K}\left(\frac{\partial K}{\partial x}+\frac{\partial K}{\partial y}y'\right)\frac{\partial\phi}{\partial y'}-\frac{1}{K}\frac{\partial K}{\partial y'}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}y'\right)-\frac{\partial\phi}{\partial y}=0,$$

c'est-à-dire, ici,

$$\frac{1}{K}\left(\frac{\partial K}{\partial x}+\frac{\partial K}{\partial y}y'\right)+\frac{1}{K}\frac{\partial K}{\partial y'}\lambda(y'-\mu)-\frac{\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial \phi}{\partial y'}}=0.$$

On en conclut d'abord: $\dfrac{\partial \rho}{\partial y}=0$, ce qui, en modifiant x, permet de prendre pour ρ une constante; nous poserons $\rho=1$ de façon à écrire

$$K=\frac{(y'-u_1)\dots(y'-u_p)}{n(y'-\omega_1)\dots(y'-\omega_{n-1})}.$$

En observant que l'on a:

$$\frac{\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial \phi}{\partial y'}}=\frac{1}{n}\frac{\partial a_1}{\partial y}+\sum\frac{A_i}{y'-\omega_i} \quad \text{où } A_i=\left|\frac{\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial^2 \phi}{dy'^2}}\right|_{y'=\omega_i}$$

on trouve:

1° que les ω_i satisfont aux conditions:

$$\frac{\partial \omega_i}{\partial x}+\omega_i\frac{\partial \omega_i}{\partial y}-\lambda(\omega_i-\mu)-A_i=0;$$

les équations $y'=\omega_i$ ne sont pas des intégrales particulières de (1): on a vu que leur solution est $\phi_i=\phi(\omega_i)=\text{const.}$

2° que les u_i satisfont aux conditions:

$$\frac{\partial u_i}{\partial x}+u_i\frac{\partial u_i}{\partial y}-\lambda(u_i-\mu)=0,$$

qui expriment que $y'-u_i=0$ est une intégrale particulière de (1).

Sa solution est donc donnée par $\phi(u_i)=\phi_i=\text{const.}$ Il reste une seule équation:

$$-\sum\frac{\partial u_i}{\partial y}+\sum\frac{\partial \omega_i}{\partial y}+\lambda(p-n+1)-\frac{1}{n}\frac{\partial a_1}{\partial y}=0,$$

qui s'intégre, sous la forme:

$$\Sigma u_i-\Sigma\omega_i+\frac{(p-n+2)}{n}a_1=f(x),$$

en raison de

$$\lambda=-\frac{1}{n}\frac{\partial a_1}{\partial y}.$$

On observera encore que le changement de y en $y+g(x)$ permettrait de prendre $f(x)=0$, et que $\Sigma\omega_i=-\dfrac{(n-1)}{n}a_1$, ce qui donne simplement:

$$u_1+\dots+u_p=-\frac{(p+1)}{n}a_1.$$

Il s'agit de trouver les intégrales du système de $(n+p)$ relations ainsi formé en $a_1, \ldots, a_n, u_1, \ldots u_p$, où les variables auxiliaires $\Phi_i=\phi(\omega_i)$, $\phi_i=\phi(u_i)$ sont les caractéristiques d'Ampère, provenant de deux sources distinctes.

Les relations qui définissent les a_i, u_k par rapport à ces caractéristique gagnent en netteté si l'on introduit explicitement les racines γ_i de $\phi(y')=0$. Posons:

$$\phi(y')=(y'-\gamma_1)\ldots(y'-\gamma_n);$$

les ω_i sont donnés par

$$\frac{1}{\omega_i-\gamma}+\ldots+\frac{1}{\omega_i-\gamma_n}=0$$

et, si l'on ajoute aux relations

$$\Phi_i=\phi(\omega_i), \qquad (i=1, \ldots, n-1),$$

la condition: $\gamma_1+\ldots+\gamma_n=\Phi$, où ϕ est une variable auxiliaire, les γ et les ω sont fonctions de $\Phi_1, \ldots, \Phi_{n-1}$ et Φ, (la variable Φ n'est autre que $-a_1$). Les relations $\phi(u_i)=\phi_i$ déterminent alors les u_i au moyen des ϕ_i et de $\Phi_1, \ldots, \Phi_{n-1}, \Phi$; enfin la relation $n(u_1+\ldots+u_p)=(p+1)\Phi$ définira Φ en $\phi_1, \ldots, \phi_p, \Phi_1, \ldots, \Phi_{n-1}$.

Si l'on regarde les γ_i $(i=1, \ldots, n)$ comme des coordonnées cartésiennes d'un espace euclidien à n dimensions, les surfaces $\Phi_i=$const., $\Phi=$const. sont deux à deux orthogonales.

On en déduit:

$$d\gamma_1^2+\ldots+d\gamma_n^2=H_1^2d\Phi_1^2+\ldots+H^2_{n-1}\,d\Phi^2_{n-1}+\frac{1}{n}d\Phi^2$$

avec

$$\frac{1}{H_i^2}=-\Phi_i\phi''(\omega_i)$$

et les formules classiques des systèmes orthogonaux donnent:

$$\frac{\partial\gamma_k}{\partial\Phi_i}=\frac{1}{\phi''(\omega_i)\ (\omega_i-\gamma_k)}, \qquad \frac{\partial\gamma_k}{\partial\Phi}=\frac{1}{n}.$$

Toute fonction y', donnée par $\phi=\phi(y')$, où le premier membre est constant, satisfait aux mêmes relations:

$$\frac{\partial y'}{\partial\Phi_i}=\frac{1}{\phi''(\omega_i)\ (\omega_i-y')}, \qquad \frac{\partial y'}{\partial\Phi}=\frac{1}{n}.$$

La dérivée $\dfrac{\partial\omega_k}{\partial\Phi_i}$, où $(k\neq i)$, s'exprime de même, et $\dfrac{\partial\omega}{\partial\Phi}=\dfrac{1}{n}$.

Le calcul des dérivées des u_i résultera des formules:

$$\frac{d\phi_i}{\phi_i}=\frac{du_i-d\gamma_1}{u_i-\gamma_1}+\ldots+\frac{du_i-d\gamma_n}{u_i-\gamma_n}$$

ou encore:

$$\frac{\phi'(u_i)}{\phi_i}du_i=\frac{d\phi_i}{\phi_i}+\sum\frac{d\gamma_k}{u_i-\gamma_k};$$

on obtient ainsi

$$\frac{\partial u_i}{\partial \Phi_h} = \frac{1}{\phi''(\omega_h)\ (\omega_h - u_i)}, \qquad \frac{\partial u_i}{\partial \Phi} = \frac{1}{n},$$

[c'est-à-dire que les u_i satisfont aux mêmes équations que y'; on a vu en effet que la solution de $y' = u_i$ est $\phi(u_i) = \text{const.}$] et en outre:

$$\frac{\partial u_i}{\partial \phi_i} = \frac{1}{\phi'(u_i)}, \qquad \frac{\partial u_i}{\partial \phi_k} = 0 \qquad (i \neq k).$$

Employons maintenant la dernière relation pour calculer $d\phi$. On trouve sans difficulté, en posant $B_h = \sum_i \frac{1}{\omega_h - u_i}$,

$$\frac{1}{n} d\Phi = \sum_h \frac{B_h}{\phi''(\omega_h)} d\Phi_h + \sum_i \frac{1}{\phi'(u_i)} d\phi_i$$

et l'on remarquera que le coefficient de $d\Phi$ n'est jamais nul.

Nous sommes maintenant en mesure d'aborder, dans le domaine des caractéristiques $\phi_1, \ldots, \phi_p, \Phi_1, \ldots, \Phi_{n-1}$ la recherche des *conditions sous lesquelles* $d\psi$, que nous écrirons:

$$d\psi = \Omega \left(\frac{M_1 d\Phi_1}{y' - \omega_1} + \ldots + \frac{N_1 d\phi_1}{y' - u_1} + \ldots \right)$$

avec:

$$\Omega = K(y' - u_1) \ldots (y' - u_p)\ (y' - \omega_1) \ldots (y' - \omega_{n-1})$$

c'est-à-dire ici: $\Omega = (y' - u_1)^2 \ldots (y' - u_p)^2$, *est une différentielle exacte pour tout* y' *donné par* $\phi = \phi(y')$, où ϕ *est constant.*

Le calcul des dérivées de $\log \Omega$ est simplifié, et celui des conditions d'intégrabilité aussi, par l'observation que $\frac{\partial}{\partial \Phi}(y' - u_i) = 0$, c'est-à-dire que les différences $(y' - u_i)$ sont indépendantes de la variable auxiliaire Φ. On a simplement

$$\frac{\partial (y' - u_i)}{\partial \Phi_h} = \frac{(y' - u_i)}{\phi''(\omega_h)\ (\omega_h - y')\ (\omega_h - u_i)}, \qquad \frac{\partial (y' - u_i)}{\partial \phi_i} = \frac{-1}{\phi'(u_i)}, \qquad \frac{\partial (y' - u_i)}{\partial \phi_k} = 0\ (i \neq k)$$

et l'on en déduit immédiatement

$$\frac{1}{2\Omega} \frac{\partial \Omega}{\partial \Phi_h} = \frac{B_h}{\phi''(\omega_h)\ (\omega_h - y')}, \qquad \frac{1}{2\Omega} \frac{\partial \Omega}{\partial \phi_k} = \frac{1}{\phi'(u_k)\ (u_k - y')}.$$

On aura de même pour $(h \neq i)$:

$$\frac{\partial (y' - \omega_i)}{\partial \Phi_h} = \frac{(y' - \omega_i)}{\phi''(\omega_h)\ (\omega_h - y')\ (\omega_h - \omega_i)}, \qquad \frac{\partial (y' - \omega_i)}{\partial \phi_k} = 0.$$

Si l'on écrit les conditions d'intégrabilité de $d\psi$, on trouve alors que l'on a:

$$M_h = \frac{\partial \theta}{\partial \Phi_h}, \qquad N_k = \frac{\partial \theta}{\partial \phi_k}$$

où la fonction θ des $(n+p-1)$ arguments satisfait à l'ensemble (T) des trois groupes d'équations de Laplace:

$$\text{(I)}\qquad (\omega_i-\omega_h)\frac{\partial^2\theta}{\partial\Phi_i\partial\Phi_h}-\frac{\partial\theta}{\partial\Phi_i}\frac{\left(2B_h-\dfrac{1}{\omega_h-u_i}\right)}{\phi''(\omega_h)}+\frac{\partial\theta}{\partial\Phi_h}\frac{\left(2B_i-\dfrac{1}{\omega_i-\omega_h}\right)}{\phi''(\omega_i)}=0,$$

$$\text{(II)}\qquad (u_i-\omega_h)\frac{\partial^2\theta}{\partial\phi_i\partial\Phi_h}-\frac{\partial\theta}{\partial\phi_i}\frac{\left(2B_h-\dfrac{1}{\omega_h-u_i}\right)}{\phi''(\omega_h)}+\frac{\partial\theta}{\partial\Phi_h}\frac{2}{\phi'(u_i)}=0,$$

$$\text{(III)}\qquad (u_i-u_h)\frac{\partial^2\theta}{\partial\phi_i\partial\phi_k}-\frac{\partial\theta}{\partial\phi_i}\frac{2}{\phi'(u_k)}+\frac{\partial\theta}{\partial\phi_k}\frac{2}{\phi'(u_i)}=0.$$

Ce système ne dépend que des différences γ_i-u_k, ω_i-u_k, u_i-u_k, $\omega_i-\omega_h$; il est donc indépendant de la variable auxiliaire Φ. En particulier chacune des équations (III) ne dépend que des deux variables ϕ_i, ϕ_k, en dehors des Φ_i. Nous signalerons la solution particulière $\theta=-\dfrac{3\Phi}{n}=\dfrac{3a_1}{n}$ que nous désignons par σ; son introduction explicite, montre aisément que (T) *exprime simplement* que

$$\Sigma(\omega_i+\sigma)\frac{\partial\theta}{\partial\Phi_i}d\Phi_i+\Sigma(u_k+\sigma)\frac{\partial\theta}{\partial\phi_k}d\phi_k=d\tau$$

est une différentielle exacte. On déduira de là et des remarques antérieures l'exis tence de solutions θ de la forme: $\theta=F_1(\phi_1)\dots F_p(\phi_p)$, où les F_i dépendent aussi de $\Phi_1,\dots,\Phi_{x-1}$. Ces solutions peuvent renfermer un paramètre et conduisent par quadratures à la solution générale.

Il s'agit maintenant de former les conditions sous lesquelles on a l'identité en y'

$$d\psi=\frac{(y'-u_1)\dots(y'-u_p)}{n(y'-\omega_1)\dots(y'-\omega_{n-1})}(dy-y'dx)=\Omega\left(\frac{M_1d\Phi_1}{y'-\omega_1}+\dots+\frac{N_1d\phi_1}{y'-u_1}+\dots\right),$$

c'est-à-dire simplement

$$dy-y'dx=R(y')\left(\frac{\dfrac{\partial\theta}{\partial\Phi_1}d\Phi_1}{y'-\omega_1}+\dots+\frac{\dfrac{\partial\theta}{\partial\phi_1}d\phi_1}{y'-u_1}+\dots\right)$$

où l'on a posé

$$R(y')P=(y').A(y'),$$

$$P(y')=(y'-u_1)\dots(y'-u_p)=y'^p+P_1y'^{p-1}+\dots+P_p,$$

$$A(y')=ny'^{n-1}+(n-1)a_1y'^{n-2}+\dots+a_{n-1}.$$

Écrivons pour un instant le polynome $R(y')$ sous la forme:

$$R(y')=n(y'^{n+p-1}+R_1y'^{n+p-2}+\dots+R_{n+p-1})$$

où l'on a, par exemple:

$$nR_1 = nP_1 + (n-1)a_1,$$
$$nR_2 = nP_2 + (n-1)a_1P_1 + (n-2)a_2,$$
$$\dots\dots\dots\dots\dots\dots$$

et remarquons que $R(y')$ est divisible par $(y'-\omega_i)$ et aussi par $(y'-u_i)$. Nous aurons, en faisant la division:

$$\frac{1}{n}(dy - y'dx)$$

$$= \sum_i \left[y'^{n+p-2} + (R_1+\omega_i)y'^{n+p-3} + \dots + R_{n+p-2} + R_{n+p-3}\omega_i + \dots + \omega_i^{n+p-2} \right] \frac{\partial\theta}{\partial\Phi_i} d\Phi_i$$

$$+ \sum \left[y'^{n+p-2} + (R_1+u_k)y'^{n+p-3} + \dots + R_{n+p-2} + R_{n+p-3}u_k + \dots + u_k^{n+p-2} \right] \frac{\partial\theta}{\partial\phi_p} d\phi_k$$

et l'on voit que l'identité, donne de proche en proche, les conditions:

$$(\Sigma) \quad \begin{cases} \sum \frac{\partial\theta}{\partial\Phi_i} d\Phi_i + \sum \frac{\partial\theta}{\partial\phi_k} d\phi_k = 0, \\ \sum \frac{\partial\theta}{\partial\Phi_i} \omega_i d\Phi_i + \sum \frac{\partial\theta}{\partial\phi_k} u_k d\phi_k = 0, \\ \dots\dots\dots\dots\dots\dots \\ \sum \frac{\partial\theta}{\partial\Phi_i} \omega_i^{n+p-4} d\Phi_i + \sum \frac{\partial\theta}{\partial\phi_k} u_k^{n+p-4} d\phi_k = 0, \\ \sum \frac{\partial\theta}{\partial\Phi_i} \omega_i^{n+p-3} d\Phi_i + \sum \frac{\partial\theta}{\partial\phi_k} u_k^{n+p-3} d\phi_k = -\frac{1}{n} dx, \\ \sum \frac{\partial\theta}{\partial\Phi_i} \omega_i^{n+p-2} d\Phi_i + \sum \frac{\partial\theta}{\partial\phi_k} u_k^{n+p-2} d\phi_k = \frac{1}{n} dy. \end{cases}$$

Ce système possède $(n+p-1)$ combinaisons intégrables que l'on peut former régulièrement. Il sera commode dans ce but de calculer d'abord, en partant des expressions des dérivées *complètes*

$$\frac{\partial\gamma_i}{\partial\Phi_h} = \frac{B_h + \frac{1}{\omega_h - \gamma_i}}{\phi''(\omega_h)}, \qquad \frac{\partial\gamma_i}{\partial\phi_k} = \frac{1}{\phi'(u_k)},$$

les dérivées de $a_1, \dots, a_n$ regardés comme donnés par

$$(y'-\gamma_1)\dots(y'-\gamma_n) = y'^n + a_1y'^{n-1} + \dots + a_n$$

$\left(\text{ceci pour éviter les dérivées } \frac{\partial\omega_i}{\partial\Phi_i}\right)$. On trouve aisément ces dérivées, en prenant comme fonctions intermédiaires les sommes de puissances: $\Sigma\gamma_i$, $\Sigma\gamma_i^2$, ...

Nous avons, par exemple:

$$\frac{\partial a_1}{\partial \Phi_h} = \frac{-nB_h}{\phi''(\omega_h)}, \qquad \frac{\partial a_1}{\partial \phi_k} = \frac{-n}{\phi'(u_k)},$$

$$\frac{\partial a_2}{\partial \Phi_h} = \frac{-(n-1)a_1B_h+n}{\phi''(\omega_h)}, \qquad \frac{\partial a_2}{\partial \phi_k} = \frac{-(n-1)a_1}{\phi'(u_k)},$$

$$\frac{\partial a_3}{\partial \Phi_h} = \frac{-(n-2)a_2B_h+(n-1)a_1+n\omega_h}{\phi''(\omega_h)}, \qquad \frac{\partial a_3}{\partial \phi_k} = \frac{-(n-2)a_2}{\phi'(u_k)},$$

. . . .

De même, en partant des formules *complètes:*

$$\frac{\partial u_i}{\partial \Phi_h} = \frac{B_h + \dfrac{1}{\omega_h - u_i}}{\phi''(\omega_h)}, \quad \frac{\partial u_i}{\partial \phi_i} = \frac{2}{\phi'(u_i)}, \quad \frac{\partial u_i}{\partial \phi_k} = \frac{1}{\phi'(u_k)} \quad (i \neq k),$$

on peut calculer les dérivées de $P_1, \ldots, P_p$ en passant encore par l'intermédiaire des sommes Σu_i, Σu_i^2, ... On trouve ainsi par exemple:

$$\frac{\partial P_1}{\partial \Phi_h} = \frac{-(p+1)B_h}{\phi''(\omega_h)}, \qquad \frac{\partial P_1}{\partial \phi_k} = \frac{-(p+1)}{\phi'(u_k)},$$

$$\frac{\partial P_2}{\partial \Phi_h} = \frac{-pP_1B_h+p-\omega_hB_h}{\phi''(\omega_h)}, \qquad \frac{\partial P_2}{\partial \phi_k} = \frac{-(pP_1+u_k)}{\phi'(u_k)},$$

$$\frac{\partial P_3}{\partial \Phi_h} = \frac{-(p-1)P_2B_h+P_1(p-1-\omega B_h)+p\omega_h-\omega_h^2B_h}{\phi''(\omega_h)}, \quad \frac{\partial P_3}{\partial \phi_k} = \frac{-((p-1)P_2+u_kP_1+u_k^2)}{\phi'(u_k)},$$

. . . .

Ces résultats établis, nous formons les combinaisons intégrales du système (Σ), de proche en proche, en combinant chacune des équations avec les précédentes, le coefficient de celle de rang le plus élevé étant l'unité. Les autres coefficients sont des combinaisons linéaires des groupements en $a_1, a_2, \ldots, a_n$ *de même poids, à coefficients constants* et des groupements analogues en $P_1, P_2, \ldots, P_p$. On attribue le poids 1 aux γ_i et ω_i et aussi aux u_k.

On a, par exemple, pour les premières combinaisons intégrables:

$$\sum \frac{\partial \theta}{\partial \Phi_h} d\Phi_h + \sum \frac{\partial \theta}{\partial \phi_k} d\phi_k = d\theta,$$

$$\sum \frac{\partial \theta}{\partial \Phi_h}\left(\omega_h + \frac{3a_1}{n}\right) d\Phi_h + \sum \frac{\partial \theta}{\partial \phi_k}\left(u_k + \frac{3a_1}{n}\right) d\phi_k = d\theta_1,$$

$$\sum \frac{\partial \theta}{\partial \Phi_h}\left(\omega_h^2 + \frac{4a_1}{n}\omega_h + \alpha\right) d\Phi_h + \sum \frac{\partial \theta}{\partial \phi_k}\left(u_k^2 + \frac{4a_1}{n}u_k + \alpha\right) d\phi_k = d\theta_2,$$

avec:

$$\alpha = 2P_2 - \frac{(2p-1)}{n} a_2 + \frac{12-2p(p+1)+(2p-1)(n-1)}{2n^2} a_1^2,$$

. . . .

Une certaine indétermination apparaît dans le calcul des coefficients numériques; elle tient à ce que l'on a:

$$P_1 = \frac{(p+1)}{n} a_1$$

On la supprimera en ne faisant figurer qu'une des quantités a_1 et P_1.

On pouvait penser que les groupements isobares à envisager seraient des fonctions des seuls coefficients $R_1, R_2, \ldots$ de $R(y') = P(y')\,A(y')$; *il n'en est rien.* L'expression de α n'est pas, dans ses termes en P_2 et a_2 proportionnelle à R_2 qui est:

$$P_2 + \frac{(n-2)}{n} a_2 + \frac{(n-1)}{n} a_1 P_1.$$

Il convient d'observer que dès que les exposants des ω_i et des u_k dans l'équation de rang le plus élevé, dépasseront respectivement $(n-1)$ et p, il faudra pour former les combinaisons isobares tenir compte des équations: $A(y') = 0$ et $P(y') = 0$. Cela entraine de légères modifications de forme.

En résumé, nous savons construire les combinaisons intégrables de (Σ): en les égalant à des constantes arbitraires, nous avons les $(n+p-1)$ relations qui définissent les caractéristiques $\phi_1, \ldots, \phi_p, \Phi_1, \ldots, \Phi_{n-1}$ en x et y. Tout est donc terminé.

Remarques générales. I. J'appellerai ici l'attention sur le caractère général de la méthode d'intégration qui nous a permis d'obtenir par quadratures les solutions générales des systèmes aux dérivées partielles rencontrés. On a vu que le problème avait été transformé par l'introduction explicite, comme variables indépendantes, des n caractéristiques d'Ampère (ou de Cauchy), chaque système de caractéristiques n'ayant ici *qu'une seule combinaison intégrable.* Le problème à n variables indépendantes dépend d'un système *linéaire* et la liaison entre les deux problèmes à n et à 2 variables x, y est donnée par un système complètement intégrable, qui s'intègre même ici par quadratures. Ces circonstances paraissent assez particulières et l'on pourrait caractériser les équations d'ordre n à une fonction inconnue z et à deux variables x, y qui s'intègrent ainsi. On peut construire *a priori* de tels systèmes d'équations en partant *d'un système linéaire quelconque.*

Par exemple, si l'on part d'un système d'équations de Laplace:

$$(T) \qquad \frac{\partial^2\theta}{\partial\phi_i\partial\phi_k} + a_{ik}\frac{\partial\theta}{\partial\phi_i} + a_{ki}\frac{\partial\theta}{\partial\phi_k} + c_{ik}\theta = 0, \qquad (i \neq k = 1, \ldots, n),$$

dont la solution générale dépend de n fonctions arbitraires d'un argument, on sait former de diverses manières (systèmes *parallèles* ou complètement conjugués, etc. . . .) des combinaisons

$$d\lambda_i = A_{i1}d\phi_1 + \ldots + A_{in}\,d\phi_n, \qquad (i = 1, \ldots, n),$$

où les A_{ij} dépendant linéairement des dérivées (même d'ordre supérieur) de θ, qui sont différentielles exactes quelle que soit la solution θ de (T).

En écrivant des équations

$$\lambda_1+x_1=\text{const.}, \ldots, \lambda_p+x_p=\text{const.}, \lambda_{p+1}=\text{const.}, \ldots, \lambda_n=\text{const.}$$

et considérant q fonctions déterminées $z_1, \ldots, z_q$ des arguments $\phi_1, \ldots, \phi_n$, ces q fonctions dépendent par l'intermédiaire des relations précédentes. de $x_1, \ldots, x_p$. L'élimination de θ donne un système (Σ) en $z_1, \ldots, z_q$ aux variables $x_1, \ldots, x_p$ dont nous avons la solution générale dès que nous connaissons la forme la plus générale de θ, c'est-à-dire que nous en avons, par exemple, des solutions dépendant de paramètres convenablement choisis. Cette méthode d'intégration va donc plus loin que les méthodes d'Ampère et de Darboux: il n'y a pour chacune des caractéristiques de Cauchy de (Σ) qu'une combinaison intégrable. Ce n'est que lorsque certaines des équations de (T) s'intégrent partiellement par la méthode de Laplace, que des fonctions arbitraires des ϕ_i apparaissent explicitement. Nous reviendrons ailleurs sur cette importante question.

II. Une dernière observation: dans la plupart des exemples traités, la forme d'équation différentielle étudiée dépend rationnellement *d'un seul* de ses arguments.

L'exemple de l'équation: $\rho'+\rho^2=\phi(x)+h$, rationnelle en ρ et en h est d'une autre nature. (Cf. Comptes Rendus Acad. Sciences, Paris, janvier 1919). En général quand on se donne une équation

$$y^{(n)}=F(x, y, y', \ldots, y^{(n)})$$

où F dépend rationnellement, sous une forme donnée, de *deux* de ses arguments au moins et que l'on recherche les conditions de réduction—aux quadratures par exemple—on trouve *un système surabondant d'équations*, dont la solution est *restreinte*. Il n'y a plus de caractéristiques d'Ampère pour un tel système. L'exemple le plus simple est celui du problème de Mécanique à deux degrés de liberté, réduit à l'intégration de l'équation: $pq=\lambda(U+h)$ où $p=\dfrac{\partial z}{\partial x}$, $q=\dfrac{\partial z}{\partial y}$. J'ai pu cependant conserver, dans l'étude de la détermination de λ et de U en x, y de manière que l'équation précédente s'intègre par quadratures, *quel que soit* h, l'esprit des méthodes précédentes, et introduire des *éléments caractéristiques*, en nombre beaucoup plus élevé, qui conduisent encore à la solution. Mais ceci nous entraînerait trop loin.

Mouvement d'un solide pesant qui a un point fixe

(Détermination du *groupe de rationalité* de l'équation différentielle du problème).

(A) Le problème du mouvement d'un solide pesant qui a un point fixe se résoud par des quadratures (elliptiques ou hyperelliptiques) dans les cas bien connus, dits d'Euler et Poinsot, de Lagrange et Poisson et de Mme Kowalewski, où il admet, en dehors de trois intégrales algébriques classiques, une nouvelle intégrale première algébrique. Proposons-nous de chercher le groupe de rationalité de l'équation différentielle du second ordre à laquelle il se ramène en général.

Les moments principaux d'inertie A, B, C relatifs au point fixe n'étant pas nuls, nous ferons dans les équations classiques le remplacement de Ap, Bq, Cr par de nouvelles variables p, q, r et celui de A, B, C par $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$. En multipliant par une constante convenable les cosinus directeurs x, y, z de la verticale par rapport aux axes principaux d'inertie au point fixe, on peut écrire les équations du problème:

$$\text{(I)} \qquad \frac{dp}{dt}=(b-c)qr+\eta z-\zeta y, \qquad \frac{dz}{dt}=cry-bqz,$$

. . . .

. . . .

et l'on en connaît les intégrales:

$$x^2+y^2+z^2=l^2, \qquad px+qy+rz=k\,,$$
$$ap^2+bq^2+cr^2+2(\xi x+\eta y+\zeta z)=2h.$$

Si Δ désigne le déterminant fonctionnel de l^2, k, h en x, y, z, Δ^2 est un polynome du sixième degré en p, q, r et les variables x, y, z sont rationnelles en p, q, r, Δ. Le système (I) se ramène donc à une équation:

$$\text{(II)} \qquad U(f)=A\frac{\partial f}{\partial p}+B\frac{\partial f}{\partial q}+C\frac{\partial f}{\partial r}=0,$$

où A, B, C sont respectivement les quantités $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$ exprimées en p,q,r,l^2,k,h; le temps t sera, après intégration de (II), donné par une quadrature.

L'équation (II) possède—comme toutes les équations rencontrées en Dynamique—un multiplicateur de Jacobi rationnel en p, q, r, Δ. Son groupe de rationalité (G) relatif aux solutions fondamentales ϕ, ψ est donc le groupe infini (Γ), défini par la relation $\frac{\partial(\Phi,\Psi)}{\partial(\phi,\psi)}=1$, ou l'un de ses sous-groupes types. Il s'agit de décider d'abord si (G) est *imprimitif*, auquel cas il existe une autre équation linéaire formant avec (II) un système complet et dont les coefficients sont rationnels ou quadratiques dans le domaine (p,q,r,Δ). Sinon le groupe primitif (G) ne pourrait se réduire qu'au groupe linéaire spécial.

(B) Remplaçons respectivement dans (I) r, p, y par ϵr, ϵp, ϵy et η par $\epsilon^2\eta$ où ϵ est un paramètre. Les expressions $\frac{dp}{dt}$, . . . , $\frac{dx}{dt}$, . . . seront du premier ou du second degré en ϵ et si l'on fait tendre ϵ vers zéro, le système (I) tendra vers un système *réduit*

$$\text{(III)} \qquad \begin{aligned} \frac{dp}{dt}&=(b-c)qr-\zeta y, & \frac{dx}{dt}&=-bqz,\\ \frac{dq}{dt}&=\zeta x-\xi z, & \frac{dy}{dt}&=apz-crx,\\ \frac{dr}{dt}&=(a-b)pq+\xi y, & \frac{dz}{dt}&=bqx,\end{aligned}$$

qui admet les intégrales premières:

$$x^2+z^2=l^2, \quad bq^2-2(\xi x+\zeta z)=2h, \quad px+qy+rz=k.$$

On voit que x et z s'expriment en fonction de q^2; y est alors linéaire en p et r. Le système (III) donne pour $\frac{dp}{dq}, \frac{dr}{dq}$ des fonctions linéaires de p et de r. Les deux intégrales de ce système linéaire sont donc définies *à un groupe linéaire près*, groupe de rationalité du système réduit. Ce groupe est-il *primitif*?

J'ai observé que si l'on fait $l^2=0$, c'est-à-dire si l'on étudie certaines solutions de (III) qui existent normalement—et qui ont été considérées par Hermite dans d'autres cas—on peut ramener le système (III) à *l'équation hypergéométrique de Gauss* et à des quadratures. Il suffit de poser:

$$q^2=\frac{2h}{b}u, \quad r+ip=u(1-u)^\nu Z, \text{ avec } \nu=-\sqrt{\left(1-\frac{c}{b}\right)\left(1-\frac{a}{b}\right)}$$

pour obtenir la forme classique

$$u(1-u)\frac{d^2Z}{du^2}+\left[\gamma-(\alpha+\beta+1)u\right]\frac{dZ}{du}-\alpha\beta Z=0$$

avec:

$$\gamma=\frac{3}{2}, \quad \alpha+\beta=\frac{3}{2}-2\nu, \quad \alpha-\beta=-2\sqrt{P+\frac{1}{4}}$$

où:

$$P=\left(1-\frac{c}{b}\right)\left(1-\frac{a}{b}\right)+\frac{\xi\left(1-\frac{c}{b}\right)+\zeta\left(1-\frac{a}{b}\right)}{2(\zeta+i\xi)}.$$

Les constantes h, k n'y figurent plus.

Dans le cas actuel, le groupe de rationalité de l'équation de Gauss est primitif; l'équation (II) possède donc aussi un groupe de rationalité (G) primitif

(C) Il reste à décider si ce groupe (G) est infini ou fini, auquel cas il se réduirait au groupe linéaire spécial. Etudions le cas d'Euler et Poinsot: il existe alors une intégrale rationnelle nouvelle ϕ et le groupe de rationalité est formé de transformations:

$$\Phi=\phi, \quad \Psi=\psi+F(\phi),$$

l'intégrale ψ étant donnée par une quadrature elliptique; la fonction $F(\phi)$ est la période d'une telle intégrale envisagée comme fonction du module. *Un tel groupe n'est pas semblable à un groupe linéaire.*

Le groupe de rationalité (G) du système (I) est donc infini et comme il est primitif, *il ne peut être que le groupe* Γ.

Les transcendantes qui permettront la détermination du mouvement d'un solide pesant autour d'un point fixe, sont donc des fonctions ϕ, ψ de p, q, r *attachées* dans le domaine (Δ) au groupe (Γ).

Tous les cas de réduction de l'équation de Gauss sont—comme cette équation elle-même—en une certaine mesure, des cas de réduction de (I). Ils correspondent à des mouvements particuliers dans lesquels certains éléments sont infiniment petits par rapport aux autres. On devra étudier dans ces divers cas la réduction possible du système (I) en développant les intégrales suivant les puissances de ϵ. Les intégrales classiques sont des polynomes en ϵ. Il convient d'observer que le système réduit (III) contient toutes les constantes du problème non particularisées—sauf la coordonnée η qui est nulle. Ce système permettrait donc d'étudier les dernières intégrales ϕ et ψ comme fonction de ces diverses constantes, et aussi de h et k.

La recherche actuelle n'est donc qu'un début.

ON THE ZEROS OF THE FUNCTIONS DEFINED BY LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1. A linear homogeneous differential equation of the second order can always be written in the form

$$w'' + G(z)w = 0. \tag{1}$$

We assume $G(z)$ to be a single-valued analytic function of z, the singularities of which shall have only a finite number of limit points. Let $w(z)$ be an arbitrary solution of (1) which is not identically zero.

The problem of integrating a linear differential equation has been interpreted in many different ways. In modern mathematics the problem is usually taken to mean the determination of the group of monodromy of the equation, inasmuch as this group gives the connection between the different expansions in series which can be obtained at the several singular points. If this problem is supposed to be solved, it is possible to construct the Riemann surface over the z-plane on which the solution $w(z)$ mentioned above is single-valued. Then the problem arises of how the values of w are distributed upon the surface. It is true that the known series expansions throw some light upon this question but only locally and in a very imperfect manner. A small but important detail of this problem is the study of the distribution of the value zero on the surface, in other words, *the problem of finding the zeros of the solutions*. The present paper is intended to call attention to a couple of methods which can be used for the purpose of elucidating this *zero point problem.*

The literature concerned with our problem is not extensive. We notice interesting investigations of Hurwitz, Van Vleck, Schafheitlin and Boutroux on various special equations. Important sidelight has been thrown on the problem through the study of irregular singular points of finite rank made by Horn, Birkhoff, Garnier and others. Finally the present writer has devoted a series of papers to this problem, some to general questions, some to particular cases*.

2. All methods which have been used for the purpose of studying the problem in question make use of one or several of the following four devices, namely:

*A paper in Proc. London Math. Soc. Ser. 2, Vol. 23 (1924) pp. 185-237: *On the zeros of Mathieu functions*, gives an outline of the general theory together with an important application.

(i) *Asymptotic representation* of the solutions in the neighbourhood of a singular point.

(ii) *Integral equalities* of the type used in the Sturmian theory of oscillation problems.

(iii) *Variation of the parameters* entering into the solutions.

(iv) *Conformal mapping* by means of the quotient of two solutions.

In the following we shall give a short account of convenient normalizations of the methods based upon the first two devices together with some reflections upon the latter two methods. For further details we have to refer the reader to the paper mentioned in the footnote.

3. In order to obtain the asymptotic representation of the solutions in the neighbourhood of a singular point we use the transformation

$$Z = Z(z;\ z_0) = \int_{z_0}^{z} \sqrt{G(z)}\, dz, \quad W = [G(z)]^{\frac{1}{4}}\, w. \tag{2}$$

The result of the transformation is a new differential equation

$$\frac{Wd^2}{dZ^2} + G^*(Z)W = 0, \tag{3}$$

where

$$G^*(Z) = g(z) = 1 - \tfrac{1}{4}\,\frac{G''(z)}{[G(z)]^2} + \tfrac{5}{16}\,\frac{[G'(z)]^2}{[G(z)]^3}\,. \tag{4}$$

This transformation has been much used in the study of oscillation problems for real variables since the time of Liouville. The importance of the transformation in the complex domain does not seem to have been generally recognized, though its normalizing power is scarcely less striking in the complex case than in the real one.

Suppose that $z = a$ is a singular point of the differential equation (1). Excluding the case in which $z = a$ is a simple pole of $G(z)$, we can always map a certain partial neighbourhood of $z = a$ upon a partial neighbourhood of $Z = \infty$ by means of the function $Z = Z(z;\ z_0)$. Suppose for instance that $z = a$ is a pole of order $k > 2$, in which case the point is an irregular singular point of finite rank. Then it is possible to find $k-2$ *principal regions* which together form a complete neighbourhood of $z = a$ and each of which can be mapped upon a half-plane

$$\mathfrak{R}(Z) = X \geqq A > 0.$$

Furthermore

$$Z^2[G^*(Z) - 1]$$

remains bounded when Z tends to infinity within the half-plane, *i.e.*, when z tends to a within the principal region in question. If the rank is not finite, we may still be able to find principal regions abutting upon $z = a$ in which $g(z)$ behaves in the same manner as in the finite case.

On the other hand, if $z=a$ is a regular singular point such that the product c of the roots of the corresponding indicial equation is positive, then

$$e^{\frac{Z}{\sqrt{c}}}\left[G^*(Z)-1+\frac{1}{4c}\right] \qquad (\sqrt{c}>0)$$

remains bounded when Z tends to infinity in the half-plane $X>A$. Such an exponential approach of $G^*(Z)$ to its limit may also occur in cases in which $z=a$ is not a regular singular point.

4. The application of the transformation (2) involves two separate steps: (i) the determination of the principal regions in the z-plane which arise at the conformal representation of the z-plane upon the Z-plane, (ii) the study of the transformed equation (3) in the corresponding half-planes over the Z-plane.

We turn our attention to the second problem. We assume that we have obtained a transformed equation (3) such that

$$F(Z)=1-G^*(Z)$$

is single-valued and analytic in any finite portion of the region D_R,

$$|Z|\geqq R \text{ if } X\geqq 0 \text{ and } |Y|\geqq R \text{ if } X<0,\ R>0,$$

and

$$|F(Z)|<\frac{M}{|Z|^2} \tag{5}$$

when Z lies in D_R.

Let us associate the following singular integral equation of the Volterra type with the transformed equation (3), namely

$$W(Z)=W_0(Z)+\int_Z^{\infty}\sin(T-Z)F(T)W(T)dT, \tag{6}$$

where

$$W_0''(Z)+W_0(Z)=0 \tag{7}$$

and the path of integration is the line $\arg(T-Z)=0$.

It can be shown that this equation possesses an analytic solution, holomorphic in any finite portion of D_R and bounded in any infinite strip $B_1\leqq Y\leqq B_2$ wholly in D_R. Moreover, this is the only solution of (6) the modulus of which admits of a finite upper bound in any finite portion of D_R. Further, the solution of the integral equation satisfies the differential equation (3) no matter how the function $W_0(Z)$ has been chosen, subject to condition (7). Conversely, every solution of the differential equation (3) satisfies an integral equation of the type (6) provided $W_0(Z)$ is properly selected from the solutions of the auxiliary differential equation (7).

Thus there exists a one-to-one correspondence between the solutions of the transformed differential equation (3) and the solutions of the sine equation. The correspondence is such that linearly independent solutions of (7) correspond to linearly independent solutions of (3). Further, the corresponding solutions are asymptotically equal in D_R in a sense upon which we shall dwell somewhat.

Let us consider that part, Δ_0, of the strip $B_1 \leqq Y \leqq B_2$ which lies in D_R. Suppose that

$$\text{Max}\,|W_0(Z)| = L;\ \text{Max}|W(Z)| = K \tag{8}$$

in the strip Δ_0. An easy calculation shows that

$$|W(Z)| < L + MK\frac{\theta}{Y},$$

where $\theta = \arg Z$. Suppose that the constant R which enters into the determination of the region D_R satisfies the condition

$$R \geqq 2\pi M. \tag{9}$$

Then $M\dfrac{\theta}{Y} \leqq \frac{1}{2}$ when Z lies in Δ_0 and consequently

$$K < 2L.$$

Hence

$$|W(Z) - W_0(Z)| < 2LM\frac{\theta}{\sin\theta}\cdot\frac{1}{|Z|}. \tag{10}$$

A similar formula holds when Z lies in D_R above Δ_0, namely

$$|e^{iZ}[W(Z) - W_0(Z)]| < 2L_+M\frac{\theta}{\sin\theta}\cdot\frac{1}{|Z|}, \tag{11}$$

where

$$L_+ = \text{Max}\,|e^{iZ}\,W_0(Z)|,\ Y \geqq B_2. \tag{12}$$

The formula for the lower half-plane is analogous.

5. These formulae express the asymptotic relationship between the corresponding functions $W(Z)$ and $W_0(Z)$. They enable us to read off the asymptotic properties of $W(Z)$ knowing those of $W_0(Z)$. Equation (7) has two different types of solutions, namely, infinitely many linearly independent oscillatory solutions of the form $\sin(Z - Z_0)$ and two linearly independent non-oscillatory solutions e^{iZ} and e^{-iZ}. To the former class of functions $W_0(Z)$ correspond oscillatory solutions of (3) which have infinitely many zeros in D_R; to the latter correspond two truncated solutions which do not vanish at all in D_R.

In the oscillatory case we can take

$$W_0(Z) = \sin(Z - Z_0).$$

Suppose that the strip Δ_0 is so chosen that $B_1 < Y_0 < B_2$ where $Z_0 = X_0 + iY_0$. Let us divide Δ_0 into rectangles R_n by means of the lines $X = X_0 + (n - \frac{1}{2})\pi$ where n is an arbitrary integer. With the aid of formula (11) and of its analogon in the lower half-plane, it is easy to choose B_1 and B_2 in such a manner that the part of D_R in which $Y \leqq B_1$ or $\geqq B_2$ cannot contain any zero of $W(Z)$. On the other hand, the theorem of Rouché shows that each of the rectangles R_n contains one and only one zero of $W(Z)$ provided either Y_0 is a sufficiently large positive or negative number, or else n is a sufficiently large positive number. In the former case there is no restriction on n, in the latter none on Y_0. When $n \to +\infty$ the zero of $W(Z)$ in R_n which we denote by A_n tends to the centre of R_n, in other words

(13) $$A_n - Z_0 - n\pi \to 0.$$

Similarly the zeros of $W'(Z)$ which we call the *extrema* of $W(Z)$ are approximated by the zeros of $\cos(Z-Z_0)$. Thus we find that the zeros and extrema of $W(Z)$ in D_R form a linear set, a *string*, in Δ_0. This set is guided by the line $Y=Y_0$; the points $Z_0+n\pi$ and $Z_0+(n-\frac{1}{2})\pi$ on this line give the approximate location of the zeros and extrema respectively when n is large.

The analysis of §§ 4 and 5 applies to the case in which $G^*(Z)$ approaches unity as a limit in the manner indicated by formula (5). If $G^*(Z)$ would approach some other limit, exponentially for instance, a similar analysis is possible. The essential condition for the success of the method is that the function $G^*(Z)$ shall be single-valued and analytic in a half-plane $X \geqq A$ and approach a finite limit a^2 when $X \to +\infty$; further the integrals

(14) $$\int_Z^{\infty} [a^2 - G^*(T)]\, dT$$

shall be absolutely convergent for every value of Z in the half-plane*.

In order to obtain the distribution of the zeros of $w(z)$ in the principal region which corresponds to the half-plane $X \geqq A$ we have only to find the images of the points A_n. These points are approximated by a set of points upon an arc of the curve

(15) $$\mathfrak{I}\left(\int_{z_0}^{z} \sqrt{G(z)}\, dz\right) = Y_0.$$

The extrema require a special discussion owing to the presence of the factor $[G(z)]^{\frac{1}{4}}$ in formula (3). Provided we can cover the Riemann surface of $w(z)$ completely with principal regions, the problem of the distribution of the zeros on the surface can be solved in a qualitative manner with the aid of the method outlined above. This solution can be given a higher degree of precision if the relations are known which express the truncated solutions belonging to one principal region in terms of the truncated solutions belonging to any other principal region.

6. The Sturmian methods are largely based upon the use of integral equalities. In order to obtain such an equality which is suitable for the complex domain, we multiply equation (1) by $\bar{w}$, the conjugate of w. If the resulting expression is integrated, a simple reduction yields the following relation

(16) $$\left[\bar{w}\, w'\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w'|^2 \overline{dz} + \int_{z_1}^{z_2} |w|^2 G(z)\, dz = 0.$$

*See the author's note in Proc. Nat. Acad. Sci. vol. 10, No. 12 (1924) pp. 488-493: *A general type of singular point.*

We call this equality *Green's transform* of the differential equation (1). In the following z_1 and z_2 shall be non-singular points of the differential equation, and the path C which joins them shall not pass through any singular point. Further, C shall be made up of a finite number of analytic arcs. Consequently, C has a continuously varying tangent except at a finite number of points where unique semi-tangents exist.

Formula (16) tells us that the necessary and sufficient condition in order that

$$w(z_2)w'(z_2)=0$$

is that

$$-\bar{w}(z_1)w'(z_1)-\int_{z_1}^{z_2}|w'(z)|^2\overline{dz}+\int_{z_1}^{z_2}|w(z)|^2G(z)dz=0, \tag{17}$$

or shorter

$$I_0+I_1+I_2=0.$$

7. The condition

$$\frac{w'(z_1)}{w(z_1)}=\lambda, \tag{18}$$

where λ is an arbitrary complex number including ∞, defines a solution of (1) uniquely save for an arbitrary multiplicative constant. We shall determine a region in which this solution $w(z;\,z_1,\,\lambda)$ cannot admit of any zeros or extrema. Let us put

$$\arg\lambda=\omega,\ \arg G(z)=\gamma_z,\ \arg dz=\theta_z. \tag{19}$$

Then we choose an arbitrary angle θ subject to the condition

$$\theta\leqq\omega+\pi\leqq\theta+\pi. \tag{20}$$

Now suppose that we choose a path C leading from $z=z_1$ to $z=z_2$ subject to the conditions mentioned in § 6 and, in addition, to the following conditions:

$$\theta\leqq\pi-\theta_z\leqq\theta+\pi, \tag{21}$$

$$\theta\leqq\gamma_z+\theta_z\leqq\theta+\pi. \tag{22}$$

Conditions (21) and (22) shall be fulfilled at all but a finite number of points on C. Further, the inequality signs shall hold either in (20) or else in one of the conditions (21) and (22) for a set of points forming a non-vanishing arc of C. Such a curve C shall be said to possess a *definite vector field* and to be *a line of influence with respect to the point* z_1 *and the solution* $w(z;\,z_1,\,\lambda)$.

It follows from the choice of C that

$$\theta\leqq\arg I_n\leqq\theta+\pi,\ |I_n|\neq 0,$$

where the relation holds for $n=0$, 1 and 2 if $\lambda\neq 0$ and ∞, and for $n=1$ and 2 if $\lambda=0$ or ∞ and the inequality signs hold for at least one value of n. Hence in every case

$$I_0+I_1+I_2\neq 0.$$

Consequently, in view of the discussion in § 6,

$$w(z_2)w'(z_2)\neq 0.$$

Now let us determine all the lines of influence with respect to $z=z_1$ and the solution $w(z;\ z_1,\ \lambda)$ which correspond to a given value of θ and then repeat the process for every value of θ compatible with condition (20). The end-points of all the lines so obtained form *the domain of influence of the point $z=z_1$ with respect to the solution $w(z;\ z_1,\ \lambda)$* which we denote by $DI(z_1,\ \omega)$. In general this domain is made up of several disconnected regions which may contain some or all of their boundary points; the domain may or may not be over-lapping, in the former case it has to be spread out upon the Riemann surface of the solution.

There are no zeros or extrema of the solution $w(z;\ z_1,\ \lambda)$ in the domain of influence of the point $z=z_1$ with respect to the solution in question.

The points of $DI(z_1;\ \omega)$ are in general contained in the region $DI(z_1)$, which corresponds to $\lambda=0$ or ∞, for every value of ω. It may happen, however, that the vector field of a path is definite only if $I_0\neq 0$; in that event the end-point of the path belongs to $DI(z_1;\ \omega)$ for certain values of ω but does not belong to $DI(z_1)$.

8. It is of importance to notice that the domain of influence $DI(z_1;\ \omega)$ does not depend upon the value of $|\lambda|$ save for the case in which this value is either 0 or ∞. Further, we do not change any of these domains, if we replace $G(z)$ in equation (1) by $k^2G(z)$, k being a real quantity. Thus the function $w_k(z;\ z_1,\ re^{i\omega})$ which satisfies the differential equation

$$w''+k^2G(z)w=0 \tag{23}$$

and the logarithmic derivative of which reduces to $re^{i\omega}$ for $z=z_1$, does not vanish in the domain $DI(z_1;\ \omega)$ for any real value of k and any value of r, $0<r<+\infty$. In particular, the functions $w_k(z;\ z_1,\ 0)$ and $w_k(z;\ z_1,\ \infty)$ do not vanish in $DI(z_1)$ for any real value of k. The same statements hold for the first derivative of the function in question.

This situation raises an interesting question. The function $w_k(z;\ z_1,\ re^{i\omega})$ does not vanish anywhere in $DI(z_1;\ \omega)$ for any real value of k and any positive value of r. *Supposing that $z=z_2$ is a point in the complementary region of $DI(z_1;\omega)$, is it possible to determine a real value of k together with a positive value of r, κ and ρ respectively, say, such that*

$$w_\kappa(z_2;\ z_1,\ \rho e^{i\omega})=0?$$

In other words, does the locus of the zeros of $w_k(z;\ z_1,\ re^{i\omega})$ for real values of k and positive values of r, coincide with *co-$DI(z_1;\ \omega)$*? Ordinary considerations of continuity indicate the plausibility of an affirmative answer to this question.

9. The complete determination in a finite number of operations of the domain of influence of a point with respect to a given solution is seldom possible. But it is always possible to reach a more or less extensive portion of this domain in a finite number of steps, if the curves which are used for the purpose of building up lines of influence are properly standardized. As proper curves we want to mention the following six types:

$$\mathfrak{R}\left[\int_{z_1}^{z} dz\right]=\text{const.};\quad \mathfrak{I}\left[\int_{z_1}^{z} dz\right]=\text{const.}; \tag{24}$$

$$\mathfrak{R}\left[\int_{z_1}^{z} G(z)\,dz\right] = \text{const.};\ \mathfrak{I}\left[\int_{z_1}^{z} G(z)\,dz\right] = \text{const.}; \tag{25}$$

$$\arg\left[\int_{z_1}^{z} dz\right] = \text{const.};\quad \arg\left[\int_{z_1}^{z} G(z)\,dz\right] = \text{const.} \tag{26}$$

None of the first four types can be used alone with any greater amount of profit; the best result is obtained by combining all four types. Each of the last two types can be used alone and does not combine easily with the other or with the first four types. For the study of truncated solutions and for purposes of ordering the zeros and extrema valuable service is rendered by the curves

$$\mathfrak{R}\left[\int_{z_0}^{z} \sqrt{G(z)}\,dz\right] = \text{const.}, \tag{27}$$

which are the orthogonal trajectories of the curves in formula (15).

10. Let us give some simple applications of this *method of zero-free regions.*

First, let us suppose that $G(z)$ is holomorphic and different from zero in a convex region B and that in addition

$$\theta_1 < \arg G(z) < \theta_2,\ 0 < \theta_2 - \theta_1 < \pi, \tag{28}$$

for every point in B. Let z_1 and z_2 be two arbitrary zeros or extrema in B of some solution $w(z)$ of the differential equation (1). Then

$$-\tfrac{1}{2}\theta_2 < \arg\,(z_1 - z_2) < -\tfrac{1}{2}\theta_1 \tag{29}$$

if the enumeration of the points has been properly chosen.

This theorem gives an easy means of arranging the zeros and extrema of a solution within a certain region into a linear sequence.

Secondly, let us assume that $G(z)$ is real positive on an interval (a, b) of the real axis. Let D be a simply-connected region symmetric to the real axis, lying between the lines $x = a$ and $x = b$, the boundary of which is cut in two points only by any line $x = c$, $a < c < b$. Finally let $G(z)$ be holomorphic in the closed region D where in addition $\arg G(z)$ shall be $\neq \pm\pi$. Then, a solution of (1) cannot have any complex zeros or extrema in D if its logarithmic derivative is real at a point of the interval (a, b). This theorem is evidently of great applicability, especially in the study of differential equations which arise in mathematical physics.

11. Finally let us say a few words about the methods which are based upon variation of parameters and upon conformal mapping. Both these methods can be attached to the Sturmian method of zero-free regions sketched in §§ 6-10.

Let us refer to the quantity λ defined by equation (18) as *the parameter of the solution at the point* z_1. It is comparatively easy to follow the effect of varying this parameter. First we notice that the zeros and extrema of the solution are analytic functions of λ. Thus, if λ is varied continuously, the zeros and extrema move in a continuous manner except if λ passes through certain singular points. Now for every value of λ we have a domain of influence $DI(z_1;\,\omega)$. Further every moving zero and extremum has its own variable domain of influence. No matter how the motion is carried out, the zeros and the extrema have to stay outside of $DI(z_1;\omega)$ and outside of each other's domains of influence. This simple rule gives a means of following the motion of the zeros and the extrema during the variation of λ.

The information so obtained can be brought to bear upon the problem of conformal representation by means of the quotient of two linearly independent solutions of (1). If we choose

$$Q(z) = -\frac{w(z;\,z_1,\,\infty)}{w(z;\,z_1,\,0)} \tag{30}$$

as the quotient, and if $Q=\lambda$ when $z=z_2$ then z_2 is a zero of

$$w(z;\,z_1,\,\lambda) = w(z;\,z_1,\,\infty) + \lambda w(z;\,z_1,\,0). \tag{31}$$

Thus, if a certain zero of $w(z;\,z_1,\,\lambda)$ describes a path S when the parameter λ is varied along a path L, then S is the map in the z-plane of the path L in the Q-plane by means of the inverse of the function $Q(z)$*.

*The author's paper *On the zeros of the functions of the parabolic cylinder*, Arkiv för Mat., Astr. o. Fys., vol. 18, No. 26, 1924, gives a simple example of the stepwise application of the four different methods mentioned in §2, beginning with the asymptotic representation and culminating in the conformal mapping.

THE ASYMPTOTIC DISTRIBUTION OF THE CHARACTERISTIC NUMBERS FOR THE SELF-ADJOINT LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER

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It is the object of this paper to obtain theorems concerning the boundary problem of the third kind, analogous to those obtained by H. Weyl* for the boundary problem of the first kind, for the linear self-adjoint equation of the second order in two variables, containing a parameter linearly. In the first part are established certain properties of a Green's function corresponding to Laplace's equation and the third boundary condition; while in the second part these properties and the general theorems of Weyl are employed in a study of the asymptotic nature of the characteristic numbers (Eigenwerte) corresponding to the self-adjoint partial differential equation and third boundary condition. It is found that the same law holds for the third boundary problem as was obtained by Weyl for the first.

1. *On the Green's function for the boundary problem of the third kind.*

Let (C) be the interior region in the (xy) plane bounded by a finite number of non-intersecting closed curves $C_1, C_2, \ldots, C_n$ each possessing continuous curvature, and without double points; the normal derivatives indicated in what follows will be interior derivatives. If on each curve C_i a point is chosen arbitrarily, and arc lengths (denoted by s or t) measured positively in a direction such that the region (C) is on the left, then if l_i is the length of C_i, an arbitrary point of C_i can be represented by s or t; $l_1+l_2\ldots+l_{i-1}\leqq s<l_1+l_2\ldots+l_i$, $(i=1, 2, \ldots, n)$. For convenience the set of curves $C_1, \ldots, C_n$ will be denoted by C, and integration over C will be understood to mean summation of the corresponding integrals over the curves C_i.

Suppose

$$(1)\qquad G(xy,\xi\eta)=\log\frac{1}{r_{xy,\xi\eta}}-\omega(xy,\xi\eta),\quad r_{xy,\xi\eta}=\sqrt{(x-\xi)^2+(y-\eta)^2}$$

where $\omega(xy, \xi\eta)$ is harmonic in (xy) in the interior of (C) and on the boundary satisfies the equation

$$(2)\qquad \frac{d}{dn_s}\,\omega(s,\xi\eta)-h(s)\omega(s,\xi\eta)=\frac{d}{dn_s}\log\frac{1}{r_{s,\xi\eta}}-h(s)\,\log\frac{1}{r_{s,\xi\eta}}=g(s,\xi\eta).$$

*Mathematische Annalen, Band 71 (1911-1912), s. 441-479.

Then

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=0, \quad \frac{dG}{dn_s}-h(s)G=0, \tag{3}$$

if (xy) is on C, where $h(s)$ is assumed to be continuous over each of the curves C_i.

It will be shown that the function ω can be represented as a potential of a single layer

$$\omega(xy,\xi\eta)=V(xy)=\int_C \rho(t)\log\frac{1}{r_{xy,t}}\,dt \tag{4}$$

in which the density $\rho(t)$ satisfies the conditions

$$\tfrac{1}{2}\left(\frac{dV}{dn_e}-\frac{dV}{dn_i}\right)=\pi\rho(s),\quad \tfrac{1}{2}\left(\frac{dV}{dn_e}+\frac{dV}{dn_i}\right)=\int_C \rho(t)\frac{\cos\psi_s}{r_{st}}\,dt. \tag{5}$$

Consequently from (2),

$$\rho(s)=-\frac{1}{\pi}g(s)+\frac{1}{\pi}\int_C \rho(t)\left[\frac{\cos\psi_s}{r_{st}}-h(s)\log\frac{1}{r_{st}}\right]dt, \tag{6}$$

where

$$g(s)=g(s,\xi\eta),\ \cos\psi_s=r_{st}\frac{d}{dn_s}\log\frac{1}{r_{st}}\cdot$$

It will be assumed that the homogeneous equation obtained from (6) by replacing $g(s)$ by zero has no solution except zero; then from the theory of integral equations the equation obtained from (6) by means of a single iteration, having a finite kernel, has a unique solution. This iterated equation becomes,

$$\rho(s,\xi\eta)=\bar{\psi}(s,\xi\eta)+\int_C K(s,t)\rho(t,\xi\eta)\,dt \tag{7}$$

in which

$$\begin{aligned}\bar{\psi}(s,\xi\eta)=&-\frac{1}{\pi}\left[\frac{d}{dn_s}\log\frac{1}{r_{s,\xi\eta}}-h(s)\log\frac{1}{r_{s,\xi\eta}}\right]\\ &-\frac{1}{\pi^2}\int_C\left[\frac{\cos\psi_s}{r_{st}}-h(s)\log\frac{1}{r_{st}}\right]\left[\frac{d}{dn_t}\log\frac{1}{r_{t,\xi\eta}}-h(t)\log\frac{1}{r_{t,\xi\eta}}\right]dt.\end{aligned} \tag{8}$$

If $k(s,t)$ is the reciprocal kernel of $K(s,t)$, the solution of (7) is given by

$$\rho(s,\xi\eta)=\bar{\psi}(s,\xi\eta)+\int_C k(s,t)\bar{\psi}(t,\xi\eta)\,dt. \tag{9}$$

The singularities of $\rho(s,\xi\eta)$ can be determined from those of the function $\bar{\psi}(s,\xi\eta)$ which can be written:

$$\bar{\psi}(s,\xi\eta) = -\frac{1}{\pi}\left[\frac{\cos\psi_s}{r_{s,\xi\eta}} - h(s)\log\frac{1}{r_{s,\xi\eta}}\right] - \frac{1}{\pi^2}\int_C \frac{\cos\psi_s}{r_{st}}\frac{\cos\psi_t}{r_{\xi\eta,t}}dt \tag{10}$$

$$-\frac{1}{\pi^2}\int_C h(s)h(t)\log\frac{1}{r_{st}}\log\frac{1}{r_{\xi\eta,t}}dt + \frac{1}{\pi^2}\int_C h(t)\frac{\cos\psi_s}{r_{st}}\log\frac{1}{r_{\xi\eta,t}}dt$$

$$+\frac{h(s)}{\pi^2}\int_C \log\frac{1}{r_{st}}\frac{\cos\psi_t}{r_{\xi\eta,t}}dt.$$

Since the curves C_i have continuous curvature $\left|\frac{\cos\psi_s}{r_{st}}\right|$ has an upper bound; consequently the singularities which can occur as $(\xi\eta)$ approaches a point of C arise from the last integral and the bracketed term. If $s \neq s_0$,

$$\lim_{(\xi\eta)\to s_0}\int_C \log\frac{1}{r_{st}}\frac{\cos\psi_t}{r_{\xi\eta,t}}dt = \pi\log\frac{1}{r_{ss_0}} + \int_C \log\frac{1}{r_{st}}\frac{\cos\psi_t}{r_{s_0t}}dt, \tag{11}$$

the last integral remaining finite for all values of s, s_0.

Hence

$$\rho(s,\xi\eta) = -\frac{1}{\pi}\left[\frac{\cos\psi_s}{r_{s,\xi\eta}} - h(s)\log\frac{1}{r_{s,\xi\eta}}\right] + \frac{h(s)}{\pi^2}\int_C \log\frac{1}{r_{st}}\frac{\cos\psi_t}{r_{t,\xi\eta}}dt + E(s,\xi\eta) \tag{12}$$

$$-\frac{1}{\pi}\int_C k(s,t)\left[\frac{\cos\psi_t}{r_{t,\xi\eta}} - h(t)\log\frac{1}{r_{t,\xi\eta}}\right]dt + \frac{1}{\pi^2}\int_C k(s,t)h(t)\int_C \log\frac{1}{r_{tt'}}\frac{\cos\psi_{t'}}{r_{t',\xi\eta}}dt'dt$$

in which $E(s,\xi\eta)$ has an upper bound independent of $(s,\xi\eta)$. If the order of integration is interchanged in the last integral, it is found that this integral is bounded, and the same is true of the one preceding it. Consequently from (4),

$$\omega(xy,\xi\eta) = \int_C \log\frac{1}{r_{xy,t}}\frac{h(t)}{\pi^2}\int_C \log\frac{1}{r_{tt'}}\frac{\cos\psi_{t'}}{r_{\xi\eta,t'}}dt'dt - \frac{1}{\pi}\int_C \log\frac{1}{r_{xy,t}}\frac{\cos\psi_t}{r_{\xi\eta,t}}dt \tag{13}$$

$$+\frac{1}{\pi}\int_C h(t)\log\frac{1}{r_{\xi\eta,t}}\log\frac{1}{r_{xy,t}}dt + \int_C E(t,\xi\eta)\log\frac{1}{r_{xy,t}}dt + \bar{E}(xy,\xi\eta)'$$

$\bar{E}$ being bounded. If the point $(\xi\eta)$ approaches the point s on C the first integral remains bounded, from (11), and the same is true of the third and fourth integrals. Since

$$\lim_{(\xi\eta)\to s} -\frac{1}{\pi}\int_C \log\frac{1}{r_{xy,t}}\frac{\cos\psi_t}{r_{t,\xi\eta}}dt = -\log\frac{1}{r_{s,xy}} - \frac{1}{\pi}\int_C \log\frac{1}{r_{xy,t}}\frac{\cos\psi_t}{r_{st}}dt$$

we obtain

$$\lim_{(\xi\eta)\to s}\omega(xy,\xi\eta) = \omega(xy,s) = -\log\frac{1}{r_{xy,s}} + \bar{E}(xy,s), \tag{14}$$

where $\overline{E}(xy, s)$ is bounded for all values of (xy) in (C) or on C, and for all values of s. It is easily shown in the usual manner that the function of Green (1) is symmetric: $G(xy, \xi\eta)=G(\xi\eta, xy)$, hence the same is true of $\omega(xy, \xi\eta)$.

2. *The characteristic numbers for the equation* $\Delta u+\lambda u=0$ *and the third boundary problem.*

If the function $u(x, y)$ is a solution of the equation and boundary condition

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\lambda u=0, \quad \frac{du}{dn_s}-h(s)u=0, \tag{15}$$

then

$$u(xy)=\frac{\lambda}{2\pi}\iint\limits_{(C)} G(xy,\xi\eta)u(\xi\eta)d\xi d\eta$$

and λ must be one of the characteristic numbers for the kernel $\frac{1}{2\pi}G(xy,\xi\eta)$, the function $G(xy,\xi\eta)$ being defined as in § 1. From (1),

$$\iint\limits_{(C)}\left[\left(\frac{\partial\omega}{\partial\xi}\right)^2+\left(\frac{\partial\omega}{\partial\eta}\right)^2\right]d\xi d\eta=-\int\limits_{C}\omega(xy,s)\frac{d\omega}{dn_s}ds$$

$$=-\int\limits_{C}\omega(xy,s)\left[h(s)\omega(xy,s)+\frac{d}{dn_s}\log\frac{1}{r_{xy,s}}-h(s)\log\frac{1}{r_{xy,s}}\right]ds$$

$$=+\int\limits_{C}\log\frac{1}{r_{xy,s}}\frac{\cos\psi_s}{r_{xy,s}}ds+E'(xy,s);$$

$$\left|\iint\limits_{(C)}\left[\left(\frac{\partial\omega}{\partial\xi}\right)^2+\left(\frac{\partial\omega}{\partial\eta}\right)^2\right]d\xi d\eta\right|\leqq C'\left|\log\frac{1}{r_{xy,s_0}}\right|+C''$$

where r_{xy,s_0} is the shortest distance from (xy) to C, and C', C'' are constants independent of x and y. It follows that the integral

$$J=\iint\limits_{(C)}\iint\limits_{(C)}\left[\left(\frac{\partial\omega}{\partial\xi}\right)^2+\left(\frac{\partial\omega}{\partial\eta}\right)^2\right]d\xi d\eta dx dy$$

converges.

It follows from the theorems of Weyl that the asymptotic law for the positive characteristic numbers for the kernel $\frac{1}{2\pi}G$ and for the kernel $\frac{1}{2\pi}\log\frac{1}{r}$ must be the same; from which, (Weyl, l. c. § 3), if $0\leqq\lambda_1\leqq\lambda_2\leqq\dots\leqq\lambda_n\leqq\dots$,

$$\lim_{n\to\infty}\frac{n}{\lambda_n}=\frac{1}{4\pi}\left[\text{area of } (C)\right]. \tag{16}$$

3. *The characteristic numbers for the general self-adjoint equation of the second order.*

In the equation and boundary condition

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right)+(\lambda k-q)u=0,\quad \frac{du}{dn_s}-\sigma(s)u=0, \tag{17}$$

suppose the functions $p(x,y)$, $k(x,y)$ positive in the interior and on the boundary of (C); the functions q and k are assumed continuous, while p has continuous first and second partial derivatives in this region. In addition suppose $\sigma(s)$ positive or zero and continuous on each curve C_i, and suppose that the equation and boundary condition (17) have no solution other than $u\equiv 0$ for $\lambda=0$. Making the substitution $v=u\sqrt{p}$, we obtain from (17)

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\left(\lambda\frac{k}{p}-l\right)v=0, \tag{18}$$

$$\frac{dv}{dn_s}+\left[\sqrt{p}\,\frac{d}{dn_s}\left(\frac{1}{\sqrt{p}}\right)-\sigma\right]v=0,$$

in which

$$l(x,y)=\frac{q}{p}+\frac{\Delta\sqrt{p}}{\sqrt{p}}\,.$$

Assume

$$-h(s)=\sqrt{p}\,\frac{d}{dn_s}\left(\frac{1}{\sqrt{p}}\right)-\sigma(s)$$

and suppose G the function of Green for the equation and boundary condition

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad \frac{du}{dn_s}-h(s)u=0. \tag{19}$$

constructed as in § 1. Then a solution of (18) must satisfy

$$v=\frac{1}{2\pi}\iint\limits_{(C)} G(xy,\xi\eta)\left[\lambda\frac{k(\xi\eta)}{p(\xi\eta)}-l(\xi\eta)\right]v(\xi,\eta)d\xi d\eta \tag{20}$$

and no solution of (20) can exist for $\lambda=0$, otherwise equations (17) would have a solution for this value of λ. Hence the non-homogeneous equation

$$\Gamma(xy,\xi\eta)+\frac{1}{2\pi}\iint\limits_{(C)} G(xy,\bar{\xi}\bar{\eta})l(\bar{\xi}\bar{\eta})\Gamma(\bar{\xi}\bar{\eta},\xi\eta)d\bar{\xi}d\bar{\eta}=G(xy,\xi\eta) \tag{21}$$

must have a solution, and it is easily verified that any solution of (20) must satisfy

$$v(xy)=\frac{\lambda}{2\pi}\iint\limits_{(C)}\Gamma(xy,\xi\eta)\frac{k(\xi\eta)}{p(\xi\eta)}v(\xi\eta)d\xi d\eta, \tag{22}$$

or if

$$w(x,y)=\sqrt{\frac{k(xy)}{p(xy)}}\,v(xy)$$

then

$$w(xy) = \frac{\lambda}{2\pi} \iint\limits_{(C)} \Gamma(xy, \xi\eta) \sqrt{\frac{k(xy)k(\xi\eta)}{p(xy)p(\xi\eta)}}\, w(\xi\eta)d\xi d\eta. \tag{23}$$

From this point on the discussion is exactly the same as that given by Weyl for the boundary relation $u=0$, if the functions G and Γ used by him are replaced by the corresponding functions introduced above. Hence, finally, if $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n \leqq \ldots$,

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{1}{4\pi} \iint\limits_{(C)} \frac{k(xy)}{p(xy)}\, dxdy.$$

It is easily shown in the usual manner that only a finite number of negative characteristic numbers λ_i can exist in virtue of the assumptions made.

In the first place, the function Γ is symmetric in its pairs of variables. For if $K(xy, \xi\eta)$ is the kernel reciprocal to $\frac{1}{2\pi} G(xy, \xi\eta)l(\xi\eta)$, then from the classical theory

$$\frac{1}{2\pi} G(xy, \xi\eta)l(\xi\eta) + K(xy, \xi\eta) + \frac{1}{2\pi} \iint\limits_{(C)} K(xy, G\bar{\xi}\bar{\eta})(\bar{\xi}\bar{\eta}, \xi\eta)l(\xi\eta)d\bar{\xi}d\bar{\eta} = 0, \tag{a}$$

$$G(xy, \xi\eta)l(\xi\eta) + K(xy, \xi\eta) + \iint\limits_{(C)} G(xy, \bar{\xi}\bar{\eta})l(\bar{\xi}\bar{\eta})K(\bar{\xi}\bar{\eta}, \xi\eta)d\bar{\xi}d\bar{\eta} = 0. \tag{b}$$

From (a) it is seen that we may suppose

$$K(xy, \xi\eta) = -\frac{1}{2\pi} \Gamma(xy, \xi\eta)l(\xi\eta),$$

hence

$$\Gamma(xy, \xi\eta) + \frac{1}{2\pi} \iint\limits_{(C)} G(xy, \bar{\xi}\bar{\eta})l(\bar{\xi}\bar{\eta})\Gamma(\bar{\xi}\bar{\eta}, \xi\eta)d\bar{\xi}d\bar{\eta} = G(xy, \xi\eta) \tag{a'}$$

$$\Gamma(xy, \xi\eta) + \frac{1}{2\pi} \iint\limits_{(C)} \Gamma(xy, \bar{\xi}\bar{\eta})l(\bar{\xi}\bar{\eta})G(\bar{\xi}\bar{\eta}, \xi\eta)d\bar{\xi}d\bar{\eta} = G(xy, \xi\eta). \tag{b'}$$

In (b') interchanging the points (xy) and $(\xi\eta)$, substracting the members of the resulting equation from the corresponding ones of (a'), and taking

$$\phi(xy) \equiv \Gamma(xy, \xi\eta) - \Gamma(\xi\eta, xy),$$

we obtain

$$\phi(xy) + \frac{1}{2\pi} \iint\limits_{(C)} G(xy, \bar{\xi}\bar{\eta})l(\bar{\xi}\bar{\eta})\phi(\bar{\xi}\bar{\eta})d\bar{\xi}d\bar{\eta} = 0. \tag{24}$$

Since the only solution of this equation is, by hypothesis, $\phi \equiv 0$, the truth of the statement follows.

Consequently the kernel of (23) is symmetric, the characteristic numbers are real and have no finite limit point. Hence if there were an infinite number of negative values of λ, these must increase indefinitely in absolute value, and for at least one value,

$$\lambda k(xy) - q(xy) < 0, \quad (xy) \text{ in } (C).$$

The impossibility of this is seen if the first equation of (17) is multiplied by u and the result integrated over (C):

$$\iint_{(C)} u\left[\frac{\partial}{\partial x}(pu_x) + \frac{\partial}{\partial y}(pu_y)\right] dxdy = -\iint_{(C)} (\lambda k - q)u^2 dxdy,$$

hence

$$-\int_C pu\frac{du}{dn} ds - \iint_{(C)} p(u_x^2 + u_y^2) dxdy = -\iint_{(C)} (\lambda k - q)u^2 dxdy,$$

$$-\int_C p\sigma u^2 ds = \iint_{(C)} \left[p(u_x^2 + u_y^2) - (\lambda k - q)u^2 \right] dxdy.$$

Since the left member is negative or zero, the right member positive, this is impossible.

If λ_1 is a value of λ such that $\lambda_1 k - q < 0$, it follows from the discussion above that the equation in which $\bar{\lambda} = \lambda - \lambda_1$, $\bar{q} = -\lambda_1 k + q$,

$$\text{(17')} \qquad \frac{\partial}{\partial x}(pu_x) + \frac{\partial}{\partial y}(pu_y) + (\bar{\lambda} k - \bar{q})u = 0, \quad \frac{du}{dn_s} - \sigma u = 0,$$

has only positive characteristic numbers $\bar{\lambda}_i$; and if this equation is substituted for (17), the corresponding equation (24) has no solution not identically zero. Hence the discussion above is valid in all cases.

SUR LES ÉQUATIONS INTÉGRO-DIFFÉRENTIELLES A PLUSIEURS VARIABLES ET LEURS SINGULARITÉS

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1. Dans une note parue aux Comptes Rendus de l'Académie des Sciences de Paris (T. 178, 26 Mai 1924, p. 1778), nous avons énoncé des généralisations du théorème de Fuchs concernant les solutions d'une équation différentielle ou intégro-différentielle linéaire à une ou plusieurs variables et indiqué les conditions pour que ces solutions soient *régulières* au voisinage de *singularités polaires* ou même de singularités, que nous avons appelées *pôles critiques*.

C'est une démonstration de ces théorèmes, que nous voulons indiquer ici dans un cas simple.

2. Soit donc (avec les mêmes notations et hypothèses que dans la note précitée et que dans une autre note des Comptes Rendus du 26 Novembre 1923) *l'équation intégrale linéaire aux dérivées partielles*, (généralisation des équations différentielles linéaires aux dérivées partielles):

$$\phi(x,y)=f(x,y)+\int_{L_x}^{m} dt^m \int_{L_y}^{p} \sum_{\nu=0}^{r} \sum_{\mu=0}^{q} K_{\nu+1}^{\mu+1}(x,y,t,v) \frac{\partial^{r-\nu+q-\mu}\phi(t,v)}{\partial t^{r-\nu}\partial v^{q-\mu}} dv^p, \tag{1}$$

où $\phi(x,y)$ désigne la fonction inconnue; $f(x,y)$ et $K_{\nu+1}^{\mu+1}(x,y,t,v)$ des fonctions analytiques données supposées holomorphes quand la variable complexe indépendante x et la variable d'intégration t ne sortent pas d'un certain domaine D_x(borné, complet et simplement connexe), et que d'autre part les variables y et v ne sortent pas d'un domaine analogue D_y; L_x et L_y deux lignes rectifiables de longueur finie joignant, l'une le point fixe x_0 au point variable x dans D_x, et l'autre y_0 à y, dans D_y; enfin $\int_{L_x}^{m}$ et $\int_{L_y}^{p}$ des intégrales m uples et p uples effectuées le long de L_x et de L_y. (Le nombre des variables pourrait, dans ce qui suit être quelconque, $>$ *ou* <2).

Si $m+p>r+q$, cette équation, (dite alors: *normale*), admet (voir la note des Comptés Rendus du 26 Nov. 1923 et le Journal de l'École Polytechnique, 24[e] Cahier) une solution holomorphe dans le même domaine (D_x, D_y) que les coefficients f et K, et cette solution $\phi(x,y)$ peut être représentée dans tout ce domaine d'existence par le développement en série $\Sigma u_n(x,y)$ dont le premier terme est $u_0(x,y)=f(x,y)$ et dont chaque autre terme u_n se déduit du précédent u_{n-1} par la formule de récurrence

$$u_n(x,y) = \int_{L_x}^{m} dt^m \int_{L_y}^{p} \sum_{\nu=0}^{r} \sum_{\mu=0}^{q} K_{\nu+1}^{\mu+1}(x,y,t,v) \frac{\partial^{r-\nu+q-\mu} u_{n-1}(t,v)}{\partial t^{r-\nu} \partial v^{q-\mu}} dv^p.$$

Cette série sera appelée *développement synthétique à termes récurrents de* $\phi(x,y)$.

3. Supposons que chaque coefficient K soit de la forme

$$(a) \qquad K_{\nu+1}^{\mu+1}(x,y,t,v) = \frac{g_{\nu+1}^{\mu+1}(x,y,t,v)}{\left[R(t,v)\right]^{\lambda_{\nu+1}^{\mu+1}}}, \qquad \begin{pmatrix} 0\leq\nu\leq r \\ 0\leq\mu\leq q \end{pmatrix},$$

cette expression étant valable dans un certain domaine (Δ_x, Δ_y), qui empiète sur le domaine (D_x, D_y) et dans lequel les fonctions $g_{\nu+1}^{\mu+1}$ et $R(t,v)$ sont holomorphes.

Les systèmes de valeurs, qui annulent la fonction R du dénominateur constituent pour $K_{\nu+1}^{\mu+1}$ des singularités (*pôles* ou *pôles critiques* suivant que les exposants $\lambda_{\nu+1}^{\mu+1}$ sont entiers ou non), au voisinage desquelles nous étudions l'allure de $\phi(x,y)$ au point de vue de son mode de croissance, allure qui dépend en partie de celle des termes $u_n(x,y)$.

Dans la partie (d_x, d_y) commune aux domaines (D_x, D_y) et (Δ_x, Δ_y) les coefficients $K_{\nu+1}^{\mu+1}$ peuvent être remplacés par leurs expressions (a).

Dans ces conditions, si les termes successifs $u_n(x,y)$ ne contiennent pas $R(t,v)$ en dénominateur sous les signes $\int$ à des puissances indéfiniment croissantes, nous dirons que la solution $\phi(x,y)$ est *régulière* au voisinage des singularités en question.

Nous nous placerons ici dans le cas très simple où les exposants $\lambda_{\nu+1}^{\mu+1}$ *sont entiers et où la fonction* $R\ (t,v)$ *se réduit à* $t-\theta$, θ étant ainsi (dans Δ_x et à l'extérieur de D_x) *un pôle fixe* d'ordre $\lambda_{\nu+1}^{\mu+1}$ pour le coefficient $K_{\nu+1}^{\mu+1}$; celui-ci admet bien alors une expression telle que (a) dans un certain cercle Δ_x de centre θ.

4. Cela étant, nous allons démontrer le théorème suivant:

Théorème. Pour que la solution $\phi(x,y)$ *soit régulière au voisinage du point fixe* θ, *qui est un pôle d'ordre* $\lambda_{\nu+1}^{\mu+1}$ *pour le coefficient* $K_{\nu+1}^{\mu+1}(x,y,t,v)$, *il suffit que l'on ait* (quelque soit μ, et pour toutes les valeurs de ν depuis o jusqu'à r).

$$(2) \qquad \lambda_{\nu+1}^{\mu+1} \leq \nu+(m-r),$$

$m-r$ pouvant être positif, négatif ou nul.

Réciproquement cette condition est NÉCESSAIRE *pour que la solution* $\phi(x,y)$ *reste régulière, quand on prend pour* $f(x,y)$ *une infinité de fonctions* ARBITRAIRES *holomorphes dans* (D_x, D_y); nous dirons alors que $\phi(x,y)$ est la *solution générale* de la *classe* d'équations intégrales constituée par l'ensemble des équations (1) correspondant à ces diverses fonctions f. (On en a un exemple par la *classe* d'équations intégrales, qui résulte d'une équation différentielle aux dérivées partielles par suite de l'introduction de certaines fonctions arbitraires).

L'analogie de ce théorème avec celui de Fuchs est évidente et accuse le parallélisme existant entre les équations intégro-différentielles *normales* et les équations différentielles linéaires ordinaires.

Si, comme dans ces dernières, on prend notamment $r=m-1$, la condition précédente devient

$$\lambda_{\nu+1}^{\mu+1} \leq \nu+1,$$

laquelle à son tour se réduit à la condition même *du théorème de Fuchs*, si on suppose que l'équation (1) ne contient plus que la seule variable x, (μ disparaissant alors).

Au contraire, dans le cas où $m<r$, posons $m=r-h$; ($m+p$ étant toujours $>r+q$); la condition énoncée devient

$$\lambda_{\nu+1}^{\mu+1} \leq \nu-h;$$

dans le cas particulier d'une seule variable x, elle exprime qu'il y a seulement $m-h$ solutions indépendantes *régulières* au voisinage du point θ [mais ce n'est plus alors qu'une *condition formelle*, car on n'est plus assuré que le développement $\Sigma u_n(x)$ soit convergent, puisque l'équation n'est plus normale en raison de $m<r$].

5. *Démonstration du Théorème.*

I Démontrons d'abord que la condition (2) est *nécessaire.*

A cet effet, développons chaque numérateur $g_{\nu+1}^{\mu+1}(x,y,t,v)$, holomorphe dans (Δ_x, Δ_y), suivant les puissances croissantes de $t-\theta$ et de $x-\theta$, et appelons $a_{\nu+1}^{\mu+1}(y,v)$ le premier terme de ce développement, indépendant de t et de x, (ce terme pouvant exceptionnellement être nul dans certains coefficients).

Soit λ_M *le plus grand* des exposants *positifs* $\lambda_{\nu+1}^{\mu+1}$, (ou l'un des plus grands) et soit $a_M(y,v)$ le premier terme du développement du numérateur g correspondant.

Soit enfin $f_0(v)$ le premier terme du développement de la fonction $D_t^{r-\nu}D_v^{q-\mu}f(t,v)$ suivant les puissances croissantes de $t-\theta$, θ étant un point quelconque de d_x (ou même plus généralement un point de Δ_x au voisinage duquel $f(t,v)$ soit holomorphe).

Dans ces conditions $u_1(x,y)$ contiendra forcément un terme de la forme

$$\psi_1(x,y)=\int_{L_x}^{m} dt_1^m \int_{L_y}^{p} \frac{a_m(y,v_1)f_0(v_1)}{(t_1-\theta)^{\lambda_M}}\, dv_1^p, \tag{3}$$

c'est-à-dire, en désignant par $b_M(y)$ l'intégrale $\int_{L_y}^{p} a_M(y,v_1)f_0(v_1)dv_1^p$, un terme de la forme

$$\psi_1(x,y)=b_M(y)\int_{L_x}^{m} \frac{dt_1^m}{(t_1-\theta)^{\lambda_M}}. \tag{4}$$

Porté dans $u_2(x,y)$, ce terme ψ_1 donnera une somme $\psi_2(x,y)$ de termes de la forme

$$\psi_2(x,y)=\sum_{\nu=0}^{r}\sum_{\mu=0}^{q}\int_{L_x}^{m} dt_2^m \int_{L_y}^{p} \frac{a_{\nu+1}^{\mu+1}(y,v_2)}{(t_2-\theta)^{\lambda_{\nu+1}^{\mu+1}}}\, D_{t_2}^{r-\nu}D_{v_2}^{q-\mu}\psi_1(t_2,v_2)dv_2^p, \tag{5}$$

ou

(6) $$\psi_2(x,y) = \sum_{\nu=0}^{r} \sum_{\mu=0}^{q} \int_{L_x}^{m} \frac{\beta_{\nu+1}^{\mu+1}(y)}{(t_2-\theta)^{\lambda_{\nu+1}^{\mu+1}}} \left[D_{t_2}^{r-\nu} \int_{L_{t_2}}^{m} \frac{dt^m}{(t_1-\theta)^{\lambda_M}} \right] dt_2^m,$$

en posant

(7) $$\beta_{\nu+1}^{\mu+1}(y) = \int_{L_y}^{p} a_{\nu+1}^{\mu+1}(y,v_2) D_{v_2}^{q-\mu} b_M(v_2) dv_2.$$

Or le crochet de $\psi_2(x,y)$, c'est-à-dire l'expression $D_{t_2}^{r-\nu} \int_{L_{t_2}}^{m} \frac{dt^m}{(t_1-\theta)^{\lambda_M}}$, a pour valeur

1° $$\int_{L_{t_2}}^{\nu-(r-m)} \frac{dt_1^m}{(t_1-\theta)^{\lambda_M}}, \quad \text{si } r-m<\nu,$$

c'est-à-dire $\frac{1}{(t_2-\theta)^{\lambda_M-\nu+(r-m)}}$ +un polynome entier en t_2, si $\lambda_m > \nu-(r-m)$,

2° $$D^{(r-m)-\nu} \frac{1}{(t_2-\theta)^{\lambda_M}} = \frac{1}{(t_2-\theta)^{\lambda_M-\nu+(r-m)}}, \text{ si } \nu < r-m,$$

3° $$\frac{1}{(t_2-\theta)^{\lambda_M}}, \quad \text{si} \quad \nu=r-m.$$

Autrement dit, si l'inégalité

(8) $$\lambda_M > \nu-(r-m)$$

a lieu quelle que soit la valeur de ν, l'expression $D_{t_2}^{r-\nu} \int_{L_{t_2}}^{m} \frac{dt_1{}^m}{(t_1-\theta)^{\lambda_M}}$ aura, pour toutes les valeurs de ν (depuis 0 jusqu'à r), la valeur

(9) $$\frac{1}{(t_2-\theta)^{\lambda_M-\nu+(r-m)}}$$

(tout au moins peut-être à un polynome entier en t_2 près).

Portons cette valeur (9) dans $\psi_2(x,y)$. On voit ainsi qu'en général $u_2(x,y)$ contiendra entre autres des termes $\Psi_2(x,y)$ de la forme*,

10) $$\Psi_2(x,y) = \sum_{\nu=0}^{r} \sum_{\mu=0}^{q} \beta_{\nu+1}^{\mu+1}(y) \int_{L_x}^{m} \frac{dt^m}{(t-\theta)^{\lambda_{\nu+1}^{\mu+1}+\lambda_M-\nu+(r-m)}}$$

Ces termes $\Psi_2(x,y)$ ne pourraient à la rigueur se détruire entre eux ou avec d'autres que pour certaines valeurs *particulières* des coefficients $\beta_{\nu+1}^{\mu+1}$, donc aussi, d'après (7), pour une valeur *particulière* de b_M, donc de $f_0(v_1)$. Ceci ne pourrait donc avoir lieu en dernière analyse que pour une fonction $f(t,v)$ choisie d'une manière *particulière* et convenable. Ce ne sera donc pas le cas pour *la solution générale*, considérée ici par hypothèse, puisqu'elle correspond à la fonction $f(x,y)$, holomorphe dans (D_x, D_y), la *plus générale*. Autrement dit la *solution générale* renferme une infinité de constantes arbitraires $c_{\mu\nu}$, qu'on peut toujours supposer

*Nous pouvons de nouveau désigner par t la variable d'intégration, au lieu de t_2, sans crainte de confusion.

ne pas satisfaire aux conditions particulières, en nombre fini, nécessaires pour que les termes Ψ_2 disparaissent de $u_2(x,y)$.

Cela étant, si l'inégalité (8) a lieu quelque soit ν, il en résulte, puisque λ_M est le plus grand des exposants $\lambda_{\nu+1}^{\mu+1}$, qu'on a, pour une valeur AU MOINS DE ν, l'inégalité

(11) $$\lambda_{\nu+1}^{\mu+1} > \nu-(r-m).$$

Si donc on supposait cette dernière inégalité vérifiée, ne serait-ce que pour *une seule* valeur de ν, *certains termes* Ψ_2 *de* $u_2(x,y)$ *renfermeraient,* en dénominateur sous les symboles d'intégration, *la quantité* $t-\theta$ *à la puissance*

$$\lambda_M+[\lambda_{\nu+1}^{\mu+1}-\nu+(r-m)],$$

c'est-à-dire à une puissance *supérieure à la puissance* λ_M *qui est la plus grande, à laquelle* $t-\theta$ figure, en dénominateur sous les symboles d'intégration, dans $u_1(x,y)$.

Le même raisonnement, bien entendu, s'applique ensuite quand on passe de $u_2(x,y)$ à $u_3(x,y)$, et d'une manière générale de $u_{n-1}(x,y)$ à $u_n(x,y)$. L'inégalité (11) ayant lieu même pour *une seule* valeur de ν, entraîne cette conséquence, que chaque terme $u_n(x,y)$ contiendra, en dénominateur sous les symboles d'intégration, l'expression $t-\theta$ à une puissance plus grande que le terme précédent $u_{n-1}(x,y)$ et cela quelque soit n; par suite dans ce cas la *solution générale* $\phi(x,y)$ ne sera pas régulière.

Donc inversement pour que cette solution soit régulière, *il est bien nécessaire que l'inégalité contraire à* (11), *c' est-à-dire la relation* (2), *ait toujours lieu.*

II *Réciproquement*, la condition est suffisante. En effet raisonnons encore *ab absurdo* et supposons que l'exposant du dénominateur $t-\theta$, sous les symboles d'intégration dans $u_n(x,y)$, croisse au-delà de toute limite avec n. Donc à partir d'une valeur suffisamment grande de l'indice n, $u_n(x,y)$ contiendra des termes qui renfermeront $t-\theta$ avec un exposant λ_M supérieur à une quantité donnée quelconque, notamment à $\nu-(r-m)$, ν ayant une valeur quelconque pouvant aller de 0 à r. Un quelconque de ces termes sera d'une forme analogue à $\psi_1(x,y)$, [formule (4)], et donnera, puisque l'inégalité (8) est ici vérifiée, dans le terme suivant $u_{n+1}(x,y)$ des termes d'une forme analogue à $\Psi_2(x,y)$ [formule (10)].

Or puisque l'exposant de $t-\theta$ doit croître, par hypothèse indéfiniment avec n, il faut évidemment que pour une infinité de valeurs de n, cet exposant croisse quand on passe du terme $u_n(x,y)$ au suivant $u_{n+1}(x,y)$. Dans ces conditions, il faut, d'après la formule (10) qui sert de type général, que l'on ait (pour *une* valeur *au moins* de ν)

$$\lambda_{\nu+1}^{\mu+1}+\lambda_M-\nu+(r-m)>\lambda_M$$

ou

$$\lambda_{\nu+1}^{\mu+1}>\nu-(r-m).$$

Donc inversement, *il suffit* qu'on ait pour toutes les valeurs de ν l'inégalité contraire (2) pour que les exposants ne puissent pas croître indéfiniment, c'est-à-dire pour que la solution $\phi(x,y)$ considérée soit régulière.—C.Q.F.D.

Remarque. On aurait une démonstration analogue en prenant $R(t,v)=t-\theta(v)$, $\theta(v)$ étant une fonction holomorphe de v.

SUR LA RÉSOLUTION DES SYSTÈMES D'ÉQUATIONS

Rot $X=A$, Grad $\Phi=A$

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1. Soit (Ω) un domaine donné, simplement connexe.

Convenons de désigner par (ω) un domaine simplement connexe, tout à fait arbitraire et contenu dans (Ω); par (σ) la surface, qui le limite; par N la direction de la normale extérieure à (σ). Supposons d'ailleurs que les frontières des domaines (ω), (Ω) satisfont aux conditions de Liapounoff. Les formules de Stokes et de Green subsistent alors pour de pareilles surfaces.

En désignant par u, v, w les fonctions intégrables et bornées dans (Ω), cherchons à résoudre au moyen des fonctions a, b, c, ϕ continues dans (Ω), les systèmes d'équations:

$$\int_{(\omega)} u d\omega = \int_{(\sigma)} (c \cos Ny - b \cos Nz) d\sigma,$$

$$(1) \qquad \int_{(\omega)} v d\omega = \int_{(\sigma)} (a \cos Nz - c \cos Nx) d\sigma,$$

$$\int_{(\omega)} w d\omega = \int_{(\sigma)} (b \cos Nx - a \cos Ny) d\sigma,$$

$$(2) \qquad \int_{(\omega)} u d\omega = \int_{(\sigma)} \phi \cos Nx d\sigma, \quad \int_{(\omega)} v d\omega = \int_{(\sigma)} \phi \cos Ny d\sigma, \quad \int_{(\omega)} w d\omega = \int_{(\sigma)} \phi \cos Nz d\sigma,$$

équivalents aux systèmes

$$(1') \qquad u = \frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}, \quad v = \frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}, \quad w = \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y},$$

$$(2') \qquad u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z},$$

dans le cas où u, v, w, ϕ, a, b, c ont des dérivées continues.

Il s'agit de trouver les conditions nécessaires et suffisantes pour que les systèmes (1), (2) aient des solutions.

2. Donnons aux fonctions

$$U(x, y, z)=\frac{1}{hkl}\int_x^{x+h}dx_0\int_y^{y+k}dy_0\int_z^{z+l}dz_0\, u(x_0, y_0, z_0),$$

$$U_m(x, y, z)=\frac{1}{h_m k_m l_m}\int_x^{x+h_m}dx_m\int_y^{y+k_m}dy_m\int_z^{z+l_m}dz_m\, U_{m-1}(x_m, y_m, z_m),$$

définies dans un certain domaine $(\Omega\delta)$ formé par les points intérieurs à (Ω) et situés à une distance de la frontière de (Ω) au moins égale à δ, le nom de fonctions de M. Stekloff.

La fonction U_m admet dans $(\Omega\delta)$ des dérivées continues d'ordre m.

Quel que soit le nombre positif ϵ, on peut déterminer δ de manière que les fonctions U_m correspondantes satisfassent à l'inégalité

$$\left|\int_{(\omega)}(U_m-u)\,\psi d\omega\right|<\epsilon,$$

$$\psi=\theta,\ \psi=\frac{\theta}{r},\ \psi=\frac{\theta(x-x^{(0)})}{r^3},$$

où r est la distance de deux points $M_0\ (x^{(0)}, y^{(0)}, z^{(0)})$ et $M(x, y, z)$ et où θ est une fonction continue.

Quand u est continue, U tend uniformément vers u, si

$$h=k=l \text{ et } \lim h=0.$$

3. Théorème. Si l'équation

$$\int_{(\omega)}ud\omega=\int_{(\sigma)}(a\cos Nx+b\cos Ny+c\cos Nz)d\sigma \tag{3}$$

est satisfaite par une fonction u intégrable et bornée dans (Ω) et par les fonctions a, b, c, continues à l'intérieur de (Ω) et si U_m, A_m, B_m, C_m sont les fonctions de Stekloff, correspondant à u, a, b, c et définies dans un domaine $(\Omega\delta)$, on a, alors, pour tous les points de (Ω),

$$U_m=\frac{\partial A_m}{\partial x}+\frac{\partial B_m}{\partial y}+\frac{\partial C_m}{\partial z}. \tag{3'}$$

La démonstration pour les fonctions U, A, B, C est immédiate; il suffit de prendre pour (ω) un parallélipipède rectangulaire, ayant deux sommets aux points $M(x, y, z)$ et $M_1(x+h, y+k, z+l)$ et ses arêtes parallèles aux axes des coordonnées.

On démontre le théorème pour les autres fonctions de proche en proche en intégrant l'identité (3').

4. Si les fonctions u, v, w sont intégrables et bornées dans (Ω), on satisfait au système d'équations

$$\int\limits_{(\omega)} u d\omega = \int\limits_{(\sigma)} (\mu \cos Nx + c \cos Ny - b \cos Nz) d\sigma,$$

(4)
$$\int\limits_{(\omega)} v d\omega = \int\limits_{(\sigma)} (\mu \cos Ny + a \cos Nz - c \cos Nx) d\sigma,$$

$$\int\limits_{(\omega)} w d\omega = \int\limits_{(\sigma)} (\mu \cos Nz + b \cos Nx - a \cos Ny) d\sigma,$$

en posant

(5)
$$\mu_0 = -\frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\int\limits_{(\Omega)}\frac{u d\omega}{r} + \frac{\partial}{\partial y}\int\limits_{(\Omega)}\frac{v d\omega}{r} + \frac{\partial}{\partial z}\int\limits_{(\Omega)}\frac{w d\omega}{r}\right\},$$

(6)
$$\begin{cases} a_0 = \dfrac{1}{4\pi}\left\{\dfrac{\partial}{\partial y}\displaystyle\int\limits_{(\Omega)}\dfrac{w d\omega}{r} - \dfrac{\partial}{\partial z}\displaystyle\int\limits_{(\Omega)}\dfrac{v d\omega}{r}\right\}, \\ b_0 = \dfrac{1}{4\pi}\left\{\dfrac{\partial}{\partial z}\displaystyle\int\limits_{(\Omega)}\dfrac{u d\omega}{r} - \dfrac{\partial}{\partial x}\displaystyle\int\limits_{(\Omega)}\dfrac{w d\omega}{r}\right\}, \\ c_0 = \dfrac{1}{4\pi}\left\{\dfrac{\partial}{\partial x}\displaystyle\int\limits_{(\Omega)}\dfrac{v d\omega}{r} - \dfrac{\partial}{\partial y}\displaystyle\int\limits_{(\Omega)}\dfrac{u d\omega}{r}\right\}, \end{cases}$$

où

$$d\omega = d\xi d\eta d\zeta, \quad r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2.$$

En supposant que les fonctions u, v w sont égales à zéro quand le point M est en dehors de (Ω), on peut les considérer comme données dans un domaine (Ω') contenant (Ω) et supposer que le domaine $(\Omega'\delta)$, où sont définies les fonctions U_2, V_2, W_2 correspondant à u, v, w et à un nombre ϵ, choisi arbitrairement, contient (Ω). On vérifie alors les équations:

(4′)
$$U_2 = \frac{\partial M}{\partial x} + \frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}, \quad V_2 = \frac{\partial M}{\partial y} + \frac{\partial A}{dz} - \frac{\partial C}{\partial x}, \quad W_2 = \frac{\partial M}{\partial z} + \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}$$

en posant

(5′)
$$M = -\frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\int\limits_{(\Omega)}\frac{U_2 d\omega}{r} + \frac{\partial}{\partial y}\int\limits_{(\Omega)}\frac{V_2 d\omega}{r} + \frac{\partial}{\partial z}\int\limits_{(\Omega)}\frac{W_2 d\omega}{r}\right\},$$

(6′)
$$A = \frac{1}{4\pi}\left\{\frac{\partial}{\partial y}\int\limits_{(\Omega)}\frac{W_2 d\omega}{r} - \frac{\partial}{\partial z}\int\limits_{(\Omega)}\frac{V_2 d\omega}{r}\right\}, \qquad B = \frac{1}{4\pi}\left\{\frac{\partial}{\partial z}\int\limits_{(\Omega)}\frac{U_2 d\omega}{r} - \frac{\partial}{\partial x}\int\limits_{(\Omega)}\frac{W_2 d\omega}{r}\right\},$$

$$C = \frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\int\limits_{(\Omega)}\frac{V_2 d\omega}{r} - \frac{\partial}{\partial y}\int\limits_{(\Omega)}\frac{U_2 d\omega}{r}\right\}.$$

On a évidemment:

$$|\mu_0 - M| < g\epsilon, \quad |a_0 - A| < g\epsilon, \quad |b_0 - B| < g\epsilon, \quad |c_0 - C| < g\epsilon$$

et, comme corollaire,

$$K_1=\left|\int_{(\omega)} ud\omega-\int_{(\sigma)}(\mu_0 \cos Nx+c_0 \cos Ny-b_0 \cos Nz)d\sigma\right|$$

$$=\left|\int_{(\omega)}|(u-U_2)d\omega-\int_{(\sigma)}\left\{(\mu_0-M)\cos Nx+(c_0-C)\cos Ny+(b_0-B) \cos Nz\right\}d\sigma\right|<a\epsilon,$$

a, g étant indépendants de ϵ, d'où il suit que K_1 est nul.

On démontre ainsi que

$$\begin{aligned}
&\int_{(\omega)} ud\omega=\int_{(\sigma)}(\mu_0 \cos Nx+c_0 \cos Ny-b_0 \cos Nz)d\sigma,\\
(7)\qquad &\int_{(\omega)} vd\omega=\int_{(\sigma)}(\mu_0 \cos Ny+a_0 \cos Nz-c_0 \cos Nx)d\sigma,\\
&\int_{(\omega)} wd\omega=\int_{(\sigma)}(\mu_0 \cos Nz+b_0 \cos Nx-a_0 \cos Ny)d\sigma.
\end{aligned}$$

5. Si les fonctions a, b, c, sont continues à l'intérieur de (Ω) et si elles satisfont au système d'équations

$$\begin{aligned}
&\int_{(\sigma)} \mu \cos Nx\, d\sigma=\int_{(\sigma)}(c \cos Ny-b \cos Nz)d\sigma,\\
(8)\qquad &\int_{(\sigma)} \mu \cos Ny\, d\sigma=\int_{(\sigma)}(a \cos Nz-c \cos Nx)d\sigma,\\
&\int_{(\sigma)} \mu \cos Nz\, d\sigma=\int_{(\sigma)}(b \cos Nx-a \cos Ny)d\sigma,
\end{aligned}$$

la fonction μ est harmonique à l'intérieur de (Ω).

Lemme. Si la fonction F dépendant d'un paramètre h est harmonique dans (Ω) et tend pour $h\longrightarrow 0$ uniformement vers la fonction f, f est harmonique dans (Ω).

Décrivons autour d'un point M de (Ω) une sphère contenue dans (Ω). Soit $\bar{f}$ la valeur de f sur cette sphère et ϕ la fonction harmonique telle que sur la sphère on a $\bar{\phi}=\bar{f}$.

Si h est assez petit, on a

$$]f-F]<\epsilon,$$

et

$$|\bar{\phi}-\bar{F}|<\epsilon,$$

car

$$|\bar{f}-\bar{F}|<\epsilon.$$

$\phi-F$ est harmonique à l'intérieur de la sphère; d'où il suit qu'à l'intérieur de la sphère, on a

$$|\phi-F|<\epsilon.$$

Or,

$$|f-F|<\epsilon,$$

d'où

$$|f-\phi|<2\epsilon$$

et par conséquent

$$\phi=f.$$

Si M_2, A_2, B_2, C_2 sont les fonctions de Stekloff correspondant à μ, a, b, c, on a:

$$\frac{\partial M_2}{\partial x}=\frac{\partial C_2}{\partial y}-\frac{\partial B_2}{\partial z},\quad \frac{\partial M_2}{\partial y}=\frac{\partial A_2}{\partial z}-\frac{\partial C_2}{\partial x},\quad \frac{\partial M_2}{\partial z}=\frac{\partial B_2}{\partial x}-\frac{\partial A_2}{\partial y}, \tag{8'}$$

d'où il suit que M_2 est harmonique, et le lemme montre que μ est harmonique en chaque point à l'intérieur de (Ω) car

$$\mu=\lim M_2.$$

6. On démontre de la même manière que si les fonctions a, b, c, satisfont au système (8), et, de plus à l'équation

$$\int_{(\sigma)}(a\cos Nx+b\cos Ny+c\cos Nz)d\sigma=0, \tag{9}$$

elles sont harmoniques à l'intérieur de (Ω).

7. En revenant au système (1), retranchons des équations (1) les identités (7); on trouve:

$$\int_{(\sigma)}\mu_0\cos Nx d\sigma=\int_{(\sigma)}\{(c-c_0)\cos Ny-(b-b_0)\cos Nz\}d\sigma,$$

$$\int_{(\sigma)}\mu_0\cos Ny d\sigma=\int_{(\sigma)}\{(a-a_0)\cos Nz-(c-c_0)\cos Nx\}d\sigma,$$

$$\int_{(\sigma)}\mu_0\cos Nz d\sigma=\int_{(\sigma)}\{(b-b_0)\cos Nx-(a-a_0)\cos Ny\}d\sigma,$$

d'où il suit que μ_0 doit être harmonique.

Si, au contraire, μ_0 est harmonique, on peut trouver les fonctions a_1, b_1, c_1 telles que

$$\frac{\partial \mu_0}{\partial x}=\frac{\partial c_1}{\partial y}-\frac{\partial b_1}{\partial z},\quad \frac{\partial \mu_0}{\partial y}=\frac{\partial a_1}{\partial z}-\frac{\partial c_1}{\partial x},\quad \frac{\partial \mu_0}{\partial z}=\frac{\partial b_1}{\partial x}-\frac{\partial a_1}{\partial y}$$

et on trouve ainsi

$$a=a_0+a_1,\quad b=b_0+b_1,\quad c=c_0+c_1.$$

Théorème I. Soient les fonctions u, v, w intégrables et bornées dans (Ω)*; pour que le système*

$$
(1)\qquad
\begin{aligned}
\int_{(\omega)} u d\omega &= \int_{(\sigma)} (c \cos Ny - b \cos Nz) d\sigma,\\
\int_{(\omega)} v d\omega &= \int_{(\sigma)} (a \cos Nz - c \cos Nx) d\sigma,\\
\int_{(\omega)} w d\omega &= \int_{(\sigma)} (b \cos Nx - a \cos Ny) d\sigma,
\end{aligned}
$$

ait des solutions exprimables au moyen des fonctions a, b, c, continues à l'intérieur de (Ω) *il faut et il suffit que la fonction*

$$\frac{\partial}{\partial x}\int_{(\Omega)} \frac{u\, d\omega}{r} + \frac{\partial}{\partial y}\int_{(\Omega)} \frac{v\, d\omega}{r} + \frac{\partial}{\partial z}\int_{(\Omega)} \frac{w\, d\omega}{r}$$

soit harmonique à l'intérieur de (Ω).

8. Reprenons à présent le système (2) et retranchons des équations (2) les identités (7); on trouve alors

$$
\begin{aligned}
\int_{(\sigma)} (\phi - \mu_0) \cos Nx d\sigma &= \int_{(\sigma)} (c_0 \cos Ny - b_0 \cos Nz) d\sigma,\\
\int_{(\sigma)} (\phi - \mu_0) \cos Ny d\sigma &= \int_{(\sigma)} (a_0 \cos Nz - c_0 \cos Nx) d\sigma,\\
\int_{(\sigma)} (\phi - \mu_0) \cos Nz d\sigma &= \int_{(\sigma)} (b_0 \cos Nx - a_0 \cos Ny) d\sigma,
\end{aligned}
$$

L'application du théorème de Stokes aux fonctions (6) montre que

$$\int_{(\sigma)} (a_0 \cos Nx + b_0 \cos Ny + c_0 \cos Nz) d\sigma = 0;$$

donc a_0, b_0, c_0 doivent être harmoniques à l'intérieur de (Ω) et l'on a

$$\frac{\partial a_0}{\partial x} + \frac{\partial b_0}{\partial y} + \frac{\partial c_0}{\partial z} = 0,$$

ce qui montre que l'on peut trouver une fonction θ telle que

$$\frac{\partial \theta}{\partial x} = \frac{\partial c_0}{\partial y} - \frac{\partial b_0}{\partial z}, \quad \frac{\partial \theta}{\partial y} = \frac{\partial a_0}{\partial z} - \frac{\partial c_0}{\partial x}, \quad \frac{\partial \theta}{\partial z} = \frac{\partial b_0}{\partial x} - \frac{\partial a_0}{\partial y}$$

et, comme conséquence, on a

$$\phi = \mu_0 + \theta.$$

Théorème II. Soient des fonctions u, v, w intégrables et bornées dans (Ω)*; pour qu'il existe une fonction* ϕ*, continue à l'intérieur de* (Ω) *et satisfaisant au système*

$$\int_{(\omega)} u d\omega = \int_{(\sigma)} \phi \cos Nx d\sigma, \quad \int_{(\omega)} v d\omega = \int_{(\sigma)} \phi \cos Ny d\sigma, \quad \int_{(\omega)} w d\omega = \int_{(\sigma)} \phi \cos Nz d\sigma \tag{2}$$

il faut et il suffit que les fonctions

$$\frac{\partial}{\partial y}\int_{(\Omega)} \frac{w\, d\omega}{r} - \frac{\partial}{\partial z}\int_{(\Omega)} \frac{v\, d\omega}{r}, \quad \frac{\partial}{\partial z}\int_{(\Omega)} \frac{u\, d\omega}{r} - \frac{\partial}{\partial x}\int_{(\Omega)} \frac{w\, d\omega}{r}, \quad \frac{\partial}{\partial x}\int_{(\Omega)} \frac{v\, d\omega}{r} - \frac{\partial}{\partial y}\int_{(\Omega)} \frac{u\, d\omega}{r}$$

soient harmoniques à l'intérieur de (Ω).

SUR UN PROBLÈME FONDAMENTAL DE L'HYDRODYNAMIQUE

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1. En désignant par (q_1, q_2, q_3) les coordonnées d'un point, supposons que l'espace (Q) de ces points soit divisé par certaines surfaces, nommées *frontières*, en régions connexes, sous les conditions que

1° en dehors d'une sphère de rayon R_0, ayant son centre à l'origine des coordonnées, il n'y ait pas de points appartenant aux frontières;

2° les frontières satisfassent aux trois conditions de Liapounoff.

Conformément à une des conditions de Liapounoff, nous supposerons que si θ est l'angle entre normales à la frontière aux points M_1 et M_2, à la distance r l'un de l'autre, on a

$$\theta < \mathrm{E} r^{1-\lambda} \quad (0<\lambda<1), \tag{1}$$

E et λ étant indépendants du choix des points M_1, M_2 ainsi que de la frontière.

2. Supposons données trois fonctions u_0, v_0, w_0 d'un point de l'espace, satisfaisant aux conditions suivantes:

(*a*) Elles sont continues en chaque point M de l'espace et dans tout l'espace vérifient les inégalités

$$|u_0|<A_0,\ \rho|u_0|<A_0,\ |v_0|<A_0,\ \ldots,\ \rho|w_0|<A_0, \tag{2}$$

ρ étant la distance de l'origine des coordonnées au point M.

(*b*) En chaque point M non situé sur la frontière, elles ont des dérivées premières par rapport à q_1, q_2, q_3 et ces dérivées sont continues dans M et dans tout l'espace vérifient les inégalités

$$\left\{\begin{array}{lll} \left|\dfrac{\partial u_0}{\partial q_i}\right|<A_0, & \left|\dfrac{\partial v_0}{\partial q_i}\right|<A_0, & \left|\dfrac{\partial w_0}{\partial q_i}\right|<A_0, \\ \rho^2\left|\dfrac{\partial u_0}{\partial q_i}\right|<A_0, & \rho^2\left|\dfrac{\partial v_0}{\partial q_i}\right|<A_0, & \rho^2\left|\dfrac{\partial w_0}{\partial q_i}\right|<A_0. \end{array}\right. \tag{3}$$

(*c*) Si le point M tend vers un point M_0 sur la frontière, les dérivées $\dfrac{\partial u_0}{\partial q_i}, \ldots$ tendent vers des limites $\left(\dfrac{\partial u_0}{\partial q_i}\right), \ldots$ dont les valeurs dépendent du côté de la frontière où est situé le point M; en outre, si r est la distance entre M et M_0, on a

$$(4) \qquad \left|\frac{\partial u_0}{\partial q_i} - \left(\frac{\partial u_0}{\partial q_i}\right)_{M_0}\right| < A_0 r^{1-\lambda}, \ldots .$$

(d) Si M et M_1 sont deux points, situés d'un même côté de la frontière et si r est leur distance,

$$(5) \qquad \left|\left(\frac{\partial u_0}{\partial q_i}\right)_M - \left(\frac{\partial u_0}{\partial q_i}\right)_{M_1}\right| < A_0 r^{1-\lambda}, \ \rho^2\left|\left(\frac{\partial u_0}{\partial q_i}\right)_M - \left(\frac{\partial u_0}{\partial q_i}\right)_{M_1}\right| < A_0 r^{1-\lambda}, \ldots$$

3. En partant des fonctions u_0, v_0, w_0, formons une suite de fonctions

$$(6) \qquad u_n, \ v_n, \ w_n, \quad (n=0, 1, 2, \ldots),$$

en posant

$$(7) \qquad \begin{cases} u_{n+1} = u_0 + \dfrac{1}{4\pi}\displaystyle\int_o^t \left(\int \frac{L_n(\xi_n - x_n)}{r_n^3} dp_1\, dp_2\, dp_3\right) dt, \\ v_{n+1} = v_0 + \dfrac{1}{4\pi}\displaystyle\int_o^t \left(\int \frac{L_n(\eta_n - y_n)}{r_n^3} dp_1\, dp_2\, dp_3\right) dt, \\ w_{n+1} = w_0 + \dfrac{1}{4\pi}\displaystyle\int_o^t \left(\int \frac{L_n(\zeta_n - z_n)}{r_n^3} dp_1\, dp_2\, dp_3\right) dt, \end{cases}$$

l'intégration $\int \ldots dp_1 dp_2 dp_3$ étant étendue à tout l'espace; de plus

$$(8) \qquad x_n = q_1 + \int_o^t u_n dt, \ y_n = q_2 + \int_o^t v_n dt, \ z_n = q_3 + \int_o^t w_n dt,$$

ξ_n, η_n, ζ_n étant formés en substituant dans x_n, y_n, z_n les variables p_1, p_2, p_3 aux variables q_1, q_2, q_3; L_n est enfin obtenue en substituant dans

$$(9) \qquad L = 2\begin{vmatrix} \dfrac{\partial u}{\partial q_1}, & \dfrac{\partial u}{\partial q_2}, & \dfrac{\partial u}{\partial q_3} \\ \dfrac{\partial v}{\partial q_1}, & \dfrac{\partial v}{\partial q_2}, & \dfrac{\partial v}{\partial q_3} \\ \dfrac{\partial z}{\partial q_1}, & \dfrac{\partial z}{\partial q_2}, & \dfrac{\partial z}{\partial q_3} \end{vmatrix} + 2\begin{vmatrix} \dfrac{\partial u}{\partial q_1}, & \dfrac{\partial u}{\partial q_2}, & \dfrac{\partial u}{\partial q_3} \\ \dfrac{\partial y}{\partial q_1}, & \dfrac{\partial y}{\partial q_2}, & \dfrac{\partial y}{\partial q_3} \\ \dfrac{\partial w}{\partial q_1}, & \dfrac{\partial w}{\partial q_2}, & \dfrac{\partial w}{\partial q_3} \end{vmatrix} + 2\begin{vmatrix} \dfrac{\partial x}{\partial q_1}, & \dfrac{\partial x}{\partial q_2}, & \dfrac{\partial x}{\partial q_3} \\ \dfrac{\partial v}{\partial q_1}, & \dfrac{\partial v}{\partial q_2}, & \dfrac{\partial v}{\partial q_3} \\ \dfrac{\partial w}{\partial q_1}, & \dfrac{\partial w}{\partial q_2}, & \dfrac{\partial w}{\partial q_3} \end{vmatrix},$$

u_n, v_n, w_n, x_n, y_n, z_n à u, v, w, x, y, z et p_1, p_2, p_3 à q_1, q_2, q_3 et l'on a

$$r_n^2 = (\xi_n - x_n)^2 + (\eta_n - y_n)^2 + (\zeta_n - z_n)^2.$$

On peut démontrer que t étant moindre qu'un certain nombre b_n, les fonctions u_n, v_n, w_n vérifient les conditions (a), (b), (c), (d) du §2 (le nombre A_0 étant, dans ces conditions, remplacé par une fonction A_n de t). Ces fonctions et leurs dérivées prises par rapport à q_1, q_2, q_3, en chaque point M non situé sur une frontière possèdent des dérivées premières par rapport à t.

Quand t est moindre que b_o, r_n n'est égal à zéro que si

$$p_1 = q_1, \qquad p_2 = q_2, \qquad p_3 = q_3,$$

et pour cela, il suffit qu'on ait

$$\int_o^t A_n dt < \frac{1}{9}. \tag{10}$$

4. La démonstration de ces assertions, qui constitue la majeure partie de mon mémoire relatif à ce sujet, est basée sur une étude détaillée des dérivées secondes du potentiel newtonien près de la frontière du domaine d'intégration; en omettant cette démonstration, remarquons seulement que, chemin faisant, on établit l'égalité

$$A_{n+1} = A_0 + k\int_o^t A_n^2 dt \tag{11}$$

où k est indépendant de n.

Or, on obtient la même égalité (11) en cherchant par la méthode des approximations successives, la solution de l'équation

$$\frac{dA}{dt} = kA^2 \tag{12}$$

égale à A_0 pour $t=0$; il en résulte que

$$A_n < \frac{A_0}{1-kA_0t} = A \tag{13}$$

et

$$\int_o^t A_n dt < \frac{1}{9}, \text{ si } t < \frac{1-e^{-\frac{k}{q}}}{A_0k} = b_0. \tag{10'}$$

5. Les fonctions u_n, v_n, w_n, ainsi que leurs dérivées $\frac{\partial u_n}{\partial q_i}$, $\frac{\partial v_n}{\partial q_i}$, $\frac{\partial w_n}{\partial q_i}$ ont des limites déterminées quand n tend vers l'infini. Formons les séries

$$\begin{cases} u_0 + (u_1-u_0) + (u_2-u_1) + \dots + (u_{n+1}-u_n) + \dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{14}$$

$$\begin{cases} \frac{\partial u_0}{\partial q_i} + \left(\frac{\partial u_1}{\partial q_i} - \frac{\partial u_0}{\partial q_i}\right) + \dots + \left(\frac{\partial u_{n+1}}{\partial q_i} - \frac{\partial u_n}{\partial q_i}\right) + \dots, \end{cases} \tag{15}$$

et posons

$$\begin{cases} |u_n - u_{n-1}| < g_n, \dots\dots, \left|\frac{\partial u_n}{\partial q_i} - \frac{\partial u_{n-1}}{\partial q_i}\right| < g_n, \dots, \\ \rho|u_n - u_{n-1}| < g_n, \dots, \rho^2\left|\frac{\partial u_n}{\partial q_i} - \frac{\partial u_{n-1}}{\partial q_i}\right| < g_n, \dots. \end{cases} \tag{16}$$

En étudiant alors les séries (14) on demontre aisément, en se basant sur les assertions du §3, que

$$|u_{n+1}-u_n|<b\int_o^t g_n dt,\quad \rho|u_{n+1}-u_n|<b\int_o^t g_n dt, \tag{17}$$

b étant indépendant de n.

6. L'étude des séries (15) est plus délicate. Nous la commencerons en supposant que le point M est à la distance δ de la frontière; dans cette étude, il faut comparer à g_n la différence T,

$$T=|(L_n-L_{n-1})_M-(L_n-L_{n-1})_{M_1}|, \tag{18}$$

des deux valeurs de L_n-L_{n-1} en deux points M et M_1 à la distance r l'un de l'autre et situés d'un même côté de la frontière.

On s'assure aisément que

$$T<ag_n,\quad T<br^{1-\lambda}, \tag{19}$$

a et b étant indépendants de n.

En élevant la première inégalité (19) à la puissance $n-1$ et en la multipliant par la seconde, on obtient

$$T<hg_n^{\frac{n-1}{n}}\, r^{\frac{1-\lambda}{n}} \tag{20}$$

où h ne dépend pas de n.

Des inégalités (20) et (17) on conclut qu'il faut poser

$$\begin{cases} g_{n+1}\leqq a\int_o^t g_n|\log\delta|dt, \text{ quand } \delta<g_n, \\ g_{n+1}\leqq an\int_o^t g_n^{1-\frac{\lambda}{n}}dt, \text{ quand } \delta>g_n, \end{cases} \tag{21}$$

ou

$$g_{n+1}\leqq b\int_o^t g_n\, dt. \tag{21'}$$

Comme on peut prendre pour g_{n+1} la plus grande des expressions (21′), et comme pour n assez grand, g_n étant borné,

$$ng_n^{1-\frac{\lambda}{n}}>g_n|\log\delta|,\quad ng_n^{1-\frac{\lambda}{n}}>g_n, \tag{22}$$

on trouve, si $n\geqq N_o$,

$$g_{n+1}=an\int_o^t g_n^{1-\frac{\lambda}{n}}dt, \tag{23}$$

N_0 étant un nombre dépendant de δ.

7. On obtient maintenant sans peine,

$$\lim g_n^{\frac{1}{n}}=[(1+\lambda)\,at]^{\frac{1}{1+\lambda}},\ (n=\infty), \tag{24}$$

d'où il suit que, pour chaque δ, les séries (14), (15) sont absolument convergentes si

$$t < \frac{1}{(1+\lambda)\, a}. \tag{25}$$

8. Ayant démontré la convergence absolue des séries (14), (15) en chaque point M non situé sur les frontières, on démontre aisément que les séries (15) restent convergentes, quand on y change $\frac{\partial u_n}{\partial q_i}$ en sa limite $\left(\frac{\partial u_n}{\partial q_i}\right)$ le point M tendant vers un point M_0 sur la frontière.

Les fonctions u, v, w étant les sommes des séries (14), jouissent des propriétés annoncées dans le §3 pour u_n, v_n, w_n (le nombre A_0 étant remplacé par A) et vérifient le système d'équations:

$$\left\{\begin{aligned} u &= u_0 + \frac{1}{4\pi}\int_o^t \left(\int \frac{L(\xi - x)}{r^3}\, dp_1\, dp_2\, dp_3\right) dt, \\ v &= v_0 + \frac{1}{4\pi}\int_o^t \left(\int \frac{L(\eta - y)}{r^3}\, dp_1\, dp_2\, dp_3\right) dt, \quad r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2, \\ w &= w_0 + \frac{1}{4\pi}\int_o^t \left(\int \frac{L(\zeta - z)}{r^3}\, dp_1\, dp_2\, dp_3\right) dt, \end{aligned}\right. \tag{26}$$

$$x = q_1 + \int_o^t u\,dt, \quad y = q_2 + \int_o^t v\,dt, \quad z = q_3 + \int_o^t w\,dt, \tag{27}$$

où ξ, η, ζ s'obtiennent en substituant dans x, y, z les variables p_1, p_2, p_3 aux variables q_1, q_2, q_3; la fonction L sous le signe d'intégration s'obtient par la même substitution effectuée sur la fonction du §3.

9. Imaginons à présent l'espace (Ξ) des points M (x, y, z). A chaque frontière de l'espace (Q), correspond une surface dans l'espace (Ξ), elle se déplace et se déforme avec t; si $t < b_0$ ces surfaces satisfont aux conditions de Liapounoff.

On peut démontrer au moyen du système (27) que si $t < b_0$ on obtient q_1, q_2, q_3 comme fonctions des x, y, z, continues dans tout espace (Ξ) et ayant des dérivées premières par rapport à x, y, z en chaque point non situé sur une de ces surfaces.

10. Soient x, y, z, les variables indépendantes dans u, v, w, les ξ, η, ζ étant les variables d'intégration; les équations (26) différenciées par rapport à t, donnent alors

$$\left\{\begin{aligned} &\frac{du}{dt} = \frac{\partial u}{dt} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{1}{4\pi}\,\frac{\partial}{\partial x}\int \frac{k\,d\xi\, d\eta\, d\zeta}{r}, \\ &\frac{dv}{dt} = \frac{1}{4\pi}\,\frac{\partial}{\partial y}\int \frac{k\,d\xi\, d\eta\, d\zeta}{r}, \quad \frac{dw}{dt} = \frac{1}{4\pi}\,\frac{\partial}{\partial z}\int \frac{k\,d\xi\, d\eta\, d\zeta}{r}, \end{aligned}\right. \tag{28}$$

où

$$k = 2\left\{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial w}{\partial z} - \frac{\partial v}{\partial z}\cdot\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\cdot\frac{\partial u}{\partial x} - \frac{\partial u}{\partial z}\cdot\frac{\partial w}{\partial x}\right\}.$$

En utilisant les équations (28) et en démontrant au préalable l'existence des dérivées nécessaires, on obtient l'identité:

$$\frac{\partial}{\partial x}\left(\frac{du}{dt}\right)+\frac{\partial}{\partial y}\left(\frac{dv}{dt}\right)+\frac{\partial}{\partial z}\left(\frac{dw}{dt}\right)=\frac{d\theta}{dt}-k+\theta^2 \tag{29}$$

$$=\frac{1}{4\pi}\left\{\frac{\partial^2}{\partial x^2}\int\frac{kd\xi d\eta d\zeta}{r}+\frac{\partial^2}{\partial y^2}\int\frac{kd\xi d\eta d\zeta}{r}+\frac{\partial^2}{\partial z^2}\int\frac{kd\xi d\eta d\zeta}{r}\right\}=-k,$$

ou

$$\frac{d\theta}{dt}+\theta^2=0$$

si

$$\theta=\frac{\partial u}{dx}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}. \tag{30}$$

Si les fonctions u_0, v_0, w_0 satisfont aussi, à la condition

$$\frac{\partial u_0}{\partial q_1}+\frac{\partial v_0}{dq_2}+\frac{\partial w_0}{\partial q_3}=0,$$

l'équation (30) donne

$$\theta=0. \tag{31}$$

Ainsi, les fonctions u, v, w et la fonction

$$\Pi=\frac{1}{4\pi}\int\frac{kd\xi d\eta d\zeta}{r} \tag{32}$$

satisfont au système d'équations

$$\begin{gathered}\frac{du}{dt}=\frac{\partial\Pi}{\partial x},\quad\frac{dv}{dt}=\frac{\partial\Pi}{\partial y},\quad\frac{dw}{dt}=\frac{\partial\Pi}{\partial z},\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\end{gathered} \tag{33}$$

et donnent la solution du problème de l'Hydrodynamique dans le cas d'un liquide remplissant tout l'espace, correspondant aux conditions initiales:

$$u=u_0,\quad v=v_0,\quad w=w_0 \text{ pour } t=0.$$

THE DIRICHLET PROBLEM FOR THE GENERAL FINITELY CONNECTED REGION

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Necessary and sufficient conditions are obtained in order that a function harmonic in a finitely connected open region, in two dimensions, none of whose boundaries are isolated points, but otherwise perfectly general, may be represented in terms of a Stieltjes integral formula involving the Green's function of the region, and also for the special case that this formula reduce to the usual integration of boundary values with respect to the conjugate of the Green's function. Under the general conditions, which may be expressed simply by saying that the function is the difference of two functions harmonic and non-negative at all points of the region, the function "takes on boundary values almost everywhere"; in the special case it is uniquely determined by those frontier values. Any bounded harmonic function comes under the special case, which however is more general than this, since a harmonic function of the special class can be found which takes on almost everywhere any set of values summable in the Lebesgue sense with respect to the conjugate of the Green's function whose pole is a fixed point of the region.

1. Open region bounded by $n+1$ circles.

Consider first an open region S whose complete boundary s consists of an external circle s_0 and n internal circles s_i not intersecting it and distinct from each other. Let P be a point of S of coordinates r_i, θ_i referred to the circle s_i of radius R_i, and let $u(P)$ be harmonic in S. We may write, as is well known,*

$$u(P) = u_0(P) + \sum_1^n (w_i(P) + m_i \log r_i)$$

where $u_0(P)$ is harmonic at points interior to s_0, and each $w_i(P)$ is harmonic at points exterior to s_i, and is regular and vanishes at ∞.

With the help of results obtained by H. E. Bray and the author† it is established that a necessary and sufficient condition that we should have

*Osgood, *Funktionentheorie*, Leipzig u. Berlin, 1912, p. 643.

†G. C. Evans and H. E. Bray, Comptes Rendus Acad. Sciences, Paris, t. 176, pp. 1042 and 1368, and t. 177, p. 241 (1923).

$$u_0(P) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R_0^2 - r_0^2)dF_0(\phi_0)}{R_0^2 + r_0^2 - 2R_0 r_0 \cos(\phi_0 - \theta_0)} \tag{1}$$

$$w_i(P) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r_i^2 - R_i^2)dF_i(\phi_i)}{R_i^2 + r_i^2 - 2R_i r_i \cos(\phi_i - \theta_i)} \tag{2}$$

in which the $F_i(\phi_i)$, $i = 0, 1, 2, \ldots n$, are functions of limited variation (not necessarily absolutely continuous or even continuous), is that there should be a constant M for which

$$\int_0^{2\pi} |u(P)| d\theta_i < M, \tag{3}$$

when the integration is extended over all circles in S concentric with a boundary circle and in its neighbourhood. It is the same thing if this uniformity extends over a denumerable set of such circles which have as limiting elements every circle of the boundary.

2. Physical Interpretation

The function given by

$$u(P) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - r^2)dF(\phi)}{1 + r^2 - 2r\cos(\phi - \theta)} \tag{4}$$

represents the potential inside the unit circle due to the most general distribution along the boundary of "circular doublets," of which the total positive and negative amounts are finite. This statement describes the degree of generality of our problem, but does not give a physical interpretation of such a general distribution of mass.

We think of such distributions as limiting cases of simpler ones. We cannot however measure the distributions directly, but only quantities like potential, attraction, etc., given in terms of those distributions,—usually by means of Stieltjes integrals like (4). An adequate interpretation therefore will be one in which not only

$$F(\phi) = \lim_{k=\infty} F_k(\phi)$$

but also

$$H = \lim_{k=\infty} H_k \tag{5}$$

where say $H = \int_a^b h(\phi) dF(\phi)$ and $H_k = \int_a^b h(\phi) dF_k(\phi)$, $h(\phi)$ being a continuous function. This will be the case if the total variation of $F_k(\phi)$ is bounded for all k.*

The Fejer trigonometric sum of order k for $F(\phi)$, the quantity $\int_0^\phi u(r, \theta) d\theta = F(r, \phi)$, and the average approximation studied by Bray† are all

*A special case of Bray's theorem, Annals of Mathematics, vol. 20 (1918), p. 180.
†Bray, Bull. Amer. Math. Soc., vol. 29 (1923).

approximations which satisfy the conditions of this theorem. More closely related to intuition are the polygonal approximations, corresponding to distributions of constant density on a finite number of intervals, and the step function approximations, corresponding to distributions of a finite number of point masses, from either of which the general $F(\phi)$ may be developed.

In this way the distribution corresponding to a continuous $F(\phi)$ with zero density almost everywhere can be written as an absolutely convergent sum in terms of a denumerable aggregate of *elementary discard distributions*, which latter correspond to the elementary discards of Vitali,— non-decreasing functions of total variation unity, continuous, but constant on each of a denumerable set of intervals whose complementary set is *perfect* and *of measure zero*. The polygonal and step function interpretation of these is at once evident. It will be interesting to watch for the first appearance of these elementary discard distributions in the field of physical applications. Is it possible that they will occur in phenomena of radiation?

3. STIELTJES INTEGRAL FORMULAE IN TERMS OF GREEN'S FUNCTIONS FOR S.

Let $G(Q, X)$ be the Green's function for S, whose pole is at a point Q of S, and $h(Q, X)$ the conjugate function, which for convenience we denote simply by h_x. In the same way that (1) is a generalization of the usual Poisson's integral, the formula

$$u(P) = \frac{1}{2\pi}\int_c \frac{dh(P, M)}{dh_M}\, dU(h_M) \tag{6}$$

is a generalization of the usual formula in terms of the Green's function

$$u(P) = \frac{1}{2\pi}\int_c f(M)dh(P,M) \tag{7}$$

M being the variable point of the complete boundary, round which the integration is extended.

By making a conformal transformation of the neighbourhood of a boundary s_i into an annular region S'' and applying the theorem of §1, it can be shown that *a necessary and sufficient condition for* (6) *is that the quantity*

$$\int_{(g(Q,X)=const.)} |u(X)|dh(Q, X) \tag{6'}$$

remain bounded as g approaches zero.

In fact under this condition we have

$$U(h_M) = \lim_{g \doteq 0}\left(\int_{\substack{H\\(g=const.)}}^{h_M} u(X)dh(Q,X)\right). \tag{8}$$

This condition can be shown to be equivalent to the assumption that $u(P)$ be the difference of two functions non-negative and harmonic in S, which is again a necessary and sufficient condition for (6). Moreover for $h(Q, \mathrm{M})$ and

$h(Q, X)$ may be substituted the functions $h(Q', M)$ and $h(Q', X)$, where the Q' is any other point of S. That is to say, if the formula (6) holds with Q as pole, it also holds with any other point Q' as pole, the $U(h)$ being properly defined.

It is sufficient if the values $g(Q, X) = \text{const.}$ over which the integral $\int |u(x)|\, dh(Q, X)$ is bounded constitute a denumerable sequence with zero as a limit.

4. Behaviour in the neighbourhood of the boundary of S.

Except for a certain set of points of measure zero on the boundary of S the function $u(P)$ approaches a unique limiting value as P approaches a boundary point M, provided that the path of approach has no contact with the boundary,— more exactly, if the collection of points P which have M as a limit are contained between any two chords joining in M.

This theorem is obtained by the conformal transformation to S'', in order to make the representation of §1 applicable, and then by deducing the result from the corresponding fact in the case of a single integral of that representation.

If $u(P)$ is given by (4), *where $F(\phi)$ is continuous at some point M and has a unique derivative f there, we draw two chords of the circle, joining at M. If P approaches M so as to remain in the region between these two chords, then* $\lim_{P \to M} u(P) = f$. The $F(\phi)$ of course has a derivative almost everywhere.

A corresponding result was established by Fatou* for the case where P approaches M along a radius. His method apparently does not yield a wider result. But if, taking M at $\phi = 0$, and taking $\theta = \theta_r$ so that $|\theta_r|/(1-r) < N$ as r tends to 1, we write

$$dF(\phi) = f(0)d\phi + d(\phi\eta(\phi))$$

where $\phi\eta(\phi)$ is a function of limited variation and $\eta(\phi)$ tends to zero as ϕ approaches 0, then we may perform an integration by parts on the Stieltje's integral corresponding to $d(\phi\eta(\phi))$. By splitting up this integral properly we obtain the desired result.

5. The usual integral formula in terms of the Green's function.

The question concerns the conditions required to reduce (6) to (7). *A necessary and sufficient condition for* (7) *is that the absolute continuity of the integral* (6′) *or the integral in* (8) *be uniform over a denumerable sequence $\{g_i\}$ of values of $g(Q, X)$, where $\lim_{i=\infty} g_i = 0$.* The theorem follows shortly by means of §3 and §4 and the well known theorem of de la Vallée-Poussin.†

In particular this condition is satisfied if $u(P)$ is bounded in S, although, as the special case of the simply connected region shows, that restriction is not necessary.

*Fatou, Acta Mathematica, vol. 30 (1907), pp. 335-400.

†de la Vallée Poussin, Trans. Amer. Math. Soc., vol. 16 (1915), p. 447.

In the theorem of §3 the harmonic function is uniquely determined by the boundary values of the integral (8), but not by its own boundary values. On the other hand, *if the conditions of the theorem of the present section are satisfied the harmonic function is uniquely determined by its own boundary values* (which are defined almost everywhere on s); *and these boundary values may be given as any set of values summable in the Lebesgue sense with respect to h_x, or with respect to the arc boundary.* The condition of this section is a necessary and sufficient condition that $u(P)$ be determined by boundary values given almost everywhere on s. This is the Dirichlet problem for S.

6. THE GENERAL OPEN REGION T OF FINITE CONNECTIVITY.

Since the conditions of §3 and §5 are expressed in terms of quantities invariant of conformal transformations, the theorems there given (with the exception of the fact that there is no longer in general any arc boundary) may be applied to any open region T of finite connectivity, none of whose boundaries is an isolated point. In this case "taking on boundary values" is interpreted as taking them on as P moves out to the boundary along curves $h(Q, X) = \text{const}$.

In order to make the theorems directly applicable to the region T it is desirable to interpret the integrals involved as Stieltjes or generalized integrals on the boundary points themselves. For this, an order of the boundary points is necessary. Since almost all values of h_x correspond to accessible boundary points a study of these accessible points alone (counted multiply if necessary) is requisite. The order is hence obtained in terms of theorems of Osgood and Taylor with respect to the values of h_x at accessible points.*

It may be noticed that the determination of a harmonic function to be attached to the values of a continuous function of two variables at the frontier of T is covered by the present investigation, since such values, being the limits of continuous functions of h_x as g approaches zero, and remaining bounded, will be integrable with respect to h_x. It is also possible to define a more general sort of continuous boundary condition where the boundary function may have different values at the same geometrical point if it happens to be a multiple point considered as a boundary point of T.

The present investigation is a development of that initiated in incomplete fashion, with a study by the author of fundamental points of potential theory.† It is unnecessary to add that Professor Wiener has obtained noteworthy results in the study of the Dirichlet problem; on the whole, however, the questions under investigation are different.‡

*Osgood and Taylor, Trans Amer. Math. Soc., vol. 14 (1913), pp. 277-298.

†Evans, Rice Institute Pamphlets, vol. 7, No. 4 (1920), pp. 252-329.

‡Wiener, Trans. Amer. Math. Soc., vol. 23 (1923), p. 307.

Journal of Math. and Physics of the Mass. Inst. of Tech., vols. II, III, and Comptes Rendus Acad. Sciences, Paris (1924).

SUL CALCOLO DELLE VARIAZIONI

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1. Oggetto Della Comunicazione.

Mi propongo di esporre brevemente il metodo seguito nei miei *Fondamenti di Calcolo delle Variazioni,* di cui furono già pubblicati i primi due volumi. In questi due primi volumi sono trattati il problema dell'estremo libero a quello isoperimetrico, relativi alle curve del piano ed agli integrali dipendenti soltanto dalla posizione dell'elemento generico della curva e dalla sua direzione; gli altri problemi del Calcolo delle Variazioni verranno considerati nei prossimi volumi.

2. Ragioni Della Ricerca di un Nuovo Metodo di Calcolo Delle Variazioni.

Sono noti gli inconvenienti che presenta il metodo classico del Calcolo delle Variazioni. Qui accennerò soltanto al fatto che, oltre alle difficoltà inerenti alla verifica, su una data curva, delle ben note condizioni sufficienti per l'estremo, vi è una difficoltà di natura assai grave, che alle volte arresta fin dall'inizio la applicazione del metodo classico, ed è che la teoria delle equazioni differenziali non offre davvero molti mezzi per assicurare o escludere anche soltanto l'esistenza di un'estremale congiungente due dati punti, che non siano convenientemente vicini. Mi proposi, pertanto, di cercare un procedimento diretto atto a risolvere i problemi del Calcolo delle Variazioni, indipendentemente dalla teoria delle equazioni differenziali. Nello stesso tempo, io ponevo alle mie ricerche altri due obbiettivi da raggiungere. Il primo era quello di ottenere risultati generali anche nel campo degli estremi assoluti, mentre, come è ben noto, il metodo classico è particolarmente indirizzato verso gli estremi relativi. In secondo luogo, si trattava di far sì che il nuovo proced mento fosse applicabile a classi di curve assai più general di quella che viene presa in considerazione negli ordinari trattati di Calcolo delle Variazioni, classe a cui in essi si è costretti a limitarsi precisamente perchè i metodi seguiti sono fondati essenzialmente sulla considerazione delle estremali. Della necessità di considerare classi di curve più generali di quella ordinaria è un primo esempio il classico problema di Newton del solido di rivoluzione di minor resistenza.

3. Concetto Fondamentale del Nuovo Metodo.

Fissati così gli scopi da raggiungere, una via si presentava naturalmente: quella del Calcolo funzionale, secondo le idee del Volterra. Gli integrali che vengono considerati nel Calcolo delle Variazioni, e che si tratta di estremare, sono numer dipendenti da una linea o superficie, e sono, pertanto, delle funzioni numeriche di linea o superficie. Essi dunque rientrano nel dominio del Calcolo funzionale; ed è evidente che i più larghi sviluppi del Calcolo delle Variazioni si otterranno costruendo, per le sue funzioni di linea o superficie, nell'ambito del Calcolo funzionale, una teoria degli estremi del tutto corrispondente a quella che, negli ordinari corsi di Analisi, si svolge per le funzioni di una o più variabili numeriche.

E' questa precisamente la via che io ho seguita.

Una prima difficoltà si è però subito presentata. Nell'ordinaria teoria dei massimi e minimi, si parte da una proprietà fondamentale delle funzioni che in essa si considerano: *la continuità*. Orbene, la *continuità* non può essere messa alla base de'la nuova teoria da costruire: le funzioni di linea o di superficie sono, infatti, funzioni genera'mente discontinue. Limitiamoci, per meglio precisare e per restare negli argomenti trattati nei primi due volumi dei miei "Fondamenti", agli integrali, funzioni di linea piana, dipendenti soltanto dall'elemento generico della curva e dalla sua direzione. Per queste funzioni, se si ponesse la condizione della continuità, bisognerebbe ridursi alla sola considerazione degli integrali la cui funzione integranda è espressa linearmente mediante le derivate del primo ordine delle coordinate del punto corrente sulla curva di integrazione; anzi, quando le curve sono date in forma parametrica, la continuità porta senz'altro che l'integrale dipenda soltanto dai punti terminali della curva d'integrazione.

Era dunque necessario scartare la *continuità* e porre, a fondamento della nuova teoria, un concetto più largo e meglio adatto alle funzioni che devono essere considerate.

L'analisi del concetto di *continuità*, per le funzioni di variabili numeriche dell'ordinario calcolo differenziale, aveva già condotto Baire a scindere tale concetto in altri due più elementari: la *semicontinuità inferiore* e *la semicontinuità superiore*. Il simultaneo presentarsi di queste due *semicontinuità* dà precisamente la *continuità*. Baire aveva anche osservato che l'elementare ragionamento in base al quale si dimostra che una funzione continua ha, in un dato campo chiuso, un minimo o un massimo, permette di giungere alla medesima conclusione pure nel caso di una funzione semicontinua inferiormente o superiormente, rispettivamente. Questa osservazione, applicata alle nostre funzioni di linea, permette di vincere la prima grave difficoltà e di porre i fondamenti della nuova teoria.

Se, assunta come variabile d'integrazione la lunghezza dell'arco, la funzione da integrare non dipende dalle derivate delle coordinate dei punti della curva di integrazione, vale a dire, è una funzione $\phi(x, y)$ delle sole coordinate x ed y, un ragionamento geometrico elementare permette subito di vedere che, quando la $\phi(x, y)$ è sempre positiva, il suo integrale è una funzione semicontinua inferiormente della curva di integrazione. Ciò trovasi già nella memorabile Tesi de Lebesgue. Ma questa semicontinuità è soltanto una proprietà degli integrali della forma particolare ora indicata, oppure appartiene ad una classe di integrali assai più vasta e tale che questa stessa proprietà possa porsi a fondamento di un metodo generale di Calcolo delle Variazioni? Le mie ricerche, iniziate nel 1911, mi hanno condotto a rispondere affermativamente a tale domanda.

Per gli integrali in forma parametrica, si ha che godono della semicontinuità tutti gli integrali *regolari*, secondo la denominazione introdotta da Hilbert, tutti quelli che io ho chiamati *quasi-regolari semi-normali*, ed anche tutti quelli semplicemente *quasi-regolari*, purchè intervenga una condizione supplementare, come per esempio quella che la funzione sotto il segno non cambi mai segno. Insieme con queste condizioni sufficienti per la semicontinuità, ho ricercato anche le condizioni necessarie, e sono giunto alla condizione necessaria e sufficiente, la quale è, per la semicontinuità inferiore, "che l'integrale sia quasi-regolare positivo, e che ogni punto *eccezionale* sia il centro di almeno un cerchio tale che l'integrale esteso ad una qualsiasi curva chiusa, tutta in esso contenuta, risulti sempre maggiore o uguale a zero". Per punto *eccezionale* si intende un punto in cui la nota funzione di Weierstrass, F_1, risulta identicamente nulla, rispetto alle x' e y'.

Nel caso degli integrali in forma ordinaria, la condizione necessaria e sufficiente per la semicontinuità assume forma assai più semplice, riducendosi a questo "che l'integrale sia quasi regolare".

La semicontinuità fino ad ora considerata è la semicontinuità in tutto il campo, ossia su ogni curva. Ho però studiato anche la semicontinuità su una data curva; e qui sono giunto alla conclusione che la condizione della semicontinuità corrisponde alla nota condizione di Weierstrass del Calcolo delle Variazioni, condizione che, come si sa, contiene in sè anche quella di Legendre. Questa conclusione mi sembra interessante perchè permette di stabilire una perfetta corrispondenza fra le condizioni che, per la ricerca degli estremi relativi, si danno nel Calcolo differenziale, e quelle che, per gli stessi estremi, si danno nel Calcolo delle Variazioni. Nel Calcolo differenziale, considerata una funzione $f(x)$, continua ed avente le due prime derivate, si ha che un punto x_0, interno al campo in cui la $f(x)$ è definita, dà un minimo relativo per la $f(x)$ se, in esso, la derivata prima della funzione è nulla e la derivata seconda è positiva. Nel Calcolo delle Variazioni una curva C_0 interna al campo in cui si considera l'integrale curvilineo da estremare, dà un minimo relativo per tale integra e se su di essa è nulla la variazione prima dell'integrale stesso, è positiva la variazione seconda, e ,di più, sono verificate le condizioni di Legendre e di Weierstrass. Ora queste condizioni di Legendre e Weierstrass stanno precisamente ad assicurare che l'integrale in questione è, sulla curva C_0, una funzione semicontinua inferiormente; condizione che, nel caso della funzione $f(x)$, era già implicitamente contenuta nell'ipotesi della continuità. Cosicchè alle condizioni del Calcolo differenziale, relative all'annullamento della derivata prima ed al segno positivo della derivata seconda, corrispondono esattamente, e con lo stesso valore, nel Calcolo delle Variazioni, le analoghe condizioni relative alla variazione prima ed alla variazione seconda. Ed il riscontro, nei due campi, delle condizioni sufficienti per il minimo, si ripete anche per le condizioni necessarie.

Gli studi ora brevemente riassunti e l'osservazione, che è bene fare esplicitamente, che cioè ogni integrale deve essere una funzione semicontinua inferiormente su ciascuna sua curva minimante, mi hanno condotto a costruire un nuovo svolgimento del Calcolo delle variazioni fondandolo essenzialmente sul concetto di semicontinuità

4. Teoremi d'esistenza dell'estremo.

Una prima immediata applicazione della semicontinuità doveva farsi alla dimostrazione dei teoremi d'esistenza dell'estremo assoluto; e ciò anche per seguire lo stesso ordine con cui si procede nel Calcolo differenziale per gli estremi delle funzioni di variabili numeriche.

Per altro, qui si presenta una difficoltà che non ha nessun riscontro in ciò che avviene nella teoria degli estremi del Calcolo differenziale. Per stabilire l'esistenza del minimo assoluto, in un dato intervallo, di una funzione $f(x)$ semicontinua inferiormente, si sceglie, come è noto, una successione minimizzante $x_1, x_2, \ldots, x_n, \ldots$; tale successione ha almeno un punto limite, per il teorema di Bolzano- Weierstrass, e in ciascuno di questi punti limiti la $f(x)$, in virtù della semicontinuità inferiore, assume il suo minimo valore. Nel caso del Calcolo delle Variazioni, la successione minimizzante di curve $C_1, C_2, \ldots C_n, \ldots$ non ha sempre necessariamente una curva limite, perchè per gli insiemi di curve non vale senz'altro il teorema di Bolzano-Weierstrass. Si ha così questo fatto nuovo, che cioè un integrale, pur supposto funzione semicontinua inferiormente e pur considerato in un campo limitato, può non avere minimo. Per poter concludere, come nel caso della funzione $f(x)$, occorre dunque l'intervento di qualche nuova condizione, la quale assicuri l'esistenza dell'ente limite per la successione minimizzante. Questa ulteriore condizione può assumere varie forme, e da ciò derivano numerosi teoremi d'esistenza per l'estremo.

Per gli integrali in forma parametrica, ho cominciato, nel secondo Volume dei miei "Fondamenti", a dimostrare l'esistenza dell'estremo relativamente agli integrali quasi regolari definiti, di cui è caso particolare l'integrale di una funzione $\phi(x, y)$, sempre di uno stesso segno, che fu trattato, come ho già accennato, dal Lebesgue nella sua Tesi. Dagli integrali quasi-regolari definiti, sono poi passato a casi più generali, i quali richiedono analisi più profonde, cercando di inquadrare in essi le più importanti applicazioni del Calcolo delle Variazioni.

Per gli integrali in forma ordinaria, la condizione che deve assicurare l'esistenza dell'ente limite per la successione minimizzante prende forme diverse da quelle assunte nel caso parametrico, inquantochè essa è legata al comportamento della funzione $f(x, y, y')$, che sta sotto il segno di integrale, al tendere all'infinito della y'. Anche qui si sono dati diversi teoremi d'esistenza, cercando di soddisfare il più possibile alle esigenze delle applicazioni; e si sono ritrovati, come casi particolari, risultati stabiliti, per altra via, dall' Hadamard.

I teoremi di esistenza, tanto per gli integrali in forma parametrica quanto per quelli in forma ordinaria, sono dati con la massima generalità per quello che riguarda la natura e la classe delle curve considerate. Per i primi, le curve si suppongono semplicemente *rettificabili*; per i secondi, le funzioni $y(x)$ che rappresentano le ordinate delle curve, sono supposte assolutamente continue, secondo la definizione del Vitali, e tali che su di esse esista finito l'integrale che si vuol studiare. Di più—ed è precisamente questa la cosa più importante da mettersi in rilievo—i teoremi di esistenza sono dati quasi tutti per le classi di curve che io ho chiamate *complete*. Nel caso parametrico, una *classe completa* è semplicemente un insieme di curve rettificabili al quale appartengono tutte quelle curve che sono, contemporaneamente, curve limiti per l'insieme medesimo e rettificabili. Nella forma ordinaria, in questa definizione, alla condizione della rettificabilità, si sostituiscono quelle due che ho indicate poco fa. I teoremi di esistenza assumono in questo modo la massima generalità; risultano del tutto indipendenti dalla considerazione delle estremali e si prestano ad essere applicati anche a problemi di estremo che i metodi già conosciuti non potrebbero in nessun modo affrontare.

E' importante osservare che il metodo di dimostrazione dell'esistenza dell'estremo mediante l'uso della semicontinuità permette pure di trattare classi di integrali che non sono ovunque funzioni semicontinue. Di ciò ho dato esempio, nel mio libro, studiando quella categoria di integrali che formò oggetto di notevoli ricerche da parte di C. Carathéodory. Tali integrali hanno generalmente come curve estremanti delle curve con punti angolosi. Per essi ho dato la dimostrazione dell'esistenza dell'estremo riconducendola a quella di altri integrali dotati ovunque della semicontinuità e in intima relazione coi primi. Il medesimo procedimento mi ha poi permesso di trattare compiutamente anche il classico problema di Newton, del solido di rivoluzione di minor resistenza, problema la cui soluzione, indicata da Newton medesimo, fu completamente giustificata soltanto nel 1900, per merito di A. Kneser.

5. ESTREMALOIDI ED ESTREMALI.

Risoluta la questione dell'esistenza dell'estremo, bisognava passare alla ricerca delle proprietà analitiche delle estremanti. E' evidente che queste proprietà sono intimamente connesse alla natura delle classi di curve rispetto a cui si ha l'estremo; e per il loro studio la generalità delle classi considerate doveva necessariamente venire limitata.

Limitandomi alle classi solitamente considerate nel Calcolo delle Variazioni, e pur mantenendo sulla natura delle curve tutta la generalità che ho più sopra indicata, sono stato condotto a introdurre il concetto di *estremaloide*, che è una generalizzazione del noto concetto di *estremale*.

Nella forma parametrica, indicando con $F(x, y, x', y')$ la funzione sotto il segno di integrale, chiamo *estremaloide* ogni curva rettificabile soddisfacente alle due equazioni

$$\int_o^s F_x ds - \frac{d}{ds}\int_o^s F_{x'} ds = c_1\,, \quad \int_o^s F_y ds - \frac{d}{ds}\int_o^s F_{y'} ds = c_2,$$

dove s rappresenta la lunghezza dell'arco. Queste equazioni generalizzano evidentemente quelle delle estremali; esse presentano il vantaggio di poter essere soddisfatte anche da una curva semplicemente rettificabile. Generalizzando un ragionamento classico, dimostro che ogni curva che sia estremante, fra tutte quelle, di un suo intorno, che congiungono gli stessi punti terminali, è, se interna al campo che si considera, un'*estremaloide*. Ottengo con ciò l'estensione della condizione di Eulero alle curve estremanti semplicemente rettificabili. Dimostro poi che, per gli integrali *quasi-regolari normali*, le estremaloidi sono necessariamente delle estremali, ed ho così il modo di trattare un caso importante in cui le estremali presentano delle singolarità derivanti dall'annullamento, nella loro equazione, del coefficiente del termine di secondo ordine. Sempre relativamente ad integrali *quasi-regolari normali*, stabilisco che, per campi di natura molto generale ma con contorno privo di punti angolosi rivolti verso l'interno del campo stesso, tutte le curve estremanti hanno sempre tangente variabile in modo continuo, pur anche nelle parti che raggiungono il contorno. Per campi speciali, o per campi illimitati, quando il comportamento della funzione F, al tendere del punto (x, y) all'infinito, soddisfa a opportune condizioni, le curve estremanti degli integrali quasi regolari normali risultano senz'altro delle estremali. Da ciò *derivano teoremi di esistenza per le estremali congiungenti due dati punti, non necessariamente vicini, e per le estremali periodiche.*

Studio, inoltre anche l'unicità delle estremanti e delle estremali congiungenti due dati punti. Va poi da sè che vengono ritrovate le note condizioni necessarie per l'estremo, a proposito delle quali osserverò che quelle di Legendre e Weierstrass sono, per quanto ho già detto, un'immediata conseguenza della semicontinuità, che deve essere verificata su ogni curva estremante.

Ritrovo pure le note condizioni sufficienti per l'estremo relativo. Qui risolvo prima la questione "in piccolo", riconducendo lo studio dell'integrale dato a quello di un altro integrale dotato ovunque della semicontinuità ed applicando i risultati già ottenuti. Lo stesso metodo permette di risolvere la stessa questione nel caso generale; il quale può anche trattarsi applicando i risultati ottenuti "in piccolo".

Nella forma ordinaria si presentano, relativamente alla condizione di Eulero, maggiori difficoltà, le quali dipendono dal fatto che non può assicurarsi a priori l'integrabilità, su ogni curva estremante, delle derivate parziali prime, rispetto ad y e y', della funzione $f(x, y, y')$ che figura nell'integrale da estremare, in modo che non può "a priori" essere assicurata l'esistenza dei due integrali che figurano nell'equazione delle estremaloidi. Tale esistenza sembra assai probabile, ma sino ad ora non mi è riuscito nè di stabilirla in tutti i casi, nè di mostrare con un esempio che essa può effettivamente mancare. Tuttavia ho dato in proposito delle proposizioni di carattere molto generale. Così ho dimostrato che ogni curva che sia estremante fra tutte quelle di un suo intorno che congiungono gli stessi punti terminali, è, se interna al campo e con rapporto incrementale limitato, un'estremaloide; ed ho anche mostrato che la condizione relativa al rapporto incrementale può senz'altro sopprimersi per una notevolissima ed estesa categoria di integrali, la quale comprende precisamente quegli integrali che più si presentano nelle applicazioni del Calcolo delle Variazioni. Per questa medesima categoria, ed anzi per un'altra ancor più generale, ho pure provato che ogni estremaloide è necessariamente un'estremale. Ho studiato in modo particolare gli integrali *regolari* e quelli *quasi regolari normali*, cercando di ottenere risultati analoghi a quelli del caso parametrico. Qui le curve estremanti possono avere la derivata prima della loro ordinata infinita; ma ho dato condizioni sufficienti perchè ciò non si presenti.

Tutti gli argomenti trattati nel caso parametrico vengono ripresi per gli integrali in forma ordinaria; e, nel modo già indicato, si ritrovano anche qui le note condizioni necessarie e quelle sufficienti per l'estremo relativo.

Il problema di cui fino ad ora ho parlato è quello dell'estremo libero; ma il metodo esposto è, nel 2° Volume dei miei "Fondamenti", applicato anche allo studio del problema isoperimetrico.

Il procedimento, come risulta da quanto ho detto, non richiede nulla della teoria delle equazioni differenziali; anzi porta ad essa un contributo, con la dimostrazione di teoremi di esistenza.

SUR LES SOLUTIONS DISCONTINUES DANS LE CALCUL DES VARIATIONS

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INTRODUCTION

Soit $f(x, y, y')$ une fonction de trois variables qui est continue, ainsi que ses dérivées partielles jusqu'à celles du troisième ordre, tant que le point (x, y) reste dans une région connexe du plan R et pour toutes les valeurs finies de $y' = \dfrac{dy}{dx}$. Soit encore donné l'ensemble $\mathfrak{M}$ des courbes continues, joignant les deux points donnés $P_1(x_1, y_1)$ et $P_2(x_2, y_2)$, qui satisfont aux conditions connues de régularité (continuité, existence des dérivées, etc.) et qui sont situées à l'intérieur de la région R.

Le problème général du calcul des variations peut être formulé ainsi:

Parmi les courbes de l'ensemble $\mathfrak{M}$ en trouver une telle que l'intégrale

$$J = \int_{x_1}^{x_2} f(x, y, y')dx,$$

prise le long de cette courbe, ait une valeur plus petite ou plus grande que le long de toute autre courbe de cet ensemble.

Il y a des problèmes où aucune courbe continue ne rend minimum l'intégrale, tandis qu'il existe des courbes discontinues, c'est-à-dire présentant des sauts brusques, qui fournissent un vrai minimum de l'intégrale dans un sens bien déterminé. C'est de ces solutions discontinues que nous allons nous occuper dans ce travail. Pour bien comprendre le problème, prenons l'exemple bien connu de Weierstrass: soit à trouver la valeur minimum de l'intégrale

$$J = \int_{-1}^{+1} x^2 y'^2 dx$$

prise le long d'une courbe continue, joignant les deux points P_1 et P_2 de coordonnées $(-1, a)$, $(+1, b)$; a étant distinct de b. Il n'existe aucune extrémale continue joignant les deux points P_1 et P_2. En outre, on peut s'assurer facilement que l'intégrale J prise le long de la courbe

$$y = \frac{a+b}{2} + \frac{b-a}{2} \frac{\operatorname{arctang} \dfrac{x}{\xi}}{\operatorname{arctang} \dfrac{1}{\xi}},$$

qui passe par les deux points P_1, P_2, tend vers zéro avec ξ. Par conséquent la limite inférieure de l'intégrale J est égale à zéro. D'autre part, quelle que soit la courbe continue joignant les deux points P_1 et P_2, la valeur de l'intégrale J prise suivant cette courbe ne peut être nulle. On en conclut que le problème proposé n'admet pas de solution continue. Mais quand ξ tend vers zéro, on a, à la limite, la fonction discontinue

$$y=a, \qquad x<0,$$
$$y=b, \qquad x>0,$$
$$y=\frac{a+b}{2}, \qquad x=0,$$

qui fournit pour l'intégrale de Weierstrass la valeur minimum zéro. On voit donc que pour le problème de Weierstrass l'extrémale est discontinue et présente un point de discontinuité de première espèce.

En général, quand le problème donné n'admet pas de solution continue vue la nature de la fonction f, il y a lieu d'élargir l'énoncé du problème par l'introduction de courbes discontinues ayant des points de discontinuité de première espèce. Ces solutions seront dites solutions discontinues*.

Le but de ce travail est de trouver pour le problème donné l'extrémale discontinue ayant *un* point de discontinuité de première espèce, qui passe par deux points donnés P_1 et P_2 et de déterminer les conditions nécessaires et suffisantes de l'extremum pour ces solutions. Mais pour que cette généralisation ait un sens et une raison, les extrémales discontinues doivent être d'une nature spéciale: la courbe discontinue ayant un point de discontinuité de première espèce est la limite d'une suite de courbes continues. Appelons *courbes d'approximation* les courbes continues qui tendent vers l'extrémale discontinue. Comparons l'intégrale J prise le long de l'extrémale discontinue à la même intégrale prise le long des courbes d'approximation. *Il ne suffit pas de s'assurer que la première intégrale est plus petite que les dernières, il faut encore que parmi celles-ci il y en ait qui s'approchent autant qu'on veut de la première.* Ce n'est que des courbes de ce genre-là que nous tenons compte comme courbes de comparaison continues. Nous les appelons courbes d'approximation admissibles. Par exemple, pour le problème de la moindre distance, il n'existe pas de courbes d'approximation admissibles et par conséquent, pour ce problème, les solutions discontinues n'ont pas de sens. Pour le problème de Weierstrass la condition énoncée est satisfaite, c'est pourquoi les solutions discontinues y ont un sens.

Nous diviserons l'exposé en deux parties. Dans la première partie nous développerons les conditions nécessaires pour le minimum. La seconde partie sera consacrée aux conditions suffisantes. A la fin de cet exposé, nous donnerons deux exemples de ce genre de problèmes. Il ne sera question dans la suite que du premier des deux extrema, minimum et maximum.

*On a l'habitude, dans le calcul des variations, d'appeler discontinues les extrémales continues à points anguleux. Cette dénomination qui a passé dans presque tous les traités ne nous semble plus tenable du moment qu'on introduit de vraies courbes discontinues. C'est pourquoi nous préférons appeler *anguleuses* ces extrémales en réservant le nom de solutions *discontinues* aux nôtres.

I. Conditions nécessaires.

A. Conditions du premier ordre.

1. Nous prenons l'équation générale des courbes discontinues sous la forme

$$y=y(x), \qquad x_1 \leqq x \leqq x_2,$$

où la fonction $y(x)$ présente dans l'intervalle (x_1, x_2) un point de discontinuité de première espèce $x=x_0$. Dans les intervalles

$$x_1 \leqq x < x_0, \quad x_0 < x \leqq x_2$$

elle est continue, aucune des tangentes n'est parallèle à l'axe des y, et le coefficient angulaire $y'(x)$ reste toujours borné.

La courbe représentée par l'équation $y=y(x)$ se compose de deux arcs de courbes continues telles que $P_1 R_0$, $\bar{R}_0 P_2$, qui ne se rejoignent pas (Fig. 1); la valeur de y qui correspond à l'abscisse x_0 du point de discontinuité peut être quelconque, parce qu'elle n'influe nullement sur la valeur de l'intégrale prise le long de la courbe correspondante. Nous appelons les points, où a lieu la rupture de ces courbes continues, *points de rupture.*

L'intégrale

$$\int_{x_1}^{x_2} f[x, y(x), y'(x)]dx,$$

prise le long d'une courbe discontinue, est la somme de deux intégrales

$$\int_{x_1}^{x_0} f[x, y(x), y'(x)]dx + \int_{x_0}^{x_2} f[x, y(x), y'(x)]dx.$$

Dans chaque intervalle partiel $y(x)$ est continue et $y'(x)$ est bornée; par conséquent l'intégrale a une valeur finie.

Dans la théorie figurent en outre des courbes continues dont nous déterminerons la forme dans le numéro suivant.

2. Définissons maintenant le voisinage de l'extrémale discontinue, dans lequel doivent être situées toute autre courbe discontinue analogue et toutes les courbes continues de comparaison joignant les deux points donnés $P_1(x_1, y_1)$, $P_2(x_2, y_2)$.

Soit

$$\mathfrak{E}_0 \qquad y=\phi_0(x) \begin{cases} =y_0(x), & x_1 \leqq x < x_0, \\ =\bar{y}_0(x), & x_0 < x \leqq x_2, \end{cases}$$

l'équation de l'extrémale discontinue du problème; $x=x_0$, le point de discontinuité; $R_0(x_0, y_0)$, $\bar{R}_0(x_0, \bar{y}_0)$, ses points de rupture. Considérons le trait brisé $P_1 R_0 \bar{R}_0 P_2$ composé des deux arcs continus $P_1 R_0$, $\bar{R}_0 P_2$ de notre extrémale et du segment vertical $R_0 \bar{R}_0$. Formons un domaine tel qu'à chacun de ses points (x, y) corresponde un point (x^0, y^0) du trait brisé dont la distance à (x, y) soit plus petite qu'une quantité donnée ρ. Ainsi nous formons la région R_ρ, ombrée sur la figure, dans laquelle doit être située toute courbe continue de comparaison et toute courbe discontinue analogue ayant ses points de rupture dans les cercles de rayon ρ décrit autour des points R_0, $\bar{R}_0$ et joignant les points extrêmes P_1, P_2.

Nous pouvons maintenant déterminer la forme des courbes continues de comparaison d'une façon précise.

Soit $\mathfrak{C}(P_1K\bar{K}P_2)$ une courbe discontinue quelconque de la région R_ρ joignant les deux points extrêmes P_1, P_2; K, $\bar{K}$ étant ses points de rupture et $\bar{x}_0$ leur abscisse commune.

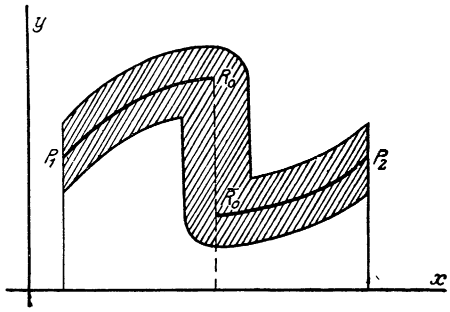

Fig. 1

La forme des courbes en question est celle des courbes continues de la région R_ρ joignant les mêmes points extrêmes et se rapprochant de plus en plus de la ligne brisée $P_1K\bar{K}P_2$ de sorte qu'aucune des tangentes ne soit parallèle à l'axe des x. Nous dirons brièvement que ces courbes *tendent* vers $\mathfrak{C}$.

Soit

$$y=\omega_n(x)$$

l'équation d'une courbe pareille λ_n; choisissons une succession de nombres positifs ϵ_n décroissant et tendant vers zéro. Il est facile de voir d'après la définition mentionnée tout à l'heure que la dérivée $\omega_n'(x)$ n'est pas bornée dans l'intervalle $(\bar{x}_0-\epsilon_n, \bar{x}_0+\epsilon_n)$ lorsque n tend vers l'infini.

L'arc de la courbe λ_n dans l'intervalle $(\bar{x}_0-\epsilon_n, \bar{x}_0+\epsilon_n)$ joue le rôle prépondérant pour la détermination des formes possibles de la fonction f correspondant au problème à résoudre. Nous appellerons cet arc *la chute* de la courbe λ_n.

2^{bis}. Les méthodes du calcul des variations ne nous permettent d'admettre comme courbes de comparaison continues que des courbes qui vérifient la condition analogue à celle mentionnée pour les courbes d'approximation dans l'Introduction. C'est-à-dire, soit $\{\lambda_n\}$ un ensemble de courbes continues de la région R_ρ tendant vers une courbe discontinue $\mathfrak{C}$ quelconque de la même région, (en particulier la courbe $\mathfrak{C}$ est l'extrémale $\mathfrak{C}_0$ lorsque $\{\lambda_n\}$ est un ensemble de courbes d'approximation): il s'agit de trouver un minimum dans le champ de tous les ensembles $\{\lambda_n\}$ et des courbes discontinues correspondantes $\mathfrak{C}$ telles que l'intégrale J prise suivant λ_n s'approche autant qu'on veut de la valeur de la même intégrale prise suivant la ligne $\mathfrak{C}$. Ce ne sont que des courbes λ_n de ce genre et des courbes discontinues $\mathfrak{C}$ correspondantes que nous admettons comme courbes de comparaison. Nous les appelons *courbes de comparaison admissibles.*

Nous dirons donc que la ligne discontinue $\mathfrak{C}_0$ fournit un minimum relatif de l'intégrale J, si la valeur de cette intégrale, prise suivant la ligne $\mathfrak{C}_0$ est plus petite *que pour toute autre courbe discontinue analogue admissible ou toute courbe continue admissible joignant les deux points donnés P_1 et P_2 et située tout entière dans la région R_ρ.*

De cette définition on déduit la première condition à laquelle doit satisfaire la ligne $\mathfrak{C}_0$.

Pour que la ligne $\mathfrak{C}_0$ donne un minimum relatif de l'intégrale J il est nécessaire que les arcs P_1R_0, $\bar{R}_0P_2$ soient des courbes extrémales (courbes d'Euler) vérifiant toutes les conditions du minimum ordinaire.

3. Nous allons tout d'abord résoudre la question fondamentale suivante:

Sous quelle condition l'intégrale J prise suivant la courbe d'approximation peut-elle approcher autant qu'on veut de la valeur de la même intégrale prise suivant l'extrémale $\mathfrak{C}_0$?

Soit $\{\lambda_n\}$ l'ensemble des courbes d'approximation admissibles et

$$y=\omega_n(x)$$

l'équation de la courbe λ_n. Dès lors on a

$$\lim_{n\to\infty} \Delta J_n=\lim_{n\to\infty} (J_{\lambda_n}-J_{\mathfrak{C}_0})=0, \tag{1}$$

ou bien

$$\lim_{n\to\infty}\int_{x_1}^{x_2} f[x,\ \omega_n(x),\ \omega_n'(x)]\,dx = \int_{x_1}^{x_2} f[x,\ \phi_0(x),\ \phi_0'(x)]dx.$$

Soient en outre $(x_0-\epsilon_n,\ x_0+\epsilon_n)$ l'intervalle de la chute et $K(x_0-\epsilon_n\ ,y_0+\delta)$, $\bar{K}(x_0+\epsilon_n,\ \bar{y}_0+\bar{\delta})$ les points de la courbe λ_n correspondant aux abscisses $x_0-\epsilon_n$, $x_0+\epsilon_n$. Joignons les points P_1, K et $\bar{K}$, P_2 par les courbes extrémales voisines P_1K, $\bar{K}P_2$ (Fig. 6, N° 21). Il est évident que la courbe $L_n(P_1K\bar{K}P_2)$ formée de ces deux arcs et ayant la même chute est une courbe d'approximation. Donc, à l'ensemble $\{\lambda_n\}$ correspond l'ensemble $\{L_n\}$ ainsi formé.

Pour que les méthodes du calcul des variations soient applicables dans le cas du problème à résoudre, nous devons nécessairement admettre que, pour le minimum, on doit avoir à la fois

$$J_{\lambda_n}-J_{\mathfrak{C}_0}\geqq J_{L_n}-J_{\mathfrak{C}_0}\geqq 0,$$

c'est-à-dire que, au point de vue de la recherche des conditions nécessaires et suffisantes, il faut que les deux champs $\{\lambda_n\}$ et $\{L_n\}$ soient équivalents.

Appelons $y=\phi_n(x)$ l'équation de la courbe L_n. En tenant compte des dernières inégalités et de l'équation (1), on en déduit la condition suivante nécessaire pour que le problème posé soit résoluble:

$$\lim_{n\to\infty} (J_{L_n}-J_{\mathfrak{C}_0})=\lim_{n\to\infty}\left[\int_{x_1}^{x_2} f(x,\ \phi_n(x),\ \phi_n'(x))\,dx - \int_{x_1}^{x_2} f(x,\ \phi_0(x),\ \phi_0'(x))\,dx\right]=0. \tag{2}$$

Donc si $\{\lambda_n\}$ est un ensemble de courbes d'approximation admissibles, l'ensemble correspondant $\{L_n\}$ doit être également admissible.

D'autre part, si η est un nombre positif arbitrairement petit, les deux courbes L_n et $\mathfrak{C}_0$, à partir d'une certaine valeur de n, auront entre elles, en dehors de l'intervalle $(x_0-\epsilon_n, x_0+\epsilon_n)$, un voisinage d'ordre 1 défini par ce nombre η*. C'est-à-dire: *à un nombre positif* η *petit qu'il soit, on peut faire correspondre un autre nombre entier* N *tel que pour* $n>N$ *on ait:*

$$(\eta) \qquad |\phi_n(x)-\phi_0(x)|<\eta, \quad |\phi_n'(x)-\phi_0'(x)|<\eta$$

pourvu que x *appartienne à l'intervalle* $(x_1, x_0-\epsilon_n)$ *ou à* $(x_0+\epsilon_n, x_2)$.

Cela posé, nous allons démontrer que la condition mentionnée ci-dessus équivaut à la suivante:

L'intégrale J *prise suivant la chute de la courbe d'approximation admissible doit tendre vers zéro.*

Il est aisé de voir que cette condition est suffisante. Démontrons qu'elle est aussi nécessaire.

Considérons l'intégrale J prise suivant la chute de la courbe λ_n. Comme la courbe L_n a la même chute, on peut écrire par suite

$$\left|\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]dx\right| \leqq \left|\int_{x_1}^{x_2} f[x, \phi_n(x), \phi_n'(x)]dx - \int_{x_1}^{x_2} f[x, \phi_0(x), \phi_0'(x)]dx\right|$$
$$+\left|\int_{x_1}^{x_0-\epsilon_n}\{f[x, \phi_n(x), \phi_n'(x)]-f[x, \phi_0(x), \phi_0'(x)]\}dx\right|$$
$$+\left|\int_{x_0-\epsilon_n}^{x_2}\{f[x, \phi_n(x), \phi_n'(x)]-f[x, \phi_0(x), \phi_0'(x)]\}dx\right|$$
$$+\left|\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \phi_0(x), \phi_0'(x)]dx\right|.$$

D'après les inégalités (η) et l'équation (2), chaque terme du second membre sera aussi petit qu'on voudra pour n assez grand. Il en sera de même du premier membre, c'est-à-dire:

$$(A) \qquad \lim_{n\to\infty}\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]dx=0$$

et notre proposition est démontrée.

Considérons maintenant l'ensemble $\{\bar{\lambda}_n\}$ des courbes continues tendant vers une courbe discontinue quelconque $\bar{\mathfrak{C}}$ de la région R_ρ.

D'une façon analogue, on peut démontrer que la condition nécessaire et suffisante pour que les courbes $\{\bar{\lambda}_n\}$ et la courbe correspondante $\bar{\mathfrak{C}}$ soient *admissibles* est la suivante:

$$(\bar{A}) \qquad \lim_{n\to\infty}\int_{\bar{x}_0-\epsilon_n}^{\bar{x}_0+\epsilon_n} f[x, \bar{\omega}_n(x), \bar{\omega}_n'(x)]dx=0,$$

*Hadamard, *Leçons sur le calcul des variations*, p. 49.

où $y=\varpi_n(x)$ est l'équation de la courbe $\bar{\lambda}_n$ et $x=\bar{x}_0$ le point de discontinuité de la courbe $\bar{\mathfrak{C}}$.

4. On voit donc que la question de l'existence des courbes continues admissibles se ramène à celle des courbes discontinues admissibles, c'est-à-dire à l'existence des valeurs de $\bar{x}_0$ pour lesquelles la condition fondamentale $(\bar{\mathrm{A}})$ est satisfaite (*points de discontinuité admissibles*).

Il y a deux cas à distinguer: ou la condition $(\bar{\mathrm{A}})$ peut être satisfaite pour un ensemble continu de valeurs de $\bar{x}_0$, ou bien elle ne l'est que pour des valeurs isolées de $\bar{x}_0$.* Par exemple pour le problème que nous donnons à la fin de notre travail,

$$\int_{x_1}^{x_2} \sin\,(yy')dx,$$

l'abscisse $\bar{x}_0$ du point de discontinuité admissible est absolument arbitraire, tandis que pour le problème de Weierstrass, il n'y a qu'un seul point de discontinuité admissible $\bar{x}_0=0$.

Il y a avantage à exprimer ce raisonnement sous forme géométrique.

Les points de rupture des courbes discontinues admissibles doivent se trouver dans deux cercles de rayon ρ décrit autour des points de rupture R_0, $\bar{R}_0$ de l'extrémale $\mathfrak{E}_0$.

Dans le premier cas (*cas général*), nous allons résoudre le problème posé lorsque chaque couple de points ayant la même abscisse, situés à l'intérieur de ces cercles peut servir de points de rupture d'une courbe discontinue admissible. †

Dans le second cas (*cas exceptionnel*) les points de rupture en restant toujours dans les mêmes cercles sont tous situés sur la droite $x=x_0$ qui passe par les points de rupture $R_0, \bar{R}_0$ de $\mathfrak{E}_0$.

4bis. Il est peu probable qu'on puisse trouver une règle générale permettant de reconnaître dans quels cas la condition $(\bar{\mathrm{A}})$ pourrait être vérifiée. On peut indiquer des règles d'une application de plus en plus étendue mais pas de règle générale.

Voici quelques-unes de ces règles.

La condition $(\bar{\mathrm{A}})$ *est vérifiée pour toutes les valeurs* $\bar{x}_0$ *dans les cas suivants*:

1° *Lorsque la valeur absolue de la fonction f pour toutes les valeurs de x, y, y' de la région R reste toujours inférieure à un nombre M.*

*On peut s'assurer qu'il n'y a que ces deux cas à distinguer, au moins en soumettant la fonction f aux conditions connues de régularité.

†Il y a cependant des cas où les deux points de rupture ne peuvent pas varier indépendamment l'un de l'autre, mais où ils sont liés par une relation. On peut le voir par l'exemple du problème

$$\int_{x_1}^{x_2} f(y,\, y')dx,$$

$f(y, y')$ étant un polynome par rapport à y' et une fonction impaire de y, c'est-à-dire

$$f(-y,\, y') = -f(y,\, y').$$

Cette remarque s'applique aussi au cas exceptionnel.

Tels sont, par exemple, les problèmes:

$$\int_{x_1}^{x_2} \frac{G(x, y)}{(1+y'^2)^k} dx, \qquad \int_{x_1}^{x_2} F(x, y) \sin y' dx,$$

où $G(x, y)$, $F(x, y)$ sont des fonctions bornées dans la région R; k est un nombre positif.

2° *Lorsque en chaque point d'une droite de coefficient angulaire* $\frac{1}{\epsilon}$ *on a*

$$|f| < \frac{M}{\epsilon^k},$$

k étant un nombre positif inférieur à l'unité.

Tel est le cas des problèmes relatifs aux intégrales

$$\int_{x_1}^{x_2} F(x, y, y') \sqrt[3]{1+y'^2}\, dx, \quad \int_{x_1}^{x_2} F(x, y, y')\ (y'^{\frac{2}{3}} - y'^{\frac{2}{5}})\, dx,$$

$F(x, y, y')$ étant une fonction bornée dans la région de régularité R.

3° Dans le cas particulier où

$$f = \phi(x, y)\ (ay'^2 + by' + c),$$

a, b, c étant des constantes, $\phi(x, y)$ étant borné. Pour x constant appartenant à un certain intervalle (ξ_1, ξ_2), $\phi(x, y)$ change de signe au moins deux fois.

C'est le cas du problème relatif á l'intégrale

$$\int_{x_1}^{x_2} (Ay^2 + By + C)(ay'^2 + by' + c) dx,$$

A, B, C étant des constantes. La chute admissible existe lorsque $B^2 - 4AC > 0$; elle n'existe pas lorsque $B^2 - 4AC \leqq 0$.

5. *Conditions fondamentales.* Nous avons dit au N° 2^{bis} qu'une solution discontinue du problème à résoudre se compose de deux arcs de courbes extrémales (courbes d'Euler). Il faut en outre que certaines conditions soient vérifiées aux points de rupture. Pour les obtenir, nous allons montrer qu'il suffit de comparer l'extrémale $\mathfrak{C}_0$ à toute autre courbe *discontinue* admissible de la région R_ρ.

En effet, soit $\{\bar{\lambda}_n\}$ un ensemble de courbes continues admissibles tendant vers une courbe discontinue $\bar{\mathfrak{C}}$ de la région R_ρ. Supposons d'abord que $\bar{\mathfrak{C}}$ est différente de $\mathfrak{C}_0$. En appliquant la definition du minimum, nous devons avoir

$$\Delta J_n = J_{\bar{\lambda}_n} - J_{\mathfrak{C}_0} \geqq 0, \qquad \Delta J = J_{\bar{\mathfrak{C}}} - J_{\mathfrak{C}_0} \geqq 0. \tag{3}$$

Mais d'autre part ΔJ_n peut être écrit ainsi:

$$\Delta J_n = J_{\bar{\mathfrak{C}}} - J_{\mathfrak{C}_0} + (J_{\lambda_n} - J_{\bar{\mathfrak{C}}}).$$

L'expression entre parenthèses tend vers zéro. Par conséquent le signe de ΔJ_n pour n suffisamment grand est déterminé par le signe de la différence $J_{\bar{\mathfrak{C}}} - J_{\mathfrak{C}_0}$, c'est-à-dire, pour n suffisamment grand, la première inégalité (3) est une conséquence de la seconde, et vice versa, la seconde est aussi une conséquence de la première. C.Q.F.D.

Donc au point de vue de la recherche des conditions nécessaires du minimum, le champ des courbes continues admissibles, tant que nous laissons de côté les courbes d'approximation, équivaut à celui des courbes discontinues admissibles.

Comparons maintenant l'intégrale J prise suivant l'extrémale discontinue $\mathfrak{E}_0$ à la même intégrale J prise suivant une courbe discontinue analogue $\overline{\mathfrak{E}}$. Plaçons-nous d'abord dans le cas général. Cette comparaison nous donnera comme condition nécessaire pour le minimum les trois équations suivantes:

$$\text{(1)} \qquad \begin{cases} f(x_0, y_0, y_0') = f(x_0, \bar{y}_0, \bar{y}_0'), \\ f_{y'}(x_0, y_0, y_0') = 0, \\ f_{y'}(x_0, \bar{y}_0, \bar{y}_0') = 0. \end{cases}$$

Dans le cas exceptionel, on aura

$$\text{(1')} \qquad f_{y'}(x_0, y_0, y_0') = 0, \quad f_{y'}(x_0, \bar{y}_0, \bar{y}_0') = 0.$$

C'est la condition fondamentale (A), déterminant l'abscisse x_0 du point de discontinuité admissible, qui remplace la première des trois équations fondamentales (1) dans le cas exceptionnel considéré.

6. Les équations fondamentales (1) ont été obtenues sans tenir compte des courbes d'approximation comme courbes de comparaison.

D'autre part, la courbe $\mathfrak{E}_0$ étant déterminée d'après les conditions initiales et les équations (1), soit $\{\lambda_n\}$ l'ensemble des courbes tendant vers $\mathfrak{E}_0$ et fournissant à l'intégrale J des valeurs aussi proches qu'on le veut de $J_{\mathfrak{E}_0}$. Pour être sûr que $\mathfrak{E}_0$ est une solution discontinue du problème et que par conséquent $\{\lambda_n\}$ est un ensemble de courbes d'approximation, il faut que la courbe $\mathfrak{E}_0$ vérifie en outre la condition du premier ordre obtenue au moyen de l'inégalité

$$\Delta J_n = J_{\lambda_n} - J_{\mathfrak{E}_0} > 0.$$

On peut montrer facilement que cette inégalité entraîne d'abord les mêmes équations (1). Mais en outre elle nous donne quelques conditions supplémentaires. Pour les trouver, développons ΔJ_n par la formule de Taylor suivant les puissances de ϵ_n. Alors, d'après une formule connue du calcul des variations*, et en tenant compte des équations (1), on doit avoir

$$\text{(4)} \qquad \Delta J_n = J_{\lambda_n} - J_{\mathfrak{E}_0} = \int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]\,dx - 2\epsilon_n f(x_0, y_0, y_0') + \epsilon_n(\epsilon_n) > 0,$$

où $y = \omega_n(x)$ est l'équation de la courbe λ_n et (ϵ_n) est une quantité tendant vers zéro avec ϵ_n.

L'inégalité (4) n'est au fond que l'énoncé de la question des conditions nécessaires et suffisantes dans le champ de courbes d'approximation. Mais dans un cas assez étendu, à savoir lorsque la limite de l'expression

$$\lim_{n\to\infty} \frac{1}{2\epsilon_n} \int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]\,dx$$

existe ou si cette limite est infiniment grande, on peut en déduire des conditions

*Bolza, *Vorlesungen über Variationsrechnung* (Leipzig, Teubner, 1909), p. 44, 45.

nécessaires précises. Pour les obtenir, divisons les deux membres de la dernière inégalité par $2\epsilon_n$ et passons ensuite à la limite: on aura la condition suivante

$$\text{(B)} \qquad \lim_{n\to\infty} \frac{1}{2\epsilon_n}\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x,\ \omega_n(x),\ \omega_n'(x)]\,dx \geqq f(x_0,\ y_0,\ y_0').$$

C'est cette condition qui est nécessaire pour le minimum dans le cas considéré.

On peut la préciser encore dans un cas intéressant, à savoir lorsque le problème à résoudre admet des chutes coupant le segment de rupture $R_0\overline{R}_0$ sous un angle arbitraire, le point d'intersection K étant quelconque sur ce segment.* Cela a lieu par exemple pour les problèmes 1°, 2°, 3° du N° 4^{bis}.

Pour obtenir la condition cherchée, prenons le cas du problème 1°:

$$|f| < M.$$

Considérons le champ des valeurs de y, y'

$$\bar{y}_0 \leqq y \leqq y_0, \quad y' \text{ quelconque.}$$

Soient y_k, y_k' des valeurs arbitraires de y, y' dans le champ. Nous allons démontrer que l'inégalité fondamentale (B) entraînera l'inégalité suivante

$$f(x_0,\ y_k,\ y_k') \geqq f(x_0,\ y_0,\ y_0').$$

En effet, menons par le point $K(x_0, y_k)$ une ligne CD de coefficient angulaire y_k'. Soit $\epsilon_n = \delta_n + \delta_n^2$. En introduisant la chute $ACKDB$ ayant les points extrêmes A, B sur les arcs de l'extrémale $\mathfrak{E}_0$, on aura, d'après la formule de la moyenne, l'intégrale

$$\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x,\ \omega_n(x),\ \omega_n'(x)]dx = 2\delta_n f(x_0,\ y_k,\ y_k') + K_0\delta_n^2,$$

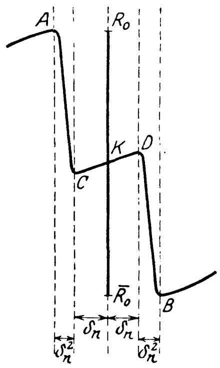

Fig. 2

*Ce cas m'a été signalé par deux lettres de M. Vessiot. La remarque avait été suggérée à M. Vessiot par une note de M. Paul Lévy (Comptes Rendus Acad. Sciences, Paris, 17 novembre 1924).

d'où l'on tire

$$\lim_{n\to\infty} \frac{1}{2\epsilon_n}\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]dx = f(x_0, y_k, y_k'),$$

et par conséquent, la condition (B) nous donne

$$(\mathrm{B}_1) \qquad f(x_0, y_k, y_k') \geqq f(x_0, y_0, y_0').$$

*Donc $f(x_0, y, y')$ doit avoir pour $y=y_0$, $y'=y'_0$ ou pour $y=\bar{y}_0$, $y'=\bar{y}_0'$ la plus petite valeur dans le champ considéré.**

Nous avons supposé $|f|<M$. Mais la démonstration est analogue pour les autres problèmes du cas considéré.

Remarque.—Il est facile de voir que le raisonnement précédent ne s'applique qu'au cas où le point K se trouve sur le segment $R_0\bar{R}_0$. Autrement, les courbes λ_n ne tendent pas vers l'extrémale discontinue $\mathfrak{E}_0$.

Exemple. Problème de Weierstrass:

$$\int_{-1}^{+1} x^2 y'^2 dx.$$

On a ici le cas exceptionnel, parce que la condition (A) n'est satisfaite que pour $\bar{x}_0=0$.

Les conditions fondamentales (I′) pour les solutions discontinues ont pour expression

$$x^2y'=0, \qquad x^2\bar{y}'=0.$$

Donc, l'extrémale discontinue se compose de deux segments de parallèles à Ox: $y=a$, $y=b$, ayant les points de rupture $R_0(0, a)$, $\bar{R}_0(0, b)$.

On peut prendre pour la chute d'une courbe continue admissible la droite joignant deux points $K(-\epsilon, a)$, $\bar{K}(\epsilon, b)$ de l'extrémale discontinue. En effet, l'intégrale J prise le long de cette droite prend la forme

$$\int_{-\epsilon}^{+\epsilon} x^2\left(\frac{b-a}{2\epsilon}\right)^2 dx = \frac{(b-a)^2 . \epsilon}{6},$$

expression qui tend vers zéro.

De plus on peut s'assurer très facilement que la condition (4) ou (B) est satisfaite au sens strict.

7. Nous admettons, d'après ce qui a été dit à la fin du N° 2^{bis}, que les conditions nécessaires du minimum ordinaire sont vérifiées:

1° Les conditions de Legendre et de Jacobi pour les arcs P_1R_0 et $\bar{R}_0P_2$

$$f_{y'y'}>0, \qquad \bar{f}_{y'y'}>0,$$
$$x_0'<x_1<x_0, \qquad x_0<x_2<\bar{x}_0',$$

au sens strict, x_0' et $\bar{x}_0'$ étant les abscisses des foyers R_0' et $\bar{R}_0'$ conjugués des points R_0 et $\bar{R}_0$ sur les arcs P_1R_0 et $\bar{R}_0P_2$ respectivement.

*Cette condition appartient à M. Vessiot. Nous l'appellerons *condition de M. Vessiot.*

2° La condition de Weierstrass pour les mêmes arcs

$$E[x,\ y_0(x),\ y_0'(x),\ \tilde{p}] \geqq 0, \qquad E[x,\ \bar{y}_0(x),\ \bar{y}_0'(x),\ \tilde{p}] \geqq 0.$$

Posons dans ces inégalités $x = x_0$. Alors, d'après les deux dernières équations (I), on aura

$$f(x_0,\ y_0,\ \tilde{p}) \geqq f(x_0,\ y_0,\ y_0'), \qquad f(x_0,\ \bar{y}_0,\ \tilde{p}) \geqq f(x_0,\ \bar{y}_0,\ \bar{y}_0').$$

Donc pour le minimum il est nécessaire que les fonctions $f(x_0, y_0, \tilde{p}), f(x_0, \bar{y}_0, \tilde{p})$ aient des minima absolus pour les valeurs $\tilde{p} = y'_0$, $\tilde{p} = \bar{y}_0'$, c'est-à-dire que nous avons retrouvé la condition (B_1) pour deux valeurs particulières de y_k, savoir: $y_k = y_0, y_k = \bar{y}_0$.

8. *Deux champs de courbes continues.* D'après ce qui précède, nous sommes amenés à distinguer deux champs de courbes continues admissibles de comparaison, outre celui des courbes discontinues analogues:

1° le champ des courbes continues tendant vers les courbes discontinues de comparaison;

2° le champ des courbes continues tendant vers l'extrémale discontinue (courbes d'approximation).

Désignons ces deux champs par $\mathfrak{F}_1$, $\mathfrak{F}_2$ respectivement, et celui des courbes discontinues analogues par $\mathfrak{F}_0$.

Au point de vue des conditions qu'on obtient en *annulant* la variation première, les deux champs $\mathfrak{F}_1$, $\mathfrak{F}_2$ nous ont donné les mêmes résultats. Ce sont les équations fondamentales (I).

Mais le champ $\mathfrak{F}_2$ nous a fourni d'autres conditions encore, à savoir l'inégalité (4).

On voit donc que le rôle de ces deux champs dans la résolution du problème posé est analogue à celui du champ de la variation faible et du champ de la variation forte pour le problème ordinaire.

Ajoutons maintenant aux champs $\mathfrak{F}_1$, $\mathfrak{F}_2$ le champ $\mathfrak{F}_0$. Alors on peut exprimer ce qui précède de la manière suivante:

Pour que le minimum soit réalisé pour le champ $\mathfrak{F}_1 + \mathfrak{F}_0$ il faut qu'il en soit de même pour le champ $\mathfrak{F}_1 + \mathfrak{F}_0$.

Dans ce qui va suivre, nous avons tout d'abord à nous occuper des conditions du minimum dans ce dernier champ.

B. Les autres conditions nécessaires du minimum. Signification des conditions fondamentales (I).

9. Prolongeons les extrémales P_1R_0 et $\bar{R}_0P_2$, la première à droite du point R_0 et l'autre à gauche du point $\bar{R}_0$. Soit X et $\bar{X}$ deux points sur ces courbes et x leur abscisse commune, qui satisfait à l'inégalité

$$|x - x_0| < \rho.$$

Nous pouvons considérer la courbe $P_1X\bar{X}P_2$ (Fig. 3) comme une courbe de comparaison, et cette comparaison nous donnera la condition suivante:

(II) $$f_x + y_0' f_y - \bar{f}_x - \bar{y}_0' \bar{f}_y \geqq 0,$$

à laquelle doit satisfaire l'extrémale discontinue aux points de rupture.

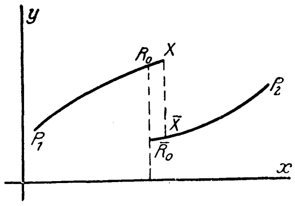

Fig. 3

10. Les deux dernières conditions (I), ou bien les conditions (I′) du cas exceptionnel

$$f_{y'} = 0, \qquad \bar{f}_{y'} = 0,$$

nous montrent que la ligne $R_0\bar{R}_0$ coupe transversalement, d'une part la branche P_1R_0, et d'autre part la branche $\bar{R}_0P_2$ de l'extrémale $\mathfrak{E}_0$. Mais la ligne $R_0\bar{R}_0$ n'est pas tangente aux courbes P_1R_0 et $\bar{R}_0P_2$ et par conséquent en vertu d'un théorème connu sur les transversales,* on peut mener par tout point M de cette ligne voisin de R_0 une seule extrémale voisine de P_1R_0, qui est transversale en M à la ligne $R_0\bar{R}_0$. L'ensemble de ces dernières extrémales constitue avec l'extrémale donnée P_1R_0 un faisceau à un seul paramètre.

Soit $\mathfrak{X}_0$ le foyer de ce faisceau sur la branche P_1R_0, et $x = \mathfrak{x}_0$ l'abscisse correspondante. Alors, en vertu du théorème connu sur les foyers, nous avons la condition suivante, nécessaire pour le minimum

(5) $$x_1 \geqq \mathfrak{x}_0.$$

En raisonnant comme tout à l'heure, nous voyons que si $\bar{\mathfrak{X}}_0$ est le foyer de la ligne $R_0\bar{R}_0$ sur la branche $\bar{R}_0P_2$ et $\bar{\mathfrak{x}}_0$, l'abscisse correspondante, l'abscisse x_2 du point P_2 doit vérifier l'inégalité

($\bar{5}$) $$x_2 \leqq \bar{\mathfrak{x}}_0.$$

En désignant maintenant par $\Delta(x_0, x)$ une intégrale de l'équation de Jacobi admettant le zéro x_0, on peut montrer facilement que la fonction

$$f_{y'y}(x_0, y_0, y_0') + f_{y'y'}(x_0, y_0, y_0') \frac{\Delta_x(x_0, x)}{\Delta\ (x_0, x)}$$

conserve un signe constant dans l'intervalle $\mathfrak{x}_0 \leqq x \leqq x_0$ et qu'il en est de même pour la fonction

$$f_{y'y}(x_0, \bar{y}_0, \bar{y}_0') + f_{y'y'}(x_0, \bar{y}_0, \bar{y}_0') \frac{\bar{\Delta}_x(x_0, x)}{\bar{\Delta}\ (x_0, x)}$$

dans l'intervalle $x_0 \leqq x \leqq \bar{\mathfrak{x}}_0$. Ce résultat va jouer un rôle important dans la théorie.

*Bolza, *Vorlesungen*, p. 322.

11. Les points $\mathfrak{X}_0$, $\bar{\mathfrak{X}}_0$ constituent le seul couple de foyers de l'extrémale discontinue $\mathfrak{E}_0$ dans le cas exceptionnel.

Donc, les raisonnements du N° 10 nous ont donné, d'une part les deux conditions nécessaires du minimum pour les extrémités P_1, P_2 dans le cas général, et d'autre part la théorie complète des foyers pour le cas exceptionnel.

On peut maintenant trouver des conditions suffisantes dans le cas exceptionnel. En effet, lorsque les conditions (5) et $(\bar{5})$ sont vérifiées au sens strict, l'extrémale discontinue $\mathfrak{E}_0$ peut être toujours entourée par un champ d'extrémales discontinues ayant les points de rupture sur la droite $x=x_0$. Les deux branches de ces extrémales ne sont pas liées entre elles. Ce sont deux groupes différents d'extrémales continues coupant transversalement la droite $x=x_0$. Par suite, le théorème de Weierstrass sera exact et nous pouvons écrire

$$\Delta J=\int_{x_1}^{x_2} E[x,\ \bar{y},\ p(x,\ \bar{y}),\ \bar{y}']dx.$$

Donc la *condition suffisante dans le cas exceptionnel* s'exprimera ainsi:
ΔJ *est essentiellement positif lorsque*

$$E[x,\ y,\ p(x,\ y),\ \tilde{p}]>0$$

pour tout point du champ et pour toute valeur de $\tilde{p}$ *distincte de* p.

Le problème du minimum dans le champ $\mathfrak{F}_0$ pour le cas exceptionnel est donc résolu. Nous nous placerons désormais dans le cas général.

C. Théorie des foyers.

12. La question se pose tout d'abord: *Sous quelle condition un nouveau couple de points de rupture* $R(x,\ y)$, $\bar{R}(x,\ \bar{y})$ *voisins des points* $R_0(x_0,\ y_0)$, $\bar{R}_0(x_0,\ \bar{y}_0)$ *peut-il servir de points de rupture d'une solution discontinue?*

En ces deux points les équations simultanées suivantes doivent être vérifiées:

$$(1'')\qquad \begin{cases} f(x,\ \bar{y},\ \bar{y}')-f(x,\ y,\ y')=0,\\ f_{y'}(x,\ y,\ y')=0,\\ f_{y'}(x,\ \bar{y},\ \bar{y}')=0,\end{cases}$$

où y', $\bar{y}'$ est le couple des directions nouvelles.

Nous avons donc un système de trois équations à cinq variables, par conséquent, parmi ces variables, il y en a deux qui sont arbitraires. Supposons que ce soient x, y, coordonnées du point R. Ces équations sont vérifiées par le système des valeurs:

$$x=x_0,\qquad y=y_0,\qquad \bar{y}=\bar{y}_0,\qquad y'=y_0',\qquad \bar{y}'=\bar{y}_0'.$$

par suite, elles feront connaître le second point $\bar{R}$ de rupture et le couple des directions nouvelles, si le jacobien

$$J=\frac{\partial(\bar{f}-f,\ f_{y'},\ \bar{f}_{y'})}{\partial(\bar{y},\ y',\ \bar{y}')}$$

n'est pas nul pour le système précédent. En désignant par J_0 la valeur de J pour ce système, on aura

$$J_0=f_y(x_0,\ \bar{y}_0,\ \bar{y}_0')f_{y'y'}\bar{f}_{y'y'}.$$

En tenant compte de la condition de Legendre nous obtenons le résultat suivant:

Pour que la détermination de $\bar{y}$, y', $\bar{y}'$, *ne soit possible que d'une seule manière il faut que*

$$f_y(x_0, \bar{y}_0, \bar{y}_0') \neq 0.$$

Si nous prenons arbitrairement les coordonnées du point $\bar{R}$ nous obtiendrons la condition analogue

$$f_y(x_0, y_0, y_0') \neq 0.$$

Or nous verrons plus tard, qu'au point de vue du minimum, ces deux conditions sont identiques et qu'elles sont étroitement liées à la condition (II).

Prenons maintenant le cas contraire où les équations (I″) entraînent

$$(6) \qquad (f_y)_0 = 0, \qquad (\bar{f}_y)_0 = 0.$$

Supposons en général que le couple R, $\bar{R}$ se déplace à partir de sa position nitiale R_0, $\bar{R}_0$. Alors, de la première équation (I″), on tire

$$(\bar{f}_x)_0 + (\bar{f}_y)_0 \frac{\delta \bar{y}}{\delta x} - (f_x)_0 - (f_y)_0 \frac{\delta y}{\delta x} = 0.$$

Dans le cas considéré cette formule se réduit à

$$f_x(x_0, \bar{y}_0, \bar{y}_0') - f_x(x_0, y_0, y_0') = 0.$$

C'est la condition nécessaire pour que, dans le cas (6), les points de rupture R, $\bar{R}$ voisins de R_0, $\bar{R}_0$ et le couple des nouvelles directions existent. Mais, bien entendu, ici, la correspondance entre les points associés R, $\bar{R}$ n'est pas univoque, et même, elle peut être arbitraire.

Nous laissons de côte le cas (6) et nous supposerons désormais

$$f_y(x_0, \bar{y}_0, \bar{y}_0') \neq 0.$$

Revenons alors aux équations (I″). Soient

$$(7) \qquad \bar{y} = F(x, y), \qquad y' = p(x, y), \qquad \bar{y}' = \bar{p}(x, y)$$

leurs solutions. Ces trois fonctions sont uniformes et continues ainsi que leurs dérivées partielles du premier ordre tant que le point R reste dans un petit cercle décrit autour du point R_0. Pour $x = x_0$, $y = y_0$, on aura

$$\bar{y}_0 = F(x_0, y_0), \qquad y'_0 = p(x_0, y_0), \qquad \bar{y}_0' = \bar{p}(x_0, y_0).$$

Par conséquent lorsque le point R décrit une courbe Γ passant par le point R_0*, le point correspondant $\bar{R}$ décrit une autre courbe $\bar{\Gamma}$ passant par le point $\bar{R}_0$.

13. Soient comme toujours P_1, P_2 les extrémités de l'extrémale discontinue $\mathfrak{E}_0$, distinctes des deux points $\mathfrak{X}_0$, $\bar{\mathfrak{X}}_0$ du N°11. Entourons maintenant la branche P_1R_0 par le faisceau des extrémales issues du point P_1. Soit

$$(\alpha) \qquad y = y(x, \alpha)$$

l'équation de ce faisceau et α_0 la valeur de α pour la branche P_1R_0.

*Cette courbe peut être arbitraire parce que la condition de Legendre est vérifiée au sen strict.

Le lieu géométrique des points de rupture R sur le faisceau (α) se détermine à l'aide de l'équation

$$f_{y'}[x, y(x, \alpha), y'(x, \alpha)]=0.$$

Cette équation est vérifiée par le système des valeurs

$$x=x_0, \quad \alpha=\alpha_0.$$

D'après la condition (5), nous pouvons résoudre cette équation par rapport à α dans le domaine de $x=x_0$, $\alpha=\alpha_0$. Désignons cette solution par

$$\alpha=\alpha(x).$$

Portons cette valeur dans la formule (α); nous aurons l'équation du lieu géométrique dont nous avons parlé plus haut

$$\Gamma \qquad y=y[x, \alpha(x)]=\gamma(x).$$

D'après ce qui précède cette courbe est distincte de la droite $x=x_0$ et par conséquent, la courbe correspondante $\overline{\Gamma}$ que nous allons obtenir est aussi distincte de la même droite*.

En portant cette valeur de y dans la première formule (7), on aura l'équation de la courbe $\overline{\Gamma}$ que décrira le point correspondant $\overline{R}$,

$$\overline{\Gamma} \qquad \bar{y}=F[x, \gamma(x)]=\bar{\gamma}(x).$$

Le couple des nouvelles directions se détermine à l'aide des équations

$$y'=p[x, \gamma(x)], \qquad \bar{y}'=\bar{p}[x, \bar{\gamma}(x)].$$

Menons maintenant par les points de Γ les courbes extrémales de coefficients angulaires $\bar{p}[x, \bar{\gamma}(x)]$. L'ensemble de ces extrémales constitue avec l'arc $\overline{R}_0P_2$ un faisceau à un seul paramètre. Soit

$$(\bar{\alpha}) \qquad y=\bar{y}(x, \bar{\alpha})$$

l'équation de ce faisceau et

$$\bar{\alpha}=\bar{\alpha}(x)$$

la valeur de $\bar{\alpha}$ le long de la courbe $\overline{\Gamma}$. Alors, nous avons les identités

$$\bar{y}=\bar{y}[x, \bar{\alpha}(x)]=\bar{\gamma}(x), \qquad \bar{y}'=\bar{p}[x, \bar{\gamma}(x)]=\bar{y}_x[x, \bar{\alpha}(x)].$$

Toutes ces valeurs de y, $\bar{y}$, y' $\bar{y}'$ en fonction de x vérifient identiquement les équations (1'').

On aura donc, pour les courbes Γ, $\overline{\Gamma}$ ainsi construites, les trois identités suivantes

$$(8) \qquad \begin{cases} f\{x, y[x, \alpha(x)], y_x[x, \alpha(x)]\} - f\ \{x, \bar{y}[x, \bar{\alpha}(x)], \bar{y}_x[x, \bar{\alpha}(x)]\} = 0, \\ \qquad f_{y'}\{x, y[x, \alpha(x)], y_x[x, \alpha(x)]\} = 0, \\ \qquad f_{y'}\{x, \bar{y}[x, \bar{\alpha}(x)], \bar{y}_x[x, \bar{\alpha}(x)]\} = 0. \end{cases}$$

*La coïncidence n'a lieu que lorsque le faisceau (α) est issu du point $\mathfrak{X}_0$.

Appelons *courbes conjuguées* les courbes Γ, $\bar{\Gamma}$ pour lesquelles ces trois équations sont toutes vérifiées. Le foyer du faisceau (α) sur l'arc P_1R_0 étant $P_1(x_1)$, désignons le foyer du faisceau correspondant $(\bar{\alpha})$ sur l'arc $\bar{R}_0P_2$ par $P_1^*(x_1^*)$. Ces deux points correspondant aux courbes conjuguées Γ et $\bar{\Gamma}$ seront euxmêmes dits *conjugués*.

Il s'agit tout d'abord de trouver la relation qui lie P_1 et P_1^*.

14. Supposons donc que les courbes Γ et $\bar{\Gamma}$ soient conjuguées. Alors les fonctions correspondantes $\alpha(x)$ et $\bar{\alpha}(x)$ satisfont identiquement aux trois équations (8). Mais dans ce cas le déterminant fonctionnel

$$\frac{\partial(f-\bar{f}, f_{y'}, \bar{f}_{y'})}{\partial(x, \alpha, \bar{\alpha})}$$

s'annule aussi identiquement pour $\alpha=\alpha(x)$, $\bar{\alpha}=\bar{\alpha}(x)$, c'est-à-dire que

$$\begin{vmatrix} \frac{\partial}{\partial x}(f-\bar{f}), & \frac{\partial}{\partial \alpha}(f-\bar{f}), & \frac{\partial}{\partial \bar{\alpha}}(f-\bar{f}) \\ \frac{\partial}{\partial x}f_{y'} & , \frac{\partial}{\partial \alpha}f_{y'} & , \frac{\partial}{\partial \bar{\alpha}}f_{y'} \\ \frac{\partial}{\partial x}\bar{f}_{y} & , \frac{\partial}{\partial \alpha}\bar{f}_{y'} & , \frac{\partial}{\partial \bar{\alpha}}\bar{f}_{y'} \end{vmatrix} = 0, \ [\alpha=\alpha(x), \bar{\alpha}=\bar{\alpha}(x)].$$

En calculant les dérivées dans cette relation et en supposant ensuite $x=x_0$, $\alpha=\alpha_0$, $\bar{\alpha}=\bar{\alpha}_0$, il en résulte

$$(9) \quad \begin{cases} \dfrac{[(f_x+y_0'f_y)f_{y'y}-f_y^2]\Delta(x_0, x_1)+(f_x+y_0'f_y)f_{y'y'}\Delta_x(x_0, x_1)}{f_{y'y}\,\Delta(x_0, x_1)+f_{y'y'}\,\Delta_x\,(x_0, x_1)} \\ = \dfrac{[(\bar{f}_x+\bar{y}_0'\bar{f}_y)\bar{f}_{y'y}-\bar{f}_y^2]\bar{\Delta}(x_0,x_1^*)+(\bar{f}_x+\bar{y}_0'\bar{f}_y)\bar{f}_{y'y'}\,\bar{\Delta}_x(x_0,x_1^*)}{\bar{f}_{y'y}\,\bar{\Delta}(x_0, x_1^*)+\bar{f}_{y'y'}\,\bar{\Delta}_x(x_0, x_1^*)}. \end{cases}$$

C'est la relation cherchée qui lie les abscisses x_1 et x_1^* des points conjugués.

Nous pouvons déduire de l'équation précédente des résultats intéressants. Le premier membre de l'équation (9) est constant lorsque $f_y(x_0, y_0, y_0')=0$, et par conséquent si cette équation est vérifiée, le second membre est aussi constant, ce qui n'est possible en général que lorsque $f_y(x_0, \bar{y}_0, \bar{y}_0')=0$. Moyennant ces hypothèses, l'équation (9) se réduit à l'équation

$$f_x(x_0, y_0, y_0')-f_x(x_0, \bar{y}_0, \bar{y}_0')=0,$$

déjà trouvée au N° 12. Elle exprime, comme nous l'avons vu, la condition nécessaire pour que les solutions discontinues voisines soient possibles dans ce cas. D'autre part elle ne contient ni x_1, ni x_1^*. Par conséquent dans le cas considéré $[f_y(x_0, y_0, y_0')=0]$, les deux points arbitraires de l'extrémale $\mathfrak{E}_0$ peuvent être considérés comme conjugués. Or nous avons laissé ce cas de côté et nous supposerons désormais comme toujours qu'on a

$$f_y(x_0, y_0, y_0')\neq 0, \quad \bar{f}_y(x_0, \bar{y}_0, \bar{y}_0')\neq 0.$$

15. Désignons les deux membres de l'équation (9) par $T(x_1)$ et $\overline{T}(x_1^*)$. Par un calcul facile, on peut montrer que T et $\overline{T}$ sont des fonctions croissantes de ces arguments. Par conséquent x_1^* est une fonction croissante de x_1. Posons

$$x_1^* = \Phi(x_1).$$

Pour les valeurs x_0' et x_0, qui correspondent aux points R_0' et R_0, la fonction $T(x_1)$ prend la même valeur $f_x + y_0' f_y$. Quant à la fonction $\overline{T}(x_1^*)$, elle prend la même valeur $\bar{f}_x + \bar{y}_0' \bar{f}_y$ pour les valeurs x_0 et $\bar{x}_0'$ de x_2 correspondant aux points $\overline{R}_0$, $\overline{R}_0'$.

Soit $H_0(h_0)$ le point de l'arc $R_0'R_0$ pour lequel $T = \bar{f}_x + \bar{y}_0' \bar{f}_y$; alors les points H_0 et $\overline{R}_0$ sont conjugués. Soit $\overline{H}_0(\bar{h}_0)$ le point de l'arc $\overline{R}_0\overline{R}_0'$ pour lequel $\overline{T} = f_x + y_0' f_y$; alors les points $\overline{H}_0(h_0)$ et R_0 sont aussi conjugués. Mais x_1^* est une fonction croissante de x_1, par conséquent:

$$T(x_0) = \overline{T}(\bar{h}_0 - 0),$$

$$T(x_0') = \overline{T}(\bar{h}_0 + 0).$$

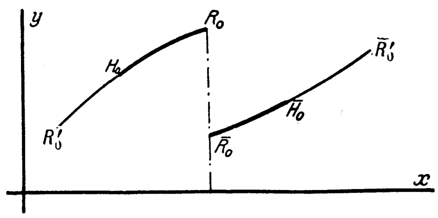

Fig. 4

Il en est de même pour la fonction $\overline{T}(x_1^*)$:

$$\overline{T}(x_0) = T(h_0 + 0),$$

$$\overline{T}(\bar{x}_0') = T(h_0 - 0).$$

De ce qui précède on conclut que si le point P_1 se déplace sur l'arc $R_0'H_0$, son foyer conjugué se déplace sur l'arc $\overline{H}_0\overline{R}_0'$ et que si le point P_1 décrit l'arc H_0R_0, son foyer conjugué décrit l'arc $\overline{R}_0\overline{H}_0$ (Fig. 4).

On voit donc que la fonction $\Phi(x_1)$ est discontinue au point h_0.

16. Supposons maintenant que le point P_2 coïncide avec le point P_1^* conjugué du point initial P_1. On peut alors démontrer par un procédé connu que *la solution discontinue* $\mathfrak{C}_0$ *aux extrémités* P_1, P_1^*, *ne peut pas fournir un minimum strict de l'intégrale* J. Par conséquent *la condition nécessaire de minimum est*

(III) $$x_2 \leqq x_1^*.$$

D'après cela et d'après ce qui a été dit au N° 14 on peut s'assurer très facilement que, dans le cas $f_y = \bar{f}_y = 0$, l'intégrale J ne peut pas avoir, en général, de minimum strict.

De plus, on en déduit la conséquence suivante:

Si l'origine P_1 *se trouve entre* R_0' *et* H_0, *l'intégrale* J *n'a pas de minimum.*

En effet, supposons d'abord que P_2 soit différent de $\overline{R}_0$. Alors tout point P de l'arc H_0R_0, suffisamment voisin de H_0, a son point conjugué sur $\overline{R}_0\overline{H}_0$ voisin de $\overline{R}_0$. Nous aurons donc au moins deux points conjugués situés entre les extrémités P_1 et P_2 de l'extrémale $\mathfrak{C}_0$ et par conséquent on peut démontrer qu'il existe des chemins donnant à l'intégrale J une valeur plus petite que $J_{\mathfrak{C}_0}$. On arrive bien simplement aux mêmes conclusions si le point P_2 coïncide avec $\overline{R}_0$, parce que nous aurons de même le point H_0, dont le point conjugué est P_2 entre les extrémités P_1 et P_2.

D'une manière générale, ce que nous venons de dire nous montre que dans le cas $x_1 < h_0$, il n'y a pas de minimum.

Nous pouvons par suite énoncer le théorème suivant:

Pour que la courbe $\mathfrak{C}_0$ rende l'intégrale J minimum, il est nécessaire que

$$x_1 \geqq h_0. \tag{IV}$$

Mais, d'après ce qui a été dit au N° 10, l'abscisse x_1 du point P_1 doit de plus vérifier l'inégalité

$$x_1 \geqq \mathfrak{x}_0. \tag{10}$$

On peut démontrer maintenant que ces deux inégalités ne sont pas contradictoires. En effet, de l'équation (9), on déduit

$$(f_x + y_0' f_y - \bar{f}_x - \bar{y}_0' \bar{f}_y) \left[f_{y'y} + f_{y'y'} \frac{\Delta_x(x_0, h_0)}{\Delta(x_0, h_0)} \right] = f_y^2.$$

Mais $f_y \neq 0$. D'où l'on voit, d'après les raisonnements du N° 10, que l'inégalité (10) est une conséquence de la condition (IV).

II. Conditions suffisantes.

A. Cas des courbes discontinues de comparaison.

17. On peut s'assurer très facilement que la méthode de Weierstrass ne nous donnerait les conditions suffisantes que dans le cas très particulier où les points de rupture de la courbe de comparaison discontinue se trouvent sur les courbes conjuguées Γ, $\overline{\Gamma}$. C'est pourquoi nous les chercherons par une méthode plus puissante, à savoir en recourant aux dérivées du second ordre des fonctions W et $\overline{W}$.*

Supposons toujours que toutes les conditions du minimum ordinaire soient vérifiées au sens strict. Construisons deux faisceaux spéciaux: l'un composé des extrémales ordinaires issues du point P_1 et entourant la branche P_1R_0 de l'extrémale $\mathfrak{C}_0$, et l'autre composé des extrémales ordinaises passant par le point P_2 et entourant la branche R_0P_2. Désignons ces deux faisceaux par $F(P_1)$, $\overline{F}(P_2)$ respectivement. Chaque point du cercle R_0 est joint à P_1 par une extrémale du premier faisceau et de même chaque point du cercle $\overline{R}_0$ est joint à P_2 par une extrémale du second faisceau (Fig. 5).

Considérons un couple $K(x, y)$, $K(x, \bar{y})$ de points de ces deux cercles ayant la même abscisse x. La courbe discontinue $\mathfrak{C}$ formée de deux extrémales P_1K, $\overline{K}P_2$ de nos faisceaux donne à l'intégrale J une valeur plus petite que pour toute

*Voir Dresden, Trans. Amer. Math. Soc. 1908.

autre courbe discontinue admissible ayant les mêmes points de rupture K, $\overline{K}$. Par conséquent, pour trouver les conditions suffisantes, il suffit de comparer l'intégrale J prise suivant la courbe $\mathfrak{C}$, à la même intégrale prise suivant l'extrémale discontinue $\mathfrak{C}_0$.

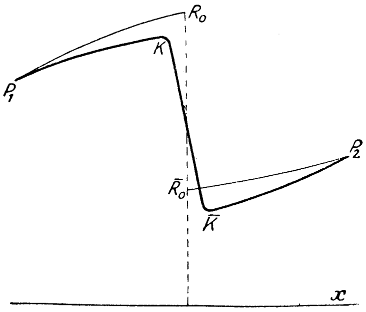

Fig. 5

Mais l'intégrale J prise suivant la courbe $\mathfrak{C}$ est la somme de deux fonctions W et $\overline{W}$ de Hamilton,

$$J_{\mathfrak{C}} = W(x, y) + \overline{W}(x, \bar{y}).$$

Par conséquent, on doit avoir

$$\Delta J = J_{\mathfrak{C}} - J_{\mathfrak{C}_0} = W(x, y) + \overline{W}(x, \bar{y}) - W(x_0, y_0) - \overline{W}(x_0, \bar{y}_0) > 0$$

pour toutes les valeurs de x, y, $\bar{y}$ suffisamment voisines de x_0, y_0, $\bar{y}_0$. Posons

$$x = x_0 + \alpha, \qquad y = y_0 + \beta, \qquad \bar{y} = \bar{y}_0 + \bar{\beta}.$$

En développant la somme $W + \overline{W}$ suivant les puissances de α, β, $\bar{\beta}$, on aura

$$\Delta J = \alpha \frac{\partial W}{\partial x_0} + \beta \frac{\partial W}{\partial y_0} + \alpha \frac{\partial \overline{W}}{\partial x_0} + \bar{\beta} \frac{\partial \overline{W}}{\partial \bar{y}_0}$$

$$+ \frac{1}{2}\left(\alpha^2 \frac{\partial^2 W}{\partial x_0^2} + 2\alpha\beta \frac{\partial^2 W}{\partial x_0 \partial y_0} + \beta^2 \frac{\partial^2 W}{\partial y_0^2} + \alpha^2 \frac{\partial^2 \overline{W}}{\partial x_0^2} + 2\alpha\bar{\beta} \frac{\partial^2 \overline{W}}{\partial x_0 \partial \bar{y}_0} + \bar{\beta}^2 \frac{\partial^2 \overline{W}}{\partial \bar{y}_0^2}\right) + (\alpha, \beta, \bar{\beta})_3,$$

où nous désignons par $(\alpha, \beta, \bar{\beta})_3$ l'ensemble des termes du troisième ordre.

Posons une fois pour toutes:

$$\Theta(\alpha, \beta, \bar{\beta}) = \alpha^2 \frac{\partial^2 W}{\partial x_0^2} + 2\alpha\beta \frac{\partial^2 W}{\partial x_0 \partial y_0} + \beta^2 \frac{\partial^2 W}{\partial y_0^2} + \alpha^2 \frac{\partial^2 \overline{W}}{\partial x_0^2} + 2\alpha\bar{\beta} \frac{\partial^2 \overline{W}}{\partial x_0 \partial \bar{y}_0} + \bar{\beta}^2 \frac{\partial^2 \overline{W}}{\partial \bar{y}_0^2}.$$

En tenant compte des formules des dérivées du premier ordre et des conditions fondamentales (I), on peut montrer que les termes du premier ordre dans le développement de ΔJ disparaissent et par conséquent on peut écrire

$$\Delta J = \frac{1}{2}\Theta(\alpha, \beta, \bar{\beta}) + (\alpha, \beta, \bar{\beta})_3. \tag{11}$$

18. Nous allons maintenant démontrer le théorème suivant:

Théorème.—*Si les conditions nécessaires* (II), (III), (IV) *sont vérifiées au sens strict, on aura toujours*

$$\Theta(\alpha, \beta, \bar{\beta}) > 0$$

pour tout système $(\alpha, \beta, \bar{\beta})$, *sauf bien entendu* $(0, 0, 0)$.

Pour avoir une démonstration générale il suffit de supposer que $\alpha \neq 0$. En effet, admettons que α soit nul. Alors

$$\Theta(0, \beta, \bar{\beta}) = \beta^2 \frac{\partial^2 W}{\partial y_0^2} + \bar{\beta}^2 \frac{\partial^2 \overline{W}}{\partial \bar{y}_0^2}.$$

Mais les formules des dérivées du second ordre de W et $\overline{W}$ nous donnent

$$\frac{\partial^2 W}{\partial \bar{y}_0^2} = f_{y'y} + f_{y'y'} \frac{\Delta_x(x_0, x_1)}{\Delta(x_0, x_1)}, \quad \frac{\partial^2 \overline{W}}{\partial \bar{y}_0^2} = -\left[\bar{f}_{y'y} + \bar{f}_{y'y'} \frac{\overline{\Delta}_x(x_0, x_2)}{\overline{\Delta}(x_0, x_2)} \right].$$

En tenant compte du raisonnement du N° 10 et des inégalités (5), ($\bar{5}$), on en déduit

$$(12) \qquad \frac{\partial^2 W}{\partial y_0^2} > 0, \qquad \frac{\partial^2 \overline{W}}{\partial \bar{y}_0^2} > 0.$$

D'autre part, le système $(\beta, \bar{\beta})$ est différent de $(0, 0)$; par conséquent on aura toujours

$$\Theta(0, \beta, \bar{\beta}) > 0.$$

Supposons donc $\alpha \neq 0$. La fonction Θ est la somme des deux formes quadratiques. D'après les inégalités (12), on peut la mettre sous la forme suivante:

$$(13) \qquad \Theta(\alpha, \beta, \bar{\beta}) = \frac{\left(\dfrac{\partial^2 W}{\partial x_0 \partial y_0} \alpha + \dfrac{\partial^2 W}{\partial y_0^2} \beta \right)^2}{\dfrac{\partial^2 W}{\partial y_0^2}} + \frac{\left(\dfrac{\partial^2 \overline{W}}{\partial x_0 \partial \bar{y}_0} \alpha + \dfrac{\partial^2 \overline{W}}{\partial \bar{y}_0^2} \bar{\beta} \right)^2}{\dfrac{\partial^2 \overline{W}}{\partial \bar{y}_0^2}}$$

$$+ \left[\frac{\dfrac{\partial^2 W}{\partial x_0^2} \dfrac{\partial^2 W}{\partial y_0^2} - \left(\dfrac{\partial^2 W}{\partial x_0 \partial y_0} \right)^2}{\dfrac{\partial^2 W}{\partial y_0^2}} + \frac{\dfrac{\partial^2 \overline{W}}{\partial x_0^2} \dfrac{\partial^2 \overline{W}}{\partial \bar{y}_0^2} - \left(\dfrac{\partial^2 \overline{W}}{\partial x_0 \partial \bar{y}_0} \right)^2}{\dfrac{\partial^2 \overline{W}}{\partial \bar{y}_0^2}} \right] \alpha^2.$$

Considérons la somme entre crochets. En portant les valeurs des dérivées secondes dans cette somme, on aura, d'après les notations du N° 15:

$$\frac{\dfrac{\partial^2 W}{\partial x_0^2} \dfrac{\partial^2 W}{\partial y_0^2} - \left(\dfrac{\partial^2 W}{\partial x_0 \partial y_0} \right)^2}{\dfrac{\partial^2 W}{\partial y_0^2}} + \frac{\dfrac{\partial^2 \overline{W}}{\partial x_0^2} \dfrac{\partial^2 \overline{W}}{\partial \bar{y}_0^2} - \left(\dfrac{\partial^2 \overline{W}}{\partial x_0 \partial \bar{y}_0} \right)^2}{\dfrac{\partial^2 \overline{W}}{\partial \bar{y}_0^2}} = T(x_1) - \overline{T}(x_2).$$

Il est facile de voir que la quantité $T(x_1) - \overline{T}(x_2)$ reste toujours positive lorsque les abscisses des points extrêmes P_1, P_2 vérifient les conditions (III) et (IV) au sens strict:

$$x_2 < x_1^*, \qquad x_1 > h_0.$$

Donc nous avons bien

$$T(x_1) - \overline{T}(x_2) > 0.$$

Revenons maintenant à l'équation (13). La somme des deux premiers termes n'est jamais négative. Appelons cette somme K^2. Donc on aura

$$\Theta(\alpha,\ \beta,\ \bar{\beta})=[T(x_1)-\overline{T}(x_2)]\alpha^2+K^2>0$$

pour tout système $(\alpha,\ \beta,\ \bar{\beta})\neq(0,\ 0,\ 0)$, de sorte que notre théorème est démontré.

Reprenons l'équation (11). On peut supposer α, β, $\bar{\beta}$ suffisamment petits pour que l'inégalité

$$\Delta J=J_{\mathfrak{C}}-J_{\mathfrak{C}_0}=\frac{1}{2}\Theta(\alpha,\ \beta,\ \bar{\beta})+(\alpha,\ \beta,\ \bar{\beta})_3>0$$

soit vérifiée au sens strict.

Tout ce qui précède peut donc se résumer dans l'énoncé suivant:

L'extrémale discontinue $\mathfrak{C}_0$ fournit à l'intégrale J un minimum fort dans le champ $\mathfrak{F}_0$ des courbes discontinues de comparaison si toutes les conditions du minimum ordinaire et les conditions (I), (II), (III), (IV), *sont vérifiées (les trois dernières au sens strict).*

19. Nous pouvons maintenant obtenir une limite inférieure de la différence

$$\Delta J=J_{\mathfrak{C}}-J_{\mathfrak{C}_0}.$$

Cette expression peut s'écrire, en appliquant la formule générale de Taylor, et poussant le développement jusqu'aux termes du second ordre:

$$\Delta J = \frac{1}{2}\,\tilde{\Theta}(\alpha,\beta,\bar{\beta}) = \frac{1}{2}\,\{[\tilde{T}(x_1)-\tilde{\overline{T}}(x_2)]\alpha^2+\tilde{K}^2\},$$

où

$$\tilde{T}(x_1)-\tilde{\overline{T}}(x_2)=[T(x_1)-\overline{T}(x_2)]_{\substack{x_0+\theta\alpha,\\ y_0+\theta'\beta,\\ \bar{y}_0+\theta''\bar{\beta}}},\qquad \tilde{K}=[K]_{\substack{x_0+\theta\alpha,\\ y_0+\theta'\beta,\\ \bar{y}_0+\theta''\bar{\beta}}}$$

c'est-à-dire que x_0, y_0, $\bar{y}_0$ doivent être remplacés dans les dérivées du second ordre par $x_0+\theta\alpha$, $y_0+\theta'\beta$, $\bar{y}_0+\theta''\bar{\beta}$.

Appelons $2g_0$ la plus petite valeur des trois quantités,

$$\tilde{T}(x_1)-\tilde{\overline{T}}(x_2),\quad \frac{\partial^2\tilde{W}}{\partial y^2_0},\quad \frac{\partial^2\tilde{\overline{W}}}{\partial \bar{y}^2_0},$$

qui sont positives pour α, β, $\bar{\beta}$ suffisamment petits. Alors on aura

$$\Delta J=\frac{1}{2}\,\tilde{\Theta}(\alpha,\ \beta,\ \bar{\beta})\geqq g_0\epsilon^2,$$

où ϵ^2 est égal à α^2 dans le cas général $(\alpha\neq0)$ et est égal à la plus grande des deux quantités β^2, $\bar{\beta}^2$ dans le cas particulier $(\alpha=0)$.

Donc nous pouvons énoncer le résultat suivant:

Si la courbe discontinue de comparaison est assujettie à avoir dans le domaine R_ρ les points de rupture $K(x_0+\alpha,\ y_0+\beta)$, $\overline{K}(x_0+\alpha,\ \bar{y}_0+\bar{\beta})$, la différence entre l'intégrale J prise suivant cette courbe et la même intégrale prise suivant l'extrémale discontinue $\mathfrak{C}_0$ a une limite inférieure positive $g_0\epsilon^2$, ϵ étant une des trois variations α, β, $\bar{\beta}$.

B. *Cas des courbes continues de comparaison.*

20. *Champ* $\mathfrak{F}_1$. Comparons l'intégrale J prise suivant la courbe continue admissible $\bar{\lambda}_n$, à la même intégrale prise suivant l'extrémale discontinue $\mathfrak{C}_0$. Soit $\bar{\mathfrak{C}}$ la courbe discontinue de la région R_ρ vers laquelle tend l'ensemble $\{\lambda_n\}$. On aura comme au N° 5:

$$\Delta J_n = J_{\bar{\lambda}_n} - J_{\mathfrak{C}_0} = J_{\bar{\mathfrak{C}}} - J_{\mathfrak{C}_0} + (J_{\bar{\lambda}_n} - J_{\bar{\mathfrak{C}}}).$$

Appelons $y = \bar{\omega}_n(x)$ l'équation de la courbe $\bar{\lambda}_n$. Soient en outre

$$K(x_0+\alpha,\ y_0+\beta),\ \bar{K}(x_0+\alpha,\ \bar{y}_0+\bar{\beta})$$

les points de rupture de la courbe $\bar{\mathfrak{C}}$.

D'après le raisonnement du N° 3 on peut mettre cette équation sous la forme

$$\Delta J_n = \frac{1}{2}\tilde{\Theta}(\alpha,\ \beta, \bar{\beta}) + \int_{\bar{x}_0-\epsilon_n}^{\bar{x}_0+\epsilon_n} f[x,\ \bar{\omega}_n\ (x),\ \bar{\omega}_n'(x)]dx + (\epsilon_n),$$

où $\bar{x}_0 = x_0 + \alpha$ et (ϵ_n) est une quantité qui tend vers zéro avec ϵ_n.

Donc la condition suffisante du minimum dans le champ $\mathfrak{F}_1$ s'exprimera ainsi,

$$\frac{1}{2}\tilde{\Theta}(\alpha,\beta,\bar{\beta}) > -\int_{\bar{x}_0-\epsilon_n}^{\bar{x}_0+\epsilon_n} f[x,\ \bar{\omega}_n(x), \bar{\omega}_n'(x)]dx + (\epsilon_n). \qquad (14)$$

Le premier membre de cette inégalité est une constante positive au moins égale $g_0\epsilon^2$; g_0, ϵ ayant les mêmes significations qu'au N° 19.

Quant au second membre, il tend vers zéro avec ϵ_n. Par conséquent, on peut trouver un nombre entier N tel qu'on ait

$$\left| \int_{\bar{x}_0-\epsilon_n}^{\bar{x}_0+\epsilon_n} f(x,\ \bar{\omega}_n,\ \bar{\omega}_n')\, dx + (\epsilon_n) \right| < g_0\epsilon^2$$

pourvu que $n > N$.

Ainsi pour des valeurs de n plus grandes que N l'inégalité (14) est toujours vérifiée.

21. *Champ* $\mathfrak{F}_2$. Comparons l'intégrale J, prise suivant la courbe d'approximation, à la même intégrale prise suivant l'extrémale discontinue.

Soit λ_n, d'équation

$$y = \omega_n(x),$$

une telle courbe. La fonction $\omega_n(x)$ satisfait à l'équation (A).

Appelons $(x_0 - \epsilon_n,\ x_0 + \epsilon_n)$ l'intervalle de la chute et $K(x_0 - \epsilon_n,\ y_0 + \beta)$, $\bar{K}(x_0+\epsilon_n,\ \bar{y}_0+\bar{\beta})$ les points de la courbe λ_n correspondant aux abscisses $x_0-\epsilon_n$, $x_0+\epsilon_n$.

Soient P_1K l'extrémale du faisceau $F(P_1)$ et $\overline{K}P_2$ celle du faisceau $\overline{F}(P_2)$. Formons la courbe d'approximation $L_n(P_1K\overline{K}P_2)$ correspondant à λ_n (N° 3).

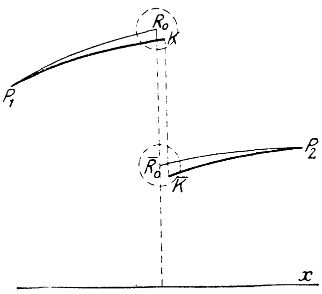

Fig. 6

Il est évident que pour trouver les conditions suffisantes dans le champ $\mathfrak{F}_2$, il suffit de comparer l'intégrale J prise suivant la courbe L_n à la même intégrale prise suivant $\mathfrak{E}_0$.

Mais

$$(15)\quad \Delta J_n = J_{L_n} - J_{\mathfrak{E}_0} = W(x_0-\epsilon_n, y_0+\beta) + \overline{W}(x_0+\epsilon_n, \overline{y}_0+\overline{\beta}) - W(x_0, y_0) - \overline{W}(x_0, \overline{y}_0) + \int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]dx.$$

En développant la somme $W+\overline{W}$ suivant les puissances de ϵ_n, β, $\overline{\beta}$, on aura

$$\Delta J_n = J_{L_n} - J_{\mathfrak{E}_0} = \int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]dx - 2\epsilon_n f(x_0, y_0, y_0') + \frac{1}{2}\tilde{\Theta}(-\epsilon_n, \beta, -\overline{\beta}).$$

Dans le cas considéré, ϵ_n est une quantité essentiellement positive; par conséquent d'après le théorème du N° 19, $\frac{1}{2}\tilde{\Theta}$ a pour limite inférieure $g_0\epsilon_n^2$:

$$\tfrac{1}{2}\tilde{\Theta}(-\epsilon_n, \beta, -\overline{\beta}) \geqq g_0\epsilon_n^2,$$

où g_0 est la valeur de la quantité positive $\tilde{T}(x_1) - \tilde{\overline{T}}(x_2)$.

On peut maintenant énoncer une condition suffisante pour le minimum dans le champ $\mathfrak{F}_2$:

ΔJ_n est essentiellement positif si, à partir d'une certaine valeur de n, on a

$$(\mathrm{C})\qquad \frac{1}{2\epsilon_n}\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]dx \geqq f(x_0, y_0, y_0')$$

pour toute fonction $\omega_n(x)$ satisfaisant à l'équation (A).

En particulier lorsque la limite du premier membre de cette inégalité existe, on peut exprimer la condition suffisante sous la forme plus précise

$$(\mathrm{C}_1)\qquad \lim_{n\to\infty} \frac{1}{2\epsilon_n}\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x, \omega_n(x), \omega_n'(x)]dx > f(x_0, y_0, y_0'),$$

pour toute fonction $\omega_n(x)$ satisfaisant à l'équation (A).

Il y a doute dans le cas où

$$\frac{1}{2\epsilon_n}\int_{x_0-\epsilon_n}^{x_0+\epsilon_n} f[x,\ \omega_n(x),\ \omega_n{}'(x)]dx - f(x_0,\ y_0,\ y_0{}')$$

tend vers zéro par des valeurs qui peuvent être négatives. Mais si cette expression tend vers zéro par valeurs positives, on peut être sûr que ΔJ sera positif.

22. Les résultats des deux N^{os} précédents peuvent se résumer de la manière suivante:

Si la courbe discontinue $\mathfrak{E}_0$ *réalise le minimum dans le champ* $\mathfrak{F}_0$, *elle donne à l'intégrale* J *une plus petite valeur*

1° *que toute autre courbe du champ* $\mathfrak{F}_1$,

2° *que toute autre courbe du champ* $\mathfrak{F}_2$ *lorsque en outre la condition* (C) *est vérifiée.*

D'autre part les courbes d'approximation fournissent à l'intégrale J des valeurs qui s'approchent autant qu'on veut de $J_{\mathfrak{E}_0}$.

Par conséquent, lorsque le minimum est réalisé dans le champ $\mathfrak{F}_0$ et lorsque en outre la condition (C) ou (C_1) est vérifiée, on aura les trois résultats suivants:

1° *Le minimum de l'intégrale* J *n'est pas réalisé par une courbe continue.*

2° *Il existe une limite inférieure des valeurs de l'intégrale* J *dans le champ des courbes continues.*

3° *C'est pour la courbe extrémale discontinue* $\mathfrak{E}_0$ *que l'intégrale* J *prend cette limite inférieure.*

Plaçons-nous maintenant seulement dans le champ des courbes continues. D'après ce qui précède, on voit alors que le rôle du champ $\mathfrak{F}_0$ des courbes discontinues de comparaison n'est qu'auxiliaire. L'introduction du champ $\mathfrak{F}_0$ nous permet de trouver toutes les conditions nécessaires et suffisantes pour qu'une limite inférieure des valeurs de l'intégrale J dans le champ des courbes continues existe.

La courbe discontinue $\mathfrak{E}_0$ est la vraie extrémale discontinue pour le champ $\mathfrak{F}_0$, tandis que pour le champ des courbes continues elle n'est que la courbe limite ou *limitale* fournissant à l'intégrale J la valeur de cette limite inférieure.

Donc dans le cas des singularités pareilles aux nôtres, on doit traiter le problème du calcul des variations au point de vue de la recherche *des conditions* nécessaires et suffisantes pour qu'une *limite inférieure* de l'intégrale J existe, c'est-à-dire pour qu'une extrémale du champ auxiliaire soit une vraie *limitale* du problème.

III. *La fonction V.*

23. Dans ce qui précède, on a supposé que, pour les points de rupture,

$$f_x + y_0{}'f_y - \bar{f}_x - \bar{y}_0{}'\bar{f}_y > 0.$$

Prolongeons les extrémales P_1R_0 et $\bar{R}_0P_2$, la première à droite du point R_0 et l'autre à gauche du point $\bar{R}_0$. D'après les conditions (I), nous aurons une nouvelle extrémale discontinue $\tilde{\mathfrak{E}}_0(\tilde{P}_1\bar{R}_0R_0\tilde{P}_2)$.

On peut s'assurer facilement que, pour les points de rupture de cette extrémale, on doit avoir

$$f_x+y_0'f_y-\bar{f}_x-\bar{y}_0'\bar{f}_y<0.$$

Admettons que les conditions nécessaires pour le minimum ordinaire le long des extrémales $P_1\tilde{P}_2$ et $\tilde{P}_1P_2$ soient vérifiées. Alors, quand la fonction

$$V=f_x+y_0'f_y-\bar{f}_x-\bar{y}_0'\bar{f}_y$$

est positive pour $L(x_0,\ y_0,\ \bar{y}_0,\ y_0',\ \bar{y}_0')$, l'extrémale $\mathfrak{E}_0$ est forte tandis que l'extrémale $\tilde{\mathfrak{E}}_0$ est faible. Si pour la même valeur cette fonction est négative, l'extrémale $\tilde{\mathfrak{E}}_0$ est forte et l'extrémale $\mathfrak{E}_0$ est faible. Il y a doute dans le cas $V=0$. Il faut alors considérer les dérivées d'ordre supérieur de la fonction $J(x)$ qui exprime la valeur de l'intégrale J prise le long de la courbe $P_1X\bar{X}P_2$ (N° 9).

IV. *Exemples*

24. *Trouver le minimum de l'intégrale*

$$J=\int_{-1}^{+1}(xy'+y)^2dx$$

parmi les courbes admissibles joignant les deux points donnés $P_1(-1,-1)$, $P_2(1,\ 1)$.

L'équation d'Euler est linéaire en y'', y':

$$x^2y''+2xy'=0.$$

Elle admet l'intégrale générale

$$y=\alpha+\frac{\beta}{x}\ ,$$

et il y a en outre une extrémale particulière $x=0$. Il n'existe donc aucune extrémale continue joignant les deux points P_1 et P_2.

Donc le problème proposé n'admet pas de solution continue.

Considérons les solutions discontinues. Nous sommes dans le cas exceptionnel, parce qu'il n'y a qu'un seul point de discontinuité admissible $\bar{x}_0=0$. La condition (A) n'est satisfaite que pour cette valeur de $\bar{x}_0$. Les équations (I'') donnent

$$x(xy'+y)=0,$$

$$x(x\bar{y}'+\bar{y})=0,$$

d'où l'on tire*

$$x=0.$$

L'extrémale discontinue cherchée se compose donc de deux segments de droites parallèles à l'axe des x:

$$\mathfrak{E}_0\quad \begin{cases} y=-1, & -1\leqq x<0, \\ y=+1, & 0<x\leqq+1. \end{cases}$$

* Nous laissons de côté les équations $xy'+y=0$, $x\bar{y}'+\bar{y}=0$, parce qu'elles ne nous donnent pas d'extrémales discontinues admissibles.

C'est le long de cette courbe discontinue que l'intégrale J atteint sa borne inférieure.

Nous allons maintenant vérifier qu'il y a des courbes continues d'approximation λ_n admissibles, c'est-à-dire satisfaisant à la condition fondamentale (A). Prenons sur les branches P_1R_0 et $\overline{R}_0P_2$ de l'extrémale $\mathfrak{E}_0$ les points $A(-\epsilon,-1)$, $B(+\epsilon,+1)$ et joignons-les par la droite AB. Nous formons ainsi la courbe continue d'approximation λ_n. L'intégrale J prise suivant λ_n a pour expression

$$J_{\lambda_n}=2+\frac{2\epsilon}{3}.$$

Donc J_{λ_n} peut être aussi voisin qu'on le veut de la valeur $J_{\mathfrak{E}_0}=2$; par conséquent la condition fondamentale (A) est satisfaite.

La dernière équation nous montre que J_{λ_n} reste toujours plus grand que $J_{\mathfrak{E}_0}$. Or nous allons voir que cette dernière propriété de $\mathfrak{E}_0$ est générale, c'est-à-dire que $J_{\mathfrak{E}_0}$ reste toujours plus petite que la même intégrale J prise suivant toute autre courbe continue quelconque joignant les mêmes points extrêmes P_1, P_2.

En effet, d'après la formule

$$\int_a^b y'^2dx \geqq \frac{(B-A)^2}{b-a},$$

où A et B sont les ordonnées des points extrêmes, on aura

$$\int_{-1}^{+1}(xy'+y)^2dx=\int_{-1}^{+1}\left[\frac{d(xy)}{dx}\right]^2dx=\int_{-1}^{0}\left[\frac{d(xy)}{dx}\right]^2dx+\int_{0}^{+1}\left[\frac{d(xy)}{dx}\right]^2dx\geqq 2.$$

D'autre part, on peut montrer facilement que l'intégrale J ne peut atteindre la valeur 2 pour une courbe continue admissible, joignant les deux points donnés P_1 et P_2.

Mais nous avons vu ci-dessus qu'il y a des courbes continues qui donnent à l'intégrale J des valeurs aussi proches qu'on veut de 2; donc $J_{\mathfrak{E}_0}=2$ est la limite inférieure de l'intégrale J. C.Q.F.D.

25. *Trouver l'extremum de l'intégrale*

$$J=\int_{x_1}^{x_2}\sin\,(yy')dx.$$

L'équation d'Euler prend la forme

$$-y\frac{d}{dx}[\cos yy']=0.$$

Elle admet l'intégrale générale

$$y^2=\alpha x+\beta$$

et en outre elle nous donne une intégrale singulière $y=0$. Proposons-nous de trouver l'extrémale discontinue de ce problème.

Il est évident que la condition fondamentale $(\overline{\mathrm{A}})$ est satisfaite pour toutes les valeurs de $\bar{x}_0$, étant donnée la nature de la fonction $f=\sin yy'$; c'est-à-dire que chaque courbe continue de comparaison vérifiant les conditions connues de la régularité est une courbe admissible au sens du N° 3. Le point de discontinuité est donc arbitraire. C'est un problème du cas général.

Les équations (I″) ont pour expressions

$$\sin yy' = \sin \bar{y}\bar{y}', \qquad y \cos yy' = 0, \qquad \bar{y} \cos \bar{y}\bar{y}' = 0,$$

d'où

$$yy' = \frac{\pi}{2} + k\pi, \qquad \bar{y}y' = \frac{5\pi}{2} + k\pi.$$

L'extrémale discontinue se compose donc de deux paraboles

$$y^2 = (2k+1)\pi x + \beta, \qquad y^2 = (2k+5)\pi x + \bar{\beta}.$$

On voit facilement d'après cela, que nous nous sommes placés ici dans le cas $f_y = 0$, $\bar{f}_y = 0$ (N° 12).

Il est facile de voir que si les extrémités P_1, P_2 se trouvent d'un même coté de l'axe des x, il y aura une infinité de courbes continues anguleuses joignant ces deux points qui donnent à l'intégrale J la valeur maximum $x_2 - x_1$ ou la valeur minimum $-(x_2 - x_1)$. Mais si les extrémités se trouvent de part et d'autre de l'axe des x, il n'y aura aucune courbe continue admissible, joignant ces points et donnant à l'intégrale J la valeur extremum. D'autre part, dans ce dernier cas, on aura toujours

$$-2\epsilon < \int_{x_0-\epsilon}^{x_0+\epsilon} \sin yy'dx < 2\epsilon.$$

Dès lors, d'après le théorème du N° 22, nous pouvons dire que lorsque les points P_1, P_2 se trouvent de part et d'autre de l'axe des x: l'extremum n'est pas réalisé parmi les courbes continues; l'intégrale J a la limite inférieure $-(x_2 - x_1)$ et la limite supérieure $x_2 - x_1$ dans le champ des courbes continues.

Ce sont les extrémales discontinues qui donnent à l'intégrale J ces valeurs limites.

THE TRANSFORMATION OF CLEBSCH IN THE CALCULUS OF VARIATIONS

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The problem of Lagrange in the calculus of variations is that of finding among the arcs

(1) $$y_i = y_i(x), \quad (i = 1, \ldots, n; \ x_1 \leqq x \leqq x_2),$$

which join two given points in the $(x, y_1, \ldots, y_n)$-space and which satisfy a system of differential equations

(2) $$\phi_a(x, y_1, \ldots, y_n, y'_1, \ldots, y_n') = 0 \quad (a = 1, \ldots, m < n),$$

one which minimizes an integral of the form

$$I = \int_{x_1}^{x_2} f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')dx.$$

In 1858 Clebsch published a complicated transformation of the so-called second variation for this problem by means of which he deduced a necessary condition for a minimizing curve, the "condition of Clebsch", analogous to one which was deduced much earlier by Legendre for simpler cases. Since the publication of Clebsch's memoir his transformation has been re-studied at length by Mayer, von Escherich, and many other writers, for the purpose of deducing the necessary conditions of both Clebsch and Jacobi, but the theory remains elaborate and complicated. In more recent years there has been a tendency to avoid the use of the second variation altogether, and to attain equivalent results by means of the Darboux-Kneser envelope theorem. It has not so far been possible, however, to secure in this way all of the generality of the theorems deduced by the older method.

Along a minimizing arc the second variation must be positive or zero. This suggests at once an auxiliary problem of minimizing the second variation, a problem which turns out to be in every respect similar to the original one. In 1916 D. M. Smith* followed a method already used in other cases by the author of this paper and showed that by applying well known theorems to the problem of minimizing the second variation the necessary conditions of Clebsch and Jacobi for the original problem can be deduced from that variation without elaborate transformations. In the present paper these ideas are pursued further in order to show that by applications of Hilbert's invariant integral and

*Trans. Amer. Math. Soc., vol. 17 (1916), p. 459.

a formula of Weierstrass, to the problem of minimizing the second variation, the transformation of Clebsch and von Escherich can be attained in relatively easy fashion. Thus the results of an elaborate chapter in the literature of the calculus of variations are simply related to formulas now regarded as classic. In an earlier paper,* which may be consulted for further references, the author has indicated that such relationships are to be expected. The following pages contain the details of the investigation. In Sections 1, 2, 3 certain well known results are presented in a form suited to the applications which are to be made of them, and in Sections 4 and 5 they are applied to the transformation of the second variation.

1. *Preliminary notions and theorems.* In the following pages the sets $(x, y_1, \ldots, y_n)$ and $(x, y_1, \ldots, y_n, y'_1, \ldots, y'_n)$ will be denoted by (x, y) and (x, y, y') when no confusion results from so doing. It will be supposed that there is a region R of sets (x, y, y') in which the functions f and ϕ_α have continuous derivatives of at least the first three orders, since these are the ones which will be needed in the following analysis. An *admissible arc* is defined to be an arc of the form (1) which is continuous, and consists of a finite number of arcs on each of which the functions $y_i(x)$ have continuous derivatives, whose elements (x, y, y') are all interior to the region R, and which satisfies the differential equations $\phi_\alpha = 0$. The problem of Lagrange may then be stated as that of finding among all admissible arcs joining two fixed points one which minimizes the integral I.

Suppose now that the arc E defined by equations (1) is a minimizing arc for this problem and that its functions $y_i(x)$ have continuous first and second derivatives. In order to avoid serious analytic difficulties it is assumed that at every element (x, y, y') of the arc E the matrix $\|\partial\phi_\alpha/\partial y'_i\|$ is of rank m so that the equations $\phi_\alpha = 0$ are independent along E. It is well known† that such a minimizing arc must satisfy a set of differential equations of the form

$$\frac{d}{dx} F_{y_i'} - F_{y_i} = 0 \quad (i = 1, \ldots, n). \tag{3}$$

where

$$F = l_0 f + l_1\phi_1 + \ldots + l_m\phi_m.$$

In this expression l_0 is a constant, the other multipliers l_α are functions of x having continuous derivatives on the interval x_1x_2, and the elements of the set l_0, $l_\alpha(x)$ $(\alpha = 1, \ldots, m)$ are not all identically zero. An arc of the type of E which satisfies equations (3) and the equations $\phi_\alpha = 0$ is called an *extremal.*

If the arc E is a member of a one-parameter family of admissible arcs

$$y_i = y_i(x, b), \qquad (i = 1, \ldots, n),$$

say for the particular parameter value $b = 0$, then the derivatives $\eta_i(x) = y_{ib}(x, 0)$ are called the *variations* of the family along E and they satisfy the differential equations

$$\Phi_\alpha(x, \eta, \eta') = \sum_i \left(\frac{\partial\phi_\alpha}{\partial y_i}\eta_i + \frac{\partial\phi_\alpha}{\partial y_i'}\eta_i'\right) = 0, \quad (\alpha = 1, \ldots, m), \tag{4}$$

*Bull. Amer. Math. Soc., vol. 26 (1920), p. 359.

†Bolza, *Vorlesungen über Variationsrechnung,* chapter XI.

as one readily sees by substituting the functions $y_i(x, b)$ in the equations $\phi_a=0$, differentiating for b, and setting $b=0$. The equations (4) are called the *equations of variation* along E. A set of *admissible variations* is a set of functions $n_i(x)$ which define in the $x\eta$-space a continuous curve consisting of a finite number of arcs on each of which the functions $\eta_i(x)$ have continuous derivatives, and which satisfy the equations of variation.

Let $\eta_{s1}, \ldots, \eta_{sn}(s=1, \ldots, 2n)$ denote $2n$ sets of admissible variations for the minimizing arc E, with a matrix

$$\begin{Vmatrix} \eta_{si}(\xi_1) \\ \eta_{si}(\xi_2) \end{Vmatrix}$$

each of whose columns consists of the values of one of these sets at two points ξ_1, ξ_2 of the interval x_1x_2. If there exists a matrix of this sort which is different from zero then E is said to be a *normal* minimizing arc* on the interval $\xi_1\xi_2$. It will be assumed from now on that E is normal on every such sub-interval $\xi_1\xi_2$ of x_1x_2.

An important property of a minimizing arc E which is normal on the whole interval x_1x_2 is that for every set of admissible variations $\eta_i(x)$ vanishing at x_1 and x_2 there is a one-parameter family $y_i=y_i(x, b)$ of admissible arcs containing E for the parameter value $b=0$, having the variations $\eta_i(x)$ along E, and such that every arc of the family passes through the points 1 and 2 of E.† It is furthermore known that every set of multipliers l_0, $l_a(x)$ for a normal minimizing arc must have $l_0\neq 0$, and if the set is reduced to the form $l_0=1$, $l_a(x)$ by dividing all of its elements through by l_0 then the resulting set is unique‡. From this point on the multipliers used for E will always be the unique set having $l_0=1$.

Consider now a one-parameter family of admissible arcs $y_i=y_i(x, b)$ containing the minimizing arc E for the parameter value $b=0$ and having all of its curves passing through the end-points of E. The values of the integral I taken along the arcs of this family define a function

$$I(b)=\int_{x_1}^{x_2} f\big(x, y(x, b), y'(x, b)\big)dx$$

and it is evident as usual that the conditions $I'(0)=0$, $I''(0)\geqq 0$ must be satisfied. Since the arc E satisfies the differential equations (3) with suitable multipliers $l_0=1$, $l_a(x)$ it follows by the usual methods that $I'(0)$ does vanish, but we are interested here primarily in the second variation. The value of $I''(0)$ is

$$I''(0)=\int_{x_1}^{x_2}\Sigma_i\left(f_{y_i}\frac{\partial^2 y_i}{\partial b^2}+f'_{y_i}\frac{\partial^2 y_i'}{\partial b^2}\right)dx+\int_{x_1}^{x_2}\Sigma_{ik}(f_{y_iy_k}\eta_i\eta_k+2f_{y_iy'_k}\eta_i\eta'_k+f'_{y_i y'_k}\eta_i'\eta_k')dx$$

in which the arguments of the derivatives of f are the functions $y_i(x)$ belonging

*Von Escherich, Wiener Berichte, vol. 108 (1899), p. 22; Bolza, *loc. cit*, p. 564; Bliss, Trans. Amer. Math. Soc., vol. 19 (1918), p. 311.

†Bolza, *loc. cit.*, p. 568

‡Bolza, *loc. cit.*, p. 565

to the arc E. At $b=0$ the second derivatives with respect to b of the equations

$$\phi_a(x,\ y(x,\ b),\ y'(x,\ b))=0, \qquad (a=1,\ \ldots,\ m),$$

give

$$\sum_i\left(\frac{\partial\phi_a}{\partial y_i}\frac{\partial^2 y_i}{\partial b^2}+\frac{\partial\phi_a}{\partial y_i'}\frac{\partial^2 y_i'}{\partial b^2}\right)+\sum_{ik}\left(\frac{\partial^2\phi_a}{\partial y_i\partial y_k}\eta_i\eta_k+2\frac{\partial^2\phi_a}{\partial y_i\partial y_k'}\eta_i\eta_k'+\frac{\partial^2\phi_a}{\partial y_i'\partial y_k'}\eta_i'\eta_k'\right)=0.$$

When these are multiplied, respectively, by the functions $l_a(x)$, integrated from x_1 to x_2, and added to the expression for $I''(0)$, the latter takes the form

$$I''(0)=2\int_{x_1}^{x_2}\omega\ (x,\ \eta,\ \eta')\ dx \tag{5}$$

where ω is the quadratic form defined by the equations

$$2\omega(x,\eta,\eta')=\Sigma_{ik}(P_{ik}\eta_i\eta_k+2Q_{ik}\eta_i\eta_k'+R_{ik}\eta_i'\eta_k'),$$

$$P_{ik}=F_{y_iy_k},\ Q_{ik}=F_{y_iy_k'}\ ,\ R_{ik}=F_{y_i'y_k'}.$$

The terms in $\partial^2y_i/\partial b^2$ have disappeared, after the usual integration by parts, since E is an extremal with the multipliers $l_0=1$, $l_a(x)$, and since the curves of the family $y_i=y_i(x,\ b)$ all pass through the fixed end-points of E so that the derivatives $\partial^2y_i/\partial b^2$ all vanish at x_1 and x_2.

Since the arc E is normal it is evident that the expression (5) must be positive or zero for every set of admissible variations $\eta_i(x)$ whose elements all vanish at x_1 and x_2. Such a set is in fact the set of variations of a one-parameter family of admissible arcs $y_i=y_i(x,\ b)$ of the type used in the preceding paragraph, and for such a family the condition $I''(0)>0$ must be satisfied. This is the situation which suggests a new minimum problem in the $x\eta$-space with the integral (5) in place of I, the equations $\Phi_a(x,\ \eta,\ \eta')=0$ in place of the original equations $\phi_a=0$, and the end-points $(x,\eta_1,\ \ldots,\ \eta_n)=(x_1,\ 0,\ \ldots,0)$, $(x,\eta_1,\ \ldots,\ \eta_n)=(x_2,\ 0,\ \ldots,\ 0)$ in place of the end-points of E.

2. *Mayer fields and a formula of Weierstrass.* Consider a region F of points $(x,\ y)$ containing only interior points and having associated with it a set of functions

$$p_i(x,\ y),\ l_a(x,\ y), \qquad (i=1,\ \ldots,\ n,\ a=1,\ \ldots,\ m), \tag{6}$$

such that the elements $(x,\ y,\ p\ (x,\ y))$ corresponding to points in F all lie interior to the region R where the continuity properties of f and the functions ϕ_a are assumed to hold. Furthermore suppose that at all points of F the functions p_i, l_a have continuous first derivatives and satisfy the equations

$$\phi_a(x,\ y,\ p)=0, \qquad (a=1,\ \ldots,\ m). \tag{7}$$

Such a region F is called a Mayer field with the slope functions and multipliers (6) if the so-called Hilbert integral

$$\mathrm{I}^*=\left\{\int F(x,\ y,\ p,\ l)+\Sigma_j(y_j'-p_j)F_{y_j'}(x,\ y,\ p,\ l)\right\}dx \tag{8}$$

is independent of the path in F. The equations

$$\frac{\partial A}{\partial y_i}=\frac{\partial B_i}{\partial x},\ \frac{\partial B_i}{\partial y_k}=\frac{\partial B_k}{\partial y_i},\qquad (i,\ k=1,\ \ldots,\ n), \tag{9}$$

where

$$A=F-\Sigma_j p_j F_{y_j'},\ B_i=F_{y_i'},$$

are the usual necessary conditions for the integral to be independent of the path. One may readily prove the identities

$$\frac{\partial A}{\partial y_i}-\frac{\partial B_i}{\partial x}=F_{y_i}-\frac{\partial}{\partial x}F_{y_i'}-\Sigma_j p_j\frac{\partial}{\partial y_i}F_{y_i'}+\Sigma_j p_j\left(\frac{\partial B_i}{\partial y_j}-\frac{\partial B_j}{\partial y_i}\right)+\Sigma_\alpha\frac{\partial l_\alpha}{\partial y_i}\phi_\alpha \tag{10}$$

in which $x,\ y_1,\ \ldots,\ y_n$ are thought of as the independent variables occurring implicitly in the functions $p_i\ (x,\ y)$, $l_\alpha\ (x,\ y)$ and also explicitly.

In the field F the solutions of the equations

$$\frac{dy_i}{dx}=p_i\ (x,\ y),\qquad (i=1,\ \ldots,\ n), \tag{11}$$

satisfy the equations $\phi_\alpha(x,\ y,\ y')=0$ on account of the relations (7), and the identity (10) with equations (7) and (9) shows that they also satisfy

$$F_{y_i}-\frac{d}{dx}F_{y_i'}=F_{y_i}-\frac{\partial}{\partial x}F_{y_i'}-\Sigma_j\, p_j\frac{\partial}{\partial y_j}F_{y_i'}=0.$$

They are therefore extremals of the integral I. They are called the *extremals of the field*, and since the equations (11) are of the first order one and only one of them passes through each point of F. Along an extremal arc of the field the value of I^* defined in equation (8) is clearly identical with that of I, since along it the differences $y_i'-p_i$ are all zero.

Theorem. If E_{12} is an extremal arc of a field F *with end-points* 1 *and* 2 *then for every admissible arc C_{12} in the field joining the same two points the formula*

$$I(C_{12})-I(E_{12})=\int_{x_1}^{x_2}E\left(x,\ y,\ p(x,\ y),\ y',\ l(x,\ y)\right)dx \tag{12}$$

holds, where

$$E(x,\ y,\ p,\ y',\ l)=F(x,\ y,\ y',\ l)-F(x,\ y,\ p,\ l)-\Sigma_j(y_j'-p_j)F_{y_j'}(x,\ y,\ p,\ l),$$

the functions x, $y(x)$, $y'(x)$ in the integrand being those belonging to C_{12}.

The formula (12) is a well known one of Weierstrass† the proof of which is now very simple. For since I^* is independent of the path in F and has the same value as I along an extremal of the field it follows that

$$I(E_{12})=I^*(E_{12})=I^*(C_{12}),$$

and hence that

$$I(C_{12})-I(E_{12})=I(C_{12})-I^*(C_{12}).$$

†Bolza, loc. cit., p. 637.

The last two terms are identical with the integral in the second member of the formula (12) when the integrand f in $I(C_{12})$ is replaced by F. This is permissible since C_{12} is by hypothesis an admissible arc and therefore satisfies the equations $\phi_\alpha=0$, so that $F=f$ along C_{12}.

3. *Methods of constructing a field.* It has been seen that through every point of a field there passes a unique extremal of the field, or, in other words, that the field is simply covered by an n-parameter family of extremals. It is not true, conversely, except in the case when $n=1$, that an n-parameter family of extremals simply covering a region F of space will always form a field there. In order to discuss further the conditions under which such an n-parameter family will form a field let us first deduce a pair of useful auxiliary formulas.

Consider a one-parameter family of extremals, with their multipliers, of the form

$$y_i=e_i(x,\ t),\ l_\alpha(x,\ t),\quad (i=1,\ \ldots,\ n;\ \alpha=1,\ \ldots,\ m). \tag{13}$$

Two functions $x_1(t)$, $x_2(t)$ defined on a common interval $t'\leqq t\leqq t''$ will determine two curves C_1 and C_2 when substituted for x in the functions $e_i(x,\ t)$ of these equations, and the integral

$$I(t)=\int_{x_1}^{x_2} F(x,\ e,\ e',\ l)dx$$

becomes a function of t. The derivative of $I(t)$ has the value

$$\begin{aligned} I'(t) &= F\frac{dx}{dt}\Big|_{x_1}^{x_2}+\int_{x_1}^{x_2}\left\{\Sigma_i\left(F_{y_i}\frac{\partial e_i}{\partial t}+F_{y_i'}\frac{\partial e_i'}{\partial t}\right)+\Sigma_\alpha F_{l_\alpha}\frac{\partial l_\alpha}{\partial t}\right\}dx \\ &= F\frac{dx}{dt}+\Sigma_i F_{y_i'}\frac{\partial e_i}{\partial t}\Big|_{x_1}^{x_2} \end{aligned}$$

since along the arcs (13) we have $F_{y_i}=dF_{y_i'}/dx$ and $F_{l_\alpha}=\phi_\alpha=0$. But on either of the curves C_1 or C_2

$$\frac{dy_i}{dt}=e_i'\frac{dx}{dt}+\frac{\partial e_i}{\partial t},\quad (i=1,\ \ldots,\ n),$$

and hence

$$I'(t)=F\frac{dx}{dt}+\Sigma_i\left(\frac{dy_i'}{dt}-e_i'\frac{dx}{dt}\right)F_{y_i'}\Big|_{x_1}^{x_2}. \tag{14}$$

If the two members of this equation are integrated from t' to t'' it follows that

$$I(t'')-I(t')=I^*(C_2)-I^*(C_1). \tag{15}$$

The last two equations will be very useful in the succeeding paragraphs.

Consider now an n-parameter family of extremals, with multipliers, of the form

$$\begin{aligned} y_i&=e_i(x,\ a_1,\ \ldots,\ a_n), &&(i=1,\ \ldots,\ n),\\ l_\alpha&=l_\alpha(x,\ a_1,\ \ldots,\ a_n), &&(\alpha=1,\ \ldots,\ m), \end{aligned} \tag{16}$$

defined for a set of values $(x, a_1, \ldots, a_n)$ satisfying the conditions

(17) $\qquad (a_1, \ldots, a_n)$ in a region A, $\quad \xi_1(a_1, \ldots, a_n) \leqq x \leqq \xi_2(a_1, \ldots, a_n)$.

In this set the functions e_i, e_{ix}, l_α are supposed to have continuous first derivatives and to define only points $(x, y, y') = (x, e, e')$ in the region R where the continuity properties of f and the functions ϕ_α are presupposed.

If the functions (16) are substituted in the integral I with limits $x_1(a_1, \ldots, a_n)$, $x_2(a_1, \ldots, a_n)$ having continuous derivatives in the region A then a function $I(a_1, \ldots, a_n)$ is defined. When all of the a's are kept fixed except $t = a_i$ the formula (14) gives the derivative I_{a_i}, and for the particular case when $x_1(a_1, \ldots, a_n)$ and $x_2(a_1, \ldots, a_n)$ are constants it has the value

$$I_{a_i} = \Sigma_j F_{y_j'} \frac{\partial y_j}{\partial a_i}\bigg|_{x_1}^{x_2}.$$

If this expression is differentiated for a_k and the two expressions for $I_{a_i a_k}$ and $I_{a_k a_i}$ equated, it turns out that

$$0 = \sum_j \left(\frac{\partial y_j}{\partial a_i}\frac{\partial F_{y_i'}}{\partial a_k} - \frac{\partial y_j}{\partial a_k}\frac{\partial F_{y_k'}}{\partial a_i}\right)\bigg|_{x_1}^{x_2}.$$

When x_1 is kept fixed and x_2 allowed to vary the following theorem becomes evident*:

Theorem. Along each extremal arc of a family of the type (16) *the expressions*

$$\sum_j \left(\frac{\partial y_j}{\partial a_i}\frac{\partial F_{y_j'}}{\partial a_k} - \frac{\partial y_j}{\partial a_k}\frac{\partial F_{y_j'}}{\partial a_i}\right) \tag{18}$$

are constants.

The extremals (16) are said to simply cover a region F of points $(x, y_1, \ldots, y_n)$ if to each point of the region there corresponds one and but one solution $[x, a_1(x, y), \ldots, a_n(x, y)]$ of equations (16) in the region of values $(x, a_1, \ldots, a_n)$ satisfying the conditions (17), and if the functions $a_i(x, y)$ so defined have continuous derivatives in F. The equations

$$\begin{aligned} p_i(x, y) &= e_i(x, a_1(x, y), \ldots, a_n(x, y)), \quad (i = 1, \ldots, n), \\ l_\alpha(x, y) &= l_\alpha(x, a_1(x, y), \ldots, a_n(x, y)), \quad (\alpha = 1, \ldots, m), \end{aligned} \tag{19}$$

define a set of slope functions and multipliers in F.

Theorem. Consider a family of extremals (16) *which simply covers a region* F *of xy-space in the sense just described. If* F *is a field with the slope functions and multipliers* (19), *then the equations*

$$\sum_j \left(\frac{\partial y_j}{\partial a_i}\frac{\partial F_{y_j'}}{\partial a_k} - \frac{\partial y_j}{\partial a_k}\frac{\partial F_{y_j'}}{\partial a_i}\right) = 0 \tag{20}$$

*For another proof of this theorem see Hahn, Rend. Circ. Mat. di Palermo, vol. 29 (1910), p. 55.

hold identically in F. *Conversely if these sums are identically zero in* F, *then for each extremal arc of the family* (16) *interior to* F *there will at least be a neighbourhood which is a field with the slope functions and multipliers* (19).

If F is a field consider first the totality of points (x, y) in F corresponding to a fixed $x=x_1$, and second the region of sets $(a_1, \ldots, a_n)$ corresponding to these points. An arc in either region defines uniquely one in the other. The Hilbert integral (8) taken along such an arc has the form

$$I^* = \int \Sigma_i F_{y_i'} dy_i = \int \Sigma_{ik} F_{y_i'} \frac{\partial y_i}{da_k} da_k \tag{21}$$

and it must be independent of the path in both spaces. The vanishing of the expression (20) at $x=x_1$, is readily seen to be a necessary condition for the invariance of the last integral in the region of points $(a_1, \ldots, a_n)$ just described.

To prove the second part of the theorem let $(x_1, a_1', \ldots, a_n')$ be a particular set of values corresponding to an interior point of F. The functions

$$A_k(a_1, \ldots, a_n) = \Sigma_i F_{y_i'} \frac{\partial y_i}{\partial a_k}$$

in the field, with $x=x_1$, substituted, satisfy the conditions

$$\frac{\partial A_i}{\partial a_k} = \frac{\partial A_k}{\partial a_i}$$

in a neighbourhood of the point $(a_1', \ldots, a_n')$, since the expressions (20) vanish, and it is readily seen that the function

$$W(a_1, \ldots, a_n) = \Sigma_k \int_{a_k'}^{a_k} A_k(a_1, \ldots, a_k, a_{k+1}', \ldots, a_n')\, da_k$$

is well defined in a neighbourhood of $(a_1', \ldots, a_n')$ and has the function $A_k(a_1, \ldots, a_n)$ as its partial derivative with respect to a_k, The integral (21) is therefore the integral of the differential of W and is independent of the path near $(a_1' \ldots, a_n')$. The extremal arc of F corresponding to the values $(a_1', \ldots, a_n')$ will have a neighbourhood N of xy-points in which every arc C_2 defines for $x=x_1$ a projection C_1 in a neighbourhood of $(a_1', \ldots, a_n')$ where the integral (21) is independent of the path, and the formula (15) will hold for C_1 and C_2. Each term in this formula is independent of the path C_2 in N except the last, and that must be also, which proves the theorem.

In the equations

$$\begin{aligned} &x = x_1(a_1, \ldots, a_n), \\ &y_i = e_i(x_1(a_1, \ldots, a_n), a_1, \ldots, a_n), \quad (i=1, \ldots, n), \end{aligned} \tag{22}$$

let $x_1(a_1, \ldots, a_n)$ have continuous first derivatives in the region A and be such that all the points $(x_1, a_1, \ldots, a_n)$ corresponding to A satisfy the conditions (17). These equations then define an n-space in the space of points (x, y) and we have the following corollary to the last theorem:

Corollary. If the family of extremals (16) *simply covers a region* F *of xy-space and is cut by an n-space of the form* (22) *on which the Hilbert integral is independent of the path, then F is a field with the slope functions and multipliers* (19).

For by means of the functions $a_i(x, y)$ every arc C_2 in F defines a corresponding projection C_1 in the n-space (22), and the equation (15) holds for these two arcs. In this equation each term except the last has the same value for all arcs in F joining the end-points of C_2, because I^* is by hypothesis independent of the path for curves C_1 in the n-space (22), and hence the last term must also be independent of the path for all arcs C_2 in F with the same end-points.

The corollary just proved suggests several ways of constructing a field out of an n-parameter family of extremals. It is not necessary that the n-space (22) should lie in the field, and one may therefore, as a first possibility, take the n-parameter family of extremals passing through a fixed point, with the fixed point as a degenerate n-space of the type (22), On such an n-space the Hilbert integral is certainly independent of the path, and every region F simply covered by the family of extremals will be a field.

A second method is to select arbitrarily an n-space

$$x = x_1(a_1, \ldots, a_n), \quad y_i = Y_i(a_1, \ldots, a_n), \quad (i = 1, \ldots, n),$$

and to determine initial values Y_i', L_α of the variables y_i', l_α as solutions of the equations

$$F\frac{\partial x}{\partial a_i} + \Sigma_j\left(\frac{\partial Y_j}{\partial a_i} - Y_j'\frac{\partial x_1}{\partial a_i}\right)F_{y_j'} = 0, \quad \phi_\alpha = 0, \tag{23}$$

$$(i = 1, \ldots, n; \ \alpha = 1, \ldots, m)$$

where the arguments of F and its derivatives, and of the functions ϕ_α, are the given functions x_1, Y_i. The variables Y_i', L_α are to be determined by the equations. The Euler-Lagrange equations (3) and the equations of condition (2) then have a unique family of solutions with multipliers of the type (16) and with the initial values Y_i, Y_i', L_α at $x = x_1$. The n-space (22) intersecting this family is said to cut the family of extremals transversally on account of the first of the equations (23), and this same equation shows that for every curve in the n-space the Hilbert integral has the value zero. Every region F of xy-space simply covered by the extremals will therefore be a field.

Finally consider a fixed value $x = x_1$ and let $W(a_1, \ldots, a_n)$ be an arbitrarily selected function of $a_1, \ldots, a_n$. The equations

$$F_{y_k'}(x_1, a, Y', L) = W_{a_k}, \quad \phi_\alpha(x_1, a, Y) = 0, \tag{24}$$

$$(k = 1, \ldots, n; \ \alpha = 1, \ldots, m),$$

will in general determine the values Y_i' and L_α also as functions of $a_1, \ldots, a_n$. The Euler-Lagrange equations (3) and the equations of condition (2) then have a unique family of solutions of the type (16) with the initial values $y_i = a_i$, $y_i' = Y_i'$,

$l_a = L_a$ at $x = x_1$. On the n-space of type (22), defined for these extremals by setting $x = x_1$, the Hilbert integral (21) is the integral of the differential dW and hence independent of the path. Every region F simply covered by the extremals is therefore a field as indicated in the corollary.

The existence theorems for implicit functions tell us that the equations (23) or (24) always have continuous families of solutions if they have a single one at which the functional determinant of their first members with respect to the dependent variables is different from zero. It can be shown that at an initial solution at which F and the determinant

$$\begin{vmatrix} F_{y_i' y_k'} & \phi_{a y_i'} \\ \phi_{a y_k'} & 0 \end{vmatrix}$$

are different from zero the functional determinant of the equations (23) is always different from zero. Furthermore at an initial solution where the last determinant is different from zero the equations (24) have also a functional determinant different from zero.

4, *The transformation of the second variation.* The problem of minimizing the second variation in which we are interested is that of finding in the class of sets of admissible variations $\eta_i(x)$ vanishing at x_1 and x_2 and satisfying the differential equations

$$\Phi_a\,(x,\,\eta,\,\eta') = \sum_i \left(\frac{\partial \phi_a}{\partial y_i} \eta_i + \frac{\partial \phi_a}{\partial y_i'} \eta_i' \right) = 0, \quad (a = 1, \ldots, m),$$

one which minimizes the integral

$$I''(0) = 2 \int_{x_1}^{x_2} \omega(x,\,\eta,\,\eta')dx. \tag{25}$$

In the derivatives of the functions ϕ_a and f which occur in the expressions Φ_a and ω the arguments are the functions $y_i(x)$ defining the arc E.

The differential equations of the minimizing arcs for this problem in the $x\eta$-space are the equations

$$\frac{d}{dx} \Omega_{\eta_i'} - \Omega_{\eta_i} = 0, \ \Omega_{\lambda_a} = \Phi_a = 0, \quad (i = 1, \ldots, n;\ a = 1, \ldots, m), \tag{26}$$

in which

$$\Omega(x,\,\eta,\,\eta',\,\lambda) = \lambda_0 \omega + \lambda_1 \Phi_1 + \ldots + \lambda_m \Phi_m$$

with λ_0 a constant and the other multipliers functions of x. They are analogous to the equations (3) and (2) for the original problem, and following von Escherich we may call them the accessory system of differential equations. Since the equations $\Phi_a = 0$ are linear in the variables η_i and η_i' it follows readily that they are their own equations of variation. The original arc E in the xy-space is normal, by hypothesis, and it follows readily from the last remark that every solution $\eta_i(x)$ of the equations (26) for the $x\eta$-problem is also normal, and its multipliers in the form $\lambda_0 = 1$, $\lambda_a(x)$ are therefore unique. We shall suppose them in this form from now on, and it is evident that the accessory equations

(26) are then linear in the variables η_i, η_i', η_i'', λ_α, λ_α'. By a solution of these equations is meant always a set of functions $\eta_i(x)$, $\lambda_\alpha(x)$ for which the functions η_i have continuous first and second derivatives on the interval x_1x_2, and the functions λ_α continuous first derivatives, and which satisfy the equations.

It will be convenient to denote by Ω_η, $\Omega_{\eta'}$, Ω_λ the derivatives of Ω with the arguments x, η, η', λ, and similar meanings are attached to the notations Ω_u, $\Omega_{u'}$, Ω_ρ and Ω_v, $\Omega_{v'}$, Ω_σ. The symbol $\eta\Omega_{\eta'}$ is a convenient symbol for the sum

$$\eta\Omega_{\eta'}=\eta_1\Omega_{\eta_1'}+\ldots+\eta_n\Omega_{\eta_n'}$$

and others of the same sort will be used which are self-explanatory. Since Ω is homogeneous and quadratic in the variables η, η', λ the equations

$$\begin{aligned}&2\Omega=\eta\Omega_\eta+\eta'\Omega_{\eta'}+\lambda\Omega_\lambda,\\ &u\Omega_v+u'\Omega_{v'}+\rho\Omega_\sigma=v\Omega_u+v'\Omega_{u'}+\sigma\Omega_\rho\end{aligned}\tag{27}$$

are identities. From the latter of these it follows readily that

$$u\left(\Omega_v-\frac{d}{dx}\Omega_{v'}\right)+\rho\Omega_\sigma-v\left(\Omega_u-\frac{d}{dx}\Omega_{u'}\right)-\sigma\Omega_\rho=-\frac{d}{dx}\left(u\Omega_{v'}-v\Omega_{u'}\right)$$

so that for every pair of solutions u_i, ρ_α and v_i, σ_α of the accessory equations the expression

$$\psi(u,\ \rho;\ v,\ \sigma)=u\Omega_{v'}-v\Omega_{u'}\tag{28}$$

is a constant. If this constant is zero the two solutions are said to be *conjugate*.

The transformation of the second variation is very simply effected with the help of the analogue for the $x\eta$-problem of the formula (12) of Weierstrass. To make this transformation consider an n-parameter family of solutions η_i, λ_α of the accessory equations of the form

$$\begin{aligned}&\eta_i=a_1u_{1i}+\ldots+a_nu_{ni}, &&(i=1,\ \ldots,\ n),\\ &\lambda_\alpha=a_1\rho_{1\alpha}+\ldots+a_n\rho_{n\alpha}, &&(\alpha=1,\ \ldots,\ m),\end{aligned}\tag{29}$$

where the a's are the parameters of the family and where the sets u_{ki}, $\lambda_{k\alpha}$ $(i=1,\ldots,n;\ \alpha=1,\ldots,m)$ are for each k solutions of the accessory equations. The first n of these equations determine uniquely the parameters a_i as functions of the variables x, η_i, and the family simply covers the region F_ω of the $x\eta$-space for which $x_1\leqq x\leqq x_2$, if the determinant $U=|u_{ik}|$ is different from zero. Suppose that this is so. Then the solutions $a_i(x,\eta)$ just mentioned when substituted in the equations

$$\eta_i'=a_1u_{1i}'+\ldots+a_nu_{ni}',\qquad (i=1,\ \ldots,\ n),$$

and in the last m of the equations (29) determine a set of slope functions and multipliers

$$\pi_i(x,\ \eta)=\frac{1}{U}\begin{vmatrix}0 & u_{1i}'\ldots u_{ni}'\\ \eta_1 & u_{11}\ldots u_{n1}\\ \ldots & \ldots\\ \eta_n & u_{1n}\ldots u_{nn}\end{vmatrix},\quad \lambda_\alpha(x,\ \eta)=\frac{1}{U}\begin{vmatrix}0 & \rho_{1\alpha}\ldots\rho_{n\alpha}\\ \eta_1 & u_{11}\ldots u_{1n}\\ \ldots & \ldots\\ \eta_n & u_{1n}\ldots u_{nn}\end{vmatrix}\tag{30}$$

for the region F_ω. The Hilbert integral for the second variation

$$2\int\{\Omega(x,\ \eta,\ \pi,\ \lambda)+\Sigma_i\ (\eta_i'-\pi_i)\Omega_{\eta_i'}(x,\ \eta,\ \pi,\ \lambda)\}dx$$

formed with the slope functions and multipliers $\pi_i(x,\ \eta)$, $\lambda_a(x,\ \eta)$ will be independent of the path and F_ω will be a Mayer field for the integral (25) with these slope functions and multipliers if and only if the equations

$$\Sigma_j\left(\frac{\partial\eta_j}{\partial a_i}\frac{\partial}{\partial a_k}\Omega_{\eta_j'}-\frac{\partial\eta_j}{\partial a_k}\frac{\partial}{\partial a_i}\Omega_{\eta_j'}\right)=0,\quad (i,\ k=1,\ \ldots,\ n),$$

analogous to the equations (20) for the original problem, are satisfied, where η_i is the function of x, a_1, $\ldots$, a_n defined by the first equation (29). But these conditions are equivalent to the system

$$\psi(u_i,\ \rho_i;\ u_k,\ \rho_k)=0,\quad (i,\ k=1,\ \ldots,\ n),$$

which express the fact that every pair of rows of the matrix $\|u_{ij},\ \rho_{i\alpha}\|$ forms a pair of conjugate solutions of the accessory equations.

When the matrix $\|u_{ij},\ \rho_{i\alpha}\|$ is a matrix of conjugate systems of this sort the Weierstrassian formula (12) can be applied in the field F_ω in the $x\eta$-space. Let the arc E_{12} of that formula be replaced by the extremal arc $\eta_i=0(i=1,\ \ldots,\ n)$, and the arc C_{12} by the curve in $x\eta$-space defined by an arbitrarily selected set $\eta_i(x)$ of admissible variations satisfying the conditions $\eta_i(x_1)=\eta_i(x_2)=0$ $(i=1,\ \ldots,\ n)$, Along the former of these arcs the second variation vanishes and the formula of Weierstrass therefore gives

$$I''(0)=\int_{x_1}^{x_2}E_\omega\ (x,\ \eta,\ \pi(x,\ \eta),\ \lambda(x,\ \eta))dx=\int_{x_1}^{x_2}\Sigma_{ik}\ R_{ik}\ (\eta_i'-\pi_i)\ (\eta_k'-\pi_k)dx, \tag{31}$$

where E_ω is the Weierstrass E-function for the function Ω belonging to the second variation. The sets $(x,\ \eta,\ \eta')$ and $(x,\ \eta,\ \pi)$ both satisfy the equations of variation $\Phi_\alpha=0$, and it follows, by subtracting corresponding ones of the equations expressing this fact, that

$$\Sigma_i\frac{\partial\phi_\alpha}{\partial y_i'}(\eta_i'-\pi_i)=0,\quad (\alpha=1,\ \ldots,\ m). \tag{32}$$

The formula (31) is the result of Clebsch's transformation of the second variation.

Theorem. If there exists a matrix $\|u_{ik},\ \rho_{i\alpha}\|$ *of conjugate systems of solutions of the accessory equations having the determinant* $|u_{ik}|$ *different from zero on the interval* x_1x_2, *then the region* F_ω *of the* $x\eta$*-space where the inequality* $x_1\leqq x\leqq x_2$ *holds is a Mayer field for the second variation* $I''(0)$ *with the slope functions and multipliers* $\pi_i(x,\ \eta)$, $\lambda_\alpha(x,\ \eta)$ *defined in equations* (30). *The integral formula of Weierstrass for this field then justifies the expression* (31) *deduced by Clebsch for the value of the second variation* $I''(0)$ *along a set of admissible variations* $\eta_i(x)$ *for which* $\eta_i(x_1)=\eta_i(x_2)=0$ $(i=1,\ \ldots,\ n)$.

It can readily be proved that the differences $\eta_i'-\pi_i$ are identical with the functions W_i of Clebsch* and the functions $\chi_i(\eta)/U$ of von Escherich†. For from equations (30) the differences $\eta_i'-\pi_i$ have the values

$$\eta_i'-\pi_i=\frac{1}{U}\begin{vmatrix}\eta_i' & u_{1i}' & \dots & u_{ni}'\\ \eta_1 & u_{11} & \dots & u_{1n}\\ \dots & \dots & \dots & \dots\\ \eta_n & u_{n1} & \dots & u_{nn}\end{vmatrix} \tag{33}$$

and these are precisely the values of the functions just referred to.

5. *A second proof of the transformation theorem and other formulae.* The transformation theorem of the preceding section can be readily proved without reference to the original Mayer field theory when once it has been viewed from the standpoint there described. For this purpose a fundamental formula will be developed which justifies at once the desired transformation of the second variation, and which may be conveniently applied to simple proofs of two important formulae of von Escherich and Hahn.

Let $||u_{ik},\rho_{i\alpha}||$ be a matrix whose rows are solutions of the accessory equations and for which the determinant $U=|u_{ik}|$ is different from zero on the interval x_1x_2. We may denote by ψ_{ik} the value of the expression (28) formed for the i-th and k-th rows of the matrix. The notations u_i, u_i', ρ_α and v_i, v_i', σ_α will be used for the expressions

$$u_i=a_1u_{ii}+\dots+a_nu_{ni},\ u_i'=a_1u_{1i}'+\dots+a_nu_{ni}',\ \rho_\alpha=a_1\rho_{1\alpha}+\dots+a_n\rho_{n\alpha},$$

$$v_i=a_1'u_{1i}+\dots+a_n'u_{ni},\ v_i'=a_1'u_{1i}'+\dots+a_n'u_{ni}',\ \sigma_\alpha=a_1'\rho_{1\alpha}+\dots+a_n'\rho_{n\alpha},$$

in which the coefficients a_k are now thought of as functions of x and the primes indicate their derivatives. If it is agreed that a prime on Ω or one of its derivatives shall indicate differentiation with respect to x with the coefficients a_k kept constant the formulae

$$u\Omega_{v'}-v\Omega_{u'}=\Sigma_{ik}\psi_{ik}a_ia_k', \tag{34}$$

$$\frac{d}{dx}\,\Omega_{u_i'}=(\Omega_{u_i'})'+\Omega_{v_i'}=\Omega_{u_i}+\Omega_{v_i'}, \tag{35}$$

will be true, since the functions $\Omega_{u_i'}$ are linear in u_i, u_i', ρ_α and since those variables satisfy the accessory differential equations.

For every system of functions $\eta_i(x)$ $(i=1,\dots,n)$ having derivatives on x_1x_2 the equations

$$\eta_i=a_1u_{1i}+\dots+a_nu_{ni},\qquad (i=1,\dots,n), \tag{36}$$

define the coefficients a_k as differentiable functions of x, and in terms of the notations of the last paragraph

$$\eta_i=u_i,\ \eta_i'=u_i'+v_i,\qquad (i=1,\dots,n). \tag{37}$$

*Journal für Math. Bd. 55 (1858), p. 266.
†Wiener Berichte, *loc. cit.*, pp. 1281, 1286.

With the help of equations (27), (26), (37), (35) it follows that

$$\begin{aligned} 2\Omega(x,\ u,\ u',\ \rho)+2(\eta'-u')\Omega_{u'} &= u\Omega_u+2\eta'\Omega_{u'}-u'\Omega_{u'} \\ &= \eta(\Omega_u+\Omega_{v'})+\eta'\Omega_{u'}+v\Omega_{u'}-u\Omega_{v'} \\ &= \frac{d}{dx}\,\eta\Omega_{u'}+v\Omega_{u'}-u\Omega_{v'} \end{aligned}$$

and from (34)

$$2\Omega(x,\ u,\ u',\ \rho)+2(\eta'-u')\Omega_{u'}=\frac{d}{dx}\,\eta\Omega_{u'}-\Sigma_{ik}\,\psi_{ik}a_i a_k'. \tag{38}$$

This is the fundamental formula referred to in the first paragraph of this section.

If the set of functions $\eta_i(x)$ is an admissible set of variations satisfying the equations $\Omega_{\lambda_a}=\Phi_a=0$ on the interval x_1x_2, then it follows from Taylor's formula and the equations (37), (38) that

$$\begin{aligned} 2\omega(x,\ \eta,\ \eta') &= 2\Omega(x,\ \eta,\ \eta',\ \rho)=2\Omega(x,\ u,\ u'+v,\ \rho) \\ &= 2\Omega(x,\ u,\ u',\ \rho)+2v\Omega_{u'}(x,\ u,\ u',\ \rho)+\Sigma_{ik}R_{ik}v_i v_k \\ &= \frac{d}{dx}\eta\Omega_{u'}-\Sigma_{ik}\psi_{ik}a_i a_k'+\Sigma_{ik}R_{ik}(\eta_i'-u_i')\ (\eta_k'-u_k'). \end{aligned} \tag{39}$$

Since u_i' and the functions $\pi_i(x,\ \eta)$ defined by equations (30) are identical this gives for the second variation the value

$$I''(0)=\int_{x_1}^{x_2}\Sigma_{ik}R_{ik}(\eta_i'-\pi_i)(\eta_k'-\pi_k)dx+\eta\Omega_u\Big|_{x_1}^{x_2}-\int_{x_1}^{x_2}\Sigma_{ik}\psi_{ik}a_i a_k'dx$$

when calculated for an arbitrarily selected set of admissible variations $\eta_i(x)$. If the rows of the matrix $||u_{ik},\ \rho_{i\alpha}||$ are conjugate in pairs the last term disappears and the formula can be shown to be equivalent to one given by Hahn*. The formula here given is much simpler than his. If the variations $\eta_i(x)$ all vanish at x_1 and x_2 the second term disappears and we have again the formula (31) of the last section.

In 1916 I published a proof† of a special case of a formula of von Escherich which is fundamental in his treatment of the second variation, and showed how two theorems whose importance has been emphasized by Bolza can be deduced from it without the transformation of the second variation which Bolza uses. A new and much simpler proof of the von Escherich formula can be made as follows.

Let the functions $\eta_i(x)$ be an admissible set of variations. By means of the equations (30) they determine functions $\pi_i(x,\ \eta)$ and multipliers $\lambda_a(x,\ \eta)$ such that the differences $\eta_i'-u_i'=\eta_i'-\pi_i$ satisfy the conditions (32), and it follows without difficulty that the equation

$$v(\Omega_{\eta'}-\Omega_{u'})=(\eta'-u')\Omega_{\eta'}(x,\ 0,\ \eta'-u',\ \lambda-\rho)=\Sigma_{ik}R_{ik}(\eta_i'-u_i')\ (\eta_k'-u_k')$$

*Rendiconti, *loc. cit.*, p. 63, formula 45.

†Bull. Amer. Math. Soc., 2d Series, vol. 22 (1916), p. 220.

holds since $\eta_i - u_i = 0$ and $\Omega_{\eta'}$, $\Omega_{u'}$ are linear in their last three arguments. If $\psi_i(\eta, \lambda)$ denoted the value of the expression (28) formed with the i-th row of the matrix $||u_{ik}, \rho_{i\alpha}||$ and the set η_i, λ_α then

$$v(\Omega_{\eta'} - \Omega_{u'}) = v\Omega_{\eta'} - \eta\Omega_{v'} + u\Omega_{v'} - v\Omega_{u'} = \Sigma_i a_i' \psi_i(\eta, \lambda),$$

since $u\Omega_{v'} - v\Omega_{u'}$ vanishes when the matrix $||u_{ik}, \rho_{i\alpha}||$ has its rows conjugate in pairs. By equating these two expressions for $v(\Omega_{\eta'} - \Omega_{u'})$ one obtains the von Escherich formula*

$$\Sigma_{ik} R_{ik}(\eta_i' - u_i')\ (\eta_k' - u_k') = \Sigma_i\, a_i'\, \psi_i\,(\eta, \lambda)$$

which holds when the rows of the matrix $||u_{ik}, \rho_{i\alpha}|$ are conjugate by pairs and the functions $\eta_i(x)$ are a set of admissible variations. The multipliers λ_α are determined by equations (30). The formula just given is not in the same notation as that of von Escherich but it may be readily identified with his by means of the equations (33) for the differences $\eta_i' - u_i' = \eta_i' - \pi_i$ and the formula

$$a_i = \frac{1}{U}\begin{vmatrix} u_{11} \dots u_{(i-1)\,1}\ \eta_1 & u_{(i+1)\,1} \dots u_{n1} \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ u_{1n} \dots u_{(i-1)\,n}\ \eta_n & u_{(i+1)\,n} \dots u_{nn} \end{vmatrix}$$

for the parameters a_i which are obtained by solving the equations (36).

From the formula (39) it follows that when the rows of the matrix $||u_{ik}, \rho_{i\alpha}||$ are conjugate in pairs and the functions $\eta_i(x)$ are a set of admissible variations we have

$$\begin{aligned}\Sigma_{ik} R_{ik}(\eta_i' - u_i')\ (\eta_k' - u_k') &= 2\Omega(x, \eta, \eta', \rho) - \frac{d}{dx}\eta\Omega_{u'} \\ &= \eta\Omega_\eta + \eta'\,\Omega_{\eta'} - \frac{d}{dx}\eta\Omega_{u'} \\ &= \eta\left(\Omega_\eta - \frac{d}{dx}\Omega_{\eta'}\right) + \frac{d}{dx}\eta(\Omega_{\eta'} - \Omega_{u'}).\end{aligned}$$

This is the formula due to Hahn† which was mentioned in the first paragraph of this section.

A matrix $||u_{ik}, \rho_{i\alpha}||$ whose rows are linearly independent solutions of the accessory equations is called a conjugate system of solutions if its rows are conjugate in pairs so that the constants ψ_{ik} are all zero. Such matrices play an important role in the theory of the second variation. In the paper above referred to Hahn has studied in detail the construction of such conjugate systems, and has shown that all of them may be obtained by differentiating with respect to the constants of integration the functions defining the extremals of a Mayer field for the original problem in xy-space. The results of the preceding pages show that the same relationship holds between conjugate systems and Mayer fields for the problem of minimizing the second variation in the $x\eta$-space.

*Wiener Berichte, *loc. cit.*, p. 1283, formula (9).

†Rendiconti, *loc. cit.*, p. 64, formula (48).

EXPANSION OF FUNCTIONS IN TERMS OF BERNOULLIAN POLYNOMIALS

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1. Taking as definition for the Bernoullian polynomials the expansion

$$\frac{xe^{xz}}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n |z| x^n, \qquad (|x| < 2\pi),$$

we have successively

$$B_0(z) = 1,$$

$$B_1(z) = z + \tfrac{1}{2},$$

$$B_2(z) = \frac{z^2}{2!} + \tfrac{1}{2}\frac{z}{1!} + \frac{1}{12},$$

$$\dots\dots\dots\dots\dots\dots$$

These functions satisfy the following conditions:

$$B'_{n+1}(z) = B_n(z),\ B_{n+1}(z) - B_{n+1}(z-1) = \frac{z^n}{n!}, \qquad (n > 0),$$

from which may be immediately deduced

$$\frac{z^n}{n!} = B_n(z) - \frac{B_{n-1}(z)}{2!} + \frac{B_{n-2}(z)}{3!} - \dots + (-1)^n \frac{B_0(z)}{(n+1)!}.$$

Moreover the following particular values may be noted:

$$B_0(0) = B_0(-1) = 1,$$

$$B_1(0) = -B_1(-1) = \tfrac{1}{2},$$

$$B_{2n}(0) = B_{2n}(-1) = (-1)^{n-1}\frac{B_n}{2n!},$$

$$B_{2n+1}(0) = B_{2n+1}(-1),$$

$$B_{2n}(-\tfrac{1}{4}) = B_{2n}(-\tfrac{3}{4}) = \frac{(-1)^n (2^{1n}-2)B_n}{(2n)!\,2^{4n}},$$

$$B_{2n+1}(-\tfrac{1}{4}) = -B_{2n+1}(-\tfrac{3}{4}) = \frac{(-1)^n E_n}{(2n)!\ 2^{4n+2}},$$

where B_n and E_n represent the Bernoullian and Eulerian numbers*.

*Nielsen, *Recherches sur les fonctions de Bernoulli,* Mém. Acad. Danemark, t. XII, N° 2, 1915

2. Assuming that any function, which is regular in a circle round the origin, may be expanded in a series of the form

$$f(z) = a_0 B_0(z) + a_1 B_1(z) + a_2 B_2(z) + \dots,$$

we can obtain the coefficients in the following way. Putting

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{z^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(0) \left\{ B_n(z) - \frac{B_{n-1}(z)}{2!} + \dots + (-1)^n \frac{B_0(z)}{(n+1)!} \right\}$$

and equating the coefficients of the several Bernoullian polynomials, we get

$$a_0 = f(0) - \frac{1}{2!} f'(0) + \frac{1}{3!} f''(0) - \frac{1}{4!} f'''(0) + \dots = \int_{-1}^{0} f(x) dx,$$

$$a_1 = f'(0) - \frac{1}{2!} f''(0) + \frac{1}{3!} f'''(0) - \frac{1}{4!} f''''(0) + .. = \int_{-1}^{0} f'(x) dx,$$

. . . .

The method by which this result has been found cannot be regarded as a proof, since the possibility of the expansion was assumed. We can, however, furnish a proof by determining directly the sum of the series obtained.

3. Comparing the series

$$a_0 B_0(z) + a_1 B_1(z) + a_2 B_2(z) + \dots,$$

wherein

$$a_n = \int_{-1}^{0} f^{(n)}(x) dx,$$

with the original definition

$$B_0(z) + x B_1(z) + x^2 B_2(z) + \dots = \frac{x e^{xz}}{1 - e^{-x}}, \quad (|x| < 2\pi),$$

it is evident that the first series is convergent if

$$|a_n| < A (2\pi)^n$$

where A is a finite constant independent of n. If this condition is satisfied we may therefore put for all values of z

$$\phi(z) = \int_{-1}^{0} [B_0(z) f(x) + B_1(z) f'(x) + B_2(z) f''(x) + \dots] dx$$

and try to determine the sum $\phi(z)$.

Remarking that

$$\phi(z) - \phi(z-1) = \int_{-1}^{0} [f'(x) + \frac{z}{1!} f''(x) + \frac{z^2}{2!} f'''(x) + \dots] dx$$

$$= \int_{-1}^{0} f'(z+x) dx = f(z) - f(z-1),$$

we see that it is sufficient to prove $\phi(z)=f(z)$ for the values of z between the limits -1 and 0.

Putting

$$B_0(z)f(x)+B_1(z)f'(x)+B_2(z)f''(x)+\ldots=\theta(x, z)$$

and differentiating with respect to x and z we get

$$B_0(z)f'(x)+B_1(z)f''(x)+B_2(z)f'''(x)+\ldots=\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial z}.$$

To establish this equation exactly we shall prove in the next article that the series in the first number is uniformly convergent. Assuming this for a moment the partial differential equation

$$\frac{\partial\theta}{\partial x}-\frac{\partial\theta}{\partial z}=0$$

shows that

$$B_0(z)f(x)+B_1(z)f'(x)+B_2(z)f''(x)+\ldots$$

is a function of the argument $x+z$, and does not vary if we interchange z and x. Therefore

$$\phi(z)=\int_{-1}^{0}[B_0(x)f(z)+B_1(x)f'(z)+B_2(x)f''(z)+\ldots]dx,$$

or finally, as

$$\int_{-1}^{0}B_n(x)dx=0, \qquad (n=1, 2, 3, \ldots),$$

we have

$$\phi(z)=f(z).$$

4. To prove that the series

$$B_0(z)f'(x)+B_1(z)f''(x)+B_2(x)f'''(x)+\ldots,$$

where both z and x are limited between -1 and 0, is uniformly convergent, we must first examine the maximum values of $B_k(z)$.

Supposing k even, then the maximum and minimum values of the function

$$y=B_{2n}(z)$$

will be found when we substitute for z the roots of

$$B_{2n-1}(z)=0.$$

Now the only roots in the interval being

$$-1,\ -\tfrac{1}{2},\ 0$$

we obtain

$$B_{2n}(-1)=B_{2n}(0)=(-1)^{n-1}\frac{B_n}{(2n)!} \quad B_{2n}(-\tfrac{1}{2})=(-1)^n\frac{2^{2n}-2}{(2n)!\,2^{2n}}\,B_n;$$

thus $B_{2n}(z)$ can never exceed the positive value

$$\frac{B_n}{(2n)!}.$$

Next suppose k odd, then the maximum and minimum values of

$$y = B_{2n+1}(z)$$

will be found by substituting for z the roots of

$$B_{2n}(z) = 0.$$

This equation has in the interval two roots equidistant from $-\frac{1}{2}$.

Let $-\frac{1}{4}+\alpha_{2n}$ be one of these roots, then we have

$$B_{2n}(-\tfrac{1}{4}) + \alpha_{2n} B_{2n-1}(-\tfrac{1}{4}) + \frac{\alpha_{2n}^2}{2!} B_{2n-2}(-\tfrac{1}{4}) + \ldots = 0.$$

Neglecting all powers of α_{2n} higher than the first, since α_{2n} is small, we get as a first approximation to the root, the value

$$-\tfrac{1}{4} - \frac{B_{2n}(-\frac{1}{4})}{B_{2n-1}(-\frac{1}{4})} = -\tfrac{1}{4} + \frac{2^{2n-1}-1}{4n(2n-1)} \frac{B_n}{E_{n-1}},$$

which for $n=1, 2, 3, 4, \ldots$ reduces to

$$-0.2083, \ -0.2403, \ -0.24754, \ -0.24938, \ldots$$

It is evident therefore that these values converge to the limit -0.25 and that for values of n sufficiently large

$$B_{2n+1}(-\tfrac{1}{4}+\alpha_{2n}) \equiv \frac{(-1)^n E_n}{2n!\, 2^{4n+2}};$$

thus $|B_{2n+1}(-\frac{1}{4}+\alpha_{2n})|$ converges to the value

$$\frac{E_n}{2n!\, 2^{4n+2}}.$$

Returning to the series and writing

$$B_0(z) f'(x) + B_1(z) f''(x) + \ldots B_{2n-1}(z) f^{(2n)}(z) + R_{2n}$$

we have

$$R_{2n} = B_{2n}(z) f^{(2n+1)}(x) + B_{2n+1}(z) f^{(2n+2)}(x) + \ldots$$

If now we replace

$|f^{(n)}(x)|$ by $C\,(2\pi-\delta)^n$, (δ small positive quantity, C constant),

$$B_{2k}(z) \text{ by } \frac{B_k}{(2k)!}, \quad B_{2k+1}(z) \text{ by } \frac{E_k}{(2k)!\; 2^{4k+2}},$$

we obtain

$$|R_{2n}(z)| < C(2\pi-\delta)\left[\frac{B_n}{(2n)!}(2\pi-\delta)^{2n} + \frac{B_{n+1}}{(2n+2)!}(2\pi-\delta)^{2n+2} + \ldots\right]$$

$$+ C\frac{(2\pi-\delta)^2}{2^2}\left[\frac{E_n}{(2n)!}\left(\frac{\pi}{2}-\frac{\delta}{4}\right)^{2n} + \frac{E_{n+1}}{(2n+2)!}\left(\frac{\pi}{2}-\frac{\delta}{4}\right)^{2n+2} + \ldots\right].$$

Comparing this with the known expressions

$$1-\frac{x}{2}\cot\frac{x}{2}=\frac{B_1}{2!}x^2+\frac{B_2}{4!}x^4+\frac{B_3}{6!}x^6+\ldots,\quad(-2\pi<x<2\pi),$$

and

$$\sec x=1+\frac{E_1}{2!}x^2+\frac{E_2}{4!}x^4+\frac{E_3}{6!}x^6+\ldots,\quad\left(-\frac{\pi}{2}<x<\frac{\pi}{2}\right),$$

it is evident that to any small positive number ϵ we can find a corresponding integer n, such that

$$|R_{2n}(z)|<\epsilon$$

for all values of z within the interval -1 to 0, which proves that the preceding series is uniformly convergent.

We have thus established the result that a regular function $f(z)$ may be expanded in the following way, which holds for all values of z:

$$f(z)=a_0B_0(z)+a_1B_1(z)+a_2B_2(z)+\ldots$$

if $a_n=\int_{-1}^{0}f^{(n)}(x)dx$ and $|f^{(n)}(z)|<C(2\pi)^n$ between the limits 0 and -1, where C is a constant independent of n.

This condition is evidently equivalent to the former condition

$$|a_n|<A(2\pi)^n,$$

for

$$|a_n|=|f^{(n-1)}(0)-f^{(n-1)}(-1)|<2C(2\pi)^{n-1}=\frac{C}{\pi}(2\pi)^n.$$

5. Finally we may remark that the Eulerian polynomials, defined by the equation

$$\frac{e^{xz}}{1+e^{-x}}=\sum_{n=0}^{\infty}E_n(z)x^n,\qquad(|x|<\pi),$$

may be treated in the same way. Here we find that a regular function $f(z)$ may be expanded in the form

$$f(z)=a_0E_0(z)+a_1E_1(z)+a_2E_2(z)+\ldots$$

if

$$a_n=f^{(n)}(0)+f^{(n)}(-1)$$

and

$$|f^{(n)}(z)|<C\pi^n.$$

ON THE ASYMPTOTIC PROPERTIES OF A CERTAIN CLASS OF TCHEBYCHEFF POLYNOMIALS

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1. Denote by

$$\phi_n(p;\, x) = a_n(p)(x^n - S_n(p)x^{n-1} + \ldots) \qquad (n = 0,\, 1,\, 2 \ldots;\, a_n > 0) \tag{1}$$

a system of orthogonal and normal "*Tchebycheff polynomials*" corresponding to a given interval $(a,\, b)$ with the "*characteristic function*" $p(x)$ integrable and not negative in $(a,\, b)$. These polynomials are uniquely determined by means of the relations:

$$\int_a^b p(x)\phi_m(p;x)\phi_n(p;x)dx = \begin{cases} 0, & m \neq n\,, \\ 1, & m = n\,. \end{cases} \tag{2}$$

In the case where $(a,\, b)$ is *finite* and $p(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}(\alpha,\, \beta > 0)$ we get Jacobi's polynomials.

Tchebycheff polynomials, it is well known, are closely connected with the continued fractions:

$$\int_a^b \frac{p(y)}{x-y}\, dy \sim \cfrac{\lambda_1(p)}{x - c_1 - \cfrac{\lambda_2(p)}{x - c_2 - \cdots}}\,. \tag{3}$$

$$\int_a^b \frac{p(y)}{x-y}\, dy \sim \cfrac{1}{l_1 x + \cfrac{1}{l_2 + \cfrac{1}{l_3 x + \cdots}}} = \cfrac{b_1(p)}{x - \cfrac{b_2(p)}{1 - \cfrac{b_3(p)}{x - \cdots}}} \qquad (0 \leq a < b). \tag{4}$$

2. Consider two characteristic functions $p(x)$ and $p_1(x)$, where

$$p_1(x) = \Pi(x)p(x) \equiv (cx^s + \ldots)p(x) = c\prod_{i=1}^{s}(x - a_i)p(x), \tag{5}$$

the polynomial $\Pi(x)$ having real non-negative coefficients in $(a,\, b)$.

We then have

$$\Pi(x)\phi_n(p_1;x) = \sum_{i=0}^{s} h_{n+i}^{(n)}\phi_{n+i}(p;\, x); \tag{6}$$

(7) $$\Pi(x)\phi_n(p_1;x) = \frac{(-1)^s h_{n+s}^{(n)}}{\Delta_{n,s}} \begin{vmatrix} \phi_n(p;x)\,\phi_{n+1}(p;x)\ldots\phi_{n+s}(p;x) \\ \phi_n(p;a_1)\ldots\ldots\ldots\ldots\phi_{n+s}(p;a_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_n(p;a_s)\ldots\ldots\ldots\ldots\phi_{n+s}(p;a_s) \end{vmatrix};$$

(8) $$\Delta_{n,i} = \begin{vmatrix} \phi_n(p;\,a_1)\ldots\phi_{n+i-1}(p;\,a_1)\;\;\phi_{n+i+1}(p;\,a_1)\ldots\phi_{n+s}(p;\,a_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_n(p;\,a_s)\ldots\phi_{n+i-1}(p;\,a_s)\;\;\phi_{n+i+1}(p;\,a_s)\ldots\phi_{n+s}(p;\,a_s) \end{vmatrix};$$
$$(i=0,\ 1,\ 2,\ldots,s).$$

[If, for example, $a_k = a_{k+1}$, then the row $\phi_n(p;\,a_{k+1})\ldots$ is to be replaced by $\phi_n'(p;\,a_k)\ldots$etc.].

Formula (8) was given by Christoffel* for Legendre's polynomials [$(a,\ b)$ finite; $p(x)\equiv 1$]. Therefore, we write:

(9) $$\frac{h_{n+i}^{(n)}}{h_{n+s}^{(n)}} = \frac{(-1)^{s-i}\Delta_{n,i}}{\Delta_{n,s}}, \quad (i=0,\ 1,\ 2,\ \ldots,s).$$

In particular, using (2), we get:

(10) $$h_n^{(n)} = \frac{a_n(p)}{a_n(p_1)},\ h_{n+s}^{(n)} = \frac{ca_n(p_1)}{a_{n+s}(p)}.$$

3. Formula (7) will be the starting point in our discussion. Consider some special cases.

1° $s=1$; $\Pi(x) = c(x-\xi)$, ξ real $<a$ or $>b$, $c=\pm 1$ (according to the value of ξ). Formula (7) gives:

(11) $$\frac{\phi_{n+1}(p;\xi)}{\phi_n(p;\xi)} = \mp\frac{a_n(p)\,a_{n+1}(p)}{a_n^2(p_1)} \quad \begin{pmatrix} -,\ \xi<a \\ +,\ \xi>b \end{pmatrix}.$$

This formula holds for any real ξ not belonging to $(a,\ b)$ and for any characteristic function $p(x)$.

2° $s=2$. $\Pi(x) = (x-\xi)\,(x-\bar{\xi})$, ξ, $\bar{\xi}$ — conjugate complex numbers. Using (7) we get:

(12) $$\begin{cases} \dfrac{K_{n+1}(p;\,\xi)}{K_n(p;\,\xi)} = \dfrac{a_{n+1}^2(p)}{a_n^2(p_1)}, \\ \dfrac{|\phi_{n+1}(p;\,\xi)|^2}{K_n(p;\,\xi)} = \dfrac{a_{n+1}^2(p)}{a_n^2(p_1)} - 1, \\ K_n(p;\,\xi) \equiv \sum\limits_{i=0}^{n} |\phi_i(p;\,\xi)|^2. \end{cases}$$

**Ueber die Gaussche Quadratur* . . ., Journal für Math. 55 (1858), 61-82, s. 77. See also G. Szegö: *Ueber die Entwickelung einer analytischen Funktion* . . ., Mathematische Annalen, Bd. 81-82 (1920-21), 188-212, s. 191.

3° $s=2$; $\Pi(x)=(x-\xi)^2$, ξ—*any real number whatever*. Making use of Darboux's formula

$$K_n(p,x)\equiv\sum_0^n{}_i\phi_i^2\,(p;x)=\frac{a_n(p)}{a_{n+1}(p)}\,[\phi'_{n+1}(x)\,\phi_n(x)-\phi'_n(x)\,\phi_{n+1}(x)]^*, \tag{13}$$

which is a particular case of the formula

$$K_n(p;x,y)\equiv\sum_0^n{}_i\phi_i(p;x)\phi_i(p;y)=\frac{a_n(p)}{a_{n+1}(p)}\,\frac{\phi_{n+1}(x)\phi_n(y)-\phi_n(x)\phi_{n+1}(y)}{x-y}, \tag{14}$$

we obtain:

$$\begin{cases}\dfrac{\phi_{n+1}^2(p;\xi)}{K_n(p;\xi)}=\dfrac{a_{n+1}^2(p)}{a_n^2(p_1)}-1\equiv\alpha_n,\\[2ex] \dfrac{\phi_{n+1}^2(p;\xi)}{\phi_n^2(p;\xi)}=\dfrac{\alpha_n}{\alpha_{n-1}}\,\dfrac{a_n^2(p)}{a_{n-1}^2(p_1)},\\[2ex] \left[\dfrac{\phi_{n+1}(p;x)}{\phi_n(p;x)}\right]'_{x=\xi}=\dfrac{a_{n+1}(p)}{a_n(p_1)}\cdot\dfrac{1}{\alpha_n}.\end{cases} \tag{15}$$

Formulae (15) *hold for any real* ξ *and for any characteristic function* $p(x)$ *and enable us to find the asymptotic expression* (*for* $n\to\infty$) *of* $\dfrac{\phi_{n+1}(p;\xi)}{\phi_n(p;\xi)}$, *also that of* $K_n(p;\xi)$ *in many cases.*

We can apply (15) also to an infinite interval, for instance, to the polynomials of Hermite-Tchebycheff $[p(x)=e^{-x^2};\ (a,b)=(-\infty,+\infty)]$.

4. *From now on we shall assume* (a, b) *finite.* Suppose $p(x)$ satisfies certain general conditions (I°, II°) given in my paper: "Sur le développement de l'intégrale $\int_a^b\frac{p(y)}{x-y}dy\ldots$"† Then‡:

$$\begin{cases}a_n(p)=\left(\dfrac{4}{b-a}\right)^nA(p)\,(1+0(1))\\[2ex] S_n(p)=n\left(\dfrac{b+a}{2}\right)+\sigma(p)+0(1),\end{cases} \tag{16}$$

where $A(p)>0$ and $\sigma(p)$ do not depend upon n.

The proof of (16) is based upon the relations (2) and upon the fact that in the case under consideration

$$a_n(p)\sim\frac{2^{2n}}{(b-a)^n}. \tag{17}$$

*Darboux, *Mémoire sur l'approximation des fonctions des grands nombres . . .*, Jour. de Math., III sér., t. IV (1878), 5-57, 377-416, p. 413.

†Rend. Circ. Mat. Palermo. t. 47 (1923), 25-46, p. 26.

‡*Ibid.*, p. 37.

In addition to (16), if $0 \leq a < b$, we have also (see (4)):

$$(18) \qquad b_{2n}(p) \to \left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)^2, \; b_{2n+1}(p) \to \left(\frac{\sqrt{b}-\sqrt{a}}{2}\right)^2 \; (n \to \infty).$$

Using formulae (15, 16), we establish the following theorems:

Theorem I. For any real ξ *and for any* $p(x)$ *subjected to the conditions* (I°, II°) *above we have, as* $n \to \infty$:

$$1° \qquad \frac{K_{n+1}(p;\,\xi)}{K_n(p;\,\xi)} \to \begin{cases} \dfrac{(\sqrt{|\xi-a|}+\sqrt{|\xi-b|})^4}{(b-a)^2}, & (\xi \leq a \text{ or } \geq b), \\ 1, & (a \leq \xi \leq b); \end{cases}$$

$$2° \qquad \frac{\phi_{n+1}^2(p;\,\xi)}{K_n(p;\,\xi)} \to \begin{cases} \left(\dfrac{4}{b-a}\right)^2 \sqrt{(\xi-a)(\xi-b)}\,[\sqrt{|\xi-a|}+\sqrt{|\xi-b|}]^2, & (\xi \leq a \text{ or} \geq b), \\ 0, & (a \leq \xi \leq b); \end{cases}$$

$$3° \qquad \frac{\phi_{n+1}(p;\,\xi)}{\phi_n(p;\,\xi)} = \pm \frac{(\sqrt{|\xi-a|}+\sqrt{|\xi-b|})^2}{b-a} = \frac{4}{b-a} z_1, \quad \begin{pmatrix} +, \; \xi \geq b \\ -, \; \xi \leq a \end{pmatrix},$$

z_1 *being the root of the equation* $z^2 - z\left(\xi - \dfrac{b+a}{2}\right) + \left(\dfrac{b-a}{4}\right)^2 = 0$ *with the larger modulus*;

$$4° \qquad \left[\frac{\phi_{n+1}(p;\,x)}{\phi_n(p;\,x)}\right]'_{x=\xi} \to \frac{1}{b-a} \frac{(\sqrt{|\xi-a|}+\sqrt{|\xi-b|})^2}{\sqrt{(\xi-a)(\xi-b)}}, \qquad (\xi < a \text{ or } > b).$$

Formula 3° proves (for ξ real) the famous Poincaré's theorem* *completely*, *i.e.*, it gives exactly the root of the characteristic equation involved. The above results are a particular case of a more general theorem.

Theorem II. In the development (6) *there exists*

$$\lim_{n=\infty} h_{n+i}^{(n)} = h_i \qquad (i = 0, 1, 2, \ldots, s),$$

provided $p(x)$ *satisfies the conditions* (I°, II°).

Making use of formulae (7, 8), from *Theorem II* we derive

Theorem III. Consider s arbitrary points $a_1, a_2, \ldots, a_s$ *(real or complex) subjected to the condition that* $\Pi(x) \equiv \pm \prod_{i=1}^{s} (x - a_i)$ *has real coefficients and is not negative in* (a, b). *Form all determinants* $\Delta_{n,i}$, *given by* (8). *Then, as* $n \to \infty$, $\dfrac{\Delta_{n,i}}{\Delta_{n,s}}$ *tends to a certain limit* $(i = 0, 1, 2, \ldots, s-1)$.

**Sur les équations linéaires* . . . , Amer. Jour. Math., v. VII, 1885, 203-285 p. 217.

5. We now use the results obtained by Szegö† and introduce the following *Definition*: *$p(x)$ is a "function (S)" if*

1°. *$p(x) \geq 0$ and is almost everywhere positive in (a, b);*

2°. *$p(x)$ and $\dfrac{\log p(x)}{\sqrt{(x-a)(b-x)}}$ are (IL), i.e., integrable in Lebesgue's sense in (a, b).*

Theorem IV. Formulae (16, 18), *and therefore Theorems I, II, III, hold for any characteristic function (S). Furthermore, under certain general conditions for $p(x)$,*

$$a_n(p) = \frac{2^{2n}}{(b-a)^n}\sqrt{\frac{2}{(b-a)\pi}}\; e^{-\frac{1}{2\pi}\int_a^b \frac{\log p(x)}{\sqrt{(x-a)(b-x)}}dx}\,(1+0(1)), \tag{19}$$

$$S_n(p) = \frac{n}{2}(b+a) + \frac{1}{2\pi}\int_a^b \frac{\{x-\frac{1}{2}(b+a)\}\log p(x)}{\sqrt{(x-a)(b-x)}}dx + 0(1). \tag{20}$$

Formula (19) was given by Szegö† and can be derived, by using (18). *Formula* (20) *seems to be a new formula, giving the asymptotic expression of the sum of the roots of $\phi_n(p;\, x)$ for $n \to \infty$.*

If we take, for example,

$$(a, b) = (-1, 1),\ p(x) = (1+x)^{\alpha-1}(1-x)^{\beta-1}q(x)$$
$$\alpha, \beta > 0,\ q(x) \equiv q(-x),\ \text{for } -1 \leq x \leq 1,$$

we get, using (20):

$$S_n(p) = \tfrac{1}{2}(\alpha - \beta),$$

a result obtained in a different way in my paper previously referred to‡.

It is interesting to notice that the comparison of Szegö's formula (19) with the results given in *Theorem I* enables us to evaluate certain definite integrals. For example:

$$\int_a^b \frac{\log[(\xi-x)^2]\,dx}{\sqrt{(x-a)(b-x)}} = \begin{cases} 4\pi \log\left\{\dfrac{\sqrt{|\xi-b|}+\sqrt{|\xi-a|}}{2}\right\}, & \xi \leq a \text{ or } \geq b, \\[2ex] 2\pi \log\left(\dfrac{b-a}{4}\right), & a \leq \xi \leq b. \end{cases} \tag{21}$$

6. Using (18) and a linear transformation of the interval we derive the following formulae:

$$\begin{cases} K_n(p;\, \xi) = \sqrt{b^*_{2n+2}(p^*)}\,\phi_n(p;\, \xi)\phi_n(p_1;\, \xi), \\ p_1(x) = \pm(x-\xi)p(x),\ p^*(x^*) \equiv p(\xi \pm x^*), & (+,\ \xi \leq a), \\ & (-,\ \xi \geq b). \\ b^*_{2n+2}(p^*) = \left(\dfrac{\sqrt{|a-\xi|}+\sqrt{|b-\xi|}}{2}\right)^2. \end{cases} \tag{22}$$

7. The following theorem gives some properties of the h_i already obtained from *Theorem II:*

†*loc. cit.*, p. 207.
‡*loc. cit.*, p. 39.

Theorem V. Suppose $p_1(x)=\Pi(x)p(x)$, *where* $p(x)$ *is a function* (*S*) *and* $\Pi(x)=c\prod_{i=1}^{m}(x-a_i)^{2k_i}\prod_{j=1}^{e}(x-a_j)^{l_j}$ $(a<a_i<b;\sum_{i=1}^{m}2k_i+\sum_{j=1}^{e}l_j=s)$. *Consider the polynomial* $\phi(x)=\sum_{i=o}^{s}h_i x^i$ *given in Theorem II. To every* a_j *there corresponds a root of* $\phi(x)$ *of the form*

$$z_j=\frac{2}{b-a}\left[a_j-\frac{b+a}{2}+\sqrt{(a_j-a)(a_j-b)}\right]$$

of multiplicity l_j. *To every* a_i *there correspond two roots of* $\phi(x)$, *each of multiplicity* k_i, *and having the form*

$$z_i^{1,2}=\cos\theta\pm\sqrt{-1}\sin\theta=\frac{2}{b-a}\left[a_i-\frac{b+a}{2}\pm\sqrt{(a_i-a)(a_i-b)}\right],$$

$$\cos\theta=\frac{2}{b-a}\left(a_i-\frac{b+a}{2}\right).$$

8. We apply the general results given above to a special case—*a generaliation of Jacobi's polynomials*. Namely, we take

$$\text{(23)}\qquad\begin{cases}p(x)=(x-a)^{\alpha-1}(b-x)^{\beta-1}\Pi(x), & (\alpha,\beta>0),\\ \Pi(x)=cx^s+\ldots-\text{polynomial}; & \Pi(a)\Pi(b)\neq 0.\end{cases}$$

Without loss of generality we assume $(a, b)=(0, 1)$.

Using the properties of Jacobi's polynomials*, we get the following results:

1° $0<\xi<1$, $\Pi(x)=(x-\xi)^{2m}\Pi_1(x)$, $(m\geq 0,\ \Pi_1(\xi)\neq 0)$;

$$\text{(24)}\qquad\begin{cases}\phi_n^{(e)}(p;\ \xi)=n^{m+e}[C_e\cos(n\theta+c_e)+\epsilon_n(\xi)], & (2\xi-1=\cos\theta),\\ K_n(p;\ \xi)=n^{2m+1}C(1+\epsilon_n(\xi)),\end{cases}$$

where C, C_e, c_e do not depend upon n, and e is any finite positive integer or zero, while $\epsilon_n(\xi)$ represents a quantity which $\longrightarrow 0$ as $n\longrightarrow\infty$.

2° $\xi=0, +1$;

$$\text{(25)}\qquad\begin{cases}\phi_n^{(e)}(p;\ 0)=(-1)^{n-e}n^{\alpha-\frac{1}{2}+2e}C_e'(1+0(1)), & (C_e'\neq 0),\\ \phi_n^{(e)}(p;\ 1)=n^{\beta-\frac{1}{2}+2e}C_e''(1+0(1)), & (C_e''\neq 0),\\ K_n(p;\ {}^{0}_{1})=C_{1,2}n^{2{}^{\alpha}_{\beta}}(1+0(1)), & (C_{1,2}\neq 0)).\end{cases}$$

3° ξ *is any real or complex number not in the interval* $(0, 1)$;

$$\text{(26)}\qquad\begin{cases}\phi_n^{(e)}(p;\ \xi)=n^e z^n f_e(z)(1+\epsilon_n(z)),\\ K_n(p;\ \xi)=|z|^{2n}f(z)(1+\epsilon_n(z)),\\ |z|\equiv|2\xi-1+\sqrt{4\xi^2-4\xi}\,|,\ z\longrightarrow\infty\ \text{as}\,\xi\longrightarrow\infty;\end{cases}$$

*Darboux, *loc. cit.*, pp. 43, 44.

where $f_e(z)$, $f(z)$ do not depend upon n and are $\neq 0$.

Our formulae permit us to find also the asymptotic expression of

$$\sum_{i=0}^{n} \phi_i(p;\ \xi)\phi_i(p;\ \eta),$$

where ξ and η are any two numbers real or complex.

9. In the case under consideration (see formula (23); $a=0$, $b=1$) we are able to make the results concerning b_n, l_n in the continued fractions (4) more precise as follows:

$$(27)\qquad \begin{cases} b_{2n}=\dfrac{1}{4}+\dfrac{2\alpha-1}{8n}+\delta_n,\ b_{2n+1}=\dfrac{1}{4}-\dfrac{2\alpha-1}{8n}+\delta_n{}',\ \ (\delta_n,\delta_n{}'\longrightarrow 0 \text{ as } n\longrightarrow\infty); \\ l_{2n+1}=\phi_n^2(0)=n^{2\alpha-1}C'(1+0(1))\ \ (C'\neq 0),\ \ l_{2n}l_{2n+1}\longrightarrow -4 \text{ as } n\longrightarrow\infty^*. \end{cases}$$

10. In conclusion we consider the $n+1$ zeros of $\phi_{n+1}(p;\ x)$. These zeros, indicated by the notation $x_i^{(n+1)}$ $(i=1,\ \ldots,\ n+1)$, satisfy the inequalities:

$$(28)\qquad 0<x_1^{(n+1)}<x_2^{(n+1)}\ldots\ <x_{n+1}^{(n+1)}<1.$$

Using formula (13), we get:

$$(29)\qquad \begin{cases} x_1^{(n+1)}<\dfrac{a_n(p)}{a_{n+1}(p)}\dfrac{|\phi_{n+1}(p;\ 0)|}{\sqrt{K_n(p;\ 0)}}, \\ 1-x_{n+1}^{(n+1)}<\dfrac{a_n(p)}{a_{n+1}(p)}\dfrac{\phi_{n+1}(p;\ 1)}{\sqrt{K_n(p;1)}}. \end{cases}$$

In any arbitrarily small interval $0<\alpha<\beta<1$ there exists at least one point $x=\lambda$ such that $\phi_n(p;\ \lambda)\neq 0$, for any n. Call $\delta_n(\lambda)$ the shortest distance from the point λ to the roots $x_i^{(n+1}(i=1,\ 2,\ \ldots,\ n+1)$. Then:

$$(30)\qquad \delta_n(\lambda)<\frac{a_n(p)}{a_{n+1}(p)}\frac{|\phi_{n+1}(p;\ \lambda)|}{\sqrt{K_n(p;\ \lambda)}}.$$

Formulae (29, 30) *hold for any* $p(x)$ and give, if $p(x)$ is a function (S):

$$(31)\qquad 1+x_1{}^{(n+1)},\ 1-x_{n+1}^{(n+1)},\ \delta_n(\lambda)\longrightarrow 0 \text{ as } n\longrightarrow\infty,$$

as is well known.

In the case of $p(x)$ given by (23), we have more precisely:

$$(32)\qquad \begin{cases} x_1{}^{(n+1)},\ 1-x_{n+1}^{(n+1)}=0\ (n^{-\frac{1}{2}}), \\ \delta_n(\lambda)=0\ (n^{-\frac{1}{2}}). \end{cases}$$

*See my paper in the Rend. Circ. Mat. Palermo, pp. 41-42.

11. Using the results given above we can find the asymptotic expressions for the coefficients of $\phi_n(p; x)$. Thus we find, for example, if $(a, b)=(-1, 1)$:

$$(33)\qquad \begin{cases} \phi_n(p;x)=a_n(p)\ (x^n-S_n(p)x^{n-1}+d_{n,\,n-2}(p)x^{n-2}+\ldots), \\ d_{n,\,n-3}(p)=n(d(p)+0(1)), \end{cases}$$

where $d(p)$ does not depend upon n*.

12. In the case $(a, b)=(-1, 1)$, $\alpha=\beta$, $\Pi(x)\equiv\Pi(-x)$ we have, if $\Pi(x)=x^{2m}\Pi_1(x)$ $(\Pi_1(0)\neq 0)$:

$$(34)\qquad \begin{cases} \phi_n^{(l)}(p;\ 0)=(-1)^{\frac{n-l}{2}}\ Cn^{m+l}(1+0(1)), \quad \text{for } n\equiv l \pmod 2, \quad (C\neq 0) \\ \phi_n^{(l)}(p;\ 0)=0 \text{ for } n\equiv l+1 \pmod 2. \end{cases}$$

The proof of (34) is based upon the following Lemma:

Lemma. Consider the interval $(-h,h)$ *(finite or infinite) with the characteristic function* $p(x)$ *such that*

$$p(x)\equiv p(-x) \text{ for } -h\leq x\leq h.$$

Consider also the interval $(0, h^2)$ *with*

$$p_1(x)\equiv\frac{p(\sqrt{x})}{\sqrt{x}},\ p_2(x)\equiv\sqrt{x}\,p(\sqrt{x}).$$

Then,

$$\phi_{2n}(p;\ x)\equiv\phi_n(p_1;\ x^2);\ \phi_{2n+1}(p;\ x)\equiv x\phi_n(p_2;\ x^2).$$

Using this Lemma we reduce in many cases the investigation of Tchebycheff polynomials corresponding to $(-\infty, \infty)$ to those corresponding to $(0, \infty)$. This is, for example, the case of the two systems of Tchebycheff polynomials: Hermite-Tchebycheff and Laguerre-Tchebycheff.

*See my paper, *loc. cit.*, p. 46.

SUR LES SÉRIES DE FONCTIONS ORTHOGONALES

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1. A. Kolmogoroff et G. Seliverstoff* ont démontré que si

$$\sum_{n=1}^{\infty}(a_n^2+b_n^2)(\log n)^{1+\delta}, \quad (\delta > 0),$$

converge, la série

$$\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converge presque partout. La méthode de ces deux auteurs peut s'appliquer à d'autres séries de fonctions orthogonales; son succès dépend de la possibilité de trouver une estimation précise de

$$\int_M \int_M \Phi_{m,n}(x,y)\,dxdy,$$

M désignant un ensemble mesurable quelconque de l'intervalle (a, b) pour lequel les fonctions continues $\phi_p(x)$ forment un système orthogonal normé, et $\Phi_{m,n}$ étant définie par

$$\Phi_{m,n}(x, y) = \underset{m<r\leqq s\leqq n}{\text{maximum}} \sum_{p=r}^{s} \phi_p(x)\phi_p(y).$$

On a, en effet, en notant

$$U_{m,n}(x) = \underset{m<r\leqq s\leqq n}{\text{maximum}} \sum_{p=r}^{s} a_p\phi_p(x),$$

$$\bar{U}_{m,n}(x) = \underset{m<r\leqq s\leqq n}{\text{maximum}} \left(-\sum_{p=r}^{s} a_p\phi_p(x)\right),$$

l'inégalité

$$\left|\int_M U_{m,n}(x)\,dx\right|^2 \leqq \sum_{p=m+1}^{n} a_p^2 \int_M \int_M \Phi_{m,n}(x,y)\,dxdy$$

et une inégalité analogue pour $\bar{U}_{m,n}(x)$. Or, si

$$U_m(x) = \lim_{n\to\infty} U_{m,n}(x), \qquad \bar{U}_m(x) = \lim_{n\to\infty} \bar{U}_{m,n}(x),$$

**Sur la convergence des séries de Fourier.* Comptes Rendus Acad. Sciences, Paris, t. 178 (1924), p. 301-303.)

la série $\sum_{p=1}^{\infty} a_p\phi_p(x)$ convergera certainement presque partout dans (a, b) lorsque, pour tout ensemble mesurable M de mesure inférieure à $b-a$, $\int_M U_m dx$ et $\int_M \bar{U}_m dx \to 0$ pour $m \to \infty$.

Les formules asymptotiques connues permettent une estimation de

$$\int_M \int_M \Phi_{m,n}(x, y)dxdy$$

dans le cas des polynomes de Legendre, des fonctions de Bessel et d'une classe étendue de polynomes orthogonaux.

On démontre ainsi, par exemple, que si

$$\sum_{n=1}^{\infty} \frac{a_n^2}{2n+1}(\log n)^{1+\delta}, \quad (\delta>0),$$

converge, la série

$$\sum_{n=0}^{\infty} a_n P_n(x)$$

converge presque partout dans $(-1, +1)$. Résultats analogues pour les séries de fonctions de Bessel et pour d'autres développements en série de polynomes orthogonaux.

2. Des considérations analogues montrent que si $f(x)^2$ est intégrable dans $(0, \infty)$ et si

$$\int_1^{\infty} f(x)^2(\log x)^{1+\delta}dx, \quad (\delta>0),$$

est finie, la limite de

$$\int_0^z f(x) \cos xydx$$

pour $z \to \infty$ existe presque partout et qu'il en est de même de celle de

$$\int_0^z xf(x)\, I_\nu(xy)dx, \quad (\nu> -1),$$

si $xf(x)^2$ est intégrable dans $(0, \infty)$ et si $\int_1^{\infty} xf(x)^2 (\log x)^{1+\delta} dx$ est finie.

3. Rademacher* a démontré, en utilisant un théorème de I. Schur, que si les fonctions $f_n(x)$ sont de carré intégrable dans (a, b) et si la forme quadratique

$$\int_a^b (z_1 f_1(x)+z_2 f_2(x)+\dots)^2 dx$$

**Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen*, Mathematische Annalen, Ed. 87 (1922), s. 126.

est bornée, la convergence de

$$\sum_{\nu=1}^{\infty} a_\nu^2 (\log \nu)^2$$

entraîne la convergence de

$$\sum_{p=1}^{\infty} a_p f_p(x)$$

presque partout dans (a, b). Il peut être utile de remarquer que ce résultat peut s'établir directement sans utiliser le théorème de Schur, et cela par la méthode qu'emploie Rademacher pour établir le théorème II de son mémoire.

SUR LA THÉORIE DES DIFFÉRENCES

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I

Étant donnée une suite de nombres quelconques, u_0, u_1, ..., u_n, ..., la théorie des différences se limite habituellement à la considération de la fonction linéaire des u_i, définie par l'égalité symbolique

$$\Delta^n u_0 = (u-1)^n. \tag{1}$$

Bien que de nombreux auteurs et notamment Laplace, dans sa *Théorie Analytique des Probabilités*, aient étudié des symboles beaucoup plus généraux, je me bornerai dans cette note à envisager, comme étant une généralisation immédiate de (1), l'expression

$$v_n = \Delta_m^n u_0 = (u-m)^n. \tag{2}$$

Cette expression donne lieu à la formule inverse

$$u_n = (v+m)^n, \tag{3}$$

et son introduction dans les calculs peut présenter quelques avantages. C'est ainsi que si les nombres u_i sont récurrents et satisfont à l'équation à coefficients constants

$$f(u) = 0,$$

les nombres v_i satisferont à l'équation

$$f(v+m) = 0.$$

En particulier, si f est du second degré, on pourra déterminer le paramètre m de telle manière que $f'(m) = 0$ ou que $f(m) = 0$. Dans les deux cas, les nombres v_i seront en progression géométrique; leur expression s'obtiendra facilement et, en la portant dans (3), on déterminera l'expression générale d'un terme d'une suite récurrente du second ordre.

II

Je me propose actuellement, connaissant la fonction

$$\phi(z) = \sum_{n=0}^{\infty} u_n \frac{z^n}{n!}, \tag{4}$$

holomorphe au voisinage de l'origine, de déterminer la somme de la série

(5) $$y=\sum_{n=o}^{\infty}\Delta_n^n u_0 \frac{\alpha^n}{n!}$$

où α désigne une nouvelle variable. D'après (4), $\frac{u_n}{n!}$ est le résidu à l'origine de $\frac{\phi(z)}{z^{n+1}}$. On peut donc écrire

$$\frac{\Delta_n^n u_0}{n!}=\mathcal{E}\frac{\phi(z)}{[z^{n+1}]}\left(1-\frac{nz}{1}+\frac{n^2z^2}{1.2}-\ldots\right)$$

ou bien, en prolongeant à l'infini la série entre parenthèses, ce qui revient à ajouter une partie entière à la fonction sous le signe:

$$\frac{\Delta_n^n u_0}{n!}=\mathcal{E}\frac{\phi(z)\,e^{-nz}}{[z^{n+1}]}.$$

On a donc

(6) $$y=\sum_{n=o}^{\infty}\alpha^n\,\mathcal{E}\frac{\phi(z)e^{-nz}}{[z^{n+1}]}.$$

Remplaçons les résidus par des intégrales curvilignes et prenons comme contour d'intégration une circonférence décrite de l'origine comme centre et telle qu'en la décrivant, on ait constamment

$$\operatorname{mod}\frac{\alpha e^{-z}}{z}<1;$$

un calcul simple montre que la valeur maximum à attribuer à mod α est $\frac{1}{e}$; la valeur correspondante de mod z est l'unité.

Supposons donc la fonction $\phi(z)$ holomorphe dans un cercle C de rayon *un;* d'une part, nous pourrons, en revenant à la notation des résidus, intervertir les signes Σ et $\mathcal{E}$ de l'égalité (6) et d'autre part, dans le cercle C, l'équation

$$ze^z=\alpha$$

n'aura qu'une seule racine que je désignerai par ζ.

On obtient donc l'égalité:

$$y=\mathcal{E}\frac{\phi(z)}{[z-\alpha e^{-z}]},$$

le résidu étant relatif cette fois à la racine ζ, ou bien

(7) $$y=\frac{\phi(\zeta)}{1+\zeta}.$$

Ce résultat s'établit, sans faire usage du calcul des résidus, en remarquant que de l'équation (4) on déduit

(8) $$\phi(z)e^{-pz}=\sum_{n=o}^{\infty}\Delta_p^n u_0\frac{z^n}{n!},$$

d'où

$$\Delta_n^n u_0 = \frac{d^n[\phi(z)e^{-nz}]}{dz^n}$$

et par suite

$$y = \sum_{n=o}^{\infty} \frac{\alpha^n}{n!} \frac{d^n[\phi(z)e^{-nz}]}{dz^n}.$$

La série de Lagrange donnera alors la formule (7).

On trouvera de même

(9) $$\phi(\zeta) = u_0 + \frac{\alpha}{1} u_1 + \ldots + \frac{\alpha^n}{n!} \Delta_n^{n-1} u_1 + \ldots$$

En supposant par exemple

$$\phi(z) = \frac{z}{e^z - 1} = \sum_o^{\infty} b_\nu \frac{z^\nu}{\nu!},$$

on aura

$$\Delta_n^n b_0 = b_n + (-1)^n n(1 + 2^{n-1} + 3^{n-1} + \ldots + n^{n-1})$$
$$= b_n + (-1)^n n\, S_{n-1}(n+1),$$

le polynome S_i étant un polynome de Bernoulli. La formule (7) donne alors le développement

$$\frac{\zeta}{(1+\zeta)(e^\zeta - 1)} = \frac{\alpha}{e^\alpha - 1} - \alpha\left[S_0(2) - \frac{\alpha}{1} S_1(3) + \frac{\alpha^2}{2!} S_2(4) - \ldots\right].$$

Si, au contraire, on suppose connue la série

$$f(z) = \sum_{n=o}^{\infty} v_n \frac{z^n}{n!}$$

et si l'on pose

(10) $$\Delta_i^i u_0 = v_i, \qquad (i = 0, 1, 2, \ldots),$$

ces équations définissent sans ambiguïté les nombres u_i et la fonction

$$\phi(z) = \sum_{n=o}^{\infty} u_n \frac{z^n}{n!}$$

s'exprimera par l'égalité

(11) $$\phi(z) = (1+z) f(ze^z).$$

Soient, par exemple, $\psi_0(x)$, $\psi_1(x)$, ..., $\psi_n(x)$, ... les polynomes dont la fonction génératrice est

$$\frac{e^{\frac{\alpha x}{1+\alpha}}}{1+\alpha} = \sum_{\nu=o}^{\infty} \psi_\nu(x) \frac{\alpha^\nu}{\nu!}$$

et posons

$$\Delta_i^i u_0 = \psi_i(x), \qquad (i = 0, 1, 2, \ldots);$$

on aura

$$\phi(z)=\sum_{n=o}^{\infty} u_n(x)\,\frac{z^n}{n!} = \frac{1+z}{1+ze^z}\,e^{\frac{xze^z}{1+ze^z}}\,.$$

Les polynomes $\psi_i(x)$ jouissent d'une propriété d'inversion analogue à celle qui est exprimée par la comparaison des formules (2) et (3). Si l'on désigne en effet, comme au paragraphe I, par $u_0, u_1, \ldots, u_n, \ldots$ des nombres arbitraires et si l'on fait, pour $n=0, 1, 2, 3, \ldots$

$$v_n=\alpha^n\psi_n\left(\frac{u}{\alpha}\right)=u_n-\frac{n^2}{1}\,\alpha\,u_{n-1}+\frac{n^2(n-1)^2}{1.2}\,\alpha^2\,u_{n-2}-\ldots\ldots,$$

on a symboliquement

$$e^{vx}=\sum_{n=o}^{\infty}\alpha^n\psi_n\left(\frac{u}{\alpha}\right)\frac{x^n}{n!} = \frac{e^{\frac{ux}{1+\alpha x}}}{1+\alpha x},$$

et en faisant le changement de variable

$$\frac{x}{1+\alpha x} = z,$$

$$e^{uz}=\frac{1}{1-\alpha z}\,e^{\frac{vz}{1-\alpha z}}=\sum_{n=o}^{\infty}(-1)^n\,\alpha^n\psi_n\left(-\frac{v}{\alpha}\right)\frac{z^n}{n!}\,.$$

Par conséquent

$$u_n=(-1)^n\psi_n\left(-\frac{v}{\alpha}\right)=v_n+\frac{n^2}{1}\,\alpha\,v_{n-1}+\frac{n^2(n-1)^2}{1.2}\,\alpha^2 v_{n-2}+\ldots\ldots$$

Pareillement, cherchons à exprimer u_n au moyen des nombres $v_i=\Delta_i^i\,u_0$; il suffit, pour y parvenir, de se reporter, à la formule (11) et de remarquer que le coefficient de z^n dans le développement de $f(ze^z)$ est égal à

$$\sum_{i=1}^{n}\frac{v_i}{i!}\,\frac{i^{n-i}}{(n-i)!}\,;$$

on obtient ainsi l'expression

$$u_n=v_n+n\sum_{i=1}^{n-1}\binom{n}{n-1}i^{n-i-1}\,v_i, \tag{12}$$

qui résoud le problème de l'inversion des équations (10).

On pourra, par exemple, faire dans l'équation (12)

$$u_0=1,\quad u_n=\cos nx;$$

alors, en observant que

$$e^{ze^{xi}} = \sum_{n=0}^{\infty}\frac{z^n}{n!}(\cos nx+i\sin nx),$$

et en s'appuyant sur la formule (8), on trouvera que

$$v_\mu=\Delta_\mu^\mu u_0=(1+\mu^2-2\mu\cos x)^{\frac{\mu}{2}}\cos\left[\mu\ \mathrm{arctg}\,\frac{\sin x}{\cos x-\mu}\right].$$

III

On obtient encore des résultats dignes d'intérêt en cherchant à sommer la série

$$f_p(x) = \sum_{n=0}^{\infty} \Delta_n^p u_0 \frac{x^n}{n!},$$

dans laquelle p est un nombre entier fixe. Dans ce cas, les fonctions linéaires, en nombre infini, $\Delta_n^p u_0$, $n=0, 1, 2, \ldots$ ne peuvent être indépendantes puisqu'elles ne renferment que $p+1$ paramètres arbitraires u_0, $u_1, \ldots, u_p$. Par suite, la fonction $\phi(z)$, définie par l'équation (4) doit s'éliminer de l'expression de $f_p(x)$.

En faisant encore usage du calcul des résidus, on trouve en effet

$$\frac{\Delta_n^p u_0}{p!} = \mathcal{E} \frac{\phi(z) e^{-nz}}{[z^{p+1}]},$$

puis

$$f_p(x) = p! \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{E} \frac{\phi(z) e^{-nz}}{[z^{p+1}]} = \mathcal{E} \frac{\phi(z) p!}{[z^{p+1}]} \sum_{n=0}^{\infty} \frac{x^n e^{-nz}}{n!},$$

ou bien

$$f_p(x) = e^x \mathcal{E}\, p! \frac{\phi(z) e^{x(e^{-z}-1)}}{[z^{p+1}]}. \tag{13}$$

Posons

$$e^{x(e^z-1)} = \sum_{n=0}^{\infty} j_n(x) \frac{z^n}{n!},$$

$$j_0(x) = 1, \quad j_1(x) = x, \quad j_2(x) = x(x+1), \quad j_3(x) = x(x^2+3x+1), \ldots\ldots$$

ou, sous forme symbolique,

$$e^{x(e^z-1)} = e^{jz}; \tag{14}$$

écrivons de même, d'après (4),

$$\phi(z) = e^{uz}$$

et l'on aura

$$\phi(z) e^{x(e^{-z}-1)} = e^{(u-j)z}.$$

Le résidu qui figure dans l'égalité (13) est donc

$$(u-j)^p = u_p\, j_0(x) - \binom{p}{1} u_{p-1} j_1(x) + \binom{p}{2} u_{p-2} j_2(x) - \ldots\ldots$$

et l'on a le résultat cherché

$$f_p(x) = e^x (u-j)^p.$$

On en déduit

$$f_p(x) e^{-x} = \sum_{n=0}^{\infty} \left[\Delta_n^p u_0 - \binom{n}{1} \Delta_{n-1}^p u_0 + \binom{n}{2} \Delta_{n-2}^p u_0 - \ldots \right] \frac{x^n}{n!} = (u-j)^p,$$

et si l'on observe que $j_p(x)$ est un polynome de degré p, on voit que l'on a constamment, lorsque n est un entier supérieur ou égal à $p+1$,

$$\Delta_n^p u_0 - \binom{n}{1}\Delta_{n-1}^p u_0 + \binom{n}{2}\Delta_{n-2}^p u_0 - \ldots = 0,$$

relation qui résulte évidemment du fait que $\Delta_n^p u_0$ est un polynome de degré p en n.

Maintenant, les polynomes $j_n(x)$ jouissent de propriétés remarquables, dont je désire signaler les plus simples.

En différenciant (14) par rapport à z, on obtient d'abord

(15) $$j_{n+1} = x(j+1)^n.$$

Différencions de même par rapport à x, il vient

$$j_{n+1}(x) = x[j_n(x) + j_n'(x)].$$

Faisons ensuite dans (14) le changement de variable

$$e^z - 1 = u,$$

d'où

$$z = \log\,(1+u).$$

On aura

$$e^{xu} = (1+u)^j$$

et par suite

(16) $$j(j-1)\ldots(j-n+1) = x^n.$$

Soit alors $\psi(z)$ un polynome entier, développé par la formule d'interpolation de Newton

(17) $$\psi(z) = \psi(0) + \frac{z}{1}\Delta\psi(0) + \frac{z(z-1)}{1.2}\Delta^2\psi(0) + \ldots..$$

Si nous substituons dans (17), pour $i = 0, 1, 2, \ldots, j_i(x)$ à la place de z^i, nous parviendrons à l'identité très générale

(18) $$\psi(j) = \psi(0) + \frac{x}{1}\Delta\psi(0) + \frac{x^2}{1.2}\Delta^2\psi(0) + \ldots\ldots$$

ou, en multipliant les deux membres par e^x,

(19) $$\psi(j)e^x = \psi(0) + \frac{x}{1}\psi(1) + \frac{x^2}{2!}\psi(2) + \ldots\ldots$$

Cette relation montre que les polynomes $j_n(x)$ jouent pour la sommation de la série

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\psi(n),$$

le même rôle que les nombres de Bernoulli dans la sommation de la suite

$$f(0) + f(1) + \ldots + f(n).$$

Soit en particulier, dans (18) et (19), $\psi(z) = z^n$, nous obtiendrons les deux expressions suivantes de $j_n(x)$:

$$j_n(x) = \frac{x}{1}\Delta 0^n + \frac{x^2}{1.2}\Delta^2 0^n + \frac{x^3}{1.2.3}\Delta^3 0^n + \ldots\ldots,$$

$$j_n(x)e^x = x\left[1^{n-1} + \frac{x}{1}2^{n-1} + \frac{x^2}{1.2}3^{n-1} + \ldots\right].$$

Si, dans le polynome $j_n(x)$, on fait $x=1$, les nombres $a_n = j_n(1)$, savoir 1, 1, 2, 5, 15, 52, 203, 877, ... dont la fonction génératrice est

$$e^{e^z-1} = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!},$$

seront tous entiers et s'exprimeront par l'une des formules

$$a_n = \frac{\Delta 0^n}{1} + \frac{\Delta^2 0^n}{1.2} + \frac{\Delta^3 0^n}{1.2.3} + \ldots\ldots,$$

$$(20) \qquad e a_{n+1} = 1^n + \frac{2^n}{1} + \frac{3^n}{1.2} + \ldots\ldots.$$

Quand on regarde n comme une variable continue, le second membre de la formule (20) définit une fonction entière qui offre, avec la fonction $\zeta(s)$ de Riemann des analogies sur lesquelles je ne puis m'étendre ici.

Posons encore

$$k_n(x) = j_n(-x), \qquad c_n = j_n(-1),$$

nous aurons, $\psi(z)$ désignant toujours un polynome, la formule suivante, déduite de (19),

$$(21) \qquad \psi(k)e^{-x} = \psi(0) - \frac{x}{1}\psi(1) + \frac{x^2}{2!}\psi(2) - \ldots\ldots,$$

de sorte que les polynomes $k_n(x)$ apparaissent comme étant analogues aux nombres d'Euler. Si, dans (21), on fait

$$\psi(z) = (\lambda + z)^n \text{ et } x = 1,$$

il viendra

$$(22) \qquad (\lambda + c)^n e^{-1} = \lambda^n - \frac{1}{1}(\lambda+1)^n + \frac{1}{1.2}(\lambda+2)^n - \ldots\ldots$$

Désignons maintenant par $P(\lambda)$ la fonction de Prym,

$$P(\lambda) = \int_0^1 e^{-t} t^{\lambda-1} dt = \frac{1}{\lambda} - \frac{1}{1}\frac{1}{\lambda+1} + \ldots\ldots;$$

nous aurons:

$$(23) \qquad \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}P(\lambda)}{d\lambda^{n-1}} = \frac{1}{\lambda^n} - \frac{1}{1}\frac{1}{(\lambda+1)^n} + \frac{1}{1.2}\frac{1}{(\lambda+2)^n} - \ldots\ldots,$$

et le rapprochement de (22) et de (23) peut servir de base pour constituer l'équivalent de l'analogie bien connue entre les polynomes de Bernoulli et les dérivées de $\log \Gamma(x)$.

SUR LES PROBLÈMES DE REPRÉSENTATION DES FONCTIONS A L'AIDE DE POLYNOMES, DU CALCUL APPROCHÉ DES INTÉGRALES DÉFINIES, DU DÉVELOPPEMENT DES FONCTIONS EN SÉRIES INFINIES SUIVANT LES POLYNOMES ET DE L'INTERPOLATION, CONSIDÉRÉS AU POINT DE VUE DES IDÉES DE TCHÉBYCHEFF.

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1. Je vais donner dans cette communication un résumé succinct de mes recherches, faites dans le courant des dernières années, sur le calcul approché des intégrales définies, la représentation approchée des fonctions à l'aide de polynomes, le développement des fonctions en séries de polynomes et l'interpolation. Bien que ces recherches n'appartiennent pas au domaine des mathématiques modernes, je crois que le champ des idées et des méthodes classiques n'est pas encore entièrement élaboré, et qu'elles méritent encore une certaine attention, au moins au point de vue de leur application pratique.

Je me permets d'affirmer, par exemple, que les méthodes de Tchébycheff, devenues classiques, paraissent loin d'être épuisées et peuvent servir de source d'applications nouvelles et de généralisations utiles.

Le but principal de ma communication consiste à montrer que plusieurs questions qui se rattachent aux problèmes indiqués plus haut, étant examinées sous l'aspect des idées de Tchébycheff, trouvent une solution simple, ce qui confirme encore une fois leur généralité et leur profondeur.

2. Commençons par le problème du calcul des intégrales définies à l'aide des quadratures.

Désignant par $\rho(x)$ une fonction non-négative dans un intervalle $(a,\ b)$, considérons la formule des quadratures

$$(1) \qquad \int_a^b \rho(x) f(x) dx = \sum_{k=1}^{n} A_k f(a_k) + R_n$$

à n ordonnées a_k, les A_k étant des constantes, R_n le reste de la formule.

Désignons par q le degré de précision de la formule (1), q étant un entier compris entre $n-1$ et $2n-1$, et définissons les A_k sous la condition que R_n soit égal à zéro toutes les fois que $f(x)$ se réduit à un polynome de degré $p \leqq q$, c'est-à-dire que

$$(2) \qquad \int_a^b \rho(x) P_p(x) dx = \sum_{k=1}^{n} A_k P_p(a_k),$$

quel que soit le polynome $P_p(x)$.

Bien que le problème ait été l'objet de nombreuses recherches pendant à peu près deux cents ans, l'expression du reste n'est connue à présent que dans quelques cas très particuliers, ce qui est une lacune essentielle, car il faut reconnaître avec L. Kronecker que sans l'expression du reste on n'a pas une formule mathématique.

Moyennant une idée appartenant à Tchébycheff et appliquée par A. Markoff au cas de la formule de Gauss, désignons par $P_p(x)$ le polynome de degré p satisfaisant à

$$a_1+a_2+\ldots+a_m=p+1 \tag{3}$$

conditions, savoir:

$$P_p(b_k)=f(b_k),\ P_p'(b_k)=f'(b_k),\ \ldots,\ P_p^{(a_k-1)}(b_k)=f^{(a_k-1)}(b_k), \tag{4}$$

où, comme on sait,

$$f(x)-P_p(x)\equiv\rho_p(x)=\frac{(x-b_1)^{a_1}(x-b_2)^{a_2}\ldots(x-b_m)^{a_m}}{(p+1)!}f^{(p+1)}(\xi), \tag{5}$$

ξ étant une quantité comprise entre a et b.

Les formules (4) et (5) donnent

$$\int_a^b\rho(x)f(x)dx=\int_a^b\rho(x)P_p(x)dx+\int_a^b\rho(x)\psi_{p+1}(x)f^{(p+1)}(\xi)dx, \tag{6}$$

où

$$\psi_{p+1}(x)=\frac{(x-b_1)^{a_1}(x-b_2)^{a_2}\ldots(x-b_m)^{a_m}}{(p+1)!}. \tag{7}$$

Les formules (6), (1), (2), (4) et (5) donnent

$$R_n=\int_a^b\rho(x)\psi_{p+1}(x)f^{(p+1)}(\xi)dx-\sum_{k=1}^n A_k\psi_{p+1}(a_k)f^{(p+1)}(\xi_k). \tag{8}$$

3. Soit

$$q=n-1+s$$

le degré de précision de la formule (1), s étant un entier compris entre 0 et n.

Supposons que le nombre r des coefficients positifs A_k satisfasse aux conditions

$$r\leqq\frac{q-1}{2}=m-1,\quad r\leqq\frac{q}{2}=m,$$

selon que q est impair ou pair. Faisant $p=q-1$, soit respectivement

$$\psi_{p+1}(x)=\frac{(x-a)(x-b_2)^2\ldots(x-b_m)^2}{(n+s-1)!},$$

ou

$$\psi_{p+1}(x)=\frac{(x-b_1)^2(x-b_2)^2\ldots(x-b_m)^2}{(n+s-1)!};$$

choisissant ensuite r des constantes arbitraires $b_1, b_2, \ldots, b_m$ sous les conditions

$$b_2 = a_1, \; b_3 = a_2, \; \ldots, \; b_{r-1} = a_r \; (q \text{ impair}),$$
$$b_1 = a_1, \; b_2 = a_2, \; \ldots, \; b_r \; = a_r \; (q \text{ pair}),$$

on s'assure, en prenant $f(x) = x^q$, que R_n est positif tandis que dans ce cas il doit être zéro.

On en tire ce théorème général:

Le nombre des coefficients positifs de toute formule des quadratures à n ordonnées et à degré de précision

$$q = n + s - 1, \qquad (n \geqq s \geqq 0),$$

***est nécessairement** supérieur ou égal à*

$$\frac{q+1}{2}, \qquad \textit{si } q \textit{ est impair,}$$

ou à

$$\frac{q}{2} + 1, \qquad \textit{si } q \textit{ est pair.}$$

Il en résulte, entre autres, immédiatement, le théorème connu que *tous les coefficients de la formule de Gauss, ainsi que des formules généralisées par Markoff, sont positifs.*

4. Au moyen du théorème précédent, on parvient aisément, en choisissant dans $\psi_{p+1}(x)$ de la formule (8) les constantes b_k d'une manière convenable, à cette expression du reste R_n de toute formule de quadratures:

$$R_n = A_{n,q} f^{(q+1)}(\xi) - B_{n,q} f^{(q+1)}(\eta). \tag{9}$$

Il faut seulement égaler les $n-r$ des $b_k (k=1, 2, 3, \ldots, m)$ à $n-r$ des ordonnées a_k correspondant aux valeurs négatives de A_k et poser:

$$\psi_{q+1}(x) = \frac{(x-a)\;(x-a_1)^2 \ldots (x-a_{n-r})^2 (x-b_{n-r+2})^2 \ldots (x-b_m)^2}{(q+1)!}, \text{ si } q \text{ est pair,}$$

et

$$\psi_{q+1}(x) = \frac{(x-a_1)^2 (x-a_2)^2 \ldots (x-a_{n-r})^2 (x-b_{n-r+1})^2 \ldots (x-b_m)^2}{(q+1)!}, \text{ si } q \text{ est impair.}$$

On aura alors:

$$A_{n,q} = \int_a^b \rho(x) \psi_{q+1}(x) dx \geqq 0,$$

$$B_{n,q} = \int_a^b \rho(x) \, [\psi_{q+1}(x) - F_{q+1}(x)] dx \geqq 0,$$

$$F_{q+1}(x) = \frac{x^{q-n+1}(x-a_1) \ldots (x-a_n)}{(q+1)!}.$$

5. Dans le cas où le nombre des coefficients positifs est précisément égal à

$$m=\frac{q+1}{2} \text{ ou à } \frac{q}{2}+1,$$

la formule (8) prend la forme simple

$$R_n=Q_{nq}f^{(q+1)}(\xi), \tag{10}$$

où

$$Q_{nq}=\int_a^b \rho(x)F_{q+1}(x)dx.$$

Les formules du reste, obtenues par Markoff pour la formule des quadratures de Gauss et pour les formules indiquées par Markoff lui-même, ne sont que des cas particuliers de celle que nous venons d'indiquer (10).

Le problème de la détermination du reste des formules des quadratures est donc résolu dans toute la généralité possible pour toute fonction $f(x)$, dérivable dans l'intervalle donné (a, b).

6. Une autre question intéressante concernant la théorie des quadratures dites *mécaniques* consiste à trouver les conditions les plus générales de la convergence des formules dont il s'agit.

Ce problème a été résolu peu T. Stieltjes, mais seulement dans le cas de la formule de Gauss. Il a établi sa convergence pour toute fonction $f(x)$ intégrable dans l'intervalle, donné (a, b).

En utilisant un théorème sur la possibilité du développement d'une fonction admettant des dérivées des deux premiers ordres en série de polynomes de Tchébycheff et la méthode analogue à celle que nous venons d'employer, nous pouvons établir d'une manière très simple quelques théorèmes plus généraux dont je veux signaler les suivants:

1. *Toute formule des quadratures dont les coefficients A_k satisfont, quel que soit l'entier n, à l'inégalité*

$$\sum_{k=1}^{n} |A_k|<A,$$

A étant un nombre fixe, converge pour toute fonction continue dans l'intervalle (a, b).

2. *Toute formule à coefficients positifs converge pourvu que la fonction à intégrer $f(x)$ soit intégrable dans l'intervalle (a, b).*

Je me permets d'attirer particulièrement l'attention non seulement sur les résultats obtenus, mais surtout sur la méthode de démonstration. Cette dernière, ainsi que celle de tous les autres résultats que j'indiquerai plus loin, est fondée sur une inégalité simple et sur l'emploi d'une certaine fonction auxiliaire que j'ai introduite pour la première fois en 1911.

Si nous désignons par $P_n(x)$ le polynome de degré n, construit suivant la loi de Fourier, à l'aide de polynomes de Tchébycheff s'écartant le moins possible de zéro, nous pouvons démontrer d'une manière élémentaire l'inégalité

$$|f(x) - P_n(x)| < 4 \frac{\epsilon}{h^2 n}, \tag{11}$$

où $f(x)$ est la fonction auxiliaire définie par la formule

$$f(x) = \frac{1}{h^2} \int_x^{x+h} d\xi \int_\xi^{\xi+h} \phi(z) dz, \tag{12}$$

$\phi(z)$ étant une fonction uniformément continue dans l'intervalle donné, c'est-à-dire que

$$|\phi(x+\delta) - \phi(x)| \leqq \frac{\epsilon}{2} \text{ pour } |\delta| \leqq 2h,$$

h désignant une constante positive arbitraire.

7. L'inégalité (11) et la fonction (12) conduisent aisément à diverses conséquences importantes dont l'une que nous venons d'indiquer, et dont nous avons exposé la démonstration un peu différente dans la note *Sopra la teoria delle quadrature dette meccaniche* (Rendiconti dei Lincei, 1923).

Soit $\phi_k (k=0, 1, 2, 3, \ldots)$ une suite de fonctions orthogonales correspondant à la fonction caractéristique $\rho(x)$ non-négative dans l'intervalle (a, b), $f(x)$ une fonction quelconque. Considérons

$$f(x) = \sum_{k=0}^{n} A_k \phi_k(x) + R_n(x), \quad A_k = \int_a^b \rho(x) f(x) \phi_k(x) dx.$$

Si l'on a: $\lim_{n \to \infty} \int_a^b \rho(x) R_n^2(x) dx = 0$ pour une classe de fonctions $f(x)$, on dira que la suite $\phi_k(x)$ est *fermée* par rapport aux fonctions $f(x)$.

En se servant de l'inégalité (11) et de la fonction (12) on démontre d'une manière fort simple et en toute rigueur ce théorème fondamental de la théorie de fermeture:

Si l'équation de fermeture a lieu pour tout polynome en x elle aura nécessairement lieu pour toute fonction assujettie à la seule condition d'être continue dans l'intervalle donné.

De ce théorème, on déduit d'une manière simple le théorème suivant:

Si un système de fonctions orthogonales $\phi_k(x)$ $(k=0, 1, 2, \ldots)$ est fermé pour tout polynome en x, il est fermé pour toute fonction intégrable dans l'intervalle donné (c'est-à-dire qu'il est absolument fermé).

8. Du théorème du N° 7 résulte immédiatement ce théorème:

La somme de la série

$$\sum_{k=0}^{\infty} A_k \phi_k(x),$$

$\phi_k(x)$ désignant une suite fermée de fonctions orthogonales correspondant à la fonction caractéristique $\rho(x)$ non-négative dans (a, b), est égale à $f(x)$ toutes les fois que cette série converge uniformément.

Je vais indiquer une application nouvelle de ce théorème un peu plus loin.

9. En vertu de la même inégalité (11) et la fonction auxiliaire de la forme

$$f(x) = \frac{1}{h}\int_x^{x+h}\phi(z)dz \tag{13}$$

nous arrivons d'une manière simple à cette proposition:

Pour tout point x de l'intervalle (a, b) où l'intégrale

$$\frac{1}{h}\int_x^{x+h}\phi(z)dz$$

tend vers une limite déterminée pour $h \to 0$, $\phi(x)$ *étant une fonction intégrable dans* (a, b), *il existe un nombre h assez petit et un nombre n assez grand tels qu'on ait:*

$$\left|\frac{1}{h}\int_x^{x+h}\phi(z)dz - P_n(x)\right| < \epsilon$$

où ϵ *est un nombre positif donné à l'avance, et*

$$P_n(x) = \sum_{k=0}^{n} T_k(x, a, b)\int_a^b f(x)T_k(x, a, b)\frac{dx}{\sqrt{(b-x)\,(x-a)}},$$

$T_k(x, a, b)$ *désignant les polynomes de Tchébycheff s'écartant le moins possible de zéro dans* (a, b).

Le théorème précédent présente une généralisation du théorème connu de Weierstrass qui en découle tout de suite si l'on suppose que $\phi(x)$ reste continue dans (a, b).

Il est important de remarquer que notre méthode de démonstration du théorème en question non seulement démontre la possibilité de représentation approchée de fonctions continues à l'aide des polynomes, mais fournit en même temps une expression simple du polynome d'approximation.

10. La connexion intime des recherches précédentes avec le problème de représentation approchée des fonctions à l'aide de toute autre suite de polynomes de Tchébycheff, ainsi qu'avec celui du développement des fonctions continues en séries procèdant suivant les dits polynomes, est évidente.

En effet, en tenant compte des théorèmes de la théorie de fermeture signalée plus haut, j'ai réussi à établir le théorème suivant:

Toute fonction $f(x)$ *satisfaisant à la condition de Cauchy,*

$$|f(x') - f(x)| < \lambda |x' - x|,$$

se développe à l'intérieur de l'intervalle donné (a, b) *en série uniformément convergente de la forme*

$$f(x) = \sum_{k=0}^{\infty} A_k\phi_k(x),$$

quels que soient les polynomes $\phi_k(x)$ *correspondant à la fonction caractéristique* $\rho(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}\theta(x)$, *où* α *et* β *sont des nombres positifs,* $\theta(x)$ *une fonction non négative dans* (a, b) *satisfaisant à la condition de Cauchy.*

De plus, ma méthode de démonstration du théorème énoncé combinée avec un théorème de M. D. Jackson conduit tout de suite à cette proposition:

Le polynome de degré n

$$P_n(x) = \sum_{k=0}^{n} A_k \phi_k(x),$$

$\phi_k(x)$ étant le polynome de Tchébycheff du théorème précédent, fournit une expression approchée de toute fonction $f(x)$ assujettie à la condition de Cauchy, dans tout intervalle intérieur à (a, b), avec une erreur moindre que $\frac{\lambda\sigma}{\sqrt{n}}$, σ désignant un nombre fixe.

Dans le cas particulier des polynomes de Tchébycheff s'écartant le moins possible de zéro, j'ai établi, sans recourir au théorème de M. D. Jackson, cette proposition:

Le polynome de degré n,

$$P_n(x) = \sum_{k=0}^{n} a_k \left(\frac{\sin hk}{hk}\right)^s \cos k \arccos x,$$

fournit pour tout point x de l'intervalle $(-a, +a)$ une expression approchée de toute fonction $f(x)$, vérifiant l'inégalité de Lipschitz,

$$|f(x') - f(x)| < \lambda |(x' - x|^\alpha, \qquad 0 < \alpha \leqq 1,$$

avec une erreur moindre que

$$\frac{\lambda a^2 \tau}{n^\sigma}$$

où

$$\sigma = \alpha - \frac{\alpha}{2(s+1)}, \quad \tau = 1 + \sqrt{2} + \frac{\sqrt{s}}{3}.$$

11. Or, dans certaines questions, comme l'a remarqué Tchébycheff, il nous arrive d'employer le polynome correspondant à la fonction caractéristique $\rho(x)$ assujettie à la seule condition d'être intégrable dans (a, b).

La méthode qui m'a conduit aux résultats que je viens de signaler ne s'applique pas à ce cas général.

J'ai étudié ce dernier cas par une autre méthode faisant usage du théorème du N° 8 et de l'inégalité de M. D. Jackson

$$|f(x) - \Pi_n(x)| < \lambda \frac{\mu}{n^2}, \quad (\mu \text{ est un nombre fixe}),$$

où $\Pi_n(x)$est un certain polynome de degré n, existant pour toute fonction $f(x)$ dont la dérivée satisfait à la condition de Cauchy

$$|f'(x') - f'(x)| < \lambda |x' - x|. \tag{14}$$

Je démontre tout d'abord que pour tout polynome de Tchébycheff, quelle que soit la fonction caractéristique $\rho(x)$, on a

$$|\phi_n(x)| < r_0\sqrt{n}, \quad |A_k| < \frac{M}{k^2} \quad (k \geqq 2), \tag{15}$$

r_0 et M étant des nombres fixes.

Puis, par application des inégalités (15), j'arrive à l'inégalité

$$|A_k\phi_k(x)| < \frac{N}{k^{3/2}},$$

N étant un nombre fixe.

Ces inégalités conduisent tout de suite à la conclusion que la série

$$\sum_{k=0}^{\infty} A_k\phi_k(x) \tag{16}$$

converge absolument et uniformément pour toute suite de polynomes de Tchébycheff, pourvu que la fonction $f(x)$ satisfasse à la condition (14).

Il s'ensuit, en vertu du théorème du N° 8, que

Toute fonction $f(x)$ dont la dérivée $f'(x)$ satisfait à la condition de Cauchy, se développe dans l'intervalle donné (a, b) en série de la forme (16) absolument et uniformément convergente à l'intérieur de l'intervalle donné quelle que soit la fonction caractéristique $\rho(x)$ positive dans (a, b), assujettie à la seule condition d'être intégrable dans (a, b).

On pourra généraliser le résultat obtenu, mais je n'insiste pas sur ce point.

12. Nous passons enfin à l'application des résultats obtenus au problème d'interpolation par la méthode de Tchébycheff. On sait que les formules d'interpolation le plus souvent employées, comme par exemple celle de Lagrange, ne convergent pas, en général, lorsque le degré du polynome d'interpolation croît indéfiniment.

Tchébycheff pose le problème sous un autre point de vue, à savoir:

On connaît la valeur de la fonction $f(x)$ pour $n+1$ valeurs de la variable $x=x_0, x_1, \ldots, x_n$ et l'on suppose que la fonction puisse être représentée par la formule

$$a+bx+cx^2+\ldots+hx^m,$$

m ne surpassant pas n. Il s'agit de trouver les coefficients $a, b, \ldots, h$ en les assujettissant à ne laisser aux erreurs des valeurs $f(x_0), f(x_1), \ldots, f(x_n)$ que la moindre influence possible sur une valeur quelconque $f(x)$.

En se servant de la théorie des fractions continues, il a trouvé ce polynome d'interpolation

$$P_m(x) = \sum_{l=0}^{m} \frac{\sum_{k=0}^{n} \rho(x_k)\psi_l(x_k)f(x_k)}{\sum_{k=0}^{n} \rho(x_k)\psi_l^2(x_k)}\,\psi_l(x)$$

où les $\psi_l(x)$ sont les polynomes définis par les conditions

$$\sum_{k=0}^{n} \rho(x_k)\psi_l(x_k)x_k^s = 0, \qquad (s=0,\ 1,\ 2,\ \ldots,\ l-1).$$

Tchébycheff a montré certains avantages de l'emploi du polynome indiqué.

En appliquant les résultats obtenus plus haut, j'ai trouvé une autre propriété remarquable de la formule considérée de Tchébycheff.

Tout d'abord l'erreur moyenne quadratique décroit indéfiniment, lorsque m tend vers l'infini.

D'autre part, *la formule d'interpolation*

$$f(x) = P_m(x) + R_m(x)$$

converge pour toute fonction $f(x)$ assujettie à la condition de Lipschitz.

En effet, on peut choisir d'abord le nombre m et puis le nombre n, plus grand, mais indépendant de m, si grand qu'on ait

$$|f(x) - P_m(x)| < \epsilon,$$

ϵ étant un nombre positif donné à l'avance.

Cela résulte de ce fait qu'en augmentant indéfiniment le nombre n on fera tendre les polynomes $\psi_m(x)$ vers les polynomes $\phi_m(x)$ de Tchébycheff correspondant à la fonction caractéristique $\rho(x)$.

Cela posé, il suffit de tenir compte des théorèmes démontrés plus haut sur la possibilité du développement des fonctions en séries suivant les polynomes $\phi_k(x)$ pour établir la proposition énoncée.

13. Désignons par h un nombre arbitraire, suffisamment petit et formons $n+1$ intervalles de la forme $(x_k - h,\ x_k + h)$, $(k=0,\ 1,\ 2,\ \ldots,\ n)$.

Partageons chacun de ces intervalles en $2p$ intervalles composants:

$$(x_k - h,\ \xi_1^{(k)}),\ (\xi_1^{(k)},\ \xi_2^{(k)}),\ \ldots,\ (\xi_{p-2}^{(k)}),\ \xi_{p-1}^{(k)}),\ (\xi_{p-1}^{(k)},\ x_k),$$
$$(x_k,\ \xi_p^{(k)}),\ (\xi_p^{(k)},\ \xi_{p+1}^{(k)}),\ \ldots,\ (\xi_{2p-2}^{(k)},\ x_k + h).$$

Cela posé, on peut démontrer ce théorème:

On peut choisir un entier m assez grand et le nombre h assez petit, puis les nombres n et p assez grands pour que le polynome

$$P_m(x) = \sum_{j=0}^{m} \frac{\sum_{k=0}^{n} \rho(x_k)\psi_j(x_k) \dfrac{1}{2p} \sum_{i=1}^{2p} f(\xi_i^{(k)})}{\sum_{k=0}^{n} \rho(x_k)\psi_j^2(x_k)}\ \psi_j(x)$$

fournisse une expression approchée de toute fonction $f(x)$ assujettie à la seule condition d'être continue dans l'intervalle donné, avec une approximation ϵ, donnée à l'avance.

Pour démontrer ce théorème, il suffit d'appliquer d'abord le théorème du numéro précédent à la fonction auxiliaire

$$\phi(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(z)dz.$$

En utilisant la définition usuelle des intégrales définies et en se rappelant que les polynomes $\psi_n(x)$ tendent vers les polynomes de Tchébycheff lorsque $n \longrightarrow \infty$ et que

$$\frac{1}{2p}\sum_{i=1}^{2p} f(\xi_i^{(k)})$$

tend vers $f(x_k)$ pour $p \longrightarrow \infty$, on obtient le théorème en question en choisissant convenablement les nombres indépendants n et p assez grands.

SUR UNE FORMULE D'INTERPOLATION

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Dans son mémoire classique* *Sur la convergence des formules d'interpolation entre ordonnées équidistantes*, M. de la Vallée Poussin a donné une formule d'interpolation comprenant comme cas particulier les formules d'interpolation trigonométrique ordinaire, ainsi que la formule d'interpolation de Lagrange (cas limite). Le mémoire de l'illustre géomètre contient la démonstration de la convergence de cette formule pour toute fonction continue à variation bornée ainsi que l'étude du degré d'approximation quand la fonction à interpoler possède une dérivée à variation bornée.

D'autre part, M. D. Jackson†, donne une autre formule d'interpolation convergente, il est vrai, pour toute fonction continue, mais construite d'après l'idée de l'interpolation généralisée, c'est-à-dire quand les formules d'interpolation se déterminent par un nombre fini de valeurs aux points donnés de la fonction qu'il s'agit de représenter, mais peut ne pas avoir la propriété interpolatoire des formules d'interpolation ordinaire, à savoir la propriété d'acquérir les valeurs prescrites aux points donnés. Par ses formules, basées au fond sur sa méthode de sommation des séries trigonométriques, M. Jackson étudie aussi le degré d'approximation quand la fonction à interpoler vérifie la condition de Cauchy-Lipschitz

$$|f(x_1)-f(x_2)|<\lambda|x_1-x_2|.$$

L'étude des diverses formules de l'interpolation généralisée basées sur les différentes méthodes de sommation des séries trigonométriques a été aussi l'objet d'une note de M. N. Kryloff‡.

Dans le travail présent nous étudions une formule généralisant celle de M. de la Vallée Poussin§. Cette formule possède la propriété interpolatoire au

*Bulletin de l'Académie Royale de Belgique, Classe des sciences, 1908, pp. 319-403.

†*On the accuracy of trigonometric interpolation*, Trans. Amer. Math. Soc., Vol. 14, 1913, pp. 453-461. Voir aussi: *A formula of trigonometric interpolation*, Rend. Circ. Mat. Palermo, 37, 1914, pp. 371-375.

‡*Sur quelques formules de l'interpolation generalisée*, Bull. Sci. Math., 2nd Ser., Vol. 41, 1917, pp. 309-320.

§Voir le mémoire de M. de la Vallée Poussin déjà mentionné.

sens strict de ce mot, dont on a parlé plus haut; elle converge en outre pour toute fonction continue et l'étude de ses diverses particularités n'est pas denuée, ce nous semble, d'un certain intérêt.

1. Commençons par l'indication d'une formule auxiliare utile pour la suite. En appliquant à la fonction

$$\frac{1}{\sin^{2r}x} \tag{1}$$

où r est un entier la méthode usuelle de décomposition en fractions simples on obtient sans peine le développement de la forme

$$\frac{1}{\sin^{2r}x} = \sum_{k=-\infty}^{+\infty}\left[\frac{1}{(x-k\pi)^{2r}} + \frac{A_1^{(r)}}{(x-k\pi)^{2r-2}} + \ldots + \frac{A_{r-1}^{(r)}}{(x-k\pi)^2}\right], \tag{2}$$

où l'expression entre crochets représente la partie infinie de la fonction (1) correspondant au pôle $x=k\pi$; les coefficients s'obtiennent immédiatement par le développement

$$\frac{1}{\sin^{2r} x} = \frac{1}{x^{2r}} + \frac{A_1^{(r)}}{x^{2r-2}} + \ \ldots + \frac{A_{r-1}^{(r)}}{x^2} + P(x), \tag{3}$$

où $P(x)$ est une série procédant suivant les puissances entières et positives de x.

Remarquons en passant, et ceci peut être utile dans bien des recherches, que tous les coefficients $A_k^{(r)}$, sont positifs; on s'en assure d'une manière tout à fait élémentaire. On tire de (2):

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(x-k\pi)^{2r}} = \frac{1}{\sin^{2r}x} + Q_{r-1}\left(\frac{1}{\sin^2 x}\right), \tag{4}$$

où $Q_{r-1}(z)$ est un polynome du degré $r-1$ en z, dépourvu de terme constant. De la formule (4) on déduit:

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(x-k\pi)^{2r}} = \frac{C_r(x)}{\sin^{2r}x}, \tag{5}$$

où $C_r(x)$ est un polynome en $\sin^2 x$ de degré $r-1$:

$$C_r(x) = 1 + b_1^{(r)} \sin^2 x + \ldots + b_{r-1}^{(r)} \sin^{2r-2} x. \tag{6}$$

Le premier membre de la formule (5) est une fonction périodique et positive (ne s'annulant pas) de la variable x, à laquelle nous donnerons désormais seulement des valeurs réelles; donc le numérateur $C_r(x)$ du second membre de (5) reste toujours positif et possède la borne inférieure g positive, c'est-à-dire

$$C_r(x) \geqq g > 0.$$

Remarquons aussi que, d'après (6),

$$C_r(k\pi) = 1,$$

et en particulier pour $r=1,\ 2$, on a

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(x-k\pi)^2} = \frac{1}{\sin^2 x},$$

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(x-k\pi)^4} = \frac{1-\frac{2}{3}\sin^2 x}{\sin^4 x}.$$

2. Soit m un nombre positif arbitraire, que nous allons ensuite faire croître indéfiniment et soit

(7) $$a_k = \frac{k\pi}{m},\ \text{où } k=0,\ \pm 1,\ \pm 2,\ \ldots$$

Prolongeons d'une manière arbitraire la fonction donnée $f(x)$ définie et bornée dans l'intervalle fini (a, b) sur tout l'intervalle $(-\infty, +\infty)$, en supposant seulement que la fonction résultante est bornée de $-\infty$ à $+\infty$ et soit M la borne supérieure du module de la fonction ainsi définie.

Posons

(8) $$F_{r,m}(x) = \frac{\sin^{2r} mx}{m^{2r} C_r(mx)} \sum_{k=-\infty}^{+\infty} \frac{f(a_k)}{(x-a_k)^{2r}};$$

alors en vertu de (7) et (8) on a

(9) $$F_{r,m}(a_k) = f(a_k),$$

c'est-à-dire que *la formule* (8) *est une formule d'interpolation au sens ordinaire de ce mot.* Démontrons à présent le théorème suivant:

En tout point de continuité $x=x_0$ de la fonction $f(x)$ on a

(10) $$\lim_{m\to\infty} F_{r,m}(x_0) = f(x_0),$$

et la convergence sera uniforme dans tout l'intervalle se trouvant à l'intérieur de l'intervalle de continuité de la fonction.

Pour la démonstration, utilisons l'identité:

(11) $$1 = \frac{\sin^{2r} mx}{m^{2r} C_r(mx)} \sum_{k=-\infty}^{+\infty} \frac{1}{(x-a_k)^{2r}},$$

qu'on obtient de (5) en y écrivant mx au lieu de x.

En multipliant les deux membres de (11) par $f(x_0)$ on obtient, en ayant égard à la formule (8) où l'on a posé $x=x_0$,

(12) $$f(x_0) - F_{r,m}(x_0) = \sum_{k=-\infty}^{+\infty} \frac{f(x_0)-f(a_k)}{(x_0-a_k)^{2r}} \frac{\sin^{2r} mx_0}{C_r(mx_0) m^{2r}}.$$

Décomposons la somme du second membre de (12) en deux Σ_1 et Σ_2; dans la somme Σ_1 se trouvent groupés les termes pour lesquels

$$|x_0 - a_k| < \delta,$$

δ étant choisi tel que

$$|f(x_0) - f(a_k)| \leq \frac{\epsilon}{2},$$

où ϵ est un nombre positif arbitrairement petit, donné d'avance; dans la somme Σ_2 sont inclus tous les autres termes.

On aura alors évidemment

$$|f(x_0) - F_{r,\,m}(x_0)| \leq |\Sigma_1| + |\Sigma_2|,$$

où

$$|\Sigma_1| < \frac{\epsilon}{2} \sum_{k=-\infty}^{+\infty} \frac{\sin^{2r} m x_0}{C_r(m x_0)(x_0 - a_k)^{2r} m^{2r}} = \frac{\epsilon}{2},$$

$$|\Sigma_2| < \frac{2M}{g m^{2r} \delta^{2r}} \frac{(b-a)m}{\pi},$$

car la valeur absolue de chaque membre de la somme Σ_2 est au plus égale à

$$\frac{2M}{g m^{2r} \delta^{2r}}$$

et le nombre des termes de Σ_2 ne surpasse pas évidemment celui des points d'abscisses $\frac{k\pi}{m}$ se trouvant à l'intérieur de l'intervalle (a, b), c'est-à-dire

$$(b-a) : \frac{\pi}{m} = \frac{(b-a)\,m}{\pi}.$$

En choisissant m assez grand pour que

$$|\Sigma_2| < \frac{\epsilon}{2},$$

on s'assure que le théorème en question est démontré.

3. L'étude du degré d'approximation fourni par la formule (8) donne le théorème suivant:

Si la fonction $f(x)$ vérifie la condition de Lipschitz:

$$|f(x_1) - f(x_2)| < \lambda |x_1 - x_2|^{\alpha}, \text{ où } 0 < \alpha < 1, \tag{13}$$

on a

$$|f(x) - F_{r,\,m}(x)| < \frac{A\lambda}{m^{\alpha}} \tag{14}$$

où A est une constante ne dépendant pas de m et de $f(x)$, c'est à-dire que l'approximation sera de l'ordre de $m^{-\alpha}$.

Pour $\alpha=1$, c'est-à-dire quand la fonction $f(x)$ vérifie la condition de Cauchy-Lipschitz, l'inégalité (14) *continue à subsister sous la condition que $r \geq 2$; pour $r=1$ l'inégalité* (14) *se trouve remplacée par une autre:*

$$|f(x)-F_{r,m}(x)| < \frac{A\lambda \log m}{m} + B\frac{M}{m}, \tag{15}$$

où B est une constante analogue à A.

En effet, d'après la formule (12) on obtient:

$$|f(x)-F_{r,m}(x)| < \lambda \sum_{k=-\infty}^{+\infty} \frac{|x-\alpha_k|^{\alpha}}{(x-\alpha_k)^{2r}} \frac{\sin^{2r} m(x-\alpha_k)}{m^{2r}}. \tag{16}$$

Prenons le cas $\alpha<1$ et dans la somme du second membre de (16) considérons deux termes correspondant aux valeurs α', α'' de α_k, les plus proches de x respectivement du côté droit et du côté gauche; désignons la somme des termes restants par Σ'. Alors, d'après la signification des nombres α_k, pour chacun de deux termes ci dessus mentionnés on a

$$|x-\alpha_k| \leq \frac{\pi}{m}; \quad \frac{|x-\alpha_k|^{\alpha}}{(x-\alpha_k)^{2r}} \cdot \frac{\sin^{2r} m(x-\alpha_k)}{m^{2r}} \leq |x-\alpha_k|^{\alpha} \leq \frac{\pi^{\alpha}}{m^{\alpha}}.$$

D'autre part, en arrangeant convenablement les termes de la somme Σ' on trouve

$$|\Sigma'| < 2\sum_{k=1}^{\infty} \frac{(k\pi)^{\alpha}}{m^{\alpha}(k\pi)^{2r}} = \frac{A_1}{m^{\alpha}}; \tag{17}$$

donc l'inégalité (14) est démontrée.

Dans tout ceci on suppose, bien entendu, que la fonction $f(x)$ est prolongée d'une manière continue dans tout l'intervalle $(-\infty, +\infty)$, y reste bornée et satisfait à la condition (13) ou à cette même condition avec $\alpha=1$, ce qui est évidemment toujours possible. Le raisonnement précédent continue à subsister même pour $\alpha=1$ sous la condition que $r \geq 2$.

Supposons à présent $r=1$, $\alpha=1$: alors tout ce qui concerne les deux termes considérés plus haut à part reste sans changement; la somme des termes restants:

$$\sum \frac{f(x)-f(\alpha_k)}{(x-\alpha_k)^2} \frac{\sin^2 mx}{m^2} \quad \text{[ici } C_r(x) \equiv 1, \text{ d'après (16)]},$$

peut être décomposée en deux parties Σ'' et Σ''', où la somme Σ'' contient $2N$ termes correspondant aux N valeurs consécutives α_k croissant à partir de α' et aux N valeurs analogues diminuant à partir de α''. Cette somme Σ'' peut être limitée de la même manière que Σ' dans la première partie du théorème et on obtient

$$|\Sigma''| < 2\lambda \sum_{k=1}^{N} \frac{k\pi}{m(k\pi)^2} < \frac{\lambda A_2 \log N}{m}.$$

D'autre part pour la somme Σ''' on a

$$|\Sigma'''| < 4M\sum_{k=N+1}^{\infty} \frac{1}{(k\pi)^2} < \frac{MA_3}{N};$$

en prenant $N=[m]$ (entier de m) on a l'inégalité (15).

En introduisant la notion du *module de continuité* de la fonction $f(x)$* on aurait pu obtenir pour les problèmes considérés des limitations analogues à celles qui ont été obtenues à propos d'une aut e question† par M. de la Vallée Poussin, ce que nous laisserons provisoirement de côté.

4. Considérons à présent comment se comporte la formule d'interpolation (8) aux points de discontinuité de la fonction $f(x)$.

Soit x_0 un point de discontinuité de première espèce de la fonction $f(x)$, c'est-à-dire tel que les limites déterminés $f(x_0 \pm 0)$ existent. Dans le cas où x_0 ne coïncide avec aucun des nombres a_k, c'est-à-dire n'est pas commensurable avec π, on peut obtenir des résultats analogues à ceux que M. de la Vallée Poussin a établi dans son mémoire déjà cité‡ pour sa formule d'interpolation.

En effet, en appliquant aux deux expressions:

(18) $$\sum_{a_k > x_0} \frac{f(x_0+0) - f(a_k)}{(x_0-a_k)^{2r}} \cdot \frac{\sin^{2r} m(x_0-a_k)}{m^{2r} C_r(mx_0)},$$

(19) $$\sum_{a_k < x_0} \frac{f(x_0-0) - f(a_k)}{(x_0-a_k)^{2r}} \cdot \frac{\sin^{2r} m(x_0-a_k)}{m^{2r} C_r(mx_0)},$$

le raisonnement du N° 2 on s'assure que ces deux expressions (18) et (19) tendent vers zéro pour $m \to \infty$; donc l'expression

$$F_{r,m}(x_0) - \frac{f(x_0+0)\ \sin^{2r} mx_0}{C_r(mx_0) m^{2r}} \sum_{a_k > x_0} \frac{1}{(x_0-a_k)^{2r}} - \frac{f(x_0-0)\ \sin^{2r} mx_0}{C_r(mx_0) m^{2r}} \sum_{a_k < x_0} \frac{1}{(x_0-a_k)^{2r}}$$

tend aussi vers zéro pour $m \to \infty$.

Soit à présent, m étant donné,

$$a_s < x_0 < a_{s+1},$$

d'où

$$\pi s < mx_0 < \pi(s+1); \quad s = \left[\frac{mx_0}{\pi}\right].$$

Alors, en posant avec M. de la Vallée Poussin

(20) $$\xi = \frac{mx_0}{\pi} - \left[\frac{mx_0}{\pi}\right] = \frac{mx_0}{\pi} - s,$$

*Ch. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, 1919, p. 7-9.

†*Ibid.*, p. 32.

‡*Loc. cit.*, p. 348-352.

on obtient sans peine:

(21) $$\sum_{a_k > x_0} \frac{1}{m^{2r}(x_0-a_k)^{2r}} = \pi^{2r}\sum_{k=1}^{\infty}\frac{1}{(\xi-k)^{2r}},$$

(22) $$\sum_{a_k < x_0} \frac{1}{m^{2r}(x_0-a_k)^{2r}} = \pi^{2r}\sum_{k=1}^{\infty}\frac{1}{(\xi-1+k)^{2r}}.$$

Par conséquent, en posant:

(23) $$\begin{cases} \psi_r(\xi) = \dfrac{\pi^{2r}\sin^{2r}\pi\xi}{C_r(\pi\xi)}\displaystyle\sum_{k=1}^{\infty}\frac{1}{(\xi-k)^{2r}}, \\ \psi_r(1-\xi) = \dfrac{\pi^{2r}\sin^{2r}\pi\xi}{C_r(\pi\xi)}\displaystyle\sum_{k=1}^{\infty}\frac{1}{(1-\xi-k)^{2r}}, \end{cases}$$

on obtient, en vertu de (11), la relation fondamentale:

(24) $$\psi_r(\xi)+\psi_r(1-\xi)=1,$$

d'où le théorème:

En tout point $x=x_0$ de discontinuité de première espèce de la fonction $f(x)$ qu'on interpole, la fonction $F_{r,m}(x_0)$ ne tend vers aucune limite déterminée, ou, en adoptant les notations (20) et (23), *on aura*

$$F_{r,m}(x_0)=f(x_0+0)\psi_r(\xi)+f(x_0-0)\psi_r(1-\xi)+\epsilon_m,$$

où chacun des coefficients $\psi_r(\xi)$ et $\psi_r(1-\xi)$ est positif, leur somme égale à un, et $\lim_{m\to\infty}\epsilon_m=0$.

Si x_0 est commensurable avec π, c'est-à-dire, par exemple, $x_0=\frac{r\pi}{s}$, où r, s n'ont pas de diviseur commun et $s\neq 0$, alors pour $m=m_\mu=\mu s$

$$F_{r,m_\mu}(x_0)=f(x_0)$$

et par conséquent

$$\lim_{\mu\to\infty} F_{r,m_\mu}(x_0)=f(x_0);$$

donc, comme première valeur limite de la fonction interpolatoire, on obtient dans ce cas $f(x_0)$.

Si on exclue de la suite des nombres positifs entiers tous les multiples de s, alors, pour les nombres restants t, le théorème précédent est vrai, c'est-à-dire que $\lim F_{r,t}(x_0)$ prend des valeurs entre $f(x_0+0)$ et $f(x_0-0)$ et les valeurs limites sont de la forme:

$$f(x_0+0)\psi_r(\xi)+f(x_0-0)\psi_r(1-\xi)$$

puisque ξ est de la forme (20). On voit que ξ ne peut prendre plus que $s-1$ valeurs, d'où le théorème:

Si $x_0 = \frac{r\pi}{s}$ *(r et s étant sans diviseur commun) la fonction* $F_{r,t}(x_0)$ *pour* $t \to \infty$ *tend vers* $f(x_0)$ *et vers* $s-1$ *diverses valeurs limites qui se trouvent entre* $f(x_0-0)$ *et* $f(x_0+0)$.

Les résultats de ce paragraphe, tout en présentant une analogie avec ceux de M. de la Vallée Poussin, sont valables cependant, non-seulement pour les fonctions à variation bornée, mais pour toutes fonctions bornées, possédant seulement les discontinuités dites de la *première espèce.* Une étude plus détaillée montre de même qu'à l'inverse de la formule de M. de la Vallée Poussin la dérivée de la fonction interpolatoire $F'_{r,m}(x)$ non-seulement ne tend pas vers $f'(x)$ pour $m \to \infty$, mais en général ne tend vers aucune limite.

5. Il semble tout naturel de poser la question du degré d'approximation, quand la fonction à interpoler, $f(x)$, possède des dérivées d'ordre supérieur. Bornons-nous à donner une courte indication à ce sujet; à cet effet, remarquons qu'en utilisant les méthodes de M. D. Jackson, et de M. de la Vallée Poussin* on peut représenter la fonction $F_{r,m}(x)$ sous la forme:

$$F_{r,m}(x) = \sum_{k=-\infty}^{+\infty} f(x-u_k) \frac{\sin^{2r} mu_k}{m^{2r} C_r(mu_k)}, \text{ où } u_k = x - a_k,$$

puis introduire les fonctions intermédiaires:

$$F_{r,m,s}(x) = \sum_{k=-\infty}^{+\infty} f\left(x - \frac{u_k}{p_s}\right) \frac{\sin^{2r} mu_k}{m^{2r} C_r(mu_k)}, \tag{25}$$

où $p_1, p_2, \ldots$ sont des nombres entiers, arbitraires, inégaux entre eux; alors en suivant la marche de M. de la Vallée Poussin on peut obtenir une combinaison linéaire des fonctions (25) de la forme

$$\sum_{s=1}^{\mu} a_s F_{r,m,s}(x) \tag{26}$$

et de telle sorte que l'approximation de $f(x)$ par la formule (26) sera d'ordre $\frac{1}{m^{n+\alpha}}$, si $f(x)$ est n fois dérivable et $f^{(n)}(x)$ vérifie la condition de Lipschitz avec l'exposant α.

6. Si la fonction $f(x)$ est périodique (de période 2π) on peut, en donnant à m les valeurs de la forme n ou $n+\frac{1}{2}$ (où n est entier), arriver à des formules de l'interpolation trigonométrique ayant cette particularité qu'elles sont représentées par le quotient de deux polynomes trigonométriques, excepté dans le cas $r=1$, où l'on a, comme d'ordinaire, le polynome trigonométrique.

En prenant par exemple le cas de m entier, on aura, d'après la périodicité de $f(x)$:

$$f(a_k) = f(a_s)$$

si

$$a_k \equiv a_s \pmod{2\pi};$$

**Loc. cit.* Chapitre III.

donc la formule d'interpolation (8) se réduit à la forme

$$(27)\qquad F_{r,m}(x)=\frac{\sin^{2r}mx}{C_r(mx)m^{2r}}\sum_{k=0}^{2m-1}f\left(\frac{\kappa\pi}{m}\right)\sum_{\mu=-\infty}^{+\infty}\frac{1}{\left(x-\frac{\kappa\pi}{m}-2\mu\pi\right)^{2r}}$$

$$=\frac{\sin^{2r}mx}{C_r(mx)m^{2r}}\sum_{k=0}^{2m-1}f\left(\frac{\kappa\pi}{m}\right)\frac{C_r\left(\frac{x}{2}-\frac{\kappa\pi}{2m}\right)}{2^{2r}\sin^{2r}\left(\frac{x}{2}-\frac{\kappa\pi}{2m}\right)};$$

l'expression dans le second membre de cette formule représente le polynome trigonométrique seulement dans le cas $r=1$.

7. En regardant de plus près notre formule d'interpolation (8) on s'assure qu'elle possède une propriété interpolatoire remarquable.

En effet, on a

$$(28)\qquad \frac{dF_{r,m}(x)}{dx}=0$$

aux points $x=\alpha_k$. Par conséquent la courbe:

$$(29)\qquad y=F_{r,m}(x)$$

représente une espèce de courbe à *escalier*, qui passe par les points donnés $x=\alpha_k$ en ayant en ces points une tangente parallèle à l'axe des x pour $r=1$; pour $r>1$, aux mêmes points, les dérivées d'ordre supérieur sont nulles. Cette circonstance rapproche la formule (8) en ce qui concerne ses propriétés interpolatoires, de la formule d'interpolation parabolique représentant une de ces paraboles à escalier (Treppenparabeln) étudiées autrefois par M. Fejer* et généralisées depuis pour des paraboles à escalier d'ordre supérieur par MM. Kryloff et Stayermann†.

Soient n et N respectivement la borne inférieure et la borne supérieure de la fonction $f(x)$ dans (a, b) alors, de (8) et (11), on obtient immédiatement:

$$n\leq F_{r,m}(x)\leq N$$

pour $a\leq x\leq b, m=1, 2, 3, \ldots$; donc, pour toute fonction $f(x)$ bornée, les $F_{r,m}(x)$ sont bornés dans leur ensemble dans (a, b). L'idée vient donc naturellement à l'esprit d'appliquer la formule (8) aux quadratures mécaniques, ce qui n'est pas dénué d'intérêt, car nous aurons ainsi pour le cas des ordonnées équidistantes des quadratures mécaniques convergentes.

**Ueber Interpolation*, Göttinger Nachrichten, Math. Phys. Kl., 1916, pp. 66-91.

†*Sur quelques formules d'interpolation convergentes pour toute fonction continue*. Bull. de l'Acad. des Sc. d'Ukraine, t. I, fas. I.

Pour le voir c'est-à-dire pour démontrer la formule

$$\lim_{m\to\infty}\int_{a}^{b} F_{r,m}(x)dx=\int_{ab,}^{r\,\approx} f(x)dx, \tag{30}$$

il suffit de remarquer que la suite des fonctions $F_{r,m}(x)$, bornées dans leur ensemble, converge vers $f(x)$ en tout point de continuité de $f(x)$; alors, en appliquant la proposition bien connue de Lebesgue on obtient l'égalité (30) pour toute fonction bornée et intégrable au sens de Riemann.

SUR QUELQUES RECHERCHES DANS LE DOMAINE DE LA THÉORIE DE L'INTERPOLATION ET DES QUADRATURES, DITES MÉCANIQUES

PAR M. N. KRYLOFF,

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Durant ces dernières années, j'ai été amené à faire, personellement et en collaboration avec d'autres personnes, quelques recherches sur la théorie de l'interpolation et sur certaines questions qui s'y rattachent. Une partie seulement des résultats de ces recherches à été publiée dans divers recueils mathématiques, en général dans des journaux mathématiques, très peu répandus en ce moment, et une autre partie n'a pas été publiée jusqu'à présent. C'est pourquoi je prends la liberté de présenter à la bienveillante attention du Congrès la simple énumération de quelques résultats obtenus, en omettant les démonstrations pour la brièveté de l'exposition, et en ne mentionnant qu'en peu de mots les articles déjà analysés ailleurs, ainsi que les travaux dont les résultats ont été depuis lors grandement dépassés par les recherches contemporaines des géomètres américains.

1. Dans une courte notice d'un caractère tout à fait élémentaire: *Sur la détermination de diverses formes du reste de la formule d'interpolation de Lagrange** on a tenté d'indiquer une méthode pour trouver les diverses formes du reste de la formule de Lagrange au moyen d'un artifice basé sur l'emploi des déterminants. La question de la détermination précise de θ dans la formule du reste, dans quelques cas bien particuliers, il est vrai, a été l'objet d'une note† où, indépendamment de M. Rothe‡ et, ce nous semble, par une méthode plus simple, le résultat suivant a été, entre autres, établi:

Pour que la formule:

$$\Delta^n f(x) = h^n f^{(n)}(x+nh\theta)$$

où $\Delta^n f(x)$ *est la différence* $n^{ième}$ *de* $f(x)$, *ait lieu avec une valeur constante de* θ, *il faut et il suffit que* $f(x)$ *soit un polynome de degré* $n+1$ *et que* $\theta=\frac{1}{2}$.

Le problème de la convergence des formules d'interpolation dont l'intérêt, toujours renouvelé, s'est encore accentué ces derniers temps, surtout après les recherches de MM. de la Vallée Poussin, D. Jackson, Faber, Féjèr et de bien d'autres, a été étudié aussi par moi dans quelques articles dont l'un a pour titre:

*Proceedings of the Math. Laboratory of the Tauric University, vol. I.
†Bull. de l'Acad. des Sc. d'Ukraine, t. I, fas. II.
‡Mathematische Zeitschrift, 1921, pp. 310-313.

Sur quelques formules d'interpolation généralisée, et sur lequel je n'insiste pas dans ce court rapport, étant donné que l'article susdit a paru dans un journal universellement répandu, le Bulletin des Sciences Mathématiques, et que certains des résultats y obtenus ont été mentionnés dans le Rapport récent de M. D. Jackson*.

Dans un autre article: *Sur quelques formules d'interpolation convergentes pour toute fonction continue* par N. Kryloff et E. Stayermann†, les recherches de L. Féjèr‡ ont été reprises et le résultat suivant a été établi: *En tout point x de continuité de la fonction $f(x)$ réelle et bornée dans l'intervalle $-1 \leq x \leq +1$, on a $\lim F_n(x) = f(x)$ où $y = F_n(x)$ représente la parabole à «escalier»* (Treppenparabel) *d'ordre supérieur*, c'est-à-dire passant par les n points donnés et où, par exemple, les dérivées des trois premiers ordres seront égales à zéro en ces points. Ce théorème a été démontré dans l'hypothèse où les points d'interpolation sont les racines du polynome trigonométrique, et tout porte à croire que le résultat subsiste pour bien d'autres polynomes hypergéométriques et qu'on peut obtenir l'évaluation du degré d'approximation de la formule d'interpolation correspondante au moyen d'un procédé analogue à celui qui a été utilisé dans l'article: *Sur une formule d'interpolation* (par N. Kryloff et J. Tamarkine) §, en essayant de généraliser les résultats d'un mémoire de M. de la Vallée Poussin ||. Les résultats principaux de cet article sont les suivants:

(1) *En tout point de continuité $x = x_0$ de la fonction $f(x)$, on a:*

$$\lim_{m \to \infty} F_{r,m}(x_0) = f(x_0)$$

où:

$$F_{r,m}(x) = \frac{\sin^{2r} mx}{m^{2r} C_r(mx)} \sum_{k=-\infty}^{+\infty} \frac{f(a_k)}{(x - a_k)^{2r}},$$

$$a_k = \frac{k\pi}{m}, \; k = 0, \pm 1, \pm 2, \ldots\ldots,$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{(x - k\pi)^{2r}} = \frac{C_r(x)}{\sin^{2r}(x)};$$

et la convergence est uniforme dans tout intervalle se trouvant à l'intérieur de l'intervalle de continuité de la fonction $f(x)$.

(2) *Si la fonction $f(x)$ vérifie la condition de Lipschitz,*

$$|f(x_1) - f(x_2)| < \lambda |x_1 - x_2|^{\alpha}, \quad \text{où } 0 < \alpha < 1,$$

*Bull. Amer. Math. Soc., vol. XXVII, 1921, p. 415.

†Bull. de l'Acad. des Sc. d'Ukraine, t. I, fas. I.

‡*Ueber Interpolation*, Göttinger Nachrichten, 1916, pp. 66-91.

§Voir les Proceedings du Congrès actuel.

||*Sur la convergence des formules d'interpolation entre ordonnées équidistantes.* Bull. de l'Académie Royale de Belgique. Classe des Sciences, 1908, pp. 319-403.

on a $|f(x)-F_{r,\,m}(x)|<\dfrac{A\lambda}{m^{\alpha}}$, [*où* A *est une constante ne dépendant pas de* m *et de* $f(x)$], *cést-à-dire que l'approximation est d'ordre* $m^{-\alpha}$. *Pour* $\alpha=1$, *c'est-à-dire quand la fonction* $f(x)$ *vérifie la condition de Cauchy-Lipschitz, l'inégalité ci-dessus écrite continue à subsister sous la condition* $r\geqq 2$; *pour* $r=1$, *elle se trouve remplacée par une autre:*

$$|f(x)-F_{r,\,m}(x)|<\frac{A\lambda\log m}{m}+B\frac{M}{m},$$

B étant une certaine constante et M la borne supérieure du module de $f(x)$.

(3^a) *En tout point* $x=x_0$, (x_0 *non commensurable avec* π), *de discontinuité de la première espèce de la fonction* $f(x)$ *qu'on interpole, la fonction* $F_{r,\,m}(x_0)$ *ne tend vers aucune limite déterminée, et on a:*

$$F_{r,\,m}(x_0)=f(x_0+0)\psi_r(\xi)+f(x_0-0)\psi_r(1-\xi)+\epsilon_m$$

où chacun des coefficients

$$\psi_r(\xi)=\frac{\pi^{2r}\sin^{2r}\pi\xi}{C_r(\pi\xi)}\sum_{k=1}^{\infty}\frac{1}{(\xi-k)^{2r}},$$

$$\psi_r(1-\xi)=\frac{\pi^{2r}\sin^{2r}\pi\xi}{C_r(\pi\xi)}\sum_{k=1}^{\infty}\frac{1}{(1-\xi-k)^{2r}},$$

est positif, leur somme égale à un et où $\epsilon_m\to 0$.

(3^b) *En tout point* $x=x_0\left(x_0=\dfrac{r\pi}{s}\right.$, *commensurable avec* $\left.\pi\right)$ *la fonction* $F_{r,\,t}(x_0)$, *quand* t *tend vers l'infini, tend vers* $f(x_0)$ *et vers* $s-1$ *diverses valeurs limites, comprises entre* $f(x_0+0)$ *et* $f(x_0-0)$.

Laissant de côté l'évaluation du degré d'approximation pour la fonction possédant des dérivées d'ordre supérieur, (ce qui peut être fait en s'inspirant des méthodes de MM. D. Jackson et de la Vallée Poussin), et l'étude du cas où $f(x)$ est périodique qui conduit aux formules d'interpolation représentées par le quotient de deux polynomes trigonométriques (pour $r=1$ on a, comme d'ordinaire, le polynome trigonométrique), remarquons seulement que la courbe $y=F_{r,\,m}(x)$ représente une espèce de courbe à escalier, qui passe par les points donnés d'abscisses $x=\alpha_k$, qui a en ces points une tangente parallèle à l'axe de x si $r=1$, et pour laquelle, si $r>1$, les dérivées d'ordre supérieur sont nulles en ces mêmes points, circonstance qui rapproche la formule d'interpolation en question des «Treppenparabeln» de M. Féjèr, dont la généralisation a été déjà exposée plus haut.

En étudiant de plus près ces paraboles à escalier, il n'était pas difficile de prévoir, que la démonstration de la convergence des formules ordinaires d'interpolation de Lagrange peut être basée sur leur étude, moyennant certaines conditions restrictives imposées à la fonction à interpoler et à la distribution des points d'interpolation. Dans un travail intitulé: *On the convergence of some interpolation formulae and in particular of M. Riesz's formula**, nous avons démontré, non-seulement la convergence de la formule d'interpolation de Riesz: le problème posé par M. Riesz lui-même†, mais encore nous avons établi les résultats suivants:

La formule d'interpolation de Lagrange est convergente quand n tend vers l'infini dans le cas où les points d'interpolation sont les racines des polynomes de Tchébycheff, pour toute fonction continue $f(x)$ admettant la représentation de la forme

$$f(x)=\int_a^x \phi(x)dx+c,$$

où $\phi(x)$ est intégrable au sens de Riemann et où c est une constante. Dans le cas où les abscisses des points d'interpolation sont les racines du polynome de Legendre, la convergence de la formule de Lagrange a lieu pour toute fonction continue $f(x)$ dont la première dérivée vérifie les conditions:

$$|\theta(x,\ h)|<M;\ \ T(x,\ h)<M,\ \ M=\text{const.}$$

où

$$f'(x+h)-f'(x)=h\theta(x,\ h)$$

et où $T(x,\ h)$ est la variation totale de $\theta(x,\ h)$.

Il faut remarquer que le cas du polynome de Tchébycheff peut être ramené, comme l'a indiqué M. Jackson, au cas trigonométrique complètement élucidé par l'illustre géomètre américain dans ses mémoires‡ et que notre résultat relatif au cas du polynome de Legendre se trouve à son tour dépassé par celui obtenu récemment par S. Bernstein au cours de ses recherches; néanmoins il nous semble, que la méthode suivie par nous mérite quelque attention, car tout porte à croire que, dans bien d'autres cas de polynomes vérifiant certaines équations différentielles et de polynomes hypergéométriques, la convergence de la formule d'interpolation peut être établie en suivant cette méthode.

2. Au problème des quadratures mécaniques, ainsi qu'à ses diverses généralisations et applications, liées intimement aux problèmes d'interpolation, nous avons consacré plusieurs travaux. Dans l'un d'eux intitulé: *Sur l'application de la théorie des quadratures mécaniques généralisées à l'évaluation par approximations successives de la solution de l'équation intégrale*, par N. Kryloff et J. Tamarkine,|| les résultats suivants ont été démontrés en se basant sur les théorèmes de ma note: *Sur quelques formules d'approximations fondées sur les généralisations des quadratures dites mécaniques:*§

(1) *Si la fonction $\phi(x)$ possède dans l'intervalle $(a,\ b)$ la dérivée continue $\phi^{(r)}(x)$ d'ordre r vérifiant la condition de Lipschitz d'ordre α et à coefficient M_r, on peut alors affirmer que le reste de la formule des quadratures mécaniques généralisées*

*Proceedings of the Math. Laboratory of the Tauric University, vol. III.

†M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichüngen für Polynome*, Jahresbericht d. Deut. Math. Vereinigung, vol. XXIII, 1919, pp. 354-368.

‡Trans. Amer. Math. Soc.

||Bull. de l'Académie des Sciences d'Ukraine, t. I, fas. I.

§Comptes Rendus Acad. Sciences, Paris, 1919.

$$\rho_n(\phi) = \int_a^b K(x,t)\phi(t)dt - \sum_{i=1}^{n} A_i\phi(x_i) = \int_a^b K(x,t)\phi(t)dt - h_n(\phi)$$

où

$$A_i = \int_a^b \frac{K(x,t)P_n(t)}{(t-x_i)P_n(x_i)}\,dt,$$

vérifie l'inégalité suivante:

$$|\rho_n(\phi)| < AM_r\left\{\frac{K_1}{n^{r+\alpha}} + \frac{K_2\sqrt{b-a}}{n^{r+\alpha-1/2}}\right\}$$

où A est une constante numérique et où K_1 et K_2 désignent respectivement les bornes supérieures dans l'intervalle (a, b) des expressions:

$$\int_a^b |K(x,t)|dt; \quad \sqrt{\int_a^b [K(x,t)]^2 dt}.$$

(2) *Si les fonctions $f(x)$ et $K(x, t)$ vérifient la condition de Lipschitz d'ordre $\alpha > \frac{1}{2}$, et si la borne supérieure K des expressions*

$$\int_a^b |K(x,t)|dt; \quad \frac{1}{\Delta x^\alpha}\int_a^b |\Delta_x K(x,t)|dt$$

est inférieure à un, la solution de l'équation intégrale

$$y\,(x) = f(x) + \int_a^b K(x,t)y(t)dt$$

peut être obtenue comme limite de la suite des fonctions:

$$\bar{y}_0,\ \bar{y}_1(x) = f(x) + h_1(\bar{y}_0),\ \ldots,\ y_n = f(x) + h_n(\bar{y}_{n-1}),$$

obtenues par la méthode des quadratures mécaniques généralisées.

Il n'est pas difficile de montrer que les formules déduites de l'algorithme précédent au moyen de la différentiation s fois par rapport à x donnent un résultat convergent vers $y^{(s)}$ (où $s \leq r$) si on suppose l'existence des dérivées des fonctions $f(x)$, $K(x, t)$ jusqu'à l'ordre r inclusivement, et si, de plus, la dérivée d'ordre r vérifie la condition de Lipschitz d'ordre α.

L'algorithme ci-dessus exposé présente, ce nous semble, certains avantages comparativement à la méthode usuelle des approximations successives, parce qu'il permet de réduire les approximations successives aux quadratures mécaniques et que, par conséquent, l'opération la plus difficile à exécuter, l'intégration, peut être faite une fois pour toutes, indépendamment de la fonction $f(x)$ et de la première approximation y_0.

En revenant au problème des quadratures mécaniques proprement dites, et à ses diverses généralisations, notons que la généralisation fondée sur l'introduction du paramètre, dont nous nous sommes occupés dans la note des Comptes Rendus déjà citée et dont l'idée revient à M. S. Pincherle,* n'est pas évidemment la seule qui peut se présenter dans ce domaine de recherches: dans un

*Atti della Academia di Scienze di Bologna, 1892.

article non encore publié, nous avons démontré la convergence pour toute fonction continue de la formule de quadrature indiquée en premier lieu, parait-il, par T. Stieltjes dans une note aux Comptes Rendus en 1884, où l'illustre analyste donne la formule suivante de quadrature comprenant, comme cas bien particulier, celle de Gauss:

$$\int_0^1 f(x)C(x)dx = A_1C(x_1)+A_2C(x_2)+\ldots+A_mC(x_m),$$

$$C(x)=a_1x^{\lambda_1}+a_2x^{\lambda_2}+\ldots+a_nx^{\lambda_n},$$

où $\lambda_1, \lambda_2, \ldots, \lambda_n$ sont des nombres positifs inégaux donnés, les A_i sont positifs, de plus $\sum_{i=1}^{m} A_i \leq \int_0^1 f(x)dx$ et les valeurs $x_1, x_2, \ldots, x_m$ sont positives, inégales et inférieures à l'unité. En se basant alors sur les résultats récents de MM. S. Bernstein*, de la Vallée Poussin†, Müntz‡, Szàsz§, concernant la généralisation du théorème de Weierstrass, on peut faire le passage à la limite dans la formule ci-dessus écrite pourvu que la série $\sum_{k=1}^{\infty} \frac{1}{\lambda_k}$ soit divergente. Il va sans dire que les considérations exposées par moi dans un article intitulé: *Sur la théorie des quadratures mécaniques et sur certaines questions qui s'y rattachent*||, ainsi que les recherches des divers auteurs qui ont traité plus amplement le même sujet peuvent être étendues à ce nouveau type de quadratures convergentes, à propos desquelles on peut donc poser les problèmes résolus pour les quadratures mécaniques ordinaires.

En omettant, pour abréger, l'exposé de mes recherches concernant l'approximation simultanée de plusieurs intégrales définies, le problème qui se présente par exemple dans les recherches de M. Pearson, je me permets de signaler seulement les quadratures fondées sur l'emploi des formules déjà indiquées dans ce rapport dans le resumé de l'article: *Sur une formule d'interpolation*, car elles donnent aussi toute une classe de quadratures convergentes

En terminant ce rapport, je m'excuse d'avoir trop parlé de mes modestes recherches relatives au sujet considéré, et je me permets seulement d'attirer l'attention sur le fait jusqu'ici parait-il, inaperçu, qu'au moyen du passage à la limite (à l'aide du théorème de Weierstrass par exemple), dans une formule de quadratures donnée autrefois par Stieltjes¶, on peut arriver à la notion de l'intégrale de Stieltjes. Ceci permet, sous bien des réserves, il va de soi, d'exprimer l'opinion que la théorie des quadratures mécaniques, ainsi que diverses théories es interpolatoires, deviendront un jour, peut-être, la source de nouvelles généralisations de la notion d'intégrale.

*Proceedings of the 5th Int. Congress of Math., vol. I, p. 267.

†Mémoires publiés par l'Acad. Royale de Belgique, 1912, p. 104.

‡*Ueber den Approximationssatz von Weierstrass*, Schwarz Festschrift, 1914, pp. 303-312.

§*Ueber die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen.* Mathematische Annalen 77, 1916, pp. 482-496.

||Annales de l'École Supérieure des Mines de Pétersbourg, 1915.

¶Nouv. Ann. Math. 1884, p. 161.

NOTE SUR L'INTERPOLATION GÉNÉRALISÉE

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Le but de cette note est une transformation simple de la formule d'interpolation

$$f(x)=\sum_{i=0}^{n}\frac{\sin \pi nz}{\pi n}\cdot\frac{(-1)^n f(x_i)}{z-x_i},\quad (x_i=i/n;\ i=0,\ 1,\ \dots,\ n-1), \tag{1}$$

due à M. de la Vallée Poussin.

En intégrant le second membre de (1) entre des limites $x-\alpha$, $x+\alpha$, où

$$\alpha=n^{2\delta-1},\ \tfrac{1}{2}>\delta>\tfrac{1}{3},$$

on obtient la proposition suivante:

Toute fonction $f(x)$ continue dans l'intervalle (0, 1) *peut être représentée sous la forme:*

$$f(x)=\lim_{n=\infty}\sum_{i=0}^{n}\frac{1}{2\alpha}\int_{x-\alpha}^{x+\alpha}\frac{\sin \pi nz}{\pi n}\ \frac{(-1)^n f(x_i)}{z-x_i}\,dz \tag{2}$$

en tout intervalle intérieur à (0, 1).

Le point essentiel de la démonstration de cette proposition consiste en la décomposition de la somme

$$\sum_{o}^{n}=\sum_{i=0}^{n}\frac{1}{2\alpha}\int_{x-\alpha}^{x+\alpha}\frac{\sin \pi nz}{\pi n}\,\frac{(-1)^n f(x_i)}{z-x_i}\,dz$$

en cinq sommes partielles:

$$\sum_{o}^{n}=\sum_{o}^{i_1}+\sum_{i_1+1}^{i_2}+\sum_{i_2+1}^{i_3}+\sum_{i_3+1}^{i_4}+\sum_{i_4+1}^{n},$$

où

$$i_1=j-[n^\delta]-[n^{2\delta}],$$
$$i_2=j+[n^\delta]-[n^{2\delta}],$$
$$i_3=j-[n^\delta]+[n^{2\delta}],$$
$$i_4=j+[n^\delta]+[n^{2\delta}],$$
$$j=[nx].$$

Le nombre n croissant infiniment, on a

$$\lim \sum_{o}^{i_1} = 0, \quad \lim \sum_{i_1+1}^{i_2} = 0, \quad \lim \sum_{i_3+1}^{i_4} = 0, \quad \lim \sum_{i_4+1}^{n} = 0,$$

$$\lim \sum_{i_2+1}^{i_3} = \lim \frac{1}{2\,[n^{2\delta}]} \cdot \sum_{i=j-[n^2\delta]}^{j+[n^2\delta]} f(x_i) = f(x),$$

ce qui donne l'égalité (2).

On peut aisément étendre cette formule au cas où $f(x)$ a des points de discontinuité isolés. Elle semble aussi avoir des applications utiles à l'approximation des fonctions par des polynomes, ainsi qu'à une transformation analogue de la formule d'interpolation de Lagrange.

SUR UN PROBLÈME GÉNÉRAL DE PROBABILITÉS ET SES DIVERSES APPLICATIONS

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I. PROBLÈME GÉNÉRAL

1. ENONCÉ DU PROBLÈME. Considérons un ensemble de n évènements $E_1, E_2, \ldots, E_n$ se produisant simultanément ou successivement et dont chacun peut être favorable ou défavorable. Choisissons $m=p+q$ d'entre eux, dans un ordre arbitraire. Appelons z_p^q la probabilité pour que les p premiers soient favorables et les q suivants défavorables. Nous supposons cette probabilité *indépendante de l'ordre* dans lequel ont été choisis lesdits évènements. Mais, nous admettons que si l'on sait que l'éventualité ci-dessus s'est produite, la probabilité d'une éventualité analogue concernant un nouveau groupe de m' évènements choisis parmi les évènements restants peut être modifiée, tout en demeurant indépendante de l'ordre dans lequel on fait ce deuxième choix. En un mot, les probabilités sont entièrement *symétriques* par rapport à $E_1, E_2, \ldots, E_n$, mais ne sont *pas indépendantes*.

Nous appellerons, en particulier, x_p la probabilité z_p^o pour que p évènements *désignés à l'avance* soient favorables et y_p la probabilité z_o^p pour qu'ils soient défavorables, sans qu'il soit rien préjugé des $n-p$ autres évènements. Nous appellerons, au contraire, t_p la probabilité z_p^{n-p} pour que les p évènements désignés soient *seuls à être favorables*.

La probabilité A_p pour qu'il y ait juste p évènements favorables, *sans que ces évènements soient désignés*, est évidemment

$$A_p = C_n^p t_p, \tag{1}$$

puisque la probabilité t_p est la même pour toutes les combinaisons possibles des n évènements p à p. On en déduit la probabilité B_p pour qu'il y ait *au plus p évènements favorables*:

$$B_p = \sum_{k=0}^{p} C_n^k \, t_k. \tag{2}$$

Enfin, la probabilité pour que le nombre des évènements favorables soit au moins égal à p et au plus égal à q est

$$\sum_{k=p}^{q} A_k = B_q - B_{p-1}.$$

2. Formules générales. D'après le théorème des probabilités composées, la probabilité z_{p+1}^q pour que les $p+1$ évènements numérotés 1, 2, . . . , p, $p+q+1$ soient favorables et que les q évènements numérotés $p+1$, $p+2$, . . . , $p+q$ soient défavorables s'obtient en multipliant z_p^q par la probabilité x pour que E_{p+q+1} soit favorable quand on sait que E_1, E_2, . . . E_p sont favorables et que E_{p+1}, . . . E_{p+q} sont défavorables. On a donc

$$z_{p+1}^q = z_p^q x.$$

De même, la probabilité pour que les p premiers évènements soient favorables et que les $q+1$ suivants soient défavorables est

$$z_p^{q+1} = z_p^q (1-x).$$

En additionnant, nous obtenons la *relation fondamentale*

$$z_{p+1}^q + z_p^{q+1} = z_p^q . \tag{3}$$

En résolvant par rapport à z_p^{q+1} et changeant q en $q-1$, ceci peut s'écrire

$$z_p^q = -(z_{p+1}^{q-1} - z_p^{q-1}) = -\Delta z_p^{q-1},$$

le symbole Δ de la théorie des différences portant sur l'indice inférieur. En appliquant $q-a$ fois de suite la même formule, on obtient

$$z_p^q = (-1)^{q-a}\Delta^{q-a} z_p^a , \tag{4}$$

où a désigne un nombre fixe quelconque compris entre 0 et q. En résolvant, au contraire, (3) par rapport à z_{p+1}^q, on obtient de même

$$z_p^q = (-1)^{p-a}\Delta^{p-a} z_a^q, \quad 0 \leq a \leq p. \tag{5}$$

Appliquons maintenant à (4) la formule classique de la théorie des différences; il vient

$$z_p^q = (-1)^{q-a} \sum_{h=0}^{q-a} (-1)^h C_{q-a}^h \, z_{p+q-a-h}^a,$$

ou, en posant $q-a-h=k$,

$$z_p^q = \sum_{k=0}^{q-a} (-1)^k C_{q-a}^k \, z_{p+k}^a.$$

De même, (5) peut s'écrire

$$z_p^q = \sum_{k=0}^{p-a} (-1)^k C_{p-a}^k z_a^{q+k}.$$

Pour $a=0$, on a

$$z_p^q = \sum_{k=0}^{q} (-1)^k C_q^k x_{p+k} , \tag{6}$$

$$z_p^q = \sum_{k=0}^{p} (-1)^k C_p^k y_{q+k} , \tag{7}$$

et, en particulier,

$$y_q = \sum_{k=0}^{q} (-1)^k C_q^k x_k, \tag{8}$$

$$x_p = \sum_{k=0}^{p} (-1)^k C_p^k y_k, \tag{9}$$

(10) $$t_p = \sum_{k=0}^{n-p} (-1)^k C_{n-p}^k x_{p+k},$$

(11) $$t_p = \sum_{k=0}^{p} (-1)^k C_p^k y_{n-p+k}.$$

3. L'éventualité dont la probabilité a été désignée par z_p^q peut se produire de telle manière que k des $n-m$ évènements non désignés à l'avance soient favorables, les $n-m-k$ autres étant défavorables. Cela fait en tout $p+k$ évènements favorables et $n-p-k$ évènements défavorables. La probabilité pour chaque combinaison k à k des $n-m$ évènements non désignés est donc t_{p+k} et l'on a, d'après le théorème des probabilités totales

(12) $$z_p^q = \sum_{k=0}^{n-m} C_{n-m}^k t_{p+k};$$

en particulier,

(13) $$x_p = \sum_{k=0}^{n-p} C_{n-p}^k t_{p+k},$$

(14) $$y_p = \sum_{k=0}^{n-p} C_{n-p}^k t_k,$$

(15) $$\sum_{k=0}^{n} C_n^k t_k = 1.$$

Cette dernière formule est un cas particulier évident de (2). On peut dire aussi qu'elle exprime que $x_0 = y_0 = 1$.

4. Nous avons maintenant les expressions explicites* de z_p^q en fonction des x_k, des y_k ou des t_k.

La formule (2) nous donne seulement B_p en fonction des t_k. Il est utile d'avoir aussi cette probabilité en fonction des x_k et des y_k. Il suffit, pour cela, de porter (10) et (11) dans (2). On a, par exemple,

$$B_p = \sum_{k=0}^{p} C_n^k \sum_{h=0}^{n-k} (-1)^h C_{n-k}^h x_{k+h},$$

ou, en posant $k+h=i$,

$$B_p = \sum\sum (-1)^{i-k} C_n^k C_{n-k}^{n-i} x_i,$$

les sommations devant être faites dans les limites suivantes:

$$0 \leq k \leq p, \quad k \leq i \leq n.$$

*En comparant entre elles ces différentes expressions, on aboutit à des identités, qui toutes se ramènent à la suivante

$$\sum_{k=0}^{p} (-1)^k C_p^k C_{q+k}^{p+h} = (-1)^p C_q^h,$$

laquelle peut être vérifiée directement, en prenant le coefficient de x^{p+h} dans

$$\sum_{k=0}^{p} (-1)^k C_p^k (1+x)^{q+k} = (1+x)^q [1-(1+x)]^p = (-1)^p x^p (1+x)^q.$$

Le coefficient de $(-1)^i x_i$ est

$$\sum_{k=0}^{p} (-1)^k C_n^k \, C_{n-k}^{n-i},$$

en faisant la convention de regarder comme nul tout coefficient binomial dont l'indice supérieur dépasse l'indice inférieur. Or, on a l'identité

$$C_n^k \, C_{n-k}^{n-i} = C_n^i C_i^k. \tag{16}$$

Dès lors, le coefficient de $(-1)^i x_i C_n^i$ est

$$\sum_{k=0}^{p} (-1)^k C_i^k = (-1)^p C_{i-1}^p, \tag{17}$$

comme on le voit en prenant le coefficient de x^p dans

$$\frac{\sum\limits_{k=0}^{i} (-1)^k C_i^k x^k}{1-x} = \frac{(1-x)^i}{1-x} = (1-x)^{i-1}.$$

Il y a toutefois exception pour $i=0$, l'expression (17) se réduisant dans ce cas au seul terme 1, obtenu pour $k=0$. On a donc, en se souvenant que $x_0=1$,

$$B_p = 1 + (-1)^p \sum_{i=p+1}^{n} (-1)^i C_n^i C_{i-1}^p \, x_i. \tag{18}$$

Utilisons de nouveau l'identité (16), après avoir remplacé C_{i-1}^p par $\frac{p+1}{i} C_i^{p+1}$; il vient

$$B_p = 1 + (-1)^p (p+1) C_n^{p+1} \sum_{i=p+1}^{n} (-1)^i \frac{1}{i} C_{n-p-1}^{n-i} \, x_i. \tag{19}$$

Si l'on porte de même (11) dans (2), un calcul analogue au précédent conduit à la formule

$$B_p = (-1)^{n-p} (n-p) C_n^p \sum_{i=n-p}^{n} (-1)^i \frac{1}{i} C_p^{n-i} \, y_i, \tag{20}$$

qui n'est valable que pour $p<n$. Pour $p=n$, on a évidemment $B_n=1$, ce qui résulte d'ailleurs aussi de la formule (19). On peut encore vérifier que $B_0=y_n$; c'est évident pour (20); pour (19), il faut se servir de la formule (8).

5. *La valeur probable* V *du nombre d'évènements favorables* est $\sum\limits_{p=1}^{n} pA_p$ ou, d'après (1) et (10),

$$V = \Sigma\Sigma (-1)^k p C_n^p C_{n-p}^k \, x_{p+k},$$

la sommation étant faite pour $1 \leq p \leq n$, $0 \leq k \leq n-p$. Posons $p+k=h$; il vient

$$V = \Sigma\Sigma (-1)^{p+h} p C_n^p C_{n-p}^{n-h} x_h = \Sigma\Sigma (-1)^{p+h} p C_n^h \, C_h^p \, x_h,$$

d'après (16); la sommation étant faite pour $1 \leq p \leq h \leq n$. Le coefficient de $(-1)^h C_n^h x_h$ est

$$\sum_{p=1}^{h} (-1)^p p C_h^p = h \sum_{p=1}^{h} (-1)^p C_{h-1}^{p-1} = -h(1-1)^{h-1} = 0,$$

si $h > 1$. Pour $h = 1$, il est égal à -1 et nous obtenons finalement

$$V = nx_1. \tag{21}$$

Ce résultat peut d'ailleurs être obtenu par un raisonnement direct. En effet, la probabilité *a priori* pour que E_i soit favorable est x_1. L'espérance mathématique de celui qui recevrait 1 franc par évènement favorable est donc égale à x_1 pour chacun des n évènements. Son espérance totale est, par suite, nx_1.

La valeur probable du nombre des évènements défavorables est, de même, ny_1.

6. Nous sommes maintenant en mesure d'exprimer à notre gré toutes les probabilités z_p^q et B_p en fonction des x_k, des y_k ou des t_k. Dans chaque problème particulier, *on emploie le système de variables dont le calcul direct paraît être le plus facile.*

On peut se demander inversement s'il existe toujours un ensemble d'évènements correspondant à des valeurs arbitrairement choisies des x_k, des y_k ou des t_k. D'abord, ces valeurs doivent évidemment être comprises entre 0 et 1. Mais, cela n'est pas suffisant. Les x_k ou les y_k doivent décroître, de par leur définition même, quand leur indice augmente. En outre, substitués dans les formules (6) et (7), ils ne doivent donner que des résultats positifs, condition dont il paraît difficile de tenir compte.

Par contre, il semble bien que l'on puisse se donner arbitrairement les t_k, sous la seule condition de vérifier (15). Ceci revient à admettre que l'on peut choisir à volonté les probabilités des différentes combinaisons relatives à l'ensemble des n évènements, pourvu que la probabilité totale soit égale à 1. Mais, c'est là une simple présomption, dont nous n'avons pas démontré rigoureusement l'exactitude.

7. Cas d'une infinité d'évènements. Supposons que n augmente indéfiniment et cherchons quelle valeur asymptotique il faut donner à t_k pour que la condition (15) ne cesse pas d'être satisfaite. On peut d'abord s'arranger pour que chaque terme ait une limite non nulle. Pour cela, il faut que $n^k t_k$ tende vers u_k tel que

$$\sum_{k=0}^{\infty} \frac{u_k}{k!} = 1.$$

Mais, les formules (13) et (14) donnent alors $x_p = 0$ si $p > 0$ et $y_p = 1$. Il y aurait donc, dans cette hypothèse, une probabilité nulle ou, plus exactement, infiniment petite pour qu' un évènement désigné à l'avance soit favorable.

Laissons de côté ce cas sans intérêt et supposons maintenant que k augmente indéfiniment en même temps que n, de telle manière que $\frac{k}{n}$ tende vers x. D'après la formule de Stirling, la valeur asymptotique de C_n^k est

$$\frac{1}{z^n\sqrt{2\pi xyn}}\,; \quad y = 1 - x, \quad z = x^x y^y. \tag{22}$$

Dès lors, les termes de (15) correspondant aux valeurs de k comprises entre nx et $n(x+dx)$ ont pour somme

$$\frac{\sqrt{n}\, t_k dx}{z^n \sqrt{2\pi xy}},$$

si l'on suppose toutefois que t_k est une fonction continue de x. On est ainsi conduit à poser

$$t_k = \sqrt{2\pi xy}\, \frac{z^n}{\sqrt{n}} f(x), \tag{23}$$

$f(x)$ étant une fonction continue positive, simplement assujettie à la condition

$$\int_0^1 f(x)dx = 1. \tag{24}$$

L'interprétation de cette fonction est d'ailleurs évidente. En effet, la somme $f(x)dx$ des termes de (15) ci-dessus considérés est la probabilité pour que le nombre total des évènements favorables soit compris entre nx et $n(x+dx)$. Donc, $f(x)dx$ est la probabilité pour que la *fréquence* des évènements favorables soit comprise entre x et $x+dx$.

Voyons maintenant ce que devient la formule (12), quand on y remplace t_k par la formule (20). Posons

$$x = \frac{k}{n-m}, \quad x' = \frac{p+k}{n} = x + \frac{\alpha}{n}, \quad \alpha = p - mx.$$

La valeur asymptotique de $C_{n-m}^k t_{p+k}$ est, d'après (22) et (23),

$$\frac{1}{n} f(x) z^m \left(\frac{z'}{z}\right)^n. \tag{25}$$

Cherchons la limite de $\left(\frac{z'}{z}\right)^n = \frac{x'^{nx'}}{x^{nx}} \cdot \frac{y'^{ny'}}{y^{ny}}$. On a

$$\frac{x'^{nx'}}{x^{nx}} = x^\alpha \left(1 + \frac{\alpha}{nx}\right)^{nx'},$$

ce qui tend vers $x^\alpha e^\alpha$. De même, $\frac{y'^{ny'}}{y^{ny}}$ tend vers $y^{-\alpha} e^{-\alpha}$. La limite de $\left(\frac{z'}{z}\right)^n$ est donc $x^\alpha y^{-\alpha} = x^{p-mx} y^{q-my} = x^p y^q z^{-m}$.

Portant dans (25), on obtient $\frac{1}{n} f(x) x^p y^q$. Donc,

$$z_p^q = \int_0^1 x^p y^q f(x) dx. \tag{26}$$

En particulier,

$$x_p = \int_0^1 x^p f(x) dx, \quad y_p = \int_0^1 y^p f(x) dx. \tag{27}$$

Il est facile, au moyen de ces formules, de vérifier (6) et (7). On a, par exemple, pour (6),

$$z_p^q = \int_0^1 f(x)\left[\sum_{k=0}^{q} (-1)^k C_q^k x^{p+k}\right]dx = \int_0^1 f(x)x^p(1-x)^q dx,$$

c'est-à-dire (26).

On peut aussi envisager directement le cas où les évènements E constituent un ensemble infini, qui peut être continu aussi bien que dénombrable. On se donne alors la loi de probabilité $f(x)$ de la fréquence des évènements favorables. La formule (26) résulte de la règle de Bayes et de la condition (24).

8. Cas où les probabilités sont indépendantes. Soit u la probabilité pour qu'un quelconque des évènements soit favorable et $v=1-u$ la probabilité complémentaire. Si nous supposons les évènements indépendants, on a évidemment, en revenant au cas où n est fini,

(28) $$z_p^q = u^p v^q, \quad x_p = u^p, \quad y_q = v^q, \quad t_p = u^p v^{n-p}.$$

La vérification des formules (6), (7), (12) est alors évidente. Voyons maintenant ce que devient la formule (23). La valeur exacte de t_k est $(u^x v^y)^n$. La valeur asymptotique de $f(x)$ est donc

$$\frac{\sqrt{n}}{\sqrt{2\pi xy}}\left(\frac{u^x v^y}{x^x y^y}\right)^n.$$

Or, le facteur entre parenthèses est < 1, sauf pour $x=u$, auquel cas il est égal à 1. On en conclut que, pour $n=\infty$, $f(x)$ est nul si $x \neq u$ et infini si $x=u$. La fréquence u a une probabilité infiniment grande par rapport à celle de toute autre fréquence. C'est le *théorème de Bernoulli.*

Posons

$$x = u + \frac{z}{\sqrt{n}}, \quad y = v - \frac{z}{\sqrt{n}}$$

et cherchons la limite du facteur $\lambda = \left(\frac{u^x v^y}{x^x y^y}\right)^n$. On a

$$\log \lambda = -n\left[x \log\left(1+\frac{z}{u\sqrt{n}}\right) + y \log\left(1-\frac{z}{v\sqrt{n}}\right)\right] = -\frac{z^2}{2uv} + \ldots$$

La limite de λ est donc $e^{-\frac{z^2}{2uv}}$ et la valeur asymptotique de $f(x)$ est

$$e^{-\frac{z^2}{2uv}}\frac{\sqrt{n}}{\sqrt{2\pi uv}}.$$

Celle de $f(x)dx$ est, par suite,

$$e^{-\frac{z^2}{2uv}}\frac{dz}{\sqrt{2\pi uv}}.$$

On retrouve la classique *loi des écarts.*

Voyons ce que devient la formule (26). On a

$$z_p^q = u^p v^q \int_{-u\sqrt{n}}^{v\sqrt{n}} \left(1+\frac{z}{u\sqrt{n}}\right)^p \left(1-\frac{z}{v\sqrt{n}}\right)^q e^{-\frac{z^2}{2uv}} \frac{dz}{\sqrt{2\pi uv}}.$$

Si l'on développe le produit des deux premiers facteurs et qu'on intègre ensuite terme à terme, on voit que l'intégrale tend vers 1 pour n infini. Donc, la limite de z_p^q est $u^p v^q$,conformément à (28).

II. Applications

9. Problème des épreuves répétées. *Soient n évènements* $E_1, E_2, \ldots, E_n$. *On fait N épreuves. A chacune d'elles, peut se produire un évènement et un seul, avec une probabilité constante x. On demande la probabilité pour que p évènements désignés se produisent chacun au moins une fois et que q autres évènements également désignés ne se produisent pas.*

Calculons y_k. La probabilité pour qu'aucun des k premiers évènements ne se produise est, pour chaque épreuve, $1\text{-}kx$. Pour les N épreuves, on a donc

$$y_k = (1-kx)^N.$$

La formule (7) nous donne ensuite

$$z_p^q = \sum_{k=0}^{p}(-1)^k C_p^k [1-(q+k)x]^N. \tag{29}$$

En faisant $q=0$, $x=\frac{1}{n}$, on a la probabilité, calculée par de Moivre*, d'amener, en N coups, p faces désignées d'un dé de n faces.

Généralisation. Supposons que les n évènements aient pour probabilités respectives $p_1, p_2, \ldots p_n$ à chaque épreuve et cherchons la probabilité *pour que* E_1 *se réalise au moins* a_1 *fois et au plus* b_1 *fois,* E_2 *au moins* a_2 *fois et au plus* b_2 *fois* $\ldots$, E_n *au moins* a_n *fois et au plus* b_n *fois.*

La probabilité pour que $E_1, E_2, \ldots, E_n$ se réalisent respectivement $c_1, c_2, \ldots, c_n$ *fois est*

$${p_1}^{c_1}{p_2}^{c_2}\ldots{p_n}^{c_n} q^c \frac{N!}{c_1!\,c_2!\ldots c_n!\,c!},$$

en posant

$$q = 1-p_1-p_2-\ldots-p_n, \quad c = N-c_1-c_2-\ldots-c_n.$$

La probabilité demandée est donc

$$P = \sum \frac{N!}{c_1!\,c_2!\ldots c_n!\,c!}{p_1}^{c_1}{p_2}^{c_2}\ldots{p_n}^{c_n} q^c, \tag{30}$$

la sommation portant sur toutes les valeurs des c_i satisfaisant aux inégalités

$$a_i \leq c_i \leq b_i.$$

*Cf. Todhunter, *History of the mathematical theory of probability*, pages 160, 250, 251. Comme cas particuliers, citons aussi les problèmes 21, 22, 30, 33 à 37, 39 à 42 du *Calcul des probabilités* de Joseph Bertrand.

Considérons les quantités

$$E_i = \sum_{h=a_i}^{b_i} \frac{p_i^h}{h!}, \quad i=1, 2, \ldots, n.$$

La probabilité P est la somme des termes de degré total N relativement à l'ensemble des variables $p_1, p_2, \ldots p_n, q$, dans le produit $N!E_1E_2 \ldots E_n e^q$*.

11. Considérons maintenant, de nouveau, le cas particulier où tous les évènements ont la même probabilité x. Cherchons la probabilité pour que p *d'entre eux se réalisent au moins a fois et au plus b fois et que q autres se réalisent moins de a fois ou plus de b fois.*

Convenons de dire qu'un évènement E_i est favorable s'il est réalisé au moins a fois et au plus b fois et qu'il est défavorable dans le cas contraire. La probabilité demandée n'est autre que z_p^q, si les $p+q$ évènements sont désignés; elle est égale à $B_q - B_{p-1}$ s'ils ne le sont pas.

Il nous suffit, dès lors, d'évaluer une fois pour toutes les x_k ou les y_k. D'après le paragraphe précédent, x_k est la somme des termes de degré total N relativement aux deux variables x et y dans $N!E^k e^y$, E désignant le polynôme $\sum_{h=a}^{b} \frac{x^h}{h!}$ et y devant être ensuite remplacé par $1-kx$.

Pour y_k, on a la même règle, à condition de remplacer E par

$$F = \sum_{h=0}^{a-1} \frac{x^h}{h!} + \sum_{h=b+1}^{N} \frac{x^h}{h!}.$$

Il y a avantage à utiliser les x_k ou les y_k, suivant que $b-a$ est inférieur ou supérieur à $\frac{N}{2}$.

Pour calculer z_p^q, nous pouvons maintenant employer l'une ou l'autre des formules (6) et (7). Mais, nous pouvons aussi résoudre le problème directement, d'après le paragraphe 10; z_p^q est la somme des termes de degré N en x, y, dans $N!E^p F^q e^y$, y devant ensuite être remplacé par $1-(p+q)x$.

En comparant ces deux méthodes de calcul, on obtient des identités, de forme générale compliquée et dont il paraît difficile de donner une démonstration directe.

12. *Cas où N augmente indéfiniment.* Un terme quelconque de degré N de $N!E^k e^y$ est de la forme

$$\frac{N!}{\alpha!\, \alpha_1! \ldots \alpha_k!} \cdot x^{N-\alpha} y^{\alpha}, \tag{32}$$

les α_i appartenant à l'intervalle (a, b) et ayant pour somme $N-\alpha$.

*Si l'on suppose $a_i=0$, $b_i=N$, on a évidemment

$$P = (p_1+p_2+ \ldots +p_n+q)^N = 1. \tag{31}$$

Si $a_i=1$, $b_i=N$, il est facile de vérifier que

$$P = 1 - \sum P_i + \sum P_{ij} - \sum P_{ijh} + \ldots + (-1)^m q^N,$$

en appelant $P_{ij\ldots k}$ ce que devient (31) quand on y annule $p_i, p_j, \ldots, p_k$. On retombe alors sur un problème de dés résolu par de Moivre (Cf. Todhunter, op. cit., p. 161).

Si l'on suppose $q=0$, $a_1=b_1$, $a_2=a_3= \ldots =a_n=0$, on se ramène aisément au *problème des partis* pour n joueurs, traité par Laplace (*Œuvres*, t. VII: p. 228).

1°. *Si a et b sont fixes ou simplement restent finis*, les α_i et $N-\alpha$ restent finis également. Dès lors, si y reste inférieur à un nombre fixe plus petit que 1, c'est-à-dire si x reste supérieur à un nombre fixe non nul, le terme (32) tend vers zéro pour N infini, car il en est ainsi de $\frac{N!}{\alpha!}y^\alpha$. Comme le nombre des termes analogues est fini, x_k tend vers zéro (sauf, bien entendu, x_0, qui est toujours égal à 1). On en conclut, d'après (8), que y_k tend vers 1.*

2°. *Si a et $N-b$ restent finis*, ce sont les termes de $N!F^k e^y$ qui restent en nombre fini. Un de ces termes est de la forme (32), tout α_i étant $< a$ ou compris entre b et N. Si tous les α_i sont $< a$, le terme tend vers zéro, comme tout-à-l'heure. Si $\alpha_1 > b$, α et les autres α_i finissent par demeurer $< a$, car $N-\alpha_1$ reste fini; donc, $\frac{N!}{\alpha_1!}x^{N-\alpha}$ tend vers zéro, ainsi par suite que le terme (32) et, par conséquent, aussi y_k. Quant à x_k, il tend vers 1, d'après (9)†.

3°. *Si $N-a$ reste fini*, $2a$ finit par demeurer supérieur à N. A partir de ce moment, x_k est rigoureusement nul, si $k > 1$. Pour $k=1$, $\frac{N!}{\alpha_1!}x^{\alpha_1}$ tend vers zéro; donc aussi x_1. On a les mêmes conclusions que dans le premier cas.

Tous ces résultats sont bien conformes aux théorèmes généraux de M. Borel sur les probabilités dénombrables‡. Les probabilités qu'il appelle p_n sont ici toutes égales à x; donc, la série Σp_n est divergente. M. Borel en conclut qu'il y a une probabilité nulle pour que le nombre des cas favorables soit fini dans une série infinie d'épreuves (ce qui s'applique à 1° et à 3°) et qu'il y a au contraire la probabilité 1 pour que le nombre des cas favorables soit infini§ (ce qui s'applique à 2°).

Si a et $N-a$ deviennent infinis, ou bien *si a reste fini et que b et $N-b$ deviennent infinis*, on ne peut rien conclure. On ne se trouve plus d'ailleurs dans les cas d'application des théorèmes de M. Borel||.

13. Supposons maintenant que x tende vers zéro en même temps que N devient infini, de telle manière *que Nx ait une limite finie non nulle u.*

*Ceci doit pouvoir se démontrer directement, à partir de $N!F^k e^y$. Mais, sauf pour $k=1$, cela paraît au premier abord assez difficile.

†Cela aussi paraît assez difficile à démontrer directement.

‡Cf. *Leçons sur la théorie des fonctions*, 2^e^ édition, pages 186 et 187.

§En réalité, la définition que donne M. Borel de cette probabilité revient, avec nos notations, à supposer a fixe et $b=N$. L'extension à l'hypothèse un peu plus générale de notre deuxième cas se fait sans peine.

||Il y aurait peut-être intérêt à les étendre dans cette voie. Un exemple simple peut donner une idée du genre de résultats que l'on pourrait attendre. Si, avec les notations de M. Borel, toutes les probabilités p_n sont comprises entre deux nombres fixes p et q, la probabilité pour que le nombre de cas favorables dépasse m tend vers 1, si $\frac{m}{n}$ reste inférieur à un nombre fixe $< p$ et vers zéro, si $\frac{m}{n}$ reste supérieur à un nombre fixe $> q$; ainsi qu'il résulte évidemment du théorème de Poisson.

Supposons a et b fixes et cherchons la limite du terme général (32). D'abord, y^a a pour limite e^{-ku}, car $\frac{a}{N}$ tend vers 1. Si maintenant nous posons $\beta = \alpha_1 + \alpha_2 + \ldots + \alpha_k = N - \alpha$, nous voyons que $\frac{N!}{\alpha!} x^{N-\alpha} = N(N-1) \ldots (N-\beta+1) x^\beta$ a même limite que $(Nx)^\beta$, soit u^β. Nous avons donc, pour N infini,

$$x_k = e^{-ku} \sum \frac{u^\beta}{\alpha_1! \, \alpha_2! \, \ldots \, \alpha_k!} = e^{-ku} \left(\sum_{h=a}^{b} \frac{u^h}{h!} \right)^k,$$

soit

$$x_k = v^k,$$

en posant

$$v = e^{-u} \left(\sum_{h=a}^{b} \frac{u^h}{h!} \right).$$

On trouverait de même que y_k et z_p^q ont pour limites respectives $(1-v)^k$ et $v^p(1-v)^q$. On retombe sur les formules (28) concernant le cas des probabilités indépendantes.

14. Problème de l'épuisement des numéros dans une loterie. *Une loterie de n numéros comporte r lots. On demande la probabilité pour que tous les numéros soient sortis après N tirages.*

Ce problème, traité par de Moivre, Laplace et Euler*, se résout très aisément par le calcul direct de y_k. La probabilité pour que k numéros désignés ne sortent pas dans un tirage est

$$\lambda_k = \frac{C_{n-k}^r}{C_n^r} = \frac{(n-k)\,(n-k-1) \, \ldots \, (n-k-r+1)}{n(n-1) \, \ldots \, (n-r+1)}.$$

La probabilité pour qu'ils ne sortent pas dans N tirages est

$$y_k = (\lambda_k)^N.$$

La formule (7) nous donne maintenant la probabilité z_p^q pour que p numéros désignés sortent et que q autres également désignés ne sortent pas. Pour $p = n$, $q = 0$, on a la probabilité de l'épuisement complet.

La probabilité, calculée par Euler†, pour que p numéros au plus soient sortis est donnée par (20).

Laplace a calculé assez péniblement une valeur asymptotique de la probabilité x_n, quand on suppose n et N très grands. On obtient beaucoup plus rapidement un résultat équivalent en se bornant à chercher la valeur asymptotique de y_k. Or, on peut écrire

$$y_k = \left(1 - \frac{k}{n}\right)^N \left(1 - \frac{k}{n-1}\right)^N \ldots \left(1 - \frac{k}{n-r+1}\right)^N.$$

*Cf. Todhunter, op. cit., page 253.

†Euler présente le résultat sous la forme de la formule (18). La formule (20) est un peu plus simple.

Si r reste fini et que $\frac{N}{n}$ tende vers u, on a , à la limite,

$$y_k = e^{-kru} = v^k,$$

en posant $v = e^{-ru}$.

On retombe encore une fois sur les probabilités indépendantes. Et ceci s'explique aisément, car la probabilité d'extraire un numéro désigné est pratiquement constante pendant chaque tirage, l'absence des numéros tirés précédemment ne modifiant pas sensiblement la composition de l'urne.

La valeur probable du nombre des numéros sortis* est ny_1, en vertu du paragraphe 5; soit $\frac{(n-r)^N}{n^{N-1}}$.

15. Problème des rencontres. Ce problème a été traité, sous différentes formes, par Montmort, de Moivre, Nicolas Bernoulli, Laplace, J. Bertrand, Andrade, Catalan, etc. Voici une première manière de l'envisager†.

Dans une urne, il y a m boules, dont nr sont numérotées de 1 *à n, chaque numéro étant répété r fois. On tire toutes les boules les unes après les autres. On dit qu'il y a rencontre si une boule sort à un rang égal à son numéro.*

La rencontre ne peut se produire qu'à un rang au plus égal à n. La probabilité pour qu'il y ait rencontre à p rangs désignés à l'avance est

$$(33) \qquad x_p = \frac{r}{m} \cdot \frac{r}{m-1} \cdot \ldots \cdot \frac{r}{m-p+1} = \frac{r^p(m-p)!}{m!}.$$

En portant dans (6), on a la probabilité z_p^q *pour qu'il y ait rencontre à p rangs désignés à l'avance et non rencontre à q autres rangs également désignés.*

En particulier, la probabilité *pour qu'il n'y ait aucune rencontre* est

$$y_n = \frac{1}{m!} \sum_{k=0}^{n} (-1)^k C_n^k r^k (m-k)!.$$

Pour $r=1$ et $m=n$, on retrouve la formule bien connue

$$y_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

La probabilité d'avoir au moins une rencontre parmi p rangs désignés à l'avance‡ est $1-y_p$.

La probabilité pour qu'il y ait *juste p rencontres* est

*Elle a été évaluée par Euler, au moyen d'un artifice élégant, mais qui exige néanmoins des calculs assez longs.

†Cf. Laplace, *Oeuvres*, tome VII, page 233. L'illustre géomètre considère seulement le cas où $m=nr$.

‡Calculé par Nicolas Bernoulli (Cf. Todhunter, op. cit., page 120).

$$A_p = C_n^p t_p = \frac{1}{m!} \sum_{k=0}^{n-p} (-1)^k C_{n-p}^k C_n^p r^{p+k} (m-p-k)!$$

$$= \frac{n!\, r^p}{m!\, p!} \sum_{k=0}^{n-p} (-1)^k r^k \frac{(m-p-k)!}{k!\,(n-p-k)!}\ ;$$

ce qui, pour $r=1$ et $m=n$, se réduit à*

$$A_p = \frac{1}{p!} \sum_{k=0}^{n-p} \frac{(-1)^k}{k!}.$$

La probabilité pour qu'il y ait *au moins p rencontres* est, d'après la formule (19).

$$1 - B_{p-1} = (-1)^p \frac{p}{m!} C_n^p \sum_{k=p}^{n} (-1)^k \frac{1}{k} C_{n-p}^{n-k} r^k (m-k)!$$

$$= \frac{n!\, r^p}{m!(p-1)!} \sum_{h=0}^{n-p} (-1)^h \frac{r^h}{h!} \frac{(m-p-h)!}{(n-p-h)!\,(p+h)},$$

formule établie par Laplace dans le cas où $m=nr$†.

La *valeur probable du nombre des rencontres* est, d'après (21),

$$V = n\frac{r}{m},$$

soit 1, lorsque $m=nr$‡.

16. Voici une autre forme du problème des rencontres, également considérée par Laplace§, ainsi que par de Moivre|| et par Catalan¶. *On a m boules dans une urne. On en tire n. On répète N fois l'opération. On demande la probabilité pour que certaines boules sortent avec le même rang dans les N tirages.*

Numérotons les boules en suivant l'ordre du premier tirage. La probabilité pour que p boules désignées sortent avec le même rang dans les N tirages est identique à la probabilité pour qu'elles donnent la rencontre en $N-1$ tirages. On a donc, en faisant $r=1$ dans la formule (33) et appliquant le théorème des probabilités composées,

$$x_p = \left[\frac{(m-p)!}{m!}\right]^{N-1}. \tag{34}$$

D'où l'on peut déduire z_p^q, y_n, A_p, B_p, V comme au paragraphe 15.

On peut généraliser la question de la manière suivante:

*Cf. Catalan: *Jour. de Math.*, tome II, 1837, page 474.

†Loc. cit., page 242.

‡J. Bertrand (*Calcul des probabilités*, page 51) a fait ce calcul pour $r=1$. Mais, son raisonnement, qui n'est autre que celui du paragraphe 5, s'étend au cas général.

§Loc. cit., page 244.

||Cf. Todhunter (op. cit., page 159). Ces deux auteurs ont seulement envisagé le cas où $n=m$.

¶Loc. cit., page 477. Il calcule A_p, dans l'hypothèse $N=2$.

Certaines boules étant numérotées de 1 à n, on fait N tirages. On considère comme boule favorable une boule qui donne la rencontre dans α tirages désignés à l'avance et ne la donne dans aucun des N − α autres tirages.

La probabilité pour que p boules désignées soient favorables est évidemment.

$$x_p = (a_p)^\alpha (b_p)^{N-\alpha},$$

en posant

$$a_p = \frac{(m-p)!}{m!}, \quad b_p = \sum_{k=0}^{p} (-1)^k C_p^k a_k.$$

On pourrait exiger seulement, pour qu'une boule soit déclarée favorable, qu'elle donne la rencontre juste dans α tirages, mais sans que ceux-ci soient désignés à l'avance. Le calcul de x_p parait alors beaucoup plus difficile.

17. Andrade* a généralisé le problème de Laplace traité au paragraphe 15, en ajoutant à la fin de l'énoncé, les mots "*à un multiple de n près*" et en exigeant, en outre, que *les boules sortent par groupes consécutifs contenant chacun n numéros différents*. Il s'est d'ailleurs borné à calculer la probabilité pour qu'il n'y ait aucune rencontre et dans le cas où $m = nr$.

Calculons, dans le cas général, la probabilité pour que, dans chaque groupe G_i, il y ait *rencontre sur p_i numéros désignés et non rencontre sur q_i autres numéros également désignés.*

Calculons d'abord la probabilité X pour que les boules sortent par groupes, comme il est exigé. La probabilité pour que les n premières boules tirées portent les n numéros, dans un ordre désigné à l'avance, est donnée par la formule (33), pour $p = n$. Comme il y a $n!$ manières de choisir cet ordre, la probabilité pour que les n premières boules comprennent les n numéros dans un ordre indéterminé est

$$n!\,x_n = \frac{r^n (m-n)!\,n!}{m!}.$$

La probabilité analogue pour les n boules suivantes s'obtient en changeant simplement r en $r-1$ et m en $m-n$ et ainsi de suite. La probabilité composée X est donc

$$X = \frac{(n!)^r (r!)^n (m-nr)!}{m!}.$$

Sachant maintenant que les boules sortent par groupes, la probabilité des rencontres imposées au groupe G_i est $z_{p_i}^{q_i}$, calculée à partir de la formule (33), dans l'hypothèse $r = 1$, $m = n$. La probabilité demandée est finalement

$$\frac{(n!)^r (r!)^n (m-nr)!}{m!} z_{p_1}^{q_1} z_{p_2}^{q_2} \ldots z_{p_r}^{q_r}.$$

*Journal de l'École Polytechnique, 64e cahier, 1894, page 225 et 2e série, 6e cahier, 1901, page 119.

En supposant les p_m nuls, les q_m égaux à n et $m=nr$, on retrouve la probabilité d'Andrade:

$$\frac{(n!)^r(r!)^n}{(nr)!}\left[\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right]^r.$$

18. PROBLÈME DES BRELANS. Ce problème a été étudié par J. Bertrand* pour le jeu de la bouillote. Nous allons le généraliser.

Supposons *n joueurs recevant chacun m cartes. Le jeu comprend r espèces de cartes, dont* α *dans chaque espèce.* On suppose $\frac{\alpha}{2}<m\leq\alpha$. *Un joueur a brelan quand ses m cartes sont de la même espèce.*

Calculons la probabilité x_p pour que p joueurs désignés aient brelan.

Supposons que l'on sache que les $p-1$ premiers ont brelan. Il reste alors $C_\alpha^m(r-p+1)$ brelans possibles, car chaque espèce de cartes ne peut donner qu'un seul brelan, à cause de l'inégalité $\alpha<2m$ et, dans une même espèce, il y a C_α^m brelans. D'autre part, il reste $r\alpha-m(p-1)$ cartes à distribuer, soit $C_{r\alpha-m(p-1)}^m$ jeux possibles pour le p^e joueur. La probabilité pour que celui-ci ait brelan est donc

$$\frac{C_\alpha^m(r-p+1)}{C_{r\alpha-m(p-1)}^m}=\frac{\alpha!}{(\alpha-m)!}(r-p+1)\frac{(r\alpha-mp)!}{[r\alpha-m(p-1)]!}.$$

Par suite, on a

$$x_p=x_{p-1}\frac{\alpha!}{(\alpha-m)!}(r-p+1)\frac{(r\alpha-mp)!}{[r\alpha-m(p-1)]!}. \tag{35}$$

En remarquant que $x_o=1$, on tire de là

$$x_p=\frac{r!}{(r\alpha)!}\frac{(r\alpha-mp)!}{(r-p)!}\left[\frac{\alpha!}{(\alpha-m)!}\right]^p.$$

Il ne reste plus maintenant qu'à appliquer les formules (6), (8), (10), (19) pour répondre à toutes les questions pouvant être posées sur les joueurs ayant brelan.

Dans le jeu de la bouillote, on a: $n=4$, $m=3$, $r=5$, $\alpha=4$. Donc,

$$x_p=\frac{5!}{20!}\frac{(20-3p)!}{(5-p)!}(4!)^p.$$

Il est d'ailleurs plus pratique d'appliquer la formule (35), qui devient ici

$$x_p=x_{p-1}\frac{8(6-p)}{(7-p)(22-3p)(23-3p)}.$$

D'où l'on tire successivement les valeurs numériques de x_1, x_2, x_3, x_4.

J. Bertrand calcule les probabilités que nous appelons t_1, t_2, t_3, t_4 et aboutit à nos formules (10), où l'on ferait $n=4$. Il cherche ensuite la probabilité pour

*Op. cit., page 33.

qu'il n'y ait aucun brelan. Cette probabilité n'est autre que

$$t_0 = y_4 = 1 - 4x_1 + 6x_2 - 4x_3 + x_4.$$

J. Bertrand obtient un résultat différent, parce que son raisonnement est inexact*. Il admet, par exemple, que la probabilité R_4 pour que le quatrième joueur ait brelan, sachant que les trois autres ne l'ont pas, est égale à t_1, c'est-à-dire à la probabilité pour que le quatrième joueur ait seul un brelan. Or, cette dernière est égale au produit de R_4 par la probabilité y_3 pour que les trois premiers joueurs n'aient pas brelan. On a donc $R_4 = \frac{t_1}{y_3}$ et non $R_4 = t_1$.

*C'est en cherchant à rectifier ce raisonnement que nous avons été conduit à la théorie générale qui fait l'objet de ce travail.

SUR LE PROBLÈME DES SÉQUENCES

PAR M. J. HAAG,
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Ce problème a été étudié, dans le cas particulier de la loterie de Gênes, par différents mathématiciens, entre autres par Euler et Jean Bernoulli*, qui en ont donné une solution inductive. Nous nous proposons, dans ces quelques lignes, de donner une solution générale directe, qui nous paraît plus simple.

Soit une loterie de n numéros. On en tire m. On dit qu'il y a une *séquence* quand, parmi les numéros tirés, se trouvent deux ou plusieurs numéros consécutifs. Euler considère la série des n numéros comme une *série linéaire,* c'est-à-dire qu'à son point de vue les numéros n et 1 ne forment pas une séquence. Bernoulli considère, au contraire, la série comme *circulaire;* les numéros n et 1 formant une séquence.

Appelons B_n^m la *probabilité pour qu'il n'y ait pas de séquence* au sens de Bernoulli et E_n^m la même probabilité au sens d'Euler. La probabilité B_n^m ne change pas si l'on fait une permutation circulaire sur tous les numéros. Dès lors, nous pouvons toujours supposer que le numéro 1 appartient aux numéros tirés. Dans ce cas, aucun des $m-1$ autres numéros tirés ne doit être 2, ni n, ce qui a pour probabilité

$$\frac{C_{n-3}^{m-1}}{C_{n-1}^{m-1}} = \frac{(n-m)\ (n-m-1)}{(n-1)\ (n-2)}.$$

Il faut ensuite et il suffit que ces $m-1$ numéros ne donnent aucune séquence au sens d'Euler. On a donc

$$B_n^m = \frac{(n-m)\ (n-m-1)}{(n-1)\ (n-2)} E_{n-3}^{m-1}. \tag{1}$$

D'autre part, si l'on sait que 1 n'est pas tiré, ce qui a pour probabilité $1-\dfrac{m}{n}$, il faut et il suffit que les m numéros tirés ne présentent pas de séquence au sens d'Euler. La probabilité de cette première éventualité est $\left(1-\dfrac{m}{n}\right)E_{n-1}^m$. Si l'on sait que 1 est tiré, ce qui a pour probabilité $\dfrac{m}{n}$, la probabilité pour qu'il n'y ait

*Cf. Encyclopédie des Sciences Mathématiques, édition française, *t.* I, vol. 4, art. 20, pages 19 et 20; Todhunter, *History of the theory of probability*, pages 245 à 247, 326 et 327.

pas séquence est B_n^m, ainsi qu'on l'a vu plus haut. La probabilité de cette deuxième éventualité est donc $\frac{m}{n} B_n^m$ et l'on a

$$B_n^m = \left(1 - \frac{m}{n}\right) E_{n-1}^m + \frac{m}{n} B_n^m$$

ou*

$$B_n^m = E_{n-1}^m. \tag{2}$$

Portons dans (1):

$$B_n^m = \frac{(n-m)\,(n-m-1)}{(n-1)\,(n-2)} B_{n-2}^{m-1}. \tag{3}$$

Changeons successivement n en $n-2$, m en $m-1$, puis n en $n-4$, m en $m-2$, etc., jusqu'à ce que nous arrivions à B_{n-2m+2}^1, qui est évidemment égal à un. Multiplions enfin toutes les égalités ainsi obtenues; il vient, en remarquant que les facteurs $n-m-1$, $n-m-2$, . . ., $n-2m+2$ se trouvent chacun deux fois au numérateur et une fois au dénominateur,

$$B_n^m = \frac{(n-m-1)\,(n-m-2)\,\ldots\,(n-2m+1)}{(n-1)\,(n-2)\,\ldots\,(n-m+1)}; \tag{4}$$

ce qui est bien la formule trouvée par Euler et par Bernoulli.

*Cette relation a été signalée, sans démonstration, par Bernoulli. Todhunter (loc. cit.) en a donné une vérification inductive assez pénible.

ABSTRACTS OF COMMUNICATIONS

SECTION I

ON THE FIRST CASE OF FERMAT'S LAST THEOREM

By Mr. H. S. Vandiver,
Cornell University, Ithaca, New York, U.S.A.

In this paper Mr. Vandiver proves a number of criteria in connection with the equation

(1) $$x^p + y^p + z^p = 0$$

where x, y and z are integers and p is an odd prime which does not divide xyz. It is known that if this relation is satisfied under these conditions then

(2) $$m^{p-1} \equiv 1 \pmod{p^2},$$

for $m = 2, 3, 5, 11, 17$, and if $p \not\equiv 1 \pmod 3$ also for $m = 7, 13$. By using simpler methods than those previously employed it is here proved that (2) holds for $m = 7$ and 13 with p any odd prime. Also it holds for $m = 19$ if $p \not\equiv 1 \pmod 9$ and $m = 23$ if $p \not\equiv 1 \pmod{11}$. A procedure is indicated for further extension of these criteria. It is also shown that if

$$-t \equiv x/y,\ y/x,\ x/z,\ z/x,\ y/z, \text{ or } z/y, \pmod p,$$

then

$$f_{p-n}(t)\, f_n(1-t) \equiv 0 \pmod p,$$

where

$$f_n(u) = \sum_{r=o}^{p-1} r^{n-1} u^r, \quad (n = 1, 2, \ldots, p-1).$$

L'ALGEBRA DELLE DERIVATE

DEL PROFESSORE PIO SCATIZZI,
Università Gregoriana, Roma, Italia.

Nel presente lavoro il Professore Scatizzi si preoccupa di risolvere un problema assai elevato, il quale constituisce un'estensione importante dei casi di dipendenza funzionale di cui mi sono occupato in altra comunicazione a questo Congresso. Il problema è definire in modo generale e trattare le derivate a indice generalizzato, cioè il cui indice sia una funzione della variabile indipendente. Queste operazioni non appartengono alla categoria di quelle che io chiamo normali, e seguono leggi meno semplici di quelle dell'algebra ordinaria.

L'A. dà un sistema di definizioni molto appropriato e che contiene come caso particolare le definizioni già conosciute pel caso di un indice costante.

Indi tratta un problema inverso, che egli chiama "problema ideale", e che è il seguente: date due funzioni $\psi(x)$, $\phi(x)$ ed essendo D il simbolo della derivazione, trovare una funzione $S(x)$, tale che

$$\phi(x) = D^{S(x)}\psi(x)$$

Seguono teoremi, e uno sviluppo della teoria ed esempi di applicazioni.

Il lavoro in esteso sarà pubblicato negli Atti della Pontificia Accademia dei Nuovi Lincei.

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ON DETERMINING THE ASYMPTOTIC DEVELOPMENTS OF A GIVEN FUNCTION

By Professor Walter B. Ford
University of Michigan, Ann Arbor, Mich., U.S.A.

If a Maclaurin series

$$a_0+a_1z+a_2z^2+\ldots.+a_nz^n+\ldots. \tag{1}$$

be given, the function of z which it defines may usually be extended analytically across the circle of convergence of the series, but the precise character of the function in these outside regions, though fully determined, is as a rule not easy to ascertain. In particular, the behaviour of such a function in distant portions of the z-plane is usually obtained only with difficulty. The author obtains a theorem of considerable generality, however, by which the behaviour of any function of z defined by a series of the type (1) may be fully determined when $|z|$ is large. Various examples and applications are cited.

ON THE ROOTS OF THE RIEMANN ZETA FUNCTION

By Professor J. I. Hutchinson,
Cornell University, Ithaca, N.Y., U.S.A.

The object of this paper is to simplify the methods and formulas hitherto used in numerical investigations connected with the Riemann function $\zeta(s)$. In particular the accurate determination of the remainder is very much simplified.

The empiric law noticed by Gram, namely that the roots of $\zeta(\sigma+ti)$ situated on the line $\sigma=\frac{1}{2}$ are separated by the remaining roots of the imaginary part of $\zeta(\frac{1}{2}+ti)$ is proved to fail when $t=282{,}455$. This separation of the roots is used, as far as it goes, for locating 59 additional roots of $\zeta(s)$ and proving that they are all situated on the line $\sigma=\frac{1}{2}$.

It is shown how, with not too large values of t, the number of roots of $\zeta(s)$ in the "critical strip" may be determined without excessive labor. This number is determined for $0<t<500$ and found to be 269.

Fourteen additional roots of $\zeta(s)$ are calculated and the last five of those calculated by Gram are determined more accurately. The method of calculating the roots, aided by the new remainder formulas, is much more expeditious than that used by Gram.

ON THE NUMERICAL SOLUTION OF INTEGRAL EQUATIONS

BY PROFESSOR GORAKH PRASAD,
Benares Hindu University, Benares, India.

The method of numerically solving integral equations investigated here is much less laborious than any other method yet known, and at the same time very general and of great accuracy. For linear integral equations it is based on the following theorem:

If the values of the solution of the integral equation

$$\phi(x) = f(x) + \int_a^x K(x, \xi)\, \phi(\xi) d\xi,$$

where $K(x,\xi)$ and $f(x)$ may be given numerically, have been computed for $x = a$, $a+w$, $a+2w$, . . . , $a+(n-1)w$, and are ϕ_0, ϕ_1, ϕ_2, . . . , ϕ_{n-1}, then its value for $x = a+nw$, viz., ϕ_n, is given by

$$\phi_n = f_n + (I_n) + cwK_{n,n}\, \frac{f_n + (I_n) - (\phi_n)}{1 - cwK_{n,n}},$$

where,

f_n *denotes* $f(a+nw)$,

$K_{r,s}$ *denotes* $K(a+rw, a+sw)$,

(ϕ_n) *is any assumed value of* ϕ_n,

$$(I_n) = w[u_0 + u_1 + u_2 + \ . \ . \ . \ + u_{n-1} + (u_n) - \tfrac{1}{2}\{(u_n) + u_o\}$$
$$- \tfrac{1}{12}\{(\Delta u_{n-1}) - \Delta u_o\} - \tfrac{1}{24}(\Delta^2 u_{n-2}) + \Delta^2 u_o\}$$

— . . . *up to the term involving the rth differences*],

$u_p = K_{n,p}\phi_p$, $(u_n) = K_{n,n}(\phi_n)$, $(\Delta u_{n-1}) = (u_n) - u_{n-1}$, . . . ,

and where c is a numerical constant whose value is ·3299, ·3156 *or* ·3042, *according as the order r of the highest order difference used in the evaluation of* (I_n) *is* 4, 5 *or* 6.

An illustrative example is worked out and it is shown how the method can, with a slight modification, be applied to non-linear integral equations of a very general type.

QUELQUES RÉCENTS TRAVAUX DE MATHÉMATICIENS DE LÉNINGRAD

Par M. N. Gunther,
Professeur à l'Université de Léningrad, Léningrad, Russie.

M. Gunther a donné un compte-rendu des communications mathématiques faites à la Société Physico-mathématique de Léningrad pendant la période 1921-1924. Il a cité les résultats obtenus et a ajouté diverses explications aux méthodes employées. Quelques-unes de ces communications n'ont pas encore été imprimées, tandis que d'autres ont été publiées par l'Académie des Sciences de Russie, ou ont paru dans divers journaux étrangers. Parmi celles-là se trouvent des memoires par MM. Coïalowitsch, Gr. M. Fichtenholz, A. F. Gavriloff et W. J. Smirnoff dont nous donnons des résumés à la suite de cette note.

Parmi les autres mathématiciens dont des travaux ont été portés à l'attention de la Section nous pouvons citer MM. A. Bezikowitsch, B. N. Delaunay, V. Fock, A. A. Friedmann, B. Izvekoff, G. A. Kroutkoff, L. Loitziansky, W. Lwowski, J. V. Uspensky, W. Speransky, J. D. Tamarkine.

M. Gunther a aussi parlé brièvement de quelques-uns de ses propres travaux et a cité plus particulièrement les résultats qu'il a obtenus relativement à la solution de l'équation

$$s=f(x, y, u, p, q, r, t).$$

Ces résultats, ont été d'ailleurs publiées depuis la clôture du Congrès dans le Recueil Mathématique de la Société Mathématique de Moscou.

SUR LES ÉQUATIONS DIFFÉRENTIELLES INDÉTERMINÉES

PAR M. B. M. COIALOWITSCH,
Professeur à l'Institut Technologique, Léningrad, Russie.

L'auteur s'occupe des formes diverses que peut recevoir l'équivalent intégral de l'équation de Monge:

$$\frac{dz}{dx} = \Phi\left(x, y, z, \frac{dy}{dx}\right). \tag{1}$$

Une des principales est

$$F(x, y, z, u) = \phi(u), \quad F'_u(x, y, z, u) = \phi'(u), \quad F''_{uu}(x, y, z, u) = \phi''(u),$$

où $F(x, y, z, u)$ est une solution de l'équation du premier ordre en F, qui s'obtient en éliminant γ entre les équations

$$\frac{\partial F}{\partial x} + \gamma \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \Phi(x, y, z, \gamma) = 0, \quad \frac{\partial F}{\partial y} + \Phi'_\gamma(x, y, z, \gamma) = 0.$$

Toutes les solutions de (1) peuvent être obtenues en attribuant à $\phi(u)$ une forme convenable, sauf certaines solutions exceptionnelles (discriminierende ou ausgelassene Lösungen de M. Hilbert). Celles-ci sont les plus importantes; on en déduit toutes les autres sans nouvelle intégration. La forme la plus générale que l'auteur ait pu trouver jusqu'ici pour l'expression intégrale de (1) est celle-ci:

$$F(x, y, z, u) = \beta, \; G(x, y, z, u) = \phi(u), \; G'_u(x, y, z, u) = \phi'(u),$$

où F et G sont liées par la relation suivante

$$F'_u = M(F, u, G, G'_u),$$

et β et ϕ par celle-ci:

$$\frac{d\beta}{du} = M(\beta, u, \phi(u), \phi'(u)).$$

SUR LA NOTION DE FERMETURE DES SYSTÈMES DE FONCTIONS

PAR M. GR. M. FICHTENHOLZ,
Professeur à l'Université de Léningrad, Léningrad, Russie.

On doit à MM. E. Schmidt* et D. Hilbert† deux définitions bien différentes de la notion de fermeture du système Ω de fonctions $\omega_n(x)$, $n=1, 2, 3, \ldots$, définies pour les valeurs de x dans un intervalle (a, b), par rapport à une classe donnée, F, de fonctions.

L'auteur a posé et résolu complètement: 1°, la question de l'équivalence de ces deux définitions et 2°, la question de savoir en quels cas et en quelle mesure il est légitime d'étendre la classe F par rapport à laquelle le système quelconque Ω est fermé, d'après la première ou d'après la seconde définition.

***Entwickelung willkürlicher Funktionen nach Systemen vorgeschriebener.* Dissertation Göttingen, 1905.

†*Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen,* Göttinger Nachrichten, Math. Phys. Klasse, 1906.

SUR L'INTÉGRATION DES ÉQUATIONS DES LIGNES GÉODÉSIQUES ET D'UN PROBLÈME DE LA DYNAMIQUE DU POINT

PAR M. A. F. GAVRILOFF,
Professeur à l'Institut Polytechnique, Léningrad, Russie.

Si, ds étant la différentielle d'un arc d'une courbe tracée sur une surface donnée, on a

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

l'équation

$$\Delta\theta = \frac{Eq^2 - 2Fpq + Gp^2}{EG - F^2} = 1 \tag{1}$$

détermine une famille de trajectoires orthogonales des lignes géodésiques de la surface.

Si $\theta(u, v, a)$ est une solution de l'équation (1), la fonction θ n'étant pas linéaire par rapport à a, l'équation des lignes géodésiques est

$$\frac{\partial\theta}{\partial a} = b.$$

Si

$$\phi(u, v, p, q) = a \tag{2}$$

est une intégrale du système d'équations

$$\frac{du}{\frac{\partial\Delta}{\partial p}} = \frac{dv}{\frac{\partial\Delta}{\partial q}} = -\frac{dp}{\frac{\partial\Delta}{\partial u}} = -\frac{dq}{\frac{\partial\Delta}{\partial v}}, \tag{3}$$

on obtient θ en associant à l'équation (2) l'équation (1) après une quadrature. Ayant deux intégrales

$$\phi_1(u, v, p, q) = a, \quad \phi_2(u, v, p, q) = b \tag{4}$$

du système (3), on déduit l'équation des lignes géodésiques sans intégration en éliminant p, q entre les équations (4) et (1). Nous disons qu'en ce cas, l'équation des géodésiques s'intègre sans quadrature.

La discussion du système (3) dans toute sa généralité étant difficile, on peut chercher les surfaces qui permettent d'avoir les intégrales (2) du système (3) d'une forme choisie.

En choisissant des coordonnées symétriques, on a

$$ds^2 = 4\lambda(u, v)dudv, \qquad \Delta\theta = \frac{pq}{\lambda} = 1, \qquad pq = \lambda.$$

Bertrand, Massieu, Koenigs, S. Lie, Bour, Bonnet ont étudié des ϕ linéaires, quadratiques, fractionnaires de formes spéciales. M. Koenigs a donné quelques théorèmes généraux sur les intégrales algébriques; Maurice Lévy a proposé de chercher les intégrales algébriques sous la forme

$$\phi = f(u, v) p^{\alpha} q^{\beta} \prod_{i=1}^{n} (pq_i - p_i q)^{m_i}, \text{ où } p_i q_i = \lambda. \tag{5}$$

Mais les équations étant trop compliquées, Maurice Lévy n'est pas venu à des résultats nouveaux.

Dans ce qui suit, ϕ est cherché sous la forme

$$\phi = f(u, v) p^{\epsilon} (pq_1 - p_1 q) \prod_{i=1}^{n} (pq_i - p_i q)^{m_i}, \tag{6}$$

qui est plus simple que celle de Maurice Lévy et aussi générale qu'elle. Le calcul nous conduit à quelques équations aux dérivées partielles du second ordre, linéaires en r, s, t, dont les coefficients sont des fonctions assez compliquées de p, q, u, v. En lui appliquant la méthode des caractéristiques de Monge-Ampère, on peut en intégrer quelques-unes, ce qui donne quelques théorèmes généraux dont voici les plus intéressants:

1° Les surfaces

$$ds^2 = 4\lambda du dv,$$

où λ satisfait à l'équation contenant une fonction arbitraire ω,

$$\lambda = \frac{\omega(\lambda) + (\epsilon - 1)u}{(\epsilon + 1)v}, \qquad \epsilon = \frac{\alpha - \beta}{m} \neq \pm 1, \tag{7}$$

ont des intégrales algébriques de la forme:

$$f p^{\alpha} q^{\beta} (pq_1 - p_1 q)^{m} \tag{8}$$

ou, ce qui est la même chose,

$$f p^{\epsilon} (pq_1 - p_1 q).$$

2° Les surfaces de Lie-Darboux, où

$$ds^2 = 4[(v + U(u))] du dv,$$

sont les seules qui donnent des intégrales de la forme

$$\phi = f p^{k\epsilon + \alpha} q^{\alpha} (pq_1 - p_1 q)^{-k(\epsilon+1)} (pq_2 - p_2 q)^{k}. \tag{9}$$

3° On peut intégrer les équations des géodésiques sur toutes les surfaces de Lie-Darboux sans quadratures.

4° La recherche des intégrales d'une forme

$$f p^{\alpha} q^{\beta} (pq_1 - p_1 q)^{m_1} (pq_2 - p_2 q)^{m_2}, \tag{10}$$

en des hypothèses plus générales pour α, β, m_1, m_2 que celles de la formule (7), ne peut pas conduire à des surfaces nouvelles.

Il est facile d'appliquer les théorèmes trouvés à la mécanique du point sur une surface fixe.

SUR LA THÉORIE DES GROUPES AUTOMORPHES

PAR M. W. J. SMIRNOFF,
Professeur à l'Institut des Ponts et Chaussées, Léningrad, Russie.

A chaque groupe fuchsien discontinu correspond un polygone fondamental normal, formé par des arcs de cercles orthogonaux à l'axe réel.

La communication est destinée à la discussion du cas, où le nombre des côtés du polygone est infini.

L'auteur donne tous les cas possibles de sommets du polygone sur l'axe réel. Il démontre, entre autres, la proposition suivante: chaque ensemble fermé partout discontinu de points sur l'axe réel peut représenter les sommets d'un polygone et leurs points d'accumulation.

Au surplus, il donne les conditions du théorème d'existence pour la fonction fuchsienne prenant dans le polygone fondamental chaque valeur le même nombre de fois.

COMMUNICATIONS

SECTION II

GEOMETRY

QUELQUES REMARQUES SUR LE PROBLÈME DES QUATRE COULEURS

PAR M. ALFRED ERRERA,

Docteur en Science Université de Bruxelles, Bruxelles, Belgique.

C'est un fait d'expérience, que les régions des cartes géographiques tracées sur la sphère ou sur le plan, peuvent être teintées en quatre couleurs, sans que deux régions adjacentes reçoivent la même teinte. On s'est demandé si cette proposition est vraie de toutes les cartes possibles.

L'origine de ce problème n'est pas certaine; la première mention imprimée que nous en connaissions, est de Cayley*.

C'est à Kempe† que l'on doit un certain nombre de réductions aujourd'hui classiques; mais sa démonstration de la proposition est illusoire.

L'on sait seulement que cinq couleurs suffisent toujours, ainsi que l'a démontré Heawood‡; et, chose curieuse, ce géomètre a résolu le même problème pour les cartes tracées sur certaines surfaces à connexion plus élevée, telles que le tore, où il faut sept couleurs.

Mentionnons aussi les travaux de Petersen§, qui a fait faire un progrès notable à la question très voisine qu'avait indiquée Tait||, de la décomposition des réseaux du troisième degré, c'est-à-dire des systèmes de lignes où chaque sommet est trièdre.

Nous n'entrerons pas dans le détail, ni de l'historique du problème, ni du peu qu'on sait de sa solution; on en trouvera des indications assez nombreuses dans notre thèse¶.

Des progrès récents ont été faits en Amérique, d'abord par Birkhoff**, dont les méthodes originales ont conduit à des réductions importantes, et enfin, tout récemment, par Franklin††, qui en a prouvé toute une série d'autres dans sa thèse.

Qu'il nous soit permis de profiter du congrès de Toronto, pour résumer des résultats nouveaux en cours de publication‡‡, dont le point de départ se trouve dans les travaux de ces deux mathématiciens.

*Cayley, Proc. London Math. Soc., IX, p. 148, 1878.

†Kempe, Amer. Jour. Math., Baltimore, II, 3, p. 193-200, 1879.

‡Heawood, Quarterly Journal of Mathematics, London, XXIV, p. 332-338, 1890.

§Petersen, Acta Math., Stockholm, XV, p. 193-220, 1891.

||Tait, Proc. R. Soc., Edinburgh, X, p. 729, 1880.

¶Errera, *Du Coloriage des Cartes*, Thèse, Bruxelles, Falk fils, Van Campenhout succ.; Paris, Gauthier-Villars, 1921.

**Birkhoff, Amer. Jour. Math., Baltimore, XXXV, 2, p. 115-128, 1913.

††Franklin, *The Four Color Problem*, Diss., Amer. Jour. Math., Baltimore, XLIV, 3, 1922.

‡‡Errera, Bull. Soc. Math. France, Paris, LIII, p. 42-55, 1925.

Traçons sur la sphère un réseau formé d'une ou de plusieurs parties connexes et sans isthme (c'est-à-dire dont les arêtes appartiennent toutes à des cycles fermés); nous obtenons un certain complexe.

Admettant que quatre couleurs ne suffisent pas à en colorier les régions, si nous appelons réseau irréductible, celui (ou l'un de ceux) qui possède le plus petit nombre de faces et qu'on ne puisse teinter en quatre couleurs, on sait qu'un réseau irréductible satisfait à des conditions nécessaires; celles-ci, lorsqu'elles ne sont pas réalisées, nous font connaître des réductions du problème; en voici quelques-unes:

(1) le réseau est connexe*, et le complexe est donc un polyèdre au sens de l'Analysis Situs;

(2) il est du troisième degré*;

(3) il ne contient aucun anneau de faces, formé de moins de cinq polygones†;

(4) il ne contient aucun anneau de cinq faces, sauf ceux qui entourent des pentagones†;

(5) il ne contient aucun anneau de pentagones entourant une face†;

(6) il ne contient aucun anneau d'hexagones entourant une face paire (d'un nombre pair de côtés)†;

(7) il ne contient aucun anneau formé d'hexagones et de paires de pentagones adjacents deux à deux, entourant une face paire‡;

(8) il ne contient aucun anneau formé de pentagones et de deux polygones quelconques adjacents, entourant une face paire‡;

(9) il ne contient aucun anneau formé de pentagones et d'un polygone quelconque, entourant une face impaire‡;

(10) il contient au moins 26 faces‡.

Birkhoff et Franklin ont donné quelques autres configurations particulières, qui ne peuvent pas exister dans un polyèdre irréductible; généralisant leurs résultats, nous avons pu supprimer, dans les cas (5) à (9) inclus, la condition que l'anneau entoure une face unique; mais notre démonstration comporte une restriction, et la voici: en appliquant la méthode de réduction indiquée, il faut obtenir un réseau réduit sans isthme, faute de quoi ce ne serait pas un polyèdre, et la réduction serait illusoire.

Ainsi, sous cette seule réserve, nous démontrons qu'un polyèdre irréductible ne contient jamais d'anneau pair d'hexagones.

En nous appuyant principalement sur ce théorème, nous établissons la proposition, qui nous paraît intéressante par sa généralité: Un polyèdre dont toutes les faces sont des pentagones et des hexagones, est toujours réductible. Autrement dit, un polyèdre irréductible contient nécessairement au moins une face ayant au moins sept côtés.

L'exemple que donne Franklin à la fin de sa thèse‡, est celui d'un polyèdre réductible par les moyens que nous indiquons.

*Kempe, *loc. cit.*
†Birkhoff, *loc. cit.*
‡Franklin, *loc. cit.*

SUR LA SPHÈRE VIDE

Par M. B. Delaunay,
Professeur à l'Université de Léningrad, Léningrad, Russie.

Soit E un système de points distribués régulièrement dans l'espace, au sens de Bravais, c'est-à-dire un système parallélipipédique de points (Fig. 1). Je me propose de considérer une sphère se mouvant entre les points de E se rétrécissant et se dilatant à volonté et assujettie à la condition d'être "vide", c'est-à-dire de ne pas contenir dans son intérieur des points de E.

Comme application de cette conception, je vais déduire quelques propriétés du domaine D de tous les points de l'espace qui sont plus près d'un point du système E que de tout autre point de E. Ce domaine D, que je propose d'appeler le *domaine de Dirichlet* de E, est en quelque sorte le domaine de prédominance d'un point de E.*

Il est facile de voir que le rayon de la sphère vide dans E est limité. Par suite, il ne peut y avoir qu'un nombre fini de sphères vides qui aient 4 points de E sur leur surface, si, bien entendu, on ne compte pas pour différentes deux sphères obtenues par translation le long d'un vecteur de E. Observons que, si sur quelques-unes de ces sphères, il y avait plus de 4 points de E, il suffirait de faire une variation infinitésimale du parallélipipède fondamental de E pour que cela n'ait plus lieu. Nous allons donc supposer qu'aucune sphère vide dans E n'a plus de 4 points sur sa surface. Nous dirons, dans ce cas, que le système E correspondant est "*non spécial*".

Il est évident que le lieu géométrique de tous les centres de toutes les sphères vides passant par le point O de E n'est autre chose que D_O. Prenons une petite sphère vide passant constamment par le point O, et éloignons son centre dans une direction donnée passant par O. La sphère, en se dilatant de la sorte, se heurtera à un point a de E (Fig. 2). Et alors, son centre sera le point de la frontière de D_O, situé dans la direction donnée. Ce point appartiendra à une face de D_O qui sera perpendiculaire à Oa (Fig. 3 et 4) et aura au point β milieu de Oa un centre de symétrie. Cela est évident si l'on remarque que β est centre de symétrie de E. A chaque couple de points aa' (Fig. 5) heurtés ainsi simultanément par cette sphère vide correspond une arête de D_O. A chaque triplet de tels points (Fig. 6), c'est-à-dire à chaque tétraèdre de E dont l'un des sommets est O et dont la sphère circonscrite est vide, correspond un sommet de D_O. Il est facile de voir que tous les tétraèdres L de E dont les sphères circonscrites sont

*Nous ne connaissons que M. Voronoï (Journal für Math. Bde. 134, 136) et M. Wulf (Zeitschr. für Krystallogr. Bd. XLV 1908) qui aient envisagé le domaine D.

vides, partagent uniformément tout l'espace. Cela provient de ce que: 1°, en déplacant le centre de la sphère vide le long de l'arête correspondant à une face Oaa' du tétraèdre L en la faisant toujours passer par les points Oaa', nous la faisons quitter le sommet opposé a'' et elle se heurte finalement à un point quelconque $\bar{a}''$ situé du côté opposé de la face Oaa'; ce nouveau point $\bar{a}''$ forme avec les points Oaa' un nouveau tétraèdre $\bar{L}$ adjacent à L par la face Oaa'; 2°, deux tétraèdres L_1 et L_2 ont en tous cas leurs sommets sur les segments de leurs sphères qui sont situés sur les côtés opposés du plan mené par le cercle d'intersection de ces sphères (Fig. 7); ils ne peuvent donc pas avoir de points communs intérieurs. En ce qui concerne les points de leurs frontières, ils ne peuvent les avoir en commun que s'ils ont un sommet commun, ou bien une arête commune, ou bien une face commune.

Soit L_{I} un de ces tétraèdres. Construisons tous les tétraèdres L_{I} correspondant à tous les points de E (Fig. 8). Ils ne remplissent pas encore tout l'espace. Construisons également tous les tétraèdres symétriques $L_{\text{I}'}$. Il ne reste maintenant comme vide que des cavernes octaédriques (Fig. 9). Ces octaèdres ont des points de E seulement à leurs sommets ce que l'on voit facilement (Fig. 10) en divisant chaque octaèdre en 16 tétraèdres en menant les hauteurs ad et $a'd'$ et les perpendiculaires de, ae, etc., et en envisageant les sphères vides ayant les arêtes de l'octaèdre pour diamètres. Tous les 16 tétraèdres seront intérieurs à l'une ou à l'autre de ces sphères vides. Comme les tétraèdres L doivent remplir tout l'espace, ces cavernes octaédriques doivent aussi être remplies. Mais elles n'ont pas de communication entre elles, étant entourées par les tétraèdres L_{I} et $L_{\text{I}'}$. Chaque caverne doit donc à elle seule être divisée en tétraèdres. Cela est possible de 3 façons (Fig. 11), en menant l'une des trois diagonales intérieures de l'octaèdre, mais seulement une seule de ces alternatives convient parce que la partition de l'espace en tétraèdres L est uniforme, chaque tétraèdre L correspondant à un sommet des domaines D et réciproquement. Admettons que, dans notre cas, ce soit la diagonale aa' qu'il faille employer. Nous obtenons ainsi deux nouveaux couples de tétraèdres symétriques L_{II}, $L_{\text{II}'}$ et L_{III}, $L_{\text{III}'}$. Les 6 tétraèdres $L_{\text{I}}L_{\text{I}'}L_{\text{II}}L_{\text{II}'}L_{\text{III}}L_{\text{III}'}$ remplissent tout l'espace.

Envisageons tous les tétraèdres L dont l'un des sommets est le point O de E (Fig. 12). Ces 24 tétraèdres constituent un polyèdre $\overline{D}$ à 24 faces ayant 14 sommets et 36 arêtes. Mais, comme ses faces triangulaires donnent deux à deux des parallélogrammes, nous obtenons, en réalité, un dodécaèdre rhomboïdal. Ce dodécaèdre a 4 paires de sommets opposés ternaires (c'est-à-dire où se rencontrent 3 faces) et trois paires de sommets quaternaires (Fig. 13). On peut l'envisager de 4 façons, respectivement aux 4 paires de sommets ternaires, comme translation d'un parallélipipède le long de sa diagonale intérieure. Si l'on mène des plans perpendiculaires par les milieux β des 14 rayons de $\overline{D}$, on obtient D.

Il faut mener les diagonales des 12 faces par les 8 sommets ternaires pour obtenir les 24 faces triangulaires (Fig. 14). A chaque sommet $\overline{D}$ correspond une face de D. A toutes les arêtes de $\overline{D}$, en comptant aussi les diagonales susdites passant par un sommet de $\overline{D}$ correspondent les faces par lesquelles sont adjacents les tétraèdres L formant $\overline{D}$ qui ont pour arête commune le rayon de

$\overline{D}$ correspondant à ce sommet, c'est-à-dire correspondant aux arêtes de la face de D relatives à ce sommet de $\overline{D}$.

A chaque *face* triangulaire de $\overline{D}$ correspond un des tétraèdres L formant $\overline{D}$, c'est-à-dire un des sommets de D. La simple inspection de $\overline{D}$ fournit les propriétés de D:

(1) D est un polyèdre à 14 faces (Fig. 15 et 16) dont 6 sont des parallèlogrammes et 8 des hexagones. En général, c'est une combinaison d'un parallélipipède avec 4 pinacoïdes, mais, dans des cas particuliers, d'un parallélipipède et d'un octaèdre.

(2) D lui-même, ainsi que ses faces, ont des centres de symétrie.

(3) Les 36 arêtes de D sont partagées en 6 zones par 6 arêtes. Toutes les arêtes d'une même zone sont égales et situées sur un même cylindre circulaire.

(4) Les 24 sommets de D sont situés sur les surfaces de 3 sphères concentriques dont les rayons sont justement les rayons des sphères circonscrites aux 3 tétraèdres L_{I}, L_{II}, L_{III}, et ainsi de suite.

Si l'on a un cas limite, c'est-à-dire si le système E devient *spécial*, il est facile de voir que les 6 arêtes d'une zone disparaissent. Nous obtenons ainsi, par conséquent encore 4 dégénérescences de D. Ce sont: un dodécaèdre allongé spécial (Fig. 17), un dodécaèdre rhomboïdal (Fig. 18), un prisme droit hexagonal dont les faces hexagonales ont des centres de symétrie et sont inscrites dans un cercle (Fig. 19), et le parallélipipède rectangulaire (Fig. 20).

Il est important de rappeler que tout corps convexe qui peut remplir uniformément l'espace en position parallèle, c'est-à-dire chaque parallèloèdre est une affinité de D ou l'une de ses dégénérescences (Théorème de Voronoï, Crelle Bd. 134).

La méthode précédente montre entre autres qu'il n'y a qu'un seul point (et son symétrique) en général, qui corresponde à un extremum (maximum) d'éloignement des points de E. C'est le centre de la sphère vide qui ne peut plus être dilatée, même si on lui permet de se mouvoir à volonté. Cette sphère est l'une des 3 sphères correspondant à l'un des tétraèdres L_{I}, L_{II}, L_{III}. Mais dans des systèmes E particuliers, qui peuvent ne pas être spéciaux, deux ou même toutes les trois de ces sphères peuvent avoir cette propriété.

La Fig. 21 donne, pour finir, la partition uniforme de l'espace par les domaines D.

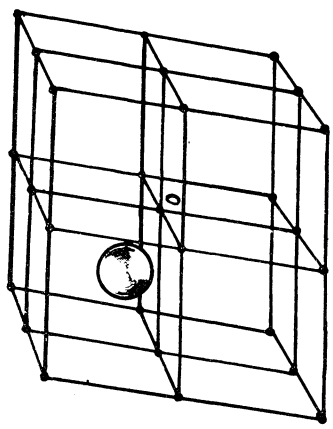


Fig. 1

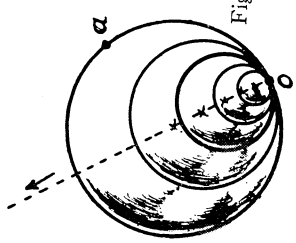


Fig. 2

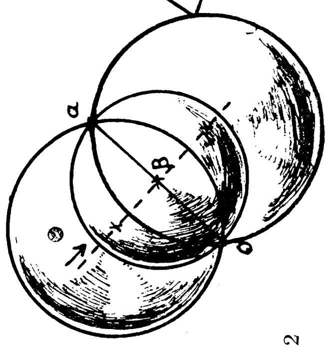


Fig. 3

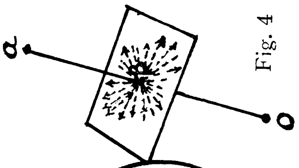


Fig. 4

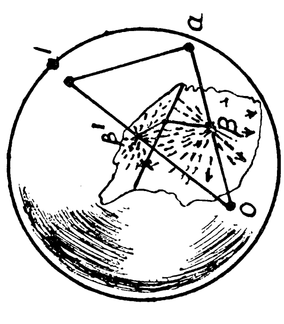


Fig. 5

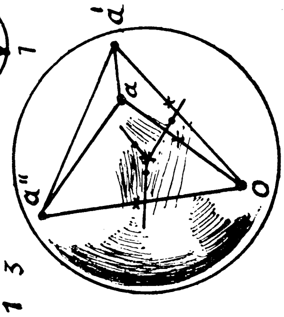


Fig. 6

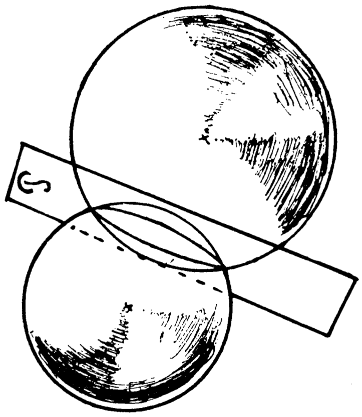


Fig. 7

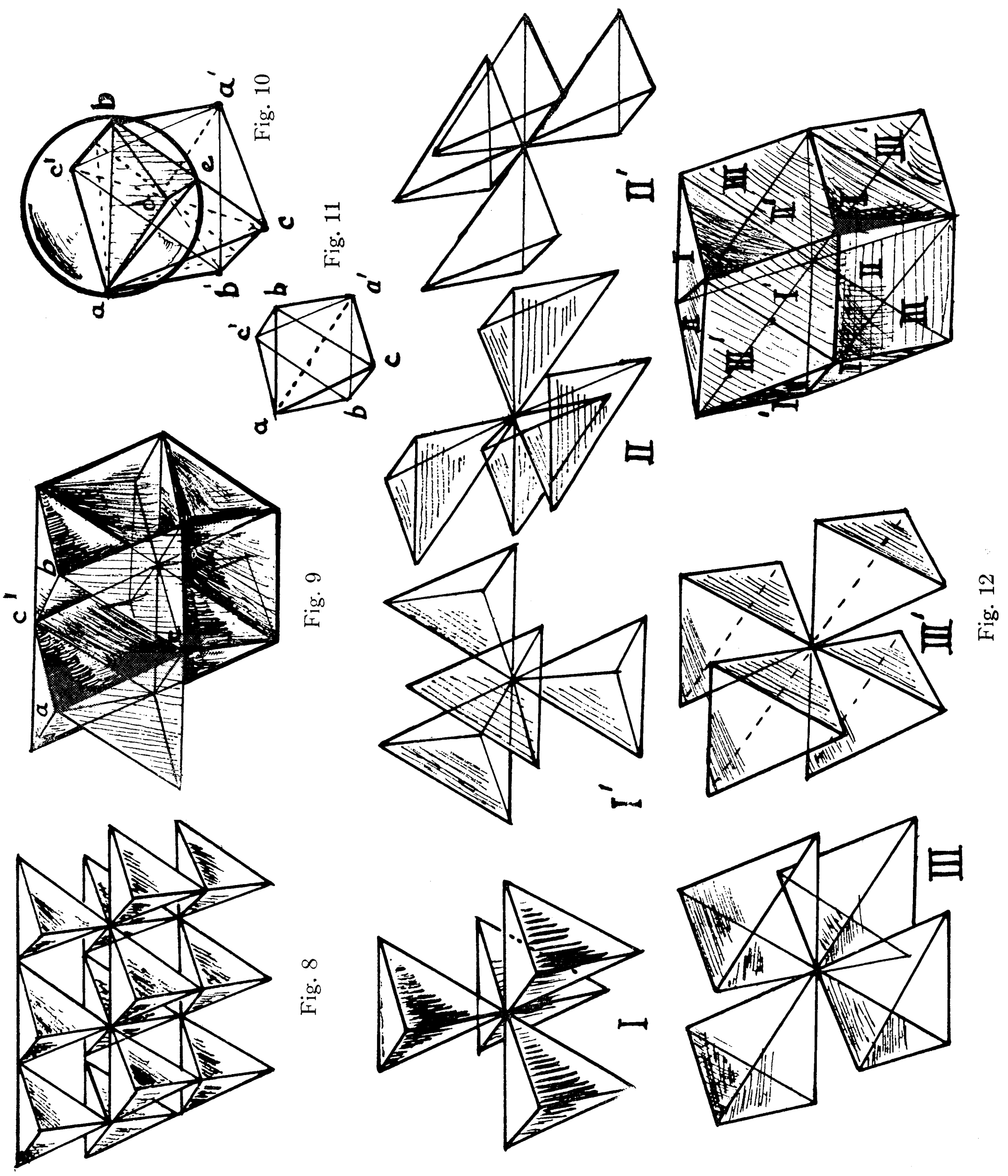


Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

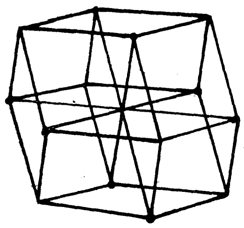


Fig. 13

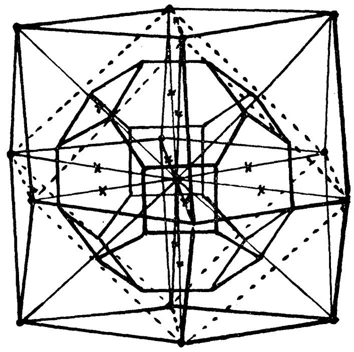


Fig. 14

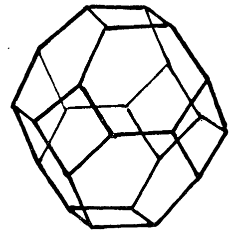


Fig. 15

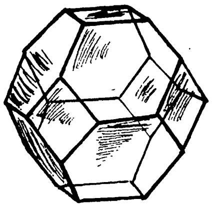


Fig. 16

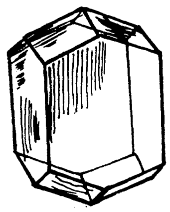


Fig. 17

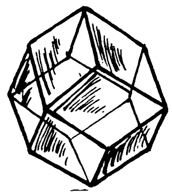


Fig. 18

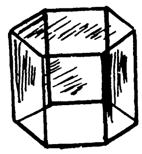


Fig. 19

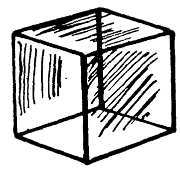
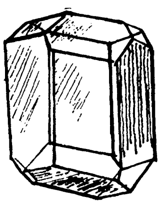
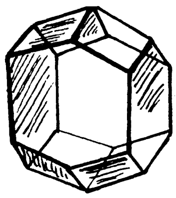
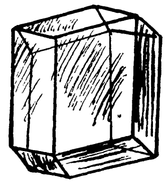
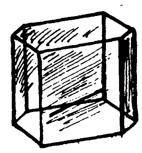


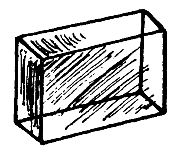
Fig. 20











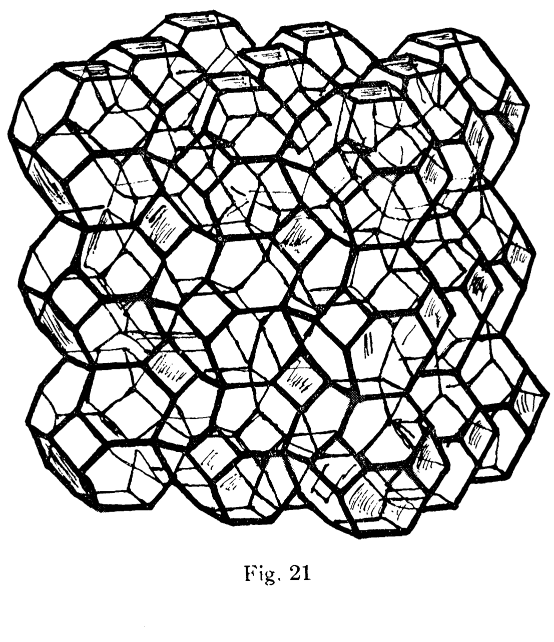


Fig. 21

CERTAIN FORMS OF THE ICOSAHEDRON AND A METHOD FOR DERIVING AND DESIGNATING HIGHER POLYHEDRA

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North High School, Worcester, Massachusetts, U.S.A.

The Five Regular Solids have afforded material for profound mathematical investigation from the days of Pythagoras to modern times. They were treated in Book XIII of Euclid's Elements and it is questionable to whom to ascribe the first knowledge of their existence.

The tetrahedron, cube and octahedron were probably known earlier than the two more complicated bodies. It is probable that Pythagoras was acquainted with the five bodies. Certainly the Pythagoreans were, since Philolaus speaks of the five bodies in the sphere. These are often referred to as the Pythagorean solids and also as the Platonic bodies, since Plato concerned himself with them.

It is not known in what connection Archimedes undertook his investigations of the half-regular solids. He knew thirteen half-regular polyhedra but did not refer to the "Archimedean Prism", so-called, nor to the antiprism.

The only point I wish to make in this connection is that Archimedes derived new solids from the five regular bodies by a systematic removal of various portions.

The existence of the inscribed and the circumscribed spheres of the five regular solids and of the thirteen Archimedean solids was discovered and many of the geometrical relations of the bodies among themselves as to inscription and circumscription were developed, but centuries passed until Kepler, by applying the method of extending the faces of a solid, obtained two new star dodecahedra. He also inscribed two tetrahedra in a cube and formed the stella octangula or group of two orthogonally intersecting concentric tetrahedra.

Whether or not Wenzel Jamitzer of Nüremberg, 1508-1586, anticipated the discovery by Kepler, 1571-1630, of regular star polyhedra does not concern us at present, for little attention was given to these new forms until Poinsot in 1809 disclosed his discovery of the four regular star polyhedra derived from the regular convex polyhedra by extending their faces until they intersect.

Mathematicians including Cauchy 1811, Bertrand 1858, Cayley 1859, Wiener 1867, Hess 1872 and others occupied themselves with these new higher polyhedra and established many theorems relating to them.

The usual "Netze" or "Blanks" for forming the five regular solids from suitable material by folding together configurations of their faces were drawn by Dürer 1525.

I find an advantage in departing from these common forms and adopting those shown in the accompanying Fig. 1.

These "Blanks" are constructed by selecting a face of a given regular solid as a first face and then joining to it a succession of adjacent faces, each connected to the next following by a common edge, and these faces are chosen to follow preferably a positive direction of rotation about the line joining the centre of the first selected face with the centre of the diametrically opposite face of the solid as an axis. They are roulettes of the regular solids.

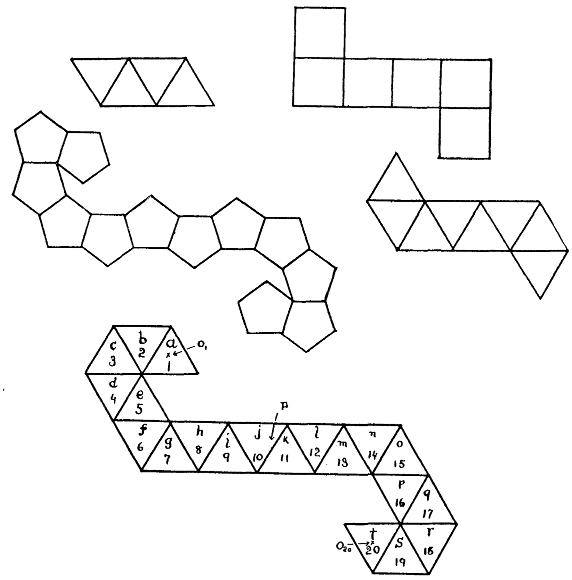

Fig. 1

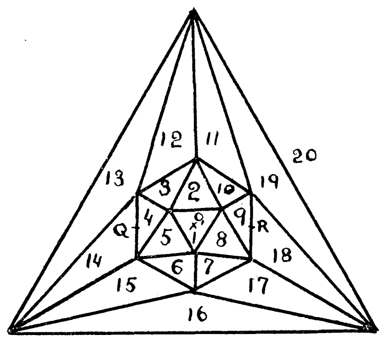

Fig. 2

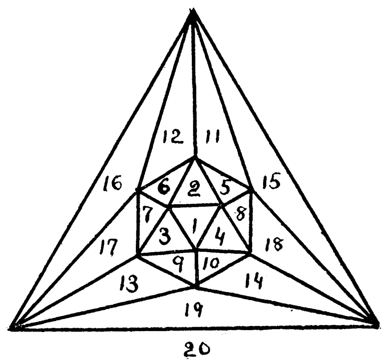

Fig. 3

These faces are designated by assigning symbols to them, such as the letters of the alphabet a, b, c, ... or by the numerals as 1, 2, 3, ...

In the present discussion of a Method for Deriving and Designating Polyhedra, I shall limit myself to a consideration of the regular convex icosahedron and some of the forms derived from it, and shall apply certain elementary principles of group theory to the derivation and designation of a few forms and also to their actual construction from certain "elements".

The numbering of the faces of an icosahedron which I shall adopt is shown in Figs. 1 and 2. The common numbering as shown by Brückner and others is shown in Fig. 3.

We will designate the faces of the regular icosahedron taken in the order shown in Fig. 1, by two sets of marks, $a, b, c, \ldots, r, s, t$, and $1, 2, 3, \ldots, 18, 19, 20$. Furthermore, a and 1 designate the first face, b and 2 the second face, c and 3 the third face, and so on in the order chosen, as shown in Fig. 1. It will be observed by referring to Fig. 1, that this blank or group of faces, $1, 2, 3, \ldots, 18, 19, 20$, consists of two sub-groups, $1, 2, 3, \ldots, 10$, and $11, 12, 13, \ldots, 20$ which possess symmetry with reference to the mid-point of the edges separating

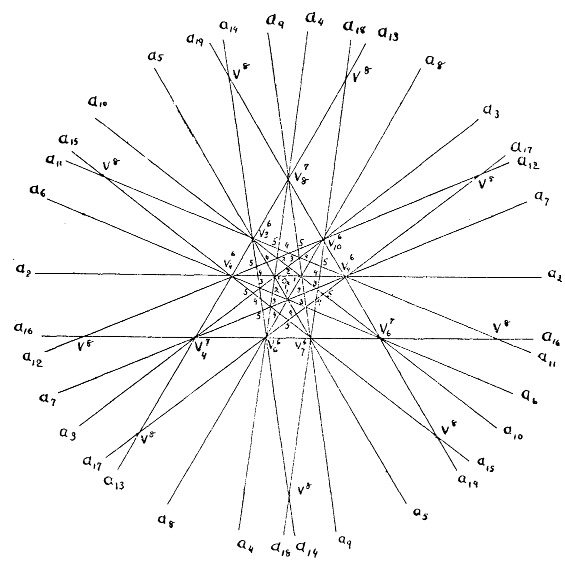

Fig. 4

the faces 10 and 11, and either group may be transformed into the other by a rotation through an angle π about this point. The first of these sub-groups $1, 2, 3, \ldots, 10$, consists of the "visible" or "upper half" of the convex icosahedron when viewed from points on the axis $O_1\, O_{20}$ perpendicular to the faces 1 and 20, Fig. 2, and the second sub-group $11, 12, 13, \ldots, 20$, constitutes the "lower half". Each of these sub-groups in position on the icosahedron can be transformed into the other by rotating the solid through an angle π about an axis passing through the points designated as Q and R, Fig. 2, which are the mid-points of the edges formed by the intersections of the faces 4 and 14, and by 9 and 18 respectively.

It should be observed that an icosahedron coincides with itself when rotated about an axis passing through the geometric centres of two diametrically opposite faces through angles of $\frac{\pi}{3}$, $\frac{2\pi}{3}$ and 2π. By referring to Fig. 2, it may be seen that the sub-group 1, 2, 3, ..., 10, can therefore be separated into the cyclic sub-groups, 1; and 2, 5, 8; 3, 6, 9; and 4, 7, 10. It also appears that the sub-group 11, 12, 13, ..., 20, can be separated into the cyclic sub-groups 11, 14, 17; 12, 15, 18; 13, 16, 19, and 20, respectively.

Omitting 1 and 20, it may be seen that the original group 1, 2, 3, ..., 18, 19, 20, can be separated into the following cyclic sub-groups:

2, 5, 8,	11, 14, 17,
3, 6, 9,	12, 15, 18,
4, 7, 10,	13, 16, 19.

Now, returning to the icosahedron, let some face as a, be selected as a plane of reference, and denote the intersections or traces of the planes of the remaining faces 2, 3, 4, ..., with a, by the notation a_2, a_3, a_4, ..., a_{18}, a_{19} as shown in Fig. 4.

It should be remembered that since the faces 1 and 20 are parallel, there will be no finite trace of 20 on 1, and hence a_{20} will not appear in the figure. Similarly for other faces as, b, we shall have the traces b_1, b_2, b_3, ..., b_{15}, b_{17}, b_{18}, b_{19}, b_{20}, and in this case there will be no trace b_{16} in the group.

Fig. 4 which shows the traces of the faces on the plane of the face a or 1, is readily constructed by observing the grouping of the faces of the icosahedron relatively to face 1 and to each other, and in particular, because of the pentagonal arrangement of groups of isosceles triangular faces about single vertices, it develops that the sides of the triangle $V_4^7\, V_6^7\, V_8^7$, Fig. 4, are divided at the six points V_n^6 in extreme and mean ratio.

It may be seen furthermore, that the 18 lines in Fig. 4 are grouped to form groups of equilateral triangles, as:

I. a_2, a_5, a_8.

II. a_3, a_6, a_9; a_4, a_7, a_{10}.

III. a_{12}, a_{15}, a_{18}; a_{11}, a_{14}, a_{17}.

IV. a_2, a_{10}, a_{13}; a_5, a_{13}, a_{16}; a_8, a_{16}, a_{19}.

V. a_{13}, a_{16}, a_{19}.

VI. a_2, a_5, a_{13}; a_5, a_8, a_{16}; a_8, a_2, a_{19}.

The triangles of these groups may be made to coincide either with themselves or with each other by rotating the entire configuration about the centre O_1 through angles of $\frac{\pi}{3}$, $\frac{2\pi}{3}$, and 2π. It follows that any configuration of lines selected from among the totality of lines a_2, a_3, a_4, ..., a_{19}, may be transformed into a congruent configuration by rotations through angles of $\frac{\pi}{3}$, $\frac{2\pi}{3}$ and 2π.

We shall now define $Fa_xa_ya_z\ldots$ to denote the plane figure formed by the lines $a_xa_ya_z\ldots$. In particular, we shall fix our attention on any portion of the plane, of which the lines $a_xa_ya_z\ldots$ taken successively in the cyclic order given, form a closed boundary. Thus, referring to Fig. 5, the figure $F\,a_2\,a_5\,a_8$ defines the triangle whose vertices are $V_2 V_3 V_1$. Also $Fa_3a_9a_4a_{10}$ defines the "kite" with one vertex at V_8^7.

We shall call any enclosed space of the plane thus defined, an element, and shall denote it by $Ea_xa_ya_z\ldots$. Thus, in Fig. 5 we have the elements $Ea_2a_5a_8$ and $Ea_3a_9a_4a_{10}$.

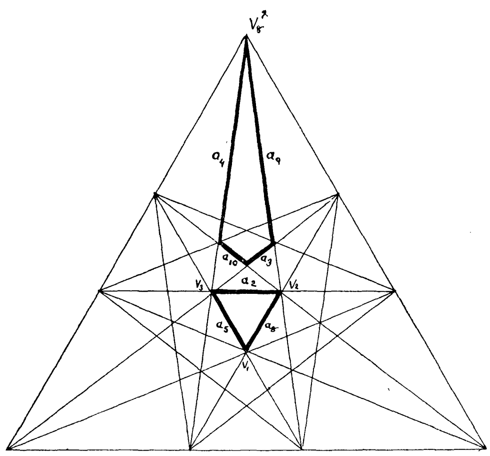

Fig. 5

The symbols in the Table are elements of faces of icosahedra, and the entire "visible" parts of these faces may be obtained by applying the substitutions

$$1,\ S,\ S^2,$$

where

$$S \equiv 2,\ 5,\ 8\cdot 3,\ 6,\ 9\cdot 4,\ 7,\ 10\cdot 11,\ 14,\ 17\cdot 12,\ 15,\ 18\cdot 13,\ 16,\ 19.$$

It should be understood that the classification of elements in the following table is not exhaustive, but it shows certain varieties and points out relations with reference to the complete figure from which they are obtained by selection.

On transforming any symbol as $Ea_pa_qa_r\ldots a_w$ by a substitution, S, we shall obtain a second element, and obviously on applying a cyclic substitution repeatedly we shall eventually reproduce the original element. The totality of such "elements" thus derived and lying in the same plane form a group and constitute the visible parts of one face of a polyhedron. Hence we may regard the Table as designating certain Icosahedra by means of Elements from which their faces may be constructed.

As Archimedes, Kepler and Poinsot derived new forms by removing portions of others or by adding to them by extending faces and edges, so we may proceed to derive additional forms by selecting groups of elements from among the lines of some complete figure, as Fig. 4.

TABLE OF ICOSAHEDRA

No.	Vertices	Elements	Varieties
1	V^1	$E_1 = Ea_2a_5a_8$,	Regular Convex Triangular.
2	V^2	$E_1 = Ea_2a_{10}a_3$,	Hexagonal.
3	V^3	$E_1 = Ea_9a_3a_4a_{10}$,	Five Intersecting Octahedra.
4	V^4	$E_1 = Ea_8a_{11}a_9$, $E_2 = Ea_3a_{11}a_{12}a_{10}$, $E_3 = Ea_4a_{12}a_5$,	Nonagonal.
5	V^5	$E_1 = Ea_2a_{18}a_{11}a_8$, $E_2 = Ea_3a_9a_{12}a_{11}a_4a_{10}$,	Archimedean Variety (A.V.) of No. 13.
6	V^6	$E_1 = Ea_{11}a_8a_2a_{18}a_{17}$,	Five Intersecting Tetrahedra (Right).
7	V^6	$E_1 = Ea_{11}a_8a_2a_{18}a_{12}$,	Five Intersecting Tetrahedra (Left).
8	V^6	$E_1 = Ea_4a_7a_{14}$, $E_2 = Ea_{17}a_{15}a_5a_8$, $E_3 = Ea_{18}a_6a_9$,	Ten Intersecting Tetrahedra,. No. 6+No. 7.
9	V^6	$E_1 = Ea_{16}a_5a_8$,	Möbius (Concave).
10	V^6	$E_1 = Ea_8a_5a_9a_6a_7a_4$,	Twenty Pointed, Six-Edged.
11	V^7	$E_1 = Ea_{16}a_{14}a_7$, $E_2 = Ea_{16}a_{15}a_{17}$, $E_3 = Ea_{16}a_6a_{18}$,	Regular Star. Poinsot.
12	V^8	$E_1 = Ea_{17}a_{16}a_{13}$, $E_2 = Ea_{18}a_{16}a_{14}$, $E_3 = Ea_{19}a_{16}a_{15}$,	Complete.
13	V^7	$E_1 = Ea_9a_4a_{10}a_3$,	Kite, Archimedean Variety of No. 11.
14	V^7	$E_1 = Ea_{16}a_4a_7$, $E_2 = Ea_{16}a_6a_9$,	Archimedean Variety of No. 11.
15	V^6, V^7	$E_1 = Ea_9a_4a_{11}a_{18}a_{12}$,	Double. No. 13+No. 7.
16	V^6, V^7	$E_1 = Ea_9a_4a_{11}a_{14}a_{12}$,	Double. No. 13+No. 6.

TABLE OF ICOSAHEDRA (*continued*)

17	V^6, V^7	$E_1 = Ea_{12}a_9a_4a_{11}$, $E_2 = Ea_{16}a_5a_{17}a_{15}a_8$,	Double. No. 13+No. 9.
18	V^6, V^7	$E_1 = Ea_{12}a_{10}a_5$, $E_2 = Ea_{12}a_{11}a_9a_4$, $E_3 = Ea_{11}a_8a_3$,	Double. No. 13+No. 10.
19	V^6, V^7	$E_1 = Ea_{18}a_8a_{11}$, $E_2 = Ea_3a_{12}a_9$, $E_3 = Ea_9a_4a_{11}a_{12}$, $E_4 = Ea_4a_{11}a_{10}$, $E_5 = Ea_{12}a_5a_{14}$,	Triple. No. 13+No. 6+No. 7.
20	V^6	$E_1 = {}^+Ea_{16}a_5a_{17}a_{15}a_8$, $E_2 = {}^-Ea_6a_{18}a_{17}a_{10}a_7a_9$,	Hollow, Labyrinth.
21	V^6	$E_1 = {}^-Ea_4a_7a_{14}$, $E_2 = {}^+Ea_{16}a_{15}a_{17}$, $E_3 = {}^-Ea_{18}a_6a_9$,	Discrete, Skeleton.
22	V^7, V^5	$E_1 = {}^+Ea_9a_4a_{11}a_{12}$, $E_2 = {}^-Ea_2a_5a_{12}a_{14}$,	Discrete. Twelve Pointed, Crown Rimmed Group.

NOTE—The notation $+E$, $-E$ (see Nos. 20, 21, 22) denotes that the elements are on "opposite sides" of a given plane.

We will apply the method of derivation and designation under consideration to the triple icosahedron composed of the group of five intersecting regular tetrahedra capable of inscription in a regular convex pentagonal dodecahedron, its enantomorph of five other intersecting regular tetrahedra, and the twelve pointed star "kite" icosahedron, the elements of a face of which are listed as No. 19 in the Table. By letting S represent the substitution above given and applying the substitutions 1, S, S^2 to the elements E_1, E_2, E_3, E_4 and E_5 we may obtain the group G_a of elements in the plane a formed by their configuration and constituting the visible portions of the face a of the polygon considered.

Hence we have

$$G_a = \begin{cases} E_1, & E_2, & E_3, & E_4, & E_5, \\ SE_1, & SE_2, & SE_3, & SE_4 & SE_5, \\ S^2E_1, & S^2E_2, & S^2E_3, & S^2E_4, & S^2E_5. \end{cases}$$

With reference to the face, a, of the polyhedron, the remaining faces are given by groups of simply isomorphic substitutions.

It will be convenient to follow the method of Klein and others and at the same time to designate or name each domain or element of a face of a polyhedron after the operation through which it is derived by a suitable substitution from one amongst a given group of elements. Thus, the elements of the group G_a of the triple icosahedron are represented in Fig. 6.

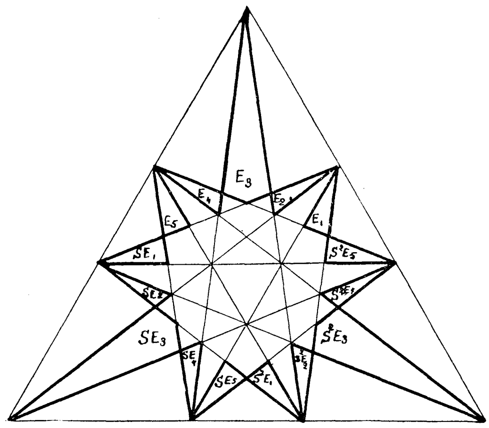

Fig. 6

After having determined the elements, G_a, of a particular polyhedron it is a simple matter to select from among them those elements which when properly grouped together about a point form the face angles of some polyhedral angle whose vertex is a point of the complete figure. By making a sufficient number of duplicates of the group of elements thus selected and arranged out of suitable material and properly assembling these groups together, models of polyhedra may be readily constructed.

THREE THEOREMS OF ANALYSIS DERIVED BY THE VECTOR METHOD AS COROLLARIES FROM A SINGLE PROPOSITION

By Dr. Almar Naess,
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Suppose we are given three functions P, Q, R of three independent variables x, y, z; if we integrate the two differential equations:

(1) $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

we can, as is well known, express the solution in the following form:

(2) $$\begin{aligned}\psi\ (x,\ y,\ z)&=c_1\\ \phi\ (x,\ y,\ z)&=c_2\end{aligned}$$

c_1 and c_2 being two arbitrary constants.

The main properties of the functions ψ and ϕ may then be stated in the following equations:

(3) $$\frac{P}{\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial z}-\frac{\partial\psi}{\partial z}\frac{\partial\phi}{\partial y}}=\frac{Q}{\frac{\partial\psi}{\partial z}\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial z}}=\frac{R}{\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}}=\Gamma$$

Γ being, of course, in general, a function of x, y, z.

We are now going to prove the following theorem:

(A) *The necessary and sufficient condition that* Γ *be expressible as a function of* ψ *and* ϕ *only is that* P, Q, R *satisfy the following equation:*

(4) $$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0.$$

Before proceeding to the proof we will state a few well-known theorems which follow immediately from (A) and which may therefore be considered simply as corollaries of the same theorem.

Corollary Ia (Jacobi): If the three functions P, Q, R *of* x, y, z *satisfy the condition*

(4′) $$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0,$$

then it is always possible to find two other functions π *and* θ *of* x, y, z *such that:*

$$(5) \qquad P=\frac{\partial\pi}{\partial y}\frac{\partial\theta}{\partial z}-\frac{\partial\pi}{\partial z}\frac{\partial\theta}{\partial y},\quad Q=\frac{\partial\pi}{\partial z}\frac{\partial\theta}{\partial x}-\frac{\partial\pi}{\partial x}\frac{\partial\theta}{\partial z},\quad R=\frac{\partial\pi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{\partial\pi}{\partial y}\frac{\partial\theta}{\partial x}.$$

This can also be expressed in the following way:

Corollary Ib: Being given any three functions F, G, H of x, y, z it is always possible to find two others, π and θ, such that:

$$(6) \qquad \begin{gathered}\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}=\frac{\partial\pi}{\partial y}\frac{\partial\theta}{\partial z}-\frac{\partial\pi}{\partial z}\frac{\partial\theta}{\partial y},\quad \frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}=\frac{\partial\pi}{\partial z}\frac{\partial\theta}{\partial x}-\frac{\partial\pi}{\partial x}\frac{\partial\theta}{\partial z},\\ \frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}=\frac{\partial\pi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{\partial\pi}{\partial y}\frac{\partial\theta}{\partial x}.\end{gathered}$$

Corollary II (The transformation of Clebsch): Being given any three functions P, Q, R, of x, y, z it is always possible to find three others, π, ϕ and ω, such that:

$$(7) \qquad P=\pi\frac{\partial\phi}{\partial x}+\frac{\partial\omega}{\partial x},\quad Q=\pi\frac{\partial\phi}{\partial y}+\frac{\partial\omega}{\partial y},\quad R=\pi\frac{\partial\phi}{\partial z}+\frac{\partial\omega}{\partial z}.$$

Corollary III: If three given functions P, Q, R of x, y, z satisfy the condition

$$(8) \qquad P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=0,$$

then the expression

$$(9) \qquad Pdx+Qdy+Rdz$$

always has an integrating factor (i.e., it is expressible as $Vd\theta$, V and θ being functions of x, y, z)

The proofs may very easily be effected by means of vector analysis conceptions and notations.

Let x, y, z be interpreted as coordinates in an ordinary cartesian 3-dimensional space, and let the functions be considered as scalar functions in this space. If then **i, j, k** designate the unit vectors and **V** the vector fie'd (or vector function of position in space)

$$(10) \qquad \mathbf{V}=\mathbf{i}P+\mathbf{j}Q+\mathbf{k}R,$$

the differential equations (1) give the *vector lines* of **V**. By (2) these are represented as intersecting curves of two families of surfaces, viz., $\psi=$const. and $\phi=$const. But obviously these curves also are the vector lines of the vector product $\nabla\psi\times\nabla\phi$, $\nabla\psi$ and $\nabla\phi$ being the potential vectors derived from the scalar functions ψ and ϕ respectively. For $\nabla\psi\times\nabla\phi$, being perpendicular to both $\nabla\psi$ and $\nabla\phi$, must be tangent both to the surface $\psi=$const. and to the surface $\phi=$const (through the origin of the vector product).

Then, the two fields **V** and $\nabla\psi\times\nabla\phi$ being collinear (from any point in space), we can put

$$(11) \qquad \mathbf{V}=\Gamma\nabla\psi\times\nabla\phi,$$

Γ being a scalar function of position in space. Formula (11) is only (3) in an abbreviated form.

Now (A) in the language of vector analysis can be expressed in the following way:

(A_1) *The necessary and sufficient condition that* Γ *be constant along the vector lines, is that* $\nabla\cdot\mathbf{V}=0$.

$\nabla\cdot\mathbf{V}$ here as usual designates the divergence of $\mathbf{V}$.

It is easily shown that the condition is sufficient. For

$$\nabla\cdot\mathbf{V}=\nabla\Gamma\cdot(\nabla\psi\times\nabla\phi)+\Gamma\nabla\cdot(\nabla\psi\times\nabla\phi)=\frac{\partial(\Gamma,\psi,\phi)}{\partial(x,y,z)}, \tag{12}$$

the divergence of $\nabla\psi\times\nabla\phi$ being zero. The last expression stands for the Jacobian, of Γ, ψ, ϕ. Hence, if $\nabla\cdot\mathbf{V}=0$, Γ is expressible as a function of ψ and ϕ and accordingly, is constant along any curve along which both ψ and ϕ are constant.

In order to show that the condition is necessary, we can proceed as follows:

We shall consider the vector line through any point in space, say O. Then at O we have:

$$\mathbf{V}=\Gamma\nabla\psi\times\nabla\phi.$$

Let the values of the same quantities at some other point, O' say, on the *same* vector line be primed, so that

$$\mathbf{V}'=\Gamma'(\nabla\psi\times\nabla\phi)'.$$

Next let us consider an infinitely small vector tube around the line OO', the cross sections of which at O and O' are $d\sigma$ and $d\sigma'$ respectively. Then $d\sigma$ and $d\sigma'$ cut out of the tube a volume τ.

Now by using the theorem of Ostrogradsky (or Gauss) and noticing that the surface integral of $\mathbf{V}$ as well as that of $\nabla\psi\times\nabla\phi$ taken over any part of the wall of the tube is zero, we get:

$$\mathbf{V}'d\sigma'-\mathbf{V}d\sigma=\iiint_\tau \nabla\cdot\mathbf{V}d\tau, \tag{13}$$

$$(\nabla\psi\times\nabla\phi)'d\sigma'-\nabla\psi\times\nabla\phi d\sigma=0. \tag{14}$$

From this we easily derive

$$(\Gamma'-\Gamma)d\sigma(\nabla\psi\times\nabla\phi)=\iiint_\tau \nabla\cdot\mathbf{V}d\tau, \tag{15}$$

As O' may be any point on the vector line through O, and O any point in space, (A_1) and consequently (A) is proved.

Corollaries *Ia* and *Ib* may be stated thus: *The curl of any vector can always be expressed as the vector product of two potential vectors.*

For: Let the vector lines of the curl of $\mathbf{V}$ (or the curl lines, "Rotorlinien", of $\mathbf{V}$) be determined by two equations of the form:

$$\psi(x,y,z)=c_1,$$

$$\phi(x,y,z)=c_2,$$

c_1 and c_2 being arbitrary constants. Then we have:

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \Gamma \nabla \psi \times \nabla \phi \tag{16}$$

where Γ, by (A_1), is expressible in terms of ψ and ϕ only, since $\nabla \cdot (\nabla \times \mathbf{V}) = 0$. Now let π and θ be two other functions of ψ and ϕ. Then it has been shown by the writer in a previous paper* that:

$$\nabla \pi \times \nabla \theta = \frac{\partial(\pi, \theta)}{\partial(\psi, \phi)} \nabla \psi \times \nabla \phi. \tag{17}$$

Here it suffices to choose π and θ such that the Jacobian $\dfrac{\partial(\pi, \theta)}{\partial(\psi, \phi)}$ is equal to the function Γ (which is given by the vector $\mathbf{V}$). This, of course, can always be done in an infinite number of ways. For example, putting $\theta = \phi$ we must then choose π so that $\dfrac{\partial \pi}{\partial \psi} = \Gamma$. Thus we get:

$$\nabla \times \mathbf{V} = \nabla \pi \times \nabla \phi, \text{ q.e.d.} \tag{18}$$

From this follows immediately Corollary II, which, in the language of vector analysis, may be expressed thus:

Any vector function $\mathbf{V}$ *can always be resolved into a sum of a potential vector times a scalar plus another potential vector.*

For, letting $\mathbf{V}$, π and ϕ be determined as above, and noting $\nabla \pi \times \nabla \phi = \nabla \times (\pi \nabla \phi)$, we get

$$\nabla \times \mathbf{V} - \nabla \pi \times \nabla \phi \equiv \nabla \times (\mathbf{V} - \pi \nabla \phi) = 0. \tag{19}$$

The difference $\mathbf{V} - \pi \nabla \phi$, having a curl equal to zero, must be expressible as a potential vector, *i.e.*, ω being a scalar function, we can put

$$\mathbf{V} = \pi \nabla \phi + \nabla \omega, \tag{20}$$

evidently nothing other than the *Clebschian transformation*, which is usually obtained in an entirely different way.

That Theorem (A) also involves what is here given as Corollary III is easily seen by means of the same transformation. For expressing P, Q, R in terms of the functions π, ϕ and ω as in (7) and inserting this in (8) the latter is, after some calculation, reduced to

$$\frac{\partial(\pi, \phi, \omega)}{\partial(x, y, z)} = 0. \tag{21}$$

That is to say, in this case any one of the three functions π, ϕ, ω is expressible as a function of the other two.
By means of this we get:

$$Pdx + Qdy + Rdz = \left(\pi + \frac{\partial \omega}{\partial \phi}\right) d\phi + \frac{\partial \omega}{\partial \pi} d\pi = \left(\pi + \frac{\partial \omega}{\partial \phi}\right)(d\phi + U d\pi), \tag{22}$$

**Einige Untersuchungen über zweifach skalare Vektoren* (Arch. f. Mathem. og Naturv. B. xxxviii) Kristiania (Oslo) 1921, where besides an analytic proof a geometric proof is also given.

where we have put

$$U = \frac{\dfrac{\partial \omega}{\partial \pi}}{\pi + \dfrac{\partial \omega}{\partial \phi}}. \tag{23}$$

Now it is always possible to determine another function θ of π and ϕ such that

$$\frac{\dfrac{\partial \theta}{\partial \pi}}{\dfrac{\partial \theta}{\partial \phi}} = U,$$

whereby (22) may be reduced to

$$Pdx + Qdy + Rdz = \frac{\pi + \dfrac{\partial \omega}{\partial \phi}}{\dfrac{\partial \theta}{\partial \phi}} d\theta,$$

which is the desired result.

Remark.—A vector of the form $\psi \nabla \phi$ is called *a double scalar vector*, because it is determined by two scalar functions only. This type of vector often proves to be useful, when introduced into vector analysis, because of its simple geometric properties. The theorems laid down above may also be stated as properties of double scalar vectors only, and treated merely from this point of view. This is done in a paper by the writer: *Zur Theorie der zweifach skalaren Vektoren*, which will be published in the Jahresbericht der deutschen Mathematiker-Vereinigung, 1924. More detailed proofs and references are given there.

CURVES AUTOPOLAR WITH RESPECT TO A FINITE NUMBER OF CONICS

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In the Nouvelles Annales, 3^{e} Série, t. 13 (1894), p. 206, Appell developed a method for deriving all curves autopolar with respect to a given conic as the envelopes of singly infinite systems of conics which are autopolar with respect to the given conic. For example, all conics autopolar with respect to

$$x^2+y^2+z^2=0 \tag{1}$$

are given by the equation

$$(x^2+y^2+z^2)\ (u^2+v^2+w^2)-2(ux+vy+wz)^2=0, \tag{2}$$

and the envelope of any system of these conics defined by the homogeneous equation

$$f(u,\ v,\ w)=0 \tag{3}$$

is autopolar with respect to (1).

In the Bulletin de la Société Philomathique for 1877, Fouret had shown that the W-curves of Klein and Lie are autopolar with respect to an infinity of conics and that they are the only curves possessing this property.

In the following paper I propose to show how Appell's method can be used to derive curves which are autopolar with respect to a *finite* number of conics.

If two conics are mutually autopolar, the product of the two corresponding correlations is a perspective collineation under which they are both invariant. For instance, the conics

$$x^2+y^2+z^2=0,$$

$$x^2+y^2-z^2=0,$$

give rise to the collineation

$$x':\ y':\ z'=x:\ y:\ -z.$$

If now $f(u,\ v,\ w)$ is invariant under this collineation, the system of conics (2) will be interchanged in pairs by the polarity with respect to

$$x^2+y^2-z^2=0,$$

and the envelope is autopolar with respect to this conic as well as with respect to (1). It follows immediately that if $f(u, v, w)=0$ contain only even powers of u, v, w the envelope will be autopolar with respect also to the conics

$$x^2-y^2+z^2=0,$$
$$-x^2+y^2+z^2=0.$$

The product of the polar transformations with respect to two conics not mutually autopolar will be a collineation which is not of period two. We are concerned only with collineations of a finite period n. In this case we consider the conics

(4) $$x^2+\eta y^2+\eta^{-1}z^2=0,$$

where $\eta^n=1$.

The products of the polarities give rise to a cyclic group of collineations

$$x':y':z'=x:\eta^k y:\eta^{-k}z,$$

The conics themselves are not mutually autopolar, but interchanged in pairs. However, they are all mutually autopolar with respect to the two conics

$$x^2+2yz=0,$$
$$x^2-2yz=0,$$

which are again mutually autopolar. Hence if from the conics autopolar to

$$x^2+2yz=0,$$

given by the equation

$$(x^2+2yz)\ (u^2+2vw)-2(ux+vy+wz)^2=0$$

we select a singly infinite system defined by $f(u, v, w)=0$, where this relation is invariant under the various collineations

$$u':v':w'=u:\eta^k w:\eta^{-k}v,$$

the envelope is autopolar with respect to all the conics (4) as well as with respect to $x^2+2yz=0$. If, furthermore, $f(u, v, w)=0$ is invariant under the collineation

$$u':v':w'=-u:v:w$$

the envelope will also be autopolar with respect to $x^2-2yz=0$.

It is true that in some cases the envelope thus obtained will reduce to factors which are polar reciprocals of each other, but that this will not always happen may be shown by two illustrations. The quartic curve with two cusps and one node is autopolar with respect to two conics, unless the two inflexions happen to coincide. The quintic curve with five cusps, investigated by Del Pezzo, is always autopolar with respect to one conic; but if the curve is (projectively) symmetric, it is autopolar with respect to a second conic; while if it is symmetric in more than one way, it is autopolar with respect to five conics besides the first.

The question of autopolarity with respect merely to a cyclic set of conics

(4) is more difficult, and I will offer only a suggestion at this time. The three conics

$$x^2+2yz=0,$$
$$y^2+2zx=0,$$
$$z^2+2xy=0$$

form a cyclic set. With respect to any one, the others are polar reciprocal. The system

$$\alpha(x^2+2yz)+\beta(y^2+2zx)+\gamma(z^2+2xy)=0$$

is converted by any one of them into a similar system. If a singly infinite set be determined by a relation

$$f(\alpha,\beta,\gamma)=0,$$

which is invariant under the three quadratic transformations

$$\alpha':\beta':\gamma'=\beta\gamma-\alpha^2:\alpha\beta-\gamma^2:\gamma\alpha-\beta^2,$$
$$\alpha':\beta':\gamma'=\alpha\beta-\gamma^2:\gamma\alpha-\beta^2:\beta\gamma-\alpha^2,$$
$$\alpha':\beta':\gamma'=\gamma\alpha-\beta^2:\beta\gamma-\alpha^2:\alpha\beta-\gamma^2$$

the set will be interchanged in pairs by each one of the polarities in question, and hence the envelope will be autopolar with respect to all three of the given conics.

SUR LA GÉOMÉTRIE DU TÉTRAÈDRE

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1. Un pentaèdre orthocentrique $\alpha\beta\gamma\delta\sigma$ est osculateur à une parabole gauche orthogonale, et réciproquement; la directrice de cette courbe passe par l'orthocentre O*. Par conséquent:

Les faces $\alpha\beta\gamma\delta$ d'un tétraèdre $ABCD$ osculateur à une parabole gauche orthogonale sont coupées par un plan σ, osculateur à la courbe, suivant les droites a, b, c, d. Si A', B', C', D', sont les projections orthogonales des sommets A, B, C, D, respectivement sur les droites a, b, c, d, les sphères décrites sur les segments AA', BB', CC', DD', comme diamètres ont pour centre radical un point O de la directrice de la parabole.

Il existe trois plans σ osculateurs à la parabole tels que chacun d'eux passe par le centre radical des sphères correspondantes (AA'), (BB'), (CC'), (DD').

En effet, la perpendiculaire abaissée du point O sur le plan σ est une directrice du système réglé des hauteurs (h_a, h_b, h_c, h_d), du tétraèdre $ABCD$; elle est l'axe de révolution d'une quadrique Σ conjuguée à ce tétraèdre; le pôle S du plan σ et le point O sont conjugués harmoniques relativement aux foyers singuliers de Σ†. Le plan σ passe donc par le point O si ce point est le centre de la quadrique. La directrice de la parabole rayon du système réglé $(h_a h_b h_c h_d)$‡ contient les centres de trois quadriques de révolution conjuguées au tétraèdre $ABCD$§; le théorème est donc démontré.

De la propriété de la directrice de la parabole||, on déduit:

Les systèmes réglés des hauteurs des cinq tétraèdres formés par les faces d'un pentaèdre orthocentrique ont un rayon commun passant par l'orthocentre.

2. L'orthocentre O du pentaèdre $\alpha\beta\gamma\delta\sigma$ est le centre de la sphère (O) conjuguée au pentaèdre. Le pôle du plan σ relativement à cette sphère est le pôle S de ce plan par rapport à la quadrique Σ¶. Le pôle X du plan ABS relativement à Σ est sur l'arête CD; la droite OX est normale au plan ABS et les deux plans ABS, CDO sont perpendiculaires. Le plan AOS contient la hauteur h_a

* Bull. de l'Académie Royale de Belgique, année 1922, p. 105.
† ibid,, 1921, p. 641.
‡ ibid., 1922, p. 55.
§ ibid., 1921, p. 166.
|| ibid., 1921, p. 55.
¶ ibid., 1921, p. 641.

(1) et est normal au plan BCD; le point S est donc l'orthocentre du pentagone $ABCDO$. Ainsi:

Quatre sommets A, B, C, D d'un pentaèdre orthocentrique $\alpha\beta\gamma\delta\sigma$ extérieurs à la face σ et l'orthocentre O sont les sommets d'un pentagone orthocentrique. L'orthocentre S de ce pentagone est le pôle du plan σ relativement à la sphère (O) conjuguée au pentaèdre $\alpha\beta\gamma\delta\sigma$.

Ce point S et ses quatre analogues sont situés sur une hyperbole gauche équilatère.

On considère les tétraèdres $\alpha\beta\gamma\delta \equiv ABCD$, $\sigma\beta\gamma\delta \equiv AB_1C_1D_1$; les plans hauteurs du trièdre $\beta\gamma\delta$ se coupent suivant l'axe orthique de ce trièdre. Cette droite s est une directrice des systèmes réglés des hauteurs $(h_a h_b h_c h_d)$, $(h_{a_1} h_{b_1} h_{c_1} h_{d_1})$ des tétraèdres $ABCD$, $AB_1C_1D_1$. La directrice d de la parabole gauche orthogonale osculatrice au pentaèdre $\alpha\beta\gamma\delta\sigma$ est un rayon de ces systèmes réglés. Le point d'intersection des droites s, d est le centre d'homothétie des hyperboloïdes équilatères $(h_a h_b h_c h_d)$, $(h_{a_1} h_{b_1} h_{c_1} h_{d_1})$. Ainsi:

Les arêtes d'un trièdre (A) osculateur à une parabole gauche orthogonale sont coupées par deux plans osculateurs aux points (B, C, D), (B_1, C_1, D_1). Les hauteurs $(h_b h_c h_d)$ $(h_{b_1} h_{c_1} h_{d_1})$ des tétraèdres $ABCD$, $AB_1C_1D_1$ déterminent sur l'axe orthique du trièdre (A) deux ponctuelles semblables.

Le centre de similitude de ces ponctuelles est sur la directrice de la parabole.

3. Le pentaèdre $\alpha\beta\gamma\delta\sigma$ étant osculateur à une parabole gauche orthogonale, une normale à la face σ est une direction asymptotique d'une hyperbole gauche équilatère circonscrite au tétraèdre $\alpha\beta\gamma\delta \equiv ABCD$. Cette face σ est donc parallèle aux deux autres directions asymptotiques de cette hyperbole et, par suite, les arêtes AB, AC, AD et les droites projetant orthogonalement les points B, C, D sur σ déterminent dans ce plan deux triangles bilogiques*. Ainsi:

Les arêtes d'un tétraèdre $ABCD$ osculateur à une parabole gauche orthogonale et les perpendiculaires abaissées des sommets sur un plan osculateur à la courbe déterminent dans ce plan les dix points d'une configuration spéciale de Desargues, telle que deux triangles homologiques quelconques de cette configuration sont orthologiques.

4. Si l'orthocentre O du pentaèdre $\alpha\beta\gamma\delta\sigma$ est dans le plan σ le point S (2) est à l'infini, et la droite OS est une direction asymptotique de l'hyperbole gauche équilatère $ABCDOS$ (2); le plan σ est donc parallèle aux deux autres directions asymptotiques de cette hyperbole, et par suite:

Si l'orthocentre O d'un pentaèdre orthocentrique $\alpha\beta\gamma\delta\sigma$ est dans la face σ, les traces sur un plan parallèle à σ, 1° des droites projetant de O les sommets A, B, C, D du tétraèdre $\alpha\beta\gamma\delta$, 2° des arêtes de ce tétraèdre, 3° des perpendiculaires menées des points A, B, C, D, O sur le plan σ, sont les quinze points d'une configuration spéciale de Cayley, telle que les triangles de l'un quelconque des ternes de triangles homologiques sont deux à deux orthologiques.

* ibid., 1923, p. 295.

5. Un plan σ normal à une directrice s du système réglé des hauteurs du tétraèdre $ABCD$ forme avec les faces α, β, γ, δ de ce solide un pentaèdre orthocentrique $\alpha\beta\gamma\delta\sigma$ et réciproquement*; par conséquent:

Etant donnés deux tétraèdres quelconques $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$ les hyperboloïdes des hauteurs de ces tétraèdres ont quatre directions asymptotiques communes. Tout plan σ normal à l'une d'elles est tel que les pentaèdres $\alpha\beta\gamma\delta\sigma$, $\alpha_1\beta_1\gamma_1\delta_1\sigma$ sont orthocentriques. Ces plans jouissent seuls de cette propriété.

Si le plan σ se déplace parallèlement à lui-même, les orthocentres O, O_1, des pentaèdres variables $\alpha\beta\gamma\delta\sigma$, $\alpha_1\beta_1\gamma_1\delta_1\sigma$ sont alignés sur un point fixe F.

En effet, la directrice s du système réglé des hauteurs du tétraèdre $\alpha\beta\gamma\delta$ normale au plan σ passe par l'orthocentre O du pentaèdre $\alpha\beta\gamma\delta\sigma$ et coupe le plan σ au point T; la droite s est l'axe de révolution d'une quadrique Σ conjuguée au tétraèdre $\alpha\beta\gamma\delta$; le pôle S du plan σ relativement à Σ est le conjugué de O relativement aux foyers singuliers de cette surface†. Les ponctuelles (O) et (S), (T) et (S), si le plan σ se déplace parallèlement à lui-même, sont donc involutives et on a la projectivité

$$(O)\,\overline{\wedge}\,(T).$$

Si S est au centre de Σ les points correspondants O, T sont tous deux à l'infini; les ponctuelles (O), (T) sont donc semblables.

Relativement au tétraèdre $\alpha_1\beta_1\gamma_1\delta_1$, on a sur la droite s_1 analogue à s les ponctuelles semblables (O_1)et (T_1); la similitude des ponctuelles (T), (T_1) entraîne celle des ponctuelles (O), (O_1) et la droite OO_1 passe par un point fixe F.

Les tétraèdres $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$ sont conjugués respectivement à deux quadriques de révolution Σ, Σ_1 dont les axes sont les droites s, s_1. Si ces quadriques sont homothétiques, le point fixe F est à l'infini.

En effet, soient E le centre de Σ; $2a$ la longueur de l'axe de révolution, $2c$ la distance des foyers singuliers. On a:

$$ES.ET=a^2, \qquad EO.ES=c^2,$$

donc

$$\frac{a^2}{c^2}.EO=ET$$

et par analogie

$$\frac{a^2}{c_1^2}E_1O_1=E_1T_1$$

pour la quadrique Σ_1. En désignant par E_1', O_1' les projections orthogonales des points E_1, O_1 sur la droite s, on déduit des deux dernières égalités la suivante:

$$\left(\frac{a_1^2}{c_1^2}-\frac{a^2}{c^2}\right)EO+\frac{a_1^2}{c_1^2}\,OO_1' = E_1'E\left(1-\frac{a_1^2}{c_1^2}\right);$$

par suite, si $\dfrac{a^2}{c^2}=\dfrac{a_1^2}{c_1^2}$ on a

$$OO_1'=\text{constante}$$

et le point F est à l'infini.

*ibid., 1922, p 104.
†ibid., 1921, p. 641.

Si Σ et Σ_1 sont deux paraboloïdes de révolution dont les paramètres principaux sont $2p$, $2p_1$, on a successivement

$$OT=p,\quad O_1T_1=p_1,\quad OO_1'=p-p_1.$$

6. Les hyperboloïdes des hauteurs $(h_ah_bh_ch_d)$ $(h_{a_1}h_{b_1}h_{c_1}h_{d_1})$ de deux tétraèdres $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$ osculateurs à une parabole gauche orthogonale sont homothétiques; la directrice d de la courbe est un rayon commun aux systèmes réglés $(h_ah_bh_ch_d)$ $(h_{a_1}h_{b_1}h_{c_1}h_{d_1})$; par suite, ces systèmes ont aussi une directrice commune s. Un plan σ osculateur à la parabole est normal à s; les pentaèdres orthocentriques $\alpha\beta\gamma\delta\sigma$, $\alpha_1\beta_1\gamma_1\delta_1\sigma$ ont pour orthocentre commun le point d'intersection O des droites d, s. Ainsi:

Deux tétraèdres $ABCD\equiv\alpha\beta\gamma\delta$, $A_1B_1C_1D_1\equiv\alpha_1\beta_1\gamma_1\delta_1$ *osculateurs à une même parabole gauche orthogonale déterminent, en général, un seul plan osculateur* σ *tel que les pentaèdres orthocentriques* $\alpha\beta\gamma\delta\sigma$, $\alpha_1\beta_1\gamma_1\delta_1\sigma$ *ont même orthocentre* O.

Ces pentaèdres sont conjugués à une même sphère (O), *si l'hyperbole gauche équilatère circonscrite au pentagone orthocentrique* $A_1B_1C_1D_1O$ *passe par l'orthocentre* S *du pentagone* $ABCDO$ (2).

7. *Si les tétraèdres* $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$ *osculateurs à une parabole gauche orthogonale sont inscrits dans une hyperbole gauche équilatère* Γ, *tout plan* σ *osculateur à la parabole est tel que les pentaèdres* $\alpha\beta\gamma\delta\sigma$, $\alpha_1\beta_1\gamma_1\delta_1\sigma$ *ont même orthocentre.*

Car dans cette hypothèse, les hauteurs des deux tétraèdres sont des rayons d'un même système réglé d'un hyperboloïde équilatère (H).

Trois quadriques de révolution Σ_1, Σ_2, Σ_3 *conjuguées au tétraèdre* $\alpha\beta\gamma\delta$ *ont leurs centres* E_1, E_2, E_3 *sur la directrice* d *de la parabole gauche orthogonale*. Ces points sont également les centres de trois quadriques de révolution* Σ_1', Σ_2', Σ_3' *conjuguées au tétraèdre* $\alpha_1\beta_1\gamma_1\delta_1$. *Les surfaces* Σ_1 *et* Σ_1', Σ_2 *et* Σ_2', Σ_3 *et* Σ_3' *sont homofocales.*

L'axe de révolution s_1 de la quadrique Σ_1, est normal à un plan σ, osculateur à la parabole et passant par le point E_1 (5). Les pentaèdres $\alpha\beta\gamma\delta\sigma$, $\alpha_1\beta_1\gamma_1\delta_1\sigma_1$ ont même orthocentre E_1 et ce point est le centre d'une quadrique Σ_1' de révolution autour de l'axe s_1 et conjuguée au tétraèdre $\alpha_1\beta_1\gamma_1\delta_1$.

L'hyperbole gauche équilatère Γ rencontre la directrice d de la parabole en un seul point; on peut donc supposer que les points E_1, E_2 du terne (E_1, E_2, E_3) ne sont pas situés sur la courbe Γ. L'axe de révolution s_1 des quadriques Σ_1, Σ_1' rencontre l'hyperbole Γ en deux points conjugués harmoniques relativement aux foyers singuliers de ces surfaces†. Par suite, les quadriques Σ_1, Σ_1', et par analogie Σ_2, Σ_2' sont homofocales.

Les surfaces Σ_3 et Σ_3' jouissent de la même propriété mais la démonstration précédente n'est plus applicable en général: on la remplacera par la suivante:

Les quadriques Σ_1, Σ_2, Σ_3 appartiennent à un même faisceau tangentiel‡: Σ_1', Σ_2', Σ_3' jouissent de la même propriété. On désigne par F_1 un foyer

*ibid., 1921, p. 166.
†ibid., 1921, p. 163.
‡ibid.. 1921, p. 497.

singulier de Σ_1 et Σ_1'; par F_2 un foyer singulier de Σ_2 et Σ_2'; les plans isotropes passant par la droite F_1F_2 sont tangents à Σ_1 et Σ_1', Σ_2 et Σ_2' et par suite, à Σ_3 et Σ_3'; ils déterminent sur l'axe de révolution de Σ_3 et Σ_3' les foyers de ces quadriques; elles sont donc homofocales.

Corollaire. Trois des plans osculateurs à la parabole gauche considérée sont tels que si σ_1 désigne l'un quelconque d'entre eux, les pentaèdres $\alpha\beta\gamma\delta\sigma_1$, $\alpha_1\beta_1\gamma_1\delta_1\sigma_1$ ont pour orthocentre commun un point E_1 de la face σ_1.

L'hyperbole gauche équilatère Γ coupe la directrice de la parabole en un seul point O'; la génératrice s' de l'hyperboloïde (H) de système contraire à celui de d, issue du point O' rencontre la courbe Γ en un second point S'. Si σ' est le plan osculateur à la parabole (Π) normal à s', les pentaèdres $\alpha\beta\gamma\delta\sigma'$, $\alpha_1\beta_1\gamma_1\delta_1\sigma'$, ont même orthocentre O'. Le point S' et le plan σ' sont pôle et plan polaire relativement aux sphères conjuguées à ces pentaèdres; ces sphères sont donc identiques, et par conséquent:

Il existe un seul plan osculateur σ' à la parabole gauche orthogonale tel que les pentaèdres $\alpha\beta\gamma\delta\sigma'$, $\alpha_1\beta_1\gamma_1\delta_1\sigma'$ sont conjugués à une même sphère. Le centre de cette sphère est le point de l'hyperbole gauche équilatère Γ situé sur la directrice de la parabole.

8. Trois des directrices l_1, l_2, l_3 du système réglé des hauteurs $(h_ah_bh_ch_d)$ d'un tétraèdre $ABCD$ sont des directions asymptotiques d'une hyperbole gauche équilatère Γ circonscrite au tétraèdre. L'une quelconque l_1, est l'axe de révolution d'une quadrique (L_1) conjuguée au tétraèdre $ABCD$. Le centre de (L_1) est un point de Γ. Le plan polaire d'un point quelconque A_1 de la courbe Γ relativement à la quadrique (L_1) coupe Γ aux points B_1, C_1, D_1.

Les tétraèdres $ABCD$, $A_1B_1C_1D_1$ inscrits dans l'hyperbole gauche équilatère Γ, sont osculateurs à une parabole gauche orthogonale et conjugués à une quadrique de révolution.

CYCLIC SYSTEMS OF SIX POINTS IN A BINARY CORRESPONDENCE

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1. Since the illumination of the binary doubly cubic forms by the theory of elliptic functions attention has been focused in general on that method of solution. Until rather recently, the direct algebraic attack by more elementary methods, upon even the simpler problems in poristic polygons, has been somewhat neglected. The present paper exhibits the nature of a special case of the (3, 3) correspondence, possessing the poristic property.

2. *The special problem.* The particular case to be considered is a closed system of 6 x's and 6 y's. The points x, on a twisted cubic C, are represented by the 6 parameters a, b, c, d, e, f; the planes y, osculating a second curve K of class 3, are represented by the parameters 1, 2, 3, 4, 5, 6 used as symbols, and the double sextette is to be subject to the cyclic substitution $S\equiv(abcdef)$ (123456). The relations between the points x and the planes y are shown in the following array I, where the point $x=a$ is associated with the triad of planes 1, 2, 3.

I

x	a	b	c	d	e	f
y	1	2	3	4	5	6
	2	3	4	5	6	1
	3	4	5	6	1	2

3. *Determination of a normal form.* For the (3, 3) correspondence admitting a cycle of six, investigation shows that the form with 15 constants may, without loss of generality, be replaced by an equivalent determinantal form which by the aid of the binary identities is reducible to an expression in 4 terms. Hence a trial form of the following type may be set up initially:

$$F(x, y)\equiv G(xb)(xc)(xd)(y1)(y2)(y3)+H(xa)(xc)(xd)(y2)(y3)(y4)$$
$$+I(xa)(xb)(xd)(y3)(y4)(y5)+J(xa)(xb)(xc)(y4)(y5)(y6)=0,$$

where (xb) is the determinant of order 2 or the simple difference $x-b$. This form visibly satisfies 12 conditions, required by the array I, all on a, b, c, d; the application of the remaining 6 conditions given by the pairs $x=e$, $y=5$, 6, 1 and $x=f$, $y=6$, 1, 2 furnishes 6 equations linear in G, H, I, J. Three of these equations determine the constants $\frac{H}{G}$, $\frac{I}{G}$, $\frac{J}{G}$, and with the binary identities allow the reduction of the form to any desired standard form such as the following:

$F(x, y) \equiv (45)\,(46)\,(ae)\,(cf)(xb)(xc)(xd)(y1)(y2)\,(y3) - (15)(46)(be)(cf)(xa)(xc)(xd)(y2)(y3)(y4) + (14)(26)(ce)(cf)(xa)(xb)(xd)(y3)\,(y4)(y5) - (14)\,(23)(ce)(df)(xa)(xb)(xc)(y4)(y5)(y6) = 0.$

The 3 remaining equations impose upon the 12 parameters 3 independent relations E_1, E_2, E_3, such as $E_1 \equiv \dfrac{(ab)\ (cd)}{(ac)\ (bd)} - \dfrac{(56)\ (12)}{(51)\ (62)} = 0$, equivalent to an *invariant set* of 15. These 15 invariant relations E_1, E_2, E_3, . . . express a unique projectivity of the x range to the y range, namely that *abcdef* is projective to 561234.

Since the 18 pairs of points of the array I determine the form $F(x, y)$ and give the 3 independent conditions E_1, E_2, E_3, these 4 forms are a sufficient basis of the whole modulus of forms of the desired type, any such form being expressible as $F(x, y) + \sum_{i=1}^{3} m_i E_i = 0$.

4. *Sets of points known to be poristic.* This correspondence contains obviously at least one cyclic set of period six, the question of interest is, does this set form an isolated closed system or does the sextette possess the poristic property like the Poncelet polygons.

Franz Meyer, among others, studied the (3, 3) correspondence of 4 points and 4 planes, and proved the double quartette to be poristic under certain conditions. One particular set on 7 points with the poristic property was established by Mr. White in 1915 and confirmed by Mr. Coble who has considered the general problem. While Mr. Coble has not attempted an exhaustive classification, he has listed 14 types old and new, which are poristic configurations of binary forms, but he has not published particular results for closed sets of 6 points. Dingeldey's summary in the Encyclopädie der Mathematischen Wissenschaften affords no enlightenment for this special case.

5. *Conditions for the porism of the double sextette.* To answer then the question concerning the poristic character of the sextette, we deviate from the initial values of the 12 parameters which express the correspondence, to the neighbouring set $a+da$, $b+db$, . . . , $6+d6$, which must express the same correspondence by the same formal equation. That is, the first differential must be a doubly cubic form identical with the form F, so that

$$F(x, y) + \frac{\partial F}{\partial a}\, da + \frac{\partial F}{\partial b} db + \ldots + \frac{\partial F}{\partial 6} d6 \equiv M.\ F(x, y).$$

Instead of comparing the coefficients of these two forms we make use again of the 18 pairs of points (x, y) given in the array I; these will furnish conditions sufficient and easy of application. For the point $(a,\ 1)$ all the partial derivatives, except two, vanish and we obtain as a first equation of condition $\left(\dfrac{\partial F}{\partial a}\right)_{\substack{a\\1}} a' + \left(\dfrac{\partial F}{\partial 1}\right)_{\substack{a\\1}} 1' = 0$ where for brevity a' replaces da, etc.

The cyclic nature of the correspondence enables us to write the 5 additional equations completing this cycle of 6 equations. Similarly the two pairs of points $(a, 2)$ and $(a, 3)$ each initiate cycles of 6 equations, so that these 3 pairs of points, together with the cyclic property will provide the whole set of 18 equations of condition to be satisfied by the 12 parameters $a', b', \ldots, 6'$.

The consistency of the 18 equations and the uniqueness of the solution is established if their number can be reduced to 11. This can be done if the two following conditions are satisfied by these equations if (1) $\frac{1'}{2'}\cdot\frac{2'}{3'}\cdot\frac{3'}{1'}\equiv 1$; (2) $\frac{1'}{2'}\cdot\frac{2'}{3'}\cdot\frac{3'}{4'}\cdot\frac{4'}{5'}\cdot\frac{5'}{6'}\cdot\frac{6'}{1'}\equiv 1$; and both of these identities are found to be satisfied if $(16)(24)(35)=(15)(26)(34)$, that is, if the 3 pairs of points 1, 4; 2, 5; 3, 6 are in a quadric involution. The identity (1) being satisfied enables us, by the adjoined cyclic property, to discard 6 of the 18 equations, while the identity (2) disposes of 1 more, leaving 11 *linear* equations for the determination of the 11 ratios $\frac{a'}{1'}, \frac{b'}{1'}, \ldots, \frac{6'}{1'}$.

6. *Theorem. Hence* 6 *points selected arbitrarily on a twisted cubic can form in a* (3, 3) *correspondence an isolated closed system of the type here sought; but if the* 6 *points are specialized as* 3 *pairs of a quadric involution on the cubic then the sextette is poristic and slides along the cubic C while the* 6 *associated planes continue to osculate the same second curve K of class* 3.

7. *Relation of the sextette to the secant-axes.* In our closed set of 6 points and 6 planes, the scheme shows a line fa joining 2 points of C and lying in 2 planes 1 and 2 of K—a line which is a secant of C and an axis of K.

Now a well known theorem of Cremona fixes the number of such secants of one twisted cubic which are at the same time axes of another, at six; and Meyer adds that two curves having more than six must have an infinity of such lines with the dual role of axes and secants, and this investigation is in entire agreement with these theorems. But a problem decidedly more interesting is presented by Meyer's further assertion that when the number of secant-axes is infinite, then the curves C and K are necessarily in the Hurwitz relation, that is, the lines in question can be grouped in sets of six to form infinitely many tetrahedrons, all inscribed in C and circumscribed to K. The Hurwitz relation exists for the Meyer double quartette, but the assertion that it is true in general is erroneous, as this double sextette shows. This relation requires at each point of C 3 secant-axes, whereas the sextette possesses 2 and no more—other essential conditions also are not fulfilled. The only harmony possible would be if for $F(x, y)=0$, we found $G\equiv 0$, $H\equiv 0$, $I\equiv 0$, $J\equiv 0$, and as each of these coefficients is a single product of differences, this can only occur when points are assumed coincident.

8. The exhibition of this type of double sextette possessing the poristic property is one concrete step towards the realization of a suggestion offered by Mr. White in a paper on Poncelet Polygons—"that all Poncelet systems are associated with linear involutions upon rational curves and that in this feature, possibly, lies even more promise of generalizations and discoveries than in Jacobi's brilliant and beautiful depiction by the aid of periodic functions".

CONTRIBUTION A LA THÉORIE DE LA SEXTIQUE A HUIT POINTS DOUBLES

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On sait qu'une sextique plane, possédant huit points doubles A_1, A_2, ..., A_8, se reproduit par une involution du 17^e ordre, nommée involution de Bertini. Les points A_1, ..., A_8 sont ses points principaux; la courbe principale correspondant au point A_i, que je désignerai par c_i, est une sextique ayant A_i pour point triple et passant doublement par les autres points A_k.

I. On peut se servir de ce fait pour trouver des familles de cette courbe, différentes au point de vue projectif. En effet, on obtient des courbes spéciales, si l'on suppose que l'ordre de cette involution s'abaisse. Cet abaissement peut s'effectuer de différentes manières; mais si l'on suppose que les points doubles de la courbe restent toujours distincts, on peut se servir de la proposition suivante, facile à établir:

L'ordre de la courbe principale c_i est réduit d'une unité, si par le point principal A_i passe une droite contenant, en tout, trois points principaux; l'ordre est réduit de deux unités, si par le point A_i passe une conique contenant six points principaux; l'ordre est réduit de trois unités, s'il existe une cubique, ayant un des points A_i pour point double et passant simplement par les autres points A_i.

En combinant ces trois cas d'une manière convenable, on peut arriver à une sextique se reproduisant par une involution d'ordre quelconque, inférieur à 17, bien entendu. Je citerai quelques cas. Si les points A_1, A_2, A_3 sont situés sur une droite, les courbes c_1, c_2, c_3 ont l'ordre 5; un calcul facile donne le nombre 16 comme l'ordre de l'involution. S'il existe une conique contenant les six points A_1, ..., A_6, l'ordre des courbes principales c_1, ..., c_6 est 4; l'ordre de l'involution est 13. S'il existe trois droites passant par un point double, soit A_1, et dont chacune contient, en outre, deux points principaux, soit, respectivement, A_2, A_3; A_4, A_5; A_6, A_7, la courbe c_1 a l'ordre 3, l'ordre des courbes c_2, ..., c_7 est 5. L'ordre de l'involution est 14. Si les points doubles sont situés, six à six, sur deux coniques se coupant en quatre points doubles, on a quatre courbes principales de l'ordre 2, quatre courbes principales de l'ordre 4; l'ordre de l'involution est 9. Si, enfin, les points doubles sont situés sur la figure bien connue de trois coniques passant par les mêmes deux points doubles, soit A_1, A_2, et se coupant, deux à deux, en deux autres points doubles, alors, les points A_1, A_2 cessent d'être principaux, et les six courbes principales c_3, ..., c_8 se réduisent à des coniques. La sextique se reproduit par une involution, d'ailleurs connue, du 5ᵉ ordre. Un cas particulièrement intéressant s'obtient, si les points doubles sont situés sur une cubique ayant un de ces points, soit A_8, pour point double. En ce cas, le point A_8, est le point résiduel du groupe $A_1, \ldots, A_8$; mais on sait que ce point est un point uni de l'involution de Bertini, sans être point principal. C'est ce qui a lieu, en ce cas, pour le point A_8. Les courbes principales, correspondant aux points $A_1, \ldots, A_7$ sont des cubiques, et l'involution de Bertini se réduit à l'involution de Geiser ayant l'ordre 8. On peut énoncer ce résultat de la manière suivante:

La condition, nécessaire et suffisante, pour que la sextique à huit points doubles se reproduise par une involution de Geiser, est que les huit points doubles forment le groupe complet de points d'intersection de deux cubiques.

Signalons enfin, parmi les cas où cette involution se réduit à une homographie, celui où les huit points doubles sont autant de points d'inflexion communs des cubiques d'un faisceau.

On pourrait obtenir des considérations analogues pour le cas où, par suite de la coïncidence de points doubles, deux à deux, la courbe possèderait des points tacnodaux.

II. En second lieu, considérons une correspondance (1, 2) entre le plan ρ de la sextique comme plan simple et un plan ρ' comme plan double. Dans ce but, rappelons sommairement quelques propriétés, d'ailleurs connues, de l'involution de Bertini. Pour obtenir le point X' correspondant à un point arbitraire X, on considère la cubique du faisceau Σ déterminé par les huit points principaux $A_1, \ldots, A_8$ et passant par X; la tangente à cette courbe au point A_9, résiduel des points $A_1, \ldots, A_8$, coupe la courbe encore en un point T; le dernier point d'intersection de la courbe avec la droite TX est le point demandé X'. Le lieu du point T est une quartique dont A_9 est un point triple et qui contient tous les points A_i. Menons, dans le plan ρ, une droite d quelconque, et choisissons, dans le second plan, ρ', deux points S', D'. Établissons une homographie entre les cubiques du faisceau Σ et les droites du faisceau (S'), ainsi qu'une homographie entre les points de la droite d et les droites du faisceau (D'). Faisons correspondre, à un point X' du plan ρ' le couple X_1, X_2 du plan ρ qu'on obtient de la manière suivante: construisons la cubique C correspondant, dans le faisceau Σ, à la droite $S'X'$, et le point Y correspondant dans la ponctuelle (d) à la droite $D'X'$. La droite joignant le point Y au point T de la cubique C, mentionné tout à l'heure, coupe encore la cubique en un couple appartenant à l'involution de Bertini; c'est le couple X_1, X_2. La construction inverse est évidente. Cette correspondance est de l'ordre 12 et elle possède, comme on trouve par une analyse approfondie, les points principaux que voici: (a) dans le plan ρ: cinq couples de points principaux du premier ordre, un point—c'est le point A_9—principal du deuxième ordre, et huit points du 4ᵉ ordre; ce sont les points $A_1, \ldots, A_8$; (b) dans le plan ρ': cinq points principaux du 3ᵉ ordre—parmi lesquels le point D'—, un point (S') du 9ᵉ ordre. Les huit courbes principales du 4ᵉ ordre, situées dans le plan ρ', ont S' pour point triple et passent par les cinq points principaux du 3ᵉ ordre. Elles forment une configuration intéressante, les derniers points de rencontre de ces courbes, deux à deux, se trouvant sur des droites du faisceau (D').

Ce qui est important pour la théorie de la sextique en question, c'est qu'elle est transformée, par cette correspondance, en une quartique ayant le point S' pour point triple et passant par les autres points principaux. Cela donne lieu à la solution de plusieurs problèmes concernant la sextique, par exemple, la sextique à huit points doubles donnés est déterminée par trois points simples. Il est facile de donner une construction quadratique de points de la courbe. En effet, cela se réduit, dans le plan ρ', à la construction d'une quartique à point triple donné, passant, en outre, par les cinq points principaux du 3^e ordre et par les trois points correspondant aux trois points donnés dans le plan ρ'. On peut passer de cette construction indirecte de la sextique à une construction directe des points de la courbe. En effet, si l'on a déterminé, au moyen de la construction de la quartique auxiliaire, neuf couples de l'involution de Bertini appartenant à la courbe demandée, il suffit de construire la tangente double de la courbe de 3^e classe, enveloppe des droites joignant les couples de l'involution de Bertini situés sur la sextique. On a alors tous les éléments nécessaires pour construire l'homographie entre les tangentes de cette courbe rationnelle et les cubiques du faisceau Σ. La construction des tangentes aux points doubles de la sextique revient à la détermination des derniers points d'intersection de la quartique, image de la sextique en question, avec une des courbes principales du 4^e ordre mentionnées plus haut. En partant de cette remarque on peut se poser des questions concernant des sextiques ayant des points de rebroussement.

SUR LES INVOLUTIONS RÉGULIÈRES D'ORDRE DEUX, APPARTENANT A UNE SURFACE IRRÉGULIÈRE

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Nous avons consacré quelques mémoires à l'étude des involutions n'ayant qu'un nombre fini de points de coïncidence, appartenant à une surface algébrique†. Nous nous proposons de faire connaître quelques résultats concernant les involutions de cette nature, régulières, d'ordre deux, appartenant à une surface irrégulière.

Quelques cas particuliers sont connus: on a étudié les involutions régulières d'ordre deux appartenant aux surfaces de Jacobi et de Picard, ce qui conduit aux surfaces de Kümmer généralisées‡. Un autre cas particulier a été considéré par G. Humbert et est en relation avec les courbes de genre trois§.

1. Soit F une surface algébrique d'irrégularité $q>0$, contenant une involution régulière I_2, d'ordre deux, n'ayant qu'un nombre fini de points de coïncidence. Désignons par Φ une surface normale, dans un espace linéaire S_s, image de cette involution et soient $|\Gamma|$ le système des sections hyperplanes de Φ, n l'ordre de Φ (c'est-à-dire le degré du système $|\Gamma|$), π le genre des courbes Γ.

Entre Φ et F existe une correspondance (1, 2) pour laquelle les points de diramation de Φ (points correspondant aux points de coïncidence de I_2) sont en nombre 4α, nécessairement multiple de 4. On peut choisir Φ de manière à ce que ces points de diramation soient des points isolés. Ce sont alors des points doubles coniques de Φ. De plus, pour s suffisamment élevé, il existe, parmi les hypersurfaces découpant sur Φ le système $|2\Gamma|$, au moins une hypersurface passant par les 4α points de diramation et touchant Φ le long d'une courbe Γ_0, d'ordre n, de genre $\pi-\alpha$ et de degré $n-2\alpha$.

Aux courbes Γ, Γ_0 correspondent, sur F, des courbes respectivement C_{01}, C_{02}, appartenant à un même système linéaire complet $|C_0|$, de genre $2\pi-1$, de degré $2n$ et de dimension $r>s$||.

†L. Godeaux, *Mémoire sur les surfaces algébriques doubles ayant un nombre fini de points de diramation* (Annales de la Faculté des Sciences de Toulouse, 1914). *Recherches sur les involutions douées d'un nombre fini de points de coïncidence, appartenant à une surface algébrique* (Bull. Soc. Math. France, 1919)—*Recherches sur les involutions cubiques appartenant à une surface algébrique* (Bull. de l'Académie Royale de Belgique, 1921).

‡Voir au sujet de ces surfaces: F. Enriques et F. Severi, *Mémoire sur les surfaces hyperelliptiques* (Acta Mathematica, 1909, vol. XXXII et XXXIII).

§G. Humbert. *Sur une surface du sixième ordre, liée aux fonctions abéliennes de genre trois* (Jour. de Math., 1896). Voir aussi L. Godeaux, *Sur une surface algébrique considérée par M. G. Humbert* (Bulletin des Sciences Math., 1921).

||Pour ces résultats, voir notre *Mémoire sur les surfaces doubles*, loc. cit.

On peut démontrer que l'on peut prendre s assez grand pour que le système continu complet $\{C\}$, déterminé par $|C_0|$, ait la dimension $r+q$.

2. Soit T la transformation birationnelle involutive de F en elle-même engendrant l'involution I_2.

La transformation T change une courbe C de $\{C\}$ en une courbe C'. Lorsque C varie d'une manière continue dans $\{C\}$ et vient coïncider avec une courbe de $|C_0|$, la courbe C' varie d'une manière continue sur F et vient coïncider également avec une courbe de $|C_0|$, ce système étant transformé en lui-même par T. Il en résulte que C' appartient au système continu complet $\{C\}$ et ce système est donc transformé en lui-même par T.

Si V_q désigne la variété de Picard attachée à F†, à T correspond donc une transformation birationnelle involutive de cette variété. On démontre que cette transformation est précisément une transformation de seconde espèce de V_q; par suite, elle laisse invariants 2^{2q} points de V_q. Il en résulte qu'il existe, dans $\{C\}$, 2^{2q} systèmes linéaires transformés en eux-mêmes par T. L'un de ces systèmes est $|C_0|$. Nous désignerons les autres par $|C_1|$, $|C_2|$, ..., $|C_k|$, où $k=2^{2q}-1$.

3. La transformation T agit, sur les courbes d'un de ces systèmes, $|C_1|$ par exemple, comme une homographie involutive. Comme ce système $|C_1|$ comprend une infinité de courbes (il est, en général, de dimension r), il contiendra deux systèmes linéaires partiels $|C_{11}|$, $|C_{12}|$ composés au moyen de l'involution I_2. A une courbe C_{11} (ou C_{12}) correspond, sur Φ, une courbe d'ordre n que nous désignerons par Γ_{11} (ou Γ_{12}). De plus, tout point de coïncidence de I_2 est un point-base de $|C_{11}|$ ou de $|C_{12}|$.

Envisageons maintenant une courbe quelconque C de $\{C\}$ et la courbe C' de ce système que T lui fait correspondre. A l'ensemble des courbes C, C' correspond, sur Φ, une courbe Γ^*, de genre effectif $2\pi-1$, possédant n points doubles.

Lorsque C varie dans $\{C\}$, Γ^* varie sur Φ et décrit un système continu qui appartient à un système linéaire, puisque Φ est régulière. Soit $|\Gamma^*|$ ce système, de genre virtuel $2\pi+n-1$.

Lorsque C vient coïncider avec une courbe C_{01} (transformée d'une courbe Γ), C' vient aussi coïncider avec cette courbe C_{01}, et par suite la courbe Γ^* se compose d'une section hyperplane Γ comptée deux fois. On a donc

$$|\Gamma^*|=|2\Gamma|.$$

Lorsque C vient coïncider avec une courbe C_{11}, Γ^* vient coïncider avec une courbe Γ_{11}, comptée deux fois, augmentée des courbes rationnelles infiniment petites représentant les domaines des points de diramation appartenant à Γ_{11}. Si l'on désigne par A la somme de ces courbes infiniment petites, on a donc

$$\Gamma^*=2\Gamma\equiv 2\Gamma_{11}+A.$$

†Voir au sujet de l'introduction de cette variété: Castelnuovo, *Sugli integrali semplici appartenenti ad una superficie algebrica* (Rendiconti dei Lincei, 1^{e} sem. 1905).

Par suite, parmi les hypersurfaces découpant sur Φ le système $|2\Gamma|$, il en existe une touchant la surface Φ le long d'une courbe Γ_{11} quelconque. Mais alors†, Φ est l'image d'une certaine involution d'ordre deux, appartenant à une certaine surface et ayant comme courbe de diramation les courbes infiniment petites composant A. Il en résulte que le nombre de ces courbes est multiple de 4. Nous le représenterons par $4\alpha_{11}$. Les courbes C_{11} passent donc, simplement, par $4\alpha_{11}$ points de coïncidence de I_2 et les courbes C_{12} par les $4\alpha_{12}= 4(\alpha-\alpha_{11})$ points de coïncidence restants.

D'une manière générale, dans $|C_i|$, il y aura deux systèmes linéaires partiels $|C_{i1}|$ ayant comme points-base $4\alpha_{i1}$ points de coïncidence de I_2, et $|C_{i2}|$, ayant comme points-base les $4\alpha_{i2}= 4(\alpha-\alpha_{i1})$ points de coïncidence restants. A ces systèmes correspondront, sur Φ, des systèmes $|\Gamma_{i1}|$, $|\Gamma_{i2}|$ possédant la propriété analogue à celle de $|\Gamma_{11}|$.

Le système $|\Gamma_{i1}|$ (ou $|\Gamma_{i2}|$) aura le genre $\pi-\alpha_{i1}$ (ou $\pi-\alpha_{i2}$) et le degré $n-2\alpha_{i1}$ (ou $n-2\alpha_{i2}$).

4. Reprenons le système $|C_{11}|$ et voyons si ce système peut être dépourvu de points-base ($\alpha_{11}= 0$). Alors, les courbes Γ_{11} correspondantes ne passent par aucun point de diramation de Φ et on a

$$|2\Gamma|= |2\Gamma_{11}|, \quad |2\Gamma_0|= |2\Gamma_{12}|.$$

Cela entraîne la condition que le diviseur de Severi‡ de la surface Φ doit être pair. L'application des résultats de M. Severi conduit alors à ces conclusions:

(1°) Il y aura au plus un des nombres α_{ij} nul, et alors le diviseur de Φ est pair;

(2°) Si le diviseur de Φ est pair, il y aura toujours deux systèmes linéaires partiels $|C_{i1}|$, $|C_{j1}|$ ayant, comme points-base, les mêmes points de coïncidence de I_2, quel que soit i, et cela ne pourra se présenter si le diviseur de Φ est impair.

(On suppose ici que l'on a toujours pris $\alpha_{i1}\leqq\alpha_{i2}$).

5. Envisageons le système $\{D\}=\{2C\}$. Il contient également 2^{2q} systèmes invariants pour T. L'un de ceux-ci est

$$|D_0|=|2C_0|=|2C_1|=\ldots=|2C_k|.$$

Les $2^{2q}-1$ autres systèmes s'obtiendront en combinant deux-à-deux des systèmes $|C_0|$, $|C_1|$, ..., $|C_k|$ distincts. Cela est possible de $2^{2q-1}(2^{2q}-1)$ manières différentes; on trouvera donc 2^{2q-1} fois le même système en faisant ces combinaisons. Les systèmes distincts obtenus sont

$$|D_i|= |C_0+C_i|, \quad (i= 1, 2, \ldots, k).$$

On a, par exemple,

†*Mémoire sur les surfaces doubles*, loc. cit.

‡Severi. *La base minima pour la totalité des courbes tracées sur une surface algébrique* (Annales de l'École Normale Supérieure, 1908). *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica* (Rend. Circ. Mat. Palermo, 1910, t. XXX).

$$|D_1| = |C_0+C_1| = |C_2+C_3| = |C_4+C_5| = \dots,$$
$$|D_2| = |C_0+C_2| = |C_3+C_1| = |C_6+C_4| = \dots,$$
$$|D_3| = |C_0+C_3| = |C_1+C_2| = |C_5+C_6| = \dots,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Dans le système $|D_1|$, il y a deux systèmes linéaires partiels $|D_{11}|$, $|D_{12}|$ composés au moyen de I_2. L'un de ces systèmes, $|D_{11}|$ par exemple, comprendra les courbes $C_{01}+C_{11}$, $C_{02}+C_{12}$, $C_{21}+C_{31}$, $C_{22}+C_{32}$, l'autre, $|D_{12}|$, comprendra les courbes $C_{01}+C_{12}$, $C_{02}+C_{11}$, $C_{22}+C_{31}$, $C_{21}+C_{32}$.

Un examen de la distribution de ces diverses courbes dans les systèmes $|D_i|$ conduit à la conclusion que, si le diviseur de Φ est impair, on a

$$\alpha_{11}=\alpha_{21}=\alpha_{31}=\dots=\alpha_{k1},\ \alpha_{12}=\alpha_{12}=\alpha_{32}=\dots=\alpha_{k2};$$

si le diviseur de Φ est pair, l'un des nombres α_{i1}, par exemple α_{11}, est nul et on a

$$\alpha_{11}=0,\ \alpha_{21}=\alpha_{31}=\dots=\alpha_{k1},\ \alpha_{12}=\alpha,\ \alpha_{22}=\alpha_{32}=\dots=\alpha_{k2}.$$

On a de plus des résultats tels que celui-ci, que nous nous bornerons à énoncer dans le cas où le diviseur de Φ est impair.

Vis-à-vis de trois systèmes $|C_i|$, $|C_j|$, $|C_l|$, les 4α points de coïncidence de I_2 se répartissent en quatre groupes:

Un groupe de $2\alpha_{11}$ points communs aux courbes C_{i2}, C_{j1}, C_{l1};

Un groupe de $2\alpha_{11}$ points communs aux courbes C_{i1}, C_{j2}, C_{l1};

Un groupe de $2\alpha_{11}$ points communs aux courbes C_{i1}, C_{j1}, C_{l2};

Un groupe de $4\alpha_{12}-2\alpha_{11}$ points communs aux courbes C_{i2}, C_{j2}, C_{l2}.

On démontre encore que dans tout système continu complet, formé de ∞^q systèmes linéaires infinis, les 2^{2q} systèmes linéaires invariants contiennent des systèmes partiels composés au moyen de I_2, ayant toujours le même nombre de points de coïncidence de I_2 comme points-base.

6. On peut maintenant se proposer l'examen d'un système continu complet $\{L\}$, 2^{2q}, formé de $\infty^{q'}(q'<q)$ systèmes linéaires, et transformé en lui-même par T. Dans ce cas, le système $\{L\}$ contient $2^{2q'}$ systèmes linéaires invariants pour T, mais il se peut que dans ces systèmes linéaires, les systèmes linéaires partiels composés au moyen de I_2 aient tous des points de coïncidence de I_2 comme points-base. C'est ce qui se présente notamment pour certains systèmes continus complets appartenant à une surface de Picard.

Dans tous les cas, le système continu complet $\{2L\}$ présentera les mêmes particularités que le système $\{C\}$ considéré plus haut.

7. Nous terminerons par l'énoncé d'un théorème concernant la surface représentant les couples de points (non ordonnés) d'une courbe de genre trois. Cette surface, considérée par G. Humbert (*loc. cit.*) a les genres $p_a=0$, $p_g=3$, $P_2=7$, $p^{(1)}=7$. Elle possède une involution régulière d'ordre deux, de genres $p_a=p_g=3$, $P_2=7$, $p^{(1)}=4$, pour laquelle il y a $4\alpha=28$ points de coïncidence.

Si $\{C\}$ est un système continu complet formé de ∞^3 systèmes linéaires infinis sur cette surface, il y a 64 systèmes linéaires invariants pour la transformation T déterminée par l'involution. Un de ces systèmes linéaires contient deux systèmes linéaires incomplets composés au moyen de l'involution, l'un est dépourvu de points-base, l'autre a comme points-base les 28 points de coïncidence de l'involution. Dans chacun des 63 autres systèmes invariants, il y a deux systèmes linéaires incomplets composés au moyen de l'involution, l'un possède 12 points-base, l'autre 16 points-base, qui sont des points unis de l'involution.

SUR LES LIGNES ASYMPTOTIQUES EN GÉOMÉTRIE INFINITÉSIMALE

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1. Nous établirons tout d'abord une formule qui exprime une propriété d'une courbe quelconque. A cet effet, attachons à chaque point M de la courbe C, un trièdre trirectangle Mx, y, z, dont l'axe des x coïncide avec la demi-tangente en M, dirigée dans le sens des arcs croissant; l'axe des y, avec la demi-normale principale, dirigée vers le centre de courbure et l'axe des z, avec la demi-binormale, choisie de telle sorte que le trièdre soit de rotations directes. La variable indépendante sera l'arc s de C compté à partir d'une origine M_0.

Des translations et rotations ξ, η, ζ; p, q, r, trois sont nulles, à savoir η, ζ et q; et ξ est égale à 1.

Cela étant, sur la binormale portons un segment variable MN. Lorsque M varie sur C, le point N, de coordonnées 0, 0, z, décrit une courbe C', dont le plan normal a pour équation

$$X - pzY + \frac{dz}{ds}(Z - z) = 0,$$

X, Y, Z désignant les coordonnées courantes. Ce plan coupe la tangente Mx en un point N', tel que le segment MN', compté sur Mx, est donné par la formule

$$(1) \qquad MN' = \frac{1}{2}\frac{dz^2}{ds}.$$

Telle est la relation que nous nous proposions d'établir. On voit que le sens dans lequel le segment MN est porté sur la binormale importe peu.

2. Considérons à présent une surface (M), lieu du point M dont les coordonnées $x(u, v)$, $y(u, v)$, $z(u, v)$, rapportées à un système d'axes rectangulaires soient telles que le déterminant $\left| \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, x \right|$ diffère de zéro.

Les coordonnées x, y, z sont solutions des équations de Gauss

$$(2) \qquad \frac{\partial^2 x}{\partial u \partial v} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{\partial x}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{\partial x}{\partial v} + D' \frac{\dfrac{\partial y}{\partial u}\dfrac{\partial z}{\partial v} - \dfrac{\partial y}{\partial v}\dfrac{\partial z}{\partial u}}{\sqrt{EG - F^2}}, \ldots, \ldots,$$

dans lesquelles le radical est pris positivement.

Désignons par p la distance de l'origine O des coordonnées au plan tangent en M; par p_1, la distance de O au plan normal qui contient la tangente à la courbe le long de laquelle u varie seul et par p_2, la distance de O au plan normal qui contient la tangente à la courbe le long de laquelle v varie seul.

De l'équation (2), nous déduisons l'équation de Laplace attachée au réseau formé par les lignes coordonnées, à savoir:

$$\frac{\partial^2\theta}{\partial u\partial v}=\left(\begin{Bmatrix}12\\1\end{Bmatrix}+\frac{D'p_2\sqrt{G}}{p\sqrt{EG-F^2}}\right)\frac{\partial\theta}{\partial u}+\left(\begin{Bmatrix}12\\2\end{Bmatrix}-\frac{D'p_1\sqrt{E}}{p\sqrt{EG-F^2}}\right)\frac{\partial\theta}{\partial v}-\frac{D'}{p}\theta, \tag{3}$$

les radicaux étant pris positivement. Les signes de p, p_1 et p_2 sont fixés de la manière suivante. Ayant adopté pour demi-normale positive, celle qui est située du côté positif du plan tangent, défini comme le fait M. Bianchi, nous convenons de donner à p, le signe contraire à celui de cos νOM, $O\nu$ ayant la direction et le sens de la demi-normale positive et OM, celui d'un mobile allant de O vers M. En M, considérons la ligne du réseau le long de laquelle u varie seul et soit MT_1, la demi-tangente positive, dans le sens des u croissants. De même, à la ligne du réseau le long de laquelle v varie seul, menons la demi-tangente positive, MT_2, dans le sens des v croissant. Par O traçons des demi-droites, $O\tau_1$, $O\tau_2$, parallèles aux demi-tangentes MT_1 et MT_2, et élevons des demi-droites $O\nu_1$, $O\nu_2$ perpendiculaires à $O\nu$ et faisant des angles de $+\frac{\pi}{2}$ avec $O\tau_1$, $O\tau_2$, respectivement. Ces constructions établies, nous donnons à p_1, un signe contraire à celui de cos $\nu_1 OM$ et à p_2, un signe contraire à celui de cos $\nu_2 OM$.

3. L'équation (3) a lieu pour un réseau quelconque.

Si le réseau est formé des asymptotiques de la surface (M), l'équation de Laplace prend la forme

$$\frac{\partial^2\theta}{\partial u\partial v}=\left(\frac{\partial\log\sqrt{\rho}}{\partial v}+\frac{p_2\sqrt{G}}{p\rho}\right)\frac{\partial\theta}{\partial u}+\left(\frac{\partial\log\sqrt{\rho}}{\partial u}-\frac{p_1\sqrt{E}}{p\rho}\right)\frac{\partial\theta}{\partial v}-\frac{\sqrt{EG-F^2}}{p\rho}\theta, \tag{4}$$

ρ étant définie, suivant l'usage, par l'équation

$$\rho^2=-r_1r_2,$$

où r_1 et r_2 désignent les rayons de courbure principaux de (M), en M. Nous convenons, en outre, de donner à ρ le signe de D'.

On peut encore fixer, comme suit, le signe de ρ. L'extrémité P du segment de composantes $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$, suivant les axes Mx, My, Mz, décrit une certaine ligne, lorsque v varie seul. Soit $O\mu$, la direction et le sens de la demi-tangente positive au lieu décrit par P, lorsque v croît. Cela étant, ρ a le signe de cos $\nu O\mu$.

Si la surface (M) est une surface de M. Tzitzéica, c'est-à-dire, si $p^4\rho^2=C^{te}$, les coefficients de $\frac{\partial\theta}{\partial u}$ et de $\frac{\partial\theta}{\partial v}$, dans (4), sont nuls*, et on a les relations géomé-

*Voir les Comptes Rendus Acad. Sciences, Paris, t. CXLIV, 1er semestre 1907, p. 1257. G. Tzitzéica, *Sur une nouvelle classe de surfaces.*

triques suivantes, caractéristiques de ces surfaces:

$$(5)\qquad \begin{aligned} &\frac{1}{2}\frac{\partial\rho}{\partial s_u}-\frac{p_1}{p}=0,\\ &\frac{1}{2}\frac{\partial\rho}{\partial s_v}+\frac{p_2}{p}=0; \end{aligned}$$

s_u et s_v désignant respectivement les arcs des courbes le long desquelles, soit u, soit v varie seul.

Sur la normale à (M), portons un segment $MN=\rho$. Quand M décrit une asymptotique M_u, le long de laquelle u varie seul, N décrit une courbe N_u. Quand M décrit une asymptotique M_v, le long de laquelle v varie seul, N décrit une courbe N_v. Les plans normaux à ces courbes, en N, coupent respectivement les tangentes à M_u et M_v en des points N' et N''. En vertu de ce qui précède, les relations (5) s'écrivent

$$(6)\qquad \begin{aligned} MN'&=\frac{2p_1\rho}{p},\\ MN''&=-\frac{2p_2\rho}{p}. \end{aligned}$$

Ces relations géométriques expriment une propriété caractéristique des surfaces de M. Tzitzéica.

4. Calculons les dérivées premières par rapport à u et à v de la distance p, de 0 au plan tangent à (M). Tenant compte des formules de Gauss et de l'équation (4), nous trouvons

$$(7)\qquad \begin{aligned} \frac{\partial p}{\partial s_u}&=-\frac{p_1}{\rho},\\ \frac{\partial p}{\partial s_v}&=\frac{p_2}{\rho}. \end{aligned}$$

Nous en concluons que, si l'on projette un point fixe O, en O', sur la normale MN, les plans normaux aux trajectoires décrites par O', lorsque u ou v varie seul, c'est-à-dire lorsqu'on se déplace sur une asymptotique de l'une ou de l'autre famille, coupent les tangentes aux asymptotiques M_u et M_v, respectivement en des points P' et P'' tels que

$$(8)\qquad \begin{aligned} MP'&=-\frac{pp_1}{\rho},\\ MP''&=\frac{pp_2}{\rho}. \end{aligned}$$

Remplaçant, dans les coefficients de $\dfrac{\partial\theta}{\partial u}$ et de $\dfrac{\partial\theta}{\partial v}$ de l'équation (4), $\dfrac{p_1}{\rho}$ et $\dfrac{p_2}{\rho}$

par leurs valeurs données par les équations (7), nous retrouvons les expressions de ces coefficients établies par M. Demoulin*.

5. La propriété exprimée par le système (7) ne caractérise pas un réseau asymptotique. Admettons, en effet, qu'elle soit satisfaite. Elle s'exprime tout aussi bien par le système (7), dans lequel nous supposons ρ exprimé en fonction des coefficients des formes différentielles fondamentales et les dérivées de p remplacées par les valeurs

$$\frac{\partial p}{\partial u} = \frac{Dp_2\sqrt{G}}{\sqrt{EG-F^2}} - \frac{D'p_1\sqrt{E}}{\sqrt{EG-F^2}}, \tag{9}$$
$$\frac{\partial p}{\partial v} = \frac{D'p_2\sqrt{G}}{\sqrt{EG-F^2}} - \frac{D''p_1\sqrt{E}}{\sqrt{EG-F^2}},$$

déduites de la définition de p et des formules de Gauss.

Les équations (7), ainsi transformées, sont vérifiées pour $D = D'' = 0$, c'est-à-dire dans le cas d'un réseau d'asymptotiques. En dehors de cette solution et de la sphère (les développables étant écartées), les seuls cas où les équations (7) sont satisfaites, sont ceux des surfaces pour lesquelles l'une des familles d'asymptotiques est formée de courbes dont les plans rectifiants passent par le point fixe O.

Ces dernières surfaces jouissent de la propriété suivante. Projetons le point fixe O sur la normale en M à la surface, nous obtenons un point O'. Sur la binormale à une ligne C, distincte des asymptotiques caractérisées, portons $MO'' = MO'$. Quand M se déplace sur C, le point O'' décrit une courbe dont le plan normal coupe la tangente à C, en un point P', tel que

$$MP' = -\frac{pp_1}{\rho}. \tag{10}$$

L'on voit que, en tout point M d'une de ces surfaces, le rapport $\dfrac{MP'}{p_1}$ est constant, quelle que soit la courbe C considérée.

6. Arrêtons-nous aux surfaces que nous venons de rencontrer en dernier lieu. En chaque point M de l'une quelconque d'entre elles, attachons un trièdre trirectangle Mx, y, z; Mz étant normal à la surface; My, tangent à l'asymptotique caractérisée, le long de laquelle nous supposons v varier seul; et prenons pour lignes le long desquelles u varie seul, les trajectoires orthogonales des asymptotiques tangentes à My.

Appliquons le tableau IV, que Darboux donne à la page 385 du tome II de ses immortelles *Leçons sur la théorie des surfaces*. Comme les lignes $u = C^{te}$ sont des asymptotiques, la rotation $p_1 = 0$. Comme le plan zMy passe par un point fixe O, dont nous désignerons les coordonnées relatives par y, z, on a:

*Voir les Comptes Rendus Acad. Sciences, Paris, t. CXLVII, 2e semestre 1908, p. 413. A. Demoulin. *Sur la théorie des lignes asymptotiques.*

$$(11)\qquad \begin{aligned} &A+qz-ry=0, && q_1z-r_1y=0,\\ &\frac{\partial y}{\partial u}-pz=0, && \frac{\partial y}{\partial v}+C=0,\\ &\frac{\partial z}{\partial u}+py=0, && \frac{\partial z}{\partial v}=0. \end{aligned}$$

Des équations (11) et des équations (A) du tableau, on déduit, en choisissant convenablement les paramètres u et v et en écartant la sphère:

$$(12)\qquad \begin{aligned} &y=\sqrt{v^2-u^2}, && z=u,\\ &A=-\frac{uv}{r_1(v^2-u^2)^{\frac{3}{2}}}, && C=-\frac{v}{\sqrt{v^2-u^2}},\\ &p=-\frac{1}{\sqrt{v^2-u^2}}, && p_1=0,\\ &q=\frac{r\sqrt{v^2-u^2}}{u}+\frac{v}{r_1(v^2-u^2)^{\frac{3}{2}}}, && q_1=\frac{r_1\sqrt{v^2-u^2}}{u},\\ &r=\frac{u}{r_1^2(v^2-u^2)}\frac{\partial r_1}{\partial v}+\frac{u(u^2+2v^2)}{vr_1(v^2-u^2)^2}, && \frac{\partial r}{\partial v}-\frac{\partial r_1}{\partial u}=-\frac{r_1}{u}. \end{aligned}$$

La détermination des translations et rotations du trièdre dépend donc d'une équation aux dérivées partielles du 2[e] ordre, de la forme

$$(13)\qquad \frac{\partial^2 \frac{1}{r_1}}{\partial v^2}=\lambda\frac{\partial \frac{1}{r_1}}{\partial u}+\mu\frac{\partial \frac{1}{r_1}}{\partial v}+\nu,$$

λ, μ, ν étant des fonctions de u, v et r_1.

Remarquons que, pour les surfaces considérées, les trajectoires orthogonales des asymptotiques $u=C^{te}$ sont des courbes sphériques situées sur des sphères concentriques, tandis que ces asymptotiques elles-mêmes sont les arêtes de rebroussement de développables circonscrites à des sphères concentriques.

Indiquons encore une autre propriété géométrique des mêmes surfaces. A cet effet, désignons par γ, l'angle formé par MO avec la tangente à l'asymptotique $u=C^{te}$, par α, l'angle que fait le plan osculateur à la trajectoire orthogonale des asymptotiques $u=C^{te}$, avec le plan tangent à (M) en M; et par β, l'angle que forme avec le plan tangent, le plan asymptote de la tangente à la trajectoire orthogonale, engendrant une série réglée, lorsque M décrit l'asymptotique $u=C^{te}$. Soient encore p la distance de l'origine au plan tangent, positive ou négative, suivant que O est du côté positif ou négatif de ce plan et H, la courbure moyenne. On a

$$\operatorname{tg}\alpha.\operatorname{tg}\beta.\operatorname{tg}^2\gamma=\frac{1}{1-(Hp)^{-1}}.$$

SUR LES LIGNES ASYMPTOTIQUES

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1. Etant donnés deux espaces réciproques (E), (E_1), à un faisceau de rayons $(m, a, b, \ldots)$ de centre M de l'espace (E), correspond dans l'espace (E_1) un faisceau de rayons $(m_1, a_1, b_1, \ldots)$ de centre M_1. Le plan $\mu \equiv (ab)$ et le point M_1, le point M et le plan $\mu_1 \equiv (a_1 b_1)$ sont des éléments homologues. Aux couples de points (A, B), (A_1, B_1) choisis respectivement sur les droites m, m_1 correspondent les couples de plans (α_1, β_1), (α, β) passant respectivement par les droites m_1, m.

A deux courbes (C), (D) de l'espace (E), tangentes en M à la droite m et osculatrices au plan μ, correspondent dans l'espace (E_1), les courbes (C_1), (D_1) tangentes en M_1 à la droite m_1 et osculatrices au plan μ_1. On désigne par ρ et τ, ρ' et τ' les rayons de courbure et de torsion au point M des courbes (C) et (D); par ρ_1 et τ_1, ρ_1' et τ_1' ceux des courbes (C_1) et (D_1) au point M_1. On a, en grandeur et en signe*,

$$(1) \qquad \tau\tau_1 = \frac{MA \,.\, MB}{AB} \, \frac{M_1A \,.\, M_1B_1}{A_1B_1} \, \frac{\sin(\alpha\beta)}{\sin(\mu\alpha) \,.\, \sin(\mu\beta)} \, \frac{\sin(\alpha_1\beta_1)}{\sin(\mu_1\alpha_1) \,.\, \sin(\mu_1\beta_1)},$$

$$(2) \qquad \frac{1}{\rho} \, \frac{MA \,.\, MB}{AB} \, \frac{\sin(ab)}{\sin(ma) \,.\, \sin(mb)} = \frac{\tau_1}{\rho_1} \, \frac{\sin(\mu_1\alpha_1) \,.\, \sin(\mu_1\beta_1)}{\sin(\alpha_1\beta_1)} \, \frac{\sin(m_1a_1) \,.\, \sin(m_1b_1)}{\sin(a_1b_1)}.$$

Ces égalités et leurs analogues pour les courbes (D) et (D_1) donnent

$$(3) \qquad \tau\tau_1 = \tau'\tau_1',$$

$$(4) \qquad \frac{\rho}{\rho'} = \frac{\rho_1}{\rho_1'} \cdot \frac{\tau_1'}{\tau_1},$$

$$(5) \qquad \frac{\rho}{\rho'} = \frac{\rho_1}{\rho_1'} \cdot \frac{\tau}{\tau'}.$$

2. Si les courbes (C) et (D) appartiennent à une même surface,

$$(6) \qquad \rho = \rho',$$

les courbes (C_1), (D_1) sont alors les arêtes de rebroussement de deux développables circonscrites à une même surface (S_1), réciproque de (S). Les égalités (4) et (6) donnent

$$\frac{\rho_1}{\tau_1} = \frac{\rho_1'}{\tau_1'}.$$

*C. Servais. *La courbure et la torsion dans la collinéation et la réciprocité.* Mémoires publiés par l'Acad. Royale de Belgique, 1898, p. 27-29.

Ainsi: *Si les arêtes de rebroussement* (C_1), (D_1) *de deux développables circonscrites à une même surface* (S_1) *sont tangentes en un point* M_1, *les rayons de courbure et de torsion* ρ_1 *et* τ_1, ρ_1' *et* τ_1' *de ces courbes au point* M_1 *sont liés par la relation*

$$\frac{\rho_1}{\tau_1}=\frac{\rho_1'}{\tau_1'}. \tag{7}$$

3. Si la courbe (C) est une asymptotique d'une surface (S) passant par la courbe (D), la courbe (C_1) est une asymptotique de la surface (S_1), réciproque de (S) et inscrite dans la développable ayant pour arête de rebroussement la courbe (D_1). Pour les courbes (C) et (D) on a*

$$\frac{\tau}{\tau'}-3=-2\frac{\rho'}{\rho}. \tag{8}$$

Des égalités (3), (4), (8), on déduit pour les courbes (C_1), (D_1)

$$\frac{\tau_1'}{\tau_1}-3=-2\frac{\rho_1'}{\rho_1}\cdot\frac{\tau_1}{\tau_1'}.$$

Donc: *Si l'arête de rebroussement* (D_1) *d'une développable circonscrite à une surface* (S_1) *est tangente en un point* M_1 *à une asymptotique* (C_1) *de* (S_1), *les rayons de courbure et de torsion* ρ_1 *et* τ_1, ρ_1' *et* τ_1' *au point* M_1 *des courbes* (C_1) *et* (D_1) *sont liés par la relation:*

$$\frac{\tau_1'}{\tau_1}-3=-2\frac{\rho_1'}{\rho_1}\cdot\frac{\tau_1}{\tau_1'}. \tag{9}$$

4. Dans le plan μ, on trace une courbe (μ) tangente en M à la droite m; R est son rayon de courbure en M; M' un point de cette courbe voisin de M; m' la droite MM'. On a

$$2R=\lim.\frac{MM'}{\sin(mm')}. \tag{10}$$

A la courbe (μ) et aux éléments M', m' de l'espace (E), correspondent dans l'espace (E_1) un cône (M_1) de sommet M_1, le plan μ_1', et la droite m_1'. Cette dernière est l'intersection des plans μ_1, μ_1' tangents au cône (M_1); la droite m_1 est la génératrice de contact du plan μ_1. On a

$$\lim.\frac{MM'}{(\mu_1\mu_1')}\,\frac{AB}{MA\,.\,MB}=\frac{\sin(\alpha_1\beta_1)}{\sin(\mu_1\alpha_1)\,.\,\sin(\mu_1\beta_1)}, \tag{11}$$

$$\lim.\frac{mm'}{(m_1m_1')}\,\frac{\sin(ab)}{\sin(ma)\,.\,\sin(mb)}=\frac{\sin(a_1b_1)}{\sin(m_1a_1)\,.\,\sin(m_1b_1)}. \tag{12}$$

Des égalités (2), (10), (11), (12), on déduit

$$\lim.\frac{(\mu_1\mu_1')}{(m_1m_1')}=\frac{2R}{\rho}\frac{\rho_1}{\tau_1}. \tag{13}$$

*G. Darboux. *Théorie générale des Surfaces*, t. II, N° 511.

Si la courbe (μ) est la section par le plan μ de la développable ayant pour arête de rebroussement la courbe (C), on a la relation, due à Ossian Bonnet,

$$\rho = \frac{3}{4} R. \tag{14}$$

Dans la figure corrélative, on a donc

$$\lim. \frac{(\mu_1\mu_1')}{(m_1m_1')} = \frac{8}{3}\frac{\rho_1}{\tau_1}.$$

Ainsi: *Soient m_1, μ_1, la tangente et le plan osculateur au point M_1 d'une courbe (C_1), ρ_1 et τ_1, les rayons de courbure et de torsion au point M_1; le cône projetant du point M_1 la courbe (C_1) est tangent au plan μ_1, le long de la génératrice m_1; ce plan est coupé par le plan tangent infiniment voisin μ_1', suivant la droite m_1'. On a*

$$\lim. \frac{(\mu_1\mu_1')}{(m_1m_1')} = \frac{8}{3}\frac{\rho_1}{\tau_1}. \tag{15}$$

5. Dans l'espace (E), la courbe (C) est supposée une asymptotique d'une surface (S); la courbe (μ) la section de (S) par le plan μ; dans l'espace (E_1), la courbe (C_1) est alors une asymptotique de la surface (S_1) réciproque de (S) et le cône (M_1) est circonscrit à (S_1). Une branche (σ_1) de la courbe de contact est tangente en M_1 à l'asymptotique (C_1). On désigne par ρ_{σ_1} et τ_{σ_1} les rayons de courbure et de torsion au point M_1 de (σ_1). D'après l'égalité (15) on a:

$$\lim. \frac{(\mu_1\mu_1')}{(m_1m_1')} = \frac{8}{3}\frac{\rho_{\sigma_1}}{\tau_{\sigma_1}}. \tag{16}$$

Le théorème de Beltrami*, appliqué à l'asymptotique (C) et à la section (μ) de la surface (S), donne

$$R = \frac{3}{2}\rho. \tag{17}$$

Des égalités (13), (16), (17), on déduit:

$$9\frac{\rho_1}{\tau_1} = 8\frac{\rho_{\sigma_1}}{\tau_{\sigma_1}}. \tag{18}$$

L'asymptotique (C_1) et la branche (σ_1) ont même plan osculateur au point M_1, donc d'après la formule (8)

$$\frac{\tau_1}{\tau_{\sigma_1}} - 3 = -2\frac{\rho_{\sigma_1}}{\rho_1}. \tag{19}$$

Les égalités (18), (19) donnent

$$\rho_{\sigma_1} = \frac{3}{4}\rho_1, \qquad \tau_{\sigma_1} = \frac{2}{3}\tau_1.$$

*Nouv. Ann. Math, 2e série, t. IV, p. 258.

Par conséquent: *Un point M_1 d'une asymptotique (C_1), d'une surface (S_1) est le sommet d'un cône (M_1) circonscrit à (S_1): une branche (σ_1) de la courbe de contact est tangente à l'asymptotique (C_1) au point M_1. Si ρ_1 et τ_1, ρ_{σ_1} et τ_{σ_1} sont les rayons de courbure et de torsion des courbes (C_1) et (σ_1) au point M_1, on a:*

$$\rho_{\sigma_1} = \frac{3}{4}\rho_1, \tag{20}$$

$$\tau_{\sigma_1} = \frac{2}{3}\tau_1. \tag{21}$$

6. A la branche (σ_1), considérée comme faisant partie de l'espace (E_1), correspond dans l'espace réciproque (E) une branche (σ) de l'arête de rebıoussement de la développable circonscrite à la surface (S) le long de la courbe plane (μ). Cette branche (σ) est tangente à l'asymptotique (C) au point M et osculatrice au plan μ. Si ρ_σ et τ_σ sont les rayons de courbure et de torsion de (σ) au point M, les égalités (4) et (5) donnent

$$\frac{\rho}{\rho_\sigma} = \frac{\rho_1}{\rho_{\sigma_1}}\frac{\tau_{\sigma_1}}{\tau_1} = \frac{\rho_1}{\rho_{\sigma_1}}\frac{\tau}{\tau_\sigma}. \tag{22}$$

Des égalités (20), (21), (22), on déduit

$$\rho_\sigma = \frac{9}{8}\rho, \qquad \tau_\sigma = \frac{3}{2}\tau.$$

Ainsi: *L'arête de rebroussement de la développable circonscrite à une surface (S) le long de la section (μ) faite par le plan tangent μ au point M a une branche (σ) tangente en M à l'asymptotique (C) passant par ce point. Si ρ et τ, ρ_σ et τ_σ sont les rayons de courbure et de torsion au point M des courbes (C) et (σ), on a*

$$\rho_\sigma = \frac{9}{8}\rho, \tag{23}$$

$$\tau_\sigma = \frac{3}{2}\tau. \tag{24}$$

7. Les tangentes à la courbe (σ_1) coupent la surface (S_1) en des points d'une courbe (T_1) dont une branche (t_1) est tangente en M_1 à l'asymptotique (C_1) et osculatrice au plan μ_1. Si ρ_{t_1}, τ_{t_1} sont les rayons de courbure et de torsion au point M_1 de la branche (t_1) on a, d'après les formules (14) et (8):

$$\rho_{\sigma_1} = \frac{3}{4}\rho_{t_1} \qquad \frac{\tau_1}{\tau_{t_1}} - 3 = -2\frac{\rho_{t_1}}{\rho_1}. \tag{25}$$

Des égalités (20), (25), on déduit

$$\rho_{t_1} = \rho_1, \quad \tau_{t_1} = \tau_1. \tag{26}$$

Par suite: *Un point M_1 de la surface (S_1) est le sommet d'un cône (M_1) circonscrit à (S_1). La développable ayant pour arête de rebroussement la courbe de contact rencontre la surface (S_1) suivant une courbe (T_1) dont une branche (t_1) est tangente en M_1 à l'asymptotique (C_1) passant par ce point. Les courbures et les torsions au point M_1 de l'asymptotique (C_1) et de la branche (t_1) sont égales et de même signe.*

8. Si la courbe (C) est une biquadratique gauche de première espèce, la tangente m est une génératrice d'une quadrique (M) circonscrite à (C). La courbe (C_1) est l'arête de rebroussement d'une développable de quatrième classe circonscrite à la quadrique (H_1) réciproque de (H). On désigne par P, S, deux points quelconques de m, par π, σ les plans tangents à la quadrique (H) en ces points: par π_1, σ_1, P_1, S_1 les éléments de l'espace (E_1) correspondant aux éléments P, S, π, σ de (E). Le rayon de torsion τ de la biquadratique au point M est donné par la formule*

$$\tau = \frac{1}{3}\,\frac{MP.MS}{PS}\,\frac{\sin(\pi\sigma)}{\sin(\mu\pi).\sin(\mu\sigma)}. \tag{27}$$

Si τ_1 désigne le rayon de torsion au point M_1 de la courbe C_1 on a (1)

$$\tau\tau_1 = \frac{MP.MS}{PS}\,\frac{M_1P_1.M_1S_1}{P_1S_1}\,\frac{\sin(\pi\sigma)}{\sin(\mu\pi).\sin(\mu\sigma)}\,\frac{\sin(\pi_1\sigma_1)}{\sin(\mu_1\pi_1).\sin(\mu_1\sigma_1)}. \tag{28}$$

Des égalités (27), (28), on tire

$$\tau_1 = 3\,\frac{M_1P_1.M_1S_1}{P_1S_1}\,\frac{\sin(\pi_1\sigma_1)}{\sin(\mu_1\pi_1)\,.\,\sin(\mu_1\sigma_1)}. \tag{29}$$

De cette formule donnant le rayon de torsion τ_1 au point M_1 de la courbe (C), on déduit par les développements corrélatifs de ceux qui ont été employés pour la biquadratique gauche† les propriétés suivantes:

Si m_1, μ_1, τ_1 et m_2, μ_2, τ_2 désignent les tangentes, les plans osculateurs et les rayons de torsion en deux points correspondants M_1 et M_2 de l'arête de rebroussement de la développable circonscrite à deux quadriques; P_1, P_2, les points $\mu_2 m_1$, $\mu_1 m_2$; S_1, S_2 les intersections des tangentes m_1, m_2 avec un plan osculateur quelconque de la courbe, on a

$$\tau_1\tau_2 = 9\,\frac{\overline{P_1P_2}^2}{\sin^2(\mu_1\mu_2)}\,\frac{\overline{M_1S_1}^2\,.\,\overline{M_2S_2}^2}{\overline{P_1S_1}^2\,.\,\overline{P_2S_2}^2}. \tag{30}$$

*C. Servais. *Sur la courbure des biquadratiques gauches de première espèce*, N° 10. Nouv. Ann. Math., t. XI, juillet 1911.

†ibid., N° 12 et N° 15.

Si la quadrique inscrite dans la développable et ayant pour génératrices les tangentes m_1, m_2 est un paraboloïde, on a la relation

$$\tau_1\tau_2 = 9\,\frac{\overline{P_1P_2}^2}{\sin^2(\mu_1\mu_2)}. \tag{31}$$

Si les points correspondants M_1, M_2 de l'arête de rebroussement (C_1) sont tels que le plan osculateur en l'un passe par l'autre, on a

$$\tau_1\tau_2 = 9\,\frac{\overline{M_1M_2}^2}{\sin^2(\mu_1\mu_2)}. \tag{32}$$

ON CERTAIN SURFACES RELATED COVARIANTLY TO A GIVEN RULED SURFACE

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1. INTRODUCTION

As the basis for the theory of ruled surfaces Wilczynski* has used a system of two ordinary linear homogeneous differential equations

$$\text{(A)} \qquad \begin{aligned} y''+p_{11}y'+p_{12}z'+q_{11}y+q_{12}z=0, \\ z''+p_{21}y'+p_{22}z'+q_{21}y+q_{22}z=0, \end{aligned}$$

with coefficients p_{ik}, q_{ik} functions of an independent variable x.

To each *fundamental system*† of simultaneous solutions y_i, z_i $(i=1\ldots4)$, of (A), there corresponds a pair of directrix curves C_y, C_z of the ruled surface, generated respectively by the points P_y, P_z whose homogeneous space coordinates are (y_1, y_2, y_3, y_4) and (z_1, z_2, z_3, z_4).

By suitable transformations‡ system (A) may be reduced to another of the same form

$$\text{(F)} \qquad \begin{aligned} y''+p_{12}z'+q_{11}y+q_{12}z=0, \\ z''+p_{21}y'+q_{21}y+q_{22}z=0, \end{aligned}$$

in which $p_{11}=p_{22}=0$; $p'_{12}=2q_{12}$, $p'_{21}=2q_{21}$; $u_{12}=u_{21}=0$.§ For system (F), which is the flecnode canonical form of (A), P_y and P_z are the *flecnode points* of the generator g joining them, and C_y, C_z the two branches of the *flecnode curve* of S.||

The *complex points*¶ of g will be denoted by $P_{\bar{y}}$ and $P_{\bar{z}}$ and the two branches of the *complex curve* of S by $C_{\bar{y}}$ and $C_{\bar{z}}$.

The osculating planes at P_y, P_z of g to the two branches C_y, C_z of the flecnode curve, and those at $P_{\bar{y}}$, $P_{\bar{z}}$ of g to the two branches $C_{\bar{y}}$, $C_{\bar{z}}$ of the complex

*Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 126 *et seq.* This text hereafter referred to as W.

†W. p. 127. In this paper D is assumed $\neq 0$; the integrating ruled surface S is not a developable, W. p. 130.

‡W. pp. 116, 117.

§W. p. 96 (20).

||Cayley, *Mathematical Papers*, vol. II, p. 29: W. p. 149. Moreover, θ_4, p_{12}, p_{21}, are all assumed $\neq 0$, so that the two branches of the flecnode curve are distinct, and neither C_y nor C_z a straight line; W. p. 150.

¶W. p. 206.

curve of S, form a tetrahedron, which will be found to be non-degenerate in general. This tetrahedron will be referred to as the *flecnode-complex tetrahedron* of the integrating ruled surface S. As x varies, the edges of this tetrahedron will generate six new ruled surfaces, while the vertices will describe space curves, each lying on three of the new ruled surfaces. The line of intersection of the osculating planes to the flecnode curve will be denoted by g_F, and the ruled surface it generates by S_F; for the complex curve, the line of intersection of the osculating planes will be denoted by g_C, and the corresponding ruled surface by S_C.

It is the purpose in this paper to make a study of these six new ruled surfaces, associated covariantly with the original ruled surface S, and to find the differential equations of form (A) defining them. Special attention will be directed to a consideration of the surfaces S_F and S_C and of their relations to one another and to the original ruled surface; in particular, a necessary and sufficient condition for coincidence will be derived.

The methods and notations of Wilczynski will be used throughout. These have become familiar to mathematicians through his many contributions. The writer wishes here to acknowledge gratefully his indebtedness to Professor Wilczynski for his helpful suggestions and encouragement in the preparation of this paper.

2. THE DIFFERENTIAL EQUATIONS OF THE RULED SURFACES S, S_F AND S_C

The Surface S.

If the surface S is referred to $C_{\bar{y}}$, $C_{\bar{z}}$ as directrix curves, the differential equations of form (A) in $\bar{y}$, $\bar{z}$ where $\bar{y}$ and $\bar{z}$ are given by*

$$\bar{y}=\sqrt{p_{21}}\, y+\sqrt{p_{12}}\, z;\quad \bar{z}=\sqrt{p_{21}}\, y-\sqrt{p_{12}}\, z; \tag{1}$$

assume the form†

$$\begin{aligned} \bar{y}''+\bar{p}_{11}\bar{y}'+\bar{p}_{12}\bar{z}'+\bar{q}_{11}\bar{y}+\bar{q}_{12}\bar{z}&=0,\\ \bar{z}''+\bar{p}_{21}\bar{y}'+\bar{p}_{22}\bar{z}'+\bar{q}_{21}\bar{y}+\bar{q}_{22}\bar{z}&=0, \end{aligned} \tag{2}$$

where

$$\begin{aligned} &\bar{p}_{11}=\sqrt{p_{12}p_{21}}-\left(\frac{q_{12}}{p_{12}}+\frac{q_{21}}{p_{21}}\right),\quad \bar{p}_{22}=-\sqrt{p_{12}p_{21}}-\left(\frac{q_{12}}{p_{12}}+\frac{q_{21}}{p_{21}}\right),\\ &\bar{p}_{12}=\bar{p}_{21}=\left(\frac{q_{12}}{p_{12}}-\frac{q_{21}}{p_{21}}\right),\\ &\bar{q}_{11}=\bar{q}_{22}=\tfrac{1}{2}\left(\frac{\Delta_2}{p^2_{21}}+\frac{\Delta_1}{p^2_{12}}\right)=\tfrac{1}{2}(\delta_2+\delta_1),\\ &\bar{q}_{12}=\bar{q}_{21}=\tfrac{1}{2}\left(\frac{\Delta_2}{p^2_{21}}-\frac{\Delta_1}{p^2_{12}}\right)=\tfrac{1}{2}(\delta_2-\delta_1).\ddagger \end{aligned} \tag{3}$$

*W. p. 207.

†A. F. Carpenter, Trans. Amer. Math. Soc., vol. 16, 1915 (p. 517): hereafter referred to as A.F.C.

‡For the values of Δ_1, Δ_2 see W. p. 230 (6), in which $p_{11}=p_{22}=0$, $p'_{12}=2q_{12}$, $p'_{21}=2q_{21}$.

The Surface S_F.

The equations of the osculating planes to the flecnode curves C_y, C_z at the points P_y and P_z of the generator g of S are, respectively,

$$|x\,y\,y'\,y''|=0, \text{ and } |x\,z\,z'\,z''|=0.* \tag{4}$$

The differential equations of form (A) for the surface S_F in homogeneous plane coordinates u, v may be found as follows:

Denote by u_1, u_2, u_3, u_4 and by v_1, v_2, v_3, v_4 the cofactors of x_1, x_2, x_3, x_4 in the two equations of (4), respectively. Omitting subscripts, these may be written $u=|yy'y''|$, $v=|zz'z''|$. Then u and v may be taken as the plane coordinates of the two planes of (4) which intersect in g_F. To find the differential equations of form (A) satisfied by u and v, we eliminate y and z and their derivatives from u and v and from the derivatives of u and v. Making use of the canonical form (F), it is found that

$$\begin{aligned} u&=-p_{12}|yy'z'|-q_{12}|yy'z|,\\ v&=-p_{21}|zz'y'|-q_{21}|zz'y|, \end{aligned} \tag{5}$$

$$\begin{aligned} u'&=-3q_{12}|yy'z'|+(p_{12}q_{22}-q'_{12})|yy'z|,\\ v'&=-3q_{21}|zz'y'|+(p_{21}q_{11}-q'_{21})|zz'y|, \end{aligned} \tag{6}$$

$$\begin{aligned} u''&=(p_{12}q_{22}-4q'_{12})|yy'z'|+(3q_{12}q_{22}+p'_{12}q_{22}+p_{12}q'_{22}-q''_{12})|yy'z|+\Delta_1|zz'y|,\\ v''&=(p_{21}q_{11}-4q'_{21})|zz'y'|+(3q_{21}q_{11}+p'_{21}q_{11}+p_{21}q'_{11}-q''_{21})|zz'y|+\Delta_2|yy'z|. \end{aligned} \tag{7}$$

Employing (5) and (6) to eliminate $|yy'z'|$, $|yy'z|$, $|zz'y'|$ and $|zz'y|$ from (7), the system of equations satisfied by u and v may be written as

$$\begin{aligned} u''+a_{11}u'+a_{12}v'+b_{11}u+b_{12}v&=0,\\ v''+a_{21}u'+a_{22}v'+b_{21}u+b_{22}v&=0, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &a_{11}=\frac{d}{dx}\log\frac{1}{\Delta_1},\quad a_{12}=-p_{21}\frac{\Delta_1}{\Delta_2},\quad a_{21}=-p_{12}\frac{\Delta_2}{\Delta_1},\quad a_{22}=\frac{d}{dx}\log\frac{1}{\Delta_2},\\ &b_{11}=q_{22}-4\frac{q'_{12}}{p_{12}}+3\frac{q_{12}}{p_{12}}\frac{d}{dx}\log\Delta_1,\quad b_{12}=3q_{21}\frac{\Delta_1}{\Delta_2},\\ &b_{21}=3q_{12}\frac{\Delta_2}{\Delta_1},\quad b_{22}=q_{11}-4\frac{q'_{21}}{p_{21}}+3\frac{q_{21}}{p_{21}}\frac{d}{dx}\log\Delta_2. \end{aligned} \tag{9}$$

* $|xyy'y''|$ is here written for $\begin{vmatrix} x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ y'_1 & y'_2 & y'_3 & y'_4\\ y''_1 & y''_2 & y''_3 & y''_4 \end{vmatrix}$.

The system of differential equations of form (A) for the surface S_F in homogeneous point coordinates may be found in the following manner:
An arbitrary point on the tangent to C_y at P_y of g is given by $y'+\lambda y$, where λ is an arbitrary function of x. This will be the point, P_r, where the tangent to C_y at P_y meets g_F when λ is chosen so that $y'+\lambda y$ lies on the osculating plane to C_z at P_z of g. That is, $|y'+\lambda y,\ z,\ z',\ z''|$ must vanish. Using the second equation of (F), this reduces to $(\lambda p_{21}-q_{21})D=0$. As p_{21}, D are assumed in §1 to be different from zero, $\lambda=\dfrac{q_{21}}{p_{21}}$, and P_r is the point $\dfrac{q_{21}}{p_{21}}y+y'$ or, written homogeneously, $q_{21}y+p_{21}y'$.

Similarly, the tangent to C_z at P_z of g meets the generator g_F in the point $q_{12}z+p_{12}z'$, which we call P_s. The curves C_r, C_s described by P_r, P_s on S_F are now taken as directrix curves, and, writing,

$$r=\lambda y+y',\quad s=\mu z+z',\ \text{where}\ \lambda=\frac{q_{21}}{p_{21}},\ \mu=\frac{q_{12}}{p_{12}}, \tag{10}$$

we obtain, on successive differentiation,

$$\begin{aligned} &r'-\lambda r+p_{12}s=-\delta_2 y,\\ &s'+p_{21}r-\mu s=-\delta_1 z, \end{aligned} \tag{11}$$

$$\begin{aligned} &r''-\lambda r'+p_{12}s'-\lambda' r+p'_{12}s=-\delta_2 y'-\delta'_2 y,\\ &s''+p_{21}r'-\mu s'+p'_{21}r-\mu' s=-\delta_1 z'-\delta'_1 z. \end{aligned} \tag{12}$$

From (12) y, z, y', z' may be eliminated by the use of (10) and (11), and there results the system

$$\begin{aligned} &r''+c_{11}r'+c_{12}s'+d_{11}r+d_{12}s=0\\ &s''+c_{21}r'+c_{22}s'+d_{21}r+d_{22}s=0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &c_{11}=\frac{d}{dx}\log\frac{1}{\delta_2},\quad c_{12}=p_{12},\quad c_{21}=p_{21},\quad c_{22}=\frac{d}{dx}\log\frac{1}{\delta_1},\\ &d_{11}=q_{11}-2\lambda'+\lambda\frac{d}{dx}\log\delta_2,\quad d_{12}=p'_{12}+\lambda p_{12}-p_{12}\frac{d}{dx}\log\delta_2,\\ &d_{21}=p'_{21}+\mu p_{21}-p_{21}\frac{d}{dx}\log\delta_1,\quad d_{22}=q_{22}-2\mu'+\mu\frac{d}{dx}\log\delta_1.* \end{aligned} \tag{14}$$

The ruled surface S_F will be a developable if and only if the determinant $D(r'_k,\ s'_k,\ r_k,\ s_k)$, $(k=1\ldots.4)$, vanishes.† Substitute the values of r and s, as given in (10), in $|r's'rs|$, and the latter reduces to $\delta_1\delta_2 D$. As D, p_{12}, p_{21} are assumed $\neq 0$, S_F will be a developable if and only if either $\Delta_1=0$ or $\Delta_2=0$. But if $\Delta_1=0$, then C_y is a plane curve, while if $\Delta_2=0$, C_z is a plane curve.‡

*For the values of δ_1, δ_2 cf. equations (3), section 2.
†W. p. 130.
‡W. p. 230.

Hence we have the result:

The ruled surface S_F will be a developable if and only if either C_y or C_z is a plane curve.

The Surface S_C.

An arbitrary point on the tangent to $C_{\bar{y}}$ at $P_{\bar{y}}$ of g is given by $\bar{\lambda}\bar{y}+\bar{y}'$. This will be the point $P_{\bar{r}}$, where the tangent to $C_{\bar{y}}$ intersects g_C, provided $\bar{\lambda}\bar{y}+\bar{y}'$ is on the osculating plane to $C_{\bar{z}}$ at $P_{\bar{z}}$ of g. Accordingly, we must have

$$|\bar{\lambda}\bar{y}+\bar{y}'\ \bar{z}\ \bar{z}'\ \bar{z}''|=0.$$

Making use of the second equation of (2), this condition reduces to

$$4(\bar{\lambda}\bar{p}_{21}-\bar{q}_{21})p_{12}p_{21}|y'z'yz|=0, \text{ and } \bar{\lambda}=\frac{\bar{q}_{21}}{\bar{p}_{21}}.$$

The point $P_{\bar{r}}$ is thus given by $\bar{r}=\bar{q}_{21}\bar{y}+\bar{p}_{21}\bar{y}'$, and, in like manner, the point $P_{\bar{s}}$, where the tangent to $C_{\bar{z}}$ at $P_{\bar{z}}$ meets the generator g_C of S_C is given by $\bar{s}=\bar{q}_{12}\bar{z}+\bar{p}_{12}\bar{z}'$.

The differential equations of form (A) for the ruled surface S_C, in which $C_{\bar{r}}$ and $C_{\bar{s}}$ are the directrix curves, are found as follows:
Writing

$$\bar{r}=\bar{\lambda}\bar{y}+\bar{y}',\ \bar{s}=\bar{\lambda}\bar{z}+\bar{z}', \text{ where } \bar{\lambda}=\frac{\bar{q}_{12}}{\bar{p}_{12}}=\frac{\bar{q}_{21}}{\bar{p}_{21}}, \text{ [cf. (3)]}, \tag{15}$$

differentiating, and using (2) and (3), it is found that

$$\begin{aligned} &\bar{r}'+(\bar{p}_{11}-\bar{\lambda})\bar{r}+\bar{p}_{12}\bar{s}=(\bar{\lambda}'-\bar{q}_{11}+\bar{\lambda}\bar{p}_{11}-\bar{\lambda}^2)\bar{y}=q_1\bar{y}, \text{where } q_1=\bar{\lambda}'-\bar{q}_{11}+\bar{\lambda}\bar{p}_{11}-\bar{\lambda}^2,\\ &\bar{s}'+\bar{p}_{21}\bar{r}+(\bar{p}_{22}-\bar{\lambda})\bar{s}=(\bar{\lambda}'-\bar{q}_{22}+\bar{\lambda}\bar{p}_{22}-\bar{\lambda}^2)\bar{z}=q_2\bar{z}, \text{ where } q_2=\bar{\lambda}'-\bar{q}_{22}+\bar{\lambda}\bar{p}_{22}-\bar{\lambda}^2, \end{aligned} \tag{16}$$

while

$$\begin{aligned} &\bar{r}''+(\bar{p}_{11}-\bar{\lambda})\bar{r}'+\bar{p}_{12}\bar{s}'+(\bar{p}'_{11}-\bar{\lambda}')\bar{r}+\bar{p}'_{12}\bar{s}=q_1\bar{y}'+q'_1\bar{y},\\ &\bar{s}''+\bar{p}_{21}\bar{r}'+(\bar{p}_{22}-\bar{\lambda})\bar{s}'+\bar{p}'_{21}\bar{r}+(\bar{p}'_{22}-\bar{\lambda}')\bar{s}=q_2\bar{z}'+q'_2\bar{z}. \end{aligned} \tag{17}$$

Using (15), (16) to eliminate $\bar{y}$, $\bar{z}$, $\bar{y}'$, $\bar{z}'$, from (17) the desired system of differential equations for S_C takes the form

$$\begin{aligned} &\bar{r}''+m_{11}\bar{r}'+m_{12}\bar{s}'+n_{11}\bar{r}+n_{12}\bar{s}=0,\\ &\bar{s}''+m_{21}\bar{r}'+m_{22}\bar{s}'+n_{21}\bar{r}+n_{22}\bar{s}=0, \end{aligned} \tag{18}$$

where

$$\begin{aligned} &m_{11}=\bar{p}_{11}-\frac{d}{dx}\log q_1,\ m_{12}=\bar{p}_{12},\ m_{21}=\bar{p}_{21},\ m_{22}=\bar{p}_{22}-\frac{d}{dx}\log q_2,\\ &n_{11}=\bar{q}_{11}-\bar{\lambda}'+q_1\left(\frac{\bar{p}_{11}-\bar{\lambda}}{q_1}\right)',\ n_{12}=\bar{q}_{12}+q_1\left(\frac{\bar{p}_{12}}{q_1}\right)', \end{aligned} \tag{19}$$

$$n_{21}=\bar{q}_{21}+q_2\left(\frac{\bar{p}_{21}}{q_2}\right)', \quad n_{22}=\bar{q}_{22}-\bar{\lambda}'+q_2\left(\frac{\bar{p}_{22}-\bar{\lambda}}{q_2}\right)',$$

and where, moreover,

$$\left(\frac{\bar{p}_{11}-\bar{\lambda}}{q_1}\right)' = \frac{d}{dx}\left(\frac{\bar{p}_{11}-\bar{\lambda}}{q_1}\right), \text{ etc.}$$

The surface S_C, characterized by equations (18), will be a developable if and only if the determinant $|\bar{r}'\bar{s}'\bar{r}\bar{s}|=0$. By the use of (15) this reduces to the condition, $q_1q_2|\bar{y}'\bar{z}'\bar{y}\bar{z}|=0$. Substituting the values of $\bar{y}$ and $\bar{z}$ in terms of y and z, as given in (1), this condition becomes $4p_{12}p_{21}q_1q_2|y'z'yz|=0$. It has been assumed, however, that p_{12}, p_{21} and $|y'z'yz|$ are all different from zero, and hence $|\bar{r}'\bar{s}'\bar{r}\bar{s}|$ vanishes only when either q_1 or q_2 vanishes.

Hence, the ruled surface S_C will be a developable if and only if either q_1 or q_2 is zero.

3. The Flecnode-Complex Tetrahedron. The Associated Ruled Surfaces

For convenience of reference, the following formulae are set down:

(20) $\bar{y}=\sqrt{p_{21}}\,y+\sqrt{p_{12}}\,z, \bar{z}=\sqrt{p_{21}}\,y-\sqrt{p_{12}}\,z$, cf. (1);

(21) $r=q_{21}\,y+p_{21}\,y',\ s=q_{12}\,z+p_{12}\,z'$, cf. (10);

(22) $\bar{r}=\bar{q}_{21}\,\bar{y}+\bar{p}_{21}\,\bar{y}',\ s=\bar{q}_{12}\,\bar{z}+\bar{p}_{12}\,\bar{z}'$, cf. (15);

wherein, by (3), it is noted that $\bar{p}_{12}=\bar{p}_{21}$, and $\bar{q}_{12}=\bar{q}_{21}$.

The point, $2\sqrt{p_{21}}\,\bar{q}_{21}\,y+2\dfrac{\bar{p}_{21}}{\sqrt{p_{21}}}\,r$, is evidently on the tangent to C_y at P_y. Using the value of r from (21),

$$2\sqrt{p_{21}}\,\bar{q}_{21}y+2\frac{\bar{p}_{21}}{\sqrt{p_{21}}}\,r$$

becomes

$$2\sqrt{p_{21}}\,\bar{q}_{21}\,y+2\frac{\bar{p}_{21}}{\sqrt{p_{21}}}(q_{21}y+p_{21}y').$$

From (20), $\bar{y}+\bar{z}=2\sqrt{p_{21}}\,y$, while

$$\bar{y}'+\bar{z}' = \frac{2}{\sqrt{p_{21}}}(q_{21}y+p_{21}\,y').$$

Thus the point, $2\sqrt{p_{21}}\,\bar{q}_{21}\,y+2\dfrac{\bar{p}_{21}}{\sqrt{p_{21}}}\,r$ is actually the point given by $\bar{q}_{21}(\bar{y}+\bar{z})+\bar{p}_{21}(\bar{y}'+\bar{z}')$, which, from (22), is evidently the point $\bar{r}+\bar{s}$. Similarly, $\bar{r}-\bar{s}$ is found to be a point on the tangent to C_z at P_z of g; and $P_{\bar{r}+\bar{s}}$ and $P_{\bar{r}-\bar{s}}$ are points on the generator g_C.

Again, it may be proved that $\bar{y}'$, a point on the tangent to $C_{\bar{y}}$ at $P_{\bar{y}}$ of g, is a point on g_F. For, from (20), on differentiation,

$$\bar{y}' = \frac{1}{\sqrt{p_{21}}}(q_{21}y+p_{21}y')+\frac{1}{\sqrt{p_{12}}}(q_{12}z+p_{12}z'),$$

which is the point $\frac{1}{\sqrt{p_{21}}} r + \frac{1}{\sqrt{p_{12}}} s$, and, accordingly, a point on g_F. Similarly, it may be shown that $\bar{z}'$ is the point $\frac{1}{\sqrt{p_{21}}} r - \frac{1}{\sqrt{p_{12}}} s$, also on g_F.

Accordingly, we have the important result:

The tangents to the flecnode curve at P_y, P_z of g, and the tangents to the complex curve at $P_{\bar{y}}$, $P_{\bar{z}}$ of g, intersect not only g_F and g_C respectively, but also both pairs intersect g_F and g_C.

Denote by A, B, C, D the vertices of the *flecnode-complex tetrahedron,* where ABC, ABD are the osculating planes to C_y, C_z at P_y, P_z respectively. The edges AB, CD of the flecnode-complex tetrahedron are the generators g_F and g_C of the ruled surfaces S_F and S_C respectively. A is the point of intersection of g_F with ACD; the osculating plane ACD contains the tangent $P_{\bar{y}}P_{\bar{y}}'$, and $P_{\bar{y}}'$ has been proved to be a point on g_F, hence the point A is the point $P_{\bar{y}}'$. In like manner, B is the point $P_{\bar{z}}'$, and, for analogous reasons, C and D are the points $P_{\bar{r}+\bar{s}}$ and $P_{\bar{r}-\bar{s}}$, respectively.

Equations (13) characterize the ruled surface S_F, where C_r, C_s are used as directrix curves, and equations (18) the ruled surface S_C with $C_{\bar{r}}$, $C_{\bar{s}}$ as directrix curves. We propose now to use as directrix curves those generated by the vertices A, B, C, D of the flecnode-complex tetrahedron, and to find the differential equations of form (A) defining the six ruled surfaces generated by the six edges of the tetrahedron $ABCD$.

As the computation involved in obtaining these systems is tedious, an outline will be given in the case of one of them, and the results alone set down for the remainder.

For brevity, denote the points A, B, C and D by P_α, P_β, P_γ and P_δ respectively, and let us find the differential equations of form (A) for the ruled surface generated by the edge AC.

We may write

$$\alpha = \bar{y}', \tag{23}$$
$$\gamma = \bar{r}+\bar{s} = \bar{\lambda}(\bar{y}+\bar{z})+\bar{y}'+\bar{z}', \text{ where, as in (15), } \bar{\lambda} = \frac{\bar{q}_{12}}{\bar{p}_{12}} = \frac{\bar{q}_{21}}{\bar{p}_{21}}.$$

Differentiating, and making use of (2), (3), (14), and (16) it is found that

$$\begin{aligned} &\alpha' + (\bar{p}_{11} - \bar{p}_{12})\alpha + \bar{p}_{12}\gamma = -\delta_1\bar{y}, \\ &\gamma' + (\bar{p}_{11} - \bar{p}_{22})\alpha + (\bar{p}_{12} + \bar{p}_{22} - \bar{\lambda})\gamma = q_2(\bar{y}+\bar{z}). \end{aligned} \tag{24}$$

A second differentiation gives

$$\begin{aligned} &\alpha'' + (\bar{p}_{11} - \bar{p}_{12})\alpha' + \bar{p}_{12}\gamma' + (\bar{p}_{11}' - \bar{p}'_{12})\alpha + \bar{p}_{12}'\gamma = -\delta'_1\bar{y} - \delta_1\bar{y}', \\ &\gamma'' + (\bar{p}_{11} - \bar{p}_{22})\alpha' + (\bar{p}_{12} + \bar{p}_{22} - \bar{\lambda})\gamma' + (\bar{p}'_{11} - \bar{p}'_{22})\alpha + (\bar{p}'_{12} + \bar{p}'_{22} - \bar{\lambda}')\gamma \\ &\quad = q'_2(\bar{y}+\bar{z}) + q_2(\bar{y}'+\bar{z}'). \end{aligned} \tag{25}$$

From (25) $\bar{y}$, $\bar{z}$, $\bar{y}'$, $\bar{z}'$, may be eliminated by the use of (23) and (24), and the resulting system for α and γ takes the form

$$
(26)\quad
\begin{aligned}
&\alpha''+(\bar{p}_{11}-\bar{p}_{12}+c_{22})\alpha'+\bar{p}_{12}\gamma'+\left\{\delta_1\left(\frac{\bar{p}_{11}-\bar{p}_{12}}{\delta_1}\right)'+\delta_1\right\}\alpha+\delta_1\left(\frac{\bar{p}_{12}}{\delta_1}\right)'\gamma=0,\\
&\gamma''+(\bar{p}_{11}-\bar{p}_{22})\alpha'+(m_{22}+\bar{p}_{12})\gamma'+\left\{q_2\left(\frac{\bar{p}_{11}-\bar{p}_{22}}{q_2}\right)'+\bar{\lambda}(\bar{p}_{11}-\bar{p}_{22})\right\}\alpha\\
&\qquad+\left\{q_2\left(\frac{\bar{p}_{22}+\bar{p}_{12}-\bar{\lambda}}{q_2}\right)'-q_2+\bar{\lambda}(\bar{p}_{12}+\bar{p}_{22}-\bar{\lambda})\right\}\gamma=0,
\end{aligned}
$$

where, as formerly, $\left(\dfrac{\bar{p}_{11}-\bar{p}_{12}}{\delta_1}\right)'=\dfrac{d}{dx}\left(\dfrac{\bar{p}_{11}-\bar{p}_{12}}{\delta_1}\right)$, etc.

In a similar manner, β and δ, α and δ, and β and γ satisfy systems (27), (28) and (29) respectively, as listed below:

$$
(27)\quad
\begin{aligned}
&\beta''+(\bar{p}_{22}+\bar{p}_{21}+c_{11})\beta'+\bar{p}_{21}\delta'+\left\{\delta_2\left(\frac{\bar{p}_{22}+\bar{p}_{21}}{\delta_2}\right)'+\delta_2\right\}\beta+\delta_2\left(\frac{\bar{p}_{21}}{\delta_2}\right)'\delta=0,\\
&\delta''+(\bar{p}_{11}-\bar{p}_{22})\beta'+(m_{11}-\bar{p}_{21})\delta'+\left\{q_1\left(\frac{\bar{p}_{11}-\bar{p}_{22}}{q_1}\right)'+\bar{\lambda}(\bar{p}_{11}-\bar{p}_{22})\right\}\beta\\
&\qquad+\left\{q_1\left(\frac{\bar{p}_{11}-\bar{p}_{21}-\bar{\lambda}}{q_1}\right)'-q_1+\bar{\lambda}(\bar{p}_{11}-\bar{p}_{21}-\bar{\lambda})\right\}\delta=0,
\end{aligned}
$$

$$
(28)\quad
\begin{aligned}
&\alpha''+(\bar{p}_{11}+\bar{p}_{12}+c_{11})\alpha'-\bar{p}_{12}\delta'+\left\{\delta_2\left(\frac{\bar{p}_{11}+\bar{p}_{12}}{\delta_2}\right)'+\delta_2\right\}\alpha-\delta_2\left(\frac{\bar{p}_{12}}{\delta_2}\right)'\delta=0,\\
&\delta''+(\bar{p}_{11}-\bar{p}_{22})\alpha'+(m_{22}-\bar{p}_{12})\delta'+\left\{q_2\left(\frac{\bar{p}_{11}-\bar{p}_{22}}{q_2}\right)'+\bar{\lambda}(\bar{p}_{11}-\bar{p}_{22})\right\}\alpha\\
&\qquad+\left\{q_2\left(\frac{\bar{p}_{22}-\bar{p}_{12}-\bar{\lambda}}{q_2}\right)'-q_2+\bar{\lambda}(\bar{p}_{22}-\bar{p}_{12}-\bar{\lambda})\right\}\delta=0,
\end{aligned}
$$

$$
(29)\quad
\begin{aligned}
&\beta''+(\bar{p}_{22}-\bar{p}_{21}+c_{22})\beta'+\bar{p}_{21}\gamma'+\left\{\delta_1\left(\frac{\bar{p}_{22}-\bar{p}_{21}}{\delta_1}\right)'+\delta_1\right\}\beta+\delta_1\left(\frac{\bar{p}_{21}}{\delta_1}\right)'\gamma=0,\\
&\gamma''+(\bar{p}_{22}-\bar{p}_{11})\beta'+(m_{11}+\bar{p}_{21})\gamma'+\left\{q_1\left(\frac{\bar{p}_{22}-\bar{p}_{11}}{q_1}\right)'+\bar{\lambda}(\bar{p}_{22}-\bar{p}_{11})\right\}\beta\\
&\qquad+\left\{q_1\left(\frac{\bar{p}_{11}+\bar{p}_{21}-\bar{\lambda}}{q_1}\right)-q_1+\bar{\lambda}(\bar{p}_{11}+\bar{p}_{21}-\bar{\lambda})\right\}\gamma=0.
\end{aligned}
$$

The differential equations of form (A) for S_F and S_C where C_r, C_s and $C_{\bar r}$, $C_{\bar s}$ were used respectively as directrix curves are given already by (13) and (18). If, however, the curves described by P_α, P_β and by P_γ, P_δ are used as directrix curves for S_F, S_C respectively, the differential equations of form (A) for these two ruled surfaces assume forms (30) and (31):

$$
(30)\quad \begin{aligned}
&\alpha''+\{\bar p_{11}+\tfrac{1}{2}(c_{11}+c_{22})\}\alpha'+\{\bar p_{12}+\tfrac{1}{2}(c_{11}-c_{22})\}\beta'+\{\bar q_{11}+\bar p'_{11}+\tfrac{1}{2}\bar p_{11}(c_{11}+c_{22})\\
&\qquad+\tfrac{1}{2}\bar p_{21}(c_{11}-c_{22})\}\alpha+\{\bar q_{12}+\bar p'_{12}+\tfrac{1}{2}\bar p_{12}(c_{11}+c_{22})+\tfrac{1}{2}\bar p_{22}(c_{11}-c_{22})\}\beta=0,\\
&\beta''+\{\bar p_{21}+\tfrac{1}{2}(c_{11}-c_{22})\}\alpha'+\{\bar p_{22}+\tfrac{1}{2}(c_{11}+c_{22})\}\beta'+\{\bar q_{21}+\bar p'_{21}+\tfrac{1}{2}\bar p_{11}(c_{11}-c_{22})\\
&\qquad+\tfrac{1}{2}\bar p_{21}(c_{11}+c_{22})\}\alpha+\{\bar q_{22}+\bar p'_{22}+\tfrac{1}{2}\bar p_{12}(c_{11}-c_{22})+\tfrac{1}{2}\bar p_{22}(c_{11}+c_{22})\}\beta=0,
\end{aligned}
$$

$$
(31)\quad \begin{aligned}
&\gamma''+\{\tfrac{1}{2}(m_{11}+m_{22})+m_{12}\}\gamma'+\tfrac{1}{2}(m_{11}-m_{22})\delta'+\tfrac{1}{2}(n_{11}+n_{12}+n_{21}+n_{22})\gamma\\
&\qquad+\tfrac{1}{2}(n_{11}-n_{12}+n_{21}-n_{22})\delta=0,\\
&\delta''+\tfrac{1}{2}(m_{11}-m_{22})\gamma'+\{\tfrac{1}{2}(m_{11}+m_{22})-m_{21}\}\delta'+\tfrac{1}{2}(n_{11}+n_{12}-n_{21}-n_{22})\gamma\\
&\qquad+\tfrac{1}{2}(n_{11}-n_{12}-n_{21}+n_{22})\delta=0.
\end{aligned}
$$

4. A Certain One-Parameter Family of Curves on S. Reciprocal Considerations for the Surface S_F. The Conjugate Family on S, and the Corresponding Laplace Transformation

An arbitrary point P of the generator g of S is given by $\alpha y+\beta z$, where the ratio $\alpha:\beta$ is arbitrary. The plane of P and g_F determines a tangent, t, to S at P. Thus, one obtains a one-parameter family of curves on S such that the tangents to them along g intersect g_F. We proceed to investigate this family of curves.

The plane determined by P and g_F is given by the equation

$$(32)\qquad |x,\ \alpha y+\beta z,\ q_{21}y+p_{21}y',\ q_{12}z+p_{12}z'|=0.$$

As the independent variable changes, P describes a curve on S and $\dfrac{d}{dx}(\alpha y+\beta z)$, which is a point on the tangent, t, to this curve at P, will lie on the plane (32), provided

$$(33)\qquad \left|\frac{d}{dx}(\alpha y+\beta z),\ \alpha y+\beta z,\ q_{21}y+p_{21}y',\ q_{12}z+p_{12}z'\right|=0.$$

On expanding, simplifying, and dividing out by $D=|y'z'yz|\neq 0$, this condition reduces to

$$(34)\qquad \frac{\alpha'}{\alpha}-\frac{\beta'}{\beta}=\frac{1}{2}\left(\frac{p'_{21}}{p_{21}}-\frac{p'_{12}}{p_{12}}\right).$$

The integration of (34) gives

$$(35)\qquad \frac{\alpha}{\beta}=c\sqrt{\frac{p_{21}}{p_{12}}},\ \text{where } c \text{ is an arbitrary constant.}$$

It may be noted that for $c=\pm 1$, the complex points, $P_{\bar y}$, $P_{\bar z}$, of g are obtained.

This important result may be stated as follows:

The one-parameter family of curves on S, characterized by the property that the tangents to them along the generator g intersect g_F, are those determined by the condition of forming a constant cross-ratio with the complex curves of S.

We may now obtain the interpretation of the point on t given by $\frac{d}{dx}(\alpha y+\beta z)$, where $\alpha:\beta$ satisfies (35). Take $\alpha=c\sqrt{p_{21}}$, $\beta=\sqrt{p_{12}}$, then

$$\frac{d}{dx}(\alpha y+\beta z)=\frac{c}{\sqrt{p_{21}}}(q_{21}y+p_{21}y')+\frac{1}{\sqrt{p_{12}}}(q_{12}z+p_{12}z')=\frac{\alpha}{p_{21}}r+\frac{\beta}{p_{12}}s,$$

which is clearly the point of intersection of t with g_F.

Again, suppose $\alpha:\beta$ satisfies the relation (35); the point on the tangent t, given by $2\bar{q}_{12}(\alpha y+\beta z)+2\bar{p}_{12}\frac{d}{dx}(\alpha y+\beta z)$, reduces, by the use of (20) and (22), to $c(\bar{r}+\bar{s})+(\bar{r}-\bar{s})$, which is seen to be a point on the generator g_C.

Thus, we have deduced the further result:

The one-parameter family of curves on S, determined by (35), *have the property that the tangents to them along g, intersect not only g_F but also g_C.*

Similarly a point P' of g_F and g determine a tangent t' to S_F at P'. Thus, in a reciprocal manner, is obtained a one-parameter family of curves on S_F such that the tangents to them along g_F intersect the generator g of S.

In this case, let P' of g_F be given by $\gamma r+\delta s$. We proceed to find the ratio $\gamma:\delta$ so that this tangent t' may intersect g.

An arbitrary point on t' is given by

$$\frac{d}{dx}(\gamma r+\delta s)+\lambda(\gamma r+\delta s). \tag{36}$$

In order that t' may intersect g, the point given by (36) must, for a proper choice of λ, be a point of g, and its coordinates expressible therefore in terms of y and z alone. If, in (36), are substituted the values of r and s in terms of y, z, y', z', and the conditions imposed that the resulting expression shall contain only terms in y and z, then γ and δ will be found to satisfy the equation

$$\frac{\gamma'}{\gamma}-\frac{\delta'}{\delta}+p_{21}\frac{\gamma}{\delta}-p_{12}\frac{\delta}{\gamma}-\frac{3}{2}\left(\frac{p'_{12}}{p_{12}}-\frac{p'_{21}}{p_{21}}\right)=0. \tag{37}$$

In (37) replace $\frac{\gamma}{\delta}$ by $\left(\frac{p_{12}}{p_{21}}\right)^{\frac{3}{2}}\pi$, and there results the Riccati equation in π,

$$\pi'+p_{12}\sqrt{\frac{p_{12}}{p_{21}}}\,\pi^2=p_{21}\sqrt{\frac{p_{21}}{p_{12}}}. \tag{38}$$

From the characteristic property of any four solutions of a Riccati equation we deduce that:

Any four points of the generator g_F, which are such that the tangents to the curves they describe (at the points where these curves meet g_F) intersect the generator g, have the property that their anharmonic ratio is constant.

The Laplace Transformation.

The one-parameter family of curves on S, characterized by the property that the tangents to them along g intersect g_F, and, as we have proved, g_C also, has associated with it a conjugate family, which we proceed to find.

An arbitrary curve of the above family is that described by P_t where $t=\bar{y}+c\bar{z}$, c being an arbitrary constant. In this investigation the complex curves are taken as directrix curves for S, and hence the system of differential equations used is that given by (2).

The two asymptotic directions at P_t are given by the generator g, and by the line joining P_t to P_α where $\alpha=2\bar{y}'+\bar{p}_{11}\bar{y}+\bar{p}_{12}\bar{z}+c(2\bar{z}'+\bar{p}_{21}\bar{y}+\bar{p}_{22}\bar{z})$.* If P_α is joined to P_t' this line will intersect g in some point P_u. To find u, it may be noted that the coordinates of an arbitrary point on the line $P_\alpha P_t'$ is given by $\alpha+\lambda t'$. In order that this may be the point P_u, $\alpha+\lambda t'$ must be expressible in terms of $\bar{y}$ and $\bar{z}$ alone; that is

$$(39)\qquad 2\bar{y}'+\bar{p}_{11}\bar{y}+\bar{p}_{12}\bar{z}+c(2\bar{z}'+\bar{p}_{21}\bar{y}+\bar{p}_{22}\bar{\ })+\lambda(\bar{y}'+c\bar{z}')=S\bar{y}+R\bar{z}.$$

From this it follows that $\lambda=-2$, and

$$(40)\qquad u=(\bar{p}_{11}+c\bar{p}_{21})\bar{y}+(\bar{p}_{12}+c\bar{p}_{22})\bar{z}.$$

Thus $\alpha=2\bar{y}'+\bar{p}_{11}\bar{y}+\bar{p}_{12}\bar{z}+c(2\bar{z}'+\bar{p}_{21}\bar{y}+\bar{p}_{22}\bar{z})=u+2t'$, and v, the harmonic conjugate of t' with respect to u and α, is found to be equal to $u+t'$. The direction conjugate to $P_tP_{t'}$ is thus given by the line P_tP_v.

If the surface S is referred to C_t, C_u as directrix curves, a system of differential equations of form (A) in t and u may be found. One of these is sufficient for present purposes.

We have

$$(41)\qquad \begin{aligned} t&=\bar{y}+c\bar{z},\\ u&=(\bar{p}_{11}+c\bar{p}_{21})\bar{y}+(\bar{p}_{12}+c\bar{p}_{22})\bar{z}. \end{aligned}$$

On differentiating we obtain,

$$(42)\qquad \begin{aligned} t'&=\bar{y}'+c\bar{z}',\\ u'&=(\bar{p}_{11}+c\bar{p}_{21})\bar{y}'+(\bar{p}_{12}+c\bar{p}_{22})\bar{z}'+(\bar{p}'_{11}+c\bar{p}'_{21})\bar{y}+(\bar{p}'_{12}+c\bar{p}'_{22})\bar{z}, \end{aligned}$$

and,

$$(43)\qquad t''=\bar{y}''+c\bar{z}''=-(\bar{p}_{11}+c\bar{p}_{21})\bar{y}'-(\bar{p}_{12}+c\bar{p}_{22})\bar{z}'-(\bar{q}_{11}+c\bar{q}_{21})\bar{y}-(\bar{q}_{12}+c\bar{q}_{22})\bar{z}.$$

This latter equation may be written, using the second from (42),

$$(44)\qquad t''+u'=(\bar{p}'_{11}+c\bar{p}'_{21}-\bar{q}_{11}-c\bar{q}_{21})\bar{y}+(\bar{p}'_{12}+c\bar{p}'_{22}-\bar{q}_{12}-c\bar{q}_{22})\bar{z}.$$

The use of (41) enables $\bar{y}$ and $\bar{z}$ to be eliminated from (44), and one of the two equations of form (A), satisfied by t and u, takes the form

$$(45)\qquad t''+u'+s_{11}t+s_{12}u=0,$$

*W. p. 146.

where

$$s_{11}=\frac{-(\bar{p}'_{11}+c\bar{p}'_{21}-\bar{q}_{11}-c\bar{q}_{21})(\bar{p}_{12}+c\bar{p}_{22})+(\bar{p}'_{12}+c\bar{p}'_{22}-\bar{q}_{12}-c\bar{q}_{22})(\bar{p}_{11}+c\bar{p}_{21})}{\bar{p}_{12}+c\bar{p}_{22}-c(\bar{p}_{11}+c\bar{p}_{21})},\tag{46}$$

$$s_{12}=\frac{c(\bar{p}'_{11}+c\bar{p}'_{21}-\bar{q}_{11}-c\bar{q}_{21})-(\bar{p}'_{12}+c\bar{p}'_{22}-\bar{q}_{12}-c\bar{q}_{22})}{\bar{p}_{12}+c\bar{p}_{22}-c(\bar{p}_{11}+c\bar{p}_{21})}.$$

The tangent $P_t\,P_v$, as x varies, will now generate a developable surface, a Laplace transform of S, and we proceed to find a point on the edge of regression.

The coordinates of an arbitrary point P_ρ on P_tP_v is given by $v+\lambda t$; this will be a point on the edge of regression if $\frac{d}{dx}(v+\lambda t)$ is again expressible in terms of t and v alone, that is, if

$$v'+\lambda t'+\lambda' t=Mt+Nv.\tag{47}$$

But $v=u+t'$, whence (47) becomes

$$u'+t''+\lambda t'+\lambda' t=Mt+\mathrm{N}v,\tag{48}$$

and, by the use of (45),

$$-s_{11}t-s_{12}u+\lambda(v-u)+\lambda' t=Mt+Nv,\tag{49}$$

whence $\lambda=-s_{12}$, and $\rho=v-s_{12}t$, and the locus of P_ρ gives the corresponding Laplace transform of S.

5. The Flecnode-Complex Quadric. Its Relation to the Osculating Hyperboloid H of S

It may be noted that the tangents along g to the one-parameter family of curves on S given in §4, equation (35), constitute the rulings of the second kind of the quadric determined by the three generators g, g_F, and g_C.

Let us use as tetrahedron of reference that determined by P_y, P_z, P_ρ, P_σ,* where $\rho=2y'+p_{12}z$, $\sigma=2z'+p_{21}y$, and define a point given by an expression of the form, $x_1y+x_2z+x_3\rho+x_4\sigma$, as having the coordinates x_1, x_2, x_3, x_4.

As already noted, an arbitrary curve of the family cuts g in the point P_t, where $t=c\sqrt{p_{21}}\,y+\sqrt{p_{12}}\,z$. A general point on the quadric determined by g, g_F and g_C will then be given by $\alpha t+\beta t'$. Since

$$\begin{aligned}\alpha t+\beta t'&=\alpha(c\sqrt{p_{21}}\,y+\sqrt{p_{12}}\,z)+\beta\left(c\sqrt{p_{21}}\,y'+\sqrt{p_{12}}\,z'+c\frac{q_{21}}{\sqrt{p_{21}}}y+\frac{q_{12}}{\sqrt{p_{12}}}z\right)\\&=\left(\alpha c\sqrt{p_{21}}+\beta c\frac{q_{21}}{\sqrt{p_{21}}}-\frac{\beta}{2}p_{21}\sqrt{p_{12}}\right)y+\left(\alpha\sqrt{p_{12}}-\frac{\beta}{2}c\,p_{12}\sqrt{p_{21}}+\beta\frac{q_{12}}{\sqrt{p_{12}}}\right)z\\&\qquad+\tfrac{1}{2}\beta c\sqrt{p_{21}}\,\rho+\tfrac{1}{2}\beta\sqrt{p_{12}}\,\sigma,\end{aligned}$$

it follows that the homogeneous point-coordinates of an arbitrary point on the quadric determined by g, g_F, and g_C are

*W. p. 191.

$$(50)\quad \begin{aligned} x_1 &= \alpha c\sqrt{p_{21}}+\beta c\frac{q_{21}}{\sqrt{p_{21}}}-\tfrac{1}{2}\beta p_{21}\sqrt{p_{12}}, \quad x_3=\tfrac{1}{2}\beta c\sqrt{p_{21}},\\ x_2 &= \alpha\sqrt{p_{12}}-\tfrac{1}{2}\beta c p_{12}\sqrt{p_{21}}+\beta\frac{q_{12}}{\sqrt{p_{12}}}, \quad x_4=\tfrac{1}{2}\beta\sqrt{p_{12}}. \end{aligned}$$

The elimination of α, β and c leads to the equation of the quadric determined by the three generators g, g_F and g_C in the form

$$(51)\quad x_1x_4-x_2x_3-\left\{p_{12}x_3^2-2\left(\frac{q_{12}}{p_{12}}-\frac{q_{21}}{p_{21}}\right)x_3x_4-p_{21}x_4^2\right\}=0,$$

which we shall call the *Flecnode-Complex Quadric** of the given integrating ruled surface S.

If from the quadratic covariants C_1†, C_2‡ and C_5§ of weights 1, 2 and 5 respectively, we form the covariant,

$$K=-C_1-\frac{\theta_9}{2\theta_{10}}C_2+\frac{C_5}{\theta_4},$$

of degree 2 and weight 1, we find that K is expressible in the form, $y\sigma-z\rho+\bar{y}\bar{z}-2\left(\frac{q_{12}}{p_{12}}-\frac{q_{21}}{p_{21}}\right)yz$. If, with each of the points P_y, P_z, P_ρ, P_σ, $P_{\bar{y}}$, $P_{\bar{z}}$ there is associated its polar plane with respect to the osculating quadric, $x_1x_4-x_2x_3=0$||, and if, from the equations of these polar planes, an expression is built up precisely as K is built up out of the coordinates of the points, this expression equated to zero gives exactly the equation of the *flecnode-complex quadric.*

The quadric (51) is tangent to the osculating quadric, $x_1x_4-x_2x_3=0$, along g, so that g, counted twice, is a part of the intersection of the two quadrics. The two rulings of the second kind on the flecnode-complex quadric, which at the same time lie on the osculating quadric, and therefore comprise the rest of the intersection, pass through the two points of g given by

$$p_{12}p^2_{21}y+\left(p_{12}q_{21}-p_{21}q_{12}\pm\sqrt{(p_{12}q_{21}-p_{21}q_{12})^2+p_{12}^3p_{21}^3}\right)z.$$

A few results will now be pointed out which follow readily from the above considerations.

An arbitrary point on g_F is given by $\lambda r+\mu s$, which may be written, using (21), in the form,

$$(\lambda q_{21}-\tfrac{1}{2}\mu p_{12}p_{21})y+(\mu q_{12}-\tfrac{1}{2}\lambda p_{12}p_{21})z+\tfrac{1}{2}\lambda p_{21}\rho+\tfrac{1}{2}\mu p_{12}\sigma.$$

The homogeneous point-coordinates of an arbitrary point on g_F, referred to the local tetrahedron of reference are therefore,

$$x_1=\lambda q_{21}-\tfrac{1}{2}\mu p_{12}p_{21},\quad x_2=\mu q_{12}-\tfrac{1}{2}\lambda p_{12}p_{21},\quad x_3=\tfrac{1}{2}\lambda p_{21},\quad x_4=\tfrac{1}{2}\mu p_{12}.$$

*A. F. Carpenter: Bull. Amer. Math. Soc., vol. XXVII, 1921, p. 3. *The Flecnode-Complex Quadric* is the quadric of the family K_6, associated with the generator g of S, whose equation is invariant in form.

†W. p. 125. ‡W. p. 125. §W. p. 208, formula (36). ||W. p. 191.

This will be a point on H if λ and μ satisfy the relation

$$\lambda^2 p_{21} - 2\lambda\mu\left(\frac{q_{12}}{p_{12}} - \frac{q_{21}}{p_{21}}\right) - \mu^2 p_{12} = 0. \tag{52}$$

The two values of $\lambda : \mu$, satisfying (52), when substituted in $\lambda r + \mu s$, determine the two points of intersection of g_F with H. These two points are harmonic conjugates with respect to P_r and P_s if $\frac{q_{12}}{p_{12}} - \frac{q_{21}}{p_{21}} = \frac{1}{2}\frac{p'_{12}}{p_{12}} - \frac{1}{2}\frac{p'_{21}}{p_{21}} = 0$, *i.e.*, if $\frac{p_{12}}{p_{21}} =$ const., that is, if S belongs to a linear complex.* The generator g_F will be a tangent to H if the two roots of (52) are equal, that is, if

$$(p_{12}q_{21} - p_{21}q_{12})^2 + p^3{}_{12}p^3{}_{21} = 0.\dagger \tag{53}$$

The osculating planes to C_y and C_z at P_y and P_z of g are given by‡

$$p_{12}x_2 + p^2{}_{12}x_3 - 2q_{12}x_4 = 0,$$

$$p_{21}x_1 - 2q_{21}x_3 + p^2{}_{21}x_4 = 0, \text{ where again } u_{12} = u_{21} = 0.$$

The null-points of these planes by means of the osculating linear complex, $p_{12}\omega_{13} + p_{21}\omega_{42} = 0$§, are respectively $(-p_{12}p_{21}, -2q_{12}, 0, -p_{12})$ and $(2q_{12}, p_{12}p_{21}, p_{21}, 0)$||, and their line of junction intersects the hyperboloid H in two points. These latter form an harmonic group with the former if S belongs to a linear complex, while their line of junction is a tangent to H if (53) holds. Since $p_{11} = p_{22} = u_{12} = u_{21} = 0$, condition (53) may be written as $\theta_4^3\theta_9^2 + 16\theta_{10}^3 = 0$.¶

6. The Osculating Planes to the Complex Curves. Relations with Respect to the Osculating Linear Complex

The osculating plane to $C_{\bar{y}}$ at $P_{\bar{y}}$ of g is given by $|x\bar{y}\bar{y}'\bar{y}''| = 0$. Using the first equation from (2), and (20), this equation becomes

$$|x\bar{y}\bar{y}'y'| + q|x\bar{y}\bar{y}'y| = 0, \text{ where } q = \frac{\bar{q}_{12}}{\bar{p}_{12}} + \frac{q_{21}}{p_{21}} = \frac{\bar{q}_{21}}{\bar{p}_{21}} + \frac{q_{21}}{p_{21}}. \tag{54}$$

From the substitution of the values of $\bar{y}$, $\bar{y}'$ in terms of y and z and of their derivatives, there results

$$p_{12}|xzy'z'| + \sqrt{p_{12}p_{21}}\,|xyy'z'| - qp_{12}|xyzz'| - 4\frac{q_{12}p_{21} - q_{21}p_{12} + qp_{12}p_{21}}{\sqrt{p_{12}p_{21}}}|xyzy'| = 0. \tag{55}$$

But, the semi-covariants of system (F), reduce to

$$\rho = 2y' + p_{12}z, \quad \sigma = 2z' + p_{21}y,$$

and, using these to determine y', z' in terms of y, z, ρ and σ, (55) may then be written

$$\begin{aligned} p_{12}|xz\rho\sigma| + \tfrac{1}{2}(\bar{p}_{11} - \bar{p}_{22})|xy\rho\sigma| - \{2qp_{12} + \tfrac{1}{2}p_{12}(\bar{p}_{11} - \bar{p}_{22})\}\,|xyz\sigma| \\ - (\bar{p}_{11} - \bar{p}_{22})\{\bar{p}_{12} + q + \tfrac{1}{4}(\bar{p}_{11} - \bar{p}_{22})\}\,|xyz\rho| = 0. \end{aligned} \tag{56}$$

*W. p. 167. †W. p. 214. ‡W. p. 214. §W. p. 206. ||W. p. 215. ¶W. p. 214.

As the tetrahedron of reference is that determined by P_y, P_z, P_ρ, P_σ the equation of the osculating plane to $C_{\bar{y}}$ at $P_{\bar{y}}$ of g thus becomes

$$(57)\qquad \begin{aligned} &p_{12}x_1-\tfrac{1}{2}(\bar{p}_{11}-\bar{p}_{22})x_2-\{2qp_{12}+\tfrac{1}{2}p_{12}(\bar{p}_{11}-\bar{p}_{22})\}x_3\\ &+(\bar{p}_{11}-\bar{p}_{22})\{\bar{p}_{12}+q+\tfrac{1}{4}(\bar{p}_{11}-\bar{p}_{22})\}x_4=0.\end{aligned}$$

In a similar manner, the equation of the osculating plane to $C_{\bar{z}}$ at $P_{\bar{z}}$ of g may be found; and, thus referred to the local tetrahedron of reference, the osculating planes to the two branches of the complex curve may be set down as

$$(58)\qquad \begin{aligned} &4p_{12}x_1-2(\bar{p}_{11}-\bar{p}_{22})x_2-\{8p_{12}q+2p_{12}(\bar{p}_{11}-\bar{p}_{22})\}x_3\\ &+(\bar{p}_{11}-\bar{p}_{22})\ (4\bar{p}_{12}+4q+\bar{p}_{11}-\bar{p}_{22})x_4=0,\\[1ex] &4p_{12}x_1+2(\bar{p}_{11}-\bar{p}_{22})x_2-\{8p_{12}q-2p_{12}(\bar{p}_{11}-\bar{p}_{22})\}x_3\\ &-(\bar{p}_{11}-\bar{p}_{22})\ (4\bar{p}_{12}+4q-\bar{p}_{11}+\bar{p}_{22})x_4=0,\end{aligned}$$

where, as already noted, $q=\dfrac{\bar{q}_{12}}{\bar{p}_{12}}+\dfrac{q_{21}}{p_{21}}=\dfrac{\bar{q}_{21}}{\bar{p}_{21}}+\dfrac{q_{21}}{p_{21}}$, and where, moreover, $u_{12}=u_{21}=0$.

Carpenter* found that, when the flecnode curves were non-rectilinear, distinct, plane-curves, so also were the two branches of the complex curve. A necessary and sufficient condition for C_y and C_z to be plane curves is $\Delta_1=\Delta_2=0$.† This condition enables equations (58) to be reduced to those given by him.‡

Suppose, however, we assume $\delta_1=\delta_2$. From (3) it follows that $\bar{q}_{12}=\bar{q}_{21}=0$, and, accordingly, q in (58) reduces to $\dfrac{q_{21}}{p_{21}}$; and, since $\bar{p}_{11}-\bar{p}_{22}=2\sqrt{p_{12}p_{21}}$, equations (58) become

$$(59)\qquad \begin{aligned} &\frac{1}{\sqrt{p_{21}}}x_1-\frac{1}{\sqrt{p_{12}}}x_2-\left(\sqrt{p_{12}}+\frac{p'_{21}}{p_{21}\sqrt{p_{21}}}\right)x_3+\left(\sqrt{p_{21}}+\frac{p'_{12}}{p_{12}\sqrt{p_{12}}}\right)x_4=0,\\ &\frac{1}{\sqrt{p_{21}}}x_1+\frac{1}{\sqrt{p_{12}}}x_2+\left(\sqrt{p_{12}}-\frac{p'_{21}}{p_{21}\sqrt{p_{21}}}\right)x_3+\left(\sqrt{p_{21}}-\frac{p'_{12}}{p_{12}\sqrt{p_{12}}}\right)x_4=0,\end{aligned}$$

the same as those given by Carpenter.§

If the first equation of (59) is subtracted from the second and the result multiplied by $\frac{1}{2}(p_{12})^{\frac{3}{2}}$, and again the two equations of (59) added and the result multiplied by $\frac{1}{2}(p_{21})^{\frac{3}{2}}$, there result, respectively,

$$(60)\qquad \begin{aligned} &p_{12}x_2+p^2_{12}x_3-p'_{12}x_4=0,\\ &p_{21}x_1-p'_{21}x_3+p^2_{21}x_4=0,\end{aligned}$$

which are the osculating planes to the two branches of the flecnode curve of S,||

*A.F.C. pp. 518, 519.

†W. p. 230.

‡A.F.C. p. 519, equations (36) and (37).

§A.F.C. p. 519.

||W. p. 214.

and we have thus seen that the two planes given by (59), and the two planes given by (60), have a common line of intersection.

Combining the results obtained above, with the harmonic property of the four points P_y, P_z, $P_{\bar{y}}$, $P_{\bar{z}}$ of the generator g, we may state in a more general form the results Carpenter* obtained (when the flecnode curves and therefore the complex curves were plane curves), as follows:

If the two branches of the flecnode curve and the two branches of the complex curve are distinct, and if the integrating ruled surface S is subject to the condition, $\delta_1-\delta_2=0$, *then the four osculating planes have a common line of intersection. Moreover, these four planes form an harmonic pencil of planes, in which the former two are separated harmonically by the latter two.*

The osculating linear complex, $p_{12}\omega_{13}+p_{21}\omega_{42}=0$,† written in the expanded form,

$$-p_{12}x_3y_1+p_{21}x_4y_2+p_{12}x_1y_3-p_{21}x_2y_4=0, \tag{61}$$

gives the point-plane correspondence,

$$u_1=-p_{12}x_3,\ u_2=p_{21}x_4,\ u_3=p_{12}x_1,\ u_4=-p_{21}x_3; \tag{62}$$

and the plane-point correspondence,

$$x_1=p_{21}u_3,\ x_2=-p_{12}u_4,\ x_3=-p_{21}u_1,\ x_4=p_{12}u_2. \tag{63}$$

The null-points of the two osculating planes to the complex curve given by (58), as determined by the osculating linear complex, (61), are, therefore, respectively,

$$\begin{aligned}&(2p_{21}\{4q+\bar{p}_{11}-\bar{p}_{22}\},\ \{\bar{p}_{11}-\bar{p}_{22}\}\{4p_{12}+4q+\bar{p}_{11}-\bar{p}_{22}\},\ 4p_{21},\ 2\{\bar{p}_{11}-\bar{p}_{22}\}),\\ &(2p_{21}\{4q-\bar{p}_{11}+\bar{p}_{22}\},-\{\bar{p}_{11}-\bar{p}_{22}\}\{4\bar{p}_{12}+4q-\bar{p}_{11}+\bar{p}_{22}\},\ 4p_{21},\ -2\{\bar{p}_{11}-\bar{p}_{22}\}).\end{aligned} \tag{64}$$

Denote these two points of (64) by P_ν and $P_{\nu'}$. An arbitrary point on their line of junction is $\lambda\nu+\mu\nu'$. This line intersects the osculating hyperboloid H in the two points resulting from the values of $\lambda:\mu$ obtained from

$$\bar{p}_{12}\lambda^2+(\bar{p}_{11}-\bar{p}_{22})\lambda\mu-\bar{p}_{12}\mu^2=0. \tag{65}$$

These points of intersection will coincide, that is, the line joining the null-points of the two osculating planes to the complex curve will be tangential to H, if $(\bar{p}_{11}-\bar{p}_{22})^2+4\bar{p}^2_{12}=0$, which condition may be expressed as $\theta^3{}_4\theta^2{}_9+16\theta^3{}_{10}=0$. This interesting result may be thus stated:

If $\theta_4^3\theta_9^2+16\theta_{10}^3=0$, *the ruled surface S has the following characteristic property. The planes, which osculate the complex curve at the two points of its intersection with any generator, intersect in a line which is tangent to the osculating hyperboloid.*

It is verified easily that the coordinates of the two points, given by (64), satisfy equation (51), so that we may add the following theorem:

The ruled surface S has the following characteristic property. The null-points of the planes, which osculate the complex curve at the two points of its intersection with any generator, as determined by the osculating linear complex, lie on the flecnode-complex quadric of S.

*A.F.C. p. 520. †W. p. 206.

7. A Necessary and Sufficient Condition for the Flecnode-Complex Tetrahedron to Become Degenerate. The Resulting Invariant Relation

The generator g_F intersects the generator g_C if the four points P_r, P_s, $P_{\bar{r}}$ and $P_{\bar{s}}$ are coplanar. If this is so, the determinant $|r^{(k)}, s^{(k)}, \bar{r}^{(k)}, \bar{s}^{(k)}|$, $(k=1\ldots4)$, must vanish. By referring to equations (21) and (22), it is seen that this condition may be expressed as

$$\left| \frac{q_{21}}{p_{21}} y+y', \frac{q_{12}}{p_{12}} z+z', \frac{\bar{q}_{21}}{\bar{p}_{21}} \bar{y}+\bar{y}', \frac{\bar{q}_{12}}{\bar{p}_{12}} \bar{z}+\bar{z}' \right| =0. \tag{66}$$

By the use of relations (20) $\bar{y}$, $\bar{z}$, $\bar{y}'$, $\bar{z}'$ are expressed in terms of y, z, y' and z'; when, in this manner, $\bar{y}$, $\bar{z}$, $\bar{y}'$, and $\bar{z}'$, are eliminated from (66), and the resulting equation simplified, the condition for the intersection of g_F and g_C reduces to

$$\sqrt{p_{12}p_{21}} \cdot \frac{\bar{q}_{12}}{\bar{p}_{12}} \cdot \frac{\bar{q}_{21}}{\bar{p}_{21}} |yzy'z'| =0. \tag{67}$$

As p_{12}, p_{21} and $D=|yzy'z'|$, are all assumed to be different from zero, it follows that

$$\bar{q}_{12}=\bar{q}_{21}=0, \tag{68}$$

which, by the use of (3), becomes

$$\delta_1-\delta_2=0. \tag{69}$$

That this necessary condition for the intersection of g_F and g_C is also sufficient is at once evident.

It may be proved, moreover, that, if $\delta_1-\delta_2=0$, then g_F and g_C coincide. For, an arbitrary point on g_C is given by $\lambda\bar{r}+\mu\bar{s}$. From (22), if $\delta_1-\delta_2=0$, that is, if $\bar{q}_{12}=\bar{q}_{21}=0$, it follows that $\lambda\bar{r}+\mu\bar{s}$ is expressible in terms of $\bar{y}'$ and $\bar{z}'$ alone. In §3 it was proved that $\bar{y}' = \frac{1}{\sqrt{p_{21}}} r + \frac{1}{\sqrt{p_{12}}} s$ and $\bar{z}' = \frac{1}{\sqrt{p_{21}}} r - \frac{1}{\sqrt{p_{12}}} s$, and accordingly $\lambda\bar{r}+\mu\bar{s}$ becomes expressible in terms of r and s alone. Thus, if $\delta_1-\delta_2=0$, an arbitrary point on g_C is a point of g_F. If necessary, it might be shown conversely that an arbitrary point on g_F is a point of g_C. *Hence, if* $\delta_1-\delta_2=0$, g_F *and* g_C *coincide.*

Condition (69) may be written, using the relations given in (3), in the form

$$\frac{p_{21}}{p_{12}} \Delta_1 - \frac{p_{12}}{p_{21}} \Delta_2 = 0.* \tag{70}$$

Carpenter† found that

$$\delta_1-\delta_2 = \frac{8}{3} \theta_4\sqrt{\theta_4}\, (6\theta_4\theta^2_{10} - \theta_4\theta_{20} - \theta_{10}\theta_{14}),$$

*Bull. Amer. Math. Soc., vol. 28, No. 6, 1922, p. 283. Notice of paper by Professor A. F. Carpenter.

†A.F.C. p. 513.

and, as θ_4 has been assumed to be different from zero, the condition expressed in (69) may be written in the form of the invariant relation,

$$6\theta_4\theta^2_{10} - \theta_4\theta_{20} - \theta_{10}\theta_{14} = 0. \tag{71}$$

Combining the results here obtained with those of section 6, we have the general theorem:

A necessary and sufficient condition, that the planes osculating the two branches of the flecnode curve at the two points where the curve intersects any generator g, may have a common line of intersection with the planes osculating the two branches of the complex curve where it meets the same generator, is that the invariant relation $6\theta_4\theta^2_{10} - \theta_4\theta_{20} - \theta_{10}\theta_{14} = 0$, *shall hold. Moreover, if this condition is satisfied, then these four planes form a harmonic pencil of planes, in which the former two are separated harmonically by the latter two.*

The writer has considered also certain questions which arise naturally in connection with the invariants of the systems of differential equations of form (A) defining the ruled surfaces S_F and S_C. In particular, when the integrating ruled surface, S, is subject to the condition $6\theta_4\theta^2_{10} - \theta_4\theta_{20} - \theta_{10}\theta_{14} = 0$, in which case S_F and S_C coincide, the possibility of S and S_F having the same relative invariants has been examined, and some preliminary results obtained. He hopes to be able to extend these considerations and develop more general properties.

THE DETERMINATION OF SURFACES CHARACTERIZED BY A REDUCIBLE DIRECTRIX QUADRIC

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Introduction

If the asymptotic curves $\Gamma(u)$, $\Gamma(v)$ of a surface Σ belong to linear complexes, then it is known that the osculating ruled surfaces $R(u)$, $R(v)$ have straight line directrices $\delta^{(\eta)}_{(u)}$, $\delta^{(\zeta)}_{(u)}$ and $\delta^{(\eta')}_{(v)}$, $\delta^{(\zeta')}_{(v)}$ respectively, and that the loci of these directrices are complementary reguli $\rho(u)$, $\rho(v)$ on a quadric surface Q*. Under certain conditions expressible by invariant equations the quadric Q is reducible. The author has already discussed elsewhere (Transactions of the Royal Society of Canada, vol. IX, 1915) the case when Q reduces to coincident planes. In this paper a study is made of the case when Q reduces to distinct planes.

The procedure employed in this study is based on the theory of a completely integrable system of partial differential equations of the second order in two independent variables, which has been developed by E. J. Wilczynski in his publications on the projective differential geometry of surfaces†. Apart from auxiliary analysis our results may be summarized under the headings: The determination of analytical criteria for the surfaces in question in the form of invariant relations; the application of these relations to transform the defining differential equations of Σ to a normal form; the integration of the normal equations, and the determination of the equations of Σ explicitly in terms of u and v, the parameters of the asymptotic curves.

I

Defining differential equations, simplification of the integrability conditions, etc.

The formulae developed in this section form the basis of the discussion in the sequel. These formulae hinge about the study of a system of partial differential equations which can be reduced to the form‡:

$$(\Sigma) \qquad y_{11}+2by_2+fy=0, \; y_{22}+2a'y_1+gy=0,$$

*C. T. Sullivan, *Properties of Surfaces whose Asymptotic Curves belong to Linear Complexes.* Trans. Amer. Math. Soc., vol. 15, 1914. When referring to this paper hereafter, we shall designate it by S for brevity.

†E. J. Wilczynski, *Projective Differential Geometry of Curved Surfaces.* Trans. Amer. Math. Soc., vols. (8-10), (1907-1909). There are five memoirs by the same author under the same title, and hereafter we shall refer to them briefly as First Memoir, Second Memoir, etc.

‡First Memoir, p. 246.

where

$$y_1=\frac{\partial y}{\partial u},\ y_2=\frac{\partial y}{\partial v},\ \text{etc.},$$

and where the coefficients satisfy the integrability conditions

$$(1)\qquad \begin{cases} a'_{11}+g_1+2ba_2'+4a'b_2=0,\\ b_{22}+f_2+2b_1a'+4a_1'b=0,\\ g_{11}-f_{22}-4a_1'f-2a'f_1+4b_2g+2bg_2=0. \end{cases}$$

Subject to these conditions the equations (Σ) possess four linearly independent solutions $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ which we interpret as the homogeneous coordinates of a point P_y in space. The locus of P_y as u and v vary is called an integral surface Σ of the system.

The four functions

$$(2)\qquad y,\ z=y_1,\ \rho=y_2,\ \sigma=y_{12}$$

determine a non-degenerate local tetrahedron of reference $P_yP_zP_\rho P_\sigma$. The local coordinate system is then defined in such a manner that the local coordinates of the point

$$x^{(i)}=x^{(1)}y^{(i)}+x^{(2)}z^{(i)}+x^{(3)}\rho^{(i)}+x^{(4)}\sigma^{(i)},\quad (i=1,2,3,4)$$

are $(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)})$.

From (Σ) and (2) we find at once the following:

$$(3)\qquad \begin{cases} z_1=-2b\rho-fy,\ z_2=\sigma,\ \rho_1=\sigma,\ \rho_2=-2a'z-gy,\\ \sigma_1=-(2b_2+f)\rho+4a'bz+(2bg-f_2)y,\\ \sigma_2=4a'b\rho-(2a_1'+g)z+(2a'f-g_1)y. \end{cases}$$

Denote by y', z', ρ', σ' and y'', z'', ρ'', σ'' the values of y, z, ρ, σ corresponding to the values $(u+\delta u, v)$ and $(u, v+\delta v)$ respectively of (u, v). Then we deduce from (3) the relations:

$$(4)\qquad \begin{cases} y'=y+z\delta u+\dots,\\ z'=z-(2b\rho+fy)\delta u+\dots,\\ \rho'=\rho+\sigma\delta u+\dots,\\ \sigma'=\sigma+\{-(2b_2+f)\rho+4a'bz+(2bg-f_2)y\}\delta u+\dots,\\ y''=y+\rho\delta v+\dots,\\ z''=z+\sigma\delta v+\dots,\\ \rho''=\rho-(2a'z+gy)\delta v+\dots,\\ \sigma''=\sigma+\{4a'b\rho-(2a'_1+g)z+(2a'f-g_1)y\}\delta v+\dots. \end{cases}$$

If the coordinates of an arbitrary point P_x be denoted by

$$x^{(i)};\ x^{(i)\prime};\ x^{(i)\prime\prime}\ (i=1,2,3,4)$$

when referred to the local tetrahedra of the points $P_{y(u,v)}$, $P_{y(u+\delta u,v)}$ and $P_{y(u,v+\delta v)}$ respectively, then from the definition of the local coordinate system and equations (4), the following equations of transformation can be calculated:

$$
(5)\quad \left\{
\begin{aligned}
kx^{(1)} &= x^{(1)\prime}+\{-fx^{(2)\prime}+(2bg-f_2)x^{(4)\prime}\}\,\delta u+\ldots,\\
kx^{(2)} &= x^{(2)\prime}+\{x^{(1)\prime}+4a'bx^{(4)\prime}\}\,\delta u+\ldots,\\
kx^{(3)} &= x^{(3)\prime}-\{2bx^{(2)\prime}+(2b_2+f)x^{(4)\prime}\}\,\delta u+\ldots,\\
kx^{(4)} &= x^{(4)\prime}+x^{(3)\prime}\delta u+\ldots,\\
lx^{(1)} &= x^{(1)\prime\prime}+\{-gx^{(3)\prime\prime}+(2a'f-g_1)x^{(4)\prime\prime}\}\,\delta v+\ldots,\\
lx^{(2)} &= x^{(2)\prime\prime}-\{2a'x^{(3)\prime\prime}+(2a_1'+g)x^{(4)\prime\prime}\}\,\delta v+\ldots,\\
lx^{(3)} &= x^{(3)\prime\prime}+\{x^{(1)\prime\prime}+4a'bx^{(4)\prime\prime}\}\,\delta v+\ldots,\\
lx^{(4)} &= x^{(4)\prime\prime}+x^{(2)\prime\prime}\delta v+\ldots,\\
mx^{(1)\prime} &= x^{(1)}+\{fx^{(2)}+(f_2-2bg)x^{(4)}\}\,\delta u+\ldots,\\
mx^{(2)\prime} &= x^{(2)}-\{x^{(1)}+4a'bx^{(4)}\}\,\delta u+\ldots,\\
mx^{(3)\prime} &= x^{(3)}+\{2bx^{(2)}+(2b_2+f)x^{(4)}\}\,\delta u+\ldots,\\
mx^{(4)\prime} &= x^{(4)}-x^{(3)}\delta u,\\
nx^{(1)\prime\prime} &= x^{(1)}+\{gx^{(3)}+(g_1-2a'f)x^{(4)}\}\,\delta v+\ldots,\\
nx^{(2)\prime\prime} &= x^{(2)}+\{2a'x^{(3)}+(2a'_1+g)x^{(4)}\}\,\delta v+\ldots,\\
nx^{(3)\prime\prime} &= x^{(3)}-\{x^{(1)}+4a'bx^{(4)}\}\,\delta v+\ldots,\\
nx^{(4)\prime\prime} &= x^{(4)}-x^{(2)}\delta v+\ldots,
\end{aligned}
\right.
$$

where k, l, m, n are factors of proportionality.

The line $P_y\ P_z$ generates the osculating ruled surface $R(u)$ as the point P_y describes the asymptotic curve $u=\text{const}$. Likewise the line $P_y\,P_\rho$ generates the osculating ruled surface $R(v)$ as P_y describes the asymptotic curve $v=\text{const}$. Two of the edges of the local tetrahedron at P_y are therefore the two generators of $R(u)$ and $R(v)$ associated with P_y.

Two invariants θ, θ' of the system (Σ) that are of fundamental importance in the study of $R(u)$ and $R(v)$ are defined as follows*:

$$
(6)\quad \left\{
\begin{aligned}
\theta &= 64a_1'^2-128a'\{a'_{11}+2a'(f+b_2)\},\\
\theta' &= 64b_2^2-128b\ \{b_{22}+2b\ (g+a_1')\}.
\end{aligned}
\right.
$$

In fact, if P_η, P_ζ are the flecnodes on the generator P_yP_z and $P_{\eta'}$, $P_{\zeta'}$ those on the generator P_yP_ρ, then†

$$
(7)\quad \left\{
\begin{aligned}
&y=16a'(\eta+\zeta)=16b(\eta'+\zeta'),\\
&z=B\eta+A\zeta,\ \rho=B'\eta'+A'\zeta',
\end{aligned}
\right.
$$

where

$$
\left\{
\begin{aligned}
&A=8a_1'-\sqrt{\theta},\ B=8a_1'+\sqrt{\theta},\\
&A'=8b_2-\sqrt{\theta'},\ B'=8b_2+\sqrt{\theta'}.
\end{aligned}
\right.
$$

*Second Memoir, pp. 81-82. †Second Memoir, pp. 83-84.

We next introduce two invariant equations characteristic of surfaces whose asymptotic curves belong to linear complexes; they are defined as follows*:

$$\frac{\partial^2 \log a'}{\partial u \partial v} = \frac{\partial^2 \log b}{\partial u \partial v} = 4a'b. \tag{8}$$

These equations lead at once to the relations

$$\begin{cases} a'a'_{112} - a'_{11}a_2' - 12a'^2a_1'b - 4a'^3b_1 = 0, \\ a'a'_{122} - a_1'a'_{22} - 12a'^2a_2'b - 4a'^3b_2 = 0, \\ bb_{112} - b_{11}b_2 - 12a'b^2b_1 - 4a_1'b^3 = 0, \\ bb_{122} - b_1b_{22} - 12a'b^2b_2 - 4a_2'b^3 = 0. \end{cases} \tag{9}$$

The integrability conditions (1) can be written in a more compact, and for our purpose a much more useful, form in virtue of (6) and (8). For (6) can be written

$$f = \frac{1}{4}\left(\frac{a_1'}{a'}\right)^2 - \frac{1}{256}\frac{\theta}{a'^2} - \frac{1}{2}\frac{a'_{11}}{a'} - b_2,$$

$$g = \frac{1}{4}\left(\frac{b_2}{b}\right)^2 - \frac{1}{256}\frac{\theta'}{b^2} - \frac{1}{2}\frac{b_{22}}{b} - a_1'.$$

By differentiating these and taking account of (8) and (9) we obtain:

$$f_1 = \frac{1}{2}\frac{a_1'}{a'}\left(\frac{a'_{11}}{a'} - \frac{a_1'^2}{a'^2}\right) - \frac{1}{256}\left(\frac{\theta}{a'^2}\right)_1 - \frac{1}{2}\left(\frac{a'_{111}}{a'} - \frac{a_1'a'_{11}}{a'^2}\right) - b_{12},$$

$$f_2 = -\frac{1}{256}\left(\frac{\theta}{a'^2}\right)_2 - 2a'b_1 - 4a_1'b - b_{22},$$

$$g_1 = -\frac{1}{256}\left(\frac{\theta'}{b^2}\right)_1 - 2a_2'b - 4a'b_2 - a'_{11},$$

$$g_2 = \frac{1}{2}\frac{b_2}{b}\left(\frac{b_{22}}{b} - \frac{b_2^2}{b^2}\right) - \frac{1}{256}\left(\frac{\theta'}{b^2}\right)_2 - \frac{1}{2}\left(\frac{b_{222}}{b} - \frac{b_2b_{22}}{b^2}\right) - a'_{12}.$$

Hence the first and second integrability conditions become

$$\frac{\partial}{\partial u}\left(\frac{\theta'}{b^2}\right) = 0, \quad \frac{\partial}{\partial v}\left(\frac{\theta}{a'^2}\right) = 0. \tag{10}$$

The expressions given above for f_2 and g_1 now reduce to

$$f_2 = -2a'b_1 - 4a_1'b - b_{22},$$

and

$$g_1 = -2a_2'b - 4a'b_2 - a'_{11}$$

respectively.

Differentiate the first of these with respect to v and the second with respect to u, and form the difference $g_{11} - f_{22}$. If cognizance be taken of (9), we find

*S, p. 175.

$$g_{11}-f_{22}=2a'b\left(\frac{a_1'a_2'}{a'}-\frac{b_1b_2}{b}\right)-(a_{111}'-b_{222}).$$

Now form the expressions

$$2(bg_2-a'f_1)=\left(\frac{a_1'^3}{a'^2}-\frac{b_2^3}{b^2}\right)-2\left(\frac{a_1'a_{11}'}{a'}-\frac{b_2b_{22}}{b}\right)$$
$$+(a_{111}'-b_{222})-2a'b\left(\frac{a_1'a_2'}{a'^2}-\frac{b_1b_2}{b^2}\right)$$
$$+\frac{1}{128}\left\{a'\left(\frac{\theta}{a'^2}\right)_1-b\left(\frac{\theta'}{b^2}\right)_2\right\},$$

and

$$4(b_2g-a_1'f)=-\left(\frac{a_1'^3}{a'^2}-\frac{b_2^3}{b^2}\right)+\frac{1}{64}\left(\frac{a_1'\theta}{a'^2}-\frac{b_2\theta'}{b^2}\right)$$
$$+2\left(\frac{a_1'a_{11}'}{a'}-\frac{b_2b_{22}}{b}\right).$$

On introducing these in the third of equations (1), we find that the third integrability condition reduces to

(11) $$\frac{1}{a'}\frac{\partial\theta}{\partial u}-\frac{1}{b}\frac{\partial\theta'}{\partial v}=0.$$

Again, we infer from equations (8) that

(12) $$\frac{\partial^2\log\left(\frac{a'}{b}\right)}{\partial u\partial v}=0;$$

hence, in virtue of (10), the relation

$$\frac{\partial^2\log\left(\frac{\theta'}{\theta}\right)}{\partial u\partial v}=0.$$

These relations imply that

$$\frac{a'}{b}=\frac{\alpha(u)}{\beta(v)},\quad \frac{\theta'}{\theta}=\frac{A(u)}{B(v)}.$$

Now a transformation of the type

$$\bar{u}=\alpha(u),\quad \bar{v}=\beta(v),\quad \bar{y}=c\sqrt{\alpha_1\beta_2}\,y$$

replaces (Σ) by a system of the same form; and the new values of a', b, θ, θ' are given by the equations*

*First Memoir, p. 256.

$$\bar{a}' = \frac{\alpha_1}{\beta_2^2} a', \qquad \bar{b} = \frac{\beta_2}{\alpha_1^2} b,$$

$$\bar{\theta}' = \frac{\theta'}{\alpha_1^4}, \qquad \bar{\theta} = \frac{\theta}{\beta_2^4}.$$

It is therefore apparent that a transformation can be found which leaves the form of (Σ) unaltered and at the same time effects either of the simplifications

$$a' = b \text{ or } \theta' = \theta. \tag{13}$$

II

Analytical criteria for directrix quadric reducible to distinct planes.

It has already been pointed out that conditions (8) imply the existence of a directrix quadric. It is also readily seen from section I that the conditions of our problem can only be satisfied provided a', b, θ', θ are distinct from zero. This statement follows from the general theory combined with (8) and the integrability conditions; it is also apparent from (8) and geometrical considerations. We infer from these conditions that $R(u)$ and $R(v)$ have distinct straight line directrices. We require the additional conditions to insure that consecutive directrices of the same regulus intersect; and these we proceed to develop. Referred to the local tetrahedron of P_y the equations of $\delta_{(u)}^{(\eta)}$, $\delta_{(u)}^{(\zeta)}$ and $\delta_{(v)}^{(\eta')}$, $\delta_{(v)}^{(\zeta')}$ are found to be*

$$\tag{14} \begin{cases} \alpha(u) \equiv 16a'x^{(1)} + (8a_1' \mp \sqrt{\theta})x^{(2)} + 32a'^2bx^{(4)} = 0, \\ \alpha'(u) \equiv 16a'x^{(3)} + (8a_1' \mp \sqrt{\theta})x^{(4)} = 0, \\ \beta(v) \equiv 16bx^{(1)} + (8b_2 \mp \sqrt{\theta'})x^{(3)} + 32a'b^2x^{(4)} = 0, \\ \beta'(v) \equiv 16bx^{(2)} + (8b_2 \mp \sqrt{\theta'})x^{(4)} = 0 \end{cases}$$

respectively, where like signs of the radical correspond. The equations of the directrices of the neighbouring scrolls are

$$\alpha(u) + \delta\alpha(u) = 0, \qquad \alpha'(u) + \delta\alpha'(u) = 0,$$

$$\beta(v) + \delta\beta(v) = 0, \qquad \beta'(v) + \delta\beta'(v) = 0$$

respectively.

In order to express these in terms of the local tetrahedron of P_y we make use of equations (5). By direct calculation it is found that

*S, p. 182.

$$
(15)\quad \begin{cases}
\delta\alpha(u)=[(16a_1'-A)x^{(1)}+(16a'f+A_1)x^{(2)}-32a'^2bx^{(3)}+\{32(a'^2b)_1-4a'bA \\
\qquad -16a'(2bg-f_2)\}x^{(4)}]\delta u+\dots, \\
\delta\alpha'(u)=[32a'bx^{(2)}+(16a_1'-A)x^{(3)}+\{16a'(2b_2+f)+A_1\}x^{(4)}]\delta u+\dots, \\
\delta\beta(v)=[(16b_2-A')x^{(1)}-32a'b^2x^{(2)}+(16bg+A_2')x^{(3)}+\{32(a'b^2)_2-4a'bA' \\
\qquad -16b(2a'f-g_1)\}x^{(4)}]\delta v+\dots, \\
\delta\beta'(v)=[(16b_2-A')x^{(2)}+32a'bx^{(3)}+\{16b(2a_1'+g)+A_2'\}x^{(4)}]\delta v+\dots,
\end{cases}
$$

and a similar set of equations obtained from these on replacing A, A' by B, B'. In the limit $\delta_{(u)}^{(\eta)}$ and $\delta_{(u+\delta u)}^{(\eta)}$ will be coplanar if the determinant of the equations

$$\alpha(u)=0,\ \alpha'(u)=0,\ \delta\alpha(u)=0,\ \delta\alpha'(u)=0$$

vanishes. The elements of this determinant have the values

$$u_{11}=16a',\ u_{12}=A,\ u_{13}=0,\ u_{14}=32a'^2b,$$

$$u_{21}=16a'_1-A,\ u_{22}=16a'f+A_1,\ u_{23}=-32a'^2b,$$

$$u_{24}=32(a'^2b)_1-4a'bA-16a'(2bg-f_2),\ u_{31}=0,$$

$$u_{32}=0,\ u_{33}=16a',\ u_{34}=A,\ u_{41}=0,$$

$$u_{42}=32a'b,\ u_{43}=16a'_1-A,\ u_{44}=16a'(2b_2+f)+A_1.$$

By using the simplified forms of the integrability conditions, and rearranging slightly, the vanishing of this determinant can be shown to be equivalent to the vanishing of the determinant the elements of which are

$$v_{11}=v_{12}=0,\ v_{13}=16a',\ v_{14}=A,$$

$$v_{21}=0,\ v_{22}=32a'b,\ v_{23}=B,\ v_{24}=\frac{AB}{16a'}-\frac{\theta_1}{2\sqrt{\theta}}+16a'b_2,$$

$$v_{31}=16a',\ v_{32}=A,\ v_{33}=0,\ v_{34}=32a'^2b,$$

$$v_{41}=B,\ v_{42}=\frac{AB}{16a'}-\frac{\theta_1}{2\sqrt{\theta}}-16a'b_2,\ v_{43}=-32a'^2b,$$

$$v_{44}=4a'b\sqrt{\theta}-\frac{1}{8}\frac{a'}{b}A'B'.$$

On expansion this condition reduces to

$$(16)\qquad \theta_1^2-16a'^2\theta\theta'=0.$$

This relation is unchanged if we replace $\sqrt{\theta}$ by $-\sqrt{\theta}$. Hence if $\delta_{(u)}^{(\eta)}$ intersects $\delta_{(u+\delta u)}^{(\eta)}$, then $\delta_{(u)}^{(\zeta)}$ will also intersect $\delta_{(u+\delta u)}^{(\zeta)}$. Since $\rho(u)$ degenerates, it is evident geometrically that $\rho(v)$ must also degenerate. It is easy to see this from the equations already developed. In short $\delta_{(v)}^{(\eta')}$ intersects $\delta_{(v+\delta v)}^{(\eta')}$ if the determinant of the equations

$$\beta(v)=0,\ \beta'(v)=0,\ \delta\beta(v)=0,\ \delta\beta'(v)=0$$

vanishes. By proceeding as above this condition is equivalent to the vanishing of the determinant the elements of which are

$$v_{11}=0,\ v_{12}=16b,\ v_{13}=0,\ v_{14}=A',$$

$$v_{21}=0,\ v_{22}=B',\ v_{23}=32a'b,\ v_{24}=\frac{A'B'}{16b}-\frac{\theta_2'}{2\sqrt{\theta'}}+16a_1'b,$$

$$v_{31}=16b,\ v_{32}=0,\ v_{33}=A',\ v_{34}=32a'b^2,$$

$$v_{41}=B',\ v_{42}=-32a'b^2,\ v_{43}=\frac{A'B'}{16b}-\frac{\theta_2'}{2\sqrt{\theta'}}-16a_1'b,$$

$$v_{44}=4a'b\sqrt{\theta'}-\frac{1}{8}\frac{b}{a'}AB;$$

and this reduces to

$$\theta_2'^2-16b^2\theta\theta'=0. \tag{17}$$

This equation is however an immediate consequence of equations (11) and (16). It is easily shown by direct calculation that (16) and (17) are invariant under transformations that leave the parametric curves of (Σ) unchanged.

We have now found in invariant form the analytical criteria for the surfaces of the problem. These are given by equation (8) and (16) or (17) together with θ and θ' different from zero.

III

Reduction of (Σ) *to normal form.*

We have seen that a transformation can be found which replaces (Σ) by a system of the same form and such that the invariants a' and b are equal. Let us assume that such a transformation has been effected already; and that the common value of these is ϕ. Conditions (8) now reduce to the single condition

$$\frac{\partial^2 \log \phi}{\partial u \partial v}=4\phi^2. \tag{18}$$

This equation can be integrated; the solution is*

$$\phi=\frac{1}{2}\frac{\sqrt{U'V'}}{(U+V)}$$

where U and V are arbitrary functions of u and v respectively, and where the primes indicate differentiation. In virtue of the simplifications we have assumed carried out, the integrability conditions become

$$\theta=\bar{U}^2\phi^2,\quad \theta'=\bar{V}^2\phi^2,\quad \frac{\partial\theta}{\partial u}=\frac{\partial\theta'}{\partial v} \tag{19}$$

where $\bar{U}^2$ and $\bar{V}^2$ are arbitrary functions of their respective arguments, and are

*S, p. 176.

to be evaluated in terms of U and V respectively. From the first and second of these it follows that

$$(20)\qquad \begin{cases} \theta_1 = 2\phi^2\left(\bar{U}^2\dfrac{\phi_1}{\phi} + \bar{U}\bar{U}'\right), \\ \theta_2' = 2\phi^2\left(\bar{V}^2\dfrac{\phi_2}{\phi} + \bar{V}\bar{V}'\right). \end{cases}$$

On substituting in the third of (19), we shall find, after some easy simplifications, the equation

$$VG(u) - UH(v) = (U'\bar{U}^2 - UG(u)) - (V'\bar{V}^2 - VH(v)),$$

where

$$G(u) = \frac{1}{2}\left(\frac{U''}{U'}\bar{U}^2 + 2\bar{U}\bar{U}'\right),$$

$$H(v) = \frac{1}{2}\left(\frac{V''}{V'}\bar{V}^2 + 2\bar{V}\bar{V}'\right).$$

Hence

$$V'G'(u) - U'H'(v) = 0,$$

and therefore

$$\frac{G'(u)}{U'} = \frac{H'(v)}{V'} = a,$$

where a is a constant.

Thus

$$G(u) = aU + k_1,$$

$$H(v) = aV + k_2,$$

where k_1 and k_2 are additional constants. When these values of $G(u)$ and $H(v)$ are substituted in the third integrability condition, there results the equation

$$(aU^2 + \overline{k_1 - k_2}U - U'\bar{U}^2) = (aV^2 - \overline{k_1 - k_2}V - V'\bar{V}^2).$$

Therefore

$$aU^2 + \overline{k_1 - k_2}U - U'\bar{U}^2 = c,$$

$$aV^2 - \overline{k_1 - k_2}V - V'\bar{V}^2 = c,$$

where c is a constant. The functions $\bar{U}^2$ and $\bar{V}^2$ introduced in (19) have therefore the values

$$(21)\qquad \bar{U}^2 = \frac{aU^2 + bU + c}{U'}, \qquad \bar{V}^2 = \frac{aV^2 - bV + c}{V'},$$

where

$$b = k_1 - k_2.$$

Equations (16) and (17) can be written

$$\theta_1^2 = 16\bar{U}^2\bar{V}^2\phi^6, \quad \theta_2'^2 = 16\bar{U}^2\bar{V}^2\phi^6.$$

From the first of these and the expressions found for $\bar{U}^2$ and $\bar{V}^2$ it follows that

$$\left(\frac{\theta_1}{2\phi^2}\right)^2 = \frac{(aU^2+bU+c)\ (aV^2-bV+c)}{(U+V)^2}.$$

From equations (20) and (21) we find also that

$$\frac{\theta_1}{2\phi^2} = G(u) - \frac{aU^2+bU+c}{U+V}.$$

Equating these values for $\left(\frac{\theta_1}{2\phi^2}\right)^2$ and simplifying we arrive at the equation

$$\psi(u, v) \equiv U[(k_2^2-ac)U-c(k_1+k_2)]+V[k_1^2-ac)V-c(k_1+k_2)]$$

$$+UV[(k_1^2+k_2^2-2ac)+a(k_1+k_2)\ (U+V)]=0.$$

Hence

$$\frac{\partial^2\psi}{\partial u\partial v} = U'V'[(k_1^2+k_2^2-2ac)+2a(k_1+k_2)\ (U+V)]=0.$$

Now since (Σ) is not a ruled surface

$$U'\neq 0,\ V'\neq 0.$$

We therefore infer that

$$k_1^2+k_2^2-2ac=0,\ a(k_1+k_2)=0.$$

These constants are assumed to be real, which appears to be equivalent to assuming the coefficients of (Σ) real. Then it follows that the possible forms of (21) are easily reducible to two, viz.:

$$(22)\qquad \begin{aligned} \bar{U}^2 &= \frac{c}{U'}, & \bar{V}^2 &= \frac{c}{V'}, \\ \bar{U}^2 &= a\frac{U^2}{U'}, & \bar{V}^2 &= a\frac{V^2}{V'}. \end{aligned}$$

It is readily seen that there is no loss of generality in assuming the constants in (22) to be unity; because a multiplicative change in the arbitrary functions introduced reduces these constants to unity and leaves undisturbed the system (Σ). Hence the normal forms to be considered correspond to the relations

$$(23)\qquad \begin{aligned} &\text{(i)}\quad \theta = \frac{\phi^2}{U'}, & \theta' &= \frac{\phi^2}{V'}, \\ &\text{(ii)}\quad \theta = \frac{U^2}{U'}\phi^2, & \theta' &= \frac{V^2}{V'}\phi^2. \end{aligned}$$

The system of equations (Σ) now takes the normal form

$$y_{11}+2\phi y_2+\left(\frac{1}{4}\frac{\phi_1^2}{\phi^2}-\frac{1}{2}\frac{\phi_{11}}{\phi}-\phi_2-\mu(u)\right)y=0,$$

$$y_{22}+2\phi y_1+\left(\frac{1}{4}\frac{\phi_2^2}{\phi^2}-\frac{1}{2}\frac{\phi_{22}}{\phi}-\phi_1-\nu(v)\right)y=0,$$

where

$$\mu(u),\ \nu(v)=\frac{1}{256U'},\ \frac{1}{256V'}\ \text{or}\ \frac{U^2}{256U'},\ \frac{V^2}{256V'}$$

according as the first or second of relations (23) holds. In the first case the system will be designated by (Σ') and in the second case by (Σ''). We find from equations (16) and (17) that for (Σ')

$$\theta_1=\theta_2'=-4\phi\sqrt{\theta\theta'},$$

and for (Σ'')

$$\theta_1=\theta_2'=+4\phi\sqrt{\theta\theta'}.$$

IV

The determination of fixed points and planes of the configuration, etc.

Consider the expressions (7) for the flecnodes on the generators of $R(u)$, $R(v)$ through the point P_y. They lead to the equations

$$32a'\sqrt{\theta}\,\zeta=By-16a'z,$$

$$-32a'\sqrt{\theta}\,\eta=Ay-16a'z,$$

$$32b\sqrt{\theta'}\,\zeta'=B'y-16b\rho,$$

$$-32b\sqrt{\theta'}\,\eta'=A'y-16b\rho.$$

Hence the flecnodal points are given by the expressions

$$\eta=-Ay+16a'z,\ \zeta=By-16a'z,$$

$$\eta'=-A'y+16b\rho,\ \zeta'=B'y-16b\rho.$$

The following useful relations are easily verified:

$$(24)\qquad\begin{cases}\theta_2=2\dfrac{\phi_2}{\phi}\theta,\ \theta_1'=2\dfrac{\phi_1}{\phi}\theta',\\ A_1=8\phi_{11}\pm2\phi\sqrt{\theta'},\ B_1=8\phi_{11}\mp2\phi\sqrt{\theta'},\\ A_2=32\phi^3+\dfrac{\phi_2}{\phi}A,\ BA_2=32\phi^3+\dfrac{\phi_2}{\phi}B,\\ A_1'=32\phi^3+\dfrac{\phi_1}{\phi}A',\ B_1'=32\phi^3+\dfrac{\phi_1}{\phi}B',\\ A_2'=8\phi_{22}\pm2\phi\sqrt{\theta},\ B_2'=8\phi_{22}\mp2\phi\sqrt{\theta}.\end{cases}$$

In case of ambiguity in the sign of the radical, the upper sign corresponds to (Σ') and the lower to (Σ'').

From the expressions given above for η, η', ζ, ζ' and (24), we deduce the relations

$$(25)\quad \begin{cases} \eta_2 = 16\phi\sigma - A\rho + 16\phi_2 z - \left(32\phi^3 + \dfrac{\phi_2}{\phi}A\right)y, \\ \zeta_2 = -16\phi\sigma + B\rho - 16\phi_2 z + \left(32\phi^3 + \dfrac{\phi_2}{\phi}B\right)y, \\ \eta_1' = 16\phi\sigma + 16\phi_1\rho - A'z - \left(32\phi^3 + \dfrac{\phi_1}{\phi}A'\right)y, \\ \zeta_1' = -16\phi\sigma - 16\phi_1\rho + B'z + \left(32\phi^3 + \dfrac{\phi_1}{\phi}B'\right)y. \end{cases}$$

Hence

$$\eta_2 - \left(\frac{3}{2}\frac{\phi_2}{\phi} - \frac{\sqrt{\theta'}}{16\phi}\right)\eta = \eta_1' - \left(\frac{3}{2}\frac{\phi_1}{\phi} - \frac{\sqrt{\theta}}{16\phi}\right)\eta',$$

$$\eta_2 - \left(\frac{3}{2}\frac{\phi_2}{\phi} + \frac{\sqrt{\theta'}}{16\phi}\right)\eta = -\zeta_1' + \left(\frac{3}{2}\frac{\phi_1}{\phi} - \frac{\sqrt{\theta}}{16\phi}\right)\zeta',$$

$$\zeta_2 - \left(\frac{3}{2}\frac{\phi_2}{\phi} + \frac{\sqrt{\theta'}}{16\phi}\right)\zeta = \zeta_1' - \left(\frac{3}{2}\frac{\phi_1}{\phi} + \frac{\sqrt{\theta}}{16\phi}\right)\zeta',$$

$$\zeta_2 - \left(\frac{3}{2}\frac{\phi_2}{\phi} - \frac{\sqrt{\theta'}}{16\phi}\right)\zeta = -\eta_1' + \left(\frac{3}{2}\frac{\phi_1}{\phi} + \frac{\sqrt{\theta}}{16\phi}\right)\eta'.$$

These relations lead to the consideration of four expressions Ω, Ω'; $\underset{\sim}{\Omega}$, $\underset{\sim}{\Omega}'$ defined as follows:

$$(26)\quad \begin{cases} \Omega = \sigma - \dfrac{A}{16\phi}\rho - \dfrac{A'}{16\phi}z + \left(\dfrac{AA'}{256\phi^2} - 2\phi^2\right)y, \\ \Omega' = \sigma - \dfrac{B}{16\phi}\rho - \dfrac{B'}{16\phi}z + \left(\dfrac{BB'}{256\phi^2} - 2\phi^2\right)y, \\ \underset{\sim}{\Omega} = \sigma - \dfrac{A}{16\phi}\rho - \dfrac{B'}{16\phi}z + \left(\dfrac{AB'}{256\phi^2} - 2\phi^2\right)y, \\ \underset{\sim}{\Omega}' = \sigma - \dfrac{B}{16\phi}\rho - \dfrac{A'}{16\phi}z + \left(\dfrac{A'B}{256\phi^2} - 2\phi^2\right)y. \end{cases}$$

If now the system (Σ') be considered, it can be shown by direct calculation that

$$(27a)\quad \begin{cases} \dfrac{\partial\underset{\sim}{\Omega}}{\partial u} = -\dfrac{A}{16\phi}\underset{\sim}{\Omega}, \quad \dfrac{\partial\underset{\sim}{\Omega}}{\partial v} = -\dfrac{B'}{16\phi}\underset{\sim}{\Omega}, \\ \dfrac{\partial\underset{\sim}{\Omega}'}{\partial u} = -\dfrac{B}{16\phi}\underset{\sim}{\Omega}', \quad \dfrac{\partial\underset{\sim}{\Omega}'}{\partial v} = -\dfrac{A'}{16\phi}\underset{\sim}{\Omega}'. \end{cases}$$

If the system (Σ'') be considered it can be shown in like manner that

$$
(27b) \qquad \begin{cases} \dfrac{\partial \Omega}{\partial u} = -\dfrac{A}{16\phi}\,\Omega, & \dfrac{\partial \Omega}{\partial v} = -\dfrac{A'}{16\phi}\,\Omega, \\[2ex] \dfrac{\partial \Omega'}{\partial u} = -\dfrac{B}{16\phi}\,\Omega', & \dfrac{\partial \Omega'}{\partial v} = -\dfrac{B'}{16\phi}\,\Omega'. \end{cases}
$$

Hence in the former case the points $\underset{\sim}{\Omega}$, $\underset{\sim}{\Omega}'$ are fixed, and in the latter case the points Ω, Ω' are fixed.

Let the planes $\underset{\sim}{\Omega}\Omega\underset{\sim}{\Omega}'$, $\underset{\sim}{\Omega}\Omega'\underset{\sim}{\Omega}'$, $\Omega\underset{\sim}{\Omega}\Omega'$ and $\Omega\underset{\sim}{\Omega}'\Omega'$ be denoted by ω, ω', $\bar{\omega}$ and $\bar{\omega}'$ respectively. The coordinates of these planes are found from (26); they turn out to be proportional to the following expressions:

$$
(28) \qquad \begin{cases} \omega = \left(\dfrac{AA'}{256\phi^2} + 2\phi^2\right)|\rho z y| + \dfrac{A}{16\phi}|\sigma\rho y| - \dfrac{A'}{16\phi}|\sigma z y| - |\sigma\rho z|, \\[2ex] \omega' = \left(\dfrac{BB'}{256\phi^2} + 2\phi^2\right)|\rho z y| + \dfrac{B}{16\phi}|\sigma\rho y| - \dfrac{B'}{16\phi}|\sigma z y| - |\sigma\rho z|, \\[2ex] \bar{\omega} = \left(\dfrac{AB'}{256\phi^2} + 2\phi^2\right)|\rho z y| + \dfrac{A}{16\phi}|\sigma\rho y| - \dfrac{B'}{16\phi}|\sigma z y| - |\sigma\rho z|, \\[2ex] \bar{\omega}' = \left(\dfrac{A'B}{256\phi^2} + 2\phi^2\right)|\rho z y| + \dfrac{B}{16\phi}|\sigma\rho y| - \dfrac{A'}{16\phi}|\sigma z y| - |\sigma\rho z|. \end{cases}
$$

If we put

$$\lambda = |\rho z y|,$$

then

$$\lambda_1 = |\sigma z y|, \quad \lambda_2 = |\rho\sigma y|,$$

$$\lambda_{12} = |\rho\sigma z| + 4\phi^2|\rho z y|.$$

Equations (28) can therefore be written

$$
\begin{aligned}
\omega &= \lambda_{12} - \frac{A}{16\phi}\lambda_2 - \frac{A'}{16\phi}\lambda_1 + \left(\frac{AA'}{256\phi^2} - 2\phi^2\right)\lambda, \\
\omega' &= \lambda_{12} - \frac{B}{16\phi}\lambda_2 - \frac{B'}{16\phi}\lambda_1 + \left(\frac{BB'}{256\phi^2} - 2\phi^2\right)\lambda, \\
\bar{\omega} &= \lambda_{12} - \frac{A}{16\phi}\lambda_2 - \frac{B'}{16\phi}\lambda_1 + \left(\frac{AB'}{256\phi^2} - 2\phi^2\right)\lambda, \\
\bar{\omega}' &= \lambda_{12} - \frac{B}{16\phi}\lambda_2 - \frac{A'}{16\phi}\lambda_1 + \left(\frac{A'B}{256\phi^2} - 2\phi^2\right)\lambda.
\end{aligned}
$$

The function λ is a solution of the system of equations adjoined to (Σ)*. The normal form of this system is

*First Memoir, pp. 257-260.

$$(\bar{\Sigma})\quad \begin{cases} \lambda_{11}-2\phi\lambda_2+\left(\dfrac{1}{4}\dfrac{\phi_1^2}{\phi^2}-\dfrac{1}{2}\dfrac{\phi_{11}}{\phi}+\phi_2-\mu(u)\right)\lambda=0, \\ \lambda_{22}-2\phi\lambda_1+\left(\dfrac{1}{4}\dfrac{\phi_2^2}{\phi^2}-\dfrac{1}{2}\dfrac{\phi_{22}}{\phi}+\phi_1-\nu(v)\right)\lambda=0. \end{cases}$$

By means of these equations it can be shown that ω and ω' are fixed when $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Omega}'$ are fixed; and $\bar{\omega}$, $\bar{\omega}'$ are fixed when Ω, Ω' are fixed. It follows from equations (14) or (26) that Ω, Ω', $\underset{\sim}{\Omega}$, $\underset{\sim}{\Omega}'$ are the vertices of the skew quadrilateral formed by $\delta^{(\eta)}$, $\delta^{(\zeta)}$, $\delta^{(\eta')}$, $\delta^{(\zeta')}$. Thus each regulus of the directrix quadric reduces to two plane pencils with different vertices and in different planes. For the system (Σ') the vertices are $\underset{\sim}{\Omega}$, $\underset{\sim}{\Omega}'$ and the planes ω, ω'; while for the system (Σ'') the vertices are Ω, Ω' and the planes $\bar{\omega}$, $\bar{\omega}'$.

V

Integration of the normal equations.

Consider the Laplace expression

$$E=\frac{\partial^2 y}{\partial u\partial v}+a(u,v)\frac{\partial y}{\partial u}+b(u,v)\frac{\partial y}{\partial v}+c(u,v)y.$$

If both of the invariants

$$h=\frac{\partial a}{\partial u}+ab-c,\qquad k=\frac{\partial b}{\partial v}+ab-c$$

vanish, then there exists a transformation

$$y=e^{-V}Y$$

where

$$V=\int bdu+adv,$$

which reduces E to the form

$$E=e^{-V}\frac{\partial^2 Y}{\partial u\partial v}. \tag{29}$$

The expressions (26) are all of the same form as E and from (24) it follows that both h and k vanish for these. We first consider the system (Σ'). The transformation which reduces Ω to the form (29) is

$$y=\sqrt{\phi}\,e^{-\frac{1}{16}\int\frac{\sqrt{\theta}\,du+\sqrt{\theta'}\,dv}{\phi}}. \tag{30}$$

From (27*a*) we find by integration that

$$\begin{cases} \underset{\sim}{\Omega}=\dfrac{\alpha}{\sqrt{\phi}}e^{\frac{1}{16}\int\frac{\sqrt{\theta}\,du-\sqrt{\theta'}\,dv}{\phi}}, \\ \underset{\sim}{\Omega}'=\dfrac{\beta}{\sqrt{\phi}}e^{-\frac{1}{16}\int\frac{\sqrt{\theta}\,du-\sqrt{\theta'}\,dv}{\phi}}, \end{cases} \tag{31}$$

where α and β are arbitrary constants. The labour of writing the subsequent formulae will be greatly reduced, and the formulae themselves rendered less formidable in appearance, if we introduce the following notation:

$$\kappa = e^{\frac{1}{16}\int \frac{\sqrt{\theta}}{\phi} du}, \quad \kappa' = e^{\frac{1}{16}\int \frac{\sqrt{\theta'}}{\phi} dv}, \quad \tau = \kappa\kappa', \ \tau' = \frac{\kappa}{\kappa'}.$$

It will be noticed that κ is a function of u alone and κ' is a function of v alone.

The transformation (30) applied to (Σ') and (31) leads to the equations

$$(32) \qquad \begin{cases} Y_{11} + \dfrac{A}{8\phi} Y_1 + 2\phi Y_2 = 0, \\ Y_{22} + 2\phi Y_1 + \dfrac{A'}{8\phi} Y_2 = 0, \\ Y_{12} - \dfrac{\sqrt{\theta'}}{8\phi} Y_1 = \alpha \dfrac{\kappa^2}{\phi}, \\ Y_{12} - \dfrac{\sqrt{\theta}}{8\phi} Y_2 = \beta \dfrac{\kappa'^2}{\phi}\,; \end{cases}$$

and therefore

$$\sqrt{\theta'}\, Y_1 - \sqrt{\theta}\, Y_2 = 8(\beta\kappa'^2 - \alpha\kappa^2).$$

The integration of the third and fourth of (32) gives

$$Y_1 = \alpha\kappa'^2 \int \frac{\tau'^2}{\phi} dv + \kappa'^2 G(u),$$

$$Y_2 = \beta\kappa^2 \int \frac{du}{\phi\,\tau'^2} + \kappa^2 H(v),$$

where $G(u)$ and $H(v)$ are arbitrary functions. Introduce these values of Y_1 and Y_2 in the fifth of (32); we find the relation

$$\alpha\sqrt{\theta'}\,\tau^2 \int \frac{dv}{\phi\,\kappa'^2} - \beta\sqrt{\theta}\,\tau^2 \int \frac{du}{\phi\kappa^2} + \sqrt{\theta'}\,\kappa'^2 G(u) - \sqrt{\theta}\,\kappa^2 H(v) = \beta\kappa'^2 - \alpha\kappa^2.$$

If cognizance be taken of (18), (23) and quadratures effected where possible, this relation becomes

$$\frac{\beta\kappa'^2 - \alpha\kappa^2}{\phi} + \frac{2\tau^2}{\sqrt{U'V'}} \left\{ \alpha \int \frac{v'}{\kappa'^2} dv - \beta \int \frac{u'}{\kappa^2} du \right\} + \tfrac{1}{8} \left\{ \frac{\tau'^2 G(u)}{\sqrt{V'}} - \frac{\tau^2 H(v)}{\sqrt{U'}} \right\}$$
$$= \beta\kappa'^2 - \alpha\kappa^2.$$

This leads to the equation

$$2\alpha \int \frac{V'dv}{\kappa'^2} - \frac{\sqrt{V'}}{8\kappa'^2} H(v) = 2\beta \int \frac{U'du}{\kappa^2} - \frac{\sqrt{U'}}{8\kappa^2} = \gamma,$$

where γ is a constant. On solving these equations for $G(u)$ and $H(v)$, we find finally that

$$G(u)=\frac{16\beta}{\sqrt{U'}}\int\frac{U'du}{\kappa^2}-8\frac{\gamma}{\sqrt{U'}}\kappa^2,$$

$$H(v)=\frac{16\alpha}{\sqrt{V'}}\int\frac{V'dv}{\kappa'^2}-8\frac{\gamma}{\sqrt{V'}}\kappa'^2.$$

The expressions for Y_1 and Y_2 now become

$$Y_1=-16\frac{\alpha}{\sqrt{U'}}(U+V)\kappa^2+\frac{8\tau^2}{\sqrt{U'}}\left[2\alpha\int\frac{V'dv}{\kappa'^2}+2\beta\int\frac{U'du}{\kappa^2}-\gamma\right],$$

$$Y_2=-16\frac{\beta}{\sqrt{V'}}(U+V)\kappa'^2+\frac{8\tau^2}{\sqrt{V'}}\left[2\alpha\int\frac{V'dv}{\kappa'^2}+2\beta\int\frac{U'du}{\kappa^2}-\gamma\right].$$

These expressions for Y_1, Y_2 satisfy the relation

$$\frac{\partial Y_1}{\partial v}=\frac{\partial Y_2}{\partial u}=\frac{2\tau^2}{\sqrt{U'V'}}\left[\alpha\int\frac{V'dv}{\kappa'^2}+\beta\int\frac{U'du}{\kappa^2}-\gamma\right].$$

Hence, since α, β, γ are arbitrary constants, the coefficient of each in the expression

$$Y_1du+Y_2dv$$

must itself be an exact differential.

Integrating each in turn, we finally obtain

$$(33)\qquad\begin{cases}Y^{(1)}=\tau^2\displaystyle\int\frac{V'dv}{\kappa'^2}+\int U'\kappa^2du-\frac{1}{2}\frac{\sqrt{U'V'}}{\phi}\kappa^2,\\ Y^{(2)}=\tau^2\displaystyle\int\frac{U'du}{\kappa^2}+\int V'\kappa'^2dv-\frac{1}{2}\frac{\sqrt{U'V'}}{\phi}\kappa'^2,\\ Y^{(3)}=\tau^2.\end{cases}$$

The corresponding fundamental set of integrals of (Σ') is therefore

$$(34)\qquad\begin{cases}y^{(1)}=\sqrt{\phi}\left[\tau\displaystyle\int\frac{V'dv}{\kappa'^2}+\frac{1}{\tau}\int U'\kappa^2du-\frac{1}{2}\frac{\sqrt{U'V'}}{\phi}\tau'\right],\\ y^{(2)}=\sqrt{\phi}\left[\tau\displaystyle\int\frac{U'du}{\kappa^2}+\frac{1}{\tau}\int V'\kappa'^2dv-\frac{1}{2}\frac{\sqrt{U'V'}}{\phi}\frac{1}{\tau'}\right],\end{cases}$$

$$y^{(3)}=\sqrt{\phi}\,\tau,\quad y^{(4)}=\sqrt{\phi}\,\frac{1}{\tau}.$$

It follows from (30) that

$$z=\frac{1}{\tau}\left(\frac{A}{16\sqrt{\phi}}Y+\sqrt{\phi}\,Y_1\right),$$

$$\rho=\frac{1}{\tau}\left(\frac{A'}{16\sqrt{\phi}}Y+\sqrt{\phi}\,Y_2\right);$$

and therefore that

$$\eta = 16\frac{\phi^{3/2}}{\tau}Y_1,\quad \zeta = \frac{1}{\tau}(2\sqrt{\phi\theta}Y - 16\phi^{3/2}Y_1),$$

$$\eta' = 16\frac{\phi^{3/2}}{\tau}Y_2,\quad \zeta' = \frac{1}{\tau}(2\sqrt{\phi\theta'}Y - 16\phi^{3/2}Y_2).$$

On using the values of Y as given by (33) we deduce the following:

$$\eta^{(2)} = 2\sqrt{\phi\theta}\,\tau\int\frac{U'du}{\kappa^2},\quad \eta^{(3)} = 2\sqrt{\phi\theta}\,\tau,\quad \eta^{(4)} = 0,$$

$$\zeta^{(1)} = 2\frac{\sqrt{\phi\theta}}{\tau}\int V'\kappa'^2 dv,\quad \zeta^{(3)} = 0,\quad \zeta^{(4)} = 2\frac{\sqrt{\phi\theta}}{\tau},$$

$$\eta'^{(1)} = 2\sqrt{\phi\theta'}\,\tau\int\frac{V'dv}{\kappa'^2},\quad \eta'^{(3)} = 2\sqrt{\phi\theta'}\,\tau,\quad \eta'^{(4)} = 0,$$

$$\zeta'^{(2)} = 2\frac{\sqrt{\phi\theta'}}{\tau}\int V'\kappa'^2 dv,\quad \zeta'^{(3)} = 0,\quad \zeta'^{(4)} = 2\frac{\sqrt{\phi\theta'}}{\tau}.$$

These equations show that the pencils of directrices $\delta^{(\eta)}$ and $\delta^{(\zeta')}$ have a common vertex $\Omega(1, 0, 0, 0)$ and the planes are $y^{(4)} = 0$ and $y^{(3)} = 0$ respectively; while the pencils of directrices $\delta^{(\zeta)}$ and $\delta^{(\eta')}$ have a common vertex Ω' (0, 1, 0, 0) and planes $y^{(3)} = 0$ and $y^{(4)} = 0$ respectively.

The line coordinates ω_{ij} of $P_\eta P_\zeta$ and $P_{\eta'} P_{\zeta'}$ are readily calculated from (33) and the expressions for η, η', ζ, ζ'. It is found that for $P_\eta P_\zeta$

$$\omega_{14} = 4\phi\theta\left[\int\frac{V'dv}{\kappa'^2} - \frac{(U+V)}{\kappa'^2}\right],$$

$$\omega_{23} = -4\phi\theta\left[\int V'\kappa'^2 dv - (U+V)\kappa'^2\right],$$

$$\omega_{34} = 4\phi\theta;$$

and that for $P_{\eta'} P_{\zeta'}$

$$\omega_{13} = -4\phi\theta'\left[\int U'\kappa^2 du - (U+V)\kappa^2\right],$$

$$\omega_{24} = 4\phi\theta'\left[\int\frac{U'du}{\kappa^2} - \frac{U+V}{\kappa^2}\right],$$

$$\omega_{34} = 4\phi\theta'.$$

It follows therefore that the curves $\Gamma(u)$ and $\Gamma(v)$ belong to the complexes

$$C(u) \equiv \frac{1}{\kappa^2}\omega_{13} + \kappa^2\omega_{24} + \left[\frac{1}{\kappa^2}\int U'\kappa^2 du - \kappa^2\int\frac{U'du}{\kappa^2}\right]\omega_{34} = 0,$$

and

$$C(v) \equiv \frac{1}{\kappa'^2}\omega_{23} + \kappa'^2\omega_{14} + \left[\frac{1}{\kappa'^2}\int V'\kappa'^2 dv - \kappa'^2\int\frac{V'dv}{\kappa'^2}\right]\omega_{34} = 0$$

respectively.

The polar plane of $\underset{\sim}{\Omega}$ in the family of complexes $C(u)$ is identical with the polar plane of $\underset{\sim}{\Omega}'$ in the family of complexes $C(v)$ and is

$$y^{(3)}=0;$$

while the polar plane of $\underset{\sim}{\Omega}'$ in the family of complexes $C(u)$ is identical with the polar plane of $\underset{\sim}{\Omega}$ in the family of complexes $C(v)$ and is

$$y^{(4)}=0.$$

We proceed next to the consideration of the system (Σ''). The transformation which reduces $\underset{\sim}{\Omega}$ to the form E is in this case

$$y=\sqrt{\phi}\, e^{-\frac{1}{16}\int\frac{\sqrt{\theta}\,du-\sqrt{\theta'}\,dv}{\phi}}. \tag{35}$$

When integrated equations (27*b*) give

$$\Omega=\frac{\alpha}{\sqrt{\phi}}\,e^{\frac{1}{16}\int\frac{\sqrt{\theta}\,du+\sqrt{\theta'}\,dv}{\phi}}, \quad \Omega'=\frac{\beta}{\sqrt{\phi}}\,e^{-\frac{1}{16}\int\frac{\sqrt{\theta}\,du+\sqrt{\theta'}\,dv}{\phi}}, \tag{36}$$

where α and β are arbitrary constants. Now apply transformation (35) to (Σ'') and (36); we find the equations

$$\left\{\begin{aligned} &Y_{11}+\frac{A}{8\phi}Y_1+2\phi Y_2=0,\\ &Y_{22}+2\phi Y_1+\frac{B'}{8\phi}Y_2=0,\\ &Y_{12}+\frac{\sqrt{\theta'}}{8\phi}Y_1=\frac{\alpha}{\phi}\kappa^2,\\ &Y_{12}-\frac{\sqrt{\theta}}{8\phi}Y_2=\frac{\beta}{\phi\kappa'^2}; \end{aligned}\right. \tag{37}$$

and therefore

$$\sqrt{\theta'}\,Y_1+\sqrt{\theta}\,Y_2=8\left(\alpha\kappa^2-\frac{\beta}{\kappa'^2}\right).$$

The third and fourth of these lead to

$$Y_1=\alpha\tau'^2\int\frac{\kappa'^2dv}{\phi}+\frac{G(u)}{\kappa'^2},$$

$$Y_2=\beta\tau'^2\int\frac{du}{\phi\kappa^2}+\kappa^2H(v),$$

where $G(u)$ and $H(v)$ are arbitrary functions. In virtue of (18) and (23) these may be written in the form

$$Y_1=2\alpha\frac{U}{\sqrt{U'}}\tau'^2\int\frac{\kappa'^2dv}{\sqrt{V'}}+16\alpha\frac{\kappa^2}{\sqrt{U'}}+\frac{G(u)}{\kappa^2},$$

$$Y_2=2\beta\frac{V}{\sqrt{V'}}\tau'^2\int\frac{du}{\sqrt{U'}\kappa^2}-\frac{16\beta}{\sqrt{V'}\kappa'^2}+H(v)\kappa^2.$$

On introducing these values of Y_1 and Y_2 in the fifth of equations (37), there results an equation which reduces to

$$\frac{\alpha}{4}\int\frac{\kappa'^2dv}{\sqrt{V'}}+\frac{\beta}{4}\int\frac{du}{\sqrt{U'}\kappa^2}+\frac{1}{U\kappa^2}\left[\frac{\sqrt{U'}}{8}G(u)+2\beta\right]+\frac{\kappa'^2}{V}\left[\frac{\sqrt{V'}}{8}H(v)-2\alpha\right]=0.$$

From this we infer that

$$\frac{1}{U\kappa^2}\left[\frac{\sqrt{U'}}{8}G(u)+2\beta\right]+\frac{\beta}{4}\int\frac{du}{\sqrt{U'}\kappa^2}=-\frac{\kappa'^2}{V}\left[\frac{\sqrt{V'}}{8}H(v)-2\alpha\right]-\frac{\alpha}{4}\int\frac{\kappa^2dv}{\sqrt{V'}}=\gamma,$$

where γ is a constant. Hence the values of $G(u)$ and $H(v)$ are given by the equations

$$G(u)=8\gamma\frac{U\kappa^2}{\sqrt{U'}}-2\beta\left[\frac{8}{\sqrt{U'}}+\frac{U\kappa^2}{\sqrt{U'}}\int\frac{du}{\sqrt{U'}\kappa^2}\right],$$

$$H(v)=-8\gamma\frac{V}{\sqrt{V'}\kappa'^2}+2\alpha\left[\frac{8}{\sqrt{V'}}-\frac{V}{\sqrt{V'}\kappa'^2}\int\frac{\kappa'^2dv}{\sqrt{V'}}\right].$$

The values of Y_1 and Y_2 are therefore

$$Y_1=2\frac{U\tau'^2}{\sqrt{U'}}\left[\alpha\int\frac{\kappa'^2dv}{\sqrt{V'}}-\beta\int\frac{du}{\sqrt{U'}\kappa^2}+4\gamma\right]+\frac{16}{\sqrt{U'}}\left[\alpha\kappa^2-\frac{\beta}{\kappa'^2}\right],$$

$$Y_2=2\frac{V\tau'^2}{\sqrt{V'}}\left[-\alpha\int\frac{\kappa'^2dv}{\sqrt{V'}}+\beta\int\frac{du}{\sqrt{U'}\kappa^2}-4\gamma\right]+\frac{16}{\sqrt{V'}}\left[\alpha\kappa^2-\frac{\beta}{\kappa'^2}\right].$$

These expressions for Y_1 and Y_2 satisfy the relation

$$\frac{\partial Y_1}{\partial v}=\frac{\partial Y_2}{\partial u}=\frac{2}{\sqrt{U'V'}}\left[\alpha U\kappa^2+\frac{\beta V}{\kappa'^2}\right]-2\frac{UV}{\sqrt{U'V'}}\tau'^2\left[\alpha\int\frac{\kappa'^2dv}{\sqrt{V'}}-\beta\int\frac{du}{\sqrt{U'}\kappa^2}+4\gamma\right].$$

Since α, β, γ are independent, the coefficient of each of these in the expression for Y_1du+Y_2dv is a complete differential. We therefore find the independent solutions of (37) to be

$$(38)\qquad\begin{cases}Y^{(1)}=\tau'^2\displaystyle\int\frac{\kappa'^2dv}{\sqrt{V'}}+\int\frac{\kappa^2du}{\sqrt{U'}},\\ Y^{(2)}=\tau'^2\displaystyle\int\frac{du}{\sqrt{U'}\kappa^2}+\int\frac{dv}{\sqrt{V'}\kappa'^2},\\ Y^{(3)}=\tau'^2.\end{cases}$$

The corresponding fundamental set of integrals of the system (Σ'') follows at once from these and (35); they are

$$(39)\qquad\begin{cases}y^{(1)}=\sqrt{\phi}\left[\tau'\displaystyle\int\frac{\kappa'^2dv}{\sqrt{V'}}+\frac{1}{\tau'}\int\frac{\kappa^2du}{\sqrt{U'}}\right],\\ y^{(2)}=\sqrt{\phi}\left[\tau'\displaystyle\int\frac{du}{\sqrt{U'}\kappa^2}+\frac{1}{\tau'}\int\frac{dv}{\sqrt{V'}\kappa'^2}\right],\\ y^{(3)}=\sqrt{\phi}\,\tau',\quad y^{(4)}=\dfrac{\sqrt{\phi}}{\tau'}.\end{cases}$$

Equation (35) leads at once to the equations

$$y=\sqrt{\phi}\,\tau' Y,$$

$$z=\tau'\left[\frac{A}{16\sqrt{\phi}}Y+\sqrt{\phi}\,Y_1\right],$$

$$\rho=\tau'\left[\frac{B'}{16\sqrt{\phi}}Y+\sqrt{\phi}\,Y_2\right],$$

and therefore to the following expressions for the flecnodes:

$$\eta=16\frac{\phi^{3/2}}{\tau'}Y_1,$$

$$\zeta=\frac{1}{\tau'}(2\sqrt{\phi\theta}\,Y-16\phi^{3/2}Y_1),$$

$$\eta'=\frac{1}{\tau'}(2\sqrt{\phi\theta'}\,Y+16\phi^{3/2}Y_2),$$

$$\zeta'=-\frac{16\phi^{3/2}}{\tau'}Y_2.$$

By using the values of Y given in (38) we find

$$\eta^{(2)}=2\sqrt{\phi\theta}\,\tau'\int\frac{du}{\sqrt{U'}\,\kappa^2}+16\sqrt{\frac{\phi^3}{U'}}\frac{1}{\tau},$$

$$\eta^{(3)}=2\sqrt{\phi\theta}\,\tau',\quad \eta^{(4)}=0,$$

$$\zeta^{(1)}=2\frac{\sqrt{\phi\theta}}{\tau'}\int\frac{\kappa^2\,du}{\sqrt{U'}}-16\sqrt{\frac{\phi^3}{U'}}\,\tau,$$

$$\zeta^{(3)}=0,\quad \zeta^{(4)}=2\frac{\sqrt{\phi\theta}}{\tau'},$$

$$\eta'^{(2)}=2\frac{\sqrt{\phi\theta'}}{\tau'}\int\frac{dv}{\sqrt{V'}\,\kappa'^2}+16\sqrt{\frac{\phi^3}{V'}}\frac{1}{\tau},$$

$$\eta'^{(3)}=0,\quad \eta'^{(4)}=2\frac{\sqrt{\phi\theta'}}{\tau'},$$

$$\zeta'^{(1)}=2\sqrt{\phi\theta'}\,\tau'\int\frac{\kappa'^2dv}{\sqrt{V'}}-16\sqrt{\frac{\phi^3}{V'}}\,\tau,$$

$$\zeta'^{(3)}=2\sqrt{\phi\theta'}\,\tau',\quad \zeta'^{(4)}=0.$$

It follows therefore that in this case the pencils of directrices $\delta^{(\eta)}$ and $\delta^{(\eta')}$ have a common vertex $\Omega(1, 0, 0, 0)$ and the planes are $y^{(4)}=0$ and $y^{(3)}=0$ respectively; while the pencils of directrices $\delta^{(\zeta)}$ and $\delta^{(\zeta')}$ have a common vertex $\Omega'(0, 1, 0, 0)$ and the planes are $y^{(3)}=0$ and $y^{(4)}=0$ respectively.

Now form the line coordinates ω_{ij} of $P_\eta P_\zeta$ and $P_{\eta'} P_{\zeta'}$. We find from equations (38) and the expressions for η, η', ζ, ζ' that for $P_\eta P_\zeta$

$$\omega_{14}=4\phi\theta\int\frac{\kappa'^2 dv}{\sqrt{V'}}+32\kappa'^2\sqrt{\frac{\phi^4\theta}{U'}},$$

$$\omega_{23}=-4\phi\theta\int\frac{dv}{\kappa'^2\sqrt{V'}}+\frac{32}{\kappa'^2}\sqrt{\frac{\phi^4\theta}{U'}},$$

$$\omega_{34}=4\phi\theta,$$

and that for $P_{\eta'}P_{\zeta'}$

$$\omega_{13}=4\phi\theta'\int\frac{\kappa^2 du}{\sqrt{U'}}+32\kappa^2\sqrt{\frac{\phi^4\theta'}{V'}},$$

$$\omega_{24}=-4\phi\theta'\int\frac{du}{\kappa^2\sqrt{U'}}+\frac{32}{\kappa^2}\sqrt{\frac{\phi^4\theta'}{V'}},$$

$$\omega_{34}=-4\phi\theta'.$$

It results from these that the curves $\Gamma(u)$ and $\Gamma(v)$ belong to the complexes

$$C(u)\equiv\frac{1}{\kappa^2}\omega_{13}-\kappa^2\omega_{24}+\left[\frac{1}{\kappa^2}\int\frac{\kappa^2 du}{\sqrt{U'}}+\kappa^2\int\frac{du}{\kappa^2\sqrt{U'}}\right]\omega_{34}=0,$$

$$C(v)\equiv\frac{1}{\kappa'^2}\omega_{14}-\kappa'^2\omega_{23}-\left[\frac{1}{\kappa'^2}\int\frac{\kappa'^2 dv}{\sqrt{V'}}+\kappa'^2\int\frac{dv}{\kappa'^2\sqrt{V'}}\right]\omega_{34}=0$$

respectively.

The polar plane of Ω in the family of complexes $C(u)$ is the same as the polar plane of Ω' in the family of complexes $C(v)$ and is

$$y^{(3)}=0;$$

while the polar plane of Ω' in the family of complexes $C(u)$ is the same as the polar plane of Ω in the family of complexes $C(v)$ and is

$$y^{(4)}=0.$$

The problem proposed at the outset is now solved. The equations of the surfaces (Σ') having the property in question are given explicitly by (34), and those of (Σ'') by (39). There are certain particular features of the general configuration here studied that merit consideration. But a study of these rests upon a general theory different from that of section one and will have to be omitted.

UN NOUVEAU PROBLÈME SUR LES SUITES DE LAPLACE

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1. Etant donné un réseau, situé dans un espace quelconque, on sait qu'en appliquant la transformation de Laplace, dans les deux sens, on obtient une configuration géométrique remarquable que l'on appelle une *suite de Laplace.*

Si l'on excepte les cas où la suite est périodique ou terminée dans les deux sens, une suite de Laplace se compose d'une infinité de réseaux; d'une manière plus précise, l'ensemble des réseaux d'une suite de Laplace est dénombrable.

Il s'agit d'étudier si cet ensemble admet une figure limite et, en général, de déterminer les figures limites de l'ensemble.

C'est ce problème, qui me semble ne pas avoir encore été envisagé par les géomètres, que j'ai pris pour sujet de ma communication. Certes, il est trop vaste et trop complexe pour que je puisse l'aborder dans toute sa généralité. Aussi, je me bornerai à considérer deux exemples intéressants qui mettront en évidence deux aspects différents de ce problème difficile.

2. Prenons tout d'abord une suite autoprojective.* Soit (x) le réseau initial,

$$(x_1),\ (x_2),\ \ldots,\ (x_i),\ \ldots$$

les réseaux obtenus en appliquant successivement la transformation de Laplace dans le sens des courbes v,

$$(x_{-1}),\ (x_{-2}),\ \ldots,\ (x_{-i}),\ \ldots$$

les transformés de Laplace dans le sens des courbes u.

On suppose qu'il y a une transformation projective P, qui transforme le réseau (x) en (x_m). Alors le réseau (x_i) devient (x_{i+m}), par conséquent la transformation P laisse la suite $[x]$ invariante.

J'ai démontré† que l'on obtient le cas général des suites autoprojectives, en supposant que le point x du réseau initial (x) et les points $x_m, x_{2m}, \ldots, x_{(r+1)m}$ qui en résultent par l'application successive de la transformation projective P, sont tous situés dans un espace linéaire L_r à r dimensions. L'espace L_r varie lorsque le point x décrit le réseau (x). On en conclut que tous les points x_{im} sont situés dans cet espace L_r.

On peut démontrer de la même manière que les points

$$x_1,\ x_{m+1},\ x_{2m+1},\ \ldots,\ x_{im+1},\ \ldots$$

*Voir pour ce qui suit ma *Géométrie différentielle* projective des réseaux, chap. IX.
†Voir le livre cité, p. 200 et suiv.

sont, eux aussi, situés dans un espace linéaire $L_t^{(1)}$ et que, en général, les points

$$x_k,\ x_{m+k},\ \ldots,\ x_{im+k},\ \ldots,\ k<m,$$

sont situés dans l'espace linéaire variable $L_t^{(k)}$.

Cela étant établi, il convient de remarquer que la transformation projective P laisse chaque espace $L_t^{(k)}$ invariant et que dans cet espace on peut la remplacer par une transformation projective $P^{(k)}$ de cet espace.

Si nous supposons que la transformation P est générale, l'ensemble des points

$$x,\ x_m,\ \ldots,\ x_{im},\ \ldots$$

a comme point limite un des points doubles ξ de la transformation P. Or, j'ai démontré que ce point ξ décrit un réseau (ξ) dont la suite de Laplace $[\xi]$ est périodique. On a donc le résultat suivant:

La figure limite de l'ensemble formé par les réseaux

$$(x),\ (x_m),\ (x_{2m}),\ \ldots,\ (x_{im}),\ \ldots$$

appartenant à la suite autoprojective [x] *est un réseau* (ξ).

Il est aisé maintenant de voir que la figure limite de l'ensemble formé par les réseaux

$$(x_k),\ (x_{m+k}),\ x_{2m+k}),\ \ldots,\ (x_{im+k}),\ \ldots,\ k<m,$$

est aussi un réseau (ξ_k), le k-ième transformé de Laplace du réseau (ξ).

Finalement, on a le résultat suivant qui me semble particulièrement intéressant:

La figure limite de la suite autoprojective considérée $[x]$ *est aussi une suite de Laplace* $[\xi]$ *périodique.*

Je ne considère pas ici le cas exceptionnel, facile à traiter, où $m=1$. Les résultats sont alors analogues à ceux du deuxième exemple que nous allons traiter.

3. Je considère comme deuxième exemple de notre problème la suite de Laplace $[x]$ déduite du réseau (x), décrit par le point dont les coordonnées projectives sont

$$x^{(i)}=A_i(u+a_i)^p(v+a_i)^q, \quad (i=1,\ 2,\ \ldots,\ n). \tag{1}$$

Le premier transformé de Laplace, dans le sens des courbes v, de ce point, a pour coordonnées

$$x_1^{(i)}=A_i(u+a_i)^{p+1}(v+a_i)^{q-1}, \quad (i=1,\ 2,\ \ldots,\ n),$$

et, en général, le k-ième transformé dans le même sens

$$x_k^{(i)}=A_i(u+a_i)^{p+k}(v+a_i)^{q-k}, \quad (i=1,\ 2,\ \ldots,\ n). \tag{2}$$

4. Avant de passer à l'étude des figures limites de la suite $[x]$, nous allons faire quelques remarques sur cette suite.

Tout d'abord, il est facile de voir que tous les réseaux de la suite ont la courbe $u=v$ commune. Cette courbe est sur chaque réseau l'enveloppe des courbes $u=\text{const.}$ et $v=\text{const.}$

Une remarque bien plus curieuse est celle-ci: si q est un nombre entier positif, le q^e transformé de Laplace du réseau (x), défini par des formules (1), se réduit à une courbe. On peut le voir sur les formules (2), en faisant $k=q$. La suite $[x]$ s'arrête donc à (x_q) dans le sens des courbes v.

Cependant les formules (2) montrent que les transformées de Laplace existent aussi pour $k>q$, c'est-à-dire *que l'on peut prolonger la suite de Laplace au delà de la courbe à laquelle elle s'était arrêtée.*

En réalité, on a deux suites de Laplace qui s'arrêtent à la même courbe, l'une par le cas de Goursat, l'autre dans le cas de Laplace*. Ce qu'il y a d'intéressant, c'est que les deux suites font partie d'une même expression analytique générale, une d'entre elles est en quelque sorte le prolongement analytique de l'autre.

5. Revenons maintenant à notre problème. Il s'agit de voir ce que devient le point x_k, défini par les formules (2), pour $\lim k=+\infty$.

Remarquons d'abord que l'on peut écrire

$$x_k{}^{(i)}=x^{(i)}\cdot\left(\frac{u+a_i}{v+a_i}\right)^k,$$

et considérons la suite

$$(3)\qquad \left|\frac{u+a_1}{v+a_1}\right|,\ \left|\frac{u+a_2}{v+a_2}\right|,\ \ldots\ldots,\left|\frac{u+a_n}{v+a_n}\right|.$$

Désignons par R_i la région du réseau (x) où doit se trouver le point (u, v) pour que le nombre $\left|\frac{u+a_i}{v+a_i}\right|$ soit le plus grand de la suite (3).

On voit ainsi que si x ou (u, v) est un point arbitraire de R_1, l'ensemble

$$(4)\qquad x,\ x_1,\ x_2,\ \ldots$$

admet pour point limite le point fondamental $O_1(1, 0, 0, \ldots, 0)$. De même, si x appartient à R_2, l'ensemble des points (4) a pour point limite $O_2(0, 1, 0, \ldots, 0)$ et ainsi de suite.

Dans le cas où x est un point tel que

$$\left|\frac{u+a_1}{v+a_1}\right| = \left|\frac{u+a_2}{v+a_2}\right| > \left|\frac{u+a_i}{v+a_i}\right|,\ i>2,$$

alors l'ensemble (4) a pour limite un point de la droite O_1O_2.

On peut énoncer le résultat suivant:

La figure limite des réseaux (x_k) *se compose des droites* O_iO_k.

J'ajoute pour finir qu'il serait interéssant d'étudier d'une manière générale dans quels cas la figure limite d'une suite de Laplace est un réseau ou une suite de Laplace.

*Voir le livre cité, chap. VI.

DÉTERMINATION DES INVARIANTS DIFFÉRENTIELS ET DES INVARIANTS INTÉGRAUX DES SURFACES POUR LE GROUPE CONFORME

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INTRODUCTION

Rappelons d'abord les notions d'invariant différentiel et d'invariant intégral des surfaces pour un groupe continu de transformations G.

Soit S une surface, lieu d'un point de coordonnées x, y, z. Une fonction d'un ou de plusieurs des éléments de l'ensemble formé par x, y, z et les dérivées de z par rapport à x et à y, qui contient une au moins de ces dérivées, est un *invariant différentiel* des surfaces pour le groupe G, si cette fonction conserve sa valeur lorsqu'on soumet la surface S à une quelconque des transformations de ce groupe.

Un invariant différentiel est dit *d'ordre n*, s'il ne contient pas de dérivée de z d'ordre supérieur à n et s'il en contient au moins une d'ordre n.

Soit

$$I=\iint F d\sigma$$

une intégrale étendue à une portion quelconque de la surface S; $d\sigma$ désigne l'élément d'aire de cette surface et F est une fonction d'un ou de plusieurs des éléments de l'ensemble formé par x, y, z et les dérivées de z par rapport à x et à y. I est un *invariant intégral* des surfaces pour le groupe G, si cette quantité conserve sa valeur lorsqu'on soumet la surface S à une quelconque des transformations de ce groupe.

M. Tresse* a donné un moyen élégant de former les invariants différentiels des surfaces pour le groupe conforme, et il a fait connaître les plus simples de ces invariants, deux du troisième ordre et cinq du quatrième ordre. Dans le présent travail, nous déterminons tous ces invariants, ainsi que les invariants intégraux des surfaces pour le groupe conforme.

Le groupe conforme admettant comme sous-groupe le groupe des déplacements, tout invariant différentiel des surfaces pour le groupe conforme est

*A. Tresse: *Sur les invariants différentiels des groupes continus de transformations* (Acta Mathematica, t. 18, 1894). *Sur les invariants différentiels d'une surface par rapport aux transformations conformes de l'espace* (Comptes Rendus Acad. Sciences Paris, t. CXIV, p. 648, 1892).

formé par les courbures principales de la surface S et leurs dérivées par rapport aux arcs des lignes de courbure.

La notion d'invariant différentiel peut être généralisée. Supposons que les coordonnées x, y, z de la surface S soient exprimées en fonction de deux paramètres u, v. Nous dirons qu'une fonction d'un ou de plusieurs des éléments de l'ensemble formé par x, y, z et leurs dérivées par rapport à u et à v est un invariant *paramétrique* des surfaces pour le groupe G, si cette fonction conserve sa valeur lorsqu'on soumet la surface S à une quelconque des transformations de ce groupe.

Il est clair qu'un invariant différentiel, au sens rappelé plus haut, est un invariant paramétrique, mais la réciproque n'est pas vraie.
nécessairement une fonction d'un ou de plusieurs des éléments de l'ensemble

Soient

$$Edu^2+2Fdudv+Gdv^2, \qquad Ldu^2+2Mdudv+Ndv^2$$

les deux formes quadratiques fondamentales de la surface S. L'invariant paramétrique le plus général pour le groupe des déplacements est une fonction d'un ou de plusieurs des éléments de l'ensemble formé par E, F, G, L, M, N et leurs dérivées par rapport à u et à v. Le groupe des déplacements étant contenu dans le groupe conforme, il en est de même des invariants paramétriques des surfaces pour le groupe conforme. Si l'on prend pour réseau (u, v) celui des lignes de courbure, F et M sont nuls; par suite, les lignes de courbure étant conservées dans une transformation conforme, tout invariant paramétrique des surfaces pour le groupe conforme, les paramètres étant ceux des lignes de courbure, est une fonction d'un ou de plusieurs des éléments de l'ensemble formé par E, G, L, N et leurs dérivées par rapport à u et à v. Les invariants de cette nature ont été déterminés par M. Calapso*. En nous appuyant sur les développements relatifs aux invariants différentiels, nous donnons une nouvelle solution de ce problème.

Signalons encore les résultats suivants, établis dans le présent travail.

Soit

$$ds^2=A^2du^2+C^2dv^2$$

la formule donnant l'élément linéaire d'une surface rapportée au réseau (u, v) de ses lignes de courbure. Cette surface possède deux invariants paramétriques Φ_1, Φ_2 définis par les égalités

$$\Phi_1=A(\rho_1-\rho_2), \quad \Phi_2=C(\rho_1-\rho_2),$$

ρ_1, ρ_2 désignant les courbures principales.

Nous démontrons que, si la surface n'est pas isothermique, Φ_1, Φ_2 sont liés par deux relations différentielles et que, réciproquement, si deux fonctions Φ_1, Φ_2, dont le rapport n'est pas le quotient d'une fonction de u par une fonction de v, satisfont à ces relations, il y a une surface, et une seule, définie à une transformation conforme près, dont ces fonctions sont les invariants Φ_1, Φ_2.

*Calapso: *Sugl' invarianti del gruppo delle transformazioni conformi dello spazio* (Rend. Circ. Mat. Palermo, t. XXII, 1906).

Si la surface est isothermique, on peut, par un choix convenable des paramètres u, v, faire en sorte que $\Phi_1 = \Phi_2$. La valeur commune Φ de ces deux fonctions satisfait à l'équation

$$2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial u\partial v}+\frac{\partial^2\Phi^2}{\partial u\partial v}=0,$$

qui a été obtenue par M. Rothe et par M. Calapso.

Si l'on convient de ne pas considérer comme distinctes deux surfaces se correspondant dans une transformation conforme, à toute solution de cette équation correspond une infinité simple de surfaces isothermiques. Nous définissons cette famille de surfaces en la rattachant à une transformation des surfaces isothermiques que M. Guichard a fait connaître, en 1903, et que M. Bianchi et nous-même avons retrouvée, indépendamment l'un de l'autre, en 1905.

Nous avons exposé les résultats établis dans ce mémoire dans une conférence faite à la Société Mathématique de Belgique, le 10 mars 1922*.

I

1. Soient S une sphère dépendant d'un paramètre u, O le centre et R le rayon de cette sphère. Supposant que la trajectoire (O) du point O n'est pas une ligne minima, nous allons calculer l'angle de deux positions de S, infiniment voisines.

Pour plus de clarté, nous examinerons d'abord les différents cas qui peuvent se présenter.

Soient x, y, z les coordonnées rectangulaires du point O, s la coordonnée curviligne de ce point et α, β, γ les cosinus directeurs de la tangente à la ligne (O) en O. L'équation de la sphère S est

$$(X-x)^2+(Y-y)^2+(Z-z)^2=R^2,$$

ou, si l'on fait usage du signe $\mathbf{S}$ de Lamé,

$$\mathbf{S}(X-x)^2=R^2.$$

La caractéristique de la sphère s'obtient en joignant à cette équation la suivante

$$\mathbf{S}(X-x)\alpha=-R\frac{dR}{ds}.$$

Les coordonnées x_0, y_0, z_0 du point d'intersection P de la tangente à (O) et du plan de la caractéristique ont des expressions de la forme

$$x_0=x+h\alpha, \quad y_0=y+h\beta, \quad z_0=z+h\gamma,$$

et l'on a évidemment

$$\mathbf{S}(x_0-x)\alpha=-R\frac{dR}{ds}.$$

*Voir Mathesis, supplément de décembre 1922.

En remplaçant, dans cette équation, x_0, y_0, z_0 par leurs valeurs, on trouve

$$h=-R\frac{dR}{ds}.$$

Donc, si $\frac{dR}{ds}$ est différent de ± 1, la caractéristique de la sphère S est un cercle de rayon non nul et si $\frac{dR}{ds}$ égale ϵ, $\epsilon=\pm 1$, cette caractéristique est un cercle de centre P et de rayon nul. Dans ce dernier cas, on a $h=-\epsilon R$ et les expressions de x_0, y_0, z_0 deviennent

$$(1)\qquad x_0=x-\epsilon R\alpha,\quad y_0=y-\epsilon R\beta,\quad z_0=z-\epsilon R\gamma.$$

On déduit de là

$$(2)\qquad \frac{dx_0}{ds}=-\epsilon R\frac{d\alpha}{ds},\quad \frac{dy_0}{ds}=-\epsilon R\frac{d\beta}{ds},\quad \frac{dz_0}{ds}=-\epsilon R\frac{d\gamma}{ds},$$

d'où

$$(3)\qquad \mathbf{S}\left(\frac{dx_0}{ds}\right)^2=R^2\mathbf{S}\left(\frac{d\alpha}{ds}\right)^2.$$

Si la trajectoire (P) du point P ne se réduit pas à un point, sa tangente est perpendiculaire à OP. En effet, les égalités (1) et (2) donnent

$$\mathbf{S}(x_0-x)\frac{dx_0}{ds}=0.$$

Deux cas peuvent se présenter.

Premier cas. La vitesse du point P n'est pas nulle. Alors, en vertu de (3), la normale principale de (O) n'est pas isotrope et, à cause de (2), la tangente à (P) est parallèle à cette droite.

Deuxième cas. La vitesse de P est nulle. On a, par suite, en vertu de (3), $\mathbf{S}\left(\frac{d\alpha}{ds}\right)^2=0$.

Si les trois quantités $\frac{d\alpha}{ds}$, $\frac{d\beta}{ds}$, $\frac{d\gamma}{ds}$ ne sont pas simultanément nulles, l'indicatrice sphérique des tangentes de (O) est une droite isotrope et (O) est située dans un plan isotrope ω. La vitesse du point P étant nulle, ce point décrit une droite isotrope i*. Toutes les sphères S passent par cette droite. En effet, en vertu d'une propriété démontrée plus haut, la droite i est perpendiculaire, en P, au rayon OP, relatif à une quelconque des sphères S; elle est donc située dans le plan tangent, en P, à la sphère considérée et, comme elle est isotrope, elle appartient à cette sphère, c.q.f.d.

Concluons de là que deux sphères S sont tangentes.

*Dans le cas présent, la binormale et la normale principale de (O) coïncident avec la droite isotrope de ω, issue de O. Donc, comme dans le premier cas, la tangente à (P) est parallèle à la normale principale de (O).

Si l'on a: $\frac{d\alpha}{ds}=0$, $\frac{d\beta}{ds}=0$, $\frac{d\gamma}{ds}=0$, le point P est fixe et le point O décrit une droite passant par P. Les sphères sont donc tangentes deux à deux en P.

2. Soit, en coordonnées pentasphériques $x_1, x_2, \ldots, x_5$,

$$\sum m_i x_i = 0$$

l'équation de la sphère S. Les coordonnées $m_1, m_2, \ldots, m_5$ de cette sphère sont des fonctions de u; supposons-les choisies de manière que l'on ait

$$\sum m_i^2 = 1. \tag{4}$$

Désignons par S' la sphère S qui correspond à la valeur $u+du$ du paramètre u et par $m_1', m_2', \ldots, m_5'$ les coordonnées de cette sphère. Si V est l'angle des sphères S, S', on a

$$\cos V = \sum m_i m_i',$$

ou, en remplaçant $\cos V$ et les m_i' par leurs développements en séries, et en tenant compte de (4),

$$V^2 - \frac{V^4}{12} + \ldots = -\sum m_i \left(2\, dm_i + d^2 m_i + \frac{1}{3} d^3 m_i + \frac{1}{12} d^4 m_i + \ldots\right). \tag{5}$$

Différentions l'égalité (4) quatre fois de suite:

$$\begin{cases} \sum m_i dm_i = 0, \\ \sum m_i d^2 m_i + \sum dm_i^2 = 0, \end{cases} \tag{6}$$

$$\begin{cases} \sum m_i d^3 m_i + 3\sum dm_i d^2 m_i = 0, \\ \sum m_i d^4 m_i + 4\sum dm_i d^3 m_i + 3\sum (d^2 m_i)^2 = 0. \end{cases} \tag{7}$$

On a, d'autre part,

$$d\sum dm_i^2 = 2\sum dm_i d^2 m_i$$
$$d^2 \sum dm_i^2 = 2\sum dm_i d^3 m_i + 2\sum (d^2 m_i)^2.$$

Si l'on porte, dans les égalités (7), les valeurs de $\sum dm_i d^2 m_i$ et de $\sum dm_i d^3 m_i$ tirées des égalités précédentes, il vient

$$\sum m_i d^3 m_i = -\frac{3}{2} d\sum (dm_i)^2,$$

$$\sum m_i d^4 m_i = -2d^2 \sum (dm_i)^2 + \sum (d^2 m_i)^2.$$

A cause de ces égalités et des égalités (6), la formule (5) s'écrit

$$V^2 - \frac{V^4}{12} + \ldots = \sum (dm_i)^2 + \frac{1}{2} d\sum (dm_i)^2 + \frac{1}{6} d^2 \sum (dm_i)^2 - \frac{1}{12} \sum (d^2 m_i)^2 + \ldots \tag{8}$$

Il y a maintenant trois cas à considérer.

Premier cas: $(\sum dm_i)^2 \neq 0$. Alors la caractéristique Γ de S est un cercle de rayon non nul. En effet, Γ est l'intersection de S et de la sphère Σ définie par l'équation $\sum dm_i . x_i = 0$. En vertu de l'hypothèse, Σ n'est pas de rayon nul et, à cause de la première des égalités (6), cette sphère est orthogonale à S. Dès lors, Γ a un rayon différent de zéro. La formule (8) donne, dans le cas présent, θ désignant la partie principale de V,

$$\theta^2 = \sum (dm_i)^2. \tag{A}$$

Exprimons θ^2 au moyen des coordonnées x, y, z du point O et de R. La sphère S a pour équation

$$X^2 + Y^2 + Z^2 - 2xX - 2yY - 2zZ + x^2 + y^2 + z^2 - R^2 = 0$$

ou

$$m_1X_1 + m_2X_2 + m_3X_3 + m_4X_4 + m_5X_5 = 0,$$

étant posé

$$X_1 = X,\ X_2 = Y,\ X_3 = Z,\ X_4 = \frac{X^2 + Y^2 + Z^2 - 1}{2},\ X_5 = i\frac{X^2 + Y^2 + Z^2 + 1}{2};$$

$$m_1 = \frac{x}{R},\ m_2 = \frac{y}{R},\ m_3 = \frac{z}{R},\ m_4 - im_5 = \frac{x^2 + y^2 + z^2 - R^2}{R},\ m_4 + im_5 = -\frac{1}{R}.$$

$X_1, X_2, \ldots, X_5$ étant liées par la relation $\sum X_i^2 = 0$ sont des coordonnées penta-sphériques; d'autre part, les coordonnées $m_1, m_2, \ldots, m_5$ de la sphère S sont liées par la relation (4); la formule (A) est donc applicable. Si l'on tient compte des expressions ci-dessus des quantités m_i, il vient

$$\theta^2 = \frac{ds^2 - dR^2}{R^2}{}^*. \tag{B}$$

Deuxième cas: $\sum (dm_i)^2 = 0$, $\sum (d^2m_i)^2 \neq 0$. Alors la caractéristique de la sphère S est un cercle de rayon nul dont le centre P, situé sur la tangente à la trajectoire du point O, a pour coordonnées $dm_1, dm_2, \ldots, dm_5$. En vertu de l'inégalité $\sum (d^2m_i)^2 \neq 0$, la vitesse du point P n'est pas nulle. La formule (8) donne, si l'on désigne par θ la partie principale de V,

$$\theta^2 = -\frac{1}{12}\sum (d^2m_i)^2. \tag{C}$$

Soit σ la coordonnée curviligne du point P. On a, R_i désignant le rayon de la sphère $x_i = 0$,

$$d\sigma^2 = \frac{(\sum d^2m_i)^2}{\sum \left(\frac{dm_i}{R_i}\right)^2}.$$

Le rayon R de la sphère S est donné par la formule

$$\frac{1}{R} = \sum \frac{m_i}{R_i}.$$

*Supposons que O décrive une courbe minima. Si R est variable, $\theta = i\frac{dR}{R}$. Si R est constant,

$$\theta = \frac{i}{4\sqrt{3}} \cdot \frac{\sqrt{Sx''^2}}{R}\, du^2.$$

Lorsque O décrit une droite isotrope, on a, si R est variable, $\theta = i\frac{dR}{R}$, et, si R est constant, $V = 0$.

On déduit de là, par différentiation,

$$d\frac{1}{R}=\sum\frac{dm_i}{R_i}.$$

Par suite, l'expression de $d\sigma^2$ peut s'écrire

$$d\sigma^2=\frac{(\Sigma d^2m_i)^2}{\left(d\dfrac{1}{R}\right)^2}.$$

Si l'on remplace, dans l'expression (C) de θ^2, $(\Sigma d^2m_i)^2$ par sa valeur tirée de l'égalité précédente, il vient

$$\text{(D)}\qquad \theta=\sqrt{-\frac{1}{12}\,d\frac{1}{R}}\,d\sigma^*.$$

Troisième cas: $\Sigma(dm_i)^2=0$, $\Sigma(d^2m_i)^2=0$. Dans ce cas, comme dans le précédent, la caractéristique de S est un cercle de rayon nul dont le centre P est situé sur la tangente à la trajectoire du point O. En vertu de l'égalité $\Sigma(d^2m_i)^2=0$, la vitesse du point P est nulle; donc ainsi qu'on l'a vu plus haut, deux sphères S se touchent et leur angle V est nul.

II

3. Rapportons une surface quelconque S au réseau (u, v) de ses lignes de courbure et attachons à tout point M de cette surface un trièdre trirectangle $Mxyz$ ou T, de rotation positive, dont les arêtes Mx, My sont respectivement tangentes aux lignes $v=\text{const.}$, $u=\text{const.}$ qui passent par ce point†. Nous conserverons toutes les notations de M. Darboux‡. Soit

$$\text{(9)}\qquad ds^2=A^2du^2+C^2dv^2$$

la formule donnant l'élément linéaire de la surface. Les translations A, C et les rotations q, p_1, r, r_1 du trièdre $Mxyz$ satisfont aux relations

*Soit R le rayon de courbure d'une ligne plane, en un point variable de coordonnée curviligne σ. Il suit de la formule (D) que l'intégrale

$$\int_{AB}\frac{\sqrt{dRd\sigma}}{R},$$

étendue à un arc quelconque AB de cette ligne, est invariante dans toute inversion. Ce résultat nous a été communiqué, sans démonstration, par M. A. Bloch, dans une lettre qu'il nous a adressée en août 1921.

†Pour la concision du discours, le trièdre T sera désigné plus bas sous le nom de trièdre principal de la surface, relatif au point M.

‡*Leçons sur la théorie des surfaces*, II^e partie.

$$(10)\qquad \begin{cases} \dfrac{\partial A}{\partial v} = -rC, \quad \dfrac{\partial C}{\partial u} = r_1 A, \\[2mm] \dfrac{\partial q}{\partial v} = rp_1, \quad \dfrac{\partial p_1}{\partial u} = -r_1 q, \\[2mm] \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = -qp_1{}^{*}. \end{cases}$$

Soient x, y, z les coordonnées d'un point rapporté au trièdre T et δx, δy, δz les parties infiniment petites du premier ordre des composantes du déplacement de ce point suivant les axes Mx, My, Mz. Si l'on pose

$$(11)\qquad \delta x = \frac{\delta x}{\partial u}du + \frac{\delta x}{\partial v}dv, \quad \delta y = \frac{\delta y}{\partial u}du + \frac{\delta y}{\partial v}dv, \quad \delta z = \frac{\delta z}{\partial u}du + \frac{\delta z}{\partial v}dv,$$

on a

$$(12)\qquad \begin{cases} \dfrac{\delta x}{\partial u} = A + qz - ry + \dfrac{\partial x}{\partial u}, & \dfrac{\delta x}{\partial v} = -r_1 y + \dfrac{\partial x}{\partial v}, \\[2mm] \dfrac{\delta y}{\partial u} = rx + \dfrac{\partial y}{\partial u}, & \dfrac{\delta y}{\partial v} = C + r_1 x - p_1 z + \dfrac{\partial y}{\partial v}, \\[2mm] \dfrac{\delta z}{\partial u} = -qx + \dfrac{\partial z}{\partial u}; & \dfrac{\delta z}{\partial v} = p_1 y + \dfrac{\partial z}{\partial v}. \end{cases}$$

Soient x, y, z les coordonnées d'un point rapporté à un trièdre T' de sommet fixe O et parallèle au trièdre T. Si δx, δy, δz ont la même signification que plus haut, les quantités $\dfrac{\delta x}{\partial u}$, $\dfrac{\delta y}{\partial u}$, ..., qui figurent dans les formules (11), sont données par les formules (12), où l'on remplacera A et C par zéro.

Soient l, l', l'' les cosinus directeurs d'une demi-droite d, rapportée au trièdre T, et OU un segment unitaire, parallèle à d. Le point U, rapporté au trièdre T', a pour coordonnées l, l', l'', et l'on a, en vertu de la remarque précédente,

$$(13)\qquad \begin{cases} \dfrac{\delta l}{\partial u} = ql'' - rl' + \dfrac{\partial l}{\partial u}, & \dfrac{\delta l}{\partial v} = -r_1 l' + \dfrac{\partial l}{\partial v}, \\[2mm] \dfrac{\delta l'}{\partial u} = rl + \dfrac{\partial l'}{\partial u}, & \dfrac{\delta l'}{\partial v} = r_1 l - p_1 l'' + \dfrac{\partial l'}{\partial v}, \\[2mm] \dfrac{\delta l''}{\partial u} = -ql + \dfrac{\partial l''}{\partial u}; & \dfrac{\delta l''}{\partial v} = p_1 l' + \dfrac{\partial l''}{\partial v}. \end{cases}$$

4. Si le point M décrit la ligne (M_u), c'est-à-dire la ligne de courbure $v=$ const., la caractéristique du plan yMz coupe Mz au centre de courbure normale C_1 et My au centre de courbure géodésique G_1 de (M_u). De même, si M décrit la ligne

*Réciproquement, six quantités A, C, q, p_1, r, r_1 qui satisfont aux relations (10) sont les translations et rotations du trièdre principal d'une surface.

(M_v), la caractéristique du plan xMz coupe Mz au centre de courbure normale C_2 et Mx au centre de courbure géodésique G_2 de (M_v).

Soient $\frac{1}{\rho_1}$, $\frac{1}{\rho_2}$ les cotes des points C_1, C_2, $\frac{1}{\gamma_1}$ l'ordonnée du point G_1 et $-\frac{1}{\gamma_2}$ l'abscisse du point G_2. On a

(14) $$\rho_1 = -\frac{q}{A}, \quad \rho_2 = \frac{p_1}{C}, \quad \gamma_1 = \frac{r}{A}, \quad \gamma_2 = \frac{r_1}{C}.$$

Si l'on tient compte de ces égalités et qu'on désigne par s_1 et s_2 les arcs des lignes (M_u) et (M_v), les sens des arcs croissants étant choisis de manière que les demi-tangentes positives de ces lignes soient Mx, My, les formules (12) et (13) s'écrivent:

(15) $$\begin{cases} \dfrac{\delta x}{\partial s_1} = 1 - \rho_1 z - \gamma_1 y + \dfrac{\partial x}{\partial s_1}, & \dfrac{\delta x}{\partial s_2} = -\gamma_2 y + \dfrac{\partial x}{\partial s_2}, \\ \dfrac{\delta y}{\partial s_1} = \gamma_1 x + \dfrac{\partial y}{\partial s_1}, & \dfrac{\delta y}{\partial s_2} = 1 + \gamma_2 x - \rho_2 z + \dfrac{\partial y}{\partial s_2}, \\ \dfrac{\delta z}{\partial s_1} = \rho_1 x + \dfrac{\partial z}{\partial s_1}; & \dfrac{\delta z}{\partial s_2} = \rho_2 y + \dfrac{\partial z}{\partial s_2}; \end{cases}$$

(16) $$\begin{cases} \dfrac{\delta l}{\partial s_1} = -\rho_1 l'' - \gamma_1 l' + \dfrac{\partial l}{\partial s_1}, & \dfrac{\delta l}{\partial s_2} = -\gamma_2 l' + \dfrac{\partial l}{\partial s_2}, \\ \dfrac{\delta l'}{\partial s_1} = \gamma_1 l + \dfrac{\partial l'}{\partial s_1}, & \dfrac{\delta l'}{\partial s_2} = \gamma_2 l - \rho_2 l'' + \dfrac{\partial l'}{\partial s_2}, \\ \dfrac{\delta l''}{\partial s_1} = \rho_1 l + \dfrac{\partial l''}{\partial s_1}; & \dfrac{\delta l''}{\partial s_2} = \rho_2 l' + \dfrac{\partial l''}{\partial s_2}, \end{cases}$$

à condition de poser, θ désignant une quelconque des quantités x, y, z, l, l', l'',

$$\frac{\delta \theta}{\partial s_1} = \frac{1}{A}\,\frac{\delta \theta}{\partial u}, \quad \frac{\delta \theta}{\partial s_2} = \frac{1}{C}\,\frac{\delta \theta}{\partial u}.$$

La signification géométrique des quantités $\frac{\delta x}{\partial s_1}$, $\frac{\delta y}{\partial s_1}$, $\dots$, $\frac{\delta z}{\partial s_2}$, $\frac{\delta l}{\partial s_1}$, $\frac{\delta l'}{\partial s_1}$, $\dots$, $\frac{\delta l''}{\partial s_2}$ est évidente.

Des égalités (14) et (10), on déduit les formules connues

(17) $$\gamma_1 = \frac{\dfrac{\partial \rho_1}{\partial s_2}}{\rho_1 - \rho_2}, \quad \gamma_2 = \frac{\dfrac{\partial \rho_2}{\partial s_1}}{\rho_1 - \rho_2},$$

(18) $$\frac{\partial \gamma_1}{\partial s_2} - \frac{\partial \gamma_2}{\partial s_1} - \rho_1\rho_2 - \gamma_1^2 - \gamma_2^2 = 0.$$

Désignons par S_{C1} et S_{G1} les sphères qui passent par le cercle osculateur de (M_u) en M et qui sont respectivement tangente et orthogonale à la surface S en ce point. Ces sphères ont respectivement pour centre les points C_1 et G_1. La première est la sphère de courbure normale de (M_u). Nous avons proposé d'appeler la seconde: *sphère de courbure géodésique* de (M_u). Désignons de même par S_{C2} et S_{G2} la sphère de courbure normale et la sphère de courbure géodésique de (M_v). En raison de leur définition, ces quatre sphères sont conservées dans toute transformation conforme.

5. Marquons sur la ligne (M_u) un point M' de coordonnées $u+du$, v et, sur la ligne (M_v) un point M'' de coordonnées u, $v+dv$. Désignons par ϕ_1, χ_1, ψ_1, ϖ_1 les parties principales des angles que les sphères S_{C1}, S_{G1}, S_{C2}, S_{G2} font respectivement avec les sphères de même définition, relatives à M', et par ϕ_2, χ_2, ψ_2, ϖ_2 les parties principales des angles que les sphères S_{C1}, S_{G1}, S_{C2}, S_{G2} font respectivement avec les sphères de même définition, relatives à M''. Nous allons évaluer ces huit quantités.

La formule (D) donne, sans calcul,

$$\text{(19)} \qquad \left\{ \begin{aligned} \sqrt{-12}\,\phi_1 &= \frac{\partial \rho_1}{\partial u} A du^2, \\ \sqrt{-12}\,\chi_1 &= \frac{\partial \gamma_1}{\partial u} A du^2, \\ \sqrt{-12}\,\psi_2 &= \frac{\partial \rho_2}{\partial v} C dv^2, \\ \sqrt{-12}\,\varpi_2 &= \frac{\partial \gamma_2}{\partial v} C dv^2. \end{aligned} \right.$$

Etablissons une formule qui donnera ψ_1 et ϕ_2. Imaginons une sphère Σ tangente à S en M et définie pour chaque position de ce point. Soit $\dfrac{1}{\rho}$ la cote de son centre C. Les coordonnées x, y, z du point C sont donc $0, 0, \dfrac{1}{\rho}$. En vertu de la formule (B), la partie principale θ de l'angle de deux positions de Σ, infiniment voisines, est donnée par l'égalité

$$\theta^2 = \left[\delta x^2 + \delta y^2 + \delta z^2 - \left(d\,\frac{1}{\rho} \right)^2 \right] \rho^2.$$

On a, d'autre part, par application des formules (11) et (12),

$$\delta x = \frac{A}{\rho}(\rho_1 - \rho_2)\,du, \;\; \delta y = \frac{C}{\rho}(\rho_1 - \rho_2)dv, \;\; \delta z = d\,\frac{1}{\rho}.$$

Par suite,

(20) $$\theta^2 = A^2(\rho-\rho_1)^2du^2 + C^2(\rho-\rho_2)^2dv^2 *.$$

Si l'on fait, dans cette formule, $\rho=\rho_2$, puis $\rho=\rho_1$, on obtient les valeurs de ψ_1 et de ϕ_2:

(21) $$\psi_1 = A(\rho_1-\rho_2)du,$$

(22) $$\phi_2 = C(\rho_1-\rho_2)dv.$$

Calculons ϖ_1. Les coordonnées x, y, z du point G_1 sont $-\dfrac{1}{\gamma_2}$, 0, 0. On a, en vertu de la formule (B),

$$\varpi_1^2 = \left[\left(\frac{\delta x}{\partial u}\right)^2 + \left(\frac{\delta y}{\partial u}\right)^2 + \left(\frac{\delta z}{\partial u}\right)^2 - \left(\frac{\partial \frac{1}{\gamma_2}}{\partial u}\right)^2\right]\gamma_2^2 du^2.$$

Or les formules (12) donnent

$$\frac{\delta x}{\partial u} = A - \frac{\partial}{\partial u}\frac{1}{\gamma_2}, \quad \frac{\delta y}{\partial u} = \frac{r}{\gamma_2}, \quad \frac{\delta z}{\partial u} = \frac{q}{\gamma_2}.$$

Par suite,

(23) $$\varpi_1^2 = \left(\rho_1^2 + \gamma_1^2 + \gamma_2^2 + 2\frac{\partial \gamma_2}{\partial s_1}\right)A^2du^2.$$

On trouvera de même

(24) $$\chi_2^2 = \left(\rho_2^2 + \gamma_1^2 + \gamma_2^2 - 2\frac{\partial \gamma_1}{\partial s_2}\right)C^2dv^2.$$

*Si l'on désigne par m le rapport anharmonique (MCC_1C_2), on a $\rho = \dfrac{\rho_2 - m\rho_1}{1-m}$. En portant cette valeur de ρ dans la formule (20), on trouve

(*a*) $$\theta^2 = \frac{(\rho_1-\rho_2)^2}{(1-m)^2}(A^2du^2 + m^2C^2dv^2)$$

ou, en introduisant les invariants Φ_1, Φ_2 définis dans l'Introduction (voir aussi le N° 12),

(*b*) $$\theta^2 = \frac{1}{(1-m)^2}(\Phi_1^2\, du^2 + m^2\Phi_2^2 dv^2).$$

Si Σ est la sphère harmonique, $m=-1$ et la formule (*b*) devient

(*c*) $$\theta^2 = \frac{1}{4}(\Phi_1^2du^2 + \Phi_2^2dv^2).$$

Si la sphère Σ est telle que $m = \dfrac{A}{C}\dfrac{V}{U}$ ou $\dfrac{\Phi_1}{\Phi_2}\dfrac{V}{U}$, U désignant une fonction de u et V une fonction de v, la même formule donne

(*d*) $$\theta^2 = \frac{U^2du^2 + V^2dv^2}{\left(\dfrac{U}{\Phi_1} - \dfrac{V}{\Phi_2}\right)^2};$$

On voit que les lignes tracées sur la surface, pour lesquelles $\theta=0$, se déterminent par quadratures.

III

6. Toutes les notations des N^os 3 et 4 étant conservées, soumettons la surface S à une inversion de pôle P et de puissance k. Soient x_0. y_0, z_0 les coordonnées relatives du point P. Les coordonnées relatives x, y, z de l'inverse M' du point M sont données par les égalités

$$x = x_0 - \frac{kx_0}{\lambda^2}, \quad y = y_0 - \frac{ky_0}{\lambda^2}, \quad z = z_0 - \frac{kz_0}{\lambda^2}, \tag{25}$$

étant posé

$$x_0^2 + y_0^2 + z_0^2 = \lambda^2. \tag{26}$$

Le point P étant fixe, on a en vertu des formules (15),

$$\left\{\begin{aligned} \frac{\partial x_0}{\partial s_1} &= \gamma_1 y_0 + \rho_1 z_0 - 1, & \frac{\partial x_0}{\partial s_2} &= \gamma_2 y_0, \\ \frac{\partial y_0}{\partial s_1} &= -\gamma_1 x_0, & \frac{\partial y_0}{\partial s_2} &= -\gamma_2 x_0 + \rho_2 z_0 - 1, \\ \frac{\partial z_0}{\partial s_1} &= -\rho_1 x_0; & \frac{\partial z_0}{\partial s_2} &= -\rho_2 y_0. \end{aligned}\right. \tag{27}$$

Dérivons l'égalité (26) par rapport à s_1, puis par rapport à s_2; il viendra, en tenant compte de (27),

$$\lambda \frac{\partial \lambda}{\partial s_1} = -x_0, \quad \lambda \frac{\partial \lambda}{\partial s_2} = -y_0. \tag{28}$$

Si l'on remplace, dans les formules (15), x, y, z par leurs expressions (25), on trouve en tenant compte des égalités (27) et (28),

$$\left\{\begin{aligned} &\frac{\delta x}{\partial s_1} = \frac{k}{\lambda^2} - \frac{2kx_0^2}{\lambda^4}, \quad \frac{\delta y}{\partial s_1} = -\frac{2kx_0y_0}{\lambda^4}, \quad \frac{\delta z}{\partial s_1} = -\frac{2kx_0z_0}{\lambda^4}, \\ &\frac{\delta x}{\partial s_2} = -\frac{2kx_0y_0}{\lambda^4}, \quad \frac{\delta y}{\partial s_2} = \frac{k}{\lambda^2} - \frac{2ky_0^2}{\lambda^4}, \quad \frac{\delta z}{\partial s_2} = -\frac{2ky_0z_0}{\lambda^4}. \end{aligned}\right. \tag{29}$$

Soit

$$ds'^2 = A'^2 du^2 + C'^2 dv^2$$

la formule donnant l'élément linéaire de la surface S', lieu du point M'. On a

$$A'^2 = \mathbf{S}\left(\frac{\delta x}{\partial u}\right)^2 = A^2\,\mathbf{S}\left(\frac{\delta x}{\partial s_1}\right)^2,$$

$$C'^2 = \mathbf{S}\left(\frac{\delta x}{\partial v}\right)^2 = C^2\,\mathbf{S}\left(\frac{\delta x}{\partial s_2}\right)^2,$$

d'où, en vertu des formules (29),

$$A'^2 = \frac{A^2k^2}{\lambda^4}, \quad C'^2 = \frac{C^2k^2}{\lambda^4}.$$

On peut donc poser:

(30) $$A' = -\frac{Ak}{\lambda^2}, \; C' = -\frac{Ck}{\lambda^2}.$$

Soient a, a', a'' les cosinus directeurs de la tangente à la ligne (M'_u); b, b', b'' les cosinus directeurs de la tangente à la ligne (M'_v); c, c', c'' les cosinus directeurs de la normale à la surface S'. En se servant des formules (29), (30) et des suivantes

$$c = a'b'' - b'a'', \; c' = a''b' - b''a', \; c'' = ab' - ba',$$

on trouve:

(31) $$\begin{cases} a = \dfrac{2x_0^2}{\lambda^2} - 1, & a' = \dfrac{2x_0y_0}{\lambda^2}, & a'' = \dfrac{2x_0z_0}{\lambda^2}, \\ b = \dfrac{2x_0y_0}{\lambda^2}, & b' = \dfrac{2y_0^2}{\lambda^2} - 1, & b'' = \dfrac{2y_0z_0}{\lambda^2}, \\ c = \dfrac{2x_0z_0}{\lambda^2}, & c' = \dfrac{2y_0z_0}{\lambda^2}, & c'' = \dfrac{2z_0^2}{\lambda^2} - 1. \end{cases}$$

Soient $M'x'$, $M'y'$, $M'z'$ les demi-droites qui ont respectivement pour cosinus directeurs a, a', a''; b, b', b''; c, c', c''. Les prolongements de ces demi-droites sont respectivement les symétriques des demi-droites Mx, My, Mz par rapport au plan mené, par le milieu de MM', perpendiculairement à ce segment.

A', C' sont les deux translations non nulles du trièdre $M'x'y'z'$. En effet, les formules (29), (30) et (31) donnent*

$$\frac{\delta x}{\partial u} = A'a, \; \frac{\delta y}{\partial u} = A'a', \; \frac{\delta z}{\partial u} = A'a'',$$

$$\frac{\delta x}{\partial v} = C'b, \; \frac{\delta y}{\partial v} = C'b', \; \frac{\delta z}{\partial v} = C'b''.$$

7. Soient, relativement au trièdre $M'x'y'z'$, $\dfrac{1}{\rho_1'}$, $\dfrac{1}{\rho_2'}$ les cotes des centres de courbure principaux de S', $\dfrac{1}{\gamma_1'}$ l'ordonnée du centre de courbure géodésique de (M'_u) et $-\dfrac{1}{\gamma_2'}$ l'abscisse du centre de courbure géodésique de (M'_v). On a, en vertu des formules d'Olinde Rodrigues et des formules analogues, relatives à la courbure géodésique des lignes de courbure,

$$\frac{\delta c''}{\partial s_1} + \rho_1' \frac{\delta z}{\partial s_1} = 0, \; \frac{\delta c''}{\partial s_2} + \rho_1'' \frac{\delta z}{\partial s_2} = 0,$$

$$\frac{\delta b'}{\partial s_1} + \gamma_1' \frac{\delta y}{\partial s_1} = 0, \; \frac{\delta a}{\partial s_2} - \gamma_2' \frac{\delta x}{\partial s_2} = 0.$$

*On remplacera, dans les formules (29), $\dfrac{\delta x}{\partial s_1}$, $\dfrac{\delta y}{\partial s_1}$, ..., par $\dfrac{1}{A}\dfrac{\delta x}{\partial u}$, $\dfrac{1}{A}\dfrac{\delta y}{\partial u}$, ...

Aux premiers membres de ces égalités, les premiers termes se calculent en faisant usage des formules (16), (31), (27) et (28). Les coefficients de ρ_1', ρ_2', γ_1', γ_2' sont donnés par le tableau (29). On trouve, tous calculs faits,:

$$(32)\qquad \begin{cases} \rho_1' = \dfrac{2z_0}{k} - \dfrac{\lambda^2}{k}\rho_1, & \rho_2' = \dfrac{2z_0}{k} - \dfrac{\lambda^2}{k}\rho_2, \\[2ex] \gamma_1' = \dfrac{2y_0}{k} - \dfrac{\lambda^2}{k}\gamma_1, & \gamma_2' = -\dfrac{2x_0}{k} - \dfrac{\lambda^2}{k}\gamma_2. \end{cases}$$

8. Posons

$$(33)\qquad \Phi_1 = A(\rho_1 - \rho_2),\ \Phi_2 = C(\rho_1 - \rho_2)^*.$$

En vertu des formules (21), (22), $|\Phi_1|$, $|\Phi_2|$ conservent leurs valeurs dans toute transformation conforme, car les sphères S_{C_1}, S_{C_2} sont conservées dans une telle transformation. Nous allons démontrer qu'il en est de même de Φ_1 et de Φ_2, le trièdre T_1 (de rotation positive) qu'il convient d'attacher au point de la surface transformée qui correspond au point M étant convenablement choisi. Il suffira de prouver que Φ_1, Φ_2 sont invariantes dans l'inversion, dans l'homothétie et dans le déplacement, car toute transformation conforme est une de ces transformations ou le produit de deux d'entre elles ou le produit des trois transformations.

Deux des formules (32) donnent, par soustraction,

$$\rho_1' - \rho_2' = -\frac{\lambda^2}{k}(\rho_1 - \rho_2).$$

On a, par suite, en tenant compte des égalités (30),

$$A'(\rho_1' - \rho_2') = A(\rho_1 - \rho_2),$$
$$C'(\rho_1' - \rho_2') = C(\rho_1 - \rho_2).$$

Donc les quantités Φ_1, Φ_2 sont invariantes dans l'inversion, si $M'x'y'z'$ est le trièdre attaché au point M'.

Soit M'' l'homologue du point M dans une homothétie quelconque. Chacun des quatre trièdres (de rotation positive) qu'on peut attacher au point M'' a ses arêtes parallèles à celles du trièdre $Mxyz$. Φ_1, Φ_2 seront invariantes dans l'homothétie considérée, si les arêtes $M''x''$, $M''y''$, $M''z''$ du trièdre $M''x''y''z''$ attaché au point M'' ont respectivement même sens que les arêtes Mx, My, Mz du trièdre $Mxyz$. En effet, si h désigne le rapport d'homothétie, les translations non nulles

*Les valeurs des quantités $A(\rho_1-\rho_2)$, $C(\rho_1-\rho_2)$ dépendent non seulement du point M, mais aussi du trièdre (de rotation positive) $Mxyz$. Soient Mx_1, My_1, Mz_1 les prolongements des demi-droites Mx, My, Mz. Si l'on substitue successivement, au trièdre $Mxyz$, les trièdres Mx_1yz_1, Mxy_1z_1, Mx_1y_1z, les quantités $A(\rho_1-\rho_2)$, $C(\rho_1-\rho_2)$ seront respectivement égales à Φ_1, $-\Phi_2$, dans le premier cas, à $-\Phi_1$, Φ_2, dans le deuxième cas, et à $-\Phi_1$, $-\Phi_2$, dans le troisième.

A'', C'' du trièdre $M''x''y''z''$ et les inverses ρ_1'', ρ_2'' des cotes des centres de courbure principaux en M'' sont donnés par les égalités

$$A''=hA,\ C''=hC,$$

$$\rho_1''=\frac{\rho_1}{h},\ \rho_2''=\frac{\rho_2}{h},$$

et l'on déduit de là

$$A''(\rho_1''-\rho_2'')=A(\rho_1-\rho_2),$$
$$C''(\rho_1''-\rho_2'')=C(\rho_1-\rho_2).$$

Enfin, il est évident que Φ_1, Φ_2 conservent leurs valeurs dans un déplacement quelconque, le trièdre attaché au point qui correspond au point M étant celui qu'on obtient en soumettant le trièdre $Mxyz$ au déplacement considéré.

Dès lors, Φ_1, Φ_2 sont invariantes dans toute transformation conforme. La détermination du trièdre T_1 attaché au point qui correspond au point M résulte clairement des considérations précédentes.* Φ_1, Φ_2 sont des invariants paramétriques, car A, C, ρ_1, ρ_2 peuvent être exprimées en fonction de dérivées de x, de y et de z par rapport à u et à v.

IV

9. Posons

$$\Omega_1=\sqrt{-12}\,\frac{\phi_1}{\psi_1^2},\ \Omega_2=\sqrt{-12}\,\frac{\psi_2}{\phi_2^2},\ \ \alpha_1=\sqrt{-12}\,\frac{x_1}{\psi_1^2},\ \ \alpha_2=\sqrt{-12}\,\frac{\varpi_2}{\phi_2^2},$$

$$\beta_1=\left(\frac{\varpi_1}{\psi_1}\right)^2,\ \beta_2=\left(\frac{\chi_2}{\phi_2}\right)^2.$$

En vertu des formules (19), (21), (22), (23) et (24), on a

$$\Omega_1=\frac{\dfrac{\partial\rho_1}{\partial s_1}}{(\rho_1-\rho_2)^2},\quad \Omega_2=\frac{\dfrac{\partial\rho_2}{\partial s_2}}{(\rho_1-\rho_2)^2},\tag{34}$$

$$\alpha_1=\frac{\dfrac{\partial\gamma_1}{\partial s_1}}{(\rho_1-\rho_2)^2},\quad \alpha_2=\frac{\dfrac{\partial\gamma_2}{\partial s_2}}{(\rho_1-\rho_2)^2},\tag{35}$$

$$\beta_1=\frac{\rho_1^2+\gamma_1^2+\gamma_2^2+2\dfrac{\partial\gamma_2}{\partial s_1}}{(\rho_1-\rho_2)^2},\quad \beta_2=\frac{\rho_2^2+\gamma_1^2+\gamma_2^2-2\dfrac{\partial\gamma_1}{\partial s_2}}{(\rho_1-\rho_2)^2}.\tag{36}$$

Les sphères S_{C1}, S_{C2}, S_{G1}, S_{G2} étant conservées dans toute transformation conforme, il est clair que les valeurs absolues des quantités Ω_1, Ω_2, α_1, α_2, β_1, β_2 sont invariantes dans une telle transformation. En vertu de la remarque faite dans la deuxième note du N° 8, il en est de même de Ω_1, Ω_2, α_1, α_2, le trièdre T_1

*On reconnaitra aisément que, pour ce choix du trièdre T_1, les quantités $\dfrac{\partial\rho_i}{\partial s_k}$, $\dfrac{\partial\gamma_i}{\partial s_k}$, $(i,k=1,2)$, conservent leurs signes. Cette remarque sera utilisée plus bas.

attaché au point qui correspond au point M étant celui qui a été défini au N° 8. Les quantités β_1, β_2 sont aussi invariantes, car elles sont positives.

Les formules (34) montrent que Ω_1, Ω_2 sont des invariants différentiels. α_1, α_2, β_1, β_2 sont aussi des invariants différentiels, car les formules (17) permettent d'exprimer ces quantités en fonction de ρ_1, de ρ_2 et de quelques-unes de leurs dérivées par rapport à s_1 et à s_2.

Les invariants β_1, β_2 ne sont pas indépendants, car on a, en vertu de la relation (18),

$$\beta_1+\beta_2=1. \tag{37}$$

Introduisons à la place de β_1, β_2, l'invariant Ω_3 défini par l'égalité

$$\Omega_3=\beta_1-\beta_2. \tag{38}$$

Ω_3 a pour expression

$$\Omega_3=\frac{\rho_1^2-\rho_2^2+2\dfrac{\partial\gamma_1}{\partial s_2}+2\dfrac{\partial\gamma_2}{\partial s_1}}{(\rho_1-\rho_2)^2}. \tag{39}$$

Des égalités (37) et (38), on déduit

$$\beta_1=\frac{1+\Omega_3}{2},\quad \beta_2=\frac{1-\Omega_3}{2}. \tag{40}$$

Les formules (17), (34), (36) et (40) donnent

$$\left\{\begin{aligned} &\frac{\partial\rho_1}{\partial s_1}=\Omega_1(\rho_1-\rho_2)^2, \quad \frac{\partial\rho_1}{\partial s_2}=\gamma_1(\rho_1-\rho_2),\\ &\frac{\partial\rho_2}{\partial s_1}=\gamma_2(\rho_1-\rho_2), \quad \frac{\partial\rho_2}{\partial s_2}=\Omega_2(\rho_1-\rho_2)^2, \end{aligned}\right. \tag{41}$$

$$\left\{\begin{aligned} &\frac{\partial\gamma_1}{\partial s_2}=\tfrac{1}{2}(\rho_2^2+\gamma_1^2+\gamma_2^2)-\tfrac{1}{4}(\rho_1-\rho_2)^2(1-\Omega_3),\\ &\frac{\partial\gamma_2}{\partial s_1}=-\tfrac{1}{2}(\rho_1^2+\gamma_1^2+\gamma_2^2)+\tfrac{1}{4}(\rho_1-\rho_2)^2(1+\Omega_3). \end{aligned}\right. \tag{42}$$

10. Soit Ω un invariant des surfaces pour le groupe conforme*. En vertu du résultat établi au N° 8, les quantités

$$\frac{\dfrac{\partial\Omega}{\partial u}}{A(\rho_1-\rho_2)},\quad \frac{\dfrac{\partial\Omega}{\partial v}}{C(\rho_1-\rho_2)}\dagger,$$

*Pour la concision du discours, nous disons *invariant* au lieu de *invariant différentiel.*

†Indiquons une interprétation géométrique de ces quantités. Soient $\Delta'\Omega$ et $\Delta''\Omega$ les accroissements que prend Ω lorsqu'on passe successivement du point M aux points M' et M''. A cause des formules (21) et (22), les quantités considérées sont respectivement les limites vers lesquelles tendent les quotients $\dfrac{\Delta'\Omega}{\psi_1}$, $\dfrac{\Delta''\Omega}{\phi_2}$ lorsque du et dv tendent vers zéro.

sont invariantes dans toute transformation conforme. On peut donc énoncer le théorème suivant:

Posons, Ω désignant une fonction de point,

(43)
$$X_1\Omega = \frac{\frac{\partial\Omega}{\partial s_1}}{\rho_1-\rho_2}, \quad X_2\Omega = \frac{\frac{\partial\Omega}{\partial s_2}}{\rho_1-\rho_2}.$$

Si Ω est un invariant, $X_1\,\Omega$ et $X_2\,\Omega$ seront aussi des invariants.

11. Nous avons obtenu deux invariants du troisième ordre, à savoir Ω_1, Ω_2; ils ont été signalés par M. Tresse. Les invariants $X_1\Omega_1$, $X_2\Omega_1$, $X_1\Omega_2$, $X_2\Omega_2$, Ω_3 sont du quatrième ordre. En effectuant, sur ces cinq invariants, les opérations X_1, X_2, on obtiendra dix invariants du cinquième ordre, et ainsi de suite. Une fonction quelconque de quelques-uns de ces invariants est évidemment un invariant. Réciproquement, *tout invariant est une fonction d'un ou de plusieurs des éléments de l'ensemble formé par Ω_1, Ω_2, Ω_3 et les quantités*

$$X_1^{\alpha_1} X_2^{\beta_1}\,\Omega_1,\ X_1^{\alpha_2} X_2^{\beta_2}\,\Omega_2,\ X_1^{\alpha_3} X_2^{\beta_3}\,\Omega_3,\ (\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \alpha_3,\ \beta_3=0,\ 1,\ 2,\ \ldots).$$

12. Pour démontrer ce théorème, nous aurons à nous appuyer sur les expressions des dérivées de ρ_1, ρ_2, γ_1, γ_2 par rapport à s_1 et à s_2. Six de ces dérivées sont données par les formules (41) et (42). Il reste à calculer $\frac{\partial\gamma_1}{\partial s_1}$, $\frac{\partial\gamma_2}{\partial s_2}$. A cet effet, nous nous servirons de l'identité bien connue

(44)
$$\frac{\partial}{\partial s_2}\frac{\partial\theta}{\partial s_1} - \frac{\partial}{\partial s_1}\frac{\partial\theta}{\partial s_2} = \gamma_1\frac{\partial\theta}{\partial s_1} + \gamma_2\frac{\partial\theta}{\partial s_2}.$$

Si l'on remplace, dans cette formule, θ par ρ_1, puis par ρ_2, et qu'on tienne compte des égalités (41) et (43), il vient

(45)
$$\begin{cases} \dfrac{\partial\gamma_1}{\partial s_1} = (X_2\Omega_1 - 2\Omega_1\Omega_2)(\rho_1-\rho_2)^2, \\ \dfrac{\partial\gamma_2}{\partial s_2} = (X_1\Omega_2 + 2\Omega_1\Omega_2)(\rho_1-\rho_2)^2. \end{cases}$$

En vertu de ces égalités, les invariants α_1, α_2, définis par les égalités (35), ont pour expressions

(46)
$$\alpha_1 = X_2\Omega_1 - 2\Omega_1\Omega_2, \quad \alpha_2 = X_1\Omega_2 + 2\Omega_1\Omega_2.$$

En remplaçant, dans l'identité (44), θ par γ_1, puis par γ_2, et en tenant compte des formules (42), (45) et (43), on obtient les relations suivantes:

(47)
$$\begin{cases} \frac{1}{4}X_1\Omega_3 - \frac{1}{2}\Omega_1(1-\Omega_3) + 2(X_2\Omega_1 - 2\Omega_1\Omega_2)\Omega_2 - X_2(X_2\Omega_1 - 2\Omega_1\Omega_2) = 0, \\ \frac{1}{4}X_2\Omega_3 - \frac{1}{2}\Omega_2(1+\Omega_3) - 2(X_1\Omega_2 + 2\Omega_1\Omega_2)\Omega_1 - X_1(X_1\Omega_2 + 2\Omega_1\Omega_2) = 0. \end{cases}$$

On a, en vertu des formules (43), A, B désignant des fonctions de point,

$$X_1AB = BX_1A + AX_1B,$$
$$X_2AB = BX_2A + AX_2B.$$

En se servant de ces formules, on reconnaîtra immédiatement que les identités (47) permettent d'exprimer les quantités $X_1^{\alpha_3} X_2^{\beta_3} \Omega_3$, $(\alpha_3, \beta_3 = 0, 1, 2, \ldots)$, au moyen de Ω_1, Ω_2, Ω_3 et des quantités $X_1^{\alpha_1} X_2^{\beta_1} \Omega_1$, $X_1^{\alpha} X_2^{\beta_2} \Omega_2$, $(\alpha_1, \beta_1, \alpha_2, \beta_2 = 0, 1, 2, \ldots)$.

13. Etablissons le théorème énoncé à la fin du N° 11. On a vu, dans l'Introduction, que tout invariant différentiel Ω des surfaces pour le groupe conforme est une fonction d'un ou de plusieurs des éléments de l'ensemble formé par ρ_1, ρ_2 et leurs dérivées par rapport aux arcs des lignes de courbure. Soit

$$\Omega = f\left(\rho_1, \ \rho_2, \ \frac{\partial \rho_1}{\partial s_1}, \ \frac{\partial \rho_1}{\partial s_2}, \ \frac{\partial \rho_2}{\partial s_1}, \ \frac{\partial \rho_2}{\partial s_2}, \ \frac{\partial^2 \rho_1}{\partial s_1^2}, \ \ldots \right).$$

A cause des formules (41), (42), (45) et (43), les dérivées de ρ_1 et de ρ_2 par rapport à s_1 et à s_2 peuvent s'exprimer au moyen de ρ_1, ρ_2, γ_1, γ_2, Ω_1, Ω_2, Ω_3 et des quantités $X_1^{\alpha_1} X_2^{\beta_1} \Omega_1$, $X_1^{\alpha_2} X_2^{\beta_2} \Omega_2$, $X_1^{\alpha_3} X_2^{\beta_3} \Omega_3$, $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 = 0, 1, 2, \ldots)$.

On peut donc mettre Ω sous la forme

$$\Omega = F(\rho_1, \rho_2, \gamma_1, \gamma_2, \Omega_1, \Omega_2, \Omega_3, X_1\Omega_1, X_1\Omega_2, X_1\Omega_3, X_2\Omega_1, \ldots).$$

Il s'agit de démontrer que F est indépendante de ρ_1, ρ_2, γ_1, γ_2. A cet effet, soumettons la surface à une inversion quelconque; nous aurons

$$F(\rho_1', \rho_2', \gamma_1', \gamma_2', \Omega_1, \Omega_2, \Omega_3, X_1\Omega_1, \ldots) = F(\rho_1, \rho_2, \gamma_1, \gamma_2, \Omega_1, \Omega_2, \Omega_3, X_1\Omega_1, \ldots).$$

En vertu de cette égalité, si F contenait une seule des quantités ρ_1, ρ_2, γ_1, γ_2, la quantité désignée par la même lettre accentuée, considérée comme fonction de x_0, y_0, z_0, k, serait constante, et si F contenait i des quantités ρ_1, ρ_2, γ_1, γ_2 $(i = 2, 3, 4)$, les quantités désignées par les mêmes lettres accentuées, considérées aussi comme fonctions de x_0, y_0, z_0, k, seraient liées par une relation. Dans tous les cas, le déterminant fonctionnel $\dfrac{\partial(\rho_1', \rho_2', \gamma_1', \gamma_2')}{\partial(x_0, y_0, z_0, k)}$ serait nul. Or, il est différent de zéro, car, des égalités (32), on déduit

$$\frac{\partial(\rho_1', \rho_2', \gamma_1', \gamma_2')}{\partial(x_0, y_0, z_0, k)} = \frac{8}{k^5}(\rho_1 - \rho_2)\lambda^2.$$

Donc F est indépendante de ρ_1, ρ_2, γ_1, γ_2, c.q.f.d.

Dès lors, *les invariants différentiels des surfaces pour le groupe conforme sont les fonctions d'un ou de plusieurs des éléments de l'ensemble formé par les invariants* Ω_1, Ω_2, Ω_3 *et les quantités*

$$X_1^{\alpha_1} X_2^{\beta_1} \Omega_1, \ X_1^{\alpha_2} X_2^{\beta_2} \Omega_2, \ X_1^{\alpha_3} X_2^{\beta_3} \Omega_3, \ (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 = 0, 1, 2, \ldots).$$

En vertu de la remarque qui termine le N° 12, ce théorème entraine le suivant:

Les invariants différentiels des surfaces pour le groupe conforme sont les fonctions d'un ou de plusieurs des éléments de l'ensemble formé par les quantités

$$\Omega_1, \ \Omega_2, \ \Omega_3, \ X_1^{\alpha_1} X_2^{\beta_1} \Omega_1, \ X_1^{\alpha_2} X_2^{\beta_2} \Omega_2, \ (\alpha_1, \beta_1, \alpha_2, \beta_2 = 0, 1, 2, \ldots).$$

V

14. Les quantités Φ_1, Φ_2 définies par les égalités (33) étant des invariants paramétriques pour le groupe conforme, l'intégrale

$$\iint(\rho_1-\rho_2)^2|AC|dudv$$

conserve sa valeur dans toute transformation conforme. C'est un invariant intégral, car, si l'on désigne par $d\sigma$ l'élément d'aire de la surface, on peut l'écrire

$$\iint(\rho_1-\rho_2)^2d\sigma. \tag{48}$$

Si Ω est un invariant différentiel, il est clair que

$$\iint\Omega(\rho_1-\rho_2)^2d\sigma \tag{49}$$

est un invariant intégral. Tout invariant intégral peut être obtenu de cette manière. En effet, si

$$\iint Fd\sigma \tag{50}$$

est un invariant intégral, $\dfrac{F}{(\rho_1-\rho_2)^2}$, quotient des éléments des intégrales (50) et (48), est un invariant différentiel. Celui-ci étant désigné par Ω, on a $F=\Omega(\rho_1-\rho_2)^2$, et, par suite, l'intégrale (50) est de la forme (49), c.q.f.d.

VI

15. Nous déterminons, dans ce paragraphe, les invariants paramétriques des surfaces pour le groupe conforme.

On connaît déja deux tels invariants, à savoir Φ_1, Φ_2. D'autre part, Ω_3 étant un invariant différentiel, est un invariant paramétrique. Il est clair que toute fonction d'un ou de plusieurs des éléments de l'ensemble formé par Φ_1, Φ_2, Ω_3 et leurs dérivées par rapport à u et à v est un invariant paramétrique. Nous allons démontrer que, réciproquement, *tout invariant paramétrique est une fonction d'un ou de plusieurs des éléments de cet ensemble.*

16. Les équations (14) et (33) permettent d'exprimer A, C, q, p_1, r, r_1 en fonction de ρ_1, ρ_2, γ_1, γ_2, Φ_1, Φ_2:

$$\left\{\begin{aligned} &A=\frac{\Phi_1}{\rho_1-\rho_2}, \quad C=\frac{\Phi_2}{\rho_1-\rho_2},\\ &q=-\frac{\Phi_1\rho_1}{\rho_1-\rho_2}, \quad p_1=\frac{\Phi_2\rho_2}{\rho_1-\rho_2},\\ &r=\frac{\Phi_1\gamma_1}{\rho_1-\rho_2}, \quad r_1=\frac{\Phi_2\gamma_2}{\rho_1-\rho_2}. \end{aligned}\right. \tag{51}$$

Exprimons les dérivées premières de ρ_1, ρ_2, γ_1, γ_2 au moyen de ces quantités et de Φ_1, Φ_2, Ω_3.

Les formules (17) et (33) donnent

$$\frac{\partial \rho_1}{\partial v}=\Phi_2\gamma_1,\quad \frac{\partial \rho_2}{\partial u}=\Phi_1\gamma_2. \tag{52}$$

Si l'on dérive par rapport à v la première des égalités (33) et par rapport à u la seconde de ces égalités, et qu'on tienne compte des égalités (33), (52), (14) et (10), il vient

$$\frac{\partial \rho_1}{\partial u}=\frac{\partial \log \Phi_2}{\partial u}(\rho_1-\rho_2),\quad \frac{\partial \rho_2}{\partial v}=-\frac{\partial \log \Phi_1}{\partial v}(\rho_1-\rho_2). \tag{53}$$

En exprimant que

$$\frac{\partial}{\partial u}\frac{\partial \rho_1}{\partial v}=\frac{\partial}{\partial v}\frac{\partial \rho_1}{\partial u},\quad \frac{\partial}{\partial u}\frac{\partial \rho_2}{\partial v}=\frac{\partial}{\partial v}\frac{\partial \rho_2}{\partial u},$$

on trouve

$$\left\{\begin{aligned}\frac{\partial \gamma_1}{\partial u}&=\frac{\rho_1-\rho_2}{\Phi_2}\left(\frac{\partial^2 \log \Phi_2}{\partial u\partial v}+\frac{\partial \log \Phi_1}{\partial v}\,\frac{\partial \log \Phi_2}{\partial u}\right),\\ \frac{\partial \gamma_2}{\partial v}&=-\frac{\rho_1-\rho_2}{\Phi_1}\left(\frac{\partial^2 \log \Phi_1}{\partial u\partial v}+\frac{\partial \log \Phi_1}{\partial v}\,\frac{\partial \log \Phi_2}{\partial u}\right).\end{aligned}\right. \tag{54}$$

Enfin, en vertu des formules (33), les égalités (42) peuvent s'écrire

$$\left\{\begin{aligned}\frac{\partial \gamma_1}{\partial v}&=\frac{\Phi_2}{2(\rho_1-\rho_2)}(\rho_2^2+\gamma_1^2+\gamma_2^2)-\tfrac{1}{4}(\rho_1-\rho_2)(1-\Omega_3)\Phi_2,\\ \frac{\partial \gamma_2}{\partial u}&=-\frac{\Phi_1}{2(\rho_1-\rho_2)}(\rho_1^2+\gamma_1^2+\gamma_2^2)+\tfrac{1}{4}(\rho_1-\rho_2)(1+\Omega_3)\Phi_1.\end{aligned}\right. \tag{55}$$

Calculons les dérivées premières de A et de C par rapport à u et à v. Les deux formules (10), (33) et (14) donnent

$$\frac{\partial A}{\partial v}=-\frac{\Phi_1\Phi_2\gamma_1}{(\rho_1-\rho_2)^2},\quad \frac{\partial C}{\partial u}=\frac{\Phi_1\Phi_2\gamma_2}{(\rho_1-\rho_2)^2}. \tag{56}$$

Dérivons par rapport à u la première des égalités (33) et par rapport à v la seconde de ces égalités; il viendra, en tenant compte des relations (52) et (53),

$$\left\{\begin{aligned}\frac{\partial A}{\partial u}&=\frac{\Phi_1}{\rho_1-\rho_2}\cdot\frac{\partial \log \dfrac{\Phi_1}{\Phi_2}}{\partial u}+\frac{\Phi_1^2\gamma_2}{(\rho_1-\rho_2)^2},\\ \frac{\partial C}{\partial v}&=\frac{\Phi_2}{\rho_1-\rho_2}\cdot\frac{\partial \log \dfrac{\Phi_2}{\Phi_1}}{\partial v}-\frac{\Phi_2^2\gamma_1}{(\rho_1-\rho_2)^2}.\end{aligned}\right. \tag{57}$$

17. On a vu, dans l'Introduction, que tout invariant paramétrique Φ des surfaces pour le groupe conforme est une fonction d'un ou de plusieurs des éléments de l'ensemble formé par E, G, L, N et leurs dérivées.

Comme on a

$$E=A^2,\ G=C^2,\ L=A^2\rho_1,\ N=C^2\rho_2,$$

Φ est une fonction d'un ou de plusieurs des éléments de l'ensemble formé par A, C, ρ_1, ρ_2 et leurs dérivées. Soit

$$\Phi=f\left(A,\ C,\ \rho_1,\ \rho_2,\ \frac{\partial A}{\partial u},\ \frac{\partial A}{\partial v},\ \frac{\partial C}{\partial u},\ \frac{\partial C}{\partial v},\ \frac{\partial \rho_1}{\partial u},\ \frac{\partial \rho_1}{\partial v},\ \frac{\partial \rho_2}{\partial u},\ \frac{\partial \rho_2}{\partial v},\ \ldots\right).$$

Il suit des égalités (52) à (57) qu'on peut exprimer A, C et les dérivées de A, C, ρ_1, ρ_2 en fonction de ρ_1, ρ_2, γ_1, γ_2, Φ_1, Φ_2, Ω_3 et de dérivées de Φ_1, Φ_2, Ω_3. On peut donc mettre Φ sous la forme

$$\Phi=F\left(\rho_1,\ \rho_2,\ \gamma_1,\ \gamma_2,\ \Phi_1,\ \Phi_2,\ \Omega_3,\ \frac{\partial \Phi_1}{\partial u},\ \frac{\partial \Phi_1}{\partial v},\ \frac{\partial \Phi_2}{\partial u},\ \frac{\partial \Phi_2}{\partial v},\ \ldots\right).$$

En vertu du raisonnement fait au N° 13, F est indépendante de ρ_1, ρ_2, γ_1, γ_2, c.q.f.d. Nous pouvons dès lors énoncer le théorème suivant: *les invariants paramétriques des surfaces pour le groupe conforme sont les fonctions d'un ou de plusieurs des éléments de l'ensemble formé par* Φ_1, Φ_2, Ω_3 *et leurs dérivées.*

18. La considération des égalités (23) et (24) donne deux invariants paramétriques J_1, J_2:

$$(58)\qquad \begin{cases} J_1=\left(\rho_1^2+\gamma_1^2+\gamma_2^2+2\dfrac{\partial \gamma_2}{\partial s_1}\right)A^2, \\ J_2=\left(\rho_2^2+\gamma_1^2+\gamma_2^2-2\dfrac{\partial \gamma_1}{\partial s_2}\right)C^2. \end{cases}$$

En se servant des formules (14) et (10), on obtient les expressions de J_1 et de J_2 en fonction de A, C, ρ_1, ρ_2 et de dérivées de A et de C:

$$(59)\qquad \begin{cases} J_1=2\dfrac{\partial^2 \log C}{\partial u^2}-2\dfrac{\partial \log A}{\partial v}\dfrac{\partial \log C}{\partial u}+\left(\dfrac{\partial \log C}{\partial u}\right)^2+\left(\dfrac{1}{C}\dfrac{\partial A}{\partial v}\right)^2+A^2\rho_1^2, \\ J_2=2\dfrac{\partial^2 \log A}{\partial v^2}-2\dfrac{\partial \log A}{\partial v}\dfrac{\partial \log C}{\partial u}+\left(\dfrac{\partial \log A}{\partial v}\right)^2+\left(\dfrac{1}{A}\dfrac{\partial C}{\partial u}\right)^2+C^2\rho_2^2. \end{cases}$$

L'invariant J_2 est celui que M. Calapso (*loc. cit.*) a désigné par la lettre J.

En vertu des égalités (36), (40), (33), les formules (58) s'écrivent

$$(60)\qquad J_1=\frac{1+\Omega_3}{2}\Phi_1^2,\ J_2=\frac{1-\Omega_3}{2}\Phi_2^2.$$

On voit que J_1, J_2 sont des fonctions de Φ_1, Φ_2, Ω_3. Des égalités (60), on déduit

$$(61)\qquad \Omega_3=\frac{J_1}{\Phi_1^2}-\frac{J_2}{\Phi_2^2}.$$

En remplaçant, dans cette égalité, Φ_1, Φ_2, J_1, J_2 par leurs valeurs (33) et (59), on obtiendra l'expression de Ω_3 en fonction de A, C, ρ_1, ρ_2 et de dérivées de A et de C.

M. Calapso a introduit (*loc. cit*) un invariant égal, en valeur absolue, à l'invariant W défini par l'égalité

$$2J_2=\Phi_2^2-\frac{\Phi_2}{\Phi_1}W.$$

Le rapprochement de cette égalité et de la deuxième des formules (60) donne

$$W=\Phi_1\Phi_2\Omega_3.$$

19. Si l'on exprime que les dérivées de γ_1 et de γ_2, données par les égalités (54) et (55), satisfont aux relations

$$\frac{\partial}{\partial v}\frac{\partial\gamma_1}{\partial u}=\frac{\partial}{\partial u}\frac{\partial\gamma_1}{\partial v},\quad \frac{\partial}{\partial v}\frac{\partial\gamma_2}{\partial u}=\frac{\partial}{\partial u}\frac{\partial\gamma_2}{\partial v},$$

on trouve, en tenant compte des égalités (52) à (55),

$$(62)\quad\left\{\begin{aligned}\frac{\partial\Omega_3}{\partial u}&=-2\frac{\partial\log\Phi_2}{\partial u}\Omega_3+2\frac{\partial\log\Phi_2}{\partial u}+\frac{4}{\Phi_1\Phi_2}\frac{\partial}{\partial v}\left[\frac{\Phi_1}{\Phi_2}\left(\frac{\partial^2\log\Phi_2}{\partial u\partial v}+\frac{\partial\log\Phi_1}{\partial v}\frac{\partial\log\Phi_2}{\partial u}\right)\right],\\ \frac{\partial\Omega_3}{\partial v}&=-2\frac{\partial\log\Phi_1}{\partial v}\Omega_3-2\frac{\partial\log\Phi_1}{\partial v}-\frac{4}{\Phi_1\Phi_2}\frac{\partial}{\partial u}\left[\frac{\Phi_2}{\Phi_1}\left(\frac{\partial^2\log\Phi_2}{\partial u\partial v}+\frac{\partial\log\Phi_1}{\partial v}\frac{\partial\log\Phi_2}{\partial u}\right)\right].\end{aligned}\right.$$

Ces équations montrent que l'on peut exprimer toutes les dérivées de Ω_3 en fonction de Ω_3, de Φ_1, de Φ_2 et de dérivées de Φ_1 et de Φ_2. Par suite, le théorème énoncé à la fin du N° 17 entraîne le suivant: *Les invariants paramétriques des surfaces pour le groupe conforme sont les fonctions d'un ou de plusieurs des éléments de l'ensemble formé par* Φ_1, Φ_2, Ω_3 *et les dérivées de* Φ_1 *et de* Φ_2. Mais il y a plus. Ecrivons la condition d'intégrabilité du système (62), considéré comme déterminant Ω_3. Cette condition est de la forme

$$(63)\qquad \frac{\partial^2\log\dfrac{\Phi_2}{\Phi_1}}{\partial u\partial v}\Omega_3=H,$$

H désignant une fonction de Φ_1, de Φ_2 et de dérivées de ces fonctions. Si la surface n'est pas isothermique, l'équation (63) permet d'exprimer Ω_3 en fonction de Φ_1, de Φ_2 et de dérivées de ces fonctions. Donc, si on laisse de côté les surfaces isothermiques, *les invariants paramétriques des surfaces pour le groupe conforme sont les fonctions d'un ou de plusieurs des éléments de l'ensemble formé par* Φ_1, Φ_2 *et leurs dérivées.*

VII

20. On a démontré que les invariants Φ_1, Φ_2, Ω_3 satisfont aux équations (62). Réciproquement, si trois fonctions Φ_1, Φ_2, Ω_3 satisfont à ces équations, le système des équations (52) à (55) est complètement intégrable et les quantités A, C, q, p_1, r, r_1, définies par les égalités (51), satisfont aux relations (10). En vertu de la remarque faite dans la troisième note du N° 3, ces quantités sont les translations et rotations du trièdre principal T d'une surface et il est clair que

cette surface admet pour invariants Φ_1, Φ_2, Ω_3 les fonctions Φ_1, Φ_2, Ω_3 considérées. Ladite surface dépend de dix paramètres: les quatre constantes arbitraires qui figurent dans les expressions de $\rho_1, \rho_2, \gamma_1, \gamma_2$ et six paramètres de position. Les ∞^{10} surfaces ainsi obtenues se correspondent deux à deux dans des transformations conformes. Soit, en effet, S_0 une quelconque de ces surfaces. Les ∞^{10} surfaces S' qui correspondent à S_0 dans des transformations conformes ont mêmes invariants Φ_1, Φ_2, Ω_3 que S_0. Or les ∞^{10} surfaces S sont les seules surfaces jouissant de cette propriété. Donc la famille des surfaces S est identique à la famille des surfaces S'. Dès lors, les surfaces S correspondent à S_0 dans des transformations conformes et se correspondent, par suite, deux à deux dans de telles transformations.

21. Le système des équations (52) à (55) n'est pas linéaire, mais on peut ramener son intégration à celle d'un système linéaire. A cet effet, introduisons la fonction σ définie par l'égalité

$$\sigma=\frac{\gamma_1^2+\gamma_2^2+\rho_1\rho_2}{\rho_1-\rho_2} \tag{64}$$

et remplaçons, dans les équations (55), la somme $\gamma_1^2+\gamma_2^2$ par son expression tirée de cette égalité; il viendra

$$\left\{\begin{aligned} \frac{\partial\gamma_1}{\partial v}&=\frac{\Phi_2}{2}(\sigma-\rho_2)-\tfrac{1}{4}(\rho_1-\rho_2)(1-\Omega_3)\Phi_2,\\ \frac{\partial\gamma_2}{\partial u}&=-\frac{\Phi_1}{2}(\sigma+\rho_1)+\tfrac{1}{4}(\rho_1-\rho_2)(1+\Omega_3)\Phi_1.\end{aligned}\right. \tag{65}$$

Si l'on calcule les dérivées premières de σ, en tenant compte des équations (52), (53), (54), (65), on trouve

$$\left\{\begin{aligned} \frac{\partial\sigma}{\partial u}&=\frac{2}{\Phi_2}\left(\frac{\partial^2\log\Phi_2}{\partial u\partial v}+\frac{\partial\log\Phi_1}{\partial v}\frac{\partial\log\Phi_2}{\partial u}\right)\gamma_1+\frac{\Phi_1}{2}(1+\Omega_3)\gamma_2\\ &\qquad+\frac{\partial\log\Phi_2}{\partial u}\rho_2-\frac{\partial\log\Phi_2}{\partial u}\sigma,\\ \frac{\partial\sigma}{\partial v}&=-\frac{\Phi_2}{2}(1-\Omega_3)\gamma_1-\frac{2}{\Phi_1}\left(\frac{\partial^2\log\Phi_1}{\partial u\partial v}+\frac{\partial\log\Phi_1}{\partial v}\frac{\partial\log\Phi_2}{\partial u}\right)\gamma_2\\ &\qquad-\frac{\partial\log\Phi_1}{\partial v}\rho_1-\frac{\partial\log\Phi_1}{\partial v}\sigma.\end{aligned}\right. \tag{66}$$

Le système des équations (52), (53), (54), (65), (66), aux inconnues $\rho_1, \rho_2, \gamma_1, \gamma_2, \sigma$, est linéaire et complètement intégrable. Il admet l'intégrale

$$\gamma_1^2+\gamma_2^2+\rho_1\rho_2-(\rho_1-\rho_2)\sigma=\text{const.} \tag{67}$$

Si l'on détermine une solution $(\rho_1, \rho_2, \gamma_1, \gamma_2, \sigma)$ de ce système, telle que, pour $u=u_0$, $v=v_0$, le premier membre de l'égalité (67) soit nul, cette expression sera nulle pour toutes les valeurs de u et de v et l'égalité (64) sera vérifiée. En portant la valeur (64) de σ dans les équations (65), on obtiendra les équations (55) et, par suite, les fonctions ρ_1, ρ_2, γ_1, γ_2 vérifieront le système des équations (52) à (55).

22. Si $\frac{\Phi_2}{\Phi_1}$ n'est pas le quotient d'une fonction de u par une fonction de v, l'équation (63) est résoluble par rapport à Ω_3. En portant, dans les égalités (62), la valeur de Ω_3, on obtient un système de deux relations différentielles entre Φ_1 et Φ_2, que nous désignerons par Σ. En vertu du résultat établi au N° 20, *deux fonctions* Φ_1, Φ_2 *dont le rapport n'est pas le quotient d'une fonction de u par une fonction de v et qui satisfont au système* Σ, *définissent, à une transformation conforme près, une surface dont les invariants* Φ_1, Φ_2 *sont les fonctions considérées.*

Le système (62) ne change pas si l'on fait une des substitutions

$$\Phi_1 | -\Phi_1, \quad \Phi_2 | -\Phi_2$$

ou ces deux substitutions. Par suite, si l'on satisfait au système Σ en posant

$$\Phi_1 = \lambda_1,\ \Phi_2 = \lambda_2,$$

on y satisfera aussi en posant

$$\Phi_1 = -\lambda_1,\ \Phi_2 = \lambda_2$$

ou

$$\Phi_1 = \lambda_1,\ \Phi_2 = -\lambda_2$$

ou

$$\Phi_1 = -\lambda_1,\ \Phi_2 = -\lambda_2.$$

Si $\frac{\lambda_2}{\lambda_1}$ n'est pas le quotient d'une fonction de u par une fonction de v, à chacun de ces couples de valeurs de Φ_1 et de Φ_2 correspondent ∞^{10} surfaces se correspondant deux à deux dans des transformations conformes. Il suit de la remarque faite dans la note du N° 8 que les quatre familles de surfaces considérées sont identiques. Dès lors, *une surface non isothermique est définie, à une transformation conforme près, par les carrés de ses invariants* Φ_1, Φ_2. A cause de la formule (c) de la note du N° 5, ce résultat est équivalent au suivant: *Si une surface n'est pas isothermique, le carré de l'angle de deux sphères harmoniques infiniment voisines définit cette surface, à une transformation conforme près.*

23. Si une surface est isothermique, on peut, par un choix convenable des paramètres u, v, faire en sorte que Φ_1 et Φ_2 soient égales. Si l'on désigne par Φ la valeur commune de ces deux fonctions, les équations (62) deviennent

$$(68) \quad \left\{ \begin{aligned} \frac{\partial \Omega_3 \Phi^2}{\partial u} &= \frac{\partial \Phi^2}{\partial u} + 4 \frac{\partial}{\partial v}\left(\frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial u \partial v}\right), \\ \frac{\partial \Omega_3 \Phi^2}{\partial v} &= -\frac{\partial \Phi^2}{\partial v} - 4 \frac{\partial}{\partial u}\left(\frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial u \partial v}\right). \end{aligned} \right.$$

On déduit de là, en écrivant la condition d'intégrabilité pour $\Omega_3\Phi^2$,

$$(69) \quad 2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial u \partial v} + \frac{\partial^2 \Phi^2}{\partial u \partial v} = 0.$$

Cette équation a été obtenue par M. Rothe et par M. Calapso*. Si l'on en connaît une solution Φ, les équations (68) donnent

$$\Omega_3\Phi^2=\Psi+\Gamma, \tag{70}$$

Ψ désignant une fonction déterminée et Γ une constante arbitraire†.

On a vu (N° 20) qu'il y a ∞^{10} surfaces admettant des invariants Φ_1, Φ_2, Ω_3 donnés, ceux-ci satisfaisant, bien entendu, aux relations (62). Dès lors, les surfaces isothermiques dont les invariants Φ_1, Φ_2 sont égaux à Φ dépendent de onze paramètres, car l'expression de leur invariant Ω_3, déduite de l'équation (70), contient la constante arbitraire Γ. Donc, à toute solution Φ de l'équation (69) correspondent ∞^{11} surfaces isothermiques. Ce résultat est équivalent à celui auquel M. Calapso a été conduit dans le mémoire que nous venons de citer. Pour toutes les surfaces considérées, la partie principale θ de l'angle de deux sphères harmoniques infiniment voisines est donnée par la formule

$$\theta^2=\frac{\Phi^2}{4}(du^2+dv^2).$$

D'autre part, en vertu du théorème établi au N° 20, les ∞^{10} surfaces pour lesquelles Γ a la même valeur se correspondent deux à deux dans des transformations conformes. Par suite, si l'on convient de ne pas considérer comme distinctes deux surfaces qui se correspondent dans une transformation conforme, on peut dire qu'à toute solution Φ de l'équation (69) correspondent ∞^1 surfaces isothermiques. Nous allons montrer que cette famille de surfaces se présente dans l'étude d'une transformation de certains couples de surfaces isothermiques.

24. Rappelons les formules relatives à la transformation des surfaces isothermiques due à M. Darboux. Nous adopterons les notations de M. Bianchi‡, en remplaçant toutefois respectivement r_1, r_2 par $-\frac{1}{\rho_2}$, $-\frac{1}{\rho_1}$.

Soit S une surface isothermique rapportée au réseau (u, v) de ses lignes de courbure. Son élément linéaire est donné par la formule

$$ds^2=e^{2\theta}(du^2+dv^2). \tag{71}$$

Soient x, y, z les coordonnées d'un point quelconque M de S, X_1, Y_1, Z_1, X_2, Y_2, Z_2 les cosinus directeurs des tangentes aux lignes de courbure $v=$const., $u=$const. qui passent par M, et X_3, Y_3, Z_3 les cosinus directeurs de la normale à la surface en ce point. Si x_1, y_1, z_1 désignent les coordonnées du point M_1 qui décrit une surface S_1 correspondant à la surface S dans une transformation D_m de Darboux, on a

*Rothe: *Untersuchungen über die Theorie der isothermen Flächen*, Berlin, 1897.
Calapso: *Sulle superficie a linee di curvatura isoterme* (Rend. Circ. Mat. Palermo, t. XVII, 1903).

†Dans le cas présent, l'invariant W, défini au N° 18, a pour expression $\Omega_3\Phi^2$. L'égalité (70) peut donc s'écrire $W=\Psi+\Gamma$. Par suite, si deux surfaces isothermiques ont même invariant Φ, leurs invariants W ne diffèrent que d'une constante.

‡L. Bianchi: *Ricerche sulle superficie isoterme* (Annali di Matematica, 3e série, t. XI).

$$(72)\quad \begin{cases} x_1=x-\dfrac{1}{m\sigma}(\lambda X_1+\mu X_2+wX_3), \\ y_1=y-\dfrac{1}{m\sigma}(\lambda Y_1+\mu Y_2+wY_3), \\ z_1=z-\dfrac{1}{m\sigma}(\lambda Z_1+\mu Z_2+wZ_3), \end{cases}$$

λ, μ, w, σ et une fonction ϕ, qui ne figure pas dans ces formules, satisfaisant au système

$$(73)\quad \begin{cases} \dfrac{\partial\lambda}{\partial u}=me^{\theta}\sigma+me^{-\theta}\phi+e^{\theta}\rho_1 w-\dfrac{\partial\theta}{\partial v}\mu, \quad \dfrac{\partial\lambda}{\partial v}=\dfrac{\partial\theta}{\partial u}\mu, \\ \dfrac{\partial\mu}{\partial u}=\dfrac{\partial\theta}{\partial v}\lambda, \quad \dfrac{\partial\mu}{\partial v}=me^{\theta}\sigma-me^{-\theta}\phi+e^{\theta}\rho_2 w-\dfrac{\partial\theta}{\partial u}\lambda, \\ \dfrac{\partial\phi}{\partial u}=e^{\theta}\lambda, \quad \dfrac{\partial\phi}{\partial v}=e^{\theta}\mu, \\ \dfrac{\partial w}{\partial u}=-e^{\theta}\rho_1\lambda, \quad \dfrac{\partial w}{\partial v}=-e^{\theta}\rho_2\mu, \\ \dfrac{\partial\sigma}{\partial u}=e^{-\theta}\lambda, \quad \dfrac{\partial\sigma}{\partial v}=-e^{-\theta}\mu, \end{cases}$$

et à la relation

$$(74)\quad \lambda^2+\mu^2+w^2=2m\phi\sigma.$$

Le carré de l'élément linéaire ds_1 de la surface S_1 a pour expression

$$(75)\quad ds_1^2=e^{-2\theta}\frac{\phi^2}{\sigma^2}(du^2+dv^2).$$

Aux surfaces S, S_1, qui se correspondent dans la transformation de Darboux la plus générale, on peut attacher, comme il suit, un couple de surfaces isothermiques $\overline{S}$, $\overline{S}_1$ se correspondant dans une transformation de Christoffel. La surface $\overline{S}$ a pour ds^2

$$(76)\quad d\overline{s}^2=\frac{e^{2\theta}}{\phi^2}(du^2+dv^2)$$

et ses courbures principales $\overline{\rho}_1$, $\overline{\rho}_2$ sont données par les formules

$$(77)\quad \overline{\rho}_1=w+\phi\rho_1, \quad \overline{\rho}_2=w+\phi\rho_2.$$

Cette correspondance entre le couple (S, S_1) et le couple $(\overline{S}, \overline{S}_1)$ a été signalée, en 1903, par M. Guichard. M. Bianchi et nous-même l'avons retrouvée, indépendamment l'un de l'autre, et sous des formes très différentes, en 1905.*

*Cl. Guichard: *Sur les systèmes orthogonaux et les systèmes cycliques* (Annales de l'École Normale Supérieure, 3[e] série t. XX).

L. Bianchi: *Complementi alle ricerche sulle superficie isoterme*(Annali di Matematica, 3 [e] série, t. XII).

A. Demoulin: *Sur les enveloppes de sphères dont les deux nappes se correspondent avec conservation des angles* (Comptes Rendus Acad. Sciences, Paris, t. CXLI).

Soit $\overline{\Phi}$ l'invariant Φ de $\overline{S}$. On a

$$\overline{\Phi} = \frac{e^\theta}{\phi}(\overline{\rho}_1 - \overline{\rho}_2)$$

ou, en vertu des égalités (77),

$$\overline{\Phi} = e^\theta(\rho_1 - \rho_2).$$

Or l'invariant Φ de S a pour expression

$$\Phi = e^\theta(\rho_1 - \rho_2). \tag{78}$$

Les surfaces S et $\overline{S}$ ont donc même invariant Φ. Par suite, (deuxième note du N° 23), leurs invariants W ne diffèrent que d'une constante. Déterminons cette constante.

Soit W_0 l'invariant W d'une surface isothermique dont le ds^2 est

$$ds^2 = A^2(du^2 + dv^2)$$

et dont ρ_1, ρ_2 sont les courbures principales.

On a (N° 23)

$$W_0 = \Omega_3 \Phi^2,$$

d'où, en vertu des formules (61) et (59),

$$\tfrac{1}{2} W_0 = \frac{\partial^2 \log A}{\partial u^2} - \frac{\partial^2 \log A}{\partial v^2} - \left(\frac{\partial \log A}{\partial u}\right)^2 + \left(\frac{\partial \log A}{\partial v}\right)^2 + \tfrac{1}{2} A^2(\rho_1^2 - \rho_2^2).$$

Soient W et $\overline{W}$ les invariants W des surfaces S, $\overline{S}$. Si l'on tient compte des expressions (71) et (76) des ds^2 de ces surfaces, la formule précédente donne

$$\tfrac{1}{2} W = \frac{\partial^2 \theta}{\partial u^2} - \frac{\partial^2 \theta}{\partial v^2} - \left(\frac{\partial \theta}{\partial u}\right)^2 + \left(\frac{\partial \theta}{\partial v}\right)^2 + \tfrac{1}{2} e^{2\theta}(\rho_1^2 - \rho_2^2), \tag{79}$$

$$\tfrac{1}{2} \overline{W} = \frac{\partial^2 \log \overline{A}}{\partial u^2} - \frac{\partial^2 \log \overline{A}}{\partial v^2} - \left(\frac{\partial \log \overline{A}}{\partial u}\right)^2 + \left(\frac{\partial \log \overline{A}}{\partial v}\right)^2 + \tfrac{1}{2} \overline{A}^2(\overline{\rho}_1^2 - \overline{\rho}_2^2), \tag{80}$$

étant posé

$$\overline{A} = \frac{e^\theta}{\phi}.$$

On déduit de la dernière égalité

$$\log \overline{A} = \theta - \log \phi,$$

d'où, en dérivant et en tenant compte des égalités (73),

$$\frac{\partial \log \overline{A}}{\partial u} = \frac{\partial \theta}{\partial u} - \frac{e^\theta \lambda}{\phi}, \quad \frac{\partial \log \overline{A}}{\partial v} = \frac{\partial \theta}{\partial v} - \frac{e^\theta \mu}{\phi},$$

$$\frac{\partial^2 \log \overline{A}}{\partial u^2} = \frac{\partial^2 \theta}{\partial u^2} - \frac{e^\theta}{\phi}\left(me^\theta \sigma + me^{-\theta}\phi + e^\theta \rho_1 w - \frac{\partial \theta}{\partial v}\mu\right) - e^\theta \frac{\partial \theta}{\partial u}\frac{\lambda}{\phi} + e^{2\theta}\frac{\lambda^2}{\phi^2},$$

$$\frac{\partial^2 \log \overline{A}}{\partial v^2} = \frac{\partial^2 \theta}{\partial v^2} - \frac{e^\theta}{\phi}\left(me^\theta \sigma - me^{-\theta}\phi + e^\theta \rho_2 w - \frac{\partial \theta}{\partial u}\lambda\right) - e^\theta \frac{\partial \theta}{\partial v}\frac{\mu}{\phi} + e^{2\theta}\frac{\mu^2}{\phi^2}.$$

A cause de ces formules et des relations (77), l'égalité (80) devient

$$\tag{81} \tfrac{1}{2}\overline{W} = \frac{\partial^2\theta}{\partial u^2} - \frac{\partial^2\theta}{\partial v^2} - \left(\frac{\partial\theta}{\partial u}\right)^2 + \left(\frac{\partial\theta}{\partial v}\right)^2 + \tfrac{1}{2}e^{2\theta}(\rho_1^2 - \rho_2^2) - 2m.$$

Des égalités (79) et (81) on déduit

$$\overline{W} = W - 4m.$$

Dès lors, la surface S étant donnée et la constante m fixée, les surfaces $\overline{S}$ obtenues au moyen des différentes solutions (λ, μ, w, ϕ, σ) des équations (83) et (84) ont mêmes invariants Φ et W et se correspondent, par suite, deux à deux dans des transformations conformes. Ces surfaces, que nous désignerons par $\overline{S}_m$, dépendent de dix paramètres: les quatre paramètres qui figurent dans les expressions les plus générales des fonctions λ, μ, w, ϕ, σ et six paramètres de position. Parmi les surfaces $\overline{S}_0$ figure la surface S, car les invariants Φ, W des surfaces $\overline{S}_0$ sont respectivement égaux aux invariants Φ, W de la surface S*.

Si m varie, l'ensemble des ∞^{10} surfaces $\overline{S}_m$ engendrera la famille de surfaces à onze paramètres que l'on a fait correspondre à toute solution de l'équation (69).

25. Démontrons que les coordonnées pentasphériques du point générateur de toute surface $\overline{S}_m$ satisfont à une équation de Moutard, indépendante de m. A cet effet, nous établirons d'abord quelques résultats concernant la théorie générale des surfaces.

Conservons toutes les notations du N° 3. Les coordonnées rectangulaires x, y, z du point M satisfont à l'équation

$$\tag{82} \frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial \log A}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial \log C}{\partial u}\frac{\partial\theta}{\partial v}.$$

Substituons à θ l'inconnue ω définie par l'égalité

$$\tag{83} \theta = \lambda\omega,$$

étant posé, pour abréger,

$$\tag{84} \lambda = \frac{1}{\rho_1 - \rho_2};$$

ω vérifie l'équation

$$\tag{85} \frac{\partial^2\omega}{\partial u\partial v} - \frac{\partial \log \Phi_1}{\partial v}\frac{\partial\omega}{\partial u} - \frac{\partial \log \Phi_2}{\partial u}\frac{\partial\omega}{\partial v} + \left(\frac{\partial^2\lambda}{\partial u\partial v} - \frac{\partial \log A}{\partial v}\frac{\partial\lambda}{\partial u} - \frac{\partial \log C}{\partial u}\frac{\partial\lambda}{\partial v}\right)\frac{\omega}{\lambda} = 0.$$

*On peut aussi établir ce résultat en raisonnant comme il suit. Si $m=0$, les équations (73) et (74) admettent la solution

$$\lambda=0,\ \mu=0,\ w=0,\ \phi=1,\ \sigma=\text{const.}$$

Pour ces valeurs de λ, μ, w, ϕ, σ les formules (76) et (77) deviennent

$$d\overline{s}^2 = e^{2\theta}(du^2+dv^2),$$

$$\overline{\rho}_1=\rho_1,\ \overline{\rho}_2=\rho_2,$$

et il suit de là qu'une des surfaces $\overline{S}_0$ coïncide avec S.

En se servant des formules (52), (53), (54), on trouve

$$\frac{\partial\lambda}{\partial u}=-\frac{\partial\log\Phi_2}{\partial u}\lambda+\gamma_2\Phi_1\lambda^2,$$

$$\frac{\partial\lambda}{\partial v}=-\frac{\partial\log\Phi_1}{\partial v}\lambda-\gamma_1\Phi_2\lambda^2,$$

$$\frac{\partial^2\lambda}{\partial u\partial v}=-\frac{\partial^2\log\Phi_1\Phi_2}{\partial u\partial v}\lambda+\left(\gamma_1\frac{\partial\Phi_2}{\partial u}-\gamma_2\frac{\partial\Phi_1}{\partial v}\right)\lambda^2-2\gamma_1\gamma_2\Phi_1\Phi_2\lambda^3.$$

D'autre part, les formules (56) et (33) donnent

$$\frac{\partial\log A}{\partial v}=-\gamma_1\Phi_2\lambda,\quad \frac{\partial\log C}{\partial u}=\gamma_2\Phi_1\lambda.$$

En vertu de ces diverses égalités, le coefficient de $\frac{\omega}{\lambda}$ dans l'équation (85) a pour valeur $-\frac{\partial^2\log\Phi_1\Phi_2}{\partial u\partial v}\lambda$. Par suite, cette équation peut s'écrire

$$\text{(86)}\qquad \frac{\partial^2\omega}{\partial u\partial v}-\frac{\partial\log\Phi_2}{\partial v}\frac{\partial\omega}{\partial u}-\frac{\partial\log\Phi_1}{\partial u}\frac{\partial\omega}{\partial v}-\frac{\partial^2\log\Phi_1\Phi_2}{\partial u\partial v}\omega=0.$$

Les invariants h, k de l'équation (86), qui sont égaux à ceux de l'équation (82), ont pour expressions

$$\text{(87)}\qquad \begin{cases} h=\dfrac{\partial^2\log\Phi_2}{\partial u\partial v}+\dfrac{\partial\log\Phi_1}{\partial v}\dfrac{\partial\log\Phi_2}{\partial u}, \\[2ex] k=\dfrac{\partial^2\log\Phi_1}{\partial u\partial v}+\dfrac{\partial\log\Phi_1}{\partial v}\dfrac{\partial\log\Phi_2}{\partial u}. \end{cases}$$

Ces formules sont dues à M. Calapso.

L'égalité (83), où l'on remplacera λ par sa valeur (84), fait correspondre, aux solutions

$$x,\ y,\ z,\ x^2+y^2+z^2,\ 1$$

de l'équation (82), les solutions

$$x(\rho_1-\rho_2),\ y(\rho_1-\rho_2),\ z(\rho_1-\rho_2),\ (x^2+y^2+z^2)(\rho_1-\rho_2),\ \rho_1-\rho_2$$

de l'équation (86).

Posons

$$\text{(88)}\qquad \begin{cases} y_1=x(\rho_1-\rho_2),\ y_2=y(\rho_1-\rho_2),\ y_3=z(\rho_1-\rho_2), \\[1ex] y_4=\dfrac{x^2+y^2+z^2-1}{2}(\rho_1-\rho_2),\ y_5=i\,\dfrac{x^2+y^2+z^2+1}{2}(\rho_1-\rho_2). \end{cases}$$

Ces quantités sont liées par la relation

$$\sum_{i=1}^{5}y_i^2=0.$$

et vérifient l'équation (86). Ce sont des coordonnées pentasphériques du point M. Si l'on effectue la transformation orthogonale

$$x_i = \sum_{k=1}^{5} c_{ik} y_k, \quad (i=1, 2, 3, 4, 5),$$

$x_1, x_2, \ldots, x_5$ seront, à un facteur commun près, les coordonnées pentasphériques du point M, les plus générales.

Pour la surface isothermique S la plus générale, considérée au N° 24, on a $\Phi_1 = \Phi_2 = \Phi$ et l'équation (86) s'écrit

$$\frac{\partial^2 \omega}{\partial u \partial v} - \frac{\partial \log \Phi}{\partial v} \frac{\partial \omega}{\partial u} - \frac{\partial \log \Phi}{\partial u} \frac{\partial \omega}{\partial v} - 2 \frac{\partial^2 \log \Phi}{\partial u \partial v} \omega = 0.$$

Posons

(89) $$\omega = \Phi \sigma;$$

σ satisfait à une équation de Moutard:

(90) $$\frac{\partial^2 \sigma}{\partial u \partial v} = k\sigma$$

et le calcul montre que cette équation admet la solution Φ*.

Il est clair qu'à l'équation (90) satisfont des coordonnées pentasphériques $z_1, z_2, \ldots, z_5$ du point générateur de la surface S, convenablement choisies†.

De même, des coordonnées pentasphériques $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_5$ du point générateur d'une surface $\bar{S}_m$, convenablement choisies, satisfont à une équation de Moutard

$$\frac{\partial^2 \tau}{\partial u \partial v} = \bar{k}\tau$$

qui admet la solution $\bar{\Phi}$. Comme $\bar{\Phi} = \Phi$, cette équation est identique à l'équation (90). Celle-ci possède donc cinq solutions $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_5$ liées par la relation

$$\sum_{i=1}^{5} \bar{z}_i^2 = 0$$

et dépendant d'une constante arbitraire m. Par suite, on peut énoncer le théorème suivant, signalé par M. Guichard (*loc. cit.*): *Si une équation de Moutard possède cinq solutions dont la somme des carrés est nulle, elle possède une infinité simple de groupes de cinq solutions jouissant de la même propriété.*

*On peut établir ce dernier résultat plus rapidement en raisonnant comme il suit. Les invariants de l'équation (90) sont égaux à k. Leur valeur est donnée par une des formules (87) où l'on fera $\Phi_1 = \Phi_2 = \Phi$. On trouve immédiatement $k = \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial u \partial v}$ c.q.f.d.

†A cause des égalités (89), (78) et (88), on peut poser

$$z_1 = xe^{-\theta}, \; z_2 = ye^{-\theta}, \; z_3 = ze^{-\theta},$$

$$z_4 = \frac{x^2+y^2+z^2-1}{2} e^{-\theta}, \quad z_5 = i \, \frac{x^2+y^2+z^2+1}{2} e^{-\theta}.$$

Deux de ces égalités donnent $z_4 + iz_5 = -e^{-\theta}$. Donc l'équation (90) admet la solution $e^{-\theta}$.

VIII

26. Des formules (53) et (33), on déduit

$$\frac{\frac{\partial \log \Phi_2}{\partial u}}{\Phi_1} = \frac{\frac{\partial \rho_1}{\partial u}}{A(\rho_1-\rho_2)^2}, \quad \frac{\frac{\partial \log \Phi_1}{\partial v}}{\Phi_2} = -\frac{\frac{\partial \rho_2}{\partial v}}{C(\rho_1-\rho_2)^2},$$

ou, à cause des égalités (34),

$$\Omega_1 = \frac{\frac{\partial \log \Phi_2}{\partial u}}{\Phi_1}, \quad \Omega_2 = -\frac{\frac{\partial \log \Phi_1}{\partial u}}{\Phi_2}. \tag{91}$$

Ces expressions des invariants différentiels Ω_1, Ω_2, considérés comme invariants paramétriques, ont été données par M. Calapso.

On a, en vertu des formules (43) et (33), Ω désignant une fonction quelconque de u et de v,

$$X_1\Omega = \frac{1}{\Phi_1}\frac{\partial \Omega}{\partial u}, \quad X_2\Omega = \frac{1}{\Phi_2}\frac{\partial \Omega}{\partial v}. \tag{92}$$

Les égalités (92) donnent

$$X_2X_1\Omega = \frac{1}{\Phi_2}\frac{\partial}{\partial v}\left(\frac{1}{\Phi_1}\frac{\partial \Omega}{\partial u}\right) = \frac{\frac{\partial^2\Omega}{\partial u\partial v}}{\Phi_1\Phi_2} - \frac{\frac{\partial \Phi_1}{\partial v}}{\Phi_1^2\Phi_2}\frac{\partial \Omega}{\partial u},$$

$$X_1X_2\Omega = \frac{1}{\Phi_1}\frac{\partial}{\partial u}\left(\frac{1}{\Phi_2}\frac{\partial \Omega}{\partial v}\right) = \frac{\frac{\partial^2\Omega}{\partial u\partial v}}{\Phi_1\Phi_2} - \frac{\frac{\partial \Phi_2}{\partial u}}{\Phi_1\Phi_2^2}\frac{\partial \Omega}{\partial v}.$$

Soustrayons l'une de l'autre les deux dernières égalités et posons

$$(X_2X_1)\Omega = X_2X_1\Omega - X_1X_2\Omega,$$

il viendra

$$(X_2X_1)\Omega = \frac{\frac{\partial \log \Phi_2}{\partial u}}{\Phi_1}\cdot\frac{1}{\Phi_2}\frac{\partial \Omega}{\partial v} - \frac{\frac{\partial \log \Phi_1}{\partial v}}{\Phi_2}\cdot\frac{1}{\Phi_1}\frac{\partial \Omega}{\partial u},$$

ou, à cause des formules (91) et (92),

$$(X_2X_1)\Omega = \Omega_2X_1\Omega + \Omega_1X_2\Omega. \tag{93}$$

Les formules (47) peuvent s'écrire

$$\begin{cases} X_1\Omega_3 + 2\Omega_1\Omega_3 + P = 0, \\ X_2\Omega_3 - 2\Omega_2\Omega_3 + Q = 0, \end{cases} \tag{94}$$

P, Q désignant des expressions ne contenant que Ω_1, Ω_2 et des quantités obtenues en effectuant sur ces invariants les opérations X_1, X_2. On déduit de là

$$X_2(X_1\Omega_3 + 2\Omega_1\Omega_3 + P) - X_1(X_2\Omega_3 - 2\Omega_2\Omega_3 + Q) = 0,$$

ou, en tenant compte de la formule (93) et en remplaçant $X_1\Omega_3$, $X_2\Omega_3$ par leurs valeurs tirées des égalités (94),

$$(X_1\Omega_2+X_2\Omega_1)\Omega_3=S, \tag{95}$$

S désignant une fonction de Ω_1, de Ω_2 et de quantités obtenues en effectuant sur ces invariants les opérations X_1 et X_2.

Démontrons que l'équation (95) peut être résolue par rapport à Ω_3, si la surface n'est pas isothermique. A cet effet, établissons de nouvelles expressions des invariants h, k de l'équation (86).

On a, en tenant compte d'une des formules (92),

$$X_2\Omega_1-2\Omega_1\Omega_2=\frac{1}{\Phi_2}\frac{\partial\Omega_1}{\partial v}-2\Omega_1\Omega_2$$

ou, en remplaçant Ω_1, Ω_2 par leurs valeurs (91),

$$X_2\Omega_1-2\Omega_1\Omega_2=\frac{1}{\Phi_1\Phi_2}\left(\frac{\partial^2\log\Phi_2}{\partial u\partial v}+\frac{\partial\log\Phi_1}{\partial v}\frac{\partial\log\Phi_2}{\partial u}\right). \tag{96}$$

On démontrera de même l'égalité

$$X_1\Omega_2+2\Omega_1\Omega_2=-\frac{1}{\Phi_1\Phi_2}\left(\frac{\partial^2\log\Phi_1}{\partial u\partial v}+\frac{\partial\log\Phi_1}{\partial v}\frac{\partial\log\Phi_2}{\partial u}\right). \tag{97}$$

Les expressions (87) de h et de k deviennent par suite

$$\begin{cases} h=\Phi_1\Phi_2(X_2\Omega_1-2\Omega_1\Omega_2), \\ k=-\Phi_1\Phi_2(X_1\Omega_2+2\Omega_1\Omega_2). \end{cases} \tag{98}$$

Elles s'écrivent encore, en vertu des égalités (45) et (33),

$$h=AC\frac{\partial\gamma_1}{\partial s_1},\quad k=-AC\frac{\partial\gamma_2}{\partial s_2}. \tag{99}$$

Si l'on divise l'une par l'autre les égalités (98), il vient

$$\frac{h}{k}=-\frac{X_2\Omega_1-2\Omega_1\Omega_2}{X_1\Omega_2+2\Omega_1\Omega_2}. \tag{100}$$

Le quotient $\frac{h}{k}$ est donc un invariant différentiel des surfaces pour le groupe conforme.

Pour que la surface soit isothermique, il faut et il suffit que l'on ait $h=k$. En vertu de (100), cette condition est équivalente à la suivante:

$$X_2\Omega_1+X_1\Omega_2=0. \tag{101}$$

Donc, si la surface n'est pas isothermique, l'équation (95) pourra être résolue par rapport à Ω_3. Par suite, si on laisse de côté les surfaces isothermiques, le théorème énoncé à la fin du N° 13 entraîne le suivant:

Les invariants différentiels des surfaces pour le groupe conforme sont les fonctions d'un ou de plusieurs des éléments de l'ensemble formé par les invariants Ω_1, Ω_2 *et les quantités*

$$X_1^{\alpha_1} X_2^{\beta_1} \Omega_1,\ X_1^{\alpha_2} X_2^{\beta_2} \Omega_2, \quad (\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2 = 0,\ 1,\ 2,\ \ldots).$$

L'équation (101) permet d'écrire l'équation aux dérivées partielles des surfaces isothermiques en coordonnées cartésiennes.

Les formules (99) montrent que les surfaces isothermiques sont caractérisées par l'égalité

$$\frac{\partial \gamma_1}{\partial s_1} + \frac{\partial \gamma_2}{\partial s_2} = 0.$$

Ce résultat est un cas particulier d'un théorème bien connu concernant les réseaux orthogonaux et isothermes.

27. On a vu (note du N° 5) que le carré de l'angle de deux sphères harmoniques infiniment voisines est égal à $\frac{1}{4}(\Phi_1^2\, du^2 + \Phi_2^2 dv^2)$. Désignons par dl^2 cette forme quadratique et appelons l'intégrale $\int dl$, étendue à un arc AB d'une courbe quelconque Γ tracée sur la surface, la l de cet arc, ou encore *l'arc angulaire de la courbe* Γ, compté depuis le point A jusqu'au point B.

Si l_1, l_2 sont les arcs angulaires des lignes (M_u), (M_v), comptés depuis des origines fixes jusqu'au point M, les formules (92) s'écrivent

$$X_1\Omega = \frac{1}{2}\frac{\partial \Omega}{\partial l_1},\ X_2\Omega = \frac{1}{2}\frac{\partial \Omega}{\partial l_2}, \tag{102}$$

et la formule (93) devient

$$\frac{\partial^2 \Omega}{\partial l_2 \partial l_1} - \frac{\partial^2 \Omega}{\partial l_1 \partial l_2} = 2\left(\Omega_2 \frac{\partial \Omega}{\partial l_1} + \Omega_1 \frac{\partial \Omega}{\partial l_2}\right). \tag{103}$$

28. A cause des formules (34), les cyclides de Dupin sont caractérisées par les égalités $\Omega_1 = 0$, $\Omega_2 = 0$*. En se servant de la formule (103), on déduit de là les conséquences suivantes:

1° Si une surface est une cyclide de Dupin, on a, pour toute fonction Ω,

$$\frac{\partial^2 \Omega}{\partial l_2 \partial l_1} = \frac{\partial^2 \Omega}{\partial l_1 \partial l_2}. \tag{104}$$

2° Si une surface est telle que l'égalité (104) soit vérifiée par deux fonctions qui ne sont pas fonction l'une de l'autre, cette surface est une cyclide de Dupin.

*En vertu des formules (91), ces conditions peuvent être remplacées par les suivantes: Φ_1 est fonction de v et Φ_2, fonction de u. Il suit de là et des formules (62) que, pour une cyclide de Dupin, Ω_3 est constant. Ce résultat est aussi une conséquence des égalités (47).

Pour qu'une surface soit un périsphère dont les lignes de courbure ont pour équation $v = \text{const.}$, il faut et il suffit que Ω_1 soit nul, condition qui peut être remplacée par celle-ci: Φ_2 est fonction de v. Pour les périsphères considérés, Ω_3 est fonction de v.

29. La cyclide de Dupin jouit encore d'une autre propriété caractéristique. On vient de démontrer que, pour une telle surface, $\Phi_1 = U$, $\Phi_2 = V$, U désignant une fonction de u et V une fonction de v. Or l'élément linéaire d'une surface rapportée à ses lignes de courbure est donné par la formule

$$ds^2 = \frac{\Phi_1^2 du^2 + \Phi_2^2 dv^2}{(\rho_1 - \rho_2)^2}.$$

Donc, pour une cyclide de Dupin,

$$ds^2 = \frac{U^2 du^2 + V^2 dv^2}{(\rho_1 - \rho_2)^2}. \tag{105}$$

On démontrera aisément que, réciproquement, si l'élément linéaire d'une surface rapportée à ses lignes de courbure est défini par la formule (105), cette surface est une cyclide de Dupin.

Pour une périsphère dont les lignes de courbure circulaires ont pour équation $v = \text{const.}$, $\Phi_2 = V$, V désignant une fonction de v. Par suite, son élément linéaire est donné par la formule

$$ds^2 = \frac{\Phi_1^2 du^2 + V^2 dv^2}{(\rho_1 - \rho_2)^2}. \tag{106}$$

Réciproquement, si l'élément linéaire d'une surface rapportée à ses lignes de courbure est donné par la formule (106), où Φ_1 désigne une fonction de u et de v, cette surface est une périsphère pour laquelle u est le paramètre des lignes de courbure circulaires.

30. Si l'on fait usage des formules (102), les relations (47) deviennent

$$(107)\quad \left\{\begin{aligned} \frac{\partial \Omega_3}{\partial l_1} &= 4\,\Omega_1(1-\Omega_3) - 8\left(\frac{\partial \Omega_1}{\partial l_2} - 4\Omega_1\Omega_2\right)\Omega_2 + 2\frac{\partial}{\partial l_2}\left(\frac{\partial \Omega_1}{\partial l_2} - 4\Omega_1\Omega_2\right), \\ \frac{\partial \Omega_3}{\partial l_2} &= 4\Omega_2(1+\Omega_3) + 8\left(\frac{\partial \Omega_2}{\partial l_1} + 4\Omega_1\Omega_2\right)\Omega_1 + 2\frac{\partial}{\partial l_1}\left(\frac{\partial \Omega_2}{\partial l_1} + 4\Omega_1\Omega_2\right). \end{aligned}\right.$$

Ces égalités, où ne figurent que des quantités ayant une signification géométrique, sont équivalentes aux relations (62).

En vertu des formules (92), le résultat énoncé à la fin du N° 13 peut être formulé comme il suit:

Les invariants différentiels des surfaces pour le groupe conforme sont les fonctions d'un ou de plusieurs des éléments de l'ensemble formé par Ω_1, Ω_2, Ω_3 *et les dérivées de* Ω_1, Ω_2 *par rapport aux arcs angulaires des lignes de courbure.*

IX

31. Dans la Note citée, M. Tresse a fait connaître cinq invariants différentiels du quatrième ordre. Nous les désignerons par I_1, I_2, I_3, I_4, I_5. M. Calapso a donné leurs expressions au moyen des invariants Φ_1, Φ_2, W. Nous transcrivons ci-après ces expressions en leur donnant une forme légèrement différente:

(108) $$I_1=\frac{1}{\Phi_1^2}\left[\frac{\partial^2 \log \Phi_2}{\partial u^2}-\frac{\partial \log \Phi_1}{\partial u}\,\frac{\partial \log \Phi_2}{\partial u}+2\left(\frac{\partial \log \Phi_2}{\partial u}\right)^2\right],$$

(109) $$I_2=\frac{1}{\Phi_2^2}\left[\frac{\partial^2 \log \Phi_1}{\partial v^2}-\frac{\partial \log \Phi_1}{\partial v}\,\frac{\partial \log \Phi_2}{\partial v}+2\left(\frac{\partial \log \Phi_1}{\partial v}\right)^2\right],$$

(110) $$I_3=\frac{1}{\Phi_1\Phi_2}\left(\frac{\partial^2 \log \Phi_2}{\partial u\partial v}+\frac{\partial \log \Phi_1}{\partial v}\,\frac{\partial \log \Phi_2}{\partial u}\right),$$

(111) $$I_4=\frac{1}{\Phi_1\Phi_2}\left(\frac{\partial^2 \log \Phi_1}{\partial u\partial v}+\frac{\partial \log \Phi_1}{\partial v}\,\frac{\partial \log \Phi_2}{\partial u}\right),$$

(112) $$I_5=\frac{W}{\Phi_1\Phi_2}.$$

Ces invariants s'expriment, comme il suit, au moyen des invariants Ω_1, Ω_2, Ω_3:

(113) $$I_1=\ \ X_1\Omega_1+2\Omega_1^2,$$

(114) $$I_2=-X_2\Omega_2+2\Omega_2^2,$$

(115) $$I_3=\ \ X_2\Omega_1-2\Omega_1\Omega_2,$$

(116) $$I_4=-X_1\Omega_2-2\Omega_1\Omega_2,$$

(117) $$I_5=\ \ \Omega_3.$$

Etablissons ces différentes formules.

On a, en tenant compte d'une des égalités (92),

$$X_1\Omega_1+2\Omega_1^2=\frac{1}{\Phi_1}\,\frac{\partial\Omega_1}{\partial u}+2\Omega_1^2,$$

ou, en remplaçant Ω_1 par sa valeur (91),

$$X_1\Omega_1+2\Omega_1^2=\frac{1}{\Phi_1}\,\frac{\partial\Omega_1}{\partial u}+2\left(\frac{\frac{\partial \log \Phi_2}{\partial u}}{\Phi_1}\right)^2$$

ou

$$X_1\Omega_1+2\Omega_1^2=\frac{1}{\Phi_1^2}\left[\frac{\partial^2 \log \Phi_2}{\partial u^2}-\frac{\partial \log \Phi_1}{\partial u}\,\frac{\partial \log \Phi_2}{\partial u}+2\left(\frac{\partial \log \Phi_2}{\partial u}\right)^2\right].$$

Le second membre de cette égalité étant égal à l'expression (108) de I_1, la formule (113) est démontrée. On établira de même l'égalité (114).

Si l'on tient compte des relations (96) et (97), les égalités (110) et (111) donnent les égalités (115) et (116).

Enfin, on obtient l'égalité (117) en remplaçant, dans l'égalité (112), W par sa valeur $\Phi_1\,\Phi_2\,\Omega_3$, obtenue au N° 18.

RIASSUNTO DI ALCUNE RICERCHE DI GEOMETRIA PROIETTIVO-DIFFERENZIALE

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Voglio qui riassumere brevemente i metodi che io ho seguito in questi ultimi anni nei miei studî di geometria differenziale. A questo campo di ricerche geometriche i lavori di Demoulin, Tzitzéica e specialmente i lavori di Wilczynski hanno portato notevolissimi contributi. I metodi di Wilczynski partono dalla considerazione di certi sistemi di equazioni differenziali, e hanno condotto l'autore a importanti scoperte. La sua teoria è abbastanza semplice per le superficie, se si usano coordinate asintotiche; le formole diventano al contrario abbastanza complicate sia per le coordinate generali, sia per le ipersuperficie ecc.

La teoria, di cui voglio parlare, parte dalla generalizzazione del metodo di Gauss per la geometria metrica, e comincia dalla definizione di applicabilità di due varietà qualsiasi rispetto a un gruppo di S. Lie. Questa definizione ci porta a nuovi problemi di analisi, di cui qui non ci occuperemo. Invece di studiare il caso generale dei gruppi di Lie, ci limiteremo al gruppo delle collineazioni nello spazio a un numero qualunque di dimensioni.

Noi diremo che *due varietà sono proiettivamente applicabili se vi è una corrispondenza biunivoca tra i loro punti e se per ogni coppia di punti corrispondenti* A, B *nelle due varietà si può trovare una collineazione, che porta i punti*

$$A,\ dA,\ d^2A$$

della prima varietà nei punti

$$B,\ dB,\ d^2B$$

della seconda.

E' ben inteso che dA è il punto, di cui le coordinate sono i differenziali primi delle coordinate di A, ecc. . . .

Si trova allora il seguente teorema per le superficie e le ipersuperficie. Nella geometria metrica due superficie od ipersuperficie sono applicabili se hanno lo stesso elemento lineare di Gauss, che è una forma quadratica nei differenziali primi. Nella geometria proiettiva si presenta un risultato affatto analogo. Ma l'elemento lineare non è più una forma intera, ma al contrario il rapporto di due forme del primo ordine: il numeratore è una forma di terzo grado, il denominatore una forma quadratica, cioè di secondo grado. Per le superficie la prima, uguagliata a zero, dà l'equazione delle linee di Darboux, la seconda dà l'equazione delle asintotiche. L'elemento lineare proiettivo è identicamente uguale a zero soltanto per le quadriche.

Se noi moltiplichiamo le coordinate omogenee di un punto della superficie per un fattore, le due forme, la forma cubica e la quadratica, vengono pure moltiplicate per uno stesso fattore. Si dimostra che vi è un solo metodo per determinare nel modo più semplice questo fattore di indeterminazione; si trovano così le coordinate normali di un punto della superficie, e due forme normali, l'una cubica, l'altra quadratica, il cui rapporto è naturalmente sempre l'elemento lineare poriettivo. Si può poi determinare una geometria metrica invariante per collineazioni, la geometria, il cui elemento lineare è la forma quadratica normale, e chiamare *geodetiche proiettive* le linee che sono geodetiche in questa geometria. Si può anche considerare l'integrale dell'elemento lineare proiettivo, che ha molte semplici interpretazioni geometriche, considerare le linee, per le quali questo integrale è un estremo, e chiamarle *pangeodetiche.* Tutte queste linee hanno delle proprietà interessanti. Si deve considerare anche il punto D, le cui coordinate sono il secondo parametro differenziale di un punto A della superficie rispetto alla forma quadratica normale. Si può dimostrare che la retta AD è la più semplice generalizzazione possibile della normale; e perciò io la ho chiamata la *normale proiettiva.* Green, dopo di me, ha pubblicato una definizione differente, ma affatto equivalente alla mia. Dal metodo di Green non si può dedurre la necessità di tale definizione, e non si può dire nulla per le ipersuperficie; la definizione, che noi qui abbiamo dato, è utile al contrario in ogni caso. Si può dare qualche proprietà di questa retta normale. Consideriamo tutte le geodetiche proiettive che escono da un punto A della superficie, e i loro piani osculatori in questo punto A. Questi piani inviluppano un cono razionale di terza classe, che ha tre piani cuspidali. Ebbene, questi tre piani passano per la normale proiettiva.

Le pangeodetiche hanno un' altra notevole proprietà. Consideriamo il Piano osculatore ad una pangeodetica in un punto A della pangeodetica. Esso taglia la superficie in una linea, che passa per A. Consideriamo su questa linea A e tre punti infinitamente vicini. I piani tangenti in essi alla superficie passano per un medesimo punto. Ciò che si può anche enunciare nel modo seguente: *se quattro punti infinitamente vicini della superficie sono in un medesimo piano, i corrispondenti piani tangenti passano per un medesimo punto, soltanto se per i tre primi di essi passa una pangeodetica.*

I piani tangenti alle pangeodetiche che escono da un punto A della superficie determinano un cono; studiando questo cono, si possono definire la normale proiettiva, e tutte le rette notevoli scoperte finora (*direttrice* di Wilczynsky, *edge* di Green, *retta* di Cech, *rette principali* di Fubini). E' anche notevole che i birapporti di quattro di queste rette hanno valori puramente numerici.

A queste considerazioni si deve portare qualche cangiamento nel caso delle superficie rigate; per queste ultime superficie il problema di *normare* le coordinate di punto si presenta in modo un po' meno semplice.

Ci si può proporre anche il problema di determinare tutte le superficie e ipersuperficie di elemento lineare dato; si deve ricorrere allora a una terza forma differenziale quadratica e costruire una teoria affatto analoga alla teoria delle equazioni di Gauss e di Codazzi della geometria metrica. E si trova che le ipersuperficie sono generalmente indeformabili; al contrario ci sono delle classi di superficie deformabili, di cui si è occupato in lavori assai importanti anche Cartan.

Si possono generalizzare la definizione di linee di curvatura, e il teorema fondamental di Gauss per la curvatura totale. Ma nella geometria proiettiva la curvatura media ha l'ufficio essenziale.

Molte sono le classi di superficie notevoli, che si presentano in queste ricerche; le superficie che Tzitzeica aveva trovato nella geometria metrica e che sono identiche a quelle che Wilczynsky aveva studiato con metodi affatto differenti, le superficie a linee di Darboux o di Segre piane, ecc.

Le più importanti sono le superficie R di Demoulin e Tzitzéica, che sono proiettivamente deformabili e per cui Démoulin e Ionas hanno trovato delle trasformazioni per congruenze W, generalizzando nel modo più completo le trasformazioni di Bianchi.

Si ha il seguente teorema: Se due superficie R sono trasformate l'una dell'altra, cioè se sono superficie focali di una congruenza W, si può deformare proiettivamente una di esse in modo tale che la congruenza, trasportata in tale deformazione, possegga come seconda superficie focale una linea retta. Se ne può dedurre che il problema della determinazione di queste congruenze è affatto equivalente al problema della deformazione proiettiva delle superficie R.

La generalizzazione di queste ricerche a varietà qualsiasi immerse in un iperspazio conduce dal lato analitico a un problema che merita di essere studiato: il problema di trovare rispetto a una forma differenziale qualsiasi l'analogo del calcolo assoluto del Ricci: questo problema, e la generalizzazione di questi studî a un gruppo qualsiasi di Lie sono stati finora soltanto sfiorati.

Si può cercare una teoria analoga rispetto al gruppo proiettivo per i sistemi di rette, congruenze e complessi, e si può studiare il problema della loro deformazione. I complessi sono generalmente indeformabili, come io ho già osservato molti anni fa; per le congruenze il problema è stato studiato dal Cartan, e più recentemente da me.

Ecco il risultato che io ho ottenuto. Se si ha una congruenza di rette, le sue sviluppabili determinano su una superficie focale un sistema coniugato; ciò che dà due equazioni di Laplace, l'una per le coordinate di punto, l'altra per le coordinate di piano tangente. Si ottengono in questo modo quattro invarianti, due per la prima equazione, due per la seconda. Chiameremo il primo e l'ultimo di questi quattro invarianti gli invarianti della congruenza. *Due congruenze sono proiettivamente applicabili soltanto se esse hanno i medesimi invarianti rispetto a due falde focali omologhe, e se su queste falde focali si corrispondono le asintotiche.* Non si è ancora risoluto completamente il problema di determinare le congruenze deformabili, che hanno come prima falda focale una superficie data. Questo problema è forse il problema più importante della teoria della deformazione delle congruenze di rette.

Finirò coll'enunciare che si possono generalizzare le teorie di Bianchi delle trasformazioni asintotiche, e i teoremi di permutabilità alle superficie, che io ho chiamato isotermo-asintotiche. Se noi ci limitiamo alle congruenze che non sono contenute in un complesso lineare, noi possiamo caratterizzare queste superficie con la proprietà seguente: *esse sono le sole superficie, che sono falda focale di una congruenza, sulle superficie focali della quale si corrispondono le asintotiche e le linee di Darboux.* Analiticamente si possono caratterizzare queste superficie con una proprietà ancora più semplice, scrivendo in modo esplicito l'elemento lineare di queste superficie.

In un trattato, che è ora in corso di stampa, il Prof. Cech dell' Università di Brno e io ci siamo proposti di svolgere completamente queste ricerche e molte altre che appartengono a questo nuovo ramo della geometria differenziale.

SUR LES SYSTÈMES LINÉAIRES D'HYPERSURFACES

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Dans un mémoire paru récemment dans les Annales Scientifiques de l'École Normale Supérieure j'ai montré comment l'étude directe des systèmes d'équations aux dérivées partielles permet de définir certains nombres *invariants* analogues aux nombres appelés par M. Cartan les *caractères* dans l'étude des systèmes d'équations aux différentielles totales.

Je me propose d'ajouter quelques compléments à cette étude: il me semble utile d'abord de dégager la forme géométrique de la notion que j'ai en vue, ensuite d'en donner quelques applications simples, enfin de montrer sur un exemple les services qu'elle peut rendre en analyse.

1. Soient $x_1, x_2, \ldots, x_n$ les n coordonnées homogènes d'un point dans l'espace projectif à $n-1$ dimensions. Un système linéaire d'hypersurfaces (Σ) est l'ensemble des surfaces représentées par l'équation:

$$\lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_r f_r = 0$$

où les f sont des formes données de même ordre p, et où les λ sont des constantes arbitraires; la dimension de ce système est le nombre des paramètres dont dépend une surface de ce système (c'est-à-dire, si les f sont linéairement indépendantes, le nombre $r-1$). Ces définitions étant rappelées, nous attacherons au système $n-1$ nombres entiers, $s_1, s_2, \ldots, s_{n-1}$ qui ont une signification invariante dans une transformation homographique arbitraire. Considérons, outre les f, les hypersurfaces d'ordre p qui contiennent un hyperplan déterminé, d'ailleurs arbitraire; nous définissons ainsi un système linéaire (Σ_1); quand on passe de (Σ) à (Σ_1), la dimension augmente de s_1. D'une manière générale, considérons, outre les f, les hypersurfaces d'ordre p contenant un quelconque de k hyperplans déterminés, d'ailleurs arbitraires; nous définissons ainsi un système (Σ_k); quand on passe de (Σ_{k-1}) à (Σ_k) la dimension augmente de s_k. On définit ainsi $n-1$ nombres qui ne vont pas en croissant:

$$s_1 \geqq s_2 \geqq \ldots \geqq s_{n-1} \geqq 0.$$

Le nombre s_n est toujours égal à 0. Nous appellerons les nombres s les *caractères* du système linéaire d'hypersurfaces.

Considérons maintenant une hypersurface d'ordre $p+1$ formée d'une des hypersurfaces $f_i=0$ et d'un hyperplan, et cela de toutes les manières possibles.

Le système linéaire (Σ') défini par ces hypersurfaces sera appelé système dérivé de (Σ). Les surfaces (Σ') sont définies par l'équation

$$\sum_{i=1}^{r}\sum_{k=1}^{n} \lambda_{ik} f_i x_k = 0$$

où les λ_{ik} sont des constantes arbitraires.

Soient $s_1', s_2', \ldots, s_{n-1}'$ les caractères de (Σ'). Je démontre que

1° $$s_1'+s_2'+\ldots+s_{n-1}' \leqq s_1+2s_2+\ldots+(n-1)s_{n-1};$$

2° si dans la relation précédente, c'est l'égalité qui est vérifiée, c'est encore l'égalité qui est vérifiée lorsque l'on part du système (Σ') au lieu de partir du système (Σ), c'est-à-dire que l'on a, en désignant par s'' les caractères du dérivé (Σ'') de (Σ'),

$$s_1''+s_2''+\ldots+s_{n-1}'' = s_1'+2s_2'+\ldots+(n-1)s_{n-1}'.$$

De plus les s' sont donnés dans ce cas en fonction des s par les relations

$$s_k' = s_k+s_{k+1}+\ldots+s_{n-1}.$$

Enfin les s satisfont aux inégalités

$$s_1 \leqq \Gamma_n^{p-1} = \frac{p(p+1)\ldots(p+n-2)}{1.2\ldots(n-1)},$$

$$s_2 \leqq \Gamma_{n-1}^{p-1} = \frac{p(p+1)\ldots(p+n-3)}{1.2\ldots(n-2)},$$

. . . .

$$s_{n-1} \leqq \Gamma_2^{p-1} = \frac{p}{1}.$$

Le théorème précédent met en évidence certains systèmes linéaires particulièrement intéressants, ceux pour lesquels *l'égalité* est vérifiée; appelons-les systèmes *involutifs*. On pourra dire: *si un système linéaire d'hypersurfaces est involutif, tous ses dérivés le sont.*

Dans le plan $(n=3)$, les systèmes involutifs de coniques* $(p=2)$ sont les suivants:

*C'est à ces différents cas que correspondent les différentes formes de systèmes d'équations aux dérivées partielles en involution, du 2^e^ ordre, à une fonction inconnue de 3 variables indépendantes, qu'a étudiées M. Cartan dans un Mémoire du Bulletin de la Société Mathématique de France de 1911.

dimension 0; une seule conique;
" 1; système de coniques décomposées, une des droites étant commune fixe, l'autre décrivant un faisceau;
" 2; système de coniques décomposées, une des droites étant commune fixe, l'autre quelconque;
(*a*) les coniques passant par 3 points,
(*b*) les coniques passant par 1 point et tangentes à une droite en un point,
(*c*) les coniques osculatrices à une conique donnée en un point,
(*d*) les coniques décomposées ayant un point double donné;
" 3; les coniques passant par deux points,
les coniques tangentes à une droite en un point;
" 4; les coniques passant par un point;
" 5; toutes les coniques du plan.

2. Supposons involutif le système linéaire d'hypersurfaces considéré. Dans chacune des f, remplaçons $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ par

$$\frac{\partial^{\alpha_1+\alpha_2+\cdots\cdots+\alpha_n} z}{\partial x_1{}^{\alpha_1}\,\partial x_2{}^{\alpha_2}\ldots\,\partial x_n{}^{\alpha_n}}.$$

Cherchons une fonction z satisfaisant, quels que soient les λ, à *l'équation* aux dérivées partielles ainsi écrite.

Pour cela, donnons-nous, dans l'espace à n dimensions, un point *arbitraire* M_0, une multiplicité linéaire à une dimension *arbitraire*, M_1, contenant M_0, des multiplicités linéaires *arbitraires* à $2, 3, \ldots, n-1$ dimensions chacune contenant la précédente. Donnons-nous, d'autre part, autant de constantes *arbitraires* qu'il y a de dérivées d'ordre inférieur à p, puis $s_1, s_2, \ldots, s_{n-1}$ fonctions *arbitraires* de $1, 2, \ldots, n-1$ variables. Si on assujettit les dérivées de z d'ordre inférieur à p à se réduire en M_0 aux constantes données, si on assujettit d'autre part certaines dérivées de z d'ordre p à se réduire aux fonctions données sur les multiplicités linéaires données, on détermine *une solution et une seule.* Le Théorème d'existence auquel on est amené a sur les théorèmes analogues de M. Riquier l'avantage d'avoir une forme invariante.* On peut dire que le degré de généralité de la solution est caractérisé par les nombres:

$$s_1,\ s_2,\ \ldots,\ s_{n-1}.$$

3. Soit maintenant un système linéaire quelconque d'hypersurfaces. Les points communs à toutes ces hypersurfaces forment une ou plusieurs multiplicités que l'on appelle *multiplicités-bases.* Quand on passe d'un système à son dérivé, ces multiplicités bases ne changent pas. J'ai démontré dans le mémoire précédemment cité que, en poussant assez loin la dérivation, *on finit par obtenir un système involutif*; les dérivés suivants le sont évidemment aussi.

Supposons que l'on connaisse les caractères $s_1, s_2, \ldots, s_{n-1}$ de l'un des dérivés involutifs, soit λ_0 l'ordre de ce système. On peut exprimer à l'aide des caractères

*Il est d'ailleurs aisé de donner une forme analogue au théorème d'existence relatif au système d'équations aux dérivées partielles à une fonction inconnue le plus général.

le nombre des conditions pour qu'un hypersurface d'ordre donné λ fasse partie du système dérivé dont l'ordre est λ; ce nombre est

$$s_1+s_2\frac{\lambda-\lambda_0+1}{1}+s_3\frac{(\lambda-\lambda_0+1)(\lambda-\lambda_0+2)}{1.2}$$
$$+\ldots\ldots+s_{n-1}\frac{(\lambda-\lambda_0+1)\ (\lambda-\lambda_0+2)\ldots(\lambda-\lambda_0+n-2)}{1.2\ldots(n-2)}.$$

C'est ce que Hilbert appelle le *polynome caractéristique** du module des formes défini par les f. On voit son lien intime avec ce que nous avons appelé les *caractères*.

Soit t l'indice du dernier des s qui soit différent de zéro; cet indice diminué de 1 est égal au degré du polynome caractéristique.

Ce nombre $t-1$ est égal à la *dimension-maximum* des multiplicités-bases; de plus s_t est égal, *en général, au degré* de la multiplicité-base *de dimension* $t-1$.

Ce résultat a été énoncé par Hilbert sans les restrictions nécessaires et sans démonstration. Supposons $t=1$, on voit immédiatement que la multiplicité-base se réduit à un nombre fini de points, et on démontre que ce nombre est au plus égal à s_1. On doit considérer le cas où il est égal à s_1 comme le cas général; c'est ainsi que, pour les coniques considérées précédemment, nous appelons singuliers les cas (b), (c), (d). Cette restriction étant faite, il est commode, pour passer du cas de $t=1$ au cas de t quelconque, d'utiliser la proposition suivante:

Soit $\chi^{(\lambda)}$ le polynome caractéristique. Joignons au système considéré toutes les hypersurfaces qui contiennent un hyperplan déterminé d'ailleurs arbitraire. Le nouveau polynome caractéristique $\chi'^{(\lambda)}$ est la différence première du précédent. D'une manière précise, on a l'égalité:

$$\chi'^{(\lambda+1)}=\chi^{(\lambda+1)}-\chi^{(\lambda)}.$$

Quand on passe d'un système à son dérivé, les nombres t, s_t ne changent pas. Au contraire les autres caractères changent. C'est ce qui fait que ce sont les deux entiers t, s_t qu'il est le plus intéressant de considérer. Si l'on considère, comme tout à l'heure, le système aux dérivées partielles correspondant au système linéaire donné, on sera amené à caractériser avant tout son degré de généralité par les deux nombres suivants: nombre maximum t des arguments qui entrent dans les fonctions arbitraires, nombre de fonctions arbitraires de t arguments.

4. Voici un exemple simple de système linéaire involutif pour lequel $t=2$. Dans le tableau carré des monomes du second degré en $x_1, x_2, \ldots, x_n$,

			......	
x_1^2	x_1x_2	x_1x_3	......	x_1x_n
x_2x_1	x_2^2	x_2x_3	......	x_2x_n
.	.	.	......	.
.	.	.	......	.
x_nx_1	x_nx_2	x_nx_3	......	x_n^2

*Considéré par les géomètres italiens sous le nom de «postulazione».

égalons les monomes situés sur une même parallèle à la seconde diagonale:

$$x_1x_3=x_2^2;\ x_1x_4=x_2x_3;\ \text{etc.}\ldots.$$

Les surfaces (I) déterminent un système linéaire (Σ). Calculons les caractères de ce système; s_1 a sa valeur maxima n, s_2 sa valeur maxima $n-1$; on le voit immédiatement en adjoignant aux hypersurfaces (1), d'abord les $x_ix_1=0$ $(i=1, 2, \ldots, n)$; ensuite les $x_kx_n=0 (k=2, 3, \ldots, n)$; s_3 est nul ainsi que les s suivants.

Considérons le système dérivé (Σ'). Pour nous représenter facilement les équations qui le composent, imaginons un tableau cubique de n^3 cases, analogue au tableau carré précédent, les cases contenant cette fois en leurs centres les monomes du 3^e ordre en $x_1, x_2, \ldots, x_n$; (les n cases de l'un des côtés correspondent aux variables $x_1, x_2, \ldots, x_n$, x_1 étant la première à partir de l'origine, et chaque case renferme les produits des trois lettres auxquelles elle se trouve respectivement correspondre sur chacun des trois côtés). Ecrire les équations de (Σ') revient à égaler les monomes qui sont sur une même parallèle à une seconde bissectrice d'un plan de coordonnées. On égale donc en somme les monomes qui se trouvent dans un même plan perpendiculaire à la diagonale principale du cube. Le nombre $s_1'+s_2'+\ldots+s'_{n-1}$ est simplement le nombre de plans distincts de cette espèce où se trouvent répartis les n^3 centres des cases; c'est donc $3n-2$. Or

$$s_1+2s_2+3s_3+\ldots=n+2(n-1)=3n-2.$$

Le système (Σ) est donc en involution.

Sa signification géométrique est simple : (Σ) est le système des hypersurfaces du 2^{e} degré qui passent par la courbe unicursale de l'espace à $n-1$ dimensions,

$$x_1=u^{n-1},\quad x_2=u^{n-2}v,\quad \ldots,\quad x_n=v^{n-1}.$$

5. L'exemple suivant montrera comment certaines des remarques faites plus haut permettent de trouver rapidement la forme générale d'un système en involution de degré et de dimension donnés.

Plaçons-nous dans l'espace projectif à 3 dimensions $(n=4)$ et étudions les systèmes involutifs de surfaces du 2^{e} degré.

Si le nombre $t=3$, la multiplicité-base contient une surface, et ce ne peut être qu'une quadrique ou un plan; si c'est une quadrique, le système se réduit à une seule quadrique; si c'est un plan, le système se réduit à un système linéaire de plans auquel on joint un plan fixe; il contient donc, suivant les cas, deux, trois ou quatre surfaces indépendantes; dans ce dernier cas (α) il est formé de l'ensemble de tous les plans de l'espace auxquels on adjoint un plan fixe.

Si le nombre $t=1$, la multiplicité base ne comprend qu'un nombre fini de points, ce nombre est au plus égal à 4 (car $s_1\leqq 4$) et le système comprend au moins 6 surfaces indépendantes.

Supposons maintenant le nombre $t=2$; la multiplicité-base ne contient pas de surface, mais contient une courbe; le degré de cette courbe est au plus 3 puisque $s_2\leqq 3$; si d'ailleurs cette courbe est effectivement une cubique gauche, on sait qu'on peut, par une transformation homographique la ramener à la forme normale:

$$x_1 = u^3, \qquad x_2 = u^2 v, \qquad x_3 = uv^2, \qquad x_4 = v^3.$$

C'est le cas précédemment étudié; le système linéaire contient exactement 3 surfaces indépendantes. Si la courbe est une droite $s_2 = 1$; et comme $s_1 \leqq 4$ il y a au moins 5 surfaces indépendantes dans le système.

Ces remarques montrent qu'en dehors du cas (α), un système en involution du second ordre de dimension 3 ne peut être qu'un système déterminé par quatre quadriques indépendantes passant par une même conique ($s_2 = s_1 = 4$); mettons en évidence dans ce système les quadriques qui contiennent le plan de la conique; on pourra en général les écrire, après une transformation homographique $x_1x_4 = 0$, $x_2x_4 = 0$, $x_3x_4 = 0$. Soit $\Psi = 0$ une quadrique du système, indépendante des précédentes.

On constate que si $\Psi = 0$ ne passe pas par $x_1 = x_2 = x_3 = 0$, le nombre des surfaces dérivées du 3^e^ degré est 13, que si elle y passe au contraire, ce nombre est 12. Or $\Gamma_4^3 - (s_1 + 2s_2) = 20 - 8 = 12$. Donc le système en involution cherché est constitué par les surfaces du 2^e^ ordre passant par une conique et un point fixe non situé dans le plan de la conique. Nous laissons de côté les cas «singuliers» qui pourraient se présenter (en particulier celui que l'on obtiendrait en faisant tendre le point isolé vers un point de la conique).

6. Indiquons maintenant sur l'exemple précédent l'utilité que peuvent avoir ces considérations dans la théorie des systèmes d'équations aux dérivées partielles.

Considérons un système d'équations aux dérivées partielles à une fonction inconnue de 4 variables indépendantes x_i et du 2^e^ ordre . En prenant* comme variables les x_i, la fonction, ses dérivées premières et secondes, on peut le ramener à un système d'équations de Pfaff à $4 + 1 + 4 + 10 = 19$ variables:

(1) $$\varpi = \varpi_1 = \varpi_2 = \varpi_3 = \varpi_4 = 0,$$

avec les conditions

(2) $$\varpi' = 0$$

(3) $$\varpi_i' \equiv \sum \omega_k \varpi_{ik} \text{ mod. } (\varpi, \varpi_1, \varpi_2, \varpi_3, \varpi_4)$$

(où $\omega_k \varpi_{ik}$ désigne un produit symbolique), les ϖ_{ik} et ω étant liées identiquement par certaines relations linéaires. Si le système de Pfaff est en involution, on peut s'arranger pour que les relations ne contiennent pas les ω; d'autre part si on y remplace ϖ_{ik} par $X_i X_k$, on obtient un système d'équations involutif au sens précédent. On aperçoit donc immédiatement, qu'il y a deux types de systèmes de 4 équations en involution, l'un où la solution dépend *d'une* fonction de 3 variables, l'autre où la solution dépend de 2 fonctions de 2 variables.

Considérons ce deuxième cas. Nous allons montrer que, tout au moins dans le cas général, chaque multiplicité intégrale est engendrée par une famille à un paramètre de multiplicités caractérisques à 3 dimensions. L'étude algé-

*Cf. Cartan, Bulletin Soc. Math., France, 1911.

brique faite précédemment montre que si l'on choisit de manière convenable les formes de Pfaff ω, ϖ, ... les conditions (3) peuvent s'écrire:

$$\begin{aligned}
\varpi_1' &\equiv \omega_1\varpi_{11}+\omega_2\varpi_{12}+\omega_3\varpi_{13},\\
\varpi_2' &\equiv \omega_1\varpi_{21}+\omega_2\varpi_{22}+\omega_3\varpi_{23},\\
\varpi_3' &\equiv \omega_1\varpi_{31}+\omega_2\varpi_{32}+\omega_3\varpi_{33},\\
\varpi_4' &\equiv \qquad\qquad\qquad\qquad \omega_4\varpi_{44},
\end{aligned}$$

où les $\varpi_{ik}(i,\ k=1,2,3)$ sont liées par *une* relation linéaire. Considérons une multiplicité intégrale et montrons que sur cette multiplicité l'équation $\omega_4=0$ est complètement intégrable. On peut supposer que sur cette multiplicitét ous les ϖ sont nuls. Supposons

$$\omega_4' = \sum a_{ik}\omega_i\omega_k + \ldots$$

les termes non écrits s'annulant avec ω_4 et les ϖ. Il suffit de montrer que les a_{ik} sont nuls.

Appliquant l'identité fondamentale au covariant ϖ_{44}' on obtient

$$\omega_4'\varpi_{44}-\omega_4\varpi_{44}'\equiv 0 \quad \text{mod } (\varpi,\ \varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4,\ \varpi_1',\ \varpi_2',\ \varpi_3',\ \varpi_4').$$

Le coefficient de $\omega_i\omega_k\varpi_{44}$ dans le premier membre doit être nul; cela montre immédiatement $a_{ik}=0$.

Les multiplicités à trois dimensions qui satisfont à l'équation

$$\omega_4=0$$

sont des caractéristiques de Monge pour le système ; sur la multiplicité intégrale considérée elles dépendent d'un paramètre.

Les exemples précédents semblent suffisants pour faire comprendre les notions introduites (caractères d'un système d'hypersurfaces, nature involutive d'un tel système) ainsi que leur utilité. Elles peuvent, je crois, éclairer ou simplifier certaines questions, dans l'étude des multiplicités algébriques, et dans celle des systèmes d'équations aux dérivées partielles, en particulier des systèmes linéaires.

CONDITIONS FOR THE INTERSECTION OF LINEAR SPACES SITUATED IN A QUADRATIC VARIETY

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The existence of (real or imaginary) linear k dimensional spaces V_k' in a quadratic variety V^2_{n-1} in S_n has been proved geometrically by Segre*, and analytically by Borel† and Darboux‡. The highest admissible value of k is

$$k=p-1 \text{ for } n=2p;\ k=p \text{ for } n=2p+1.$$

For $n=2p$ the generators V_k' form a single system; for $n=2p+1$ they split up into two systems, two generators of the same system having, generally, no intersection if p is odd, but having, generally, one point in common if p is even. Two generators belonging to different systems, generally, intersect in one point, if p is odd, but do not intersect if p is even.

If, for n odd, in the general case there is no intersection, the two generators can be specially chosen to have a line V_1', a 3-space V_3', ..., a V'_{2q+1} in common $(2q+1<k)$; if in the general case there is intersection in a point, special situations may cause intersections along a plane V_2', a 4-space V_4', ..., a V_{2q}' $(2q<k)$. For $n=2p$, two generators in general do not intersect, but they may in special cases have in common a point, a line V_1', a plane V_2', ..., a V'_{p-2}.

We propose to find, making use of the analysis of Borel and Darboux, the condition for the intersection of two generators in those cases where they do not generally intersect, the special intersection thus being a line-intersection if $n=2p+1$, or a point-intersection if $n=2p$.

Case A. $n=2p+1$, p odd, $n=4q+3$.

The equation of the V^2_{n-1} can be written

$$x_1y_1+x_2y_2+\ldots+x_{2q+2}\,y_{2q+2}=0.$$

*C. Segre. *Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni.* (Mem. R. Accad. di Torino, serie 2ª, T. XXXVI, 1885, p. 3).

†E. Borel. *Sur l'équation adjointe et sur certains systèmes d'équations différentielles.* (Annales de l'École Normale Supérieure, série 3, T. IX, 1892, p. 63).

‡G, Darboux. *Sur certains systèmes d'équations differentielles linéaires.* (Annales de l'École Normale Supérieure, série 3, T. XXVI, 1909, p. 449).

Also G. Fano. *Sul sistema ∞^3 di rette contenuto in una quadrica dello spazio a quattro dimensioni.* (Giornale di Matem. di Battaglini, XLIII, 1905, p. 1).

The equations of a generator V'_{2q+1} are

$$(1)\qquad \left\{\begin{array}{l} x_1+\sum_{j=1}^{2q+2} a_{1,j}\, y_j=0,\\ x_2+\sum a_{2,j}\, y_j=0,\\ \dots\dots\dots\dots\dots\dots\\ x_i+\sum a_{i,j}\, y_j=0,\\ \dots\dots\dots\dots\dots\dots\\ x_{2q+2}+\sum a_{2q+2,j}\, y_j=0, \end{array}\right\} a_{ij}=0 \text{ for } i=j,\ a_{ij}=-a_{ji}.$$

This generator a is evidently determined by the elements a_{ij} of a skew determinant of order $(2q+2)$; another generator b of the same system in the same way by b_{ij}; the condition for their intersection is the vanishing of the skew determinant

$$(a_{ij}-b_{ij}),$$

that is the vanishing of its square root, the Pfaffian function $P(a_{ij}-b_{ij})$; so that

$$(2)\qquad P(a_{ij}-b_{ij})=0$$

is the required condition.

Taking the coefficients a_{ij} $(i<j)$ as Cartesian coordinates, we represent the V'_{q+1} of one system of the V^2_{4q+2} by points of a point-space $S_{(q+1)\,(2q+1)}$, and two generators intersect if, and only if, the line joining the points representing them has a direction satisfying (2); that is, if it is parallel to a certain cone of degree $(q+1)$. This representation, of course, is not one-to-one.

The equations (1) may be given another form (comp. Darboux, l. c.).

$$(3)\qquad \left\{\begin{array}{l} y_1+\sum_{j=1}^{2q+2} \alpha_{1,j}\, x_j=0\\ \dots\dots\dots\dots\dots\dots\\ y_{2q+2}+\sum \alpha_{2q+2,j}\, x_j=0, \end{array}\right.$$

where in consequence of well known properties of skew determinants

$$\alpha_{ij}=\frac{{}_{ij}P(a_{kl})}{P(a_{kl})},$$

the symbol ${}_{ij}P(a_{kl})$ denoting the Pfaffian function of the a_{kl}, where the indices k, l are made to run from (1) to $(2q+2)$, *excluding the values* i *and* j.

Inversely

$$a_{ij}=\frac{{}_{ij}P(\alpha_{kl})}{P(\alpha_{kl})}.$$

The intersection of two generators α and β now depends on the vanishing of a determinant

$$(4)\qquad \begin{vmatrix} 1 & 0 & 0 & . & . & 0 & a_{12} & a_{13} & . & . \\ 0 & 1 & 0 & . & . & a_{21} & 0 & a_{23} & . & . \\ 0 & 0 & 1 & . & . & a_{31} & a_{32} & 0 & . & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \beta_{12} & \beta_{13} & . & . & 1 & 0 & 0 & . & . \\ \beta_{21} & 0 & \beta_{23} & . & . & 0 & 1 & 0 & . & . \\ \beta_{31} & \beta_{32} & 0 & . & . & 0 & 0 & 1 & . & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

This determinant is a polynomial Q of degree $(4q+4)$, which contains the a's and the β's to degrees of at most $(2q+2)$ each. This polynomial vanishes, according to (2), when

$$(5)\qquad P\left\{a_{ij} - \frac{{}_{ij}P(\beta_{kl})}{P(\beta_{kl})}\right\} = 0,$$

the left-hand expression becoming, after clearing of fractions, a polynomial R containing the a's, and likewise the β's, to the degree of $P(a_{ij})$, which is $(q+1)$. So Q contains a factor of this degree, and as the second factor is identical with the first, *Q is a perfect square**. In consequence of the properties of minors of skew determinants, R can be expressed in Pfaffian functions of degrees $(q+1)$ and lower.

Case B. $n=4q+1$.

$$x_1y_1+x_2y_2+\ldots+x_{2q+1}\,y_{2q+1}=0.$$

Two generators V'_{2q} of *different systems* are now given by

$$a\,x_i+\sum_{j=1}^{2q+1} a_{ij}y_j=0, \qquad \beta\,y_i+\sum_{j=1}^{2q+1} \beta_{ij}\,x_j=0.$$

They do not intersect generally; the condition for their intersection is the vanishing of a determinant of type (4), but now of order $2(2p+1)$. Such a determinant arises from (4), when for every i we put:

$$a_{i,\,2q+2}=\beta_{i,\,2q+2}=0;$$

so that (4) is a perfect square, not only when its order is $4p$, as in case A, but also when it is $2(2p+1)$.

The conditions for the intersection of a and β is therefore

$$R=0,$$

where R is a polynomial of degree $2q$, containing a_{ij} and β_{ij} each to a degree of at most q and expressible in terms of Pfaffian functions.

*It becomes indeed, as Mr. J. A. Prins remarks, a skew determinant by properly interchanging the four quadrants and changing signs in two of them.

Case C. $n=4q+2$.

We obtain this case, when in case A we identify x_{2q+2} with $y_{2q+2}(\equiv z)$; thus

$$x_1y_1+x_2y_2+\ldots+x_{2q+1}\,y_{2q+1}+z^2=0.$$

Two generators V'_{2q} obviously intersect if, and only if,

$$P(a_{ij}-b_{ij})=0;$$

for any of the $(4q+3)$ determinants in the matrix of the coefficients of the $(4q+4)$ equations (1), for a and b respectively and modified for case C, contains the factor P. But substracting the equations for a from those for b we get

$$P^2=0.$$

Thus $P=0$ is the necessary and sufficient condition of intersection.

Case D. $n=4q$.

Identifying $x_{2q+1}\equiv y_{2q+1}\equiv z$ in the equations of case B we get

$$x_1y_1+x_2y_2+\ldots+x_{2q}\,y_{2q}+z^2=0.$$

Two generators a and b are given by two sets of linear equations like α and β in case B. These generators intersect if all the determinants of order $(4q+1)$ in the matrix of the coefficients of the $(4q+2)$ linear equations vanish. Now each of these determinants splits up into two factors, the first of which is a Pfaffian function of order q, contained in the skew determinant of order $(2q+1)$ of the $(a_{ij}-b_{ij})$. These Pfaffians not being identical, it is the remaining common factor of degree $(q+1)$ in the a_{ij}, b_{ij}, the vanishing of which gives the required condition.

Applying the preceding theorems to the lower values of n, we obtain the following results:

$n=5$. The condition of intersection of two generators V_2',

$$\left\|\begin{matrix}\lambda_1 & 0 & 0 & 0 & a_1 & b_1\\ 0 & \lambda_1 & 0 & -a_1 & 0 & c_1\\ 0 & 0 & \lambda_1 & -b_1 & -c_1 & 0\end{matrix}\right\| \text{ and } \left\|\begin{matrix}\lambda_2 & 0 & 0 & 0 & a_2 & b_2\\ 0 & \lambda_2 & 0 & -a_2 & 0 & c_2\\ 0 & 0 & \lambda_2 & -b_2 & -c_2 & 0\end{matrix}\right\|$$

is

$$a_1a_2+b_1b_2+c_1c_2+\lambda_1\lambda_2=0.$$

The generators V_2' of one system can be brought into one-to-one correspondence with the points, those of the other system into one-to-one correspondence with the planes of projective 3-dimensional space; and the intersection of two generators V_2' belonging to different systems is represented by the incidence of the corresponding point and plane.

This result however is nothing other than *F. Klein's* well known S_5 — representation of 3-dimensional line-space, now obtained in the inverse way.

$n=4$. Two generators V_1' intersect if the matrix

$$\begin{Vmatrix} \lambda_1 & 0 & 0 & a_1 & b_1 \\ 0 & \lambda_1 & -a_1 & 0 & c_1 \\ 0 & 0 & -b_1 & -c_1 & \lambda_1 \\ \lambda_2 & 0 & 0 & a_2 & b_2 \\ 0 & \lambda_2 & -a_2 & 0 & c_2 \\ 0 & 0 & -b_2 & -c_2 & \lambda_2 \end{Vmatrix}$$

has the rank 4, the conditions being

$$a_1\lambda_2 - a_2\lambda_1 + c_1b_2 - c_2b_1 = 0.$$

Thus interpreting $(abc\lambda)$ as a point of projective S_3, two generators V_1' intersect if the line joining the points representing them, viz.

$$\begin{Vmatrix} a_1 & b_1 & c_1 & \lambda_1 \\ a_2 & b_2 & c_2 & \lambda_2 \end{Vmatrix}$$

belongs to the linear complex

$$p_{14} - p_{23} = 0.$$

This result once more can be immediately derived from *Klein's* representation.

$n=6$. Two generators V_2' intersect if the matrix

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & a_1 & b_1 & c_1 \\ 0 & 1 & 0 & -a_1 & 0 & d_1 & e_1 \\ 0 & 0 & 1 & -b_1 & -d_1 & 0 & f_1 \\ 0 & 0 & 0 & -c_1 & -e_1 & -f_1 & 1 \\ 1 & 0 & 0 & 0 & a_2 & b_2 & c_2 \\ 0 & 1 & 0 & -a_2 & 0 & d_2 & e_2 \\ 0 & 0 & 1 & -b_2 & -d_2 & 0 & f_2 \\ 0 & 0 & 0 & -c_2 & -e_2 & -f_2 & 1 \end{Vmatrix}$$

is of rank 6, that is if

(6) $$AF - BE + CD = 0,$$

where $$A = a_1 - a_2, \quad B = b_1 - b_2, \quad C = c_1 - c_2,$$

$$F = f_1 - f_2, \quad E = e_1 - e_2, \quad D = d_1 - d_2.$$

The generators V_2' of the V_5^2 in S_6 can now be represented by the points $(abcdef)$ of 6-dimensional space; two generators V_2' intersect if the line joining the points representing them has a direction satisfying (6). This representation however is not one-to-one. We may derive from it a one-to-one representation by considering an S_7 containing the representing S_6, and imagining a V_6^2, that intersects the S_6 along the V_5^2 locus of points at infinity of the directions (6). Now taking a centre of projection in the V_6^2 we may project the points of the S_6 on this variety. In this way every generator V_2':

(7) $$\begin{Vmatrix} \lambda & 0 & 0 & 0 & a & b & c \\ 0 & \lambda & 0 & -a & 0 & d & e \\ 0 & 0 & \lambda & -b & -d & 0 & f \\ 0 & 0 & 0 & -c & -e & -f & \lambda \end{Vmatrix}$$

for which $\lambda \neq 0$, is uniquely represented.

The generator

$$y_1 = y_2 = y_3 = z = 0,$$

that is

$$\lambda = 0, \; af - be + cd \neq 0,$$

we represent by the centre of projection itself. There remains to establish a correspondence between those V_2', the representation of which cannot be brought under the form (7) on the one hand, and the points, common to the V_6^2 and its tangent $-V_6'$ at the centre of projection, on the other. A generator not representable by (7) can however be expressed in some analogous way, obtained by interchanging x_1 and y_1, x_2 and y_2, or x_3 and y_3. Thus the generator

$$\left\| \begin{matrix} \lambda & 0 & b & 0 & a & 0 & c \\ 0 & \lambda & d & -a & 0 & 0 & e \\ 0 & 0 & 0 & -b & -d & \lambda & f \\ 0 & 0 & -f & -c & -e & 0 & \lambda \end{matrix} \right\|$$

is the same as

$$\left\| \begin{matrix} f\lambda & 0 & 0 & 0 & af-be+cd & -c\lambda & b\lambda \\ 0 & f\lambda & 0 & -af+be-cd & 0 & -e\lambda & d\lambda \\ 0 & 0 & f\lambda & c\lambda & e\lambda & 0 & -\lambda^2 \\ 0 & 0 & 0 & -b\lambda & -d\lambda & \lambda^2 & f\lambda \end{matrix} \right\|.$$

Putting

$$\lambda^2 = \xi_1, \; \lambda a = \xi_2, \quad \lambda b = \xi_3, \quad \lambda c = \xi_4,$$

$$af - be + cd = \xi_8, \; \lambda f = \xi_7, \quad \lambda e = \xi_6, \quad \lambda d = \xi_5$$

we see that the generators V_2' of a V_5^2 can be brought into one-to-one correspondence with the points of projective point space

$$(\xi_1 \; \xi_2 \; \xi_3 \; \xi_4 \; \xi_5 \; \xi_6 \; \xi_7 \; \xi_8),$$

that belong to a quadratic variety V_6^2:

$$\xi_1 \xi_8 - \xi_2 \xi_7 + \xi_3 \xi_6 - \xi_4 \xi_5 = 0. \tag{8}$$

Two generators of the V_5^2 intersect if the line joining the points ξ and η representing them belongs to (8), this condition:

$$(\xi_1 \eta_8 + \xi_8 \eta_1) - (\xi_2 \eta_7 + \xi_7 \eta_2) + (\xi_3 \eta_6 + \xi_6 \eta_3) - (\xi_4 \eta_5 + \xi_5 \eta_4) = 0$$

being in fact identical with (6).

$n = 7$. Reasoning similar to that of the preceding case gives the result that the ∞^6 generators V_3' belonging to one of the systems in a V_6^2 can be made to correspond one-to-one to the points of a V_6^2; two generators intersecting if the line joining their corresponding points belongs to the representing V_6^2 (which of course may be chosen the same as the original one).

The theory of linear spaces situated in a quadratic variety is connected with that of skew, or Clifford, parallelism in elliptic space. In fact, choosing

$$x_1^2+x_2^2+\dots+x_n^2+x_{n+1}^2=0$$

as the absolute base for elliptic measure in S_n (Cayley), the locus of points at a constant distance d to a linear space S_p:

$$x_1=x_2=\dots=x_{n-p}=0$$

which is the same as the locus of points at complementary distance $\left(\frac{\pi}{2}-d\right)$ to the S_q:

$$x_{n-p+1}=x_{n-p+2}=\dots=x_{n+1}=0,$$

where $p+q=n-1$, S_p and S_q being absolutely perpendicular) is the quadratic variety V_{n-1}^2:

(9) $$x_1^2+x_2^2+\dots+x_{n-p}^2-\lambda(x_{n-p+1}^2+x_{n-p+2}^2+\dots+x_{n+1}^2)=0,$$

the relations between λ and d being

$$\lambda=(\mathrm{tg}d)^2.$$

Through any point P in this V_{n-1}^2 passes one line perpendicular to both S_p and S_q; and normal to this line there is a linear space Σ_p in the S_{p+1}containing S_p and P, and likewise a space Σ_q in the S_{q+1} containing S_q and P; Σ_p and Σ_q being moreover absolutely perpendicular to each other.

Now (supposing $p\leqq q$, or, if not so, interchanging p with q) in consequence of the theory of Clifford parallelism any linear p-dimensional space through P in the $(n-1)$-space containing Σ_p and Σ_q, having its angles of position with Σ_p all equal to the distance d, is skew parallel to S_p and is thus contained in the V_{n-1}^2 given under (9).

CONTRIBUTO ALLA TEORIA DELLE VARIETÀ RIEMANNIANE

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I metodi di Calcolo Differenziale assoluto si applicano allo studio delle linee tracciate in una varietà V_n, in quanto lo studio di una linea è fatto attraverso le sue equazioni differenziali e quindi la linea stessa è inserita nell'insieme di ∞^{n-1} linee constituenti una congruenza. Se questa è normale un tale studio poi coincide con quello delle $\infty^1 V_{n-1}$, che le linee della congruenza incontrano sotto angolo retto.

Analogamente lo studio di una V_{n-m} di V_n può farsi inserendola (come è sempre possibile se si tratta di una V_{n-m} dotata di rappresentazione analitica regolare) in un insieme di $\infty^m V_{n-m}$ rappresentata da $n-m$ congruenze ortogonali di linee, che le generano, o mediante m congruenze costituenti con quelle una ennupla ortogonale.

Molti anni fa in uno studio pubblicato tra le Memorie della Reale Accademia dei Lincei esposi la teoria delle ennuple ortogonali nelle sue linee fondamentali ed introdussi quei coefficienti di rotazione, che occupano nella teoria stessa un posto preminente. Più tardi ne feci alcune applicazioni allo studio delle varietà considerate come generate da n congruenze ortogonali e precisamente alla ricerca di varietà dotate di proprietà caratteristiche assegnate a priori.

Mi onoro di comunicare al Congresso matematico di Toronto ulteriori applicazion di quella teoria allo studio delle sotto-varietà riemanniane; applicazioni che, si chiudono con una dimostrazione diretta e semplice del teorema fondamentale di Riemann sulla curvatura delle varietà.

Dalle considerazioni svolte risulterà poi come lo studio di una V_{n-m} considerata come appartenente a sistemi di $\infty^m V_{n-m}$ consenta anche di indagarne i comportamenti locali.

1. In una varietà qualunque ad n dimensioni si considerino n congruenze di linee [1], [2], ..., [n] costituenti una ennupla ortogonale Λ, le cui equazioni canoniche siano

$$l_h^{(r)} = \frac{dx_r}{ds_h}, \ (h=1, 2, \ldots, n),$$

il che vale quanto dire che $l_h^{(r)}$ sono gli elementi del sistema coordinato contro-variante (paràmetri di direzione) della congruenza [h]; ds_h misurando la distanza di un punto qualunque P di V_n (di coordinate $x_1, x_2, \ldots x_n$) da quello, che gli succede lungo la linea, percorsa in senso positivo, della detta congruenza*.

Come si dice normale la congruenza $[n]$, se è costituita dalle traiettorie ortogonali di ∞^1 sottovarietà V_{n-1} di V_n; così, in generale, diremo che le congruenze $[n-m+1]$, $[n-m+2]$, $\ldots$, $[n]$ costituiscono una *emmupla normale* di V_n, se esistono ∞^m sottovarietà V_{n-m} di V_n tali che le linee delle dette congruenze incontrino sotto angolo retto tutte le linee tracciate sopra una V_{n-m} qualunque. Evidentemente perchè ciò avvenga è necessario e basta che esistano m funzioni indipendenti $f_1, f_2, \ldots, f_m$ di $x_1, x_2, \ldots, x_n$ tali che le linee delle congruenze $[1], [2], \ldots, [n-m]$ appartengano alle V_{n-m} di equazioni

$$(1) \qquad f_1 = c_1, f_2 = c_2, \ldots, f_m = c_m$$

($c_1, c_2, \ldots, c_m$ rappresentando delle costanti arbitrarie) ovvero, ciò che vale o stesso, tali che il sistema di equazioni a derivate parziali

$$(\mathrm{S}) \qquad \sum_r l_h^{(r)} \frac{\partial f}{\partial x_r} = 0, \quad (h = 1, 2, \ldots, n-m)\dagger,$$

ammetta le soluzioni $f = f_i (i = 1, 2, \ldots, m)$ e di conseguenza sia completo. Ed è poi anche chiaro che, ciò essendo, ed $f_1, f_2, \ldots, f_m$ costituendo un sistema fondamentale di integrali per il sistema S), e equazioni (1) definiscono ∞^m sottovarietà V_{n-m} *di* V_n secate ortogonalmente dalle linee delle congruenze $[n-m+1]$, $[n-m+2]$, $\ldots$, $[n]$; e che perciò possono considerarsi come generate dalle congruenze $[1], [2], \ldots, [n-m]$.

Le condizioni di completa integrabilità del sistema S) sono poi rappresentate dalle relazioni

$$\sum_s l_{i/s} \sum_r \left(l_h^{(r)} \frac{\partial l_k^{(s)}}{\partial x_r} - l_k^{(r)} \frac{\partial l_h^{(s)}}{\partial x_r} \right) \equiv 0,$$

che equivalgono alle

$$\sum_{rs} l_h^{(r)} l_k^{(s)} \left(\frac{\partial l_{i/r}}{\partial x_s} - \frac{\partial l_{i/s}}{\partial x_r} \right) \equiv 0,$$

e anche alle

$$\sum_{rs} l_h^{(r)} l_k^{(s)} (l_{i/rs} - l_{i/sr}) \equiv 0\ddagger,$$

$$\left(\begin{matrix} h \\ k \end{matrix} \Big\{ = 1, 2, \ldots, n-m; \; i = n-m+1, \ldots, n \right).$$

*Farò in questo scritto uso degli indici $f, g, h, \ldots, l$, per distinguere i diversi sistemi variabili; e degli indici $r, s, t, \ldots$ per distinguere gli elementi di uno stesso sistema. Per questi ultimi rimane sottinteso che essi possono assumere tutti i valori da 1 fino ad n.

†Qualora la cosa non sia espressamente chiarita in modo diverso, ogni sommatoria Σ si intenderà estesa ai termini, che si traggono da quello, che segue il simbolo Σ, dando i valori $1, 2, \ldots, n$ all'indice, cui essa si riferisce.

‡Le derivazioni covarianti si intendono fatte rispetto alla forma fondamentale

$$\phi = \Sigma_{rs} a_{rs} dx_r dx_s, \quad (a_{rs} = \Sigma_h l_{h/r} l_{h/s}),$$

che definisce la metrica di V_n.

Se infine introduciamo i coefficienti di rotazione γ_{ihk} della ennupla Λ e poniamo

$$(2) \qquad 2\delta_{ihk} = \gamma_{ihk} - \gamma_{ikh}$$

le condizioni stesse vengono sostituite dalle

$$(3) \qquad \delta_{ihk} \equiv 0, \quad \left(\begin{matrix} h \\ k \end{matrix}\right\} = 1, 2, \ldots, n-m; \; i = n-m+1, n-m+2, \ldots, n\Big).$$

Queste sono dunque altresì le condizioni necessarie e sufficienti perchè la emmupla $[n-m+1]$, $[n-m+2]$, ..., $[n]$ sia normale; o, in altri termini, perchè le congruenze $[1], [2], \ldots, [n-m]$ generino ∞^m sottovarietà V_{n-m} di V_n.
Siano

$$(4) \qquad f_1 = 0, f_2 = 0, \ldots, f_m = 0$$

le equazioni di una sottovarietà di V_n. Essendo $f_1, f_2, \ldots, f_m$ funzioni indipendenti di $x_1, x_2, \ldots, x_n$, a queste si potranno sostituire altre coordinate $y_1, y_2, \ldots, y_n$ scegliendo $y_1 = f_1$, $y_2 = f_2$, ..., $y_m = f_m$. La sottovarietà di equazioni (4), potrà così riguardarsi come facente parte dell'insieme di ∞^m sottovarietà V_{n-m} di V_n di equazioni

$$y_1 = c_1, \; y_2 = c_2, \; \ldots, \; y_m = c_m;$$

e sarà possibile determinare una ennupla ortogonale Λ di V_m tale che queste varietà siano generate dalle congruenze $[1]$, $[2]$, ..., $[n-m]$; e quindi le relazioni (3) risultino soddisfatte.

2. Si riferisca ora ad una ennupla qualunque Λ di V_n una congruenza μ di equazioni canoniche

$$\mu^{(r)} = \frac{dx_r}{ds},$$

e si indichino con $A_i (i = 1, 2, \ldots, n)$ i coseni degli angoli che le linee di questa congruenza fanno con quelle della congruenza $[i]$. Varranno le relazioni

$$\mu_r = \sum_i A_i l_{i/r}$$

e le

$$\mu_{rs} = \sum_i A_i l_{i/rs} + \sum_i A_{i/s} l_{i/r}$$

che ne scendono per derivazione covariante; e che, tenuto conto delle note relazioni tra le $l_{i/rs}$ e i coefficienti di rotazione della ennupla Λ assumono la forma

$$\mu_{rs} = \sum_{hki} A_i \gamma_{ihk} l_{h/r} l_{k/s} + \sum_i A_{i/s} l_{i/r}.$$

Da queste si passa alle

$$\sum_s \mu^{(s)} \mu_{rs} = \sum_i l_{i/r} \left(\frac{dA_i}{ds} - \sum_{hk} \gamma_{ihk} A_h A_k \right).$$

in conseguenza delle quali le equazioni

$$\sum_s \mu^{(s)} \mu_{rs} = 0$$

delle congruenze geodetiche assumono la forma

$$\frac{dA_i}{ds} = \Sigma_{hk}\gamma_{ihk}A_hA_k;$$

o anche

$$\text{(A)} \qquad \frac{dA_i}{ds} = \Sigma_{hk}(\delta_{hki}+\delta_{khi})A_hA_k, \quad (i=1,\,2,\,\ldots,\,n),$$

in quanto dalle (2) e dalle note relazioni tra le γ_{ihk} scendono le

$$(5) \qquad \gamma_{ihk}+\gamma_{ikh}=2(\delta_{hki}+\delta_{khi}).$$

Diremo che le congruenze [1], [2], ..., $[n-m]$ costituiscono una $(n-m)^{pla}$ *geodetica di* V_n se esistono ∞^{n-m+1} congruenze geodetiche di V_n, le cui linee siano ortogonali a quelle delle congruenze $[n-m+1]$, $[n-m+2]$, ..., $[n]$. Segue ora dalle (A) essere per ciò necessario e sufficiente che siano verificate le relazioni

$$(6) \qquad \delta_{hki}+\delta_{khi}\equiv 0, \quad \left(\begin{matrix} h \\ k \end{matrix}\right\}=1,2,\,\ldots,\,n-m;\ i=n-m+1,\,\ldots,\,n\Big),$$

verificate le quali si ottengono le congruenze geodetiche di V_n normali alle congruenze $[n-m+1]$, $[n-m+2]$, ..., $[n]$ integrando il sistema di equazioni

$$\frac{dA_i}{ds} = \sum_1^{n-m}{}_{hk}\,(\delta_{hki}+\delta_{khi})A_hA_k, \quad (i=1,\,2,\ldots,\,n-m).$$

Perchè di più le congruenze [1], [2], ..., $[n-m]$ generino ∞^{n-m} sottovarietà V_{n-m} di V_n sarà poi necessario e sufficiente che siano insieme soddisfatte le (3) e le (6), le quali, tenuto conto delle (5), si possono riassumere nelle

$$(7) \qquad \gamma_{ihk}=0, \quad \left(\begin{matrix} h \\ k \end{matrix}\right\}=1,\,2,\,\ldots,\,n-m;\ i=n-m+1,\,\ldots,\,n\Big).$$

3. Si consideri ora una geodetica di V_n uscente da un determinato punto P tangenzialmente alla sottovarietà V_{n-m} definita dalle linee delle congruenze [1], [2], ..., $[n-m]$ uscenti dallo stesso punto.

In P avremo

$$A_{n-m+1}=A_{n-m+2}=\ldots=A_n=0$$

e dalle (A) seguiranno le

$$\frac{dA_i}{ds} = \sum_1^{n-m}{}_{hk}\,(\delta_{hki}+\delta_{khi})A_hA_k, \quad (i=n-m+1,\,n-m+2,\,\ldots,\,n),$$

e le equazioni

$$(8) \qquad \sum_1^{n-m}{}_{hk}\,(\delta_{hki}+\delta_{khi})A_hA_k=0, \quad (i=n-m+1,\,n-m+2,\,\ldots,\,n),$$

soddisfatte in P esprimeranno le condizioni necessarie e sufficienti perchè la geodetica considerata esca dal punto P' *successivo* a P tangenzialmente alla varietà V'_{n-m} definita dalle linee delle congruenze [1], [2], ..., $[n-m]$ uscenti da P'.

Diremo che queste congruenze determinano intorno a P *una giacitura geodetica*, se ciò si verifica per tutte le geodetiche di V_n uscenti da P tangenzialmente a V_{n-m}; e in particolare se V_{n-m} è una sottovarietà geodetica di V_n. E perciò sarà necessario e sufficiente che le (8) siano identicamente soddisfatte rispetto ad $A_1, A_2, \ldots A_{n-m}$; che val quanto dire che in P siano soddisfatte le equazioni

$$\delta_{hki}+\delta_{khi}=0, \quad \left(\begin{matrix} h \\ k \end{matrix}\right\{ =1, 2, \ldots, n-m; \ i=n-m+1, n-m+2, \ldots, n \Big). \tag{9}$$

E se si suppone che le congruenze [1], [2], ..., $[n-m]$ generino ∞^m sottovarietà V_{n-m} di V_n alle (9) si potranno sostituire le

$$\gamma_{ihk}=0, \quad \left(\begin{matrix} h \\ k \end{matrix}\right\{ =1, 2, \ldots, n-m; \ i=n-m+1, \ldots, n \Big). \tag{10}$$

4. Essendo $a_{rt, su}$ i simboli di Riemann di 1^a. specie relativi alla forma fondamentale, si facciano le posizioni

$$\gamma_{ih, jk}=\sum_{rstu} a_{rt, su}\, l_i^{(r)} l_j^{(s)} l_h^{(t)} l_k^{(u)}.$$

Gli invarianti $\gamma_{ih, jk}$ sono legati ai coefficienti di rotazione della ennupla Λ dalle relazioni

$$\gamma_{ih, jk}=\frac{\partial \gamma_{ihj}}{\partial s_k}-\frac{\partial \gamma_{ihk}}{\partial s_j}+\sum_f \gamma_{ihf}(\gamma_{fjk}-\gamma_{fkj})+\sum_f(\gamma_{fik}\gamma_{fhj}-\gamma_{fij}\gamma_{fhk});$$

le quali per valori degli indici i, f, h, k compresi tra 1 ed $n-m$ assumeranno la forma

$$\gamma_{ih, jk}=\frac{\partial \gamma_{ihj}}{\partial s_k}-\frac{\partial \gamma_{ihk}}{\partial s_j}+\sum_{1}^{n-m}{}_f\ \gamma_{ihf}(\gamma_{fjk}-\gamma_{fkj})+\sum_{1}^{n-m}{}_f\ (\gamma_{fik}\gamma_{fhj}-\gamma_{fij}\gamma_{fhk}) \tag{10'}$$

(sempre che le congruenze [1], [2], ..., $[n-m]$ generino ∞^m sottovarietà V_{n-m} di V_n) nei punti P di V_n intorno ai quali esse determinino una giacitura geodetica.

In particolare, fissato un punto P si consideri una superficie geodetica σ di V_n uscente da P, si consideri σ come inserita in un insieme di ∞^{n-2} sottovarietà V_2 di V_n e si scelga l'ennupla Λ in modo che le congruenze [1], [2] generino questo insieme. Le (10′) assumeranno la forma

$$\gamma_{12, 12}=\frac{\partial \gamma_{121}}{\partial s_2}-\frac{\partial \gamma_{122}}{\partial s_1}-\gamma^2{}_{211}-\gamma^2{}_{122}$$

e basta ancora ricordare che i coefficienti di rotazione γ_{121}, γ_{122} non cambiano valore se invece che alla varietà V_n ci riferiamo alla sottovarietà V_2, per riconoscere nel 2° membro la espressione della curvatura gaussiana della superficie σ, e quindi nel risultato ottenuto il teorema fondamentale di Riemann concernente la curvatura delle varietà.

NORMALS AND CURVATURES OF A CURVE IN THE RIEMANNIAN MANIFOLD

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1. In this paper there is defined for the general curve in Riemannian N-space a system of $N-1$ mutually perpendicular normals and $N-1$ corresponding curvatures*. The mode of development yields a completely real system even in the case where the line-element is not a positive definite form. Formulae are developed which reduce to the Frenet-Serret formulae in the particular case of the Euclidean 3-space. The derivation of the first normal and first curvature are given by Bianchi (*Lezioni di Geometria Differenziale*: 3rd ed., vol. 2, p. 456).

The following conventions will be adopted:

(1) repeated small italic indices are to be summed from 1 to N; if not repeated, a range of values from 1 to N is understood;

(2) capital indices, even when repeated, are not to be summed; when not repeated, no range of values is understood unless specifically indicated.

2. The manifold is characterized by a fundamental symmetric covariant tensor g_{jk}. The expression $g_{jk}dx^j dx^k$ is invariant. The direction dx^i will be called *positive*, *nul* or *negative* according as $g_{jk}dx^j dx^k$ is positive, zero or negative. We assume that none of the directions encountered in the course of the development are nul-directions; if such were the case, the process of development could not be further continued.

We shall write

$$ds^2 = \epsilon_0 g_{jk} dx^j dx^k, \tag{2.1}$$

where $\epsilon_0 = \pm 1$ according as the direction dx^i is positive or negative, thus ensuring a real value for

$$ds = \sqrt{\epsilon_0 g_{jk} dx^j dx^k} \tag{2.2}$$

which we shall call the *line-element* of the manifold. The number ϵ_0 we shall call the *indicator* of the direction in question.

Any contravariant vector X^i defines a direction $dx^i = \theta X^i$, where θ is indeterminate. Writing $X = \epsilon g_{jk} X^j X^k$, where ϵ is the indicator of the direction defined,

*Cf. Blaschke, Mathematische Zeitschrift, vol. 6 (1920), pp. 94-99, where a somewhat different process is employed and the line-element is a positive definite form.

we shall call the quantities $\frac{X^i}{\sqrt{X}}$ the *directors* of this direction. These quantities reduce to direction cosines in Euclidean 3-space.

It is well known, and easily proved by direct transformation, that if X^i is a contravariant vector given at all points along a curve, then

$$Y^i = \dot{X}^i + \left\{ {jk \atop i} \right\} X^j \dot{x}^k \tag{2.3}$$

is a contravariant vector, the point superposed indicating differentiation with respect to the arc of the curve. We shall call Y^i the *contravariant derivative of* X^i *along the given curve* and write

$$Y^i = \overline{X}^i. \tag{2.4}$$

3. For any curve defined by equations $x^i = x^i(s)$, we have

$$\epsilon_0 g_{jk} \dot{x}^j \dot{x}^k = 1, \tag{3.1}$$

and therefore, if we put

$$\underset{0}{\lambda}^i = \dot{x}^i, \tag{3.2}$$

$\underset{0}{\lambda}^i$ are the directors of the tangent. Let us define $\underset{1}{C}^i$ by the equations

$$\underset{1}{C}^i = \underset{0}{\bar{\lambda}}^i = \ddot{x}^i + \left\{ {jk \atop i} \right\} \dot{x}^j \dot{x}^k \tag{3.3}$$

and let ϵ_1 denote the indicator of the direction defined by $\underset{1}{C}^i$, so that

$$\epsilon_1 g_{jk} \underset{1}{C}^j \underset{1}{C}^k > 0. \tag{3.4}$$

Then let

$$\underset{1}{C} = \epsilon_1 g_{jk} \underset{1}{C}^j \underset{1}{C}^k, \tag{3.5}$$

and

$$\underset{1}{\lambda}^i = \frac{\underset{1}{C}^i}{\sqrt{\underset{1}{C}}}, \tag{3.6}$$

so that $\underset{1}{\lambda}^i$ will be the directors of the direction defined by $\underset{1}{C}^i$. This direction can easily be shown to be normal to the curve; it is, in fact, well known as the *principal* or *first* normal (Bianchi: *loc. cit.*).

We therefore have definitively:

Components of first curvature	$\underset{1}{C}^i$,
First curvature	$\sqrt{\underset{1}{C}}$,
Indicator of first normal	ϵ_1,
Directors of the first normal	$\underset{1}{\lambda}^i$.

The following theorem is obvious:

If the first curvature ($\sqrt{\underset{1}{C}}$) *vanishes, then either the curve is a geodesic or the first normal has a nul-direction.*

4. We shall have occasion to define certain contravariant vectors

$$\underset{M}{C^i}, \quad (M=2, 3, \ldots, N-1).$$

It will be convenient to put

$$\underset{M}{C} = \epsilon_M g_{jk} \underset{M}{C^j} \underset{M}{C^k}, \qquad (M=2, 3, \ldots, N-1), \tag{4.1}$$

where ϵ_M is the indicator of the direction defined by $\underset{M}{C^i}$; the directors of these directions are

$$\underset{M}{\lambda^i} = \frac{\underset{M}{C^i}}{\sqrt{\underset{M}{C}}}, \qquad (M=2, 3, \ldots, N-1). \tag{4.2}$$

Equations (4.1) and (4.2) already exist for $M=1$.

Let us consider the equations of definition

$$\underset{2}{C^i} = \underset{1}{\bar{\lambda}^i} + \epsilon_1 \epsilon_0 \underset{0}{\lambda^i} \sqrt{\underset{1}{C}}. \tag{4.3}$$

The quantities on the right hand side have already been defined in §3. It is evident that $\underset{2}{C^i}$ is a contravariant vector. Further let

$$\underset{3}{C^i} = \underset{2}{\bar{\lambda}^i} + \epsilon_2 \epsilon_1 \underset{1}{\lambda^i} \sqrt{\underset{2}{C}}, \tag{4.4}$$

the quantities on the right hand side being determined by (4.1), (4.2), (4.3) and earlier equations. Proceeding in this manner by means of defining equations

$$\underset{M}{C^i} = \underset{M-1}{\bar{\lambda}^i} + \epsilon_{M-1} \epsilon_{M-2} \underset{M-2}{\lambda^i} \sqrt{\underset{M-1}{C}}, \quad (M=2, 3, \ldots, N-1), \tag{4.5}$$

we obtain all the contravariant vectors

$$\underset{M}{C^i}, \qquad (M=1, 2, 3, \ldots, N-1),$$

the corresponding invariants $\underset{M}{C}$ and the corresponding directors $\underset{M}{\lambda^i}$, defining in conjunction with $\underset{0}{\lambda^i}$ a set of N directions at any point on the given curve.

5. We shall now proceed to prove the following theorem by mathematical induction from the lemma below:

Theorem T (M). *The* M *directions defined by* $\underset{P}{\lambda^i}(P=0, 1, 2, \ldots, M-1)$ *are mutually perpendicular.*

Lemma. If $T(M)$ *be true, then* $T(M+1)$ *is also true* $(M>1)$.

Since $T(M)$ is true by hypothesis,

$$g_{jk} \underset{P}{\lambda^j} \underset{Q}{\lambda^k} = 0, \qquad (P, Q=0, 1, \ldots, M-1;\ P \neq Q); \tag{5.1}$$

also by definition of indicators and directors,

$$(5.2)\qquad g_{jk}\underset{P}{\lambda}^{j}\underset{P}{\lambda}^{k}=\epsilon_{P},\qquad (P=0,\ 1,\ \dots,\ M-1).$$

Now for $P=0,\ 1,\ \dots,\ M-1$,

$$(5.3)\qquad \sqrt{\underset{M}{C}}\cdot g_{jk}\underset{P}{\lambda}^{j}\underset{M}{\lambda}^{k}=g_{jk}\underset{P}{\lambda}^{j}\underset{M}{C}^{k}\qquad \text{by (4.2)},$$

$$=g_{jk}\underset{P}{\lambda}^{j}\left(\underset{M-1}{\bar{\lambda}}^{k}+\epsilon_{M-1}\epsilon_{M-2}\underset{M-2}{\lambda}^{k}\sqrt{\underset{M-1}{C}}\right)\ \text{by (4.5)},$$

$$=g_{jk}\underset{P}{\lambda}^{j}\underset{M-1}{\bar{\lambda}}^{k}+\phi,$$

$$\text{where } \phi=\begin{cases}0 \text{ when } P\neq M-2, \text{ by (5.1).}\\ \epsilon_{M-1}\sqrt{\underset{M-1}{C}} \text{ when } P=M-2, \text{ by (5.2).}\end{cases}$$

But by the definition of the contravariant derivative (2.3),

$$g_{jk}\underset{P}{\lambda}^{j}\underset{M-1}{\bar{\lambda}}^{k}=g_{jk}\underset{P}{\lambda}^{j}\left(\underset{M-1}{\dot{\lambda}}^{k}+\left\{{st \atop k}\right\}\underset{M-1}{\lambda}^{s}\dot{x}^{t}\right),$$

$$=\frac{d}{ds}\left(g_{jk}\underset{P}{\lambda}^{j}\underset{M-1}{\lambda}^{k}\right)-\underset{M-1}{\lambda}^{k}\frac{d}{ds}\left(g_{jk}\underset{P}{\lambda}^{j}\right)+\left[{st \atop j}\right]\underset{P}{\lambda}^{j}\underset{M-1}{\lambda}^{s}\dot{x}^{t}.$$

The first term on the right hand side vanishes, by (5.1) and (5.2); therefore

$$g_{jk}\underset{P}{\lambda}^{j}\underset{M-1}{\bar{\lambda}}^{k}=-g_{jk}\underset{M-1}{\lambda}^{k}\underset{P}{\dot{\lambda}}^{j}-\underset{P}{\lambda}^{j}\underset{M-1}{\lambda}^{k}\left(\frac{\partial g_{jk}}{\partial x^{t}}-\left[{kt \atop j}\right]\right)\dot{x}^{t},$$

$$=-g_{jk}\underset{M-1}{\lambda}^{k}\left(\underset{P}{\bar{\lambda}}^{j}-\left\{{st \atop j}\right\}\underset{P}{\lambda}^{s}\dot{x}^{t}\right)-\underset{P}{\lambda}^{j}\underset{M-1}{\lambda}^{k}\left[{jt \atop k}\right]\dot{x}^{t},$$

$$=-g_{jk}\underset{M-1}{\lambda}^{k}\left(\underset{P+1}{C}^{j}-\epsilon_{P}\epsilon_{P-1}\underset{P-1}{\lambda}^{j}\sqrt{\underset{P}{C}}\right),\ \text{by (4.5)},$$

$$=-g_{jk}\underset{M-1}{\lambda}^{k}\underset{P+1}{\lambda}^{j}\sqrt{\underset{P+1}{C}},$$

$$=\begin{cases}0 \text{ when } P\neq M-2, \text{ by (5.1)*.}\\ -\epsilon_{M-1}\sqrt{\underset{M-1}{C}} \text{ when } P=M-2, \text{ by (5.2).}\end{cases}$$

Thus

$$g_{jk}\underset{P}{\lambda}^{j}\underset{M-1}{\bar{\lambda}}^{k}=-\phi,$$

and, substituting in (5.3), we obtain

$$(5.4)\qquad \sqrt{\underset{M}{C}}\cdot g_{jk}\underset{P}{\lambda}^{j}\underset{M}{\lambda}^{k}=0,\qquad (P=0,\ 1,\ \dots,\ M-1).$$

*This argument fails for $P=M-1$, but it is easily seen that the statement is nevertheless true in that case. In fact, it is only necessary to differentiate (5.2) with respect to the arc in order to prove that any contravariant unit vector is perpendicular to its contravariant derivative along any curve.

Therefore the Lemma is true. But Theorem $T(2)$ is true (cf. §3). Therefore, by induction, Theorem $T(N)$ is true and we have the result:

The N directions defined by $\underset{M}{\lambda}^i$ $(M=0, 1, \ldots, N-1)$ *are mutually perpendicular.*

We therefore have definitively:

Components of Mth curvature	$\underset{M}{C}^i$,
Mth curvature	$\sqrt{\underset{M}{C}}$,
Indicator of Mth normal	ϵ_M,
Directors of the Mth normal	$\underset{M}{\lambda}^i$.

6. Let us now change slightly the form of the equations (3.3), (4.3), (4.4) and (4.5), writing them

$$
(6.1)\quad \left\{
\begin{aligned}
\underset{0}{\bar{\lambda}}^i &= \underset{1}{\lambda}^i \sqrt{\underset{1}{C}}, \\
\underset{1}{\bar{\lambda}}^i &= \underset{2}{\lambda}^i \sqrt{\underset{2}{C}} - \epsilon_1\epsilon_0 \underset{0}{\lambda}^i \sqrt{\underset{1}{C}}, \\
\underset{2}{\bar{\lambda}}^i &= \underset{3}{\lambda}^i \sqrt{\underset{3}{C}} - \epsilon_2\epsilon_1 \underset{1}{\lambda}^i \sqrt{\underset{2}{C}}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\underset{M-1}{\bar{\lambda}}^i &= \underset{M}{\lambda}^i \sqrt{\underset{M}{C}} - \epsilon_{M-1}\epsilon_{M-2} \underset{M-2}{\lambda}^i \sqrt{\underset{M-1}{C}}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\underset{N-2}{\bar{\lambda}}^i &= \underset{N-1}{\lambda}^i \sqrt{\underset{N-1}{C}} - \epsilon_{N-2}\epsilon_{N-3} \underset{N-3}{\lambda}^i \sqrt{\underset{N-2}{C}}.
\end{aligned}
\right.
$$

Now since there can exist at a point at most N mutually perpendicular directions, it follows that when we substitute N for M in (4.5) the right hand side must vanish for $i=1, 2, \ldots, N$; otherwise an Nth normal direction would exist, forming with the tangent and normals of lower order a set of $N+1$ mutually perpendicular directions. Thus we obtain

$$
(6.2)\quad \underset{N-1}{\bar{\lambda}}^i = -\epsilon_{N-1}\epsilon_{N-2} \underset{N-2}{\lambda}^i \sqrt{\underset{N-1}{C}}.
$$

We have in (6.1) and (6.2) a generalized form of the Frenet-Serret Formulae. For, in the case of Euclidean 3-space, they reduce to

$$
(6.3)\quad \left\{
\begin{aligned}
\underset{0}{\bar{\lambda}}^i &= \underset{1}{\lambda}^i \sqrt{\underset{1}{C}}, \\
\underset{1}{\bar{\lambda}}^i &= \underset{2}{\lambda}^i \sqrt{\underset{2}{C}} - \underset{0}{\lambda}^i \sqrt{\underset{1}{C}}, \\
\underset{2}{\bar{\lambda}}^i &= \qquad\quad - \underset{1}{\lambda}^i \sqrt{\underset{2}{C}},
\end{aligned}
\right.
$$

which are immediately identified with the classical formulae.

If we write A instead of i as a general superscript in equations (6.1) and (6.2), multiply them by $\epsilon_0 \lambda_0^A, \epsilon_1 \lambda_1^A, \epsilon_2 \lambda_2^A, \ldots, \epsilon_{M-1} \lambda_{M-1}^A, \ldots, \epsilon_{N-2} \lambda_{N-2}^A, \epsilon_{N-1} \lambda_{N-1}^A$ in order and add them together, we obtain the interesting relations

$$\epsilon_0 \lambda_0^A \bar{\lambda}_0^A + \epsilon_1 \lambda_1^A \bar{\lambda}_1^A + \ldots + \epsilon_{N-1} \lambda_{N-1}^A \bar{\lambda}_{N-1}^A = 0, \; (A = 1, 2, \ldots, N). \tag{6.4}$$

7. In Euclidean 3-space the tangent and the binormal have no definite senses intrinsically definable, whereas the principal normal has a definite sense, viz., that in the direction of the concavity of the projection of the curve on its osculating plane. In the system of normals defined above it is not difficult to see that every normal of *odd order* has an intrinsic definite sense, whereas those of *even order* have not. For if we change the direction on the curve in which s is measured, we change the signs of $\lambda_0^i, \lambda_2^i, \lambda_4^i, \ldots$, but the signs of $\lambda_1^i, \lambda_3^i, \ldots$ remain unaltered. We may therefore state that the intrinsic definite sense of $\lambda_{1\,i}$ (for example) is that defined by the equations $dx^i = \theta \lambda_1^i$ where θ is a *positive* indeterminate quantity.

QUADRATIC FLAT-COMPLEXES IN ODD n-SPACE AND THEIR SINGULAR SPREADS, FLAT-SPHERE TRANSFORMATION

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The generalized Kummer's surface in odd n-space

1. In the geometry of the special self-dual flat in odd space which has been treated by the author in several papers*, we adopt as space-elements the ∞^n flats in S_{n-1} (n even):

$$(1)\qquad \begin{cases} \sigma_0 x_j - \sigma_{n/2} y_j + \rho_j = 0, & (j=1,2,\ldots,(n-2)/2) \\ \sigma_0 z + \sum \sigma_k y_k + \rho_{n/2} = 0, & \\ & (k=1,2,\ldots,(n-2)/2), \\ \sigma_{n/2} z + \sum \sigma_k x_k - \rho_0 = 0, & \end{cases}$$

where $\sum \rho_j \sigma_j = 0$, $(j=0,1,\ldots,n/2)$. Consider the quadratic complex C_2:

$$\sum a_{jk}\rho_j\rho_k + \sum b_{jk}\sigma_j\sigma_k + \sum c_{jk}\rho_j\sigma_k = 0, \quad (j,k=0,1,\ldots,n/2),$$

We shall suppose that this bilinear form is non-singular. In the most general case, that of distinct elementary divisors, this complex may be reduced to the canonical form

$$(2)\qquad \phi = \sum a_j(\rho_j^2 + \sigma_j^2) + 2\sum b_j\rho_j\sigma_j = 0, \ \omega = \sum \rho_j\sigma_j = 0, \quad (j=0,1,\ldots,n/2).$$

or, using Klein's coordinates,

$$(3)\qquad \begin{cases} \sigma_j = y_{2j-1} - iy_{2j}, \quad \sigma_0 = y_{n+1} - iy_{n+2}, \\ \rho_j = y_{2j-1} + iy_{2j}, \quad \rho_0 = y_{n+1} + iy_{n+2}, \\ (j=1,2,\ldots,n/2), \end{cases}$$

$$(4)\qquad \phi = \Sigma k_j y_j^2 = 0, \qquad \omega = \Sigma y_j^2 = 0, \quad (j=1,2,\ldots,n+2),$$

*J. Eiesland. *On a Flat-Sphere Geometry in Odd Dimensional Space.* Amer. Jour. Math., Vol. 35, pp. 202-228.

Flat-Sphere Geometry, Amer. Jour. Math., Vol. 40, pp. 1-44. Tohôku Math. Jour., Vols. 16 and 17 (third and fourth papers).

$$(5)\qquad \begin{cases} 2k_{2j-1}=b_j+a_j, & 2k_{n+1}=b_0+a_0, \\ 2k_{2j}=b_j-a_j, & 2k_{n+2}=b_0-a_0, \end{cases}$$
$$(j=1,2,\ldots,n/2).$$

The linear flat-complex

$$(6)\qquad \sum\left\{\rho_j\left[\mu\frac{\partial\phi}{\partial\sigma_j}+\nu\sigma_j\right]+\sigma_j\left[\mu\frac{\partial\phi}{\partial\rho_j}+\mu\rho_j\right]\right\}=0$$
$$(j=0,1,\ldots,n/2),$$

is the polar complex of a flat (ρ, σ) for the quadratic complex C_2, and if the flat belongs to the complex, this linear complex contains (ρ, σ) and also the flat consecutive to it, that is, the flat $(\rho+d\rho, \sigma+d\sigma)$; for,

$$\sum\left(\rho_j\frac{\partial\phi}{\partial\rho_j}+\sigma_j\frac{\partial\phi}{\partial\sigma_j}\right)=0,\quad \sum(\rho_jd\sigma_j+\sigma_jd\rho_j)=0,\quad \sum\left(\frac{\partial\phi}{\partial\rho_j}d\rho_j+\frac{\partial\phi}{\partial\sigma_j}d\sigma_j\right)=0,$$
$$(j=0,1,\ldots,n/2).$$

In this case the linear complex (6) is said to be *the tangent complex* to ϕ.

2. The points of space for which the tangent complex is special are *the singular points* of the complex C_2, and the directrices of the tangent special complexes are the singular flats of C_2. We shall prove that these points form an $(n-2)$-spread in S_{n-1}, *the singular spread of the complex,* and that a singular flat touches the surface at every point of it.

The condition that the tangent complex to C_2 shall be special is

$$\sum\frac{\partial\phi}{\partial\sigma_j}\,\frac{\partial\phi}{\partial\rho_j}=\sum(a_j\rho_j+b_j\sigma_j)(a_j\sigma_j+b_j\rho_j)=0,$$
$$(j=0,1,\ldots,n/2),$$

hence all the flats of the pencils

$$(7)\qquad \bar{\rho}_j=\rho_j+\alpha(a_j\sigma_j+b_j\rho_j),\quad \bar{\sigma}_j=\sigma_j+\alpha(a_j\rho_j+b_j\sigma_j),$$

meet the singular flat (ρ, σ). The locus of the vertices of the pencils (7) will be a spread Γ, and the envelope of the $(n-2)$-flats containing the same pencils will be a spread Γ'. As will be proved later, these two spreads coincide, and hence $\Gamma=\Gamma'$ is the singular spread of the complex.

The singular flats of the complex belong to a complex of rank 2*:

$$(8)\qquad \phi=0,\quad \psi=\sum a_jb_j(\rho_j^2+\sigma_j^2)+\sum(a_j^2+b_j^2)\rho_j\sigma_j=0,\quad (j=0,1,\ldots,n/2),$$

*A flat complex of rank r consists of all the flats common to r complexes. If $r=n/2$, we have the analogue of a congruence in 3-space. In the case of such a "congruence", a finite number of flats will pass through every point of S_{n-1}. This number is the *order* of the "congruence". Likewise, in any $(n-2)$-flat will be imbedded a finite number of flats; this number may be called the *class* of the congruence. If $r>n/2$ we have a regulus $R^{(n-r)}$ which in general will generate a spread of $(3n-2)/2-r$ dimensions.

which we shall call *the singular complex.* In the pencil at a point P on Γ one flat (ρ, σ) is singular. The flats $(\bar{\rho}, \bar{\sigma})$ for which α takes any one of ∞^1 values, except zero, we shall call *the adjoint flats.*

3. The equation of the singular spread Γ is obtained by eliminating ρ_j and σ_j from the two flats

$$(9)\quad \begin{cases} \sigma_0 x_j - \sigma_{n/2}\, y_j + \rho_j = 0, \quad (a_0\rho_0 + b_0\sigma_0)x_j - (a_{n/2}\,\rho_{n/2} + b_{n/2}\sigma_{n/2})y_j + a_j\sigma_j + b_j\rho_j = 0, \\ \sigma_0 z + \sum \sigma_k y_k + \rho_{n/2} = 0, \quad (a_0\rho_0 + b_0\sigma_0)z + \sum (a_k\rho_k + b_k\sigma_k)y_k + a_{n/2}\sigma_{n/2} + b_{n/2}\rho_{n/2} = 0, \\ \sigma_{n/2} z + \sum \sigma_j x_j - \rho_0 = 0, \quad (a_{n/2}\rho_{n/2} + b_{n/2}\sigma_{n/2})z + \sum (a_k\rho_k + b_k\sigma_k)x_k - (a_0\sigma_0 + b_0\rho_0) = 0, \\ \qquad (j, k = 1, 2, \ldots, (n-2)/2). \end{cases}$$

We obtain the following four equations:

$$(10)\quad \begin{cases} \alpha_1\rho_{n/2} + \beta_1\rho_0 + \gamma_1\sigma_{n/2} + \delta_1\sigma_0 = 0, \\ \beta_1\rho_{n/2} + \beta_2\rho_0 + \gamma_2\sigma_{n/2} + \delta_2\sigma_0 = 0, \\ \gamma_1\rho_{n/2} + \gamma_2\rho_0 + \gamma_3\sigma_{n/2} + \delta_3\sigma_0 = 0, \\ \delta_1\rho_{n/2} + \delta_2\rho_0 + \delta_3\sigma_{n/2} + \delta_4\sigma_0 = 0, \end{cases}$$

where

$$(11)\quad \begin{cases} \alpha_1 = \dfrac{1}{a_0}\left[1 + a_{n/2}\sum \dfrac{y_j^2}{a_j}\right], \quad \beta_1 = -\sum \dfrac{x_j y_j}{a_j}, \quad \gamma_1 = \dfrac{1}{a_0}\left[\sum \dfrac{b_{n/2} - b_j}{a_j}\, y_j^2\right], \\ \delta_1 = \dfrac{1}{a_0}\left[z + \sum \dfrac{b_j - b_0}{a_j}\, x_j y_j\right], \quad \beta_2 = \dfrac{1}{a_{n/2}}\left[1 + a_0 \sum \dfrac{x_j^2}{a_j}\right], \quad \gamma_2 = \dfrac{1}{a_{n/2}}\left[\sum \dfrac{b_j - b_{n/2}}{a_j} x_j y_j - z\right], \\ \delta_2 = \dfrac{1}{a_{n/2}} \sum \dfrac{b_0 - b_j}{a_j}\, x_j^2, \quad \gamma_3 = \dfrac{1}{a_0 a_{n/2}}\left[\sum \dfrac{(b_j - b_0)^2 - a_j^2}{a_j}\, y_j^2 - a_{n/2}\right], \\ \delta_3 = \dfrac{1}{a_0 a_{n/2}}\left[b_{n/2} - b_0)z + \sum \dfrac{a_j^2 - (b_j - b_{n/2})\,(b_j - b_0)}{a_j}\, x_j y_j\right], \\ \delta_4 = \dfrac{1}{a_0 a_{n/2}}\left[\sum \dfrac{(b_{n/2} - b_0)^2 - a_j^2}{a_j}\, x_j^2 - a_0\right], \\ \qquad (j = 1, 2, \ldots, (n-2)/2). \end{cases}$$

Through a fixed point $P(x_i, y_i, z)$ on Γ such that the determinant of the system (10) is of rank 3 there will pass one and only one flat of the complex whose coordinates are found by solving (10) for $\dfrac{\rho_{n/2}}{\sigma_0}, \dfrac{\rho_0}{\sigma_0}, \dfrac{\sigma_{n/2}}{\sigma_0}$ and the system (9) for the remaining coordinates $\dfrac{\sigma_i}{\sigma_0}, \dfrac{\rho_i}{\sigma_0}$. We shall call such a point a regular point on Γ. If the point P is such that the determinant of (10) is of rank r inferior to 3, an infinite number of singular flats will pass through P and be tangent to Γ. Thus, if the rank is 2 we have a single infinity of such flats; if $r = 1$, or 0, an

∞^2 or an ∞^3 of singular flats respectively will pass through P. All such points we shall call *singular points of* Γ. The discussion of the nature of these points will be reserved for a later opportunity.

The equation of Γ is obtained by equating to zero the determinant $|\alpha_1\beta_2\gamma_3\delta_4|$ of (10). This determinant is symmetrical, and by a proper transformation which is not hard to find, it may be reduced to the form:

$$(12)\quad \Gamma=\begin{vmatrix} \dfrac{1+a_{n/2}D}{a_0} & A & \dfrac{b_0+a_{n/2}F}{a_0} & B+z \\ A & \dfrac{1+a_0C}{a_{n/2}} & B-z & \dfrac{b_0+a_0E}{a_{n/2}} \\ \dfrac{b_{n/2}+a_{n/2}F}{a_0} & B-z & \dfrac{b_{n/2}^2-a_{n/2}^2+a_{n/2}(M-G)}{a_0} & K-J \\ B+z & \dfrac{b_0+a_0E}{a_{n/2}} & K-J & \dfrac{b_0^2-a_0^2+a_0(L-H)}{a_{n/2}} \end{vmatrix}=0,$$

where

$$(13)\quad \begin{gathered} A=\sum\frac{x_jy_j}{a_j},\quad B=\sum\frac{b_jx_jy_j}{a_j},\quad C=\sum\frac{x_j^2}{a_j},\quad D=\sum\frac{y_j^2}{a_j},\quad E=\sum\frac{b_jx_j^2}{a_j},\quad F=\sum\frac{b_jy_j^2}{a_j}, \\ M-G=\sum\frac{b_j^2-a_j^2}{a_j}y_j^2,\quad K-J=\sum\frac{b_j^2-a_j^2}{a_j}x_jy_j,\quad L-H=\sum\frac{b_j^2-a_j^2}{a_j}x_j^2. \\ (j=1,2,\ldots,(n-2)/2). \end{gathered}$$

Γ is therefore of the eighth order. *It is also of the eighth class.* To prove this we shall consider the surface Γ', the envelope of the singular $(n-2)$-flats. Writing the equation of the tangent $(n-2)$-flat to Γ' in the form

$$-z+\sum u_jx_j+\sum v_jy_j+w=0,\quad (j=1,2,\ldots,(n-2)/2),$$

the equations of the flats (9) in tangential coordinates u_j, v_j, w are:

$$(14)\quad \begin{gathered} \sigma_0v_j+\sigma_{n/2}u_j+\sigma_j=0,\quad (a_0\rho_0+b_0\sigma_0)v_j+a_{n/2}\rho_{n/2}+b_{n/2}\sigma_{n/2})u_j+(a_j\sigma_j+b_j\rho_j)=0, \\ \sigma_0w-\sum\rho_ku_k+\rho_{n/2}=0,(a_0\rho_0+b_0\sigma_0)w-\sum(a_k\sigma_k+b_k\rho_k).\,u_j+(a_{n/2}\sigma_{n/2}+b_{n/2}\rho_{n/2})=0, \\ \sigma_{n/2}w+\sum\rho_kv_k-\rho_0=0,\ (a_{n/2}\rho_{n/2}+b_{n/2}\sigma_{n/2})w+\sum(a_k\sigma_k+b_k\rho_k).\,v_j-(a_0\sigma_0+b_0\rho_0)=0. \\ (j,k=1,2,\ldots,(n-2)/2). \end{gathered}$$

Eliminating σ_j and ρ_j from these equations we have the equation of the spread Γ' in tangential coordinates. This spread is of the eighth class.

4. If now we can show that Γ' is identical with Γ, the above statement is proved. The equation of the tangent $(n-2)$-flat to Γ' is

$$(15)\quad \begin{gathered} \sum(\bar\sigma_j\sigma_0-\sigma_j\bar\sigma_0)x_j+\sum(\bar\sigma_{n/2}\sigma_j-\sigma_{n/2}\bar\sigma_j)y_j-(\sigma_{n/2}\bar\sigma_0-\bar\sigma_{n/2}\sigma_0)z+\sum\rho_k\bar\sigma_k=0, \\ (j=1,2,\ldots,(n-2)/2;\ k=0,1,\ldots,n/2), \end{gathered}$$

where $\bar{\rho}_k$, $\bar{\sigma}_k$ are the coordinates of the adjoint flats. This $(n-2)$-flat passes through the point on Γ whose coordinates are those of the vertex of the adjoint pencil, viz.:

$$(16) \qquad \bar{x}_j = \frac{\rho_j\bar{\sigma}_{n/2} - \sigma_{n/2}\bar{\rho}_j}{\bar{\sigma}_0\sigma_{n/2} - \sigma_0\bar{\sigma}_{n/2}},\ \bar{y}_j = \frac{\rho_j\bar{\sigma}_0 - \sigma_0\bar{\rho}_j}{\bar{\sigma}_0\sigma_{n/2} - \sigma_0\bar{\sigma}_{n/2}},\ \bar{z} = \frac{\Sigma\,\bar{\rho}_k\sigma_k + \bar{\sigma}_0\rho_0 + \bar{\sigma}_{n/2}\rho_{n/2}}{\bar{\sigma}_0\sigma_{n/2} - \sigma_0\bar{\sigma}_{n/2}},$$
$$(j, k = 1,2,\ldots, (n-2)/2).$$

Take a point on Γ consecutive to $P(\bar{x}_i, \bar{y}_i, \bar{z})$; then if we can prove that the distance of this point from the flat (15) is an infinitesimal of the second order, the distance $\overline{PP'}$ being taken as one of the first order, the flat (15) will be a tangent $(n-2)$-flat to Γ, and Γ will be identical with Γ'.

It is evident that the following relation must hold:

$$(17) \qquad \Sigma(\bar{\sigma}_j\sigma_0 - \sigma_j\bar{\sigma}_0)d\bar{x}_j + \Sigma(\bar{\sigma}_j\sigma_{n/2} - \sigma_{n/2}\bar{\sigma}_j)d\bar{y}_j + (\sigma_0\bar{\sigma}_{n/2} - \bar{\sigma}_0\sigma_{n/2})d\bar{z} = 0,$$
$$(j = 1,2,\ldots, (n-2)/2),$$

to within infinitesimals of order higher than the first. Differentiating the equations (16) we have

$$D^2d\bar{x}_j = (\bar{\sigma}_0\sigma_{n/2} - \sigma_0\bar{\sigma}_{n/2})(\bar{\sigma}_{n/2}d\rho_j - \sigma_{n/2}d\bar{\rho}_j) + (\sigma_0\bar{\rho}_j - \bar{\sigma}_0\rho_j)(\bar{\sigma}_{n/2}d\sigma_{n/2} - \sigma_{n/2}d\bar{\sigma}_{n/2})$$
$$+ (\bar{\sigma}_{n/2}\rho_j - \sigma_{n/2}\bar{\rho}_j)(\bar{\sigma}_{n/2}d\sigma_0 - \sigma_{n/2}d\bar{\sigma}_0),$$
$$(18) \qquad D^2d\bar{y}_j = (\bar{\sigma}_0\sigma_{n/2} - \sigma_0\bar{\sigma}_{n/2})(\bar{\sigma}_0 d\rho_j - \sigma_0 d\bar{\rho}_j) + (\bar{\sigma}_{n/2}\rho_j - \sigma_{n/2}\bar{\rho}_j)(\bar{\sigma}_0 d\sigma_0 - \sigma_0 d\bar{\sigma}_0)$$
$$+ (\sigma_0\bar{\rho}_j - \bar{\sigma}_0\rho_j)\ (\bar{\sigma}_0 d\sigma_{n/2} - \sigma_0 d\bar{\sigma}_{n/2}),$$
$$D^2d\bar{z} = (\bar{\sigma}_0\sigma_{n/2} - \sigma_0\bar{\sigma}_{n/2})\left\{\Sigma\bar{\rho}_k d\sigma_k + \Sigma\sigma_k d\bar{\rho}_k + \bar{\sigma}_0 d\rho_0 + \rho_0 d\bar{\sigma}_0 + \sigma_{n/2}d\bar{\rho}_{n/2} + \rho_{n/2}d\bar{\sigma}_{n/2}\right\}$$
$$- \left\{\Sigma\bar{\rho}_k\sigma_k + \bar{\sigma}_0\rho_0 + \bar{\sigma}_{n/2}\rho_{n/2}\right\}\left\{\bar{\sigma}_0 d\sigma_{n/2} + \sigma_{n/2}d\bar{\sigma}_0 - \sigma_0 d\bar{\sigma}_{n/2} - \bar{\sigma}_{n/2}d\sigma_0\right\}.$$
$$(j, k = 1,2,\ldots, (n-2)/2).$$

From equations (2) and (8) we find by differentiation,

$$(18') \qquad \Sigma(\rho_j d\sigma_j + \sigma_j d\rho_j) = 0,\ \Sigma(\bar{\sigma}_j d\rho_j + \bar{\rho}_j d\sigma_j) = 0,\ \Sigma(\sigma_j d\bar{\rho}_j + \rho_j d\bar{\sigma}_j) = 0,$$
$$(j = 0,1,\ldots, n/2).$$

Substituting the values of $d\bar{x}_j$, $d\bar{y}_j$ and $d\bar{z}$ from (18) in (17) we find that this equation is identically satisfied, account being taken of the identities (18').

5. The tangential equation of Γ may be obtained from (12) by substituting v_j, $-u_j$, w for x_j, y_j and z respectively. For, making this substitution in (9), we obtain a set of flats which is transformed into the set (14) by interchanging each

ρ_j and its corresponding σ_j, $(j=1, 2, 3, \ldots, (n-2)/2)$ and since the singular complex $\phi=0$, $\psi=0$ is invariant by this interchange, the statement follows.

6. We now write the duality transformation

$$z'=z-\sum u_k x_k-\sum v_k y_k,\ x_j'=v_j,\ y_j'=-u_j,\ u_j'=-y_j,\ v_j'=x_j,$$

$$(j, k=1,2,\ldots, (n-2)/2).$$

It is the product of $(n-2)/2$ polarizations with respect to the complexes $\rho_j=\sigma_j$*. The effect of this transformation is merely to interchange the ρ's and the σ's, and the pencil of flats (7) passing through $P(x_i, y_i, z)$ is therefore transformed into a pencil whose $(n-2)$-flat is also tangent to Γ at its vertex. The coordinates of this point are therefore, interchanging the ρ's and the σ's in (16),

$$(16'')\qquad \bar{\bar{x}}_j=\frac{\sigma_j\bar{\sigma}_{n/2}-\sigma_{n/2}\bar{\sigma}_j}{\bar{\sigma}_0\sigma_{n/2}-\sigma_0\bar{\sigma}_{n/2}},\ \bar{\bar{y}}_j=\frac{\bar{\sigma}_j\sigma_0-\sigma_j\bar{\sigma}_0}{\bar{\sigma}_0\sigma_{n/2}-\sigma_0\bar{\sigma}_{n/2}},\ \bar{\bar{z}}=\frac{\Sigma\rho_k\bar{\sigma}_k}{\bar{\sigma}_0\sigma_{n/2}-\sigma_0\bar{\sigma}_{n/2}}.$$

$$(j, k=0,1,\ldots, n/2).$$

Comparing these values with u_j, v_j, w in (15) we have

$$\bar{\bar{x}}_j=v_j,\ \bar{\bar{y}}_j=-u_j,\ \bar{\bar{z}}=w.$$

We may then say: *The product of the $\frac{1}{2}(n-2)$ polarizations with respect to the $\frac{1}{2}(n-2)$ fundamental complexes $\rho_j=\sigma_j$ transforms, in a dualistic sense, the generalized Kummer's surface Γ into itself.* We shall show later that *the surface is not self-dual except in 3-space* $(n=4)$, when Γ becomes the ordinary Kummer's surface.

7. Consider the $n+2$ fundamental complexes $\rho_j-\sigma_j=0$, $\rho_j+\sigma_j=0$ $(j=0, 1, 2, \ldots, n/2)$. Polarizations in these complexes are equivalent to the $n+2$ contact-transformations

$$(19)\qquad \begin{cases} S_j:\ z'=z-u_jx_j-v_jy_j,\ x_j'=v_j,\ y_j'=-u_j,\ x_k'=x_k,\ y_k'=y_k, \\ \qquad\qquad x_j=v_j',\ y_j=-u_j',\ v_k=v_k',\ u_k=u_k', \\ T_j:\ z'=z-u_jx_j-v_jy_j,\ x_j'=-v_j,\ y_j'=u_j,\ x_k=x_k',\ y_k=y_k, \\ \qquad\qquad x_j=-v_j',\ y_j=u_j',\ v_k=v_k',\ u_k=u_k', \\ (j=1, 2,\ldots, (n-2)/2;\ k=1, 2, \ldots, j-1, j+1, \ldots, (n-2)/2). \end{cases}$$

*J. Eiesland, *Flat-Sphere Geometry, Fourth Paper.* Tohôku Math. Jour., vol. 17, pp. 302-304.

$$
(20)\quad \begin{cases}
S_{n/2}: \; z' = \dfrac{\Sigma v_k y_k}{\Sigma u_k y_k}, \; x_j' = x_j + \dfrac{y_j[\Sigma v_k y_k - z]}{\Sigma u_k y_k}, \; y_j' = \dfrac{u_j}{\Sigma u_k y_k}, \\
\qquad u_j' = \dfrac{u_j}{\Sigma u_k y_k}, \; v_j' = v_j - \dfrac{u_j[\Sigma v_k y_k - z]}{\Sigma u_k y_k}, \\
T_{n/2}: \; z' = -\dfrac{\Sigma v_k y_k}{\Sigma u_k y_k}, \; x_j' = x_j + \dfrac{y_j[\Sigma v_k y_k - z]}{\Sigma u_k y_k}, \; y_j' = -\dfrac{u_j}{\Sigma u_k y_k}, \\
\qquad u_j' = -\dfrac{u_j}{\Sigma u_k y_k}, \; v_j' = v_j - \dfrac{u_j[\Sigma v_k y_k - z]}{\Sigma u_i y_i}, \\
S_0: \; z' = -\dfrac{\Sigma u_k x_k}{\Sigma v_k x_k}, \; x_j' = -\dfrac{x_j}{\Sigma v_k x_k}, \; y_j' = y_j + \dfrac{x_j[\Sigma u_k x_k - z]}{\Sigma v_k x_k}, \\
\qquad u_j' = u_j - \dfrac{v_j[\Sigma u_k x_k - z]}{\Sigma v_k x_k}, \; v_j' = -\dfrac{v_j}{\Sigma v_k x_k}, \\
T_0: \; z' = \dfrac{\Sigma u_k x_k}{\Sigma v_k x_k}, \; x_j' = \dfrac{x_j}{\Sigma v_k x_k}, \; y_j' = y_j + \dfrac{x_j[\Sigma u_k x_k - z]}{\Sigma v_k x_k}, \\
\qquad u_j' = u_j - \dfrac{v_j[\Sigma u_k x_k - z]}{\Sigma v_k x_k}, \; v_j' = \dfrac{v_j}{\Sigma v_k x_k}, \\
\qquad (j, k = 1, 2, \ldots, (n-2)/2),
\end{cases}
$$

where S_j and T_j denote the polarizations with respect to $\rho_j - \sigma_j = 0$ and $\rho_j + \sigma_j = 0$ respectively. Since the transformation S_j interchanges the two coordinates ρ_j and σ_j, and T_j the coordinates ρ_j and $-\sigma_j$, and since the complex $\phi = 0$, $\psi = 0$ is invariant under these transpositions, it follows that Γ is invariant under the $n+2$ polarizations S_j and T_j. Since all the $n+2$ fundamental complexes are mutually in involution (or apolar), $S_jT_j = T_jS_j$, $S_jS_k = S_kS_j$, $T_jT_k = T_kT_j$, $S_j^2 = 1$, $T_j^2 = 1$. The product $S_0S_1 \ldots S_{n/2}T_0T_1 \ldots T_{n/2} = 1$. Forming then all possible combinations we have 2^{n+1} transformations of a group G_{n+2} (necessarily a discrete group) which will carry a given surface-element on Γ into $2^{n+1}-1$ different surface-elements on the same spread. The 2^{n+1} elements form a closed system. The transformations S_jT_j and their combinations are point-transformations, namely the reflexions:

$$
\begin{aligned}
S_jT_j:\; & x_j' = -x_j,\; y_j' = -y_j,\; x_k' = x_k,\; y_k' = y_k,\; z' = z, \\
& (j = 1, 2, \ldots, (n-2)/2; \quad k = 1, 2, \ldots, j-1, j+1 \ldots, (n-2)/2), \\
S_0T_0:\; & x_j' = -x_j,\; y_j' = y_j,\; z' = -z;\; S_{n/2}T_{n/2}\colon x_j' = x_j,\; y_j' = -y_j,\; z' = -z.
\end{aligned}
$$

Hence, if (x_i, y_i, z) is a point on Γ, all the $2^{n/2}-1$ points, obtained by combining the $(n+2)/2$ reflexions

$$
\begin{aligned}
(21)\quad & (x_1, x_2, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_{\frac{n-2}{2}}, y_1, y_2, \ldots, y_{j-1}, -y_j, y_{j+1}, \ldots, y_{\frac{n-2}{2}}), \\
& (-x_j, y_j, -z),\; (x_j, -y_j, -z),
\end{aligned}
$$

lie on Γ also. Dually, if (u_j, v_j, w) is a tangent $(n-2)$-flat to Γ, the $2^{n/2}-1$ flats, obtained by combining the $(n+2)/2$ dual reflexions

$$(21') \quad \begin{cases} (u_1, u_2, \ldots u_{j-1}, -u_j, u_{j+1}, \ldots, u_{\frac{n-2}{2}}, v_1, v_2, \ldots v_{j-1}, -v_j, v_{j+1}, \ldots, v_{\frac{n-2}{2}}), \\ (-u_j, v_j, -w), (u_j, -v_j, -w), \end{cases}$$

are also tangent flats to Γ. If $n>4$ these are all the point-transformations in the group G_{n+2}. Γ is therefore in general not self-dual. If $n=4$, the transformations S_jT_k, T_jT_k, S_jS_k are all point-transformations, and the remaining 16 are point-to-plane transformations, that is 16 collineations and 16 correlations.

8. The reciprocal of Γ is also a spread $\bar{\Gamma}$ which belongs to the quadratic complex obtained by interchanging in $\phi=0$ the coefficients a_0 and $a_{n/2}$ and also b_0 and $b_{n/2}$. If $a_0=a_{n/2}$, $b_0=b_{n/2}$ the spread is transformed into itself by (21) and those obtained from them by interchanging all the x_j's and their corresponding y_j's and at the same time changing z into $-z$, so that it admits of $2^{(n+2)/2}$ reflexions. The spread is therefore invariant under the duality transformations*:

$$z'+z=\Sigma u_k x_k+\Sigma v_k y_k,\ x_j'=u_j,\ y_j'=v_j,\ u_j'=x_j,\ v_j'=y_j.$$

9. Before we consider the singularities of Γ in the finite space it will be convenient to get an idea of the relation of Γ to the space at infinity. For this purpose it will be necessary to expand the determinant (12). Using Laplace's method and arranging in ascending order we get the following equation for Γ:

$$\begin{aligned}
(22)\quad 0 = {} & 1+A_{0n/2}z^2+\Sigma A_{0j}\,x_j^2+\Sigma A_{n/2j}y_j^2+2z\Sigma B_{0n/2j}x_jy_j+z^4+\Sigma x_j^4+\Sigma y_j^4 \\
& +\Sigma A_{jk}(x_j^2x_k^2+y_j^2y_k^2)+\Sigma A_{0j}A_{n/2k}x_j^2y_k^2+A_{0n/2}\Sigma x_j^2y_j^2 \\
& +\Sigma(A_{jk}A_{0n/2}-A_{0j}A_{n/2k}-A_{0k}A_{n/2j})x_jy_jx_ky_k+z^2[\Sigma A_{0j}y_j^2+A\Sigma_{n/2j}x_j^2] \\
& +2z[\Sigma B_{n/2kj}x_jx_k(x_jy_k-x_ky_j)+\Sigma B_{0kj}y_jy_k(x_jy_k-x_ky_j)] \\
& +\Sigma A_{0k}A_{js}(x_jy_k-x_ky_j)\ (x_sy_k-x_ky_s)y_jy_s+\Sigma A_{0s}A_{jk}(x_ky_s-x_sy_k)\ (x_jy_s-x_sy_j)y_jy_k \\
& +\Sigma A_{0j}A_{sk}(x_jy_k-x_ky_j)\ (x_jy_s-x_sy_j)y_sy_k+\Sigma A_{n/2k}A_{js}(x_jy_k-x_ky_j)\ (x_sy_k-x_ky_s)x_jx_s \\
& +\Sigma A_{n/2s}A_{jk}(x_ky_s-x_sy_k)(x_jy_s-x_sy_j)x_jx_k+\Sigma A_{n/2j}A_{sk}(x_jy_s-x_sy_j)(x_jy_k-x_ky_j)x_sx_k \\
& +2z\Sigma B_{jks}(x_jy_k-x_ky_j)(x_jy_s-x_sy_j)(x_ky_s-x_sy_k) \\
& +\Sigma(x_jy_k-x_ky_j)^4+\Sigma A_{jk}(x_jy_s-x_sy_j)^2(x_sy_k-x_jy_s)^2 \\
& +\Sigma A_{js}(x_jy_k-x_ky_j)^2(x_ky_s-x_sy_k)^2+\Sigma A_{ks}(x_ky_j-x_jy_k)^2(x_jy_s-x_sy_j)^2 \\
& +\Sigma A_{jt}A_{ks}(x_jy_s-x_sy_j)(x_ky_t-x_ty_k)(x_jy_k-x_ky_j)(x_sy_t-x_ty_s) \\
& +\Sigma A_{js}A_{kt}(x_jy_k-x_ky_j)(x_sy_t-x_ty_s)(x_jy_t-x_ty_j)(x_ky_s-x_sy_k) \\
& +\Sigma A_{jk}A_{st}(x_jy_t-x_ty_j)(x_sy_k-x_ky_s)(x_jy_s-x_sy_j)(x_ty_k-x_ky_t)
\end{aligned}$$

where in the summations j, k, s, and t take the values 1, 2, ..., $(n-2)/2$ and

*S. Lie, *Drei Kapitel zur Geometrie der Berûhrungstransformationen*, Kap. I,§ 6, pp. 236-237, Mathematische Annalen, Bd. 59. This is also evident from the equation (22).

$j<k<s<t$. The coefficients have the following values:

$$A_{jk}=\frac{a_j^2-b_j^2+a_k^2-b_k^2+2b_jb_k}{a_ja_k},$$

$$B_{jks}=\frac{(b_j-b_k)a_s^2+(b_k-b_s)a_j^2+(b_s-b_j)a_k^2+(b_j-b_k)(b_k-b_s)(b_s-b_j)}{a_ja_ka_s}.$$

These coefficients are not independent, since the following relations hold:

$$(23)\quad (a)\qquad 4+B_{jks}^2-A_{jk}^2-A_{js}^2-A_{ks}^2+A_{jk}A_{js}A_{sk}=0.$$

Moreover, an extended calculation, which we shall not repeat here, will show that the determinant

$$(23)\ (b)\qquad \begin{vmatrix} 2 & A_{n/2\,0} & A_{n/2\,1} & A_{n/2\,2} & \cdots\cdots & A_{n/2\frac{n-2}{2}} \\ A_{n/2\,0} & 2 & A_{0\,1} & A_{0\,2} & \cdots\cdots & A_{0\frac{n-2}{2}} \\ A_{n/2\,1} & A_{0\,1} & 2 & A_{1\,2} & \cdots\cdots & A_{1\frac{n-2}{2}} \\ A_{n/2\,2} & A_{0\,2} & A_{1\,2} & 2 & \cdots\cdots & A_{2\frac{n-2}{2}} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ A_{n/2\frac{n-2}{2}} & A_{0\frac{n-2}{2}} & A_{1\frac{n-2}{2}} & A_{2\frac{n-2}{2}} & \cdots\cdots & 2 \end{vmatrix}$$

vanishes and is of rank 3. Since there are $n(n+2)/8$ A's and the conditions that the symmetrical determinant (b) shall be of rank 3 are $(n-2)(n-4)/8$ in number, there will be $n(n+2)/8-(n-2)(n-4)/8=n-1$ independent coefficients in the spread Γ. The number of coefficients in the most general spread is $n-1+(n+1)(n+2)/2=n(n+5)/2$, since the group of the quadric $\Sigma\rho_k\sigma_k=0$ contains precisely $(n+1)(n+2)/2$ parameters. For $n=6$ the determinant (b) gives us one relation between the six A's, namely:

$$(24)\qquad \begin{vmatrix} 2 & A_{30} & A_{31} & A_{32} \\ A_{30} & 2 & A_{01} & A_{02} \\ A_{31} & A_{01} & 2 & A_{12} \\ A_{32} & A_{02} & A_{12} & 2 \end{vmatrix} = -2B_{012}[2B_{012}-A_{30}B_{312}-A_{31}B_{032}-A_{32}B_{031}]\equiv 0.$$

The equation of Γ in 5-space is:

$$\begin{aligned}(25)\qquad 0=\;&1+A_{03}z^2+A_{01}x_1^2+A_{02}x_2^2+A_{31}y_1^2+A_{32}y_2^2+2z(B_{031}x_1y_1+B_{032}x_2y_2)\\ &+z^2(A_{01}y_1^2+A_{02}y_2^2+A_{31}x_1^2+A_{32}x_2^2)+z^4+x_1^4+x_2^4+y_1^4+y_2^4\\ &+A_{12}(x_1^2x_2^2+y_1^2y_2^2)+A_{03}(x_1^2y_1^2+x_2^2y_2^2)+A_{01}A_{32}x_1^2y_2^2+A_{02}A_{31}x_2^2y_1^2\\ &+(A_{13}A_{02}-A_{01}A_{32}-A_{02}A_{31})x_1y_1x_2y_2+(x_1y_2-x_2y_1)^4\\ &+2z(B_{321}x_1x_2+B_{021}y_1y_2)\ (x_1y_2-x_2y_1)\\ &+(A_{23}x_1^2+A_{13}x_2^2+A_{02}y_1^2+A_{01}y_2^2+A_{12}z^2)\ (x_1y_2-x_2y_1)^2.\end{aligned}$$

Returning to the general equation (22) we see at once that the locus

$$x_j y_k - x_k y_j = 0, \quad t = 0 \tag{26}$$

is a locus of quadruple points on Γ, since the line

$$x_j = \alpha_j z + \beta_j,$$
$$y_j = r\alpha_j z + \gamma_j$$

meets the surface at infinity in 4 coincident points. This locus, which is an $n/2$-dimensional quadric cone, we shall denote by Q_∞.

The singular loci on Γ

10. We have seen that through a regular point on Γ passes one singular flat which is tangent to it at the point. Suppose that a point exists such that ∞^1 singular flats pass through the point. The determinant of the system (10) must be of rank 2, and all the points for which this will happen form a locus Δ on Γ, the equations of which are obtained by equating to zero all the first minors of the determinant of (10). But these equations are not independent. In fact, we shall prove the following:

Theorem. The spread Γ *has a nodal locus* Δ_{n-4} *of order* 80 *and dimension* $n-4$. Let

$$\begin{aligned} A &= \beta_2\gamma_3 - \gamma_2^2, \quad B = \gamma_3\alpha_1 - \gamma_1^2, \quad C = \alpha_1\beta_2 - \beta_1^2, \\ A' &= \gamma_1\beta_1 - \alpha_1\gamma_2, \ B' = \beta_1\gamma_2 - \gamma_1\beta_2, \ C' = \gamma_2\gamma_1 - \beta_1\gamma_3 \end{aligned} \tag{27}$$

and let Δ_{44} be the minor of δ_4; we also write

$$\frac{\partial\Gamma}{\partial\delta_4} = \Delta_{44}, \quad \frac{\partial\Gamma}{\partial\delta_1} = -\Delta_{14}, \quad \frac{\partial\Gamma}{\partial\delta_2} = \Delta_{24}, \quad \frac{\partial\Gamma}{\partial\delta_3} = -\Delta_{34}, \tag{28}$$

Δ_{14}, Δ_{24}, Δ_{34} being the minors of δ_1, δ_2 and δ_3, and

$$\Delta_{11} = \begin{vmatrix} \beta_2 & \gamma_2 & \delta_2 \\ \gamma_2 & \gamma_3 & \delta_3 \\ \delta_2 & \delta_3 & \delta_4 \end{vmatrix}.$$

By (27) and (28) we have

$$\begin{aligned} &B'C' - AA' = \Delta_{44}\gamma_2, \ A'C' - BB' = \Delta_{44}\gamma_1, \ A'B' - CC' = \Delta_{44}\beta_1, \\ &BC - A'^2 = \Delta_{44}\alpha_1, \ AC - B'^2 = \Delta_{44}\beta_2, \ AB - C'^2 = \Delta_{44}\gamma_3, \end{aligned} \tag{29}$$

$$\begin{aligned} &\Delta_{44} = A\alpha_1 + B'\gamma_1 + C'\beta_1, \ \Delta_{14} = A\delta_1 + C'\delta_2 + B'\delta_3, \\ &-\Delta_{24} = C'\delta_1 + B\delta_2 + A'\gamma_3, \ \Delta_{34} = B'\delta_1 + A'\delta_2 + C\delta_3. \end{aligned} \tag{30}$$

Eliminating δ_1 from the second and the third of (30) we have, taking account of the first and sixth of (29),

$$\Delta_{24}A + \Delta_{14}C' = \Delta_{44}(\delta_3\gamma_2 - \delta_2\gamma_3).$$

Hence, if Q is any point (x_i, y_i, z) for which two of the minors Δ_{14}, Δ_{24}, Δ_{44} vanish, the third will vanish also, provided none of the minors A, C' and $(\delta_3\gamma_2-\delta_2\gamma_3)$ vanishes at Q. But the equations $A=0$, $C'=0$, $\delta_2\gamma_3-\delta_3\gamma_2=0$ are equivalent to

$$\frac{\beta_1}{\gamma_1}=\frac{\beta_2}{\gamma_2}=\frac{\gamma_2}{\gamma_3}=\frac{\delta_2}{\delta_3},$$

so that the exceptional points, denoted by P, are points of intersection of the six spreads of the matrix

$$\left\|\begin{matrix}\beta_1 & \beta_2 & \gamma_2 & \delta_2\\ \gamma_1 & \gamma_2 & \gamma_3 & \delta_3\end{matrix}\right\|=0, \tag{31}$$

forming an $(n-4)$-spread which we shall denote by ϕ_{n-4}. This spread is of order **32**. Proof: we write the six spreads in two groups:

$$(31') \qquad \begin{aligned}&(a)\quad \beta_1\gamma_2-\beta_2\gamma_1=0,\ \beta_2\gamma_3-\gamma^2{}_2,\ =0,\ \beta_1\gamma_3-\gamma_2\gamma_1=0,\\ &(b)\quad \gamma_2\delta_3-\delta_2\gamma_3=0,\ \delta_2\gamma_2-\beta_2\delta_3=0,\ \delta_2\gamma_1-\beta_1\delta_3=0.\end{aligned}$$

The three spreads (a) intersect in a spread of order 12 and dimension $n-3$; for, since the spread $\gamma_2=0$, $\gamma_3=0$ lies on the last two, but not on the first, we must eliminate this spread of order 4 to get the complete intersection of the two spreads defined by (a). The order is therefore $4\cdot 4-4=12$. The intersection of the spreads of the matrix (31) is therefore the intersection of this spread with any one of (b) say $\delta_2\gamma_1-\beta_1\delta_3=0$. But the two spreads of $n-4$ dimensions

$$\gamma_2=0,\ \gamma_3=0,\ \beta_2=0;\ \gamma_3=0,\ \gamma_2=0,\ \gamma_1=0,$$

evidently lie on all the spreads (a) but not on all of (b), hence they cannot count as part of complete locus of intersection. We have then, order of ϕ_{n-4} equals $12.4-8-8=32$.

This exceptional locus (points P) lies on Γ; it also lies on $\Delta_{11}=0$. We have thus proved that if Q be a point for which $\Delta_{11}=0$, $\Delta_{14}=0$, then Δ_{24} and Δ_{34} will vanish also, provided Q does not lie on ϕ_{n-4}.

If we write Γ in the form

$$\Gamma=\begin{vmatrix}\delta_4 & \delta_1 & \delta_2 & \delta_3\\ \delta_1 & \alpha_1 & \beta_1 & \gamma_1\\ \delta_2 & \beta_1 & \beta_2 & \gamma_2\\ \delta_3 & \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix}=0,$$

and let $\dfrac{\partial\Gamma}{\partial\gamma_3}=\Delta_{33}$, it follows that since $\dfrac{\partial\Gamma}{\partial\delta_3}=-\Delta_{34}$ has been shown to vanish, $\dfrac{\partial\Gamma}{\partial\gamma_1}=\Delta_{13}$ and $\dfrac{\partial\Gamma}{\partial\gamma_2}=-\Delta_{23}$ will vanish also. Proceeding in this way we arrive at the following conclusion: if any two of the differential coefficients of Γ with respect to the four elements α_1, β_2, γ_3, δ_4 and any one of those with respect to the six elements δ_1, δ_2, δ_3, β_1, γ_1, γ_2 also vanish, then all the differential coefficients will vanish, provided the points at which they vanish are not on the locus ϕ_{n-4}.

Thus, let the three equations be

$$\frac{\partial\Gamma}{\partial\alpha_1}=0,\quad \frac{\partial\Gamma}{\partial\delta_4}=0,\quad \frac{\partial\Gamma}{\partial\delta_1}=0;$$

they will represent an $(n-4)$-spread, and if Q be a point on it which does not have a general position on ϕ_{n-4} all the remaining 3rd order minors of Γ will vanish.

We shall now show that the points Q are nodes on Γ. Introducing homogeneous coordinates, x_j, y_j, z, t we put

$$\begin{aligned}&\alpha_1=a_{01}t^2+a_{11}t+a_{21}, &&\beta_k=b_{0k}t^2+b_{1k}t+b_{2k}, &&(k=1,\,2),\\ &\gamma_l=c_{0l}t^2+c_{1l}t+c_{2l}, &&\delta_j=d_{0j}t^2+d_{1j}t+d_{2j}, &&(l=1,\,2,\,3), \quad (j=1,2,3,4),\end{aligned}$$

where a_{11}, b_{1k}, c_{1l}, d_{1j} are linear forms in the variables x_j, y_j, z. We have then

$$\Gamma=\Gamma_0 t^8+\left(\frac{\partial\Gamma_0}{\partial a_{01}}a_{11}+\ldots+\frac{\partial\Gamma_0}{\partial d_{04}}d_{14}\right)t^7+\ldots=0.$$

Let the coordinate simplex be so chosen that the vertex $A(0,0,\ldots,0,1)$ coincides with the point Q. Then Γ_0 and all its differential coefficients vanish and the highest power of t is t^6, which shows that A is a node on Γ. The nodes are therefore on the intersection of the three spreads

$$(32)\qquad (\mathrm{I})\quad \Delta_{11}=0,\qquad (\mathrm{II})\quad \left\|\begin{matrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \beta_1 & \beta_2 & \gamma_2 & \delta_2\\ \gamma_1 & \gamma_2 & \gamma_3 & \delta_3\end{matrix}\right\|=0.$$

But the four determinantal spreads (II) intersect in a spread of order 24, and dimension $n-3$. In fact, the four spreads are $|\beta_1\ \gamma_2\ \delta_3|=0$, $|\alpha_1\ \gamma_2\ \delta_3|=0$, $|\alpha_1\ \beta_2\ \gamma_3|=0$, $|\alpha_1\ \beta_2\ \delta_3|=0$; but the spread defined by the determinants of the matrix

$$\left\|\begin{matrix}\alpha_1 & \beta_1 & \gamma_1\\ \beta_1 & \beta_2 & \gamma_2\end{matrix}\right\|=0,$$

which is of order 12, will evidently lie on the last two spreads but not on the first two. Hence the order is $6.6-12=24$, q.e.d.

11. The locus ϕ_{n-4} lies on the spread defined by (II) and also on $\Delta_{11}=0$. We shall show that the points P are not nodes in general on Γ and that the locus of points Q is a spread of order 80. We say "in general", for we shall show that a locus on ϕ_{n-4} of dimension $n-5$ exists for which points P are nodes.

Let A be a point P on ϕ_{n-4}; it is easily seen that we may put

$$\begin{aligned}&\beta_1=t^2+b_{11}t+b_{21},\ \beta_2=t^2+b_{12}t+b_{22},\quad \gamma_1=t^2+c_{11}t+c_{21},\\ &\gamma_3=t^2+c_{13}t+c_{23},\ \delta_2=t^2+d_{12}t+d_{22},\quad \delta_3=t^2+d_{13}t+d_{23},\\ &\gamma_2=t^2+c_{12}t+c_{22},\ \delta_1=d_{01}t^2+d_{11}t+d_{21},\ \delta_4=d_{04}t^2+d_{14}t+d_{24},\\ &\alpha_1=a_{01}t^2+a_{11}t+a_{21}.\end{aligned}$$

Substituting in Γ and Δ_{11} and developing, we find

$$\Gamma=\{(a_{01}-1)\ (d_{04}-1)-(d_{01}-1)^2\}\{c_{13}+b_{12}-2c_{12}\}t^7+\ldots,$$
$$\Delta_{11}=(d_{04}-1)\ (c_{13}+b_{12}-2c_{12})t^5+\ldots.$$

Hence, *the point A is not a node, provided*

$$D=\begin{vmatrix} a_{01}-1 & d_{01}-1 \\ d_{01}-1 & d_{04}-1 \end{vmatrix}\neq 0,$$

and Γ has contact at A with $\Delta_{11}=0$, if $d_{04}\neq 1$. In the same way we find,

$$\Delta_{14}=(d_{01}-1)\ (c_{13}+b_{12}-2c_{12})t^5+\ldots,$$
$$\Delta_{44}=(a_{01}-1)\ (c_{13}+b_{12}-2c_{12})t^5+\ldots,$$

that is Γ has contact with the spread (31), a_{01} and d_{01} differing from unity. It follows therefore that the nodal locus Δ_{n-4} is of order $6.24-2.32=80$, q.e.d.

If $D=0$, Γ has a double point on ϕ_{n-4} at A. This means that A also lies on the spread $\Delta_{33}=|\alpha_1\beta_2\delta_4|=0$, in which case all the remaining minors of the third order will vanish for the point A. $\Delta_{33}=0$ intersects ϕ_{n-4} in a locus of dimension $n-5$ and order $32.6-2^4=176$, since the spread $\beta_1=0$, $\beta_2=0$, $\delta_2=0$, $\gamma_2=0$ lies on ϕ_{n-4} and $\Delta_{33}=0$ but not on the spread represented by equating all the remaining minors of the third order to zero. We shall denote this new locus of double points by Δ_{n-5}. It is an $(n-5)$-dimensional branch of Δ_{n-4}.

12. Let D be $\neq 0$, and consider the net of spreads

$$\Phi=h_1\Delta_{44}+h_2\Delta_{14}+h_3\Delta_{11}=0$$
$$(33)\quad =\{h_1(a_{01}-1)+h_2(d_{01}-1)+h_3(d_{04}-1)\}(c_{13}+b_{12}-2c_{12})t^5$$
$$+\{[h_1(a_{01}-1)+h_2(d_{01}-1)+h_3(d_{04}-1)]A_1+h_1B_1+h_2C_1+h_3D_1\}t^4+\ldots=0$$

for points P on ϕ_{n-4}. We also have for these points

$$\Gamma=D(c_{13}+b_{12}-2c_{12})t^7+[DA_1+(d_{04}-1)B_1+2(d_{01}-1)C_1+(a_{04}-1)D_1]t^6+\ldots,$$

where

$$A_1=c_{23}+b_{22}-2c_{22}+b_{12}c_{13}-c_{12}^2,$$
$$B_1=(c_{12}-c_{13})\ (c_{11}-b_{11})-(b_{11}-a_{11})\ (c_{13}-c_{12})$$
$$+(b_{11}-a_{11})\ (c_{12}-b_{12})-(b_{12}-b_{11})(c_{11}-b_{11})$$
$$+(c_{11}-c_{12})\ (c_{12}-b_{12})-(b_{11}-b_{12})(c_{13}-c_{12}),$$
$$C_1=(c_{13}-c_{12})\ (b_{11}-d_{11})+(c_{12}-c_{11})\ (c_{12}-d_{13})$$
$$+(c_{11}-c_{13})\ (b_{12}-d_{13})+d_{11}(c_{12}-b_{12})+d_{12}(b_{11}-c_{11})+d_{11}(b_{12}-b_{11}),$$
$$D_1=(c_{13}-c_{12})\ (d_{14}-d_{12})-(d_{13}-c_{12})\ (d_{13}-d_{14})+(c_{12}-b_{12})(d_{13}-c_{12})$$
$$-(c_{13}-c_{12})\ (d_{12}-b_{12})+(d_{13}-d_{12})\ (d_{12}-b_{12})-(c_{12}-b_{12})(d_{14}-b_{12}).$$

The net Φ has for base locus the locus of double points on Γ (points Q) and on ϕ_{n-4} (points P) it has contact of the first order with Γ, that is, *ϕ_{n-4} is a tac-locus for the spreads* $\Phi=0$.

If $D=0$, that is, for points P on Δ_{n-5}, there will be two spreads of the net having A as node and the same tangent cone as Γ at this point; for, if we take

$$h_1: h_2: h_3 = d_{04}-1: \mp 2\sqrt{(a_{01}-1)(d_{04}-1)}: a_{01}-1,$$

we have

$$(d_{04}-1)\Delta_{44} \mp 2\sqrt{(a_{01}-1)(d_{04}-1)}\Delta_{14} + (a_{01}-1)\Delta_{11} = 0,$$
$$[(d_{04}-1)B_1 \pm 2\sqrt{(a_{01}-1)(d_{04}-1)}C_1 + (a_{01}-1)D_1]t^4 + \ldots = 0,$$
$$\Gamma = [(d_{04}-1)B_1 \pm 2\sqrt{(a_{01}-1)(d_{04}-1)}C_1 + (a_{01}-1)D_1]t^6 + \ldots = 0.$$

13. If $D=0$ and $d_{01}=1$, then either $a_{01}=1$ or $d_{04}=1$. In the first case $\Delta_{14}=0$ and $\Delta_{44}=0$ have a double point at A and Γ has the same tangent cone as $\Delta_{44}=0$. In the second case $\Delta_{11}=0$, and $\Delta_{14}=0$ have a double point at A and Γ has the same tangent cone as $\Delta_{11}=0$.

The locus of points A for which $d_{01}=a_{01}=1$ is of dimension $n-6$; it is on the ntersection of Δ_{n-5} with the spread $\alpha_1\beta_2-\beta_1^2=0$, that is, it is on the locus of intersection of the spreads of the two matrices:

$$(34) \qquad (\mathrm{M}_1) \begin{Vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \beta_1 & \beta_2 & \gamma_2 & \delta_2 \end{Vmatrix} = 0, \quad (\mathrm{M}_2) \begin{Vmatrix} \beta_1 & \beta_2 & \gamma_2 & \delta_2 \\ \gamma_1 & \gamma_2 & \gamma_3 & \delta_3 \end{Vmatrix} = 0,$$

which is of order 32.16 and dimension $n-6$. But the five spreads

$$(35) \qquad \begin{gathered} \beta_1=\beta_2=\gamma_2=\delta_2=\alpha_1=0, \quad \beta_1=\beta_2=\gamma_2=\delta_2=\gamma_1=0, \\ \beta_1=\beta_2=\gamma_2=\delta_2=\delta_1=0, \quad \beta_1=\beta_2=\gamma_2=\delta_2=\gamma_3=0, \\ \beta_1=\beta_2=\gamma_2=\delta_2=\delta_3=0, \end{gathered}$$

although belonging to the complete intersection of the spreads of the matrices (M_1) and (M_2) must not be counted as part of the locus of points A under consideration. Hence the locus is of order $16.32-4.2^5-2^4=368$, the third locus of (35) being of order 2^4 as appears from (11). We shall denote this locus by $\Delta_{n-6}^{(1)}$.

In the same way we prove that the locus of points A for which $d_{01}=d_{04}=1$ will lie on the intersection of Δ_{n-5} with $\gamma_3\delta_4-\delta_3^2=0$, that is, on the intersection of the spreads of the two matrices

$$(36) \quad (\mathrm{M}_3) \begin{Vmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \delta_3 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{Vmatrix} = 0, \quad (\mathrm{M}_2) \begin{Vmatrix} \beta_1 & \beta_2 & \gamma_2 & \delta_2 \\ \gamma_1 & \gamma_2 & \gamma_3 & \delta_3 \end{Vmatrix} = 0,$$

which is also of order 386. We shall denote it by $\Delta_{n-6}^{(2)}$.

14. We shall now prove the following:

Theorem. The locus of triple points T_{n-7} on Γ lies on the intersection of $\Delta_{n-6}^{(1)}=0$ and $\Delta_{n-6}^{(2)}=0$ *and is of order* 7.2^8 *and dimension* $n-7$.

Proof: If A is a point on the intersection of $\Delta^{(1)}_{n-6}$ and $\Delta^{(2)}_{n-6}$, we have $a_{01}=d_{01}=d_{04}=1$; hence all the minors of the second order vanish. The spread may therefore be represented by the six equations

$$\begin{aligned}(37)\qquad &\beta_1\gamma_2-\beta_2\gamma_1=0, \quad \beta_2\gamma_3-\gamma_2^2=0, \quad \gamma_2\delta_3-\delta_2\gamma_3=0,\\ &\alpha_1\beta_2-\beta_1^2=0, \quad \gamma_1\delta_2-\delta_1\gamma_2=0, \quad \gamma_3\delta_4-\delta_3^2=0,\end{aligned}$$

provided the point A does not make any one of the following four groups of elements vanish:

$$\begin{aligned}&\gamma_1,\ \gamma_2,\ \gamma_3,\ \alpha_1,\ \beta_1,\ \delta_3; \qquad \gamma_1,\ \gamma_2,\ \gamma_3,\ \delta_3,\ \beta_1,\ \beta_2;\\ &\beta_1,\ \beta_2,\ \gamma_2,\ \delta_2,\ \delta_3,\ \delta_4; \qquad \beta_1,\ \beta_2,\ \gamma_2,\ \delta_2,\ \delta_3,\ \gamma_3;\end{aligned}$$

since in any one of these cases the six determinants (37) will all vanish, but not all the remaining minors of the second order. The order of the locus T_{n-7} is then $32.64-4.2^6=7.2^8$, q.e.d.

The net Φ which has Δ_{n-4} as a base locus with multiplicity 1 has also T_{n-7} as base locus with multiplicity 2; for $\Delta_{44}=0$, $\Delta_{11}=0$, $\Delta_{14}=0$ have nodes on T_{n-7}, as appears also from (33), putting $a_{01}=d_{01}=d_{04}=1$. This triple locus exists if $n\geqq 8$.

There exist on Γ four extraneous loci of double points which we shall now consider. Suppose that $A(0, 0, \ldots, 1)$ lies on the intersection of the six quadrics

$$\alpha_1=\beta_1=\gamma_1=\gamma_2=\gamma_3=\beta_2=0.$$

It is easily seen that Γ may be written

$$\Gamma=\begin{vmatrix} a_{11}t+a_{21} & b_{11}t+b_{21} & c_{11}t+c_{21} & t^2+d_{11}t+d_{21}\\ b_{11}t+b_{21} & b_{12}t+b_{22} & c_{12}t+c_{22} & t^2+d_{12}t+d_{22}\\ c_{11}t+c_{21} & c_{12}t+c_{22} & c_{13}t+c_{23} & t^2+d_{13}t+d_{23}\\ t^2+d_{11}t+d_{21} & t+d_{12}t+d_{22} & t^2+d_{13}t+d_{23} & t+d_{14}t+d_{24}\end{vmatrix}=0,$$

which shows that A is a double point In the same way we prove that the three spreads

$$\beta_1=\gamma_2=\delta_2=\gamma_3=\delta_3=\delta_4=0, \quad \alpha_1=\gamma_1=\delta_3=\delta_4=\delta_1=\gamma_3=0,$$

$$\alpha_1=\beta_1=\delta_1=\beta_2=\delta_2=\delta_4=0,$$

are loci of double points on Γ. We shall denote them by $\Delta^{(j)}_{n-7}$ $(j=1, 2, 3, 4)$ they are of order 64. $\Delta^{(1)}_{n-7}$ lies on the intersection of the spreads of the matrices (34) and $\Delta^{(2)}_{n-7}$ on the intersection of the spreads of the matrices (36). They may be considered as $(n-7)$-dimensional branches of the $(n-6)$-dimensional spreads defined by the matrices (34) and (36). In the same way $\Delta^{(3)}_{n-7}$ and $\Delta^{(4)}_{n-7}$ may be considered as $(n-7)$-dimensional branches of the two $(n-6)$-dimensional spreads

$$\left\|\begin{matrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \gamma_1 & \gamma_2 & \gamma_3 & \delta_3\end{matrix}\right\|=\left\|\begin{matrix}\gamma_1 & \gamma_2 & \gamma_3 & \delta_3\\ \delta_1 & \delta_2 & \delta_3 & \delta_4\end{matrix}\right\|=0, \quad \left\|\begin{matrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \beta_1 & \beta_2 & \gamma_2 & \delta_2\end{matrix}\right\|=\left\|\begin{matrix}\beta_1 & \beta_2 & \gamma_2 & \delta_2\\ \delta_1 & \delta_2 & \delta_3 & \delta_4\end{matrix}\right\|=0.$$

We may also note here that a generic spread of the net Φ has A as an ordinary non-singular point.

There are 4 extraneous loci of triple points $T^{(j)}_{n-10}$ of dimension $n-10$ and order 2^8. In fact, if A lies on the intersection of any one of the four groups of 9 quadrics

$$\alpha_1=\beta_1=\gamma_1=\delta_1=\beta_2=\gamma_2=\gamma_3=\delta_2=\delta_3=0, \quad \alpha_1=\beta_1=\gamma_1=\delta_1=\beta_2=\gamma_2=\delta_2=\delta_3=\delta_4=0,$$
$$\alpha_1=\beta_1=\gamma_1=\delta_1=\gamma_2=\delta_2=\delta_3=\delta_4=\gamma_3=0, \quad \beta_1=\delta_1=\beta_2=\gamma_2=\delta_2=\gamma_3=\delta_3=\gamma_1=\delta_4=0,$$

this point will be a triple point on Γ. We shall prove the following:

Theorem. If $n\geqq 12$, Γ has a locus of quadruple points Q_{n-11} which is the intersection of any two of the extraneous loci of triple points $T^{(j)}_{n-10}$, or, the intersection of any three of the extraneous loci of double points $\Delta^{(j)}_{n-7}$. The locus is of order 2^9 and dimension $n-11$.

The locus is represented by the intersection of the 10 quadrics

$$\alpha_1=\beta_1=\gamma_1=\delta_1=\beta_2=\gamma_2=\delta_2=\gamma_3=\delta_3=\delta_4=0,$$

which is of order 2^9; it is immersed in the space $z=0$, as is easily seen from equations (11). To show that it is a locus of quadruple points we need only observe that for a point $A(0, 0, \ldots, 0, 1)$ on Q_{n-11}, Γ may be put in the form

$$\Gamma=\begin{vmatrix} a_{11}t+a_{21} & b_{11}t+b_{21} & c_{11}t+c_{21} & d_{11}t+d_{21} \\ b_{11}t+b_{21} & b_{12}t+b_{22} & c_{12}t+c_{22} & d_{12}t+d_{22} \\ c_{11}t+c_{21} & c_{12}t+c_{22} & c_{13}t+c_{23} & d_{13}t+d_{23} \\ d_{11}t+d_{21} & d_{12}t+d_{22} & d_{13}t+d_{23} & d_{14}t+d_{24} \end{vmatrix}=0.$$

A generic spread of the net Φ has Q_{n-11} for locus of triple points, since Δ_{44}, Δ_{14} and Δ_{11} pass through this locus with multiplicity 3. It is not difficult to prove that it has the quadruple locus Q_∞ at infinity as nodal locus.

15. We shall arrange the results obtained in the following table:

Γ: Singular spread of a quadratic flat-comples. Order and class 8.

Δ_{n-4}: Principal nodal locus. Order and class 80.

ϕ_{n-4}: Tac-locus. Order 32.

Δ_{n-5}: Branch of nodal locus on ϕ_{n-4} Order 176.

$\Delta^{(1)}_{n-6}$ and $\Delta^{(2)}_{n-6}$: Two nodal loci on Δ_{n-5} Order 368.

T_{n-7} Locus of triple points, intersection of $\Delta^{(1)}_{n-6}$ and $\Delta^{(2)}_{n-6}$ Order 7.2^8

$\Delta^{(j)}_{n-7}$ $(j=1, 2, 3, 4)$: Extraneous loci of double points. Order 2^6.

$T^{(j)}_{n-10}$ $(j=1, 2, 3, 4)$: Extraneous loci of triple points. Order 2^8.

Q_{n-11}: Locus of quadruple points, intersection of any three of $\Delta^{(j)}_{n-7}$ or any two of $T^{(j)}_{n-10}$. Order 2^9.

Q_∞: Locus of quadruple points at infinity. A quadric cone of dimension $n/2$.

The net Φ: Base loci: Δ_{n-4}, Δ_{n-5}, $\Delta^{(j)}_{n-6}$, $\Delta^{(j)}_{n-7}$,

T_{n-7}, $T^{(j)}_{n-10}$ with multiplicity 2,

Q_{n-11} with multiplicity 3,

Q_{∞} with multiplicity 2.

It has contact at all points of ϕ_{n-4}, not on base loci.

Flat-sphere Correspondence

16. By means of the general flat-sphere transformation* a generalized Kummer surface Γ in odd space S_{n-1} is transformed into another surface Φ in the space $\overline{S}_{n-1}$, the coordinates of which we denote by X_k ($k=1, 2, \ldots, n-1$). This new spread will be the singular spread of a quadratic sphere-complex in $\overline{S}_{n-1}$. To the asymptotic curves on Γ will correspond the lines of curvature on Φ. The Cartesian equation of this spread we now proceed to deduce. As far as known it has not been produced by this method, even in 3-space. A special case for n-space has been treated by V. Snyder† who finds the singular spread of a special sphere-complex in n-space, namely the case where the centres of the generating spheres lie on a quadric spread. For 3-space J. L. Coolidge and P. F. Smith‡ arrive at a surface of the 24th order and class, but do not give its Cartesian equation for very good reasons.

17. The equations of the flat-sphere transformation (Amer. Jour. Math. vol. 35, p. 214) are

$$(38)\quad \begin{cases} X_{2j-1}= \dfrac{(y_j-p_j)(\Sigma q_k y_k - z)}{2(1-\Sigma p_k y_k)} - \tfrac{1}{2}(x_j+q_j), \\[2ex] X_{2j}= -i\,\dfrac{(y_j+p_j)(\Sigma q_k y_k - z)}{2(1-\Sigma p_k y_k)} - \tfrac{1}{2}i(x_j-q_j), \\[2ex] X_{n-1}= \dfrac{\Sigma q_k y_k - z}{1-\Sigma p_k y_k}, \end{cases}$$

$$(j, k=1,2, \ldots\ldots, (n-2)/2).$$

*J. Eiesland, *On a Flat-Spread Sphere Geometry in Odd-dimensional Space*, Amer. Jour. Math., vol. 35, pp. 201-229.

†V. Snyder, *On Cyclical Surfaces in Space of n Dimensions*, Bull. Amer. Math. Soc., vol. 6, 1900.

‡P. F. Smith, *On Surfaces enveloped by Spheres belonging to a Spherical complex*, Trans. Amer. Math. Soc., vol. 1, pp. 372-390.

J. L. Coolidge, *A Treatise on the Circle and the Sphere*, p. 445.

On Γ in S_{n-1} we have the following parametric equations for the coordinates x_j, y_j, z, p_j, q_j, w (equations (15) and (16)):

$$(16'')\quad x_j=\frac{\rho_j\bar\sigma_{n/2}-\sigma_{n/2}\bar\rho_j}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},\quad y_j=\frac{\rho_j\bar\sigma_0-\sigma_0\bar\rho_j}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},\quad z=\frac{\Sigma\bar\rho_k\sigma_k+\bar\sigma_0\rho_0+\bar\sigma_{n/2}\rho_{n/2}}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},$$

$$(j,k=1,2,\ldots,(n-2)/2),$$

$$(16''')\quad p_j=\frac{\bar\sigma_j\sigma_0-\sigma_j\bar\sigma_0}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},\quad q_j=\frac{\bar\sigma_{n/2}\sigma_j-\sigma_{n/2}\bar\sigma_j}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},\quad w=\frac{\Sigma\rho_k\bar\sigma_k}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},$$

$$(j=1,2,\ldots,(n-2)/2;\ k=0,1,\ldots,n/2),$$

in which the parameters satisfy the relations

$$(39)\qquad \Sigma\rho_k\sigma_k=0,\ \Sigma\bar\rho_k\bar\sigma_k=0,\ \Sigma(\rho_k\bar\sigma_k+\bar\rho_k\sigma_k)=0,\ \bar\rho_j=a_j\sigma_j+b_j\rho_j,\ \bar\sigma_j=a_j\rho_j+b_j\sigma_j,$$

$$(k=0,1,\ldots,n/2;\ j=1,2,\ldots,(n-2)/2).$$

We find

$$x_j+q_j=\frac{\bar\sigma_{n/2}(\rho_j+\sigma_j)-\sigma_{n/2}(\bar\rho_j+\bar\sigma_j)}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},\quad y_j-p_j=\frac{\bar\sigma_0(\rho_j+\sigma_j)-\sigma_0(\bar\rho_j+\bar\sigma_j)}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},$$

$$x_j-q_j=\frac{\bar\sigma_{n/2}(\rho_j-\sigma_j)-\sigma_{n/2}(\bar\rho_j-\bar\sigma_j)}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},\quad y_j+p_j=\frac{\bar\sigma_0(\rho_j-\sigma_j)-\sigma_0(\bar\rho_j-\bar\sigma_j)}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},$$

$$1-\Sigma p_ky_k=\frac{\sigma_0(\bar\rho_{n/2}-\bar\sigma_{n/2})-\bar\sigma_0(\rho_{n/2}-\sigma_{n/2})}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}},\quad \Sigma q_ky_k-z=\frac{\sigma_{n/2}\bar\rho_{n/2}-\bar\sigma_{n/2}\rho_{n/2}}{\bar\sigma_0\sigma_{n/2}-\sigma_0\bar\sigma_{n/2}}.$$

Substituting in (38) we derive,

$$X_{2j-1}=\frac{(\rho_j+\sigma_j)\ (\bar\rho_{n/2}-\bar\sigma_{n/2})-(\bar\rho_j+\bar\sigma_j)\ (\rho_{n/2}-\sigma_{n/2})}{2[\sigma_0(\bar\rho_{n/2}-\bar\sigma_{n/2})-\bar\sigma_0(\rho_{n/2}-\sigma_{n/2})]},$$

$$(40)\qquad X_{2j}=-i\frac{(\rho_j-\sigma_j)\ (\bar\rho_{n/2}-\bar\sigma_{n/2})-(\bar\rho_j-\bar\sigma_j)\ (\rho_{n/2}-\sigma_{n/2})}{2[\sigma_0(\bar\rho_{n/2}-\bar\sigma_{n/2})-\bar\sigma_0(\rho_{n/2}-\sigma_{n/2})]},$$

$$X_{n-1}=\frac{\sigma_{n/2}\bar\rho_{n/2}-\bar\sigma_{n/2}\rho_{n/2}}{\sigma_0(\bar\rho_{n/2}-\bar\sigma_{n/2})-\bar\sigma_0(\rho_{n/2}-\sigma_{n/2})}.$$

From the last two of (39) we derive

$$\bar\rho_j-\bar\sigma_j=(b_j-a_j)\ (\rho_j-\sigma_j),\ \bar\rho_j+\bar\sigma_j=(b_j+a_j)\ (\rho_j+\sigma_j).$$

Hence (40) becomes

$$X_{2j-1} = \frac{(\rho_j+\sigma_j)\ (b_{n/2}-a_{n/2}-b_j-a_j)}{2[(b_{n/2}-a_{n/2}-b_0)\ \sigma_0-a_0\rho_0]},$$

$$X_{2j} = -i\,\frac{(\rho_j-\sigma_j)\ (b_{n/2}-a_{n/2}-b_j+a_j)}{2[(b_{n/2}-a_{n/2}-b_0)\sigma_0-a_0\rho_0]},$$

$$X_{n-1} = \frac{-a_{n/2}(\rho_{n/2}+\sigma_{n/2})}{(b_{n/2}-a_{n/2}-b_0)\sigma_0-a_0\rho_0},$$

$$(j=1,2,\ldots,(n-2)/2),$$

which may be written in either one of the forms

$$X_{2j-1}=\frac{(\rho_j+\sigma_j)A_j}{2D},\quad X_{2j}=-i\,\frac{(\rho_j-\sigma_j)B_j}{2D},\quad X_{n-1}=-\frac{a_{n/2}(\rho_{n/2}+\sigma_{n/2})}{2D},$$

$$X_{2j-1}=\frac{(\bar\rho_j+\bar\sigma_j)A_j}{2(a_j+b_j)D},\quad X_{2j}=-i\,\frac{(\bar\rho_j-\bar\sigma_j)B_j}{2(b_j-a_j)D},\quad X_{n-1}=-\frac{a_{n/2}(\bar\rho_{n/2}+\bar\sigma_{n/2})}{(a_{n/2}+b_{n/2})D},$$

Solving we have

(41)

(a) $$\rho_j+\sigma_j=\frac{2D\,.\,X_{2j-1}}{A_j},\quad i(\rho_j-\sigma_j)=-\frac{2D\,.\,X_{2j}}{B_j},\quad \rho_{n/2}+\sigma_{n/2}=-\frac{D\,.\,X_{n-1}}{a_{n/2}},$$

(b) $$\bar\rho_j+\bar\sigma_j=\frac{2D\,.\,X_{2j-1}(b_j+a_j)}{A_j},\quad i(\bar\rho_j-\bar\sigma_j)=-\frac{2D\,.\,X_{2j}(b_j-a_j)}{B_j},$$

$$\bar\rho_{n/2}+\bar\sigma_{n/2}=-\frac{D\,.\,X_{n-1}(a_{n/2}+b_{n/2})}{a_{n/2}},$$

$$(j=1,2,\ldots,(n-2)/2).$$

From these equations we easily derive the following, taking account of (39):

(42)

$$\sum\rho_k\sigma_k=D^2\sum\left[\frac{X_{2j-1}^2}{A_j^2}+\frac{X_{2j}^2}{B_j^2}\right]+\rho_0\sigma_0+\rho_{n/2}\sigma_{n/2}=0,$$

$$\sum\bar\rho_k\bar\sigma_k=D^2\left[\sum\frac{(b_j+a_j)^2X_{2j-1}^2}{A_j^2}+\sum\frac{(b_j-a_j)^2X_{2j}^2}{B_j^2}\right]+\bar\rho_0\bar\sigma_0+\bar\rho_{n/2}\bar\sigma_{n/2}=0,$$

$$\sum(\rho_k\bar\sigma_k+\bar\rho_k\sigma_k)=2D^2\left[\sum\frac{(b_j+a_j)^2X_{2j-1}^2}{A_j^2}+\sum\frac{(b_j-a_j)^2X_{2j}^2}{B_j^2}\right]+\rho_0\bar\sigma_0+\bar\rho_0\sigma_0$$
$$+\rho_{n/2}\bar\sigma_{n/2}+\bar\rho_{n/2}\sigma_{n/2}=0.$$

$$(k=0,1,\ldots,n/2;\ j=1,2,\ldots,(n-2)/2).$$

18. We shall now introduce Klein's coordinates:

$$(43)\qquad \begin{gathered} \rho_j+\sigma_j=2y_{2j-1},\ \rho_{n/2}+\sigma_{n/2}=2y_{n-1},\ \rho_0+\sigma_0=2y_{n+1},\\ -i(\rho_j-\sigma_j)=2y_{2j},\ -i(\rho_{n/2}-\sigma_{n/2})=2y_n,\ -i(\rho_0-\sigma_0)=2y_{n+2}.\\ (j=1,2,\ldots,(n-2)/2). \end{gathered}$$

If furthermore we put

$$(44)\qquad \begin{gathered} b_j+a_j=2k_{2j-1},\ b_{n/2}+a_{n/2}=2k_{n-1};\ b_0+a_0=2k_{n+1},\\ b_j-a_j=2k_{2j},\ b_{n/2}-a_{n/2}=2k_n,\ b_0-a_0=2k_{n+2},\\ (j=1,2,\ldots,(n-2)/2), \end{gathered}$$

the equations of the quadratic complex and the singular complex become

$$(45)\qquad \begin{gathered} \sum y_j^2=0,\ \sum k_jy_j^2=0,\ \sum k_j^2y_j^2=0,\\ (j=1,2,\ldots,n+2). \end{gathered}$$

In particular we have

$$(46)\qquad \begin{aligned} &\rho_0\sigma_0+\rho_{n/2}\sigma_{n/2}=y_{n-1}^2+y_n^2+y_{n+1}^2+y_{n+2}^2,\\ &\bar\rho_0\bar\sigma_0+\bar\rho_{n/2}\bar\sigma_{n/2}=4[k_{n-1}^2y_{n-1}^2+k_n^2y_n^2+k_{n+1}^2y_{n+1}^2+k_{n+2}^2y_{n+2}^2],\\ &\rho_0\bar\sigma_0+\bar\rho_0\sigma_0=\rho_{n/2}\bar\sigma_{n/2}+\bar\rho_{n/2}\sigma_{n/2}=4[k_{n-1}y_{n-1}^2+k_ny_n^2+k_{n+1}y_{n+1}^2+k_{n+2}y_{n+2}^2]. \end{aligned}$$

From the last of (41, a), using (43) and (44),

$$(47)\qquad 2y_{n-1}=\frac{-2X_{n-1}}{k_{n-1}-k_n}[(k_n-k_{n+1})y_{n+1}-i(k_n-k_{n+2})y_{n+2}],$$

or, transposing and squaring,

$$(48)\qquad \begin{aligned} &(k_{n-1}-k_n)^2y_{n-1}^2+(k_n-k_{n+1})^2X_{n-1}^2y_{n+1}^2+(k_n-k_{n+2})^2X_{n-1}^2y_{n+2}^2\\ &\qquad+2(k_{n-1}-k_n)(k_n-k_{n+1})X_{n-1}y_{n+1}y_{n-1}=0. \end{aligned}$$

Equations (42) now become, simplifying and taking account of (46) and (47),

$$(49)\qquad \begin{aligned} &\frac{(k_n-k_{n-1})^2}{X_{n-1}^2}\sum \alpha_jX_j^2y_{n-1}^2+y_n^2+y_{n+1}^2+y_{n+2}^2=0,\\ &\frac{(k_n-k_{n-1})^2}{X_{n-1}^2}\sum \beta_jX_j^2y_{n-1}^2+k_ny_n^2+k_{n+1}y_{n+1}^2+k_{n+2}y_{n+2}^2=0,\\ &\frac{(k_n-k_{n-1})^2}{X_{n-1}^2}\sum \gamma_jX_j^2y_{n-1}^2+k_n^2y_n^2+k_{n+1}^2y_{n+1}^2+k_{n+2}^2y_{n+2}^2=0, \end{aligned}$$

where

$$(50)\qquad \begin{gathered} \alpha_j=\frac{1}{(k_n-k_j)^2},\quad \beta_j=\frac{k_j}{(k_n-k_j)^2},\quad \gamma_j=\frac{k_j^2}{(k_n-k_j)^2},\\ (j=1,2,\ldots,n-1). \end{gathered}$$

If we eliminate y_{n-1}, y_n, y_{n+1} and y_{n+2} from (48) and (49), we shall obtain the transform of the generalized Kummer surface in the sphere-space $\overline{S}_{n-1}$. To do this we solve (49) for y_{n+1}^2 and y_{n+2}^2 in terms of y_{n-1}^2 and substitute in (48). We find

(51)
$$y_{n+1}^2 = \frac{\begin{vmatrix} 1 & \sum \alpha_j X_j^2 & 1 \\ k_n & \sum \beta_j X_j^2 & k_{n+2} \\ k_n^2 & \sum \gamma_j X_j^2 & k_{n+2}^2 \end{vmatrix} \dfrac{(k_n - k_{n-1})^2 y_{n-1}^2}{X_{n-1}^2}}{(k_n - k_{n+1})(k_n - k_{n+2})(k_{n+2} - k_{n+1})} = \frac{B(k_n - k_{n+1})^2 y_{n-1}^2}{X_{n-1}^2},$$
$$y_{n+2}^2 = \frac{-\begin{vmatrix} 1 & 1 & \sum \alpha_j X_j^2 \\ k_n & k_{n+1} & \sum \beta_j X_j^2 \\ k_n^2 & k_{n+1}^2 & \sum \gamma_j X_j^2 \end{vmatrix} \dfrac{(k_n - k_{n-1})^2 y_{n-1}^2}{X_{n-1}^2}}{(k_n - k_{n+1})(k_n - k_{n+2})(k_{n+2} - k_{n+1})} = \frac{C(k_n - k_{n-1})^2 y_{n-1}^2}{X_{n-1}^2}.$$

Substituting in (48), squaring and dividing through by $y_{n-1}^2(k_{n-1}-k_n)^2$ we have

(52)
$$1 + (k_n - k_{n+1})^4 B^2 + (k_n - k_{n+2})^4 C^2 + 2(k_n - k_{n+1})^2 (k_n - k_{n+2})^2 CB$$
$$- 2(k_n - k_{n+1})^2 B + 2(k_n - k_{n+2})^2 C = 0.$$

Expanding the determinants in (51) and introducing the values of $\alpha_j, \beta_j, \gamma_j$ from (50) we have

$$B = -\frac{\sum \dfrac{k_{n+2} - k_j}{k_n - k_j} X_j^2}{(k_n - k_{n+1})(k_{n+2} - k_{n+1})}, \quad C = \frac{\sum \dfrac{k_{n+1} - k_j}{k_n - k_j} X_j^2}{(k_n - k_{n+2})(k_{n+2} - k_{n+1})},$$
$$(j = 1, 2, \ldots, n-1).$$

Introducing these values in (52) we get the final result

(53)
$$(k_{n+2} - k_{n+1})^2 + (k_n - k_{n+1})^2 \left[\sum \frac{k_{n+2} - k_j}{k_n - k_j} X_j^2 \right]^2 + (k_n - k_{n+2})^2 \left[\sum \frac{k_{n+1} - k_j}{k_n - k_j} X_j^2 \right]^2$$
$$- 2(k_n - k_{n+1})(k_n - k_{n+2}) \sum \frac{k_{n+2} - k_j}{k_n - k_j} X_j^2 \sum \frac{k_{n+2} - k_j}{k_n - k_j} X_j^2$$
$$+ 2(k_n - k_{n+1})(k_{n+2} - k_{n+1}) \sum \frac{k_{n+1} - k_j}{k_n - k_j} X_j^2$$
$$+ 2(k_n - k_{n+2})(k_{n+2} - k_{n+1}) \sum \frac{k_{n+1} - k_j}{k_n - k_j} X_j^2 = 0,$$
$$(j = 1, 2, \ldots, n-1).$$

The surface is therefore a quartic. Since the equation may be put in the form

$$(54)\qquad \left[(k_n-k_{n+1})\sum\frac{k_{n+2}-k_j}{k_n-k_j}X_j^2-(k_n-k_{n+2})\sum\frac{k_{n+1}-k_j}{k_n-k_j}X_j^2\right]^2$$

$$+(k_{n+2}-k_{n+1})^2+2(k_n-k_{n+1})(k_{n+2}-k_{n+1})\sum\frac{k_{n+2}-k_j}{k_n-k_j}X_j^2$$

$$+2(k_n-k_{n+2})(k_{n+2}-k_{n+1})\sum\frac{k_{n+1}-k_j}{k_n-k_j}X_j^2=0,$$

$$(j=1,2,\ldots,n-1),$$

it appears that it has a locus of double points at infinity. If we contract the terms in the squared expression it reduces to

$$(k_{n+2}-k_{n+1})^2[\sum X_j^2]^2,\quad (j=1,2,\ldots,n-1);$$

hence the locus of double points is the absolute. It is easily seen that the equation (54) is reducible to that of a generalized Darboux Cyclide in n-space. We find by rearranging the equation (54),

$$(55)\qquad \sum\frac{X_j^2}{k_j-k_n}+\frac{(1-\sum X_j^2)^2}{k_{n+1}-k_n}-\frac{(1+\sum X_j^2)^2}{k_{n+2}-k_n}=0,$$

$$(j=1,2,\ldots,n-1).$$

We have thus proved the following:

Theorem. The Flat-Sphere Transformation (36) *carries a surface* Γ *into the generalized Darboux Cyclide**.

As a corollary we have: *The Line-Sphere transformation of Lie carries the Kummer surface into a cyclide having the absolute as nodal conic.*

This last proposition has been proved in an indirect way by V. Snyder in Bulletin of the Am. Math. Soc., vol. 4, pp. 150-152†.

*G. Darboux, *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*, pp. 172-173

†J. L. Coolidge, *loc. cit.*, p. 444 in a footnote refers to Snyder's result as incorrect. He reduces the general sphere-complex to the canonical form $\Sigma a_i z_i^2=0$ by means of a contact-transformation of the group $\overline{G}_{15}$, but he does not seem to have noticed that a transformation in $\overline{G}_{15}$ can be found which will not only do this, but also make the linear equation of the null-spheres take the form $z_6=0$. The cyclide must therefore be considered just as "general" as his surface of the 24th order and class, precisely as Kummer's surface in its simplest form (see Jessop's treatise, p. 98, to which Coolidge refers in a footnote on the same page), considered as a singular surface of the line complex $\Sigma a_i z_i^2=0$, is just as general as any of its projective equivalents. True enough, the group $\overline{G}_{15}$ does not preserve order and class, but for that reason to refuse calling the cyclide the most general singular surface of a quadratic sphere-complex is to abandon the standpoint of sphere-space for that of a projective space. In odd $(n-1)$-space, $n>4$, the group $G_{\frac{(n+1)(n+2)}{2}}$ of contact-transformations in flat-space does not preserve order and class (although the particular surface Γ suffers no change in this respect) precisely like its corresponding group $\overline{G}_{\frac{(n+1)(n+2)}{2}}$ in sphere-space. Only certain projective subgroups of $G_{\frac{(n+1)(n+2)}{2}}$ will preserve order and class.

There is however no objection to Coolidge's presentation, but it should be made clear that by so doing we no longer place ourselves on the basis of a strict group-geometry. There is then no reason to say that Snyder's statement is incorrect.

If k_n is variable in (55) we have: *The* ∞^1 *sphere-complexes*

$$\sum y_l^2=0,\ \sum k_j y^2+k_{n+1}y^2{}_{n+1}+k_{n+2}y^2_{n+2}+\lambda y^2_n=0,$$
$$(l=1,2,\ldots,n+2;\ j=1,2,\ldots,n-1),$$

have for singular surfaces a system of confocal cyclides.

19. The lines of curvature of the cyclide are known. If we introduce the coordinates λ as Klein has done (Gesammelte Werke, Bd, I, p. 144) we have

$$\rho y_j^2=\frac{(k_j-\lambda_1)(k_j-\lambda_2)\ldots\ldots(k_j-\lambda_{n-2})}{f'(k_j)}, \tag{56}$$

where $f(\lambda)=(k_1-\lambda)\ (k_2-\lambda),\ldots\ldots(k_{n+2}-\lambda)$.

These are the coordinates of the singular spheres since they satisfy the equations $\Sigma y_j^2=0,\ \Sigma k_j y_j^2=0,\ \Sigma k_j^2 y_j^2=0$. Substituting in the equations

$$X_{2j-1}=\frac{y_{2j-1}\,.\,A_j}{D},\quad X_{2j}=\frac{y_{2j}\cdot B_j}{D},\ X_{n-1}=\frac{-2y_{n-1}(k_{n-1}-k_n)}{D}, \tag{57}$$
$$(j=1,\ 2,\ \ldots,(n-2)/2),$$

where $D=2\,[(k_n-k_{n+1})y_{n+1}-i(k_n-k_{n+2})y_{n+2}]$ we have the parametric equation of the cyclide. The lines of curvature are here parametric, so that any one of the ∞^{n-3} lines is represented by the equations $\lambda_j=c_j;\ j=1,2,\ldots,k-1,\ k+1,\ldots,n-2$. The asymptotic lines on Γ are therefore also known and they are also coordinate asymptotic lines*. The parametric equations of Γ are obtained by substituting in (37) the values of ρ_j, σ_j as calculated from (43) and (56).

20. The investigation of the various properties of Γ may now be continued along the following lines: The equations of the coordinates of a flat are

$$z_j^2=\frac{(k_j-\lambda_1)(k_j-\lambda_2)\ldots\ldots(k_j-\lambda_n)}{f'(k_j)}, \tag{58}$$

from which it follows that

$$\sum z_j^2=0,\ \sum\frac{z_j^2}{k_j-\lambda_1}=0,\ \sum\frac{z_j^2}{k_j-\lambda_2}=0,\ \ldots,\ \sum\frac{z_j^2}{k_j-\lambda_n}=0, \tag{59}$$
$$(j=1,2,\ldots,n+2).$$

Hence, a flat belongs to n quadratic cosingular complexes $\lambda=\lambda_j$; each one is a member of the ∞^1 cosingular complexes

$$\sum\frac{z_j^2}{k_j-\lambda}=0,\quad (j=1,2,\ldots,n+2),$$

and the n λ's are the roots of this equation. For $\lambda=\infty$ we have the complex $\sum k_j z_j^2=0$, considered in the preceeding pages.

*J. Eiesland, *Flat-Sphere Geometry, Second Paper*, Amer, Jour. Math., vol. 40, pp. **31-32**. For definition of and conditions for coordinate lines of curvature, see Darboux, *loc. cit.*, pp. **135-137**.

21. Any $n/2+r$ complexes, $r=1,2,\ldots,(n-2)/2$ have $\infty^{n/2-r}$ flats in common which generate a surface which we shall call *a principal surface of the complex* (*type* r). Those of type $r=\frac{1}{2}(n-2)$ are the analogues of the lines of curvature of a spread in n-space. (See Jessop's treatise pp. 165-254). It would be worth while to investigate these surfaces and their analogues in n-space.

22. Any two complexes $\lambda=\lambda_r$, $\lambda=\lambda_s$ are in involution. The condition for this is

$$\sum \frac{\partial \phi}{\partial z_j} \frac{\partial \psi}{\partial z_j}=0,$$

when $\phi=0$, $\psi=0$; which is satisfied, $\phi=0$, $\psi=0$, being any two of the complexes (59).

23. The singular flats are obtained by putting any two roots equal, say $\lambda_{n-1}=\lambda_n=$const. and the coordinates are therefore

$$y_j^2=\frac{(k_j-\lambda_1)(k_j-\lambda_2)\ldots(k_j-\lambda_{n-2})}{f'(k_j)}=\frac{z_j^2}{(k_j-\lambda_{n-1})^2}.$$

If $\lambda_{n-1}=\lambda_n$ varies, the remaining λ's being constant, we get the ∞^1 tangent flats of the adjoint pencil at the point $P(\lambda_1, \lambda_2, \ldots, \lambda_{n-2})$on Γ, that is $z_j=(k_j-\lambda_{n-1})y_j$. The y's are the coordinates of the singular flats in the complex $\lambda=\infty$, while the z's are those of the complex $\lambda=\lambda_{n-1}$. Each flat of the pencil of adjoints will belong to one of the ∞^1 cosingular complexes.

24. If $\lambda_{n-1}=\lambda_n=k_r$, k_r being any one of the $n+2$ k's, we have *the singular bitangent flats* of the complex $\lambda=k_r$. The proof is identical with that for 3-space given in Jessop's treatise, p. 101. The coordinates of these flats are

$$z_j=0,\ z_j=(k_j-k_r)y_j.$$

In each pencil there will be $n+2$ bitangent flats one of which will be a singular flat belonging to the complex $\lambda_{n-1}=k_r$. All the ∞^{n-2} bitangent flats belong to $n+2$ complexes of rank 2 viz.,

$$z_j=0,\ \sum\frac{z_l^2}{k_l-k_r}=0,$$

$$(j=1,2,\ldots,n+2;\ l=1,\ 2,\ \ldots,\ r-1,\ r+1,\ \ldots,\ n+2);$$

they belong therefore to the $n+2$ fundamental complexes $z_j=0$.

25. If $\lambda_{n-2}=\lambda_{n-1}=\lambda_n(\lambda_n$ varying) and the remaining λ's constant we get the asymptotic tangent flats along an asymptotic line λ_{n-2}. These flats we shall call *principal tangent flats*; they have three-point contact with the curve and any two consecutive principal flats intersect. A principal tangent flat must therefore

lie in the osculating tangent $(n-2)$-flat at P*. Through P will pass $n-2$ such flats and they will belong respectively to the $n-2$ singular complexes $\lambda_j=\lambda_{n-1}=\lambda_n$, $j=1, 2, \ldots, n-2$.

26. The condition that two consecutive flats z_j, z_j+dz shall intersect is $\sum dz_l^2=0$, or,

$$0=d\lambda_1^2\,\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}{f(\lambda_1)}+d\lambda_2^2\,\frac{(\lambda_2-\lambda_3)\ldots(\lambda_2-\lambda_n)(\lambda_2-\lambda_1)}{f(\lambda_2)}+\ldots+d\lambda_n^2\,\frac{(\lambda_n-\lambda_1)(\lambda_n-\lambda_2)\ldots\ldots(\lambda_n-\lambda_{n-1})}{f(\lambda_n)}. \tag{60}$$

Among the many obvious solutions of this differential equation the solution $\lambda_{n-2}=\lambda_{n-1}=\lambda_n$, $\lambda_j=c_j$ is the one that gives the asymptotic line λ_{n-2}. We have then: *For any one point of the asymptotic curve λ_k, the principal tangent flats of the remaining $n-3$ curves belong respectively to the complexes $\lambda_j=c_j$, $(j=1, 2, \ldots, k-1, k+1, \ldots, n-3)$.*

Conclusion

27. We shall not pursue the study of the spread Γ any further on this occasion. The investigation of the singular loci Δ_{n-4}, T_{n-7} and Q_{n-11} by means of the λ coordinates and equation (60) is suggested. This being no easy task on account of the number of variables, it will be better to limit oneself to a special case, say $n=6$ (5-space), The flat-sphere correspondence as established in the third paper (Tohôku Math. Jour., vol. 16, pp. 185-235) will aid in the further study of the Darboux cyclide in odd n-space.

There is no doubt that quadratic complexes other than that of the general case [111...11] will give us a number of interesting surfaces in odd n-space such as the generalization of the wave surface and the tetrahedroid. In this connection it may be pointed out that the complex analogous to the harmonic complex in 3-space, viz.,

$$\sum a_j(\rho_j^2+\sigma_j^2)=0$$

will give a generalization of the tetrahedroid, and for a special case, that of the wave surface. In all cases the method of transformation into sphere-space given in this paper will give corresponding surfaces in this space. The generalized Dupin cyclide has already been obtained as the correspondent of a quartic surface in flat-space, this quartic playing the rôle analogous to the quadric in 3-space†.

A generalization of the tetrahedral complex may be obtained geometrically by considering all the ∞^{n-1}flats that intersect corresponding flats of two projective pencils having no flat in common. This complex and its singular surface deserve also to be studied.

*J. Eiesland, *Flat-Sphere Geometry, Third Paper, loc. cit.*, p. 201.

†Amer. Jour. Math., vol. 40, pp. 1-46 (Second Paper). Also Third Paper, *loc. cit.*, pp. 210-235.

SUR LES MOUVEMENTS A DEUX PARAMÈTRES DOUBLEMENT DÉCOMPOSABLES

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1. La théorie des Courbes et des Surfaces a été ramenée par Ribaucour et Darboux à la considération du mouvement d'un trièdre mobile par rapport à un autre, c'est-à-dire au mouvement relatif de deux corps solides, ce mouvement étant, dans un cas, à un paramètre et, dans l'autre, à deux paramètres. Pour employer une locution reçue, on peut dire que, le trièdre fixe et le trièdre mobile représentent deux corps, A et B, qui forment un système binaire soit à *un* degré de liberté soit à *deux* degrés de liberté.

Pour indiquer que, dans ce système binaire, on considère plus particulièrement le mouvement de A par rapport au corps B regardé comme fixe, on emploiera la notation $\boxed{A, B}$ qui se lira, mouvement de A par rapport à B. La notation $\boxed{B, A}$ représenterait le mouvement inverse de B par rapport à A regardé, à son tour, comme fixe.

Un grand nombre d'auteurs, en vue de généraliser les théories géomètriques, se sont laissé inspirer par l'analyse qui les a aiguillés vers la géomètrie non-euclidienne et les espaces à N dimensions.

Mais ce qui précède montre une voie plus géomètrique, au coeur même de l'espace euclidien. La continuation naturelle de la Géomètrie apparait alors dans l'étude des systèmes non plus de deux corps, mais de plusieurs corps dont les positions mutuelles dépendent de un ou de plusieurs paramètres. Un tel système est ce que l'on appelle une chaîne cinématique et le nombre des paramètres indépendants est le degré de liberté de la chaîne.

L'exemple d'une chaîne particulière à quatre membres que je traite ici, est bien propre à montrer que la Géomètrie des courbes et des surfaces trouve bien son compte dans cette sorte de problèmes.

2. Soit deux corps B_1 et B_2 formant un système binaire à deux paramètres ou de liberté 2. Ce système est dit décomposable s'il est possible de trouver un corps A_1 tel que les corps B_1 et A_1 forment un système de liberté 1 dont le paramètre de position soit u_1, tandis que les corps A_1 et B_2 forment, de leur coté, un système binaire de liberté 1 dont le paramètre de position soit v_1; ce dernier paramètre est naturellement indépendant de u_1. Dans ces conditions, les positions de B_1 par rapport à B_2 dépendent des paramètres u_1 et v_1, à travers le corps A_1.

Nous disons que, dans ce cas, le mouvement $\boxed{B_1, B_2}$ est un mouvement à deux paramètres *décomposable.* Ses paramètres sont ceux des mouvements $\boxed{B_1, A_1}$ et $\boxed{A_1 B_2}$. Il suffit de se donner à priori ces deux mouvements pour avoir un mouvement à deux paramètres décomposable.

Mais le problème devient plus difficile si l'on veut qu'il existe encore un autre corps A_2 possédant la même propriété que A_1, en sorte que le mouvement $\boxed{B_1, A_2}$ dépende d'un paramètre u_2 et que le mouvement $\boxed{A_2, B_2}$ dépende d'un paramètre v_2, indépendant de u_2.

Le système binaire $\boxed{B_1, B_2}$ apparait ainsi comme décomposable d'une autre manière. Il est dit dans ce cas, DOUBLEMENT DÉCOMPOSABLE.

3. Il est clair que les paramètres nouveaux u_2, v_2 sont des fonctions des premiers u_1, et v_1 attendu que à tout système de valeurs de u_1, v_1 il correspond une position de B_1 par rapport à B_2 et par suite un système de valeurs de u_2, v_2 et réciproquement.

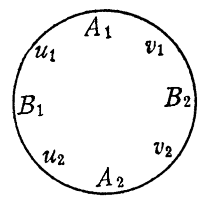
De la sorte on peut écrire les équations

$$(1) \qquad u_2 = \phi(u_1, v_1), \; v_2 = \psi(u_1, v_1)$$

ou, tout aussi bien

$$(2) \qquad u_1 = \omega(u_2, v_2), \; v_1 = \chi(u_2, v_2).$$

En général, l'équation $u_2 = \phi(u_1, v_1)$ qui exprime u_2 en fonction de u_1, v_1 ne se réduit pas à une simple relation entre u_1 et u_2. Alors u_1 et u_2 sont des variables indépendantes et le système binaire $\boxed{A_1, A_2}$ est, lui aussi, à deux paramètres et doublement décomposable.



Traçons un cercle et marquons les lettres B_1 et B_2 aux extrèmités d'un diamètre horizontal, puis les lettres A_1 et A_2 aux extrèmités d'un diamètre vertical. Plaçons ensuite les lettres u_1, v_1, u_2, v_2 sur les arcs respectifs B_1A_1, A_1B_2, A_2B_1, B_2A_2. Ce schéma représente une chaîne de 4 membres dans laquelle les membres consécutifs forment des systèmes binaires à un paramètre, savoir: le système $\boxed{B_1 A_1}$, au paramètre u_1; le système $\boxed{A_1 B_2}$, au paramètre v_1; le système $\boxed{B_2 A_2}$, au paramètre v_2 et enfin le système $\boxed{A_2 B_1}$, au paramètre u_2. Par contre, les corps opposés forment des systèmes binaires à deux paramètres, savoir: $\boxed{B_1 B_2}$ aux paramètres u_1, v_1 ou u_2, v_2; ces deux systèmes de paramètres sont équivalents d'après les formules (1) ou (2). Les corps A_1, A_2, forment aussi un système à deux paramètres $\boxed{A_1, A_2}$; les paramètres sont u_1, u_2 ou v_1, v_2.

Ces deux mouvements à deux paramètres sont l'un et l'autre, doublement décomposables.

4. Pour montrer que cette question n'est pas indifférente à la Géomêtrie telle que la comprenaient les Maîtres d'hier, faisons la remarque que le point P_1, pris quelconque dans le corps B_1, décrit dans le corps B_2 une surface trajectoire S_2 qui est le lieu de deux familles de courbes égales.

En effet, dans le mouvement $\boxed{B_1, A_1}$, le point P_1 décrit dans A_1 une courbe C_1 qui, dans le mouvement $\boxed{A_1, B_2}$ décrit à son tour la surface S_2. De même, dans le mouvement $\boxed{B_1, A_2}$, le point P_1 décrit dans A_2 une courbe C_2 qui engendre la surface S_2 au cours du mouvement $\boxed{A_2, B_2}$. Il peut arriver que le mouvement $\boxed{B_1, B_2}$ soit décomposable de plus de deux manières et même d'une infinité. La surface S_2 sera autant de fois le lieu d'une famille de courbes égales. Cette question des surfaces qui sont le lieu de plusieurs familles de courbes égales s'est déjà offerte à l'occasion des surfaces de translation qui sont normalement le lieu de deux familles de courbes égales. Ces surfaces servent à diriger le mouvement le plus général de translation à deux paramètres.

On sait que Sophus Lie, le grand géomètre norwègien, a déterminé les surfaces qui sont de translation de deux manières, ou, plus exactement de quatre, attendu que la génération par la translation d'une courbe entraîne un second mode de génération du même genre. Henri Poincaré* et Gaston Darboux† sont revenus successivement sur le problème de Lie qui met en jeu, dans sa solution les fonctions abéliennes. Ces surfaces servent de directrices à des mouvements à deux paramètres de translation qui sont *doublement décomposables.*

5. Quelques années avant sa mort, G. Darboux avait attiré l'attention des géomètres sur cette question des surfaces qui sont le lieu de plusieurs familles de courbes égales et quelques chercheurs avaient fait connaître quelques cas particuliers; mais la guerre qui est survenue a appelé ailleurs leur activité, l'un d'eux est même mort et Darboux ayant disparu, à son tour, la question de ces surfaces qui avait été présentée comme sujet de concours par l'Académie des Sciences de Paris, a été retirée comme n'ayant provoquée aucune recherche.

La considération des mouvements à deux paramètres plusieurs fois décomposables, non-seulement donne une solution de cette question, mais en manifeste une extension possible dans le sens de la dualité, car, en face de ces surfaces remarquables, se placent celles qui sont les enveloppes de deux ou de plusieurs développables égales. En effet, tout plan Π_1 solidaire du corps B_1 enveloppe dans A_1 une développable Γ_1 qui enveloppe à son tour dans B_2 la surface Σ_2 qui est l'enveloppe dans B_2 du plan Π_1. Il apparait ainsi que si le mouvement $\boxed{B_1, B_2}$ est plusieurs fois décomposable, toutes les surfaces enveloppes des plans de B_1 dans B_2 sont les enveloppes de deux familles au moins de développables égales, et que si le mouvement $\boxed{B_1, B_2}$ est n fois déconposable, ces surfaces seront les enveloppes de n familles de développables égales. On verrait de même que

*Henri Poincaré, *Remarques diverses sur les fonctions abéliennes* (Jour. de Math. 5° série, 1, 1895, p. 219): *Sur les surfaces de translation et les fonctions abéliennes* (Bull. Soc. Math. France, 29, 1901, p. 61).

†G. Darboux, *Sur les surfaces de translation* (Comptes Rendus Acad. Sciences, Paris, 155, 1912, p. 1449) et *Leçons sur la Théorie des Surfaces*, 2e Edition, T. 1, p. 151.

les congruences lieux, dans le corps B_2, d'une droite quelconque de B_1 sont les lieux d'autant de surfaces règlées égales.

Et l'on reconnait ainsi que ces mouvements à deux paramètres plusieurs fois décomposables ont dans la géomètrie pure, je veux dire dans celle qui se passe dans l'espace à trois dimensions euclidien, un rôle à jouer qui n'est pas dépourvu d'intérêt.

Il est permis d'ajouter qu'au point de vue mécanique, les chaînes spéciales à quatre membres que nous venons de définir amènent la conception de dispositifs tout à fait nouveaux, car on ne connait guère actuellement que le couple verrou qui soit décomposable de plusieurs manières, et même d'une infinité, en couples vis-écrou. C'est, au fond, sur cette multiple décomposition que reposent certains mécanismes différentiels.

Proposition fondamentale

6. Nos raisonnements et nos calculs reposeront sur une proposition essentielle qu'il nous faut d'abord démontrer.

Soit deux corps A et B formant un système binaire $\boxed{A,B}$ à n paramètres $u_1, u_2, u_3, \ldots, u_n$, ou du degré n de liberté. Soit T_A un trièdre trirectangle lié au corps A et x, y, z, les coordonnées d'un point P de A par rapport à ce trièdre. Si les paramètres subissent des variations représentées par leurs différentielles, le point P subit un déplacement PP' dont nous désignerons par D_x, D_y, D_z, les projections sur les axes de coordonnées. Or on sait que PP' est le moment par rapport au point P d'un système de vecteurs-rotations dont les projections sur les axes sont:

$$(3) \qquad \left\{\begin{array}{l} P_1du_1+P_2du_2+\ldots+P_ndu_n=\sum P_idu_i, \\ Q_1du_1+Q_2du_2+\ldots+Q_ndu_n=\sum Q_idu_i, \\ R_1du_1+R_2du_2+\ldots+R_ndu_n=\sum R_idu_i, \end{array}\right\} \quad (i=1, 2, \ldots, n)$$

et dont les moments résultants pris par rapport à ces mêmes axes sont:

$$(4) \qquad \left\{\begin{array}{l} \Xi_1du_1+\Xi_2du_2+\ldots+\Xi_ndu_n=\sum \Xi_idu_i, \\ \mathrm{H}_1du_1+\mathrm{H}_2du_2+\ldots+\mathrm{H}_ndu_n=\sum \mathrm{H}_idu_i, \\ \mathrm{Z}_1du_1+\mathrm{Z}_2du_2+\ldots+\mathrm{Z}_ndu_n=\sum \mathrm{Z}_idu_i. \end{array}\right\} \quad (i=1, 2, \ldots, n)$$

Dans ces formules, les quantités P_i, Q_i, R_i, Ξ_i, H_i, Z_i sont des fonctions des paramètres u.

On a ainsi les formules générales bien connues:

$$(5) \qquad \left\{\begin{array}{l} D_x=\sum \Xi_idu_i+z\sum Q_idu_i-y\sum R_idu_i, \\ D_y=\sum \mathrm{H}_idu_i+x\sum R_idu_i-z\sum P_idu_i, \\ D_z=\sum \mathrm{Z}_idu_i+y\sum P_idu_i-x\sum Q_idu_i. \end{array}\right.$$

De plus, en vertu de la composition des mouvements, si le point P, au lieu d'être solidaire de A, était mobile dans le corps A, il faudrait ajouter aux expressions précédentes les différentielles respectives dx, dy, dz, en sorte qu'il viendrait alors:

$$(6)\qquad \begin{cases} D_x = \sum \Xi_i du_i + z\sum Q_i du_i - y\sum R_i du_i + dx, \\ D_y = \sum \mathrm{H}_i du_i + x\sum R_i du_i - z\sum P_i du_i + dy, \\ D_z = \sum \mathrm{Z}_i du_i + y\sum P_i du_i - x\sum Q_i du_i + dz. \end{cases}$$

7. Considérons maintenant un trièdre trirectangle, T_B, lié au corps B; si l'on appelle X, Y, Z les coordonnées par rapport à ce trièdre du point P dont les coordonnées par rapport au trièdre T_A sont x, y, z, on passera des coordonnées par rapport à T_B aux coordonnées par rapport à T_A par les formules de la forme suivante où a, b, c, et les 9 cosinus α, α', α''; β, β', β''; γ, γ', γ'' sont des fonctions des paramètres u_1, u_2, u_3, . . . , u_n,

$$(7)\qquad \begin{cases} x = a + \alpha X + \alpha' Y + \alpha'' Z, \\ y = b + \beta X + \beta' Y + \beta'' Z, \\ z = c + \gamma X + \gamma' Y + \gamma'' Z. \end{cases}$$

Supposons alors que le point P soit solidaire du corps B. Les projections de son déplacement (soit D_x, D_y, D_z, données par les formules (6)), sont nulles, en sorte que les formules (7), où X, Y, Z, sont des constantes (coordonnées de P par rapport au trièdre T_B) donnent à x, y, z des valeurs qui vérifient les équations aux différentielles totales:

$$(8)\qquad \left.\begin{cases} D_x = \sum \Xi_i du_i + z\sum Q_i du_i - y\sum R_i du_i + dx = 0, \\ D_y = \sum \mathrm{H}_i du_i + x\sum R_i du_i - z\sum P_i du_i + dy = 0, \\ D_z = \sum \mathrm{Z}_i du_i + y\sum P_i du_i - x\sum Q_i du_i + dz = 0. \end{cases}\right\} \quad (i = 1, 2, \ldots, n)$$

Ces équations sont, du reste, équivalentes au système des $3n$ équations aux dérivées partielles

$$(9)\qquad \left.\begin{cases} \dfrac{\partial x}{\partial u_i} + \Xi_i + Q_i z - R_i y = 0, \\ \dfrac{\partial y}{\partial u_i} + \mathrm{H}_i + R_i x - P_i z = 0, \\ \dfrac{\partial z}{\partial u_i} + \mathrm{Z}_i + P_i y - Q_i x = 0. \end{cases}\right\} \quad (i = 1, 2, \ldots, n)$$

8. Ces faits connus étant rappelés, nous allons démontrer la proposition suivante:

Pour que le nombre n des paramètres indépendants puisse être réduit, il faut et il suffit qu'il existe au moins un système de multiplicateurs, non tous nuls, g_1, g_2, g_3, . . ., g_n, tels que l'on ait, en même temps:

$$(10)\qquad \begin{cases} \sum g_i P_i = 0, \ \sum g_i Q_i = 0, \ \sum g_i R_i = 0, \\ \sum g_i \Xi_i = 0, \ \sum g_i \mathrm{H}_i = 0, \ \sum g_i \mathrm{Z}_i = 0. \end{cases}$$

Supposons, par exemple, que le nombre des paramètres puisse être réduit d'une unité. On pourra introduire $n-1$ paramètres nouveaux $v_1, v_2, v_3, \ldots, v_{n-1}$, fonctions de $u_1, u_2, \ldots, u_n$, tels que, dans les équations (7), les quantités a, b, c et les 9 cosinus soient fonctions seulement des v.

Formons alors l'équation différentielle du 1er ordre

$$g_1 \frac{\partial \theta}{\partial u_1} + g_2 \frac{\partial \theta}{\partial u_2} + g_3 \frac{\partial \theta}{\partial u_3} + g_4 \frac{\partial \theta}{\partial u_4} + \ldots + g_n \frac{\partial \theta}{\partial u_n} = 0, \tag{11}$$

où les coëfficients g_i sont définis par les conditions que les $v_1, v_2, \ldots, v_{n-1}$ soient solutions de cette équation.

On aura donc, par hypothèse, puisque x, y, z sont des fonctions des v_i,

$$\Sigma g_i \frac{\partial x}{\partial u_i} = 0, \Sigma g_i \frac{\partial y}{\partial u_i} = 0, \Sigma g_i \frac{\partial z}{\partial u_i} = 0, \tag{12}$$

c'est-à-dire, d'après les équations (9):

$$\left\{ \begin{array}{l} \Sigma g_i \Xi_i + z \Sigma g_i Q_i - y \Sigma g_i R_i = 0, \\ \Sigma g_i \mathrm{H}_i + x \Sigma g_i R_i - z \Sigma g_i P_i = 0, \\ \Sigma g_i Z_i + y \Sigma g_i P_i - x \Sigma g_i Q_i = 0. \end{array} \right. \tag{13}$$

Ces équations doivent avoir lieu, pour des valeurs quelconques des paramètres et pour des valeurs quelconques des quantités X, Y, Z, c'est-à-dire pour des valeurs numériques quelconques de x, y, z. Ces équations devant ainsi avoir lieu quels que soient x, y, z, on en conclut précisément les équations (10).

Supposons, réciproquement, que l'on puisse trouver des multiplicateurs g_i, non tous nuls, de façon à vérifier les équations (10). Par le moyen des équations (9), on en déduira les équations (12) qui expriment que x, y, z sont des fonctions de $n-1$ solutions $v_1, v_2, \ldots, v_{n-1}$, indépendantes entr'elles de l'équation (11), ce qui équivaut à dire que les coefficients $a, b, c, \alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma''$ des équations (7) ne dépendent que de $n-1$ paramètres ou que le système de corps A, B, ne présente que le degré $(n-1)$ de liberté. Nous ajouterons la remarque suivante.

9. Nous avons supposé que les coordonnées du système des vecteurs-rotations pour le mouvement du corps A par rapport au corps B etaient rapportées à un trièdre T_A, solidaire du corps A. Si l'on rapporte les coordonnées du même système de vecteurs-rotations à un trièdre T quelconque, ces nouvelles coordonnées sont des fonctions linéaires et homogènes des premières, en sorte que des équations du type (10) entre les premières coordonnées en entraînent d'analogues entre les coordonnées nouvelles et réciproquement. On reconnait ainsi que la proposition précédente peut être étendue au cas où les coordonnées du système des vecteurs-rotations sont rapportées à un trièdre trirectangle quelconque.

On verra dans un instant l'utilité de cette remarque évidente.

Équations générales du Problème

10. Nous conserverons ici toutes les notations précédentes.

Nous avons déjà fait la remarque que dans la chaîne considérée on peut prendre comme variables indépendantes les paramètres u_1 et u_2. En adoptant dans l'étude du mouvement $\boxed{B_1, B_2}$ le trièdre mobile T_{B_1} pour y rapporter les coordonnées du système des vecteurs-rotations de ce mouvement, ces coordonnées seront de la forme suivante;

$$(14)\qquad \begin{cases} P_1 du_1 + P_2 du_2, \quad Q_1 du_1 + Q_2 du_2, \quad R_1 du_1 + R_2 du_2, \\ \Xi_1 du_1 + \Xi_2 du_2, \quad \mathrm{H}_1 du_1 + \mathrm{H}_2 du_2, \quad \mathrm{Z}_1 du_1 + \mathrm{Z}_2 du_2, \end{cases}$$

où P_1, P_2, Q_1, Q_2, R_1, R_2, Ξ_1, Ξ_2, H_1, H_2, Z_1, Z_2 sont des fonctions de u_1, u_2.

D'autre part, le mouvement $\boxed{B_1, A_1}$ dépend du paramètre u_1; en sorte que, par rapport au trièdre T_{B_1}, les coordonnées de son système de vecteurs-rotations sont de la forme suivante,

$$(15)\qquad p_1 du_1, \ q_1 du_1, \ r_1 du_1, \ \xi_1 du_1, \ \eta_1 du_1, \ \zeta_1 du_1,$$

où p_1, q_1, r_1, ξ_1, η_1, ζ_1 sont des fonctions de u_1.

Pareillement, le mouvement $\boxed{B_1, A_2}$ qui dépend du paramètre u_2, en prenant le même trièdre T_{B_1} pour y rapporter le système des vecteurs-rotations, a comme coordonnées de ceux-ci:

$$(16)\qquad p_2 du_2, \ q_2 du_2, \ r_2 du_2, \ \xi_2 du_2, \ \eta_2 du_2, \ \zeta_2 du_2,$$

où p_2, q_2, r_2, ξ_2, η_2, ζ_2 sont des fonctions de u_2.

Si l'on se reporte au mouvement $\boxed{A_1, B_2}$, que l'on peut regarder comme résultant des mouvements $\boxed{A_1, B_1}$ et $\boxed{B_1, B_2}$, comme le premier de ces mouvement est l'inverse du mouvement $\boxed{B_1, A_1}$, les coordonnées de son système de vecteurs-rotations seront égales et de signes contraires à celles du système (15).

En conséquence, le système des vecteurs-rotations pour le mouvement $\boxed{A_1, B_2}$, en prenant encore le trièdre T_{B_1}, aura comme coordonnées, d'après la théorie de la composition des rotations:

$$(17)\qquad \begin{cases} (P_1 - p_1) du_1 + P_2 du_2, \ (Q_1 - q_1) du_1 + Q_2 du_2, \ (R_1 - r_1) du_1 + R_2 du_2, \\ (\Xi_1 - \xi_1) du_1 + \Xi_2 du_2, \ (\mathrm{H}_1 - \eta_1) du_1 + \mathrm{H}_2 du_2, \ (\mathrm{Z}_1 - \zeta_1) du_1 + \mathrm{Z}_2 du_2. \end{cases}$$

Mais comme le mouvement $\boxed{A_1, B_2}$ ne dépend que d'un paramètre, il faut, d'après le théorème généralisé donné plus haut, qu'il existe une même relation linéaire entre les coëfficients partiels des coordonnées, en sorte que l'on peut trouver des coëfficients λ et μ tels que l'on ait les relations

$$(18)\qquad \begin{cases} \lambda(P_1 - p_1) + \mu P_2 = 0, \ \lambda(Q_1 - q_1) + \mu Q_2 = 0, \ \lambda(R_1 - r_1) + \mu R_2 = 0, \\ \lambda(\Xi_1 - \xi_1) + \mu \Xi_2 = 0, \ \lambda(\mathrm{H}_1 - \eta_1) + \mu \mathrm{H}_2 = 0, \ \lambda(\mathrm{Z}_1 - \zeta_1) + \mu \mathrm{Z}_2 = 0. \end{cases}$$

L'hypothèse $\lambda=0$ entraînerait la conséquence de $P_2=0$, $Q_2=0$, $R_2=0$, $\Xi_2=0$, $H_2=0, Z_2=0$; résultat inadmissible. Puisque λ n'est pas nul, on peut supposer $\lambda=1$.

Et l'on peut alors écrire les relations sous la forme suivante où m_2 est un coefficient auxiliaire:

$$(19)\quad \begin{cases} P_1-p_1=m_2P_2,\ Q_1-q_1=m_2Q_2,\ R_1-r_1=m_2R_2, \\ \Xi_1-\xi_1=m_2\Xi_2,\ H_1-\eta_1=m_2H_2,\ Z_1-\zeta_1=m_2Z_2. \end{cases}$$

La considération du mouvement $\boxed{A_2, B_2}$ conduira de même à poser les équations analogues:

$$(20)\quad \begin{cases} P_2-p_2=m_1P_1,\ Q_2-q_2=m_1Q_1,\ R_2-r_2=m_1R_1, \\ \Xi_2-\xi_2=m_1Q_1,\ H_2-\eta_2=m_1H_1,\ Z_2-\zeta_2=m_1Z_1, \end{cases}$$

où m_1 est encore coefficient auxiliaire.

Si nous posons alors:

$$(21)\quad \theta=\frac{1}{1-m_1m_2},\ \theta_1=\frac{m_1}{1-m_1m_2},\ \theta_2=\frac{m_2}{1-m_1m_2},$$

en sorte que

$$(22)\quad \theta(\theta-1)-\theta_1\theta_2=0,$$

nous trouverons pour les quantités P_1, Q_1, R_1, Ξ_1, H_1, Z_1, P_2, Q_2, R_2, Ξ_2, H_2, Z_2 les expressions suivantes

$$(23)\quad \begin{cases} P_1=\theta p_1+\theta_2p_2, & P_2=\theta p_2+\theta_1p_1, \\ Q_1=\theta q_1+\theta_2q_2, & Q_2=\theta q_2+\theta_1q_1, \\ R_1=\theta r_1+\theta_2r_2, & R_2=\theta r_2+\theta_1r_1, \\ \Xi_1=\theta\xi_1+\theta_2\xi_2, & \Xi_2=\theta\xi_2+\theta_1\xi_1, \\ H_1=\theta\eta_1+\theta_2\eta_2, & H_2=\theta\eta_2+\theta_1\eta_1, \\ Z_1=\theta\zeta_1+\theta_2\zeta_2, & Z_2=\theta\zeta_2+\theta_1\zeta_1. \end{cases}$$

11. On sait maintenant que les douze fonctions P_1, Q_1, . . . , H_2, Z_2 doivent vérifier six conditions d'intégrabilité. Ces équations ont les formes suivantes:

$$(24)\quad \begin{cases} \dfrac{\partial P_1}{\partial u_2}-\dfrac{\partial P_2}{\partial u_1}=Q_1R_2-R_1Q_2, \\ \dfrac{\partial Q_1}{\partial u_2}-\dfrac{\partial Q_2}{\partial u_1}=R_1P_2-P_1R_2, \\ \dfrac{\partial R_1}{\partial u_2}-\dfrac{\partial R_2}{\partial u_1}=P_1Q_2-Q_1P_2, \end{cases}$$

$$(24^{bis})^* \quad \begin{cases} \dfrac{\partial \Xi_1}{\partial u_2} - \dfrac{\partial \Xi_2}{\partial u_1} = Q_1 Z_2 - Q_2 Z_1 - R_1 H_2 + R_2 H_1, \\ \dfrac{\partial H_1}{\partial u_2} - \dfrac{\partial H_2}{\partial u_1} = R_1 \Xi_2 - R_2 \Xi_1 - P_1 Z_2 + P_2 Z_1, \\ \dfrac{\partial Z_1}{\partial u_2} - \dfrac{\partial Z_2}{\partial u_1} = P_1 H_2 - P_2 H_1 - Q_1 \Xi_2 + Q_2 \Xi_1. \end{cases}$$

En remplaçant dans ces équations d'intégrabilité les quantités P_1, Q_1, . . ., H_2, Z_2 par leurs valeurs (23), posant pour abréger:

$$(25) \qquad L = \frac{\partial \theta}{\partial u_2} - \frac{\partial \theta_1}{\partial u_1}, \quad M = \frac{\partial \theta}{\partial u_1} - \frac{\partial \theta_2}{\partial u_2}$$

et désignant par l'accent prime les dérivées de p_1, q_1, r_1, ξ_1, η_1, ζ_1 par rapport à u_1 et celles de p_2, q_2, r_2, ξ_2, η_2, ζ_2 prises par rapport à u_2, on trouvera les équations suivantes que j'ai aussi données en mars 1925 aux Comptes Rendus de l'Académie des Sciences, t. 180, p. 621; savoir:

$$(26) \quad \begin{cases} Lp_1 - Mp_2 + \theta(r_1 q_2 - r_2 q_1) - \theta_1 p_1' + \theta_2 p_2' = 0, \\ Lq_1 - Mq_2 + \theta(p_1 r_2 - p_2 r_1) - \theta_1 q_1' + \theta_2 q_2' = 0, \\ Lr_1 - Mr_2 + \theta(q_1 p_2 - q_2 p_1) - \theta_1 r_1' + \theta_2 r_2' = 0, \end{cases}$$

$$(27) \quad \begin{cases} L\xi_1 - M\xi_2 + \theta(q_2 \zeta_1 - q_1 \zeta_2 - r_2 \eta_1 + r_1 \eta_2) - \theta_1 \xi_1' + \theta_2 \xi_2' = 0, \\ L\eta_1 - M\eta_2 + \theta(r_2 \xi_1 - r_1 \xi_2 - p_2 \zeta_1 + p_1 \zeta_2) - \theta_1 \eta_1' + \theta_2 \eta_2' = 0, \\ L\zeta_1 - M\zeta_2 + \theta(p_2 \eta_1 - p_1 \eta_2 - q_2 \xi_1 + q_1 \xi_2) - \theta_1 \zeta_1' + \theta_2 \zeta_2' = 0. \end{cases}$$

Il convient de rappeler, à coté de ces équations l'équation (22) déjà trouvée:

$$\theta(\theta - 1) = \theta_1 \theta_2.$$

12. Remarques au sujet de ces équations. Appelons $A_1°$, $B_1°$, $A_2°$, $B_2°$ quatre corps ayant un même point commun K et en translation avec les corps A_1, B_1, A_2, B_2 respectivement. Dans la chaîne formée par ces quatre nouveaux corps les translations ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2, seront nulles; les équations (27) auront lieu identiquement et les équations (26) et (22) subsisteront seules. On est dans le cas d'une chaîne sphérique à deux paramètres doublement décomposable.

Une seconde remarque a trait à la détermination des paramètres v_1 et v_2 des mouvements respectifs à un paramètre $\boxed{A_1, B_2}$ et $\boxed{A_2, B_2}$.

Par rapport au trièdre T_{B_1} le système des vecteurs-rotations du mouvement $\boxed{A_1, B_2}$ a pour coordonnées, ainsi qu'on l'a déjà vu:

$$(P_1 - p_1)du_1 + P_2 du_2, \quad (Q_1 - q_1)du_1 + Q_2 du_2, \quad (R_1 - r_1)du_1 + R_2 du_2,$$
$$(\Xi_1 - \xi_1)du_1 + \Xi_2 du_2, \quad (H_1 - \eta_1)du_1 + H_2 du_2, \quad (Z_1 - \zeta_1)du_1 + Z_2 du_2;$$

*Pour ces conditions d'intégrabilité, voir: G. Darboux, *Leçons sur la Théorie des Surfaces*, t. 1, 2[e] édition, p. 67 et 71; et aussi G. Koenigs, *Leçons de Cinématique théorique*, p. 230.

et, en utilisant les equations (19), ces coordonnées deviennent:

$$P_2(m_2du_1+du_2),\ Q_2(m_2du_1+du_2),\ R_2(m_2du_1+du_2),$$
$$\Xi_2(m_2du_1+du_2),\ \mathrm{H}_2(m_2du_1+du_2),\ \mathrm{Z}_2(m_2du_1+du_2).$$

Si l'on appelle alors $\frac{1}{\rho_2}$ un facteur intégrant de $m_2du_1+du_2$ en sorte que

$$m_2du_1+du_2=\rho_2dv_1, \tag{28}$$

il est clair que v_1 est précisément le paramètre dont dépend le mouvement $\boxed{A_1, B_2}$. Les coordonnées par rapport au trièdre T_{B_2} du système des vecteurs-rotations de ce même mouvement seront donc:

$$P_2\rho_2dv_1,\ Q_2\rho_2dv_1,\ R_2\rho_2dv_1,\ \Xi_2\rho_2dv_1,\ \mathrm{H}_2\rho_2dv_1,\ \mathrm{Z}_2\rho_2dv_1. \tag{29}$$

De même, si $\frac{1}{\rho_1}$ est un facteur intégrant de $m_1du_2+du_1$, en sorte que

$$m_1du_2+du_1=\rho_1dv_2, \tag{30}$$

v_2 est le paramètre du mouvement $\boxed{A_2, B_2}$. Il n'y a pas lieu de s'arrêter sur la possibilité de remplacer dans ce cas, comme dans le précédent, v_2 ou v_1 par des fonctions de l'un ou l'autre de ces paramètres.

Par rapport au même trièdre que plus haut, les coordonnées du système des vecteurs-rotations pour le mouvement $\boxed{A_2, B_2}$ seront, en vertu de raisonnements analogues:

$$P_1\rho_1dv_2,\ Q_1\rho_1dv_2,\ R_1\rho_1dv_2,\ \Xi_1\rho_1dv_2,\ \mathrm{H}_1\rho_1dv_2,\ \mathrm{Z}_1\rho_1dv_2. \tag{31}$$

Ces résultats nous seront utiles pour les déductions qui vont suivre.

Cas où tous les corps sont en translation

13. Supposons que les corps A_1 et A_2 soient en translation avec B_1; les coëfficients p_1, q_1, r_1 sont alors nuls ainsi que p_2, q_2, r_2. D'après les formules (23), P_1, Q_1, R_1, P_2, Q_2, R_2 sont tous nuls et les quatre corps du système sont deux à deux en translation.

Les corps non contigüs du système formeront des systèmes binaires à deux paramètres de translation doublement décomposables. On doit s'attendre à ce que les surfaces trajectoires soient des surfaces de Sophus Lie à quadruple génération par translation. C'est en effet ce qui a lieu.

D'abord, les équations d'intégrabilité, réduites ici aux formules (24^{bis}), expriment que les formes linéaires $\Xi_1du_1+\Xi_2du_2$, $\mathrm{H}_1du_1+\mathrm{H}_2du_2$, $\mathrm{Z}_1du_1+\mathrm{Z}_2du_2$ sont des différentielles exactes, en sorte qu'il existe trois fonctions A, B, C de u_1, u_2 telles que l'on ait:

$$\Xi_1du_1+\Xi_2du_2+dA=0,\quad \mathrm{H}_1du_1+\mathrm{H}_2du_2+dB=0,\quad \mathrm{Z}_1du_1+\mathrm{Z}_2du_2+dC=0. \tag{32}$$

Ces équations expriment que le point M_2 dont A, B, C sont les coordonnées par rapport au trièdre T_{B_1} est fixe dans le corps B_2, en sorte que A, B, C en tant que fonctions de u_1, u_2, donnent une représentation paramétrique de la surface trajectoire du point M_2 dans le corps B_1. Il est aisé de se rendre compte que c'est une surface de Sophus Lie. Remarquons d'abord que d'après les tableaux des valeurs des coordonnées des systèmes de vecteurs-rotations pour les mouvements $\boxed{A_1, B_2}$ et $\boxed{A_2, B_2}$, formules (29) et (31), nous pouvons écrire:

$$\Xi_2 \rho_2 dv_1 = \Xi_1 du_1 + \Xi_2 du_2 - \xi_1 du_1 = -dA - \xi_1 du_1,$$
$$\mathrm{H}_2 \rho_2 dv_1 = \mathrm{H}_1 du_1 + \mathrm{H}_2 du_2 - \eta_1 du_1 = -dB - \eta_1 du_1,$$
$$\mathrm{Z}_2 \rho_2 dv_1 = \mathrm{Z}_1 du_1 + \mathrm{Z}_2 du_2 - \zeta_1 du_1 = -dC - \zeta_1 du_1.$$

On voit que les seconds membres de ces équations sont des différentielles exactes; il en est de même pour les premiers membres, en sorte qu'en appelant u_3 une fonction de v_1 et ξ_3, η_3, ζ_3 des fonctions de u_3, nous pouvons écrire:

$$(33) \qquad \begin{cases} \Xi_2 \rho_2 dv_1 = \xi_3 du_3 = -dA - \xi_1 du_1, \\ \mathrm{H}_2 \rho_2 dv_1 = \eta_3 du_3 = -dB - \eta_1 du_1, \\ \mathrm{Z}_2 \rho_2 dv_1 = \zeta_3 du_3 = -dC - \zeta_1 du_1. \end{cases}$$

On aura de la même façon:

$$\Xi_1 \rho_1 dv_2 = -dA - \xi_2 du_2,$$
$$\mathrm{H}_1 \rho_1 dv_2 = -dB - \eta_2 du_2,$$
$$\mathrm{Z}_1 \rho_1 dv_2 = -dC - \zeta_2 du_2.$$

Ici encore, les seconds membres et, par suite aussi les premiers sont des différentielles exactes et, en appelant u_4 une fonction de v_2 et ξ_4, η_4, ζ_4 des fonctions de u_4, on pourra écrire:

$$(34) \qquad \begin{cases} \Xi_1 \rho_1 dv_2 = \xi_4 du_4 = -dA - \xi_2 du_2, \\ \mathrm{H}_1 \rho_1 dv_2 = \eta_4 du_4 = -dB - \eta_2 du_2, \\ \mathrm{Z}_1 \rho_1 dv_2 = \zeta_4 du_4 = -dC - \zeta_2 du_2. \end{cases}$$

Rapprochons alors les équations (33) et (34), nous aurons:

$$(35) \qquad \begin{cases} dA = -\xi_1 du_1 - \xi_3 du_3 = -\xi_2 du_2 - \xi_4 du_4, \\ dB = -\eta_1 du_1 - \eta_3 du_3 = -\eta_2 du_2 - \eta_4 du_4, \\ dC = -\zeta_1 du_1 - \zeta_3 du_3 = -\zeta_2 du_2 - \zeta_4 du_4. \end{cases}$$

Posons alors:

$$(36) \qquad \begin{cases} a_1 = -\int \xi_1 du_1, \ b_1 = -\int \eta_1 du_1, \ c_1 = -\int \zeta_1 du_1, \\ a_2 = -\int \xi_2 du_2, \ b_2 = -\int \eta_2 du_2, \ c_2 = -\int \zeta_2 du_2, \\ a_3 = -\int \xi_3 du_3, \ b_3 = -\int \eta_3 du_3, \ c_3 = -\int \zeta_3 du_3, \\ a_4 = -\int \xi_4 du_4, \ b_4 = -\int \eta_4 du_4, \ c_4 = -\int \zeta_4 du_4; \end{cases}$$

il viendra:

$$(37)\quad \begin{cases} A = a_1 + a_3 = a_2 + a_4, \\ B = b_1 + b_3 = b_2 + b_4, \\ C = c_1 + c_3 = c_2 + c_4. \end{cases}$$

Ces équations fonctionnelles sont précisément celles qui ont servi de point de départ à Sophus Lie et à G. Darboux dans leurs recherches sur les surfaces à translations multiples. Le passage des formules (35) aux formules (37) introduit une constante additive à A, une autre à B, une troisième à C. A chaque choix de ces trois constantes il correspond un point différent du corps B_2; et en effet, tous les points solidaires du corps B_2 décrivent dans B_1 des surfaces de translations multiples égales entr'elles. Nous ne pouvons que renvoyer aux mémoires de Sophus Lie et de G. Darboux pour la solution du système d'équations fonctionnelles (37).

Cas de deux translations

14. Nous traiterons actuellement le cas où il existe seulement deux mouvements de translation. Nous avons vu que si trois corps consécutifs sont en translation, tous les mouvements de la chaîne sont des translations.

Il ne peut donc exister exactement deux mouvements de translation que si un tel mouvement étant réalisé entre deux des quatre corps, le second mouvement de translation est réalisé entre les deux corps opposés à ceux-là. Par exemple, entre B_1 et A_1 ainsi qu'entre A_2 et B_2.

Si l'on se reporte au tableau (31) des coordonnées du système des vecteurs-rotations pour ce dernier mouvement, on voit que notre hypothèse se traduit par les six équations suivantes:

$$(38)\quad p_1 = 0,\ q_1 = 0,\ r_1 = 0,\ P_1 = 0,\ Q_1 = 0,\ R_1 = 0.$$

Les équations (23) montrent alors que, puisqu'il faut exclure l'hypothèse de $p_2 = 0$, $q_2 = 0$, $r_2 = 0$, il est nécessaire que l'on ait:

$$(39)\quad \theta_2 = 0.$$

Alors l'équation (22) nous donne $\theta(\theta - 1) = 0$. Mais si θ était nul, les formules (23) donneraient zéro pour Ξ_1, H_1, Z_1, ce qui est inadmissible puisque P_1, Q_1, R_1 sont déjà nuls. C'est donc la valeur $\theta = 1$ qu'il faut prendre. Les formules (23) donnent alors

$$(40)\quad \begin{cases} P_1 = 0,\ \ Q_1 = 0,\ \ R_1 = 0,\ \ \Xi_1 = \xi_1,\ \mathrm{H}_1 = \eta_1,\ \mathrm{Z}_1 = \zeta_1, \\ P_2 = p_2,\ Q_2 = q_2,\ R_2 = r_2,\ \Xi_2 = \xi_2 + \theta_1 \xi_1,\ \mathrm{H}_2 = \eta_2 + \theta_1 \eta_1,\ \mathrm{Z}_2 = \zeta_2 + \theta_1 \zeta_1. \end{cases}$$

Les équations (26), M étant nul, sont identiquement vérifiées et les équations (27) prennent la forme simple:

$$(41)\qquad \begin{cases} \dfrac{\partial(\theta_1\xi_1)}{\partial u_1} = q_2\zeta_1 - r_2\eta_1, \\ \dfrac{\partial(\theta_1\eta_1)}{\partial u_1} = r_2\xi_1 - p_2\zeta_1, \\ \dfrac{\partial(\theta_1\zeta_1)}{\partial u_1} = p_2\eta_1 - q_2\xi_1. \end{cases}$$

Multiplions ces équations par $\theta_1\xi_1$, $\theta_1\eta_1$, $\theta_1\zeta_1$ et ajoutons, il viendra, en intégrant:

$$(42)\qquad \theta_1^2(\xi_1^2+\eta_1^2+\zeta_1^2)=\omega_2^2$$

où ω_2 est une fonction arbitraire de u_2. Posons

$$(43)\qquad \sqrt{\xi_1^2+\eta_1^2+\zeta_1^2}=\omega_1;$$

l'équation (42) donnera

$$(44)\qquad \theta_1=\frac{\omega_2}{\omega_1}.$$

En multipliant les mêmes équations (41) par p_2, q_2, r_2 respectivement, puis ajoutant et intégrant, on trouve:

$$(45)\qquad \theta_1(p_2\xi_1+q_2\eta_1+r_2\zeta_1)=\phi_2=\text{fonction de } u_2.$$

Si l'on remplace alors θ_1 par sa valeur (44), il vient:

$$(46)\qquad p_2\frac{\xi_1}{\omega_1}+q_2\frac{\eta_1}{\omega_1}+r_2\frac{\zeta_1}{\omega_1}=\frac{\phi_2}{\omega_2}=s_2$$

où s_2 est une fonction quelconque, finie, non nulle de u_2 que l'on substitue à la fonction transitoire ϕ_2 de u_2.

On simplifie les formules en substituant aux variables u_1, u_2 les variables u_1', u_2', définies par les intégrales:

$$(47)\qquad u_1'=\int\omega_1 du_1,\quad u_2'=\int\omega_2 du_2,$$

transformation permise car ω_1 et ω_2 ne sont jamais nuls ni infinis. Ce changement de variables revient à supposer ω_1 et ω_2 égaux à unité.

On a dans ces conditions:

$$(48)\qquad \theta_1=1,\ m_1=1,\ \text{tandis que } m_2=0.$$

Les formules (40), (41) et (42) deviennent:

$$(40^{\text{bis}})\qquad \begin{cases} P_1=0,\ Q_1=0,\ R_1=0,\ \Xi_1=\xi_1,\ \mathrm{H}_1=\eta_1,\ \mathrm{Z}_1=\zeta_1, \\ P_2=p_2,\ Q_2=q_2,\ R_2=r_2,\ \Xi_2=\xi_1+\xi_2,\ \mathrm{H}_2=\eta_1+\eta_2,\ \mathrm{Z}_2=\zeta_1+\zeta_2; \end{cases}$$

$$(41^{\text{bis}})\qquad \frac{d\xi_1}{du_1}=q_2\zeta_1-r_2\eta_1,\quad \frac{d\eta_1}{du_1}=r_2\xi_1-p_2\zeta_1,\quad \frac{d\zeta_1}{du_1}=p_2\eta_1-q_2\xi_1;$$

$$(42^{\text{bis}}) \qquad \xi_1^2+\eta_1^2+\zeta_1^2=1.$$

De même l'équation (46) se simplifie et devient,

$$(46^{\text{bis}}) \qquad p_2\xi_1+q_2\eta_1+r_2\zeta_1=s_2.$$

Mais là où apparaît le mieux l'utilité du changement de variables, c'est dans la simplicité des expressions de v_1 et de v_2 qui sont les paramètres des mouvements $\boxed{A_1, B_2}$ et $\boxed{A_2, B_2}$. On trouve, en effet, par application des formules (28) et (30) au cas actuel:

$$(49) \qquad v_1=u_2, \; v_2=u_1+u_2.$$

Première hypothèse

15. La discussion nous amène à faire une première hypothèse, celle où ξ_1, η_1, ζ_1 seraient constants.

Appelons d la droite qui porte le vecteur ξ_1, η_1, ζ_1. On peut toujours supposer que l'axe Oz du trièdre T_{B_1} est confondu avec cette droite, ce qui entraîne $\xi_1=0$, $\eta_1=0$ et, en vertu de (42^{bis})

$$\zeta_1=1.$$

Les équations (41^{bis}) se réduisent alors à $p_2=0$, $q_2=0$ et d'après (46^{bis}), on a $s_2=r_2$ et r_2 est une fonction quelconque de u_2. Alors les expressions (40^{bis}) deviennent

$$(50) \qquad \begin{cases} P_1=0, \; Q_1=0, \; R_1=0, \; \Xi_1=0, \; \mathrm{H}_1=0, \; \mathrm{Z}_1=1, \\ P_2=0, \; Q_2=0, \; R_2=r_2, \; \Xi_2=\xi_2, \; \mathrm{H}_2=\eta_2, \; \mathrm{Z}_2=\zeta_2+1. \end{cases}$$

Formons alors les équations (8) rappelées plus haut, qui donnent les coordonnées variables x, y, z par rapport au trièdre T_{B_1}, d'un point M_2, fixe par rapport au corps B_2; nous trouvons ici le système d'équations aux différentielles totales:

$$(51) \qquad \begin{cases} dx+(\xi_2-r_2y)du_2=0, \\ dy+(\eta_2+r_2x)du_2=0, \\ dz+du_1+(\zeta_2+1)du_2=0. \end{cases}$$

Introduisons une fonction c_2 de u_2 dont la dérivée par rapport à u_2 soit $-r_2$; le système précédent est équivalent à celui-ci:

$$(51^{\text{bis}}) \qquad \frac{\partial x}{\partial u_2}=-\xi_2-\frac{dc_2}{du_2}y, \quad \frac{\partial y}{\partial u_2}=-\eta_2+\frac{dc_2}{du_2}x, \quad \frac{\partial x}{\partial u_1}=0, \quad \frac{\partial y}{\partial u_1}=0,$$

$$(51^{\text{ter}}) \qquad \frac{\partial z}{\partial u_1}=-1, \quad \frac{\partial z}{\partial u_2}=-(\zeta_2+1).$$

Les équations (51^{bis}) montrent que x et y ne dépendent que de u_2. Appelons A_2 et B_2 deux fonctions de u_2 solutions particulières des équations (51^{bis}). Ces fonctions sont quelconques comme le sont ξ_2 et η_2.

On constate alors que les fonctions $x-A_2$, $y-B_2$ de u_2 vérifient les équations (51^{bis}) où les termes ξ_2 et η_2 seraient absents en sorte que x, y, z, solutions générales des systèmes (51) ou (51^{bis}), (51^{ter}) ont les expressions suivantes:

$$(52)\qquad \begin{cases} x=A_2+X\cos c_2-Y\sin c_2,\\ y=B_2+X\sin c_2+Y\cos c_2,\\ z=-u_1-\int(\zeta_2+1)du_2+Z. \end{cases}$$

Dans ces formules, X, Y, Z sont des constantes d'intégration qui représentent les coordonnées par rapport au trièdre T_{B_2} du point M_2, solidaire de B_2, en sorte que les formules (52) expriment les coordonnées de M_2 dans T_{B_1} en fonction de ses coordonnées constantes dans T_{B_2}. Ceci conformément aux remarques du N° 7.

Voici maintenant l'interprétation de ces formules. Les trièdres T_{B_1} et T_{A_1}, solidaires des corps en translation relative ont leurs axes de même nom parallèles. Si l'on appelle M_1' le point de coïncidence du point M_2 dans le corps A_1, et si l'on désigne par X', Y', Z' les coordonnées, par rapport au trièdre T_{A_1}, du point M_1'; les équations (52) se laissent décomposer en les suivantes:

$$(53)\qquad x=X',\ y=Y',\ z=Z'-u_1,$$

$$(54)\qquad \begin{cases} X'=A_2+X\cos c_2-Y\sin c_2,\\ Y'=B_2+X\sin c_2+Y\cos c_2,\\ Z'=-\int(\zeta_2+1)du_2+Z. \end{cases}$$

Les équations (53) représentent le mouvement de translation $\boxed{A_1, B_1}$ lequel a pour amplitude $-u_1$ suivant la direction fixe d, parallèle aux axes des z des deux trièdres T_{A_1} et T_{B_1}.

Les équations (54) représentent le mouvement du corps B_2 par rapport au corps A_1. Une première décomposition du mouvement $\boxed{B_2, B_1}$ est ainsi réalisée.

Le mouvement $\boxed{B_2, A_1}$, représenté par les équations (54), est le mouvement le plus général à un paramètre au cours duquel une droite du corps B_2 reste parallèle à une droite du corps A_1, en l'espèce, les axes des z des deux trièdres T_{B_2} et T_{A_1}.

On peut ajouter une représentation plus expressive de ces divers mouvements.

Appelons toujours d la direction de droite qui est constante, à la fois dans les trois corps B_2, A_1, B_1. Il y a deux cylindres parallèles à d, de sections droites quelconques, l'un W_{B_2} solidaire de B_2, l'autre W_{A_1} solidaire de A_1. Le mouvement de ces deux corps se réalise par la viration de ces deux cylindres l'un sur l'autre.

La loi du glissement infiniment petit suivant la génératrice de contact est donnée par la formule:

$$(55)\qquad -(\zeta_2+1)du_2.$$

La génératrice de contact des deux cylindres qui est aussi le siège du glissement de viration, est le lieu des points dont la vitesse est parallèle à d. D'après les résultats classiques de cinématique, les équations de cette génératrice de contact seront:

$$(56) \qquad \begin{cases} \dfrac{dA_2}{du_2} - \dfrac{dc_2}{du_2}(y-B_2)=0, \\[2ex] \dfrac{dB_1}{du_2} + \dfrac{dc_2}{du_2}(x-A_2)=0. \end{cases}$$

Nous appellerons g cette génératrice de contact. Passons au mouvement de translation $\boxed{A_1, B_1}$. Ici encore le cylindre W_{A_1} intervient utilement. En effet, au cours de la translation rectiligne, ce cylindre forme glissière avec un cylindre égal W_{B_1}, solidaire du corps B_1, sur lequel il est appliqué.

La décomposition du mouvement $\boxed{B_2, B_1}$ prend donc cette forme de la viration de W_{B_2} sur W_{A_1}, tandis que ce dernier cylindre forme glissière rectiligne avec le cylindre W_{B_1}.

Cherchons actuellement les éléments du second mode de décomposition du mouvement $\boxed{B_2, B_1}$. Rappelons d'abord que les variables des mouvements $\boxed{B_2, A_2}$ et $\boxed{A_2, B_1}$ sont u_2 et v_2; de plus, d'après les formules (49) du N° 14, on a

$$(57) \qquad v_2 = u_1 + u_2.$$

Pour mettre en évidence la seconde décomposition il faut introduire la variable v_2 au lieu de la variable u_1 dans les équations (52). La simplicité de l'équation (57) rend la chose facile et les équations (52) deviennent

$$(58) \qquad \begin{cases} x = A_2 + X\cos c_2 - Y\sin c_2, \\ y = B_2 + X\sin c_2 + Y\cos c_2, \\ z = -v_2 - \int \zeta_2 du_2 + Z. \end{cases}$$

Nous allons suivre ici une marche toute pareille à celle qui a été suivie plus haut.

Appelons M_2'' le point de coïncidence du point M_2 dans le corps A_2 et désignons par X'', Y'', Z'' les coordonnées de M_2'' par rapport au trièdre T_{A_2} solidaire du corps A_2. On décomposera les équations (58) de la façon suivante:

$$(59) \qquad X''=X, \; Y''=Y, \; Z''=Z-v_2;$$

$$(60) \qquad \begin{cases} x = A_2 + X''\cos c_2 - Y''\sin c_2, \\ y = B_2 + X''\sin c_2 + Y''\cos c_2, \\ z = -\int \zeta_2\, du_2 + Z''. \end{cases}$$

Les équations (59) représentent le mouvement de translation $\boxed{B_2, A_2}$ que nous reconnaissons être rectiligne et dirigé suivant la droite d considérée plus haut. Au cours de cette translation, le cylindre W_{B_2}, déjà considéré, forme glissière avec un cylindre égal solidaire du corps A_2 et que nous appellerons W_{A_2}. Ces deux cylindres sont parallèles à d. Enfin, les équations (60) représentent le mouvement $\boxed{A_2, B_1}$. Ce mouvement se manifeste comme étant une viration de deux cylindres l'un sur l'autre. Ces deux cylindres ne sont autres que les cylindres W_{A_2} et W_{B_1}.

Si, en effet, nous cherchons la génératrice de contact, nous trouvons que cette droite est fournie par les mêmes équations (56) qui sont celles de la droite g. Le cylindre solidaire de B_1 étant engendré par la droite g, comme le cylindre déjà considéré W_{B_1} coïncide avec lui et, pour la même raison, le cylindre solidaire de A_2 étant engendré par g dans A_2, tout comme W_{A_2}, coïncide avec ce cylindre.

La représentation de cette nouvelle décomposition est donc la suivante. Le cylindre W_{B_2}, solidaire de B_2, forme glissière rectiligne avec un cylindre égal W_{A^2}, solidaire de A_2, tandis que W_{B_2} vire sur le cylindre W_{B_1} solidaire de B_1.

Il est à noter que dans cette dernière viration, le glissement infiniment petit suivant la génératrice est égal à

$$-\zeta_2 du_2, \tag{61}$$

formule différente de la formule (55).

Il suffit du reste de rapprocher les deux premières équations (54) et les deux premières équations (58) pour constater l'identité des deux cylindres W_{B_2} et W_{A_2}, ainsi que celle des cylindres W_{A_1} et W_{B_1}. En effet les deux mouvements $\boxed{B_2, A_1}$ et $\boxed{A_2, B_1}$ ne diffèrent que par les expressions (55) et (68) des glissements, en sorte que les sections droites des cylindres virants sont identiques.

Du reste l'identité des cylindres W_{B_2} et W_{A_2} ressort déjà du fait que ces cylindres forment glissière rectiligne, et de même pour les cylindres W_{A_1} et W_{B_1}.

On a déjà dû remarquer que la génératrice g de contact est la même pour les deux couples de cylindres virants.

Faisons enfin remarquer que, dans le cas actuel, les surfaces trajectoires étant des cylindres, la question de ces surfaces est ici sans intérêt.

Seconde hypothèse

16. Revenons actuellement sur les hypothèses faites au N° 15. Si ξ_1, η_1, ζ_1 ne sont pas tous constants, le point U, de coordonnées ξ_1, η_1, ζ_1 décrit, en vertu de l'équation (42^{bis}), une courbe sphérique. D'autre part, si l'on attribue à u_2 dans l'équation (46^{bis}), une valeur numérique, cette équation linéaire en ξ_1, η_1, ζ_1, assujettit le point U à décrire un cercle; on voit en même temps que si l'on ne veut pas retomber sur l'hypothèse précédente, il faudra que le plan défini par l'équation (46^{bis}) soit indépendant de u_2. Dès lors, puisque les rapports des quantités p_2, q_2, r_2 sont constants, on peut toujours supposer que la droite d de direction fixe qui porte le vecteur p_2, q_2, r_2 coïncide avec l'axe Oz du trièdre T_{B_1}, en sorte que l'on aura alors

$$p_2=0,\ q_2=0, \tag{62}$$

et l'équation (42^{bis}) devient ainsi

$$\zeta_1=\frac{s_2}{r_2}=\cos\alpha, \tag{63}$$

où cos α représente une constante plus petite que l'unité, en vertu de l'équation (42^{bis}). On peut poser dès lors, en vertu de cette même équation,

$$\xi_1 = \sin\alpha . \cos\psi_1, \quad \eta_1 = \sin\alpha . \sin\psi_1, \quad \zeta_1 = \cos\alpha, \tag{64}$$

où ψ_1 est un angle auxiliaire, fonction de u_1. En portant ces valeurs de ξ_1, η_1, ζ_1 dans les équations (41^{bis}), on trouve qu'elles se réduisent à

$$\frac{d\psi_1}{du_1} = r_2 = n = \text{constante}, \tag{65}$$

d'où, en faisant rentrer dans u_1 une constante additive,

$$\psi_1 = nu_1. \tag{66}$$

Il en résulte les expressions suivantes des coordonnées des systèmes des vecteurs-rotations

$$\begin{cases} P_1 = 0, \ Q_1 = 0, \ R_1 = 0, \\ \Xi_1 = \xi_1 = \sin\alpha \cos nu_1, \\ \mathrm{H}_1 = \eta_1 = \sin\alpha \sin nu_1, \\ \mathrm{Z}_1 = \zeta_1 = \cos\alpha, \\ P_2 = 0, \ Q_2 = 0, \ R_2 = r_2 = n, \\ \Xi_2 = \xi_2 + \xi_1 = \xi_2 + \sin\alpha \cos nu_1, \\ \mathrm{H}_2 = \eta_2 + \eta_1 = \eta_2 + \sin\alpha \sin nu_1, \\ \mathrm{Z}_2 = \zeta_2 + \zeta_1 = \zeta_2 + \cos\alpha. \end{cases} \tag{67}$$

On peut alors former le système d'équations aux différentielles totales (8) du N° 7. On trouve:

$$\begin{cases} \sin\alpha \cos nu_1 . du_1 + (\sin\alpha \cos nu_1 + \xi_2) du_2 - ny du_2 + dx = 0, \\ \cos\alpha \sin nu_1 . du_1 + (\sin\alpha \sin nu_1 + \eta_2) du_2 + nx du_2 + dy = 0, \\ \cos\alpha \, du_1 + (\cos\alpha + \zeta_2) du_2 + dz = 0. \end{cases} \tag{68}$$

On constate en premier lieu sur les équations (68) elles-mêmes que x, y, z ont la forme suivante:

$$\begin{cases} x = -\dfrac{1}{n} \sin\alpha . \sin nu_1 + E, \\ y = \dfrac{1}{n} \sin\alpha . \cos nu_1 + F, \\ z = -\cos\alpha . u_1 + G, \end{cases} \tag{69}$$

où E, F, G sont trois fonctions de u_2 assujetties à vérifier les équations

$$(70)\qquad \left\{\begin{aligned} &\frac{dE}{du_2}-nF=-\xi_2,\\ &\frac{dF}{du_2}+nE=-\eta_2,\\ &\frac{dG}{du_2}+\cos\alpha=-\zeta_2. \end{aligned}\right.$$

Si l'on se reporte au N° 7, on y voit que les équations (8) aux différentielles totales admettent un système de solutions comportant trois constantes arbitraires qui s'interprètent comme des coordonnées rectangulaires par rapport au trièdre T_{B_2}. Ce sont les coordonnées du point M_2, solidaire du corps B_2, dont x, y, z sont les coordonnées variables par rapport au trièdre T_{B_1}. Il y a donc intérêt à suivre l'intégration du système (70) auquel se réduit le système (68) par la transformation (69).

Appelons A_2, B_2, C_2 trois fonctions de u_2 constituant un système particulier de solutions des équations (70). Ces trois fonctions sont quelconques comme ξ_2, η_2, ζ_2 elles mêmes. On obtiendra le système général des expressions de E, F, G on ajoutant à A_2, B_2, C_2 les solutions générales des équations (70) privées de seconds membres, savoir:

$$(71)\qquad \left\{\begin{aligned} &\frac{dE}{du_2}-nF=0,\\ &\frac{dF}{du_2}+nE=0,\\ &\frac{dG}{du_2}=0. \end{aligned}\right.$$

On trouve ainsi, pour valeurs générales de E, F, G,

$$(72)\qquad \left\{\begin{aligned} &E=X\cos nu_2+Y\sin nu_2+A_2,\\ &F=-X\sin nu_2+Y\cos nu_2+B_2,\\ &G=Z+C_2; \end{aligned}\right.$$

d'où enfin, par les formules (69), les expressions générales de x, y, z, solutions du système (68);

$$(73)\qquad \left\{\begin{aligned} &x=-\frac{1}{n}\sin\alpha\sin nu_1+A_2+X\cos nu_2+Y\sin nu_2,\\ &y=\frac{1}{n}\sin\alpha\cos nu_1+B_2-X\sin nu_2+Y\cos nu_2,\\ &z=-\cos\alpha . u_1+C_2+Z. \end{aligned}\right.$$

La discussion et l'interprétation de ce résultat se fera aisément en suivant une méthode identique à la précédente.

Considérons d'abord un trièdre T_{A_1}, solidaire du corps A_1. Comme les corps A_1 et B_1 sont en translation, on peut supposer les axes du trièdre T_{A_1} parallèles respectivement aux axes de même nom du trièdre T_{B_1}. Désignons par X', Y', Z', les coordonnées par rapport au trièdre T_{A_1} du point M_2 solidaire du corps B_2 et dont les coordonnées par rapport à T_{B_2} sont X, Y, Z et, par rapport au trièdre T_{B_1}, x, y, z.

Les équations (73) peuvent être décomposées de la façon suivante:

$$(74)\qquad \left\{\begin{aligned} x &= -\frac{1}{n}\sin\alpha.\sin nu_1 + X', \\ y &= \frac{1}{n}\sin\alpha.\cos nu_1 + Y', \\ z &= -\cos\alpha.u_1 + Z', \end{aligned}\right.$$

$$(75)\qquad \left\{\begin{aligned} X' &= A_2 + X\cos nu_2 + Y\sin nu_2, \\ Y' &= B_2 - X\sin nu_2 + Y\cos nu_2, \\ Z' &= C_2 + Z. \end{aligned}\right.$$

Les équations (74) représentent le mouvement de translation de A_1 par rapport à B_1. Au cours de ce mouvement, tous les points du corps A_1 décrivent des hélices égales à base circulaire, dont les axes ont une même direction d, parallèle aux axes des z des trièdres T_{A_1} et T_{B_1}; leur pas est égal à

$$(76)\qquad h = \frac{\cos\alpha}{n};$$

on constate du reste aisément que n est l'inverse d'une ligne.

Les formules (75) représentent le mouvement du corps B_2 par rapport au corps A_1. C'est le mouvement le plus général dans lequel une droite du corps B_2 (l'axe des z du trièdre T_{B_2}) reste parallèle à une direction fixe dans le corps A_1 à savoir, la direction d à laquelle est parallèle l'axe des z du trièdre T_{A_1}. Telle est la forme simple de la première décomposition du mouvement $\boxed{B_2, B_1}$.

Pour faire apparaître le second mode de décomposition, il faut introduire les variables u_2, v_2 en utilisant la formule (57), déjà employée plus haut,

$$(77)\qquad v_2 = u_1 + u_2.$$

Les formules (73) deviennent ainsi:

$$(78)\qquad \left\{\begin{aligned} x &= -\frac{1}{n}.\sin\alpha.\sin n(v_2-u_2) + A_2 + X\cos nu_2 + Y\sin nu_2, \\ y &= \frac{1}{n}\sin\alpha\cos n(v_2-u_2) + B_2 - X\sin nu_2 + Y\cos nu_2, \\ z &= -\cos\alpha.(v_2-u_2) + C_2 + Z. \end{aligned}\right.$$

Nous introduirons, ici encore, les coordonnées X'', Y'', Z'' du point M_2 par rapport au trièdre T_{A_2}, solidaire du corps A_2. Les corps A_2 et B_2 étant en translation, on peut supposer que les trièdres T_{A_2} et T_{B_2} ont leurs axes de même nom parallèles. D'après cela, le mouvement $\boxed{B_2, A_2}$ de translation, au paramètre v_2, sera représenté par des équations de la forme

$$X''=f+X,\ Y''=g+Y,\ Z''=h+Z, \tag{79}$$

où f, g, h, sont des fonctions de v_2. Elles devront être telles qu'en transportant dans les formules (78) les valeurs de X, Y, Z tirées des équations (79), les formules (78) deviennent les équations représentant le mouvement $\boxed{A_2, B_1}$. On trouve fort aisément:

$$f=-\frac{1}{n}\sin\alpha.\sin nv_2,\ g=\frac{1}{n}\sin\alpha.\cos nv_2,\ h=-v_2\cos\alpha. \tag{80}$$

Le mouvement $\boxed{B_2, B_1}$ ce trouve décomposé dans les deux mouvements suivants: Mouvement $\boxed{B_2, A_2}$, de translation,

$$\left\{\begin{aligned} X''&=-\frac{1}{n}\sin\alpha\sin nv_2+X,\\ Y''&=\frac{1}{n}\sin\alpha\cos nv_2+Y,\\ Z''&=-v_2\cos\alpha+Z,\end{aligned}\right. \tag{81}$$

et mouvement $\boxed{A_2, B_1}$, représenté par les formules:

$$\left\{\begin{aligned} x&=A_2+X''\cos nu_2+Y''\sin nu_2,\\ y&=B_2-X''\sin nu_2+Y''\cos nu_2,\\ z&=u_2.\cos\alpha+C_2+Z''.\end{aligned}\right. \tag{82}$$

La comparaison des formules (81) et (74) manifeste que les deux mouvements $\boxed{A_1, B_1}$ et $\boxed{B_2, A_2}$ quoique ne dépendant pas du même paramètre, sont des mouvements hélicoidaux de translation guidés par deux hélices identiques.

Mais il n'en est pas tout à fait de même pour les deux mouvements $\boxed{B_2, A_1}$ et $\boxed{A_2, B_1}$ malgré qu'ils dépendent de la même variable u_2 du fait que $v_1=u_2$.

Les trièdres T_{A_2} et T_{B_1} ont leurs axes des z parallèles tout comme les trièdres T_{B_2} et T_{A_1}.

D'après les formules (75), le mouvement $\boxed{B_2, A_1}$ résulte de la viration d'un cylindre W_{B_2} solidaire du corps B_2, sur un cylindre W_{A_1} solidaire du corps A_1. Ces deux cylindres ont leurs génératrices parallèles à la droite d, déjà définie, à laquelle sont parallèles aussi les axes des z de ces trièdres. Les sections droites de ces cylindres sont définies par les deux premières des équations (75). Par rapport au trièdre T_{B_2}, les équations de la génératrice de contact sont

$$(83)\qquad \left\{\begin{aligned} X &= \frac{1}{n}\left[\frac{dA_2}{du_2}\sin nu_2 + \frac{dB_2}{du_2}\cos nu_2\right],\\ Y &= \frac{1}{n}\left[-\frac{dA_2}{du_2}\cos nu_2 + \frac{dB_2}{du_2}\sin nu_2\right].\end{aligned}\right.$$

La dernière équation (75) donne la loi de glissement suivant la génératrice de contact g. Les équations (82) définissent, de même, le mouvement $\boxed{A_2, B_1}$; il existe un cylindre W_{A_1}, solidaire de A_1 qui vire sur un cylindre W_{B_2}, cylindre solidaire de B_1. Les génératrices de ces cylindres sont parallèles à la direction d à laquelle sont parallèles les axes des z des trièdres T_{A_2} et T_{B_1}. La comparaison des deux premières formules (82) avec les deux premières formules (75) manifeste l'identité des cylindres W_{B_2} et W_{A_1} respectivement avec W_{A_2} et W_{B_1}. Mais ces mouvements diffèrent par les lois de glissements qui ne sont pas les mêmes, comme en témoigne le rapprochement des troisièmes formules (75) et (82).

Dans le cas actuel, un point quelconque du corps B_2 décrit dans le corps B_1 une surface $S(B_2B_1)$ qui est le lieu de la courbe $C(B_2A_1)$ que ce même point décrit dans le corps A_1. Cette courbe engendre une première famille de courbes égales tracées sur la surface.

D'autre part, le point M_2 du corps B_2, considéré déjà, décrit dans A_2 une hélice $C(B_2A_2)$ qui, au cours du mouvement $\boxed{A_2, B_2}$ engendrent sur la surface $S(B_2B_1)$ une seconde famille de courbes égales.

On obtient donc ainsi une solution non banale, avec plusieurs fonctions arbitraires du problème des surfaces contenant plusieurs familles de courbes égales.

Mais il y a ici une remarque particulière à présenter. Puisque le mouvement $\boxed{A_1, B_1}$ est une translation au cours de laquelle tous les points du corps A_1 décrivent des hélices égales, il en est ainsi pour tous les points de la courbe $C(B_2A_1)$ qui seront des hélices égales, en translation les unes par rapport aux autres. Ces hélices, que nous désignerons par H, sont égales aux hélices $C(B_2A_2)$ mais elles ne coïncident pas avec elles car ces dernières ont bien leurs axes parallèles entr'eux et aux axes des hélices H; mais elles ne sont en translation ni entr'elles ni avec les H.

La surface $S(B_2B_1)$ contient donc trois familles de courbes égales, savoir: les hélices désignées par $C(B_2A_2)$; puis les courbes $C(B_2A_1)$; puis enfin les hélices H qui, comme il est aisé de le voir sur les formules (73), forment un système conjugué sur la surface.

On voit que l'existence de cette troisième famille, formée des hélices H, tient au fait que la courbe $C(B_2A_1)$ admet pour trajectoires de chacun de ses points des courbes toutes égales.

La question qui se pose ici est une autre face du problème des surfaces qui sont le lieu de plusieurs familles de courbes égales.

Mais, pour ne pas abuser de l'hospitalité qui est accordée ici à l'exposition de ces nouvelles idées, je me bornerai là et reprendrai dans une autre circonstance leur développement.

SUR LES BASES NOUVELLES DE LA THÉORIE DES SYSTÈMES ARTICULÉS

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Après l'invention du parallélogramme de Watt la théorie des systèmes articulés fut poursuivie avec beaucoup d'intérêt par plusieurs savants, et même par des géomètres tels que Tchébycheff et Sylvester. Mais pendant la dernière trentaine d'années cet intérêt s'est considérablement amoindri. Et je crois que c'est parce qu'on a basé cette théorie sur des idées stériles, en imaginant qu'on pourrait construire des nouveaux mécanismes compliqués par l'addition l'un à l'autre de quadrilatères articulés. On peut le faire, mais cette méthode, excluant l'essentiel, le but du mécanisme, aboutit à des systèmes qui ne servent à rien.

Dans la présente note, je veux exposer une méthode qui rapproche la théorie des systèmes articulés de quelques autres parties des mathématiques pures et appliquées.

Je pense que ce n'est pas le *mécanisme* articulé qui doit figurer au premier plan de la théorie mais la *chaîne cinématique fermée* (suivant la nomenclature de Reuleaux). L'inverseur de Peaucellier, par exemple, est un système très intéressant parce qu'il se rattache à beaucoup de disciplines mathématiques en donnant l'inverse d'une courbe quelconque. Mais le même inverseur, adapté, comme mécanisme, à la description d'une droite, devient un appareil trop spécial. Le pantographe est aussi un appareil très intéressant pour la théorie parce qu'il donne la transformation homothétique. Mais lorsque le pantographe devient un mécanisme, c'est-à-dire *un système à liaisons complètes*, il résout le problème beaucoup plus restreint de donner une homothétie d'une courbe spéciale. En pratique, on arrive toujours au mécanisme, mais pour la théorie, la chaîne cinématique fermée présente un champ beaucoup plus vaste. En résumé, j'énonce la proposition suivante: la base de la théorie doit consister en la transformation $x'=f(x, y)$; $y'=F(x, y)$ des coordonnées de deux points d'une chaîne cinématique fermée, transformation ponctuelle (suivant la classification de Sophus Lie) qui joue un rôle considérable dans la théorie des équations différentielles et dans d'autres disciplines mathématiques.

On peut classer toute cette théorie suivant ces transformations. Je vais en donner des exemples.

Transformation $x'=-x$, $y'=-y$: *pantographe isoscèle,* qui donne aussi le milieu de la distance variable entre les points (x', y'); (x, y) de la chaîne. Il peut aussi servir pour la transformation $\omega'=+\omega$ des rotations.

Transformation $x'=kx$; $y'=ky$: *pantographe*, qui donne la transformation homothétique et peut donner aussi une transformation des rotations.

Transformation $r=\dfrac{\mu^2}{\rho}$; $\phi'=\phi$: *inverseur*, qui donne l'inversion et peut servir à la construction de plusieurs mécanismes.

Transformation $x'=x$; $y'=ky$: mon *projecteur*, qui donne une projection orthogonale d'une courbe plane quelconque et, entre autres, l'ellipse comme projection d'une circonférence.

Transformation $x'^2=x^2-(m^2-n^2)$; $y'=y$, qui donne mon *hyperbolographe*.

Transformation $x'^2=\dfrac{[a^2-(x^2+y^2)]y^2}{x^2+y^2}$; $y'^2=\dfrac{[a^2-(x^2+y^2)]x^2}{x^2+y^2}$, qui donne le mécanisme de Tchébycheff pour le mouvement rectiligne approximatif. Et ainsi de suite.

Ainsi, au lieu d'une théorie stérile des mécanismes articulés, nous avons la théorie féconde des *transformateurs articulés*, qui se rattache intimement à beaucoup de questions analytiques. Au lieu d'un encombrement, sans but, de *quadrilatères* articulés il vaut mieux construire des mécanismes, en composant entre eux des *transformateurs* articulés présentants des chaînes cinématiques fermées.

En combinant, par exemple, l'inverseur avec mon projecteur, on peut construire un mécanisme traçant des limaçons de Pascal. En combinant deux pantographes avec un inverseur, on peut faire la clôture (l'anéantissement des points morts) d'un antiparallélogramme, qui ne donne pas de percussions comme le font les fourchettes de Reuleaux.

Le quadrilatère articulé réalise une courbe de 6^e degré, et en combinant ces quadrilatères on agit à l'aveuglette, tandis que chaque transformateur articulé réalise une idée nette de telle ou telle transformation de coordonnées, et en combinant ces idées, on marche sur un terrain sûr vers un but bien marqué.

La conclusion est que:

(1) la théorie des systèmes articulés doit être envisagée comme théorie des *transformateurs articulés*.

(2) chacun de ces transformateurs est une chaîne cinématique fermée et doit produire une transformation $x'=f(x, y)$; $y'=F(x, y)$ des coordonnées.

On peut espérer même qu'une telle élaboration de la théorie pourrait conduire à la théorie des transformateurs articulés à trois dimensions qui donneraient la transformation $x'=f(x, y, z)$; $y'=F(x, y, z)$; $z'=\phi(x, y, z)$*.

L'essentiel de ma proposition est que les transformateurs articulés sont intimement liés à la discipline fondamentale des mathématiques modernes, à la théorie des groupes des transformations (*Transformationsgruppen* de Sophus Lie et Klein).

*Par exemple, l'ellipsoïdographe (Bull. Sci. Math., t. XIX, 1895).

NOTA. J'ai réalisé l'exécution de quelques uns des mécanismes de mon invention:

(1) l'ellipsographe (Bull. Sci. Math., t. XIX, 1895;

(2) l'hyperbolographe (*ibid.*);

(3) le duplicateur articulé des rotations (Séances de la Soc. Française de Phys. 1895);

(4) la clôture cinématique d'antiparallélogramme;

(5) la *distribution par tiroir* de Hackworth, modifiée de telle façon qu'elle n'a plus d'excentrique.

ABSTRACTS OF COMMUNICATIONS

SECTION II

THE REPEATING PATTERNS OF THE REGULAR POLYGONS AND THEIR RELATION TO THE ARCHIMEDEAN BODIES

By Professor D'Arcy W. Thompson,
University of St. Andrews, St. Andrews, Scotland.

Take any one of the plane repeating patterns (or continuous symmetrical assemblages) of the regular polygons—as enumerated by Kepler,—as (for instance) a continuous sheet of hexagons. Remove alternate hexagons (in this case), so as to leave symmetrically arranged fenestrae, each surrounded by a symmetrical ring of hexagons; and suppose each ring to be slit, so that one hexagon may *slide over* another. The hexagonal fenestra will thus be converted into a pentagon; or you may slide two over two, converting the fenestra into a square; or three over three, converting it into a triangle.

When we proceed in this manner over the entire sheet, or entire assemblage, the result is to produce, successively, three semi-regular, isogonal solids (or Archimedean bodies), viz. (*a*) that with 20 hexagonal and 12 pentagonal faces; (*b*) that with eight hexagons and four squares; and (*c*) that with four hexagons and four triangles.

The particular case of the sheet of hexagons the author exhibited to the British Association two years ago. He has since discovered that precisely the same method may be applied to the entire series of plane repeating patterns (except the continuous sheet of equilateral triangles), and that by so doing the entire series of regular and semi-regular isogonal polyhedra (Platonic and Archimedean bodies) can be developed.

RECHERCHES GÉOMÉTRIQUES SUR LE PROBLÈME ISOPÉRIMÉTRIQUE

PAR M. T. BONNESEN,
Professeur à l'Université de Copenhague, Copenhague, Danemark.

Dans un mémoire célèbre Steiner a démontré que parmi les courbes planes ayant un périmètre donné c'est le cercle qui a la plus grande aire; mais dans sa démonstration Steiner suppose tacitement l'existence de la courbe présentant un aire maximum. Cette difficulté peut être surmontée en appliquant les méthodes analytiques du calcul des variations. Mais ce moyen étant d'une nature trop compliquée pour un problème géométrique d'une nature aussi simple, on a cherché d'autres démonstrations qui sont résumées dans la communication. La propriété en question du cercle, que nous appelerons sa propriété isopérimétrique, est exprimé par l'inégalité suivante valable pour une courbe plane quelconque de périmètre p et ayant l'aire f:

$$\frac{p^2}{4\pi} - f \geqq 0,$$

le signe d'égalité n'ayant lieu que pour le cercle. Étant donné qu'il suffit de considérer des courbes convexes, l'inégalité resulte immédiatement du théorème suivant:

Etant donnée une courbe convexe O, de périmètre p et d'aire f, on peut construire une autre courbe, non-convexe, entourant la courbe donnée et ayant l'aire $\frac{p^2}{4\pi}$.

Les deux courbes ne sont identiques que si O est un cercle. En utilisant la construction conduisant à la démonstration de ce théoreme, on arrive aisment à établir cette autre inégalité:

$$\frac{p^2}{4\pi} - f \geqq \frac{\pi}{4}\left(\frac{B-b}{2}\right)^2,$$

où B et b désignent respectivement la plus grande et la plus petite largeur de la courbe. On peut trouver une inégalité plus précise, conséquence du théorème suivant:

Etant donnée une courbe convexe O, il existe deux cercles concentriques de rayons R et r, respectivement, tels que la courbe donnée se trouve incluse dans l'aire limitée par les deux circonférences, tout en ayant deux points A et B communs avec la plus grande circonférence et deux points a et b communs avec la *plus petite, de telle sorte que sur chacun des deux arcs AB de la courbe O, il se trouve un, et un seul des points a et b. Dans ces conditions, les cercles sont déterminés d'une manière univoque et l'inégalité suivante*

$$\frac{p^2}{4\pi} - f \geqq \frac{\pi}{4}\left(R - r\right)^2$$

aura lieu.

Ces théorèmes s'établissent par des considérations de géométrie élèmentaire.

On arrive à l'inégalité encore plus précise

$$\frac{p^2}{4\pi} - f \geqq (R-r)^2$$

en cherchant la courbe inscrite dans l'anneau circulaire de rayons R et r, pour laquelle $\frac{p^2}{4\pi} - f$ est minimum. On peut démontrer de plus qu'une inégalité de cette forme ne pourrait pas exister pour toute courbe plane, si le coefficient de $(R-r)^2$ avait une valeur supérieure à l'unité.

Les mêmes recherches peuvent être étendues avec succès à des courbes sphériques, et les considérations sont même plus aisées que dans le cas des courbes planes*.

*Comptes Rendus Acad. Sciences, Paris, 1920; Mathematische Annalen Bde. 84, 91.

ON THE GENERALIZATION OF THE VECTOR PRODUCT TO S_n*

BY DR. ALMAR NAESS,
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1. In an ordinary n-space let there be given an orthogonal set of unit vectors $\mathbf{e}_i$. If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ be p given vectors independent of one another $(\mathbf{a}_t = \mathbf{e}_i\, a_{ti})$ then, by the *space complement* of these vectors, we understand the following *polyadic* of *order* $n-p$:

$$(1) \qquad \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots\cdots & \mathbf{e}_n \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots\cdots & \mathbf{e}_n \\ a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{p1} & a_{p2} & \cdots\cdots & a_{pn} \end{vmatrix}$$

This quantity we shall denote by $\langle^{p}\, \mathbf{a}_1 \ldots \mathbf{a}_p$ or $\mathbf{a}_1 \ldots \mathbf{a}_p\, ^{p}\rangle$ or $\mathbf{a}_1 \ldots \mathbf{a}_r\, {}^{r}\!\times^{t}\, \mathbf{a}_{r+1} \ldots \mathbf{a}_p$, $t = p - r$. If r or t is equal to 1 it is omitted. Hence $\langle^{2}\, \mathbf{a}_1\mathbf{a}_2 = \mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{a}_1\mathbf{a}_2\, ^{2}\rangle$.

2. The space complement is invariant with regard to orthogonal transformations of coordinates, and is expressible by vectors lying in the $(n-p)$-space which is absolutely perpendicular to the p-space containing the p primary vectors. It may be considered as a function of the polyad $\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_p$ and is independent of the particular form in which the polyad is expressed. The space complement of a polyadic is equal to the sum of the space complements of each of its polyads, whereby the validity of the distributive law is obtained.

The space complement of the space complement of a polyadic is equal to the (primary) polyadic times a scalar.

If we put $n=3$, $p=2$ the space complement is equal to the ordinary vector product of two vectors in S_3. We can show that all the fundamental vector product equations are obtained as special cases from equations expressing properties of the space complement, which thus may be considered as the generalization of the vector product of S_3.

*In this abstract 1-6 constitute a recapitulation of results which will be found in a paper published by the author in the Videnskabsselskabetskrifter, I. Math. Naturv. Klasse, 1922, No. 13, Oslo 1923. The indications given in 7 may be regarded as a brief preliminary report on questions which will be handled more in detail in a paper which will appear in a later number of the same publication. (I. Math. Naturv. Klasse, 1925, No. 9, Oslo, 1926).

We may, *e.g.*, mention:

$$(2)\qquad \boldsymbol{v}\times s\,(\langle_p\, \boldsymbol{a}_1\ldots\boldsymbol{a}_p) = -(-1)^{np}(n-p)!\,\boldsymbol{v}.\begin{vmatrix} \boldsymbol{a}_1 \ldots\ldots \boldsymbol{a}_p \\ \ldots\ldots\ldots\ldots \\ \boldsymbol{a}_1 \ldots\ldots \boldsymbol{a}_p \end{vmatrix}$$

where $s=n-p$. Putting $n=3$, $p=2$ we obtain the well-known expansion for the Gibbsian triple vector product:

$$(3)\qquad \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = -\boldsymbol{a}.(\boldsymbol{bc}-\boldsymbol{cb}).$$

3. Of other theorems shall only be indicated the following: let there be given an ordered set of n vectors in S_n, viz. $\boldsymbol{f}_i=\boldsymbol{e}_j f_{ij}$. This is equivalent to a matrix (linear transformation) f_{ij}. The set $\kappa_i=\boldsymbol{e}_j f_{ji}$ is called the *conjugate set.* The "*Ergänzungssystem*" to $\boldsymbol{f}_i$ is defined by:

$$(4)\qquad \boldsymbol{w}_i=(-1)^{i-1}\langle_m\, \boldsymbol{f}_1\ldots\boldsymbol{f}_{i-1}\boldsymbol{f}_{i+1}\ldots\boldsymbol{f}_n$$

where $m=n-1$. Then we have:

$$(5)\qquad \frac{1}{|f_{ij}|}\boldsymbol{f}_i\boldsymbol{w}_i=\boldsymbol{I}$$

where I is the idemfactor. That is: $\frac{1}{|f_{ij}|}\boldsymbol{w}_i$ is the *reciprocal system* to $\boldsymbol{f}_i$ in S_n, often denoted by $\boldsymbol{f}_i^*$.

The conjugate set to $\boldsymbol{w}_i$ is the Ergänzungssystem to κ_i and is denoted by $\boldsymbol{w}_i^\kappa$

In dyadic form the transformation f_{ij} is written: $\phi=\boldsymbol{e}_i\boldsymbol{f}_i$, and its *inverse* transformation: $\phi_c^*=\boldsymbol{f}_i^*\boldsymbol{e}_i=\frac{1}{|f_{ij}|}\boldsymbol{w}_i\,\boldsymbol{e}_i=\frac{1}{|f_{ij}|}\boldsymbol{e}_i\boldsymbol{w}_i^\kappa$. That is: $\boldsymbol{w}_i$ determines the inverse transformation of that given by the primary vector system.

4. In the n linear equations

$$(6)\qquad f_{i1}x_1+f_{i2}x_2+\ldots+f_{in}x_n=v_i$$

where the symbols x_i are the unknowns, f_{ij} and v_i the known quantities; put $f_{ij}\boldsymbol{e}_j=\boldsymbol{f}_i$, $v_i\boldsymbol{e}_i=\boldsymbol{v}$. We then have *Cramer's formula*:

$$(7)\qquad x_i=\frac{\boldsymbol{v}.\boldsymbol{w}_i^\kappa}{|f_{ij}|}.$$

Writing $x_i\boldsymbol{e}_i=\boldsymbol{x}$, these n equations may be stated in the form of a single formula:

$$(8)\qquad \boldsymbol{x}=\phi_c^*.\boldsymbol{v}.$$

5. Let there be given two sets of independent vectors, $\boldsymbol{f}_i$ and $\boldsymbol{f}_i'$. We will find the matrix x_{ij} which transforms the vectors $\boldsymbol{f}_i$ into $\boldsymbol{f}_i'$ respectively. Then

$$(9)\qquad x_{ij}=\frac{1}{|f_{ij}|}\kappa_i'.\boldsymbol{w}_j^\kappa$$

where κ_i' stands for the system conjugate to $\boldsymbol{f}_i'$.

6. For the generalized Gibbsian "vector" of a dyadic we get:

$$\phi_\times = \boldsymbol{e}_i \times \boldsymbol{f}_i = \sum_{j,l} E_{jl} d_{jl} \tag{10}$$

where:

$$\sum_{j,l} -(-1)^{j+l} E_{jl} \begin{vmatrix} \boldsymbol{e}_j & \boldsymbol{e}_l \\ \boldsymbol{e}_j & \boldsymbol{e}_l \end{vmatrix} = \begin{vmatrix} \boldsymbol{e}_1 \ldots\ldots \boldsymbol{e}_n \\ \ldots\ldots\ldots\ldots \\ \boldsymbol{e}_1 \ldots\ldots \boldsymbol{e}_n \end{vmatrix}, \tag{11}$$

and

$$-(-1)^{j+l}(f_{jl} - f_{lj}) = d_{jl}. \tag{12}$$

The $\frac{n(n-1)}{2}$ quantities d_{jl} are called *the symmetric differences* of the matrix f_{ij} and appear rather frequently in formulas concerning the space complement.

We will mention another theorem. The two matrices which can be formed from the second minors of a primary matrix and from the second minors of the conjugate of this, transform the symmetric differences of the primary matrix into the same set of quantities.

Moreover, we have,

$$\boldsymbol{v} \times^\nu \phi_\times = -\nu!\ \boldsymbol{v}.(\phi - \phi_c), \tag{13}$$

and

$$\boldsymbol{e}_i \times^\nu \phi_\times = -\nu!\ (\boldsymbol{f}_i - \kappa_i) \tag{14}$$

where $\nu = n-2$. Putting $n=3$ we derive some well-known formulae of vector analysis. For example, (14) says, that the vectors $\frac{\partial \boldsymbol{v}}{\partial x} - \nabla P$, $\frac{\partial \boldsymbol{v}}{\partial y} - \nabla Q$, $\frac{\partial \boldsymbol{v}}{\partial z} - \nabla R$ are coplanar and perpendicular to the curl of $\boldsymbol{v}$ if $\boldsymbol{v} = \boldsymbol{i}P + \boldsymbol{j}Q + \boldsymbol{k}R$ in S_3.

7. *Remark on the generalization of the curl of* $\boldsymbol{v}$ *to* S_4 (preliminary report). Gibbs considered the curl of $\boldsymbol{v}$ as being the "vector" of a special dyadic $\nabla\boldsymbol{v}$. From this formal point of view the generalized curl in S_4 (and S_n) must be the space complement of the dyadic $\nabla \boldsymbol{v} = \boldsymbol{e}_i \frac{\partial \boldsymbol{v}}{\partial x_i}$, which is easily determined by §1; *e.g.*, the curl in S_4 is

$$\text{curl } \boldsymbol{v} = \nabla \times \boldsymbol{v} = \boldsymbol{e}_i \times \frac{\partial \boldsymbol{v}}{\partial x_i} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ v_1 & v_2 & v_3 & v_4 \end{vmatrix}. \tag{15}$$

But it is rather interesting that from a purely geometric view-point we arrive at the same generalization.

The curl in S_3 determines those points which by the rotation (of each point of the field) are left invariant (axis of rotation) and a scalar (angular velocity). By a vector field in S_4 is analogously in each point determined a rotation, which appears to be general. That is: it may be uniquely resolved into two simple ones, the axis planes being absolutely perpendicular to one another, and these planes and two associated scalars (the corresponding angular velocities) determine the rotation. Now we can show that the generalized curl (15) vector analytically represents these two axis planes and their associated angular velocities. That is: it determines the general rotation of the field in each point completely in the same way as the curl of $\mathfrak{v}$ determines the rotation of the field in S_3.

In tensor analysis a rotation is also defined by a skew-symmetric tensor of the second order, determining the transformation which each point is subjected to by the rotation. This tensor and the curl then in different ways determine the same (geometric) phenomenon, and these two representations of the rotation are, of course, closely related to one another. We can prove that this skew-symmetric tensor in S_4 (and S_n) as well as in S_3 is equal to the space complement of the curl (the term curl used in the same sense as in the Gibbsian vector analysis and above (15)).

In the more extended paper referred to in the note at the foot of the first page of this abstract the theory of the space complement is applied to the study of rotations in general in S_4.

THE BEHAVIOUR OF THE RATIONAL PLANE SEXTIC AND ITS RELATED CAYLEY SYMMETROID UNDER REGULAR CREMONA TRANSFORMATION

By Professor A. B. Coble*,
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A rational plane sextic is conjugate to a rational space sextic whose catalectic plane sections envelop a symmetroid. The symmetroid, however, is determined in this way by two projectively distinct rational space sextics and thus a pairing of rational plane sextics is determined. A rational plane sextic can be transformed by Cremona transformation into only a finite number of projectively distinct types; a symmetroid can be transformed into only a finite number of projectively distinct symmetroids. It is the purpose of the author's paper to trace the effect of such Cremona transformation on the projective relations between the members of a pair of rational plane sextics, and between the plane sextic and the symmetroid.

*Now a member of the mathematical staff of the Johns Hopkins University.

THE CONDITION THAT THE CURVES OF A NET HAVE A COMMON POINT

By Professor Frank Morley,
Johns Hopkins University, Baltimore, Maryland, U.S.A.

The question is how to tell from the given equations of three curves, when they have a common point.

In the case when the curves are of the same order, a process was given by Sylvester, and explained in Salmon, *Higher Algebra*, Lesson 9. This process is evidently not in final form. The object of the paper is to obtain this final form, and hence to determine the common point when there is one.

ON SURFACES WHOSE ASYMPTOTIC CURVES ARE CUBICS

By Professor Charles H. Sisam,
Colorado College, Colorado Springs, Colorado, U.S.A.

The author's paper is devoted to the determination of the algebraic surfaces whose asymptotic curves constitute two pencils of cubic curves such that curves of opposite systems intersect in one variable point.

ON A SYSTEM OF TRIANGLES RELATED TO A PORISTIC SYSTEM

By Professor J. H. Weaver,
Ohio State University, Columbus, Ohio, U.S.A.

In the last half century considerable work has been done on the modern geometry of triangles by such men as Feuerbach, Brocard, Lemoine, Casey and others. Some of this work has been an investigation of the properties of a poristic system of triangles, *i.e.*, a system of triangles having a fixed circumcircle and a fixed incircle. In the present paper the author studies some of the properties of a system of triangles having a fixed circumcircle and a fixed nine point circle. The following properties are some of the results of this study:

(1) This system of triangles envelops a conic having the circumcentre and orthocentre for foci.

(2) The incentre and the excentres lie on a quartic curve C_4 which has the nine point centre for focus. C_4 consists of two ovals, one lying within the other, and so related that each is the inverse of the other with respect to the nine point centre.

(3) The Simson lines of the extremities of any chord of the circumcircle intersect on a circle equal to the nine point circle.

THE GENERALISED KRONECKER SYMBOL

By Professor F. D. Murnaghan,
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In a space S_n of n dimensions the generalised Kronecker symbol $\delta^{r_1 \ldots r_m}_{s_1 \ldots s_m}$ $(m \leq n)$ is defined by the following properties: (*a*) it is alternating in both the superscripts and subscripts, *i.e.*, an interchange of any two of the superscripts $r_1, \ldots, r_m$, each of which may take independently any one of the values $1, 2, \ldots, n$, changes the sign but not the numerical value of the symbol. Similarly for the subscripts $s_1, \ldots, s_m$. (*b*) If the values assigned to $r_1, \ldots, r_m$ are all distinct, as also are the values assigned to $s_1, \ldots, s_m$, the symbol has the value zero if the group $(r_1, \ldots, r_m)$ of m out of the n numbers $1, \ldots, n$ is different from the group $(s_1, \ldots, s_m)$; if these groups are the same, the symbol has the value ± 1 according as the arrangements $(r_1, \ldots, r_m)$, $(s_1, \ldots, s_m)$ are of the same class or not.

It is readily shown that $\delta^{r_1 \ldots r_m}_{s_1 \ldots s_m}$ is an "arithmetical" tensor contravariant of rank m and covariant of rank m. It may be used to present the theory of determinants in a brief and elegant manner. Conversely it may be expressed as the determinant

$$\begin{vmatrix} \delta^{r_1}_{s_1} & \cdots & \delta^{r_1}_{s_m} \\ \delta^{r_2}_{s_1} & \cdots & \delta^{r_2}_{s_m} \\ \cdots & \cdots & \cdots \\ \delta^{r_m}_{s_1} & \cdots & \delta^{r_m}_{s_m} \end{vmatrix}$$

whose elements are components of the ordinary Kronecker tensor δ^r_s. The whole theory of outer multiplication of tensors follows at once through use of the generalised Kronecker symbol.

Thus if we have any two covariant tensors $A_{r_1 \ldots r_p}$, $B_{s_1 \ldots s_q}$ of ranks p and q respectively $(p+q \leq n)$ the alternating tensor of rank $p+q$, $C_{r_1 \ldots r_p s_1 \ldots s_q} = \delta^{\rho_1 \ldots \rho_p \sigma_1 \ldots \sigma_q}_{r_1 \ldots r_p s_1 \ldots s_q} A_{\rho_1 \ldots \rho_p} B_{\sigma_1 \ldots \sigma_q}$ is the outer product. Similarly we have the outer product $C^{r_1 \ldots r_p s_1 \ldots s_q} = \delta^{r_1 \ldots r_p s_1 \ldots s_q}_{\rho_1 \ldots \rho_p \sigma_1 \ldots \sigma_q} A^{\rho_1 \ldots \rho_p} B^{\sigma_1 \ldots \sigma_q}$ of two contravariant tensors. It may be pointed out that these are theorems of tensor *algebra* rather than tensor *analysis*. Neither differential nor metrical properties of the S_n are postulated and the results hold for the space S_n of Analysis Situs. The application of the tensor $\delta^{r_1 \ldots r_m}_{s_1 \ldots s_m}$ to the discussion of "oriented cells" in Analysis Situs is apparent.

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