PROCEEDINGS

OF THE

INTERNATIONAL CONGRESS

OF

MATHEMATICIANS

Cambridge, Massachussetts, U. S. A.

August 30—September 6, 1950

PUBLISHED BY THE

AMERICAN MATHEMATICAL SOCIETY

80 Waterman Street, Providence, R. I.

1952

PROCEEDINGS OF THE

INTERNATIONAL CONGRESS OF MATHEMATICIANS

Cambridge, Massachusetts, U. S. A.

1950

Under the Auspices of the

AMERICAN MATHEMATICAL SOCIETY

HOST

Harvard University

CO-HOSTS

American Academy of Arts and Sciences
Boston College
Boston University
Massachusetts Institute of Technology
Tufts College

VOLUME I

OFFICERS AND MEMBERS

REPORT OF THE SECRETARY

STATED ADDRESSES

ADDRESSES AND COMMUNICATIONS IN SECTIONS

HONORARY PRESIDENTS

GUIDO CASTELNUOVO
Accademia Nazionale dei Lincei

JACQUES HADAMARD
Collège de France and École Polytechnique

CHARLES DE LA VALLÉE POUSSIN
Université de Louvain

OFFICERS OF THE CONGRESS

President: OSWALD VEBLEN, Institute for Advanced Study

Secretary: J. R. KLINE, University of Pennsylvania

Associate Secretary: R. P. BOAS, JR., Mathematical Reviews and Northwestern University

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

COMMITTEES OF THE CONGRESS

ORGANIZING COMMITTEE

GARRETT BIRKHOFF, Chairman, Harvard University
W. T. MARTIN, Vice Chairman, Massachusetts Institute of Technology

A. A. ALBERT, University of Chicago
J. L. DOOB, University of Illinois
G. C. EVANS, University of California
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
J. R. KLINE, University of Pennsylvania
SOLOMON LEFSCHETZ, Princeton University
SAUNDERS MACLANE, University of Chicago
MARSTON MORSE, Institute for Advanced Study
JOHN VON NEUMANN, Institute for Advanced Study
OSWALD VEBLEN, Institute for Advanced Study
J. L. WALSH, Harvard University
HASSLER WHITNEY, Harvard University
D. V. WIDDER, Harvard University
R. L. WILDER, University of Michigan

FINANCIAL COMMITTEE

JOHN VON NEUMANN, Chairman, Institute for Advanced Study
W. L. G. WILLIAMS, Vice Chairman, McGill University

J. L. COOLIDGE, Harvard University
B. P. GILL, City College of New York
M. H. INGRAHAM, University of Wisconsin
A. E. MEDER, JR., Rutgers University

EDITORIAL COMMITTEE

L. M. GRAVES, Chairman, University of Chicago

EINAR HILLE, Yale University
P. A. SMITH, Columbia University
OSCAR ZARISKI, Harvard University

SUBCOMMITTEES OF THE ORGANIZING COMMITTEE

BUDGET COMMITTEE

W. T. MARTIN, Chairman, Massachusetts Institute of Technology
J. R. KLINE, University of Pennsylvania
A. E. MEDER, JR., Rutgers University

G. B. Price, University of Kansas
Oswald Veblen, Institute for Advanced Study
Oscar Zariski, Harvard University

COOPERATION COMMITTEE

Samuel Eilenberg, Chairman, Columbia University
E. F. Beckenbach, University of California, Los Angeles
Hassler Whitney, Harvard University
S. S. Wilks, Princeton University
J. W. T. Youngs, Indiana University

ENTERTAINMENT COMMITTEE

L. H. Loomis, Chairman, Harvard University
C. R. Adams, Brown University
Mrs. L. V. Ahlfors, Harvard University
Mrs. G. D. Birkhoff, Harvard University
J. A. Clarkson, Tufts College
Mrs. W. C. Graustein, Tufts College
F. B. Hildebrand, Massachusetts Institute of Technology
Evelyn M. Hull, American Mathematical Society
J. R. Kline, University of Pennsylvania
E. R. Lorch, Columbia University
Mrs. W. T. Martin, Massachusetts Institute of Technology
E. B. Mode, Boston University
G. A. O'Donnell, Boston College
Mrs. H. B Phillips, Massachusetts Institute of Technology
Helen G. Russell, Wellesley College
J. H. Van Vleck, Harvard University
Mrs. J. L. Walsh, Harvard University
Mrs. D. V. Widder, Harvard University
Mrs. Norbert Wiener, Massachusetts Institute of Technology

PROGRAM COMMITTEE

E. G. Begle, Chairman, Yale University
A. A. Albert, University of Chicago
Garrett Birkhoff, Harvard University
R. P. Boas, Jr., Mathematical Reviews and Northwestern University
Richard Courant, New York University
J. L. Doob, University of Illinois
Samuel Eilenberg, Columbia University
G. C. Evans, University of California
J. R. Kline, University of Pennsylvania
L. H. Loomis, Harvard University
Marston Morse, Institute for Advanced Study

John von Neumann, Institute for Advanced Study
C. V. Newsom, New York State Education Department
H. A. Rademacher, University of Pennsylvania
Alfred Tarski, University of California
Hassler Whitney, Harvard University

PUBLICITY COMMITTEE

R. P. Boas, Jr., Co-Chairman, Mathematical Reviews and Northwestern University
G. W. Mackey, Co-Chairman, Harvard University
A. A. Bennett, Brown University
J. A. Clarkson, Tufts College
C. O. Oakley, Haverford College
R. M. Thrall, University of Michigan

COMMITTEE ON TRANSPORTATION GRANTS

Garrett Birkhoff, Chairman, Harvard University
Samuel Eilenberg, Columbia University
J. R. Kline, University of Pennsylvania
W. T. Martin, Massachusetts Institute of Technology
J. L. Walsh, Harvard University

COMMITTEE ON CONFERENCE IN ALGEBRA

A. A. Albert, Chairman, University of Chicago
Richard Brauer, University of Michigan
Nathan Jacobson, Yale University
Saunders MacLane, University of Chicago
Oscar Zariski, Harvard University

COMMITTEE ON CONFERENCE IN ANALYSIS

Marston Morse, Chairman, Institute for Advanced Study
L. V. Ahlfors, Harvard University
Salomon Bochner, Princeton University
G. C. Evans, University of California
Einar Hille, Yale University

COMMITTEE ON CONFERENCE IN APPLIED MATHEMATICS

John von Neumann, Chairman, Institute for Advanced Study
Richard Courant, Vice Chairman, New York University
Walter Bartky, University of Chicago
R. V. Churchill, University of Michigan
G. C. Evans, University of California
William Prager, Brown University
Mina Rees, Office of Naval Research

COMMITTEE ON CONFERENCE IN TOPOLOGY

HASSLER WHITNEY, Chairman, Harvard University
DEANE MONTGOMERY, Institute for Advanced Study
N. E. STEENROD, Princeton University

CHAIRMEN OF SECTIONS

Section I	Algebra and Theory of Numbers: H. A. RADEMACHER, University of Pennsylvania
Section II	Analysis: G. C. EVANS, University of California
Section III	Geometry and Topology: SAMUEL EILENBERG, Columbia University
Section IV	Probability and Statistics, Actuarial Science, Economics: J. L. DOOB, University of Illinois
Section V	Mathematical Physics and Applied Mathematics: RICHARD COURANT, New York University
Section VI	Logic and Philosophy: ALFRED TARSKI, University of California
Section VII	History and Education: C. V. NEWSOM, New York State Education Department

DONORS

Carnegie Corporation
Rockefeller Foundation

United Nations Educational, Scientific and Cultural Organisation

Boston College
Boston University
Brown University
Harvard University
Institute for Advanced Study
Massachusetts Institute of Technology
University of Notre Dame
Tufts College
Yale University

American Academy of Arts and Sciences
American Mathematical Society
Mathematical Association of America
National Research Council

Aetna Life Insurance Company
Allied Chemical and Dye Corporation
American Cyanamid Company
American Viscose Corporation
Bell Telephone Laboratories, Inc.
Connecticut General Life Insurance Company
Corning Glass Works
Allen B. DuMont Laboratories, Inc.
Eastman Kodak Company
Ford Motor Company, Engineering Division
General Electric Company
General Motors
Hercules Powder Company
International Business Machines Corporation
Jackson and Moreland
John Hancock Mutual Life Insurance Company
Johns-Manville Corporation
Leeds and Northrup Company
Lincoln National Life Insurance Company
Massachusetts Mutual Life Insurance Company
Midvale Company
Mutual Benefit Life Insurance Company
National Life Insurance Company
New England Mutual Life Insurance Company

Northwestern National Life Insurance Company
Northwestern Mutual Life Insurance Company
Philco Corporation
Provident Mutual Life Insurance Company
Prudential Insurance Company
Radio Corporation of America
Raytheon Manufacturing Company
Register and Tribune Company
Shell Oil Company
Standard Oil Development Company
State Mutual Life Assurance Company
The Travelers Insurance Company
The Union Central Life Insurance Company
United Fruit Company
United States Steel Corporation of Delaware
D. Van Nostrand Company
John Wiley and Company

R. W. Byerly
Donald Cody
J. L. Coolidge
T. J. Coolidge
Otto Dunkel
B. N. Everett
W. B. Ford
A. L. Lowell
A. L. Miller
Philip W. Pillsbury
G. E. Roosevelt
Philip Stockton

LIST OF DELEGATES

(The following is a list of delegates appointed by universities, colleges, etc.)

International Council of Scientific Unions
Emile Borel

International Union of Theoretical and Applied Mechanics
Johannes Martinus Burgers
J. C. Hunsaker
George Frederick James Temple

Foreign

ANGLO-EGYPTIAN SUDAN

Gordon Memorial College, Khartoum
T. H. Fallows

ARGENTINA

Academia Nacional de Ciencias Exactas, Físicas y Naturales de Buenos Aires
Alberto González Domínguez
Universidad de Cuyo, Facultad de Ciencias de la Educación
Gino Turrin
Unión Mathemática Argentina
Julio Rey Pastor
Universidad de Buenos Aires, Facultad de Ciencias Exactas, Físicas y Naturales
Alberto González Domínguez
Juan Carlos Vignaux
Universidad Nacional de Córdoba
Alfredo Tecera Martinez

AUSTRALIA

Royal Society of Western Australia
Ray Davis
University of Melbourne
T. M. Cherry
University of Sydney
Frederick Chong
Thomas G Room
University of Western Australia, Perth
Albert Laurence Blakers

AUSTRIA

Mathematische Gesellschaft in Wien
RUDOLF INZINGER

BELGIUM

Académie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique
FLORENT BUREAU
LUCIEN GODEAUX
THÉOPHILE H. J. LEPAGE
FERNAND SIMONART
Le Centre Belge de Recherches Mathématiques, Liège
FLORENT BUREAU
ALFRED J. J. H. ERRERA
LUCIEN GODEAUX
FERNAND SIMONART
Le Comité Belge de Mathématiques
LUCIEN GODEAUX
L'École Royale Militaire, Bruxelles
E. FRANCKX
La Faculté Polytechnique de Mons
GEORGES BOULANGER
La Société Belge de Statistique
PAUL PIERRE GILLIS
La Société Mathématique
THÉOPHILE H. J. LEPAGE
GEORGE LÉOPOLD PAPY
Université Catholique de Louvain
FERNAND SIMONART
Université de Liège
FLORENT BUREAU
LUCIEN GODEAUX
Université Libre de Bruxelles
ALFRED J. J. H. ERRERA
PAUL PIERRE GILLIS
GUY C. HIRSCH
THÉOPHILE H. J. LEPAGE
GEORGE LÉOPOLD PAPY
LÉON CHARLES VAN HOVE
Universiteit te Gent
PAUL DRUMAUX

BRAZIL

Academia Brasileiro de Ciênicas
IGNAZIO M. AZEVEDO DO AMARAL

Leopoldo Nachbin
Mauricio Matos Peixoto
Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro
Leopoldo Nachbin
Mauricio Matos Peixoto

BURMA

University of Rangoon
U Aung Hla

CHILE

Universidad Técnica Federico Santa Maria
Roberto Frucht

CHINA

National Taiwan University, Formosa
Cheng-Jung Hsu

COLOMBIA

Universidad del Cauca, Popayán
Carlos E. Obando V.

CUBA

Sociedad Cubana de Ciencias Físicas y Mathemáticas
Mario O. González
Lino G. Gutiérrez Novoa
Universidad de la Habana
Mario O. González
Lino G. Gutiérrez Novoa

DENMARK

Københavns Universitet
Harald Bohr
Børge Jessen
N. E. Nørlund
Kongelige Danske Videnskabernes Selskab
Harald Bohr
Werner Fenchel
Børge Jessen
N. E. Nørlund
Matematisk Forening i København
Asger Aaboe
Thøger Bang
Julie Marie Vinter Hansen

EGYPT

Institut d'Egypte, Cairo
Ismail Ratib Bey

ENGLAND

Birkbeck College, London
Albert Cyril Offord
British Association for the Advancement of Science
Daniel Pedoe
John Todd
Olga Taussky Todd
Dorothy Wrinch
Cambridge Philosophical Society
William Vallance Douglas Hodge
Robert Alexander Rankin
Imperial College of Science and Technology, London
George Frederick James Temple
King's College, London
Richard Rado
London Mathematical Society
Edward Foyle Collingwood
Hans Arnold Heilbronn
Maxwell Herman Alexander Newman
Albert Cyril Offord
Frank Smithies
Military College of Science, Shrivenham
C. J. Tranter
Ministry of Supply
Robert Arthur Fairthorne
C. K. Thornhill
Oxford University
Kathleen Sarginson
John Henry Constantine Whitehead
Queen Mary College, London
George Cunliffe McVittie
Royal Holloway College, Surrey
Mary Bradburn
Royal Society
Mary Lucy Cartwright
Harold Davenport
William Vallance Douglas Hodge
Louis Joel Mordell
George Frederick James Temple
Herbert W. Turnbull

University College, London
HAROLD DAVENPORT
University College, Southampton
EVAN TOM DAVIES
University College of the South-West of England, Exeter
WALTER KURT HAYMAN
University of Birmingham
FREEMAN J. DYSON
University of Bristol
HANS ARNOLD HEILBRONN
LESLIE HOWARTH
Cambridge University
MARY LUCY CARTWRIGHT
WILLIAM VALLANCE DOUGLAS HODGE
LOUIS JOEL MORDELL
ROBERT ALEXANDER RANKIN
SHAUN WYLIE
FRANK SMITHIES
University of Durham
KURT AUGUSTUS HIRSCH
FRANK F. BONSALL
WERNER WOLFGANG ROGOSINSKI
JEFFREY DENNIS WESTON
University of Liverpool
LOUIS ROSENHEAD
University of London
HAROLD DAVENPORT
GEORGE FREDERICK JAMES TEMPLE
Victoria University of Manchester
MAXWELL HERMAN ALEXANDER NEWMAN
Westfield College, London
DANIEL PEDOE

FINLAND

Åbo Akademi
GUSTAV ELFVING
The Academy of Finland, Helsinki
ROLF NEVANLINNA

FRANCE

Académie des Sciences
EMILE BOREL
JEAN CHAZY
ARNAUD DENJOY

Jacques Hadamard
Gaston Maurice Julia
Paul Montel

Association Française pour l'Avancement des Sciences
Albert Monjallon
Georges Valiron

Collège de France, Paris
Jean Leray
Szolem Mandelbrojt

École Polytechnique, Paris
Gaston Maurice Julia
Paul Lévy

Université de Bordeaux
Henri Milloux

Université Catholique de Lyon
Gabriel Vial

Université de Lille
Joseph Kampé de Fériet

GERMANY

Albert-Ludwigs-Universität Freiburg
Wilhelm Süss

Bayerische Akademie der Wissenschaften
Frank Löbell
Robert Sauer

Deutsche Mathematiker Vereinigung
Lothar Collatz
Erich Kamke
Konrad Knopp
Helmut Wielandt

Eberhard-Karls-Universität Tübingen
Erich Kamke
Hellmuth Kneser
Konrad Knopp

Georg-August-Universität Göttingen
Franz Rellich
Theodor Schneider

Sächsische Akademie der Wissenschaften
Ernst Hölder

Technische Hochschule, Darmstadt
Alwin Walther

Technische Hochschule München
Robert Sauer

GREECE

Academy of Athens
PANAGIOTIS ZERVOS
Polytechneion, Athens
CONSTANTINOS P. PAPAIOANNOU
ALEXANDER A. TRYPANIS
PHILON VASSILIOU
Société Mathématique de Grèce
MRS. ANTIGONI APOSTOLIDOU
MILTIADIS GERMANOS
NICOLAOS MICHALOPOULOS
CONSTANTINOS P. PAPAIOANNOU
NILOS SAKELLARIOU
SPYRIDION B. SARANTOPOULOS
PHILON VASSILIOU
University of Athens
NILOS SAKELLARIOU
SPYRIDION B. SARANTOPOULOS

INDIA

Calcutta Mathematical Society
RAJ CHANDRA BOSE
SAMARENDRA NATH ROY
N. R. SEN
BHOJ RAJ SETH
Indian Mathematical Society
SARVADAMAN CHOWLA
RAM BEHARI
A. NARASINGA RAO
R. VAIDYANATHASWAMI
THIRUKKANNAURAM VIJAYARAGHAVAN
National Institute of Sciences of India
RAJ CHANDRA BOSE
SARVADAMAN CHOWLA
PRASANTA CHANDRA MAHALANOBIS
SAMARENDRA NATH ROY
Tata Institute of Fundamental Research, Bombay
KOMARAVOLU CHANDRASEKHARAN
D. D. KOSAMBI
F. W. LEVI
University of Allahabad
H. K. SEN
University of Delhi
BHOJ RAJ SETH

University of Lucknow
A. N. Singh
University of Madras
S. Sivasankaranarayana Pillai

INDONESIA

Universiteit van Indonesie, Bandung
Dr. Goenarso
H. Th. M. Leeman

IRELAND

Dublin Institute for Advanced Studies
Albert Joseph McConnell
National University of Ireland
James McConnell
University of Dublin
Albert Joseph McConnell
The Royal Irish Academy
Albert Joseph McConnell

ISRAEL

Cercle Mathématique de Tel-Aviv
Binyamin A. Amirà
Michael Fekete
Hebrew Technical College, Haifa
Sydney Goldstein
The Hebrew University, Jerusalem
Michael Fekete
Jakob Levitzki
Giulio Racah
Menahem Schiffer
The Weizmann Institute of Science
Chaim L. Pekeris

ITALY

Accademia delle Scienze di Napoli
Mauro Picone
Accademia delle Scienze, Torino
Gino Fano
Accademia Nazionale dei Lincei
Enrico Bompiani
Gino Fano
Mauro Picone
Giovanni Sansone

Beniamino Segre
Francesco Severi
Antonio Signorini

Accademia Nazionale delle Scienze
Francesco Severi

Consiglio Nazionale delle Ricerche, Roma
Gaetano Fichera
Mauro Picone

Istituto Lombardo di Scienze e Lettere
Gino Fano
Giovanni Ricci

Istituto Nazionale di Alta Matematica
Francesco Severi

Politecnico di Milano
Luigi Amerio

Pontificia Academia Scientiarum
Francesco Severi

Unione Matematica Italiana
Enrico Bompiani
Giovanni Sansone
Beniamino Segre

Università degli Studi di Bologna
Lamberto Cesari
Dario Graffi
Beniamino Segre

Università di Milano
Giovanni Ricci

Università di Firenze
Giovanni Sansone

Università di Modena
Antonio Pignedoli

Università di Roma
Enrico Bompiani

Università di Torino
Alessandro M. Terracini
Francesco Tricomi

JAPAN

The Japan Academy
Zyoiti Suetuna

Kyôto University
Yasuo Akizuki

Mathematical Society of Japan
Yasuo Akizuki

Kenkichi Iwasawa
Shôkichi Iyanaga
Shizuo Kakutani
Kunihiko Kodaira
Sigeiti Moriguti
Tadasi Nakayama
Kenjiro Shôda
Zyoiti Suetuna
Kôsaku Yosida

Nagoya University
Tadasi Nakayama
Kôsaku Yosida

Osaka University
Kenjiro Shôda

Tokyo University
Kunihiko Kodaira

LEBANON

Centre d'Études Mathématiques, Beirut
Jean Léon Paul Chevrier

MALAYA

University of Malaya, Singapore
John Caister Cooke
Alexander Oppenheim

MEXICO

Sociedad Matemática Mexicana
Alberto Barajas
Javier Barros Sierra
Carlos Graef Fernández
Alfonso Nápoles Gándara
Francisco Zubieta Russi

Universidad de Guadalajara
Edmundo Ponce Adame
Miguel Rodríguez Puga

Universidad Nacional Autónoma de México, Mexico City (Instituto de Físicas)
Antonio Romero Juarez

Universidad Nacional Autónoma de México, Mexico City (Instituto de Matemáticas)
Juan Morcos Salman
Alfonso Nápoles Gándara
Félix Recillas Juárez

Enrique Valle Flores
Roberto Vázquez García

NETHERLANDS

Koninklijke Nederlandsche Akademie van Wetenschappen
L. E. J. Brouwer
Johannes Martinus Burgers
Johannes Gualtherus van der Corput
J. A. Schouten
Rijksuniversiteit te Groningen
Johan Cornelis Hendrik Gerretsen
Jacob Ridder
Rijksuniversiteit te Leiden
Hendrik D. Kloosterman
Rijksuniversiteit te Utrecht
Jan Popken
Vrije Universiteit, Amsterdam
J. F. Koksma
Wiskundig Genootschap te Amsterdam
Oene Bottema
Johannes Gualtherus van der Corput
Hendrik D. Kloosterman
J. F. Koksma
J. A. Schouten

NEW ZEALAND

University of New Zealand, Wellington
Sainsbury Lee Strack

NORTHERN IRELAND (United Kingdom)

Queen's University of Belfast
Harry Raymond Pitt

NORWAY

Norges Tekniske Høgskole, Trondheim
Sigmund Selberg
Norske Videnskaps Akademi i Oslo
Thoralf Skolem
Norwegian Mathematical Society
Thoralf Skolem
Universitetet i Oslo
Thoralf Skolem
Universitetet i Bergen
Oddvar Bjørgum
Wilhelm Ljunggren

PANAMA

Universidad de Panamá, Panama City
Nariño Rivera C.

PERU

Escuela Nacional de Ingenieros, Lima
Pedro Abel Labarthe
José Tola Pasquel
Universidad Nacional Mayor de San Marcos
José Tola Pasquel

PHILIPPINES

Silliman University, Dumaguete
Juan B. Escarda
University of Manila
Alfonso G. Garcia
Faustino R. Lozada

POLAND

Polskie Towarzystwo Matematyczne
Andrzej Mostowski

PORTUGAL

Junta de Investigação Matemática
Ruy Luis Gomes
Sociedade Portuguesa de Matemática
Antonio Gião
Manuel Zaluar Nunes

SCOTLAND

Royal Society of Edinburgh
Edward Thomas Copson
University College, Dundee
Edward Thomas Copson
University of Glasgow
Thomas Smith Graham
University of St. Andrews, Dundee
Herbert W. Turnbull
Edward Thomas Copson

SOUTH AFRICA

University of Capetown
Laurence Chisholm Young

SPAIN

Instituto Nacional de Estadística
José Antonio de Artigas Sanz
Enrique S. Cansado Maceda
Real Sociedad Matemática Española
Thomas Rodriguez Bachiller

SWEDEN

Chalmers Tekniska Högskola, Göteborg
Harald Bergström
Henry Wallman
Kungl. Tekniska Högskola, Stockholm
Folke K. G. Odqvist
Kungl. Vetenskapsakademien
Arne Beurling
Kungl. Vetenskaps- och Vitterhets-Samhället i Göteborg
Henry Wallman
Kungl. Vetenskapssocieteten i Uppsala
Arne Beurling
Matematiska Föreningen, Uppsala
Göran Borg
Lennart Carleson
Bo Kjellberg
Stockholms Högskola
Harald Cramér
University of Uppsala
Arne Beurling

SWITZERLAND

Eidgenössische Technische Hochschule, Zürich
Beno Eckmann
Heinz Hopf
Albert Pfluger
Walter Saxer
Swiss Mathematical Society
Charles Blanc
Johann Jakob Burckhardt
Félix Fiala
Max Gut
Albert Pfluger
Université de Lausanne
Charles Blanc

Université de Neuchâtel
Félix Fiala
Universität Zürich
Rolf Nevanlinna

TURKEY

Ankara Universitesi
Hans Ludwig Habmurger
Istanbul Universitesi
Cahit Arf
Kerim Erim
Nazim Terzioğlu

URUGUAY

Asociación Uruguaya para el Progreso de la Ciencia
Rafael Laguardia
Universidad de la República, Facultad de Humanidades y Ciencias
Rafael Laguardia
Universidad de la República, Facultad de Ingeniería (Instituto de Matemática y Estadística)
Rafael Laguardia

VENEZUELA

Academia de Ciencias Físicas, Matemáticas y Naturales
Francisco José Duarte

YUGOSLAVIA

Council of the Academies of the Federative People's Republic of Yugoslavia
Jovan Karamata
Univerza v Ljubljana
Ivan Vidav
University of Zagreb
George Kurepa

Canada and the United States

CANADA

Acadia University, Wolfville
Ernest W. Sheldon
Laval University, Quebec
Adrien Pouliot
Loyola College, Montreal
Hugh John MacPhee
R. Eric O'Connor

McGill University, Montreal
Herbert Tate
W. Lloyd G. Williams
McMaster University, Hamilton
F. R. Britton
Alfred E. Johns
William James McCallion
Memorial University of Newfoundland, St. John's
William John Blundon
Albert George Hatcher
National Research Council, Ottawa
H. H. Clayton
Ta-You Wu
Royal Society of Canada
Richard Brauer
W. Lloyd G. Williams
Saint Francis Xavier University, Antigonish
Francis I. Ginivan
University of British Columbia, Vancouver
Ralph D. James
S. A. Jennings
University of Manitoba, Winnipeg
William H. McEwen
Université de Montréal
Maurice L'Abbé
University of New Brunswick, Fredericton
Fritz Rothberger
Ralph A. Staal
University of Ottawa
Rev. Gérard Cloutier
Rev. Fr. Thibaud Gagné
University of Saskatchewan, Saskatoon
Herman Harvey Ferns
Peter Scherk
University of Toronto
Samuel Beatty
A. R. Gordon
University of Western Ontario, London
Gordon R. Magee
H. R. Kingston

ALABAMA

Howard College
George Wellman Hess

Talladega College
Warren Hill Brothers, Jr.
University of Alabama
Herbert S. Thurston

ARIZONA

University of Arizona
Edwin J. Purcell

ARKANSAS

University of Arkansas
Bernard Hugo Gundlach
Stacy L. Hull
Clay L. Perry, Jr.
Davis Payne Richardson

CALIFORNIA

California Institute of Technology
Henri Frederic Bohnenblust
Arthur Erdélyi
Morgan Ward
Fresno State College
Frank Ray Morris
Pomona College
Chester George Jaeger
Elmer Beaumont Tolsted
Stanford University
George Pólya
Gabor Szegö
University of Southern California
Herbert Busemann
Whittier College
Henry Randolph Pyle

COLORADO

Colorado School of Mines
William James Berry
Ivan L. Hebel
University of Colorado
Albert B. Farnell
Burton Jones
University of Denver
Kenneth L. Noble
Homer C. Peterson

CONNECTICUT

Connecticut College
Julia Wells Bower
William Eugene Ferguson
Trinity College
Harold Laird Dorwart
University of Connecticut
William Fitch Cheney
Charles Hill Wallace Sedgewick
Wesleyan University
Burton H. Camp
Malcolm Cecil Foster
Yale University
Gustav Arnold Hedlund
Einar Hille

DELAWARE

University of Delaware
Carl John Rees
George Cuthbert Webber

DISTRICT OF COLUMBIA

Catholic University of America
Otto Joseph Ramler
George Washington University
Francis Edgar Johnston
David Nelson
Georgetown University
William Herbert Schweder
Frederick Wyatt Sohon
Howard University
Jeremiah Certaine
William S. Claytor

FLORIDA

Florida State University
Dwight B. Goodner
Thomas Leonard Wade
University of Florida
Franklin Wesley Kokomoor
Thomas Marshall Simpson

GEORGIA

Georgia Institute of Technology
Herman Kyle Fulmer

University of Georgia
TOMLINSON FORT
GERALD BOONE HUFF

HAWAII

University of Hawaii, Honolulu
HUGH E. STELSON

ILLINOIS

Illinois Institute of Technology
LESTER R. FORD
GORDON PALL

Illinois Wesleyan University
NORMAN A. GOLDSMITH

Knox College
ROTHWELL STEPHENS

North Central College
MARY ANICE SEYBOLD

Rosary College
SISTER M. PHILIP STEELE

Southern Illinois University
AMOS HALE BLACK
W. C. MCDANIEL

University of Chicago
WALTER BARTKY
SAUNDERS MACLANE

University of Illinois
STEWART SCOTT CAIRNS
JOSEPH LEO DOOB

INDIANA

Ball State Teachers College
CHARLES BRUMFIEL
PRENTICE DEARING EDWARDS
LEVI S. SHIVELY

Indiana University
EBERHARD HOPF
JOHN WILLIAM THEODORE YOUNGS

Purdue University
WILLIAM LEAKE AYRES
RALPH HULL

Rose Polytechnic Institute
THEODORE P. PALMER

University of Notre Dame
KY FAN
JOSEPH P. LASALLE

Wabash College
JOSEPH CRAWFORD POLLEY

IOWA

Grinnell College
H. G. Apostle
Raymond Benedict McClenon
Iowa State College of Agriculture and Mechanic Arts
Dio Lewis Holl
State University of Iowa
Edward Wilson Chittenden

KANSAS

Kansas State College
Rodney Whittemore Babcock
Sidney Thomas Parker
University of Kansas
G. Baley Price
Guy Watson Smith
Ellis Bagley Stouffer

KENTUCKY

University of Louisville
Walter Lee Moore
Guy Stevenson

LOUISIANA

Tulane University of Louisiana
William Larkin Duren, Jr.
Marie Johanna Weiss

MAINE

Bowdoin College
Edward Sanford Hammond
University of Maine
Esther Comegys
Spofford Harris Kimball

MARYLAND

Goucher College
Verena Haefeli Dyson
Marian Marsh Torrey
Johns Hopkins University
Wei-Liang Chow
Daniel C. Lewis
United States Naval Academy
Richard Pennington Bailey
Levi Thomas Wilson

United States Naval Postgraduate School
Charles C. Torrance
A. Boyd Mewborn
University of Maryland
Monroe Harnish Martin
Alexander Weinstein

MASSACHUSETTS

Amherst College
Atherton Hall Sprague
Boston College
Rev. Anthony Joseph Eiardi
George Anthony O'Donnell
Boston University
Elmer Beneken Mode
Andrew Sobczyk
Clark University
Charles Theodore Bumer
John S. Stubbe
College of the Holy Cross
Vincent Owen McBrien
Thomas J. Smith
Harvard University
Percy Williams Bridgman
Paul H. Buck
Massachusetts Institute of Technology
William Ted Martin
Norbert Wiener
Mount Holyoke College
Marie Litzinger
Simmons College
Leland David Hemenway
Smith College
Richard Edward Johnson
Neal Henry McCoy
University of Massachusetts
Allen E. Anderson
Harold D. Boutelle
Wellesley College
Helen Gertrude Russell
Marion Elizabeth Stark
Wheaton College
Hilda Geiringer

Williams College
DONALD EVERETT RICHMOND
Worcester Polytechnic Institute
RAYMOND K. MORLEY

MICHIGAN

Hope College
ALBERT EUGENE LAMPEN
EVERETT WILMERS
Michigan State College of Agriculture and Applied Science
JAMES SUTHERLAND FRAME
VERNON GUY GROVE
University of Michigan
GEORGE YURI RAINICH
RAYMOND LOUIS WILDER
Wayne University
HARVEY COHN
MAX CORAL

MINNESOTA

Carleton College
KENNETH OWNSWORTH MAY
College of St. Catherine
SISTER ADA MARIE BOEHM
SISTER SERAPHIM GIBBONS
College of St. Teresa
DOROTHY SCHRADER
College of St. Thomas
LAURENCE EARLE BUSH
FRANKLIN SMITH
Macalester College
EZRA JOHN CAMP
University of Minnesota
RAYMOND WOODARD BRINK
ROBERT HORTON CAMERON
WILLIAM LEROY HART

MISSISSIPPI

University of Mississippi
THOMAS ALTON BICKERSTAFF

MISSOURI

St. Louis University
JOHN D. ELDER

Francis Regan
Washington University
Walter Leighton
William H. Roever

MONTANA

Montana State College
Henry A. Antosiewicz

NEBRASKA

University of Nebraska
Miguel Antonio Basoco
William Grenfell Leavitt

NEVADA

University of Nevada
Edward Maurice Beesley

NEW HAMPSHIRE

Dartmouth College
Fred William Perkins
Robin Robinson
University of New Hampshire
Dennis B. Ames
William L. Kichline

NEW JERSEY

College of St. Elizabeth
Sister Dorothy Clare Cannon
Sister Anna Concilio O'Neill
Institute for Advanced Study
Deane Montgomery
Marston Morse
New Jersey State Teachers College, Upper Montclair
Virgil Sampson Mallory
Princeton University
Norman Earl Steenrod
Albert William Tucker
Rutgers University
C. A. Nelson
Malcolm Slingsby Robertson
Stevens Institute of Technology
Nicholas J. Rose

NEW MEXICO

University of New Mexico
Morris S. Hendrickson
Lincoln LaPaz

NEW YORK

Alfred University
Charles Everett Rhodes
Colgate University
Carl Wallace Munshower
College of New Rochelle
Bernard Cioffari
Cornell University
Wallie Abraham Hurwitz
Robert John Walker
Elmira College
Mary Clegg Suffa
Fordham University
Rev. John Joseph Lynch
Domhnall Arthur Steele
Hobart and William Smith Colleges
Robert L. Beinert
Walter Hetherington Durfee
Hunter College
Jewell Hughes Bushey
New York State College for Teachers, Albany
Ralph Alexander Beaver
Caroline Avery Lester
New York University
Richard Courant
Kurt O. Friedrichs
James J. Stoker, Jr.
Polytechnic Institute of Brooklyn
Ronald Martin Foster
Queens College
Thomas Freeman Cope
Arthur Sard
Syracuse University
Donald E. Kibbey
Charles Loewner
Union College
Orin James Farrell
David Sherman Morse

United States Military Academy
William Weston Bessell
University of Buffalo
Harry M. Gehman
Harriet Frances Montague
Vassar College
Frances E. Baker
Janet McDonald

NORTH CAROLINA

Duke University
Francis G. Dressel
John Jay Gergen
North Carolina College at Durham
Marjorie Lee Browne
University of North Carolina
Harold Hotelling
William M. Whyburn
Woman's College of the University of North Carolina
Helen Barton
Anne Louise Lewis

NORTH DAKOTA

University of North Dakota
Sewell L. Mason

OHIO

Baldwin-Wallace College
Dean L. Robb
Case Institute of Technology
Louis J. Green
Robert Fross Rinehart
College of Wooster
Melcher Prince Fobes
Charles Owen Williamson
Denison University
Chosaburo Kato
Marion Wetzel
Hiram College
Elbert Howard Clarke
Kenyon College
William Darragh Berg
Otton Martin Nikodým
Edward Silverman
William Reagle Transue

Marietta College
Theodore Bennett
Miami University
Ward C. Sangren
Mount Union College
William Glenn Clark
Frances Freese
Oberlin College
Elbridge Putnam Vance
Chester Yeaton
Ohio State University
Marshall Hall
Tibor Radó
Ohio University
Victor Goedicke
George W. Starcher
Ohio Wesleyan University
Sidney Archie Rowland
University of Akron
Samuel Selby
University of Cincinnati
Louis Brand
Western Reserve University
John Rogers Musselman
Webster Godman Simon
Wittenberg College
Raymond L. Krueger

OKLAHOMA

Oklahoma Agricultural and Mechanical College
Nachman Aronszajn
Ainsley Herbert Diamond
University of Oklahoma
Casper Goffman
Charles E. Springer

OREGON

Oregon State College
William Edmund Milne

PENNSYLVANIA

Allegheny College
Frederick Henry Steen
Eric Avern Sturley

Bryn Mawr College
John Corning Oxtoby
Dickinson College
Frank Ayres, Jr.
Roy Raymond Kuebler
Franklin and Marshall College
Donald W. Western
Haverford College
Carl B. Allendoerfer
Cletus Odia Oakley
Muhlenberg College
Luther J. Deck
Everett W. Holt
Truman Lester Koehler
Thorman Archer Nelson
Pennsylvania College for Women
Helen Calkins
Pennsylvania State College
Haskell Brooks Curry
Orrin Frink
Seton Hill College
Sister M. Thaddeus McManama
Sister Marie Gertrude McNeil
Swarthmore College
Heinrich Wilhelm Brinkmann
Arnold Dresden
University of Pennsylvania
John Robert Kline
Hans Adolph Rademacher
Wilson College
Roberta Frances Johnson
Joan Elizabeth Robinson

RHODE ISLAND

Brown University
Raymond Clare Archibald
Otto Eduard Neugebauer
Rhode Island State College
Edward Monroe Joseph Pease
Jacob Reiff Kob Stauffer

SOUTH CAROLINA

Furman University
Reese Croxton Blackwell

TENNESSEE

University of Chattanooga
Ruth Perry
University of Tennessee
Wallace Givens
Walter S. Snyder
Vanderbilt University
Moffatt Grier Boyce
John A. Hyden
E. Baylis Shanks

TEXAS

Rice Institute
Hubert Evelyn Bray
Floyd Edward Ulrich
Texas Technological College
Lida B. May

VERMONT

University of Vermont and State Agricultural College
James Atkins Bullard
Jules A. Larrivee

VIRGINIA

College of William and Mary
Harold R. Phalen
Mary Baldwin College
Mildred Ellen Taylor
Randolph-Macon Woman's College
M. Gweneth Humphreys
Gillie A. Larew
Sweet Briar College
Mary Ann Lee
University of Richmond
Charles H. Wheeler, III
University of Virginia
Edward J. McShane
Gordon T. Whyburn
Washington and Lee University
Charles Wiley Williams

WASHINGTON

State College of Washington
Morris Samuel Knebelman
John Russell Vatnsdal

University of Washington
LEE HORACE MCFARLAN
ROY MARTIN WINGER

WEST VIRGINIA

Bethany College
BRADFORD TYE
West Virginia University
CLARENCE N. REYNOLDS, JR.
MARVIN LEWIS VEST

WISCONSIN

Marquette University
HARVEY P. PETTIT
L. V. TORALBALLA
University of Wisconsin
CYRUS C. MACDUFFEE

ACOUSTICAL SOCIETY OF AMERICA

R. CLARK JONES
PHILIP M. MORSE

AMERICAN ACADEMY OF ARTS AND SCIENCES

CLARENCE RAYMOND ADAMS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

JOSEPH LEONARD WALSH

AMERICAN ASTRONOMICAL SOCIETY

SUBRAHMANYAN CHANDRASEKHAR
ZDENĚK KOPAL

AMERICAN INSTITUTE OF ELECTRICAL ENGINEERS

MURRAY F. GARDNER
MICHEL G. MALTI

AMERICAN INSTITUTE OF PHYSICS, INC.

HOWARD PERCY ROBERTSON
JOHN C. SLATER

AMERICAN PHILOSOPHICAL SOCIETY

LUTHER P. EISENHART

AMERICAN PHYSICAL SOCIETY

Howard Percy Robertson
John C. Slater

AMERICAN PSYCHOLOGICAL ASSOCIATION

Philip J. Rulon

AMERICAN SOCIETY OF CIVIL ENGINEERS

Albert Haertlein

AMERICAN SOCIETY OF MECHANICAL ENGINEERS

Gleason H. MacCullough
Hillel Poritsky

EASTERN NORTH AMERICAN REGION BIOMETRIC SOCIETY

John W. Tukey
Charles P. Winsor

ECONOMETRIC SOCIETY

Maurice Fréchet
Arthur Smithies

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

Lawrence M. Graves
Rudolph Ernest Langer

NATIONAL ACADEMY OF SCIENCES

Griffith C. Evans
Marston Morse
Edwin B. Wilson
Oscar Zariski

NATIONAL RESEARCH COUNCIL

Stewart Scott Cairns
Theophil Henry Hildebrandt

LIST OF MEMBERS

A

AABOE, Asger H., Adjunkt, Cand. Mag.
(Statsskolen, Birkerød, Denmark)
Mrs. Aaboe
ABBOTT, James Crawford, Prof.
(United States Naval Academy, Md., U. S. A.)
ABELLANAS, Pedro, Prof.
(University of Madrid, Madrid, Spain)
ADAMS, Charles William, Research Engineer
(Massachusetts Institute of Technology, Mass., U. S. A.)
ADAMS, Clarence Raymond, Prof.
(Brown University, R. I., U. S. A.)
Mrs. Adams
ADAMS, George, Lecturer
(Gothern Science Foundation, Worcestershire, England)
ADAMSON, Iain Thomas Arthur Carpenter
(Princeton University, N. J., U. S. A.)
ADKISSON, Virgil William, Prof.
(University of Arkansas, Ark., U. S. A.)
ADNEY, J. E., Jr., Assistant
(Ohio State University, Ohio, U. S. A.)
ADSHEAD, John Geoffrey, Prof.
(Dalhousie University, N. S., Canada)
AGAZARIAN, Karekin
(Boston University, Mass., U. S. A.)
AGMON, Shmuel, Lecturer
(Rice Institute, Tex., U. S. A.)
AGNEW, Ralph Palmer, Prof.
(Cornell University, N. Y., U. S. A.)
Mrs. Agnew
Mr. Palmer Agnew
AHEART, Andrew Norwood, Instr.
(West Virginia State College, W. Va., U. S. A.)
AHLFORS, Lars Valerian, Prof.
(Harvard University, Mass., U. S. A.)
Mrs. Ahlfors
Miss Caroline Ahlfors
Miss Cynthia Ahlfors
Miss Vanessa Ahlfors
AISSEN, Michael Israel, Instr.
(University of Pennsylvania, Pa., U. S. A.)
AKIZUKI, Yasuo, Prof.
(Kyôto University, Kyôto, Japan)
AKUTOWICZ, Edwin James, Prof.
(Pennsylvania State College, Pa., U. S. A.)
ALAOGLU, Leonidas, Operations Analyst
(United States Air Force, D. C., U. S. A.)
Mrs. Alaoglu
Master Angelo W. Alaoglu
Miss Ann E. Alaoglu
ALBERT, A. Adrian, Prof.
(University of Chicago, Ill., U. S. A.)
Mrs. Albert
ALBERT, Richmond G., Instr.
(Brown University, R. I., U. S. A.)
ALEXANDER, Howard Wright, Prof.
(Adrian College, Mich., U. S. A.)
ALGER, Philip Langdon, Consulting Engineer
(General Electric Company, N. Y., U. S. A.)
ALLEGRI, Linda
(Hunter College High School, N. Y., U. S. A.)
ALLEN, Edward S., Prof.
(Iowa State College, Iowa, U. S. A.)
ALLEN, Edwin Brown, Prof.
(Rensselaer Polytechnic Institute, N. Y., U. S. A.)
Mrs. Allen
ALLEN, William Robert, Instr.
(University of Illinois, Ill., U. S. A.)
Mrs. Allen
ALLENDOERFER, Carl B., Prof.
(Haverford College, Pa., U. S. A.)
Mrs. Allendoerfer
VAN ALSTYNE, John Pruyn, Instr.
(Hamilton College, N. Y., U. S. A.)
ALT, Franz L., Dr.
(National Bureau of Standards, D. C., U. S. A.)
AMERIO, Luigi, Prof.
(Politecnico di Milano, Università di Milano, Milan, Italy)
AMES, Dennis Burley, Prof.
(University of New Hampshire, N. H., U. S. A.)
AMIRÀ, Binyamin A., Prof.
(Hebrew University, Jerusalem, Israel)
ANDERSON, Allan George, Instr.
(Oberlin College, Ohio, U. S. A.)

ANDERSON, Allen Emil, Prof.
(University of Massachusetts, Mass., U. S. A.)
ANDERSON, Richard Davis, Prof.
(University of Pennsylvania, Pa., U. S. A.)
ANDERSON, Theodore Wilbur, Jr., Prof.
(Columbia University, N. Y., U. S. A.)
ANDREOTTI, Aldo, Prof.
(Università di Roma, Rome, Italy)
ANKENY, Nesmith Cornett
(Princeton University, N. J., U. S A.)
Mrs. Ankeny
ANTHONY, Maurice Lee, Associate Engineer
(Armour Research Foundation, Illinois Institute of Technology, Ill., U. S. A.)
ANTOSIEWICZ, Henry Albert, Prof.
(Montana State College, Mont., U. S. A.)
Mrs. Antosiewicz
APOSTOLIDOU, Mrs. Antigoni
(Greek Mathematical Society, Athens, Greece)
ARCHIBALD, Raymond Clare, Prof.
(Brown University, R. I., U. S. A.)
ARENS, Richard F., Prof.
(University of California, Los Angeles, Calif., U. S. A.)
Mrs. Arens
ARF, Cahit, Prof.
(University of Istanbul, Istanbul, Turkey)
ARMSTRONG, Beulah May, Prof.
(University of Illinois, Ill., U. S. A.)
ARNOLD, B. H., Prof.
(Oregon State College, Ore., U. S. A.)
Mrs. Arnold
ARNOLD, Kenneth James, Prof.
(University of Wisconsin, Wis., U. S. A.)
AROIAN, Leo Avedis, Research Physicist
(Howard Hughes Aircraft Company, Calif., U. S. A.)
ARONSZAJN, Nachman, Prof.
(Oklahoma Agricultural and Mechanical College, Okla., U. S. A.)
ARSOVE, Maynard Goodwin, Dr.
(Brown University, R. I., U. S. A.)
ARSOVE, Melvin W., Physicist
(Harvard University, Mass., U. S. A.)
ARTIAGA, Santiago, Dr.
(University of Michigan, Mich., U. S. A.)
DE ARTIGAS, José Antonio, Prof.
(Instituto de Ampliación de Estudios e Investigación Industrial, Madrid, Spain)
Dr. Maria Carmen de Artigas
ARTIN, Emil, Prof.
(Princeton University, N. J., U. S. A.)
Mrs. Artin
Mr. Michael Artin
ASOFSKY, Samuel, Statistician
(National Jewish Welfare Board, N. Y., U. S. A.)
ATCHISON, William Franklin, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Atchison
Master Allen Atchison
Master Glen Atchison
ATKINSON, F. V., Prof.
(University College, Ibadan, Nigeria, West Africa)
AUDE, Herman T. R., Prof.
(Colgate University, N. Y., U. S. A.)
Mrs. Aude
AULUCK, Faqir Chand, Reader in Physics
(University of Delhi, Delhi, India)
AURORA, Silvio
(Columbia University, N. Y., U. S. A.)
AYER, Miriam C., Prof.
(Wellesley College, Mass., U. S. A.)
AYOUB, Raymond G., Instr.
(Harvard University, Mass., U. S. A.)
Mrs. Ayoub
Mrs. Anne Williams
AYRES, Frank, Jr., Prof.
(Dickinson College, Pa., U. S. A.)
Mrs. Ayres
AYRES, William Leake, Dean, School of Science
(Purdue University, Ind., U. S. A.)
AYYANGAR, Krishnaswami Attipat Asuri, Reader in Statistics
(Andhra University, Waltair, India)

B

BABCOCK, Rodney Whittemore, Dean, School of Arts and Sciences
(Kansas State College, Kan., U. S. A.)
BACHILLER, T. R., Prof.
("Jorge Juan" Institute of Mathematics, Madrid, Spain)
Mrs. Bachiller

BACON, Harold Maile, Prof.
(Stanford University, Calif., U. S. A.)
BAER, Reinhold, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Baer
BAHADUR, R. R., Instr.
(University of Chicago, Ill., U. S. A.)
Mrs. Bahadur
BAIADA, Emilio
(Università di Pisa, Pisa, Italy)
BAILEY, Richard Pennington, Prof.
(United States Naval Academy, Md., U. S. A.)
BAKER, Frances Ellen, Prof.
(Vassar College, N. Y., U. S. A.)
BALLIEU, Robert, Prof.
(Université de Louvain, Louvain, Belgium)
BALLOU, Donald Henry, Prof.
(Middlebury College, Vt., U. S. A.)
BANG, Thøger, Dr.
(Universitetets Matematiske Institut, Copenhagen, Denmark)
BARAJAS, Alberto, Prof.
(University of Mexico, Mexico, D. F. Mexico)
BARANKIN, Edward William, Prof.
(University of California, Calif., U. S. A.)
BARGMANN, Valentine, Prof.
(Princeton University, N. J., U. S. A.)
Mrs. Bargmann
BARLAZ, Joshua, Prof.
(Rutgers University, N. J., U. S. A.)
BARNARD, Raymond W., Prof.
(University of Chicago, Ill., U. S. A.)
Mrs. Barnard
BARNETT, Isaac Albert, Prof.
(University of Cincinnati, Ohio, U. S. A.)
BARNETT, Joseph, Jr., Prof.
(Oklahoma Agricultural and Mechanical College, Okla., U. S. A.)
BARRAR, Richard Blaine
(University of Michigan, Mich., U. S. A.)
BARSOTTI, Iacopo, Prof.
(University of Pittsburgh, Pa., U. S. A.)
BARTELS, Robert C. F., Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Bartels
Mr. Richard H. Bartels
BARTLE, Robert Gardner
(University of Chicago, Ill., U. S. A.)
BARTLETT, Thomas J., Prof.
(University of Denver, Colo., U. S. A.)
BARTON, Helen, Prof.
(Woman's College of the University of North Carolina, N. C., U. S. A.)
BATCHELDER, Paul Mason, Prof.
(University of Texas, Tex., U. S. A.)
BATEMAN, Felice Davidson, Dr.
(University of Illinois, Ill., U. S. A.)
BATEMAN, Paul Trevier, Prof.
(University of Illinois, Ill., U. S. A.)
BATEN, William Dowell, Prof.
(Michigan State College, Mich., U. S. A.)
BATES, Grace E., Prof.
(Mount Holyoke College, Mass., U. S. A.)
BATTIN, Richard Horace, Instr.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Battin
BAUSCH, Augustus Francis, Instr.
(University of Chicago, Ill., U. S. A.)
BEARMAN, Jacob E., Prof.
(University of Minnesota, Minn., U. S. A.)
BEASLEY, S. Louise, Prof.
(Lindenwood College for Women, Mo., U. S. A.)
BEATLEY, Ralph, Prof.
(Harvard University, Mass., U. S. A.)
BEATTY, Samuel, Prof.
(University of Toronto, Ont., Canada)
Mrs. Beatty
BEAVER, Ralph Alexander, Prof.
(New York State College for Teachers at Albany, N. Y., U. S. A.)
BECKENBACH, Edwin F., Prof.
(University of California, Los Angeles, Calif., U. S. A.)
BECKER, Harold William, Instr.
(Electronic Radio-Television Institute, Neb., U. S. A.)
BECKER, Richard Logan
(Harvard University, Mass., U. S. A.)
BECKSTROM, Agnes Josephine, Prof.
(State Teachers College, N. D., U. S. A.)
BEER, Ferdinand Pierre, Prof.
(Lehigh University, Pa., U. S. A.)
BEESLEY, Edward Maurice, Prof.
(University of Nevada, Nev., U. S. A.)
BEGLE, Edward Griffith, Prof.
(Yale University, Conn., U. S. A.)
BEHNKE, Heinrich A. L., Prof.
(Universität Münster, Münster, Germany)
BEIMAN, Henry
(University of Wisconsin, Wis., U. S. A.)

BELL, John Clarence, Research Mathematician
(Battelle Memorial Institute, Ohio, U. S. A.)
BELL, Philip Osborne, Prof.
(University of Kansas, Kan., U. S. A.)
BELL, Raymond Frank, Prof.
(Eastern Washington College of Education, Wash., U. S. A.)
BELLAMY, Ben C., Civil Engineer
(Bellamy and Sons, Wyo., U. S. A.)
BELLIN, Albert I., Prof.
(Harvard University, Mass., U. S. A.)
BENAC, Theodore Joseph, Prof.
(United States Naval Academy, Md., U. S. A.)
BENDER, John
(Rutgers University, N. J., U. S. A.)
BENEDICT, Donald Lee, Dr.
(Stanford Research Institute, Calif., U. S. A.)
BENNETT, Albert Arnold, Prof.
(Brown University, R. I., U. S. A.)
BENNETT, Theodore, Prof.
(Marietta College, Ohio, U. S. A.)
Mrs. Bennett
BERG, Paul W., Research Assistant
(New York University, N. Y., U. S. A.)
BERG, William Darragh, Prof.
(Kenyon College, Ohio, U. S. A.)
BERGMAN, Stefan, Dr.
(Harvard University, Mass., U. S. A.)
Mrs. Bergman
BERGMANN, Peter G., Prof.
(Syracuse University; Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
BERGSTRÖM, Harald Bror Evert, Prof.
(Chalmers' Tekniska Högskola, Gothenburg, Sweden)
BERKER, Ratip, Prof.
(Technical University of Istanbul, Istanbul, Turkey)
BERNHARD, Herbert Ashley, Electronic Engineer
(Evans Signal Corps Laboratory, N. J., U. S. A.)
BERNHART, Arthur, Prof.
(University of Oklahoma, Okla., U. S. A.)
Mrs. Bernhart
Mr. Frank Bernhart
BERNSTEIN, Dorothy L., Prof.
(University of Rochester, N. Y., U. S. A.)
BERNSTEIN, Jeremy
(Harvard University, Mass., U. S. A.)
BERRY, William James, Actuary
(Metropolitan Life Insurance Company, N. Y., U. S. A.)
BERS, Lipman, Prof.
(Syracuse University, N. Y., U. S. A.)
Mrs. Bers
Miss Ruth Bers
Master Victor Bers
BESSELL, William Weston, Prof.
(United States Military Academy, N. Y., U. S. A.)
Mrs. Bessell
Miss Constance Bessell
BETH, Richard A., Prof.
(Western Reserve University, Ohio, U. S. A.)
BETTS, Barbara Bassett, Mathematics Editor
(D. C. Heath and Company, Mass., U. S. A.)
BETZ, Ebon Elbert, Prof.
(United States Naval Academy, Md., U. S. A.)
BETZ, William, Specialist in Mathematics
(Public School System of Rochester, N. Y., U. S. A.)
BEURLING, Arne, Prof.
(University of Uppsala, Uppsala, Sweden)
BEYER, William A.
(Pennsylvania State College, Pa., U. S. A.)
BIESELE, Ferdinand Charles, Prof.
(University of Utah, Utah, U. S. A.)
BILLINGS, Bruce, Director of Research
(Baird Associates, Inc., Mass., U. S. A.)
BING, Kurt
(Harvard University, Mass., U. S. A.)
BING, R. H., Prof.
(University of Wisconsin, Wis., U. S. A.)
BIRKHOFF, Garrett, Prof.
(Harvard University, Mass., U. S. A.)
Mrs. Birkhoff
*BIRKHOFF, Margaret G. (Associate Member)
(Cambridge, Mass., U. S. A.)
BJØRGUM, Oddvar, Lecturer
(Universitetet i Bergen, Bergen, Norway)
BLACK, Amos Hale, Prof.
(Southern Illinois University, Ill., U. S. A.)
Mrs. Black

---

* Special Guest of Organizing Committee.

BLACK, Harold Lyle, Prof.
(Pennsylvania State College, Pa., U. S. A.)
BLACKETT, Donald Watson, Instr.
(Princeton University, N. J., U. S. A.)
BLACKWELL, Reece Croxton, Prof.
(Furman University, S. C., U. S. A.)
Mrs. Blackwell
BLAKE, Archie, Consultant in Applied Mathematics
(Mechanical Research Corp., Ill., U. S. A.)
BLAKERS, Albert Laurence, Prof.
(Lehigh University, Pa., U. S. A.)
T. Blakers
BLANC, Charles, Prof.
(Université de Lausanne, Lausanne, Switzerland)
BLANCH, Gertrude, Dr.
(National Bureau of Standards, Calif., U. S. A.)
BLAU, Julian H., Prof.
(Pennsylvania State College, Pa., U. S. A.)
Mrs. Blau
BLOCK, Henry David, Prof.
(Iowa State College, Iowa, U. S. A.)
BLOCK, I. Edward
(Harvard University, Mass., U. S. A.)
BLOMQVIST, Nils Gösta, Instr.
(Boston University, Mass., U. S. A.)
BLUMBERG, Henry, Prof.
(Ohio State University, Ohio, U. S. A.)
BLUMEN, Isadore, Prof.
(Cornell University, N. Y., U. S. A.)
BLUMENTHAL, Leonard Mascot, Prof.
(University of Missouri, Mo., U. S. A.)
BLUNDON, William John, Prof.
(Memorial University of Newfoundland, Newfoundland, Canada)
BOAS, Ralph Philip, Jr., Executive Editor
(Mathematical Reviews, Providence, R. I.; Northwestern University, Ill., U. S. A.)
Dr. Mary L. Boas
Miss Anne Louise Boas
Master Ralph Layne Boas
BOBONIS, Augusto, Prof.
(City College, Hato Rey, Puerto Rico)
BOBROFF, David L., Research Engineer
(Massachusetts Institute of Technology, Mass., U. S. A.)
BOCHNER, Salomon, Prof.
(Princeton University, N. J., U. S. A.)
Mrs. Bochner
BODE, Hendrik W., Dr.
(Bell Telephone Laboratories, N. J., U. S. A.)
BOEHM, Sister Ada Marie
(College of St. Catherine, Minn., U. S. A.)
BOHNENBLUST, Henri Frederic, Prof.
(California Institute of Technology, Calif., U. S. A.)
Mrs. Bohnenblust
BOHR, Harald August, Prof.
(University of Copenhagen, Copenhagen, Denmark)
BOMPIANI, Enrico, Prof.
(Università di Roma, Rome, Italy)
BONSALL, Frank Featherstone, Lecturer
(King's College, University of Durham, Newcastle-on-Tyne, England)
Mrs. Bonsall
BOOS, Pierre Edmond, Doyen, Faculté Libre des Sciences
(Institut Catholique de Paris, Paris, France)
BOOTON, Richard Crittenden, Jr., Research Assistant
(Massachusetts Institute of Technology, Mass., U. S. A.)
BORCH, Karl Henrik, Science Officer
(UNESCO, Cairo, Egypt)
BORG, Göran Lars, Docent
(Matematiska Institutionen, Uppsala, Sweden)
Mrs. Borg
BOSCHAN, Paul, Econometrist
(Econometric Institute, N. Y., U. S. A.)
Mrs. Boschan
BOSE, Raj Chandra, Prof.
(University of North Carolina, N. C., U. S. A.)
Mrs. Bose
Miss Purabi Bose
Miss Sipra Bose
BOTHWELL, Frank E., Prof.
(Northwestern University, Ill., U. S. A.)
BOTT, Raoul
(Institute for Advanced Study, N. J., U. S. A.)
BOTTEMA, Oene, Prof.
(Technische Hogeschool, Delft, Netherlands)
BOULANGER, Georges Robert, Prof.
(Faculté Polytechnique de Mons and Université de Bruxelles, Brussels, Belgium)

BOURGIN, David Gordon, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Bourgin
Miss Marcia Bourgin
Master Richard Bourgin
BOURNE, Samuel, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
BOUTELLE, Harold Danforth, Prof.
(University of Massachusetts, Mass., U. S. A.)
BOWER, Julia Wells, Prof.
(Connecticut College, Conn., U. S. A.)
BOWER, Oliver K., Prof.
(University of Illinois, Ill., U. S. A.)
BOWMAN, John Ridgway, Dr.
(Mellon Institute of Industrial Research, Pa., U. S. A.)
BOYCE, Moffatt Grier, Prof.
(Vanderbilt University, Tenn., U. S. A.)
Mrs. Boyce
Miss Elizabeth Boyce
Miss Mary Frances Boyce
BOYER, Carl B., Prof.
(Brooklyn College, N. Y., U. S. A.)
BOYER, Lee Emerson, Dr.
(Millersville State Teachers College, Pa., U. S. A.)
Mrs. Boyer
Miss Dolores E. Boyer
BRADBURN, Mary, Lecturer
(Royal Holloway College, University of London, Englefield Green, England)
BRAHANA, Henry Roy, Prof.
(University of Illinois, Ill., U. S. A.)
Mr. Thomas Brahana
BRAMBLE, Charles Clinton, Director of Computation and Ballistics
(Naval Proving Ground, Va., U. S. A.)
BRAND, Louis, Prof.
(University of Cincinnati, Ohio, U. S. A.)
BRAUER, Alfred Theodor, Prof.
(University of North Carolina, N. C., U. S. A.)
BRAUER, Richard, Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Brauer
Mr. George U. Brauer
Mr. Fred G. Brauer
BRAVO, Salvador de la Torre, Industrial Engineer
(Military Academy of Mexico, Mexico, D. F., Mexico)
BRAY, Hubert E., Prof.
(Rice Institute, Tex., U. S. A.)
BRELOT, Marcel Emile, Prof.
(Université de Grenoble, Grenoble, France)
BREUSCH, Robert H., Prof.
(Amherst College, Mass., U. S. A.)
BRIGHAM, Nelson Allen, Prof.
(University of Maryland, Md., U. S. A.)
BRILLOUIN, Leon
(New York, N. Y., U. S. A.)
BRIND, Abraham, Prof.
(New York, N. Y., U. S. A.)
BRINK, Raymond Woodard, Prof.
(University of Minnesota, Minn., U. S. A.)
Mrs. Brink
BRINKMANN, Heinrich Wilhelm, Prof.
(Swarthmore College, Pa., U. S. A.)
BROCK, Paul, Mathematician
(Reeves Instrument Corporation, N. Y., U. S. A.)
Mrs. Brock
BROOKS, Foster, Prof.
(Kent State University, Ohio, U. S. A.)
BROTHERS, Warren H., Jr., Prof.
(Talladega College, Ala., U. S. A.)
BROUWER, Luitzen Egbertus Jan, Prof.
(University of Amsterdam, Amsterdam, Netherlands)
BROWDER, Felix E., Instr.
(Massachusetts Institute of Technology, Mass., U. S. A.)
BROWN, Arthur Barton, Prof.
(Queens College, N. Y., U. S. A.)
Mrs. Brown
Mr. Alan Brown
BROWN, Bailey, Prof.
(Amherst College, Mass., U. S. A.)
BROWN, Rev. Ferdinand Louis, Prof.
(University of Notre Dame, Ind., U. S. A.)
BROWN, Herbert Kapfel, Prof.
(Northeastern University, Mass., U. S. A.)
BROWNE, Marjorie Lee, Instr.
(North Carolina College at Durham, N. C., U. S. A.)
BROWNELL, Frank Herbert, III, Prof.
(University of Washington, Wash., U. S. A.)
Mrs. Brownell
BRUCK, Richard Hubert, Prof.
(University of Wisconsin, Wis., U. S. A.)
Mrs. Bruck

BRUDNO, Charlotte, Editor
(Naval Research Laboratory, D. C., U. S. A.)
BRUINS, Evert Marie, Lecturer
(University of Amsterdam, Amsterdam, Netherlands)
BRUMFIEL, Charles Francis, Prof.
(Ball State Teachers College, Ind., U.S.A.)
Mrs. Brumfiel
BRYAN, Joseph Gerard
(Massachusetts Institute of Technology, Mass., U. S. A.)
BUCK, Charles Carpenter, Instr.
(University of Nebraska, Neb., U. S. A.)
BUCK, Elsie M., Prof.
(Boise Junior College, Idaho, U. S. A.)
BÜCKNER, Hans F., Dr.
(Minden, Germany)
BULLARD, James Atkins, Prof.
(University of Vermont and State Agricultural College, Vt., U. S. A.)
BUMER, Charles Theodore, Prof.
(Clark University, Mass., U. S. A.)
BURCHAM, Paul Baker, Prof.
(University of Missouri, Mo., U. S. A.)
BURCKHARDT, Johann Jakob, Prof.
(University of Zürich, Zürich, Switzerland)
BURDICK, Wayne E., Research Associate
(Aeronautical Research Center, University of Michigan, Mich., U. S. A.)
BUREAU, Florent Joseph, Prof.
(Université de Liège, Liège, Belgium)
BURGERS, Johannes Martinus, Prof.
(Technische Hogeschool, Delft, Netherlands)
Mrs. Burgers
BURINGTON, Richard Stevens, Chief Mathematician
(Bureau of Ordnance, Navy Department, D. C., U. S. A.)
BURKE, Sister Mary Leonarda, Prof.
(Regis College, Mass., U. S. A.)
Sister M. Justine Sullivan
BURROUGHS, Winifred Kip
(University of Michigan, Mich., U. S. A.)
BURROW, M. D., Lecturer
(McGill University, Que., Canada)
BURTON, Lindley J., Prof.
(Bryn Mawr College, Pa., U. S. A.)
BUSEMANN, Herbert, Prof.
(University of Southern California, Calif., U. S. A.)
BUSH, Kenneth Arthur, Prof.
(Champlain College, N. Y., U. S. A.)
BUSH, Laurens Earle, Prof.
(College of St. Thomas, Minn., U. S. A.)
Mrs. Bush
Miss Patricia Bush
Mr. Edward Bush
BUSHEY, Jewell Hughes, Prof.
(Hunter College, N. Y., U. S. A.)
BUSINGER, Roger Thomas, Chemical Engineer
(Pittsburgh Plate Glass Company, Pa., U. S. A.)
Mrs. Businger
BUSSEY, William Henry, Prof.
(University of Minnesota, Minn., U. S. A.)
BUTCHER, George H., Jr., Instr.
(Howard University, D. C., U. S. A.)
BUTLER, Loren Gardner, Prof.
(State College of Washington, Wash., U. S. A.)
Mrs. Butler
BUTZER, Paul Leo
(University of Toronto, Ont., Canada)

C

CAIRNS, Stewart S., Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Cairns
CAIRNS, William DeWeese, Prof.
(Oberlin College, Ohio, U. S. A.)
Mrs. Cairns
CALABI, Eugenio, Dr.
(Princeton University, N. J., U. S. A.)
CALDERÓN, Alberto Pedro
(University of Chicago, Ill., U. S. A.)
CALKINS, Helen, Prof.
(Pennsylvania College for Women, Pa., U. S. A.)
CALLAHAN, Ethel B., Prof.
(Hartwick College, N. Y., U. S. A.)
CALLOWAY, Jean Mitchener, Instr.
(University of Pennsylvania, Pa., U. S. A.)
CALVIN, Robert Leitzel, Assistant Engineer
(Armour Research Foundation, Illinois Institute of Technology, Ill., U. S. A.)
CAMERON, Robert Horton, Prof.
(University of Minnesota, Minn., U. S. A.)
CAMP, Burton Howard, Prof. Emeritus
(Wesleyan University, Conn., U. S. A.)
Mrs. Camp

CAMP, Ezra John, Prof.
(Macalester College, Minn., U. S. A.)
Mrs. Camp
CAMPAIGNE, Howard Herbert, Dr.
(Armed Forces Security Agency, D. C., U. S. A.)
CAMPBELL, Howard E., Instr.
(University of Pennsylvania, Pa., U. S. A.)
CAMPBELL, John William, Prof.
(University of Alberta, Alta., Canada)
Mrs. Campbell
CANEPA, Michele, Engineer
(Instituto per le Applicazioni del Calcolo, Rome, Italy)
CANSADO MACEDA, Enrique S., Dr.
(Consejo Superior de Estadística, Madrid, Spain)
CARIS, Perry Aquila, Prof.
(University of Pennsylvania, Pa., U. S. A.)
Mrs. Caris
CARLEN, Mildred E., Instr.
(Brown University, R. I., U. S. A.)
CARLESON, Lennart Axel Edvard, Docent
(University of Uppsala, Uppsala, Sweden)
Mrs. Carleson
CARLSON, Bengt G.
(Los Alamos Scientific Laboratory, N. M., U. S. A.)
CARR, John W., III
(Institut Henri Poincaré, Paris, France)
CÂRSTOIU, Ion, Prof.
(Johns Hopkins University, Md., U. S. A.)
CARTAN, Henri Paul, Prof.
(Université de Paris à la Sorbonne, Paris, France)
Mrs. Cartan
CARTER, Hobart C., Prof.
(Mary Washington College, University of Virginia, Va., U. S. A.)
CARTER, William Caswell, Dr.
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
CARTWRIGHT, Mary Lucy, Dr.
(Girton College, Cambridge University, Cambridge, England)
CARVER, Walter Buckingham, Prof.
(Cornell University, N. Y., U. S. A.)
CASTELLANI, Maria, Prof.
(University of Kansas City, Mo., U. S. A.)
CATON, Willis B., Prof.
(DePaul University, Ill., U. S. A.)
CAULFIELD, John Franklin
(Indiana University, Ind., U. S. A.)
Mr. Walter J. Feeney
CERNUSCHI, Félix, Prof.
(University of Uruguay, Montevideo, Uruguay)
CESARI, Lamberto, Prof.
(Università di Bologna, Bologna, Italy)
Mrs. Cesari
CHAMBERLIN, Richard Eliot, Prof.
(University of Utah, Utah, U. S. A.)
CHAND, Uttam, Prof.
(Boston University, Mass., U. S. A.)
CHANDRASEKHAR, Subrahmanyan, Prof.
(University of Chicago, Ill., U. S. A.)
CHANDRASEKHARAN, Komaravolu, Reader in Mathematics
(Tata Institute of Fundamental Research, Bombay, India)
CHANG, Chieh Chien, Prof.
(Johns Hopkins University, Md., U. S. A.)
CHANG, Harold Hsiao-Li, Prof.
(West China Union University, Chengtu, China)
CHANLER, Josephine Hughes, Prof.
(University of Illinois, Ill., U. S. A.)
CHARNES, Abraham, Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
Mrs. Charnes
CHATLAND, Harold, Prof.
(University of Montana, Mont., U. S. A.)
CHAZY, Jean François, Prof.
(Université de Paris à la Sorbonne, Paris, France)
Mrs. Chazy
CHELLEVOLOD, John O., Prof.
(Lehigh University, Pa., U. S. A.)
CHEN, Kuo-Tsai, Dr.
(Columbia University, N. Y., U. S. A.)
CHEN, Yu Why, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
Mrs. Chen
Mr. Victor Chen
CHENEY, William Fitch, Jr., Prof.
(University of Connecticut, Conn., U. S. A.)
CHERLIN, George Y., Instr.
(Rutgers University, N. J., U. S. A.)
CHERN, Shiing Shen, Prof.
(University of Chicago, Ill., U. S. A.)
Mrs. Chern

CHERRY, Thomas MacFarland, Prof.
(University of Melbourne, Melbourne, Australia)
CHEVALLEY, Claude Charles, Prof.
(Columbia University, N. Y., U. S. A.)
Mrs. Chevalley
CHEVRIER, Jean Léon Paul, Director
(Centre d'Études Mathématiques, Beirut, Lebanon)
CHINITZ, Manuel Paul
(Eckert-Mauchly Computer Corporation, Pa., U. S. A.)
CHITTENDEN, Edward Wilson, Prof.
(State University of Iowa, Iowa, U. S. A.)
CHONG, Frederick, Lecturer
(University of Sydney, Sydney, Australia)
CHOQUET, Gustave, Maître de Conférences
(Université de Paris, Paris, France)
CHOW, Wei-Liang, Prof.
(Johns Hopkins University, Md., U. S. A.)
CHOWLA, Sarvadaman, Visiting Prof.
(University of Kansas, Kan., U. S. A.)
Mrs. Chowla
Miss Paromita Chowla
CHRISTENSEN, Sabinus Hoegsbro, Aerodynamicist
(Harvard University, Mass., U. S. A.)
CHRISTIAN, Robert R., Instr.
(Clark University, Mass., U. S. A.)
CHUNG, Kai-Lai, Prof.
(Cornell University, N. Y., U. S. A.)
CHURCH, Alonzo, Prof.
(Princeton University, N. J., U. S. A.)
Mrs. Church
CHURCH, Randolph, Prof.
(United States Naval Postgraduate School, Md., U. S. A.)
CHURCHILL, Ruel Vance, Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Churchill
CIOFFARI, Bernard, Prof.
(College of New Rochelle, N. Y., U. S. A.)
CLARK, Frank Eugene, Prof.
(Rutgers University, N. J., U. S. A.)
CLARK, Helen M., Instr.
(Northwestern University, Ill., U. S. A.)
CLARK, William Glenn, Prof.
(Mount Union College, Ohio, U. S. A.)
CLARKE, Elbert Howard, Prof.
(Hiram College, Ohio, U. S. A.)
CLARKE, F. Marion, Instr.
(University of Nebraska, Neb., U. S. A.)
CLARKSON, Helen Elizabeth
(Equitable Life Assurance Society, N. Y., U. S. A.)
CLARKSON, James A., Prof.
(Tufts College, Mass., U. S. A.)
Mrs. Clarkson
CLAYTON, Henry Hubert, Associate Research Physicist
(National Research Council of Canada, Ont., Canada)
CLAYTOR, William W. S., Prof.
(Howard University, D. C., U. S. A.)
CLEMENT, Mary Dean, Dr.
(University of Chicago, Ill., U. S. A.)
CLEMENTS, Guy R., Prof.
(United States Naval Academy, Md., U. S. A.)
CLIFFORD, Alfred H., Prof.
(Johns Hopkins University, Md., U. S. A.)
CLIPPINGER, Richard F., Assistant Chief, Computing Laboratory
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
CLOUTIER, Rev. Gerard, Prof.
(University of Ottawa, Ont., Canada)
COBLE, Arthur Byron, Prof.
(University of Illinois, Ill., U. S. A.)
COHEN, Alonzo Clifford, Jr., Prof.
(University of Georgia, Ga., U. S. A.)
COHEN, Carl, Instr.
(Cambridge Junior College, Mass., U. S. A.)
COHEN, Irvin S., Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
COHEN, Leon W., Prof.
(Queens College, N. Y., U. S. A.)
COHN, Harvey, Prof.
(Wayne University, Mich., U. S. A.)
COHN, Richard M., Prof.
(Rutgers University, N. J., U. S. A.)
COLCHAGOFF, George D., Aeronautical Engineer (Associate Member)
(Wright-Patterson Air Force Base, Ohio, U. S. A.)
COLE, Nancy, Prof.
(Syracuse University, N. Y., U. S. A.)
COLEMAN, Albert John, Lecturer
(University of Toronto, Ont., Canada)
COLLATZ, Lothar, Prof.
(Technische Hochschule Hannover, Hannover, Germany)

COLLINGWOOD, Edward Foyle, Dr.
(Alnwick, England)
COLVIN, Burton Houston, Prof.
(University of Wisconsin, Wis., U. S. A.)
COMBA, Paola
(California Institute of Technology, Calif., U. S. A.)
COMEGYS, Esther, Prof.
(University of Maine, Me., U. S. A.)
COMFORT, Edwin G. H., Prof.
(Illinois Institute of Technology, Ill., U. S. A.)
CONFORTO, Fabio, Prof.
(Università di Roma, Rome, Italy)
CONNELLY, Brother Damian, Prof.
(LaSalle College, Pa., U. S. A.)
CONSTABLE, Mary Louise
(Philadelphia High School for Girls, Pa., U. S. A.)
CONWELL, George Macfeely, Prof.
(University of Georgia, Ga., U. S. A.)
Mrs. Conwell
COOKE, John Caister, Prof.
(University of Malaya, Singapore, Malaya)
COOLEY, James A., Prof.
(University of Tennessee, Tenn., U. S. A.)
COOLIDGE, Julian Lowell, Prof.
(Harvard University, Mass., U. S. A.)
Mrs. Coolidge
Miss Theresa Coolidge
COPE, Thomas Freeman, Prof.
(Queens College, N. Y., U. S. A.)
Mrs. Cope
Miss Elizabeth Cope
Miss Mary Cope
COPELAND, Arthur Herbert, Jr.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Copeland
COPELAND, Arthur Herbert, Sr., Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Copeland
COPELAND, Lennie Phoebe, Prof.
(Wellesley College, Mass., U. S. A.)
COPPOCK, Stephen William, Director, Technical Services
(British Joint Services Mission, D. C., U. S. A.)
Mrs. Coppock
COPSON, Edward Thomas, Prof.
(University College, University of St. Andrews, Dundee, Scotland)
CORAL, Max, Prof.
(Wayne University, Mich., U. S. A.)
CORLISS, John Johnson, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Corliss
VAN DER CORPUT, Johannes Gualtherus, Prof.
(Municipal University of Amsterdam, Amsterdam, Netherlands)
Mrs. van der Corput
COURANT, Richard, Prof.
(New York University, N. Y., U. S. A.)
COURT, L. M., Prof.
(Rutgers University, N. J., U. S. A.)
COVEYOU, Robert R., Mathematician
(Oak Ridge National Laboratory, Tenn., U. S. A.)
COWLES, William Henry Harrison, Prof.
(Pratt Institute, N. Y., U. S. A.)
COWLING, Vincent Frederick, Prof.
(University of Kentucky, Ky., U. S. A.)
COX, Gertrude M., Director
(Institute of Statistics, N. C., U. S. A.)
COX, Mary Jane, Mathematician
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
COXETER, H. S. M., Prof.
(University of Toronto, Ont., Canada)
CRABTREE, James Bruce, Prof.
(University of New Hampshire, N. H., U. S. A.)
CRAIG, William
(Princeton University, N. J., U. S. A.)
CRAMÉR, Harald, President
(University of Stockholm, Stockholm, Sweden)
Mrs. Cramér
CRANDALL, Robert Wilson (Associate Member)
(McGraw-Hill Book Company, Inc., N. Y., U. S. A.)
CRANDALL, Stephen Harry, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
CRAWFORD, Jane E., Engineer
(General Electric Company, N. Y., U. S. A.)
CRAWFORD, William S. H., Prof.
(Mount Allison University, N. B., Canada)
CREESE, Franklyn George
(San Leandro, Calif., U. S. A.)

CREMER, Hubert Heinrich, Prof.
(Rheinisch-Westfälische Technische Hochschule, Aachen, Germany)
CROSBY, William James Robert, Prof.
(University of Toronto, Ont., Canada)
CROUSE, Charles William, Actuary
(Manufacturers Casualty Insurance Company, Pa., U. S. A.)
CULLEN, Helen Frances, Prof.
(University of Massachusetts, Mass., U. S. A.)
CUNNINGHAM, A. B., Prof.
(West Virginia University, W. Va., U.S.A.)
CURRIE, John Cecil, Prof.
(Georgia Institute of Technology, Ga., U. S. A.)
CURRY, Haskell Brooks, Prof.
(Pennsylvania State College, Pa., U. S. A.)
Mrs. Curry
Mr. Robert Curry
CURTIS, Morton Landers, Instr.
(Northwestern University, Ill., U. S. A.)
CURTISS, John Hamilton, Administrator
(National Bureau of Standards, D. C., U. S. A.)
CUTLER, Edward H., Prof.
(Lehigh University, Pa., U. S. A.)

D

DAHLGREN, Thorild Sture Hialmar, Dr.
(Skåne Life Insurance Company, Malmö, Sweden)
DALKEY, Norman C., Dr.
(RAND Corporation, Calif., U. S. A.)
DALLA VOLTA, Vittorio, Prof.
(Università di Roma, Rome, Italy)
DANSKIN, John Moffatt, Operations Analyst
(Navy Department, D. C., U. S. A.)
DANTZIG, George B., Mathematical Adviser to the Comptroller
(Headquarters United States Air Force, D. C., U. S. A.)
DARKOW, Marguerite D., Prof.
(Hunter College, N. Y., U. S. A.)
DARWIN, Sir Charles Galton
(Newnham Grange, Cambridge, England)
Miss Cecily Darwin
DAUM, John Andrew, Prof.
(Agricultural and Mechanical College of Texas, Tex., U. S. A.)
Mrs. Daum
Master Johnny Daum
DAUS, Paul H., Prof.
(University of California, Los Angeles, Calif., U. S. A.)
Mrs. Daus
DAVENPORT, Harold, Prof.
(University College, University of London, London, England)
Mrs. Davenport
DAVIDSON, Luis J., Prof.
(Institute of Secondary Education of Matanzas, Matanzas, Cuba)
DAVIES, Evan Tom, Prof.
(University College, Southampton, England)
DAVIS, Chester Lee
(General Motors Institute, Mich., U. S. A.)
DAVIS, David R., Prof.
(State Teachers College, N. J., U. S. A.)
Mrs. Davis
DAVIS, Horace Chandler, Instr.
(University of Michigan, Mich., U. S. A.)
DAVIS, James Elmer, Prof.
(Drexel Institute of Technology, Pa., U. S. A.)
Mrs. Davis
Mr. John Davis
DAVIS, Margaret Rae, Prof.
(Southeastern Louisiana College, La., U. S. A.)
DAVIS, Martin, Instr.
(University of Illinois, Ill., U. S. A.)
Mrs. Davis
DAVIS, Philip, Dr.
(Harvard University, Mass., U. S. A.)
DAVIS, Ray, Principal
(Technical College, Perth, Western Australia)
DAWKINS, Walter Sindall, Prof.
(Loyola College, Md., U. S. A.)
DAY, Mahlon Marsh, Prof.
(University of Illinois, Ill., U. S. A.)
DEAN, Thomas Scott
(Sherman, Tex., U. S. A.)
Mrs. Dean
DEBAGGIS, Rev. Henry F., Prof.
(University of Notre Dame, Ind., U. S. A.)
DECICCO, John, Prof.
(DePaul University, Ill., U. S. A.)
DECK, Luther J., Prof.
(Muhlenberg College, Pa., U. S. A.)
DECLEENE, Rev. Louis A. V., Prof.
(St. Norbert College, Wis., U. S. A.)

DEDERICK, L. S., Associate Director
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
DEIHL, D. George, Statistician
(S.C.A.P., Programs and Statistics Division, Calif., U. S. A.)
DELANEY, Andrew, Actuary
(Union Central Life Insurance Company, Ohio, U. S. A.)
DELANGE, Hubert, Prof.
(Université de Clermont-Ferrand, Clermont-Ferrand, France)
DELGLEIZE, Augustin Arthur, Prof.
(Université de Liège, Liège, Belgium)
DENJOY, Arnaud, Prof.
(Université de Paris, Paris, France)
DEPRIMA, Charles Raymond, Prof.
(California Institute of Technology, Calif., U. S. A.)
DEVOL, David Lester
(University of Colorado, Colo., U. S. A.)
DIAMOND, Ainsley H., Prof.
(Oklahoma Agricultural and Mechanical College, Okla., U. S. A.)
DIAS, Candido Silva, Prof.
(University of São Paulo, São Paulo, Brazil)
DIAZ, Joaquin B., Research Associate
(University of Maryland, Md., U. S. A.)
Mrs. Diaz
DICE, Stanley F.
(University of Pittsburgh, Pa., U. S. A.)
DICKINSON, David James
(University of Michigan, Mich., U. S. A.)
Mrs. Dickinson
DIEUDONNÉ, Jean Alexandre, Prof.
(Université de Nancy, Nancy, France)
Mrs. Dieudonné
DILIBERTO, Stephen P., Prof.
(University of California, Calif., U. S. A.)
DINES, Lloyd L., Prof. Emeritus
(Carnegie Institute of Technology, Pa., U. S. A.)
DIXON, Wilfrid J., Prof.
(University of Oregon, Ore., U. S. A.)
DOLCHER, Mario, Prof.
(Università di Trieste, Trieste)
DOLCIANI, Mary Patricia, Prof.
(Vassar College, N. Y., U. S. A.)
Mrs. Peter Dolciani
DOLPH, Charles Laurie, Prof.
(University of Michigan, Mich., U. S. A.)
DONCHIAN, Paul S.
(Samuel Donchian Rug Company, Conn., U. S. A.)
DONOGHUE, William F., Jr., Assistant
(University of Wisconsin, Wis., U. S. A.)
DONSKER, Monroe David, Prof.
(University of Minnesota, Minn., U. S. A.)
DOOB, Joseph Leo, Prof.
(University of Illinois, Ill., U. S. A.)
DORWART, Harold Laird, Prof.
(Trinity College, Conn., U. S. A.)
DOSSEY, Maurice P., Prof.
(Southwestern Louisiana Institute, La., U. S. A.)
DOTO, Irene L., Instr.
(Temple University, Pa., U. S. A.)
DOUGLIS, Avron, Prof.
(Institute for Mathematics and Mechanics, New York University, N. Y., U. S. A.)
DOWNIE, Edwin John, Prof.
(Colgate University, N. Y., U. S. A.)
Mrs. Downie
Miss Edelgard Downie
Miss Irmgard Downie
DOWNS, Thomas L., Prof.
(Washington University, Mo., U. S. A.)
DOYLE, Thomas Carlson, Prof.
(Dartmouth College, N. H., U. S. A.)
DRAGONETTE, Lelia Ann, Instr.
(University of Pennsylvania, Pa., U. S. A.)
DRESCH, Francis William, Dr.
(Naval Proving Ground, Va., U. S. A.)
DRESDEN, Arnold, Prof.
(Swarthmore College, Pa., U. S. A.)
DRESHER, Melvin, Dr.
(RAND Corporation, Calif., U. S. A.)
DRESSEL, F. G., Prof.
(Duke University, N. C., U. S. A.)
DRIBIN, Daniel M., Dr.
(Armed Forces Security Agency, D. C., U. S. A.)
DRUMAUX, Paul, Prof.
(Université de Ghent, Ghent, Belgium)
DUARTE, Francisco José, Civil Engineer
(Caracas, Venezuela)
Mrs. Duarte
Mr. Carlos Duarte
DUBÉ, Georges Roland
(Yale University, Conn., U. S. A.)
DUBREIL, Paul Jacques, Prof.
(Université de Paris à la Sorbonne, Paris, France)

DUDMAN, John Almon, Research Assistant
(Bureau of Social Science Research, Columbia University, N. Y., U. S. A.)
DUFF, George Francis Denton
(Princeton University, N. J., U. S. A.)
DUFFIN, Richard James, Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
Mrs. Duffin
DUGUNDJI, James, Prof.
(University of Southern California, Calif., U. S. A.)
DUNFORD, Nelson, Prof.
(Yale University, Conn., U. S. A.)
DUPARC, Herman J. A., Assistant
(University of Amsterdam, Amsterdam, Netherlands)
DURAND, Janet C.
(Philadelphia, Pa., U. S. A.)
DUREN, William L., Jr., Prof.
(Tulane University of Louisiana, La., U. S. A.)
DURFEE, Walter H., Provost
(Hobart and William Smith Colleges, N. Y., U. S. A.)
DURFEE, William H., Prof.
(Dartmouth College, N. H., U. S. A.)
Mrs. Durfee
DURLACH, Nathaniel Isaac
(Bard College, N. Y., U. S. A.)
DU VAL, Patrick, Prof.
(University of Georgia, Ga., U. S. A.)
Mrs. DuVal
Mr. Nicholas S. DuVal
Miss Paula G. DuVal
DVORETZKY, A., Instr.
(Hebrew University, Jerusalem, Israel)
DYER-BENNET, John, Prof.
(Purdue University, Ind., U. S. A.)
DYSON, Freeman John, Research Fellow
(University of Birmingham, Birmingham, England)
DYSON, Verena Esther Haefeli, Dr.
(Goucher College, Md., U. S. A.)

E

EARL, James Mayberry, Prof.
(University of Omaha, Neb., U. S. A.)
Mrs. Earl
EAVES, Edgar D., Prof.
(University of Tennessee, Tenn., U. S. A.)
EBERHART, Paul, Prof.
(Washburn University, Kan., U. S. A.)
EBERLEIN, William F., Prof.
(University of Wisconsin, Wis., U. S. A.)
EBIN, Alexander Weaver, President
(American Institute of Man, Ill., U. S. A.)
Mrs. Ebin
ECKMANN, Beno, Prof.
(Eidgenössische Technische Hochschule, Zürich, Switzerland)
EDREI, Albert, Prof.
(University of Saskatchewan, Sask., Canada)
Mrs. Edrei
Miss Marilene Edrei
EDWARDS, Prentice Dearing, Prof.
(Ball State Teachers College, Ind., U. S. A.)
Mrs. Edwards
EHRESMANN, Charles, Prof.
(Université de Strasbourg, Strasbourg, France)
EIARDI, Rev. Anthony Joseph, Prof.
(Boston College, Mass., U. S. A.)
EILENBERG, Samuel, Prof.
(Columbia University, N. Y., U. S. A.)
EISELE, Carolyn, Instr.
(Hunter College, N. Y., U. S. A.)
Dr. Morris Halpern
EISENHART, Churchill, Chief, Statistical Engineering Laboratory
(National Bureau of Standards, D. C., U. S. A.)
Mrs. Eisenhart
Miss Evelyn Eisenhart
EISENHART, Luther Pfahler, Executive Officer
(American Philosophical Society, Pa., U. S. A.)
EISENMAN, Richard Leo, Instr.
(University of Maryland, Md., U. S. A.)
ELDER, John D., Prof.
(St. Louis University, Mo., U. S. A.)
Mrs. Elder
Mr. John W. Elder
Miss Mary L. Elder
Mr. Richard C. Elder
ELFVING, Gustav, Prof.
(Cornell University, N. Y., U. S. A.)
Mrs. Elfving
ELJOSEPH, Nathan, Librarian
(Mathematical Circle, Tel-Aviv, Israel)
ELKINS, Thomas Anthony, Geophysicist
(Gulf Research and Development Company, Pa., U. S. A.)

ELLIOTT, H. Margaret, Instr.
(Washington University, Mo., U. S. A.)
ELLIOTT, Joanne, Dr.
(Cornell University, N. Y., U. S. A.)
ELLIS, David, Prof.
(University of Florida, Fla., U. S. A.)
ELLIS, Terrell, Prof.
(North Texas State College, Tex., U. S. A.)
ELSTON, James Strode, Assistant Actuary
(Travelers Insurance Company, Conn., U. S. A.)
ENEQUIST, Lars Niclas, Mathematician and Philologist
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
ENGEL, Joseph H., Dr.
(Operations Evaluation Group (M.I.T.), U. S. A.)
Mrs. Engel
Miss Wendy Melissa Engel
EPSTEIN, Bernard, Prof.
(University of Pennsylvania, Pa., U. S. A.)
EPSTEIN, Marvin P., Lecturer
(Columbia University, N. Y., U. S. A.)
ERDÉLYI, Arthur, Prof.
(California Institute of Technology, Calif., U. S. A.)
Mrs. Erdélyi
ERDÖS, Paul, Research Fellow
(University of Aberdeen, Aberdeen, Scotland)
ERIM, Kerim, Prof.
(University of Istanbul, Istanbul, Turkey)
ERRERA, Alfred J. J. H., Prof.
(Université Libre de Bruxelles, Brussels, Belgium)
Mrs. Errera
ESCARDA, Juan Basas, Civil Engineer
(Silliman University, Dumaguete City, Philippines)
ESTERMANN, Theodor, Reader in Mathematics
(University College, University of London, London, England)
EVANS, Griffith C., Prof.
(University of California, Calif., U. S. A.)
Mrs. Evans
EVANS, Herbert P., Prof.
(University of Wisconsin, Wis., U. S. A.)
EVANS, Trevor, Lecturer
(University of Manchester, Manchester, England)
EVES, Howard Whitley, Prof.
(Oregon State College, Ore., U. S. A.)
EWING, George McNaught, Prof.
(University of Missouri, Mo., U. S. A.)

F

FAGERSTROM, William Henry, Prof.
(City College of New York, N. Y., U. S. A.)
FAIRTHORNE, Robert Arthur, Principal Scientific Officer
(Royal Aircraft Establishment, Hampshire, England)
FAN, Ky, Prof.
(University of Notre Dame, Ind., U. S. A.)
FANO, Gino, Prof.
(Università di Torino, Turin, Italy)
Mrs. Fano
Mr. R. Fano
FANTAPPIÈ, Luigi, Prof.
(Università di Roma, Rome, Italy)
FARNELL, Albert B., Prof.
(University of Colorado, Colo., U. S. A.)
FARRELL, Orin James, Prof.
(Union College, N. Y., U. S. A.)
FASS, Arnold L., Lecturer
(Columbia University, N. Y., U. S. A.)
FAULKNER, Frank David, Prof.
(United States Naval Postgraduate School, Md., U. S. A.)
FAWCETT, Harold Pascoe, Prof.
(Ohio State University, Ohio, U. S. A.)
FEDER, Walter, Instr.
(City College of New York, N. Y., U. S. A.)
FEHR, Henri, Prof.
(University of Geneva, Geneva, Switzerland)
FEIBLEMAN, James Kern, Prof.
(Tulane University of Louisiana, La., U. S. A.)
FEKETE, Michael, Prof.
(Hebrew University, Jerusalem, Israel)
FELLER, William, Prof.
(Princeton University, N. J., U. S. A.)
Mrs. Feller
FENCHEL, Werner, Dr.
(Technical University of Denmark, Copenhagen, Denmark)

Mrs. Fenchel
Mr. Tom Fenchel
FERGUSON, William Eugene, Instr.
(Connecticut College, Conn., U. S. A.)
FERNS, Herman Harvey, Prof.
(University of Saskatchewan, Sask., Canada)
Mrs. Ferns
FIALA, Félix, Prof.
(Université de Neuchâtel, Neuchâtel, Switzerland)
FIALKOW, Aaron, Prof.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
FICHERA, Gaetano, Prof.
(Università di Roma, Rome, Italy)
FICKEN, Frederick Arthur, Prof.
(University of Tennessee, Tenn.; Institute for Mathematics and Mechanics, New York University, N. Y., U. S. A.)
Mrs. Ficken
Mr. William H. Ficken
FIELDS, Raymond Ira, Prof.
(University of Louisville, Ky., U. S. A.)
FINE, Nathan Jacob, Prof.
(University of Pennsylvania, Pa., U. S. A.)
Mrs. Fine
DE FINETTI, Bruno, Prof.
(Università di Trieste, Trieste)
FINKBEINER, Daniel T., Instr.
(Yale University, Conn., U. S. A.)
FINSTON, Morton, Research Associate in Aeronautical Engineering
(Massachusetts Institute of Technology, Mass., U. S. A.)
FISCHER, Wilhelm, Dr.
(University of Pennsylvania, Pa., U. S. A.)
FISHBACK, William Thompson
(Harvard University, Mass., U. S. A.)
FLANDERS, Donald A., Senior Mathematician
(Argonne National Laboratory, Ill., U. S. A.)
Mrs. Flanders
Mr. Steven Flanders
FLANDERS, Harley, Research Fellow
(California Institute of Technology, Calif., U. S. A.)
Mrs. Flanders
FLEDDERMANN, Harry T., Prof.
(Loyola University, La., U. S. A.)
FOBES, Melcher Prince, Prof.
(College of Wooster, Ohio, U. S. A.)
FOOTE, Joe Reeder, Prof.
(Iowa State College, Iowa, U. S. A.)
Mrs. Foote
FORD, Lester R., Prof.
(Illinois Institute of Technology, Ill., U. S. A.)
Mrs. Ford
FORSYTHE, George Elmer, Mathematician
(National Bureau of Standards, Calif., U. S. A.)
Mrs. Forsythe
FORT, Tomlinson, Prof.
(University of Georgia, Ga., U. S. A.)
FORTET, Robert, Prof.
(Université de Caen, Caen, France)
FOSTER, Alfred L., Prof.
(University of California, Calif., U. S. A.)
FOSTER, Malcolm Cecil, Prof.
(Wesleyan University, Conn., U. S. A.)
FOSTER, Ronald Martin, Prof.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
Mrs. Foster
FOX, Augustus Henry, Prof.
(Union College, N. Y., U. S. A.)
FOX, Charles, Prof.
(McGill University, Que., Canada)
FOX, Leslie, Dr.
(National Physical Laboratory, Teddington, England)
FOX, Ralph Hartzler, Prof.
(Princeton University, N. J., U. S. A.)
FRAME, James Sutherland, Prof.
(Michigan State College, Mich., U. S. A.)
Mrs. Frame
FRANCKX, E., Major
(École Royale Militaire, Brussels, Belgium)
FRANK, Arnold R.
(George Washington University, D. C., U. S. A.)
FRANKEL, Edward T., Budget Analyst
(Health and Welfare Federation of Allegheny County, Pa., U. S. A.)
FRASER, Wallace C. G., Prof.
(Dartmouth College, N. H., U. S. A.)
FREESE, Frances, Prof.
(Mount Union College, Ohio, U. S. A.)
FRENKIEL, François N., Consultant
(Naval Ordnance Laboratory, Md., U.S.A.)
FRIEDLEN, Donald Mandel
(Illinois Institute of Technology, Ill., U. S. A.)

FRIEDMAN, Bernard, Prof.
(New York University, N. Y., U. S. A.)
FRIEDMAN, Edward Louis
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Friedman
FRIEDMAN, Joyce, Mathematician
(Logistics Research Project, George Washington University, D. C., U.S.A.)
FRIEDRICHS, Kurt Otto, Prof.
(New York University, N. Y., U. S. A.)
FRINK, Orrin, Prof.
(Pennsylvania State College, Pa., U.S.A.)
FRUCHT, Roberto, Dr.
(Vina del Mar, Chile)
FUCHS, Wolfgang Heinrich, Lecturer
(University of Liverpool, Liverpool, England)
FUDGE, Helen G., Dr.
(Lincoln High School, Pa., U. S. A.)
FUEYO MEULENER, Julia, Prof.
(Institute Cienfuegos, Los Villas, Cuba)
Mrs. Araceli Artiles
FUGLEDE, Bent
(University of Copenhagen, Copenhagen, Denmark)
FULLER, Donald Leask, Associate Director, Central Research Laboratory
(General Analine and Film Corporation, Pa., U. S. A.)
FULLER, Francis Brock
(Princeton University, N. J., U. S. A.)
FULLERTON, Robert Elston, Prof.
(University of Wisconsin, Wis., U. S. A.)
FULMER, Herman Kyle, Prof.
(Georgia Institute of Technology, Ga., U. S. A.)
Mrs. Fulmer
FULTON, Dawson Gerald, Prof.
(Tufts College, Mass., U. S. A.)

G

GADSDEN, Christopher Paul, Instr.
(Tulane University of Louisiana, La., U. S. A.)
GAFFNEY, Matthew Page, Jr., Instr.
(Northwestern University, Ill., U. S. A.)
GAGNÉ, Rev. Fr. Thibaud
(University of Ottawa, Ont., Canada)
GÁL, István Sándor
(Institut Henri Poincaré, Paris, France)
GALBRAITH, Alan Stuart, Dr.
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
Mrs. Galbraith
Miss Judy Galbraith
Mr. James Galbraith
GALE, David, Prof.
(Brown University, R. I., U. S. A.)
GALLAGHER, Rev. Edward H., Prof.
(Providence College, R. I., U. S. A.)
GARABEDIAN, Carl A., Prof.
(Wheaton College, Mass., U. S. A.)
GARABEDIAN, Henry Leslie, Reactor Physicist
(Westinghouse Electric Corporation, Pa., U. S. A.)
Mrs. Garabedian
GARABEDIAN, Paul Roesel, Prof.
(Stanford University, Calif., U. S. A.)
GARCIA, Alfonso Gonzales, Registrar
(University of Manila, Manila, Philippines)
GARCÍA, Hilda Santamarina, Dr.
(Institute of Secondary Education of Matanzas, Matanzas, Cuba)
GÅRDING, Lars, Docent
(University of Lund, Lund, Sweden)
Mrs. Gårding
GARDNER, Clifford Speer
(New York University, N. Y., U. S. A.)
GARDNER, Murray Frank, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
GARDNER, Robert Wayne, Prof.
(Drake University, Iowa, U. S. A.)
GARRETT, George Alvin, Dr.
(Carbide and Carbon Chemicals Corporation, Tenn., U. S. A.)
GASPAR, Fernando Luis, Prof.
(Universidad Nacional del Litoral, Rosario, Argentina)
GEHMAN, Harry Merrill, Prof.
(University of Buffalo, N. Y., U. S. A.)
Mrs. Gehman
GEIRINGER, Hilda, Prof.
(Wheaton College, Mass., U. S. A.)
GELBART, Abe, Prof.
(Syracuse University, N. Y., U. S. A.)
Mrs. Gelbart
GELBAUM, Bernard Russell, Prof.
(University of Minnesota, Minn., U.S.A.)
GEORGE, T. S., Dr.
(Oreland, Pa., U. S. A.)

GEPHART, F. L., Mathematician
(Office of Air Research, Wright-Patterson Air Force Base, Ohio, U. S. A.)
GERE, Brewster H., Prof.
(Hamilton College, N. Y., U. S. A.)
Mrs. Gere
GERGEN, John Jay, Prof.
(Duke University, N. C., U. S. A.)
Mrs. Gergen
Mr. Stephen L. Gergen
Mr. David R. Gergen
GERMAY, Rodolphe-Henri, Prof.
(Université de Liège, Liège, Belgium)
GERMOND, Hallett H., Dr.
(United States Navy)
Mrs. Germond
Miss Roberta Pelton
GERRETSEN, Johan Cornelis Hendrik, Prof.
(University of Groningen, Groningen, Netherlands)
GERST, Irving, Dr.
(Control Instrument Company, N. Y., U. S. A.)
GERSTENHABER, Murray
(University of Chicago, Ill., U. S. A.)
GETCHELL, Bassford Case, Research Analyst
(Armed Forces Security Agency, D. C., U. S. A.)
GHAFFARI, Abolghassem, Prof.
(University of Tehran, Tehran, Iran)
GHURYE, Sudhish Govind
(University of North Carolina, N. C., U. S. A.)
GIBBONS, Sister Seraphim
(College of St. Catherine, Minn., U. S. A.)
GIDDINGS, Horace Alpheus, Prof.
(New York University, N. Y., U. S. A.)
Mrs. Giddings
GIESE, John H., Dr.
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
GIEVER, John Bertram, Instr.
(Boston University, Mass., U. S. A.)
Mrs. Giever
GILBARG, David, Prof.
(Indiana University, Ind., U. S. A.)
Mrs. Gilbarg
Master Daniel Gilbarg
GILBERT, Edgar Nelson, Dr.
(Bell Telephone Laboratories, N. J., U. S. A.)
GILBERT, Paul W., Prof.
(Syracuse University, N. Y., U. S. A.)
GILLAM, Basil Early, Prof.
(Drake University, Iowa, U. S. A.)
GILLIES, Donald Bruce
(University of Illinois, Ill., U. S. A.)
GILLIS, Paul Pierre, Prof.
(Université de Bruxelles, Brussels, Belgium)
GILLMAN, Leonard
(Massachusetts Institute of Technology, Mass., U. S. A.)
GILMAN, Ray E., Prof.
(Brown University, R. I., U. S. A.)
GINIVAN, Francis Joseph
(St. Francis Xavier University, N. S., Canada)
GINSBURG, Jekuthiel, Prof.
(Yeshiva University, N. Y., U. S. A.)
Mrs. Ginsburg
GIVENS, James Wallace, Jr., Prof.
(University of Tennessee, Tenn., U. S. A.)
Mrs. Givens
GLAUZ, Robert Doran, Instr.
(Montana State College, Mont., U. S. A.)
GLEASON, Andrew Mattei, Prof.
(Harvard University, Mass., U. S. A.)
GLENN, Oliver Edmunds, Prof.
(Lansdowne, Pa., U. S. A.)
GODEAUX, Lucien Auguste, Prof.
(Université de Liège, Liège, Belgium)
GÖDEL, Kurt, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
GODEMENT, Roger Jean, Maître de Conférences
(Université de Nancy, Nancy, France)
GOFFMAN, Casper, Prof.
(University of Oklahoma, Okla., U. S. A.)
Mrs. Goffman
Misses Goffman
GOHEEN, Harry Earl, Prof.
(Syracuse University, N. Y., U. S. A.)
GOKHALE, Vishnu Dattatreya, Prof.
(Atlanta University, Ga., U. S. A.)
GOLDBERG, Michael, Ordnance Engineer
(Navy Department, D. C., U. S. A.)
Mrs. Goldberg
Mr. Jeremy R. Goldberg
Miss Susan D. Goldberg
GOLDBERG, Samuel Irving, National Research Council Scholar
(National Research Council, Canada)

GOLDHABER, Jacob K., Dr.
(University of Connecticut, Conn., U. S. A.)
GOLDMAN, Charles C., Instr.
(Purdue University, Ind., U. S. A.)
GOLDMAN, Lawrence, Lecturer
(Hunter College, N. Y., U. S. A.)
GOLDMAN, Malcolm
(University of Minnesota, Minn., U. S. A.)
GOLDSMITH, Norman A., Prof.
(Illinois Wesleyan University, Ill., U. S. A.)
GOLDSTEIN, Max
(Los Alamos Scientific Laboratory, N. M., U. S. A.)
GOLDSTEIN, Sydney, Prof.
(The Technion, Haifa, Israel)
GOLDSTINE, Herman, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
GOLOMB, Michael, Prof.
(Purdue University, Ind., U. S. A.)
GONSHOR, Harry
(Harvard University, Mass., U. S. A.)
GONZÁLEZ, Mario O., Prof.
(University of Havana, Havana, Cuba)
GONZÁLEZ-DOMÍNGUEZ, Alberto, Prof.
(University of Buenos Aires, Buenos Aires, Argentina)
Mrs. González-Domínguez
Master Alvaro González-Domínguez
GOODMAN, Adolph Winkler, Prof.
(University of Kentucky, Ky., U. S. A.)
GOODNER, Dwight Benjamin, Prof.
(Florida State University, Fla., U. S. A.)
Mrs. Goodner
GORDON, Robert Dean, Prof.
(University of Buffalo, N. Y., U. S. A.)
GORDON, Walter Ora, Prof.
(Pennsylvania State College, Pa., U. S. A.)
GORENSTEIN, Daniel, Dr.
(Newton, Mass., U. S. A.)
GOTTLIEB, Morris J., Prof.
(University of Chicago, Ill., U. S. A.)
GOTTSCHALK, Walter Helbig, Prof.
(University of Pennsylvania, Pa., U. S. A.)
GOUGH, Lillian, Instr.
(University of Buffalo, N. Y., U. S. A.)
GOULD, Sydney H., Prof.
(Purdue University, Ind., U. S. A.)
GRAD, Arthur, Dr.
(Office of Naval Research, D. C., U. S. A.)
GRAD, Harold, Prof.
(New York University, N. Y., U. S. A.)
GRAEF-FERNÁNDEZ, Carlos, Director of the Institute of Physics
(University of Mexico, Mexico, D. F., Mexico)
GRAFF, Frederick George, Prof.
(Oberlin College, Ohio, U. S. A.)
GRAFFI, Dario, Prof.
(Università di Bologna, Bologna, Italy)
GRAHAM, Thomas Smith, Lecturer
(University of Glasgow, Glasgow, Scotland)
GRAS, Edwin Culbert, Prof.
(United States Naval Academy, Md., U. S. A.)
GRAUSTEIN, Mary Curtis, Prof.
(Tufts College, Mass., U. S. A.)
GRAVALOS, Fausto Garcia, Prof.
(Rensselaer Polytechnic Institute, N. Y., U. S. A.)
GRAVES, Lawrence M., Prof.
(University of Chicago, Ill., U. S. A.)
Mrs. Graves
Mr. Robert L. Graves
Miss Anne L. Graves
Mr. John Graves
GRAVES, Ross Edwin, Prof.
(University of Minnesota, Minn., U. S. A.)
Mrs. Graves
GRAY, Marion C., Dr.
(Bell Telephone Laboratories, N. J., U. S. A.)
GREEN, John W., Prof.
(University of California, Los Angeles, Calif., U. S. A.)
Mrs. Green
GREEN, Lowell Forrest, Physicist
(Naval Ordnance Laboratory, Md., U. S. A.)
GREENBERG, Herbert Julius, Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
GREER, Edison, Prof.
(Kansas State College, Kan., U. S. A.)
GREGG, John Burt, Civil Engineer
(Arlington, Mass., U. S. A.)
DE GREIFF, B. Luis, Civil Engineer
(National University of Colombia, Colombia)
Mrs. de Greiff
GREVILLE, Thomas Nall Eden, Actuary
(Public Health Service, D. C., U. S. A.)

GRIFFIN, Frank Loxley, Prof.
(Reed College, Ore., U. S. A.)
GROSSWALD, Emil, Lecturer
(University of Saskatchewan, Sask., Canada)
GROVE, Vernon Guy, Prof.
(Michigan State College, Mich., U. S. A.)
Mrs. Grove
GRUDIN, Arnold
(University of Colorado, Colo., U. S. A.)
GRUNSKY, Helmut, Dozent
(Universität Tübingen, Tübingen, Germany)
GUARD, Harris T., Prof.
(Colorado Agricultural and Mechanical College, Colo., U. S. A.)
GUGGENBUHL, Laura, Prof.
(Hunter College, N. Y., U. S. A.)
GUMBEL, Emil Julius, Visiting Prof.
(New School of Social Research, N. Y., U. S. A.)
GUNDLACH, Bernard Hugo, Prof.
(University of Arkansas, Ark., U. S. A.)
GUT, Max, Prof.
(University of Zürich, Zürich, Switzerland)
GUTIÉRREZ NOVOA, Lino G., Prof.
(University of Havana, Havana, Cuba)
Mrs. Gutiérrez Novoa
GUTTERMAN, Milton M.
(University of Chicago, Ill., U. S. A.)
GUY, William Thomas, Jr.
(California Institute of Technology, Calif., U. S. A.)

H

HAANTJES, Johannes, Prof.
(University of Leiden, Leiden, Netherlands)
Mrs. Haantjes
HACHTROUDI, Mohsen, Prof.
(University of Tehran, Tehran, Iran)
HADAMARD, Jacques, Prof.
(Académie des Sciences, Paris, France)
HAEFELI, Hans Georg, Prof.
(Boston College, Mass., U. S. A.)
HAERTLEIN, Albert, Prof.
(Harvard University, Mass., U. S. A.)
Mrs. Haertlein
HAHN, Samuel Wilfred, Prof.
(Wittenberg College, Ohio, U. S. A.)
HAILPERIN, Theodore, Prof.
(Lehigh University, Pa., U. S. A.)
HAIMO, Franklin, Prof.
(Washington University, Mo., U. S. A.)
Mrs. Haimo
Miss Zara Haimo
Mr. Ethan Haimo
HALL, Dick Wick, Prof.
(University of Maryland, Md., U. S. A.)
HALL, Marguerite Franklyn, Dr.
(New Jersey State Department of Health, N. J., U. S. A.)
HALL, Marshall, Jr., Prof.
(Ohio State University, Ohio, U. S. A.)
HALMOS, Paul R., Prof.
(University of Chicago, Ill., U. S. A.)
Mrs. Halmos
HALVERSON, Frederick, Physical Chemist
(American Cyanamid Company, Conn., U. S. A.)
Mrs. Halverson
HAMBURGER, Hans Ludwig, Prof.
(University of Ankara, Ankara, Turkey)
HAMILTON, Olan Harvey, Prof.
(Oklahoma Agricultural and Mechanical College, Okla., U. S. A.)
Mrs. Hamilton
HAMMER, Preston C., Dr.
(Los Alamos Scientific Laboratory, N. M., U. S. A.)
HAMMING, Richard W., Dr.
(Bell Telephone Laboratories, N. J., U. S. A.)
Mrs. Hamming
HAMMOND, Edward Sanford, Prof.
(Bowdoin College, Me., U. S. A.)
HAMSTROM, Mary-Elizabeth, Instr.
(University of Texas, Tex., U. S. A.)
HANDELMAN, George Herman, Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
HANDSFIELD, Hugh W. (Associate Member)
(McGraw-Hill Book Company, Inc., N. Y., U. S. A.)
HANNAN, James Francis, Instr.
(Catholic University of America, D. C., U. S. A.)
HANNER, Olof, Dr.
(University of Stockholm, Stockholm, Sweden)
HANSEN, Arthur G., Instr.
(University of Maryland, Md., U. S. A.)
HANSEN, Richard W., Associate Editor (Associate Member)
(Prentice-Hall, Inc., N. Y., U. S. A.)

HANSON, James Edward
(Harvard University, Mass., U. S. A.)
HARARY, Frank, Instr.
(University of Michigan, Mich., U. S. A.)
HARISH-CHANDRA, Dr.
(Harvard University, Mass., U. S. A.)
HARPER, Floyd S., Prof.
(Drake University, Iowa, U. S. A.)
HARRISON, Gerald, Prof.
(Wayne University, Mich., U. S. A.)
HARROLD, Orville G., Jr., Prof.
(University of Tennessee, Tenn., U. S. A.)
HART, Bertha Irene, Mathematician
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
Mrs. Mildred H. Goetz
HART, William L., Prof.
(University of Minnesota, Minn., U. S. A.)
HARTMAN, Philip, Prof.
(Johns Hopkins University, Md., U. S. A.)
HARVEY, George Graham, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
HASKELL, Ray Nelson, Prof.
(University of Texas, Tex., U. S. A.)
HASKIND, Mina
(Washington, D. C., U. S. A.)
HASKINS, Elizabeth Morgan, Prof.
(State Teachers College, Mass., U. S. A.)
HATCHER, Albert George, President
(Memorial University of Newfoundland, Newfoundland, Canada)
Mrs. Hatcher
HAVELOCK, Thomas Henry, Prof.
(King's College, University of Durham, Newcastle-on-Tyne, England)
HAY, George Edward, Prof.
(University of Michigan, Mich., U. S. A.)
HAYMAN, Walter Kurt, Lecturer
(University College of the South-West, Exeter, England)
Mrs. Hayman
Miss Daphne Hayman
HAYNES, Nola Anderson, Prof.
(University of Missouri, Mo., U. S. A.)
Prof. Eli S. Haynes
HAZARD, Clifton Terrell, Prof.
(Purdue University, Ind., U. S. A.)
HAZARD, Katherine Elizabeth, Prof.
(New Jersey College for Women, N .J., U. S. A.)
HEBBERT, Clarence Mark, Engineer
(Bell Telephone Laboratories, N. J., U. S. A.)
Mrs. Hebbert
HEBEL, Ivan Lee, Prof.
(Colorado School of Mines, Colo., U. S. A.)
HEDLUND, Gustav Arnold, Prof.
(Yale University, Conn., U. S. A.)
HEEREMA, Nicholas
(University of Tennessee, Tenn., U. S. A.)
HEILBRONN, Hans Arnold, Prof.
(University of Bristol, Bristol, England)
HEINS, Albert Edward, Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
HEINS, Maurice Haskell, Prof.
(Brown University, R. I., U. S. A.)
HEISENBERG, Werner Carl, Prof.
(Max Planck Institute for Physics, Göttingen, Germany)
HELLER, Alex, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
HELMER, Olaf, Dr.
(RAND Corporation, Calif., U. S. A.)
HELSEL, Robert Griffith, Prof.
(Ohio State University, Ohio, U. S. A.)
Mrs. Helsel
HELSON, Henry, Dr.
(Harvard University, Mass., U. S. A.)
HEMMINGSEN, Erik, Prof.
(Syracuse University, N. Y., U. S. A.)
HENDERSON, Emma Marie, Lecturer
(University of Southern California, Calif., U. S. A.)
Mr. Keith W. Henderson
HENDRICKSON, Morris S., Prof.
(University of New Mexico, N. M., U. S. A.)
HENKIN, Leon Albert, Prof.
(University of Southern California, Calif., U. S. A.)
HENRIKSEN, Melvin
(University of Wisconsin, Wis., U. S. A.)
HENRIQUES, Anna Stafford, Prof.
(University of Utah, Utah, U. S. A.)
Mr. D. E. Henriques
Mr. H. E. Henriques
HERBST, Albert Franklin, Prof.
(LaVerne College, Calif., U. S. A.)
HERRIOT, John George, Prof.
(Stanford University, Calif., U. S. A.)
Mrs. Herriot

Miss S. Jean Herriot
Mr. James W. Herriot
Mr. Robert G. Herriot
HERSCHFELD, Aaron
(New York University, N. Y., U. S. A.)
HERSHNER, Ivan Raymond, Jr., Prof.
(University of North Carolina, N. C., U. S. A.)
Mrs. Hershner
HERSTEIN, Israel Nathan, Instr.
(University of Kansas, Kan., U. S. A.)
Mrs. Herstein
HERTZ, Hans G., Associate Astronomer
(United States Naval Observatory, D. C., U. S. A.)
HERWITZ, Paul Stanley
(University of Cincinnati, Ohio, U. S. A.)
Mrs. Herwitz
HERZBERGER, Maximilian J., Dr.
(Eastman Kodak Company Laboratory, N. Y., U. S. A.)
HERZOG, Fritz, Prof.
(Michigan State College, Mich., U. S. A.)
Mrs. Herzog
HESS, George Wellman, Prof.
(Howard College, Ala., U. S. A.)
HESTENES, Arnold D., Chief, Analysis Section
(Franklin Institute Laboratories, Pa., U. S. A.)
Mrs. Hestenes
Miss Beverly Ann Hestenes
Mr. Marshall D. Hestenes
HESTENES, Magnus R., Prof.
(University of California, Los Angeles, Calif., U. S. A.)
HEWITT, Edwin, Prof.
(University of Washington, Wash., U. S. A.)
HEWSON, William N.
(University of Pennsylvania, Pa., U. S. A.)
HILDEBRAND, Francis Begnaud, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Hildebrand
HILDEBRANDT, Emanuel Henry C., Prof.
(Northwestern University, Ill., U. S. A.)
HILDEBRANDT, Theophil Henry, Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Hildebrandt
HILDBRANDT, Theodore Ware
(Massachusetts Institute of Technology, Mass., U. S. A.)
HILL, Lester S., Prof.
(Hunter College, N. Y., U. S. A.)
HILLE, Einar, Prof.
(Yale University, Conn., U. S. A.)
Mrs. Hille
Mr. Harald Hille
Mr. Bertil Hille
HINRICHSEN, J. J. L., Prof.
(Iowa State College, Iowa, U. S. A.)
HIRSCH, Guy Charles, Prof.
(Université de Bruxelles, Brussels, Belgium)
HIRSCH, Kurt August, Dr.
(King's College, University of Durham, Newcastle-on-Tyne, England)
HIRSCHMAN, Isidore Isaac, Prof.
(Washington University, Mo., U. S. A.)
HIŻ, Henry, Dr.
(Harvard University, Mass., U. S. A.)
HLA, U Aung, Prof.
(University of Rangoon, Rangoon, Burma)
HLAVATÝ, Václav, Prof.
(Indiana University, Ind., U. S. A.)
HOCHSCHILD, Gerhard P., Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Hochschild
HOCKING, John Gilbert
(University of Michigan, Mich., U. S. A.)
HODGE, Philip G., Jr., Prof.
(University of California, Los Angeles, Calif., U. S. A.)
HODGE, William Vallance Douglas, Prof.
(Cambridge University, Cambridge, England)
Mrs. Hodge
Mr. M. R. Hodge
Miss G. J. Hodge
Mr. G. A. R. Broadbent
HODGES, Joseph Lawson, Jr., Prof.
(University of California, Calif., U. S. A.)
HOEFFDING, Wassily, Prof.
(University of North Carolina, N. C., U. S. A.)
HOFFMAN, Alan Jerome, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
Mrs. Hoffman
HOFFMANN, Banesh, Prof.
(Queens College, N. Y., U. S. A.)
Mrs. Hoffmann

Mr. Laurence D. Hoffmann
Miss Deborah A. Hoffmann
HOLL, D. L., Prof.
(Iowa State College, Iowa, U. S. A.)
Mrs. Holl
Miss Elizabeth Holl
Mr. William W. Holl
HOLLCROFT, Temple Rice, Prof.
(Wells College, N. Y., U. S. A.)
Mrs. Hollcroft
HOLT, Everett W., Instr.
(Muhlenberg College, Pa., U. S. A.)
HOLT, H. Kendrick, Prof.
(Union College, N. Y., U. S. A.)
HOPF, Eberhard, Prof.
(Indiana University, Ind., U. S. A.)
Mrs. Hopf
Miss Barbara Hopf
HOPF, Heinz, Prof.
(Eidgenössische Technische Hochschule, Zürich, Switzerland)
Mrs. Hopf
HOSTINSKY, L. Aileen, Dr.
(Temple University, Pa., U. S. A.)
HOTELLING, Harold, Prof.
(University of North Carolina, N. C., U. S. A.)
HOUSEHOLDER, Alston Scott, Dr.
(Oak Ridge National Laboratory, Tenn., U. S. A.)
HOWARTH, Leslie, Prof.
(University of Bristol, Bristol, England)
HOWE, Anna M., Prof.
(Cazenovia Junior College, N. Y., U. S. A.)
HSIUNG, Chuan Chih, Dr.
(Northwestern University, Ill., U. S. A.)
HSU, Kuo-Chih
(University of Kansas, Kan., U. S. A.)
HU, Sze Tsen, Prof.
(Tulane University of Louisiana, La., U. S. A.)
Mrs. Hu
HUBBARD, John F., Prof.
(University of Massachusetts, Mass., U. S. A.)
HUGHES, Howard Kibble, Prof.
(Purdue University, Ind., U. S. A.)
Mr. Malcolm Gleason
HUKA, Richard, Dr.
(Vienna, Austria)
HULL, Evelyn M.
(American Mathematical Society, N. Y., U. S. A.)
HULL, Ralph, Prof.
(Purdue University, Ind., U. S. A.)
HULLINGHORST, David Warren, Director
(South Pacific Division, Corps of Engineers, Calif., U. S. A.)
HUMPHREYS, Mabel Gweneth, Prof.
(Randolph-Macon Woman's College, Va., U. S. A.)
HUMPHREYS, Thomas Roland, Prof.
(State Teachers College, N. J., U. S. A.)
HUNT, Burrowes
(University of Colorado, Colo., U. S. A.)
Mrs. Hunt
HUREWICZ, Witold, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
HURWITZ, Wallie Abraham, Prof.
(Cornell University, N. Y., U. S. A.)
HUSKEY, Harry D., Dr.
(National Bureau of Standards, Calif., U. S. A.)
Mrs. Huskey
HUTCHERSON, William Robert, Prof.
(University of Florida, Fla., U. S. A.)
HUTCHINSON, Leroy Charles, Prof.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
HUTCHINSON, Martha C.
(New York, N. Y., U. S. A.)
HYDEMAN, William Robert
(Armed Forces Security Agency, D. C., U. S. A.)
HYDEN, John Albert, Prof.
(Vanderbilt University, Tenn., U. S. A.)
HYERS, Donald Holmes, Prof.
(University of Southern California, Calif., U. S. A.)
HYSLOP, James M., Prof.
(University of Witwatersrand, Johannesburg, South Africa)

I

IGLESIAS, Isabel María
(Instituto de la Víbora, Havana, Cuba)
IKEHARA, Shikao, Prof.
(Tokyo Institute of Technology, Tokyo, Japan)
INGRAHAM, Mark H., Prof.
(University of Wisconsin, Wis., U. S. A.)
INVERARITY, W. M., Lecturer
(Newton Mearns, Renfrewshire, England)

INZINGER, Rudolf, Prof.
(Technische Hochschule Wien, Vienna, Austria)
ITO, Wally, Instr.
(University of Minnesota, Minn., U. S. A.)
IVANOVICH, Dragisha, Docent
(University of Belgrade, Belgrade, Yugoslavia)
IWASAWA, Kenkichi, Prof.
(Tokyo University, Tokyo, Japan)
IYANAGA, Shôkichi, Prof.
(Tokyo University, Tokyo, Japan)

J

JABOTINSKY, Eri, Engineer
(Ha Keneset, Jerusalem, Israel)
JACKSON, Lloyd Kenneth, Prof.
(University of Nebraska, Neb., U. S. A.)
JACKSON, Stanley Bartlett, Prof.
(University of Maryland, Md., U. S. A.)
JACOBS, Walter W.
(George Washington University, D. C., U. S. A.)
Mrs. Jacobs
JACOBSON, Nathan, Prof.
(Yale University, Conn., U. S. A.)
JAEGER, Chester George, Prof.
(Pomona College, Calif., U. S. A.)
JAMES, Alan T., Research Officer
(University of Adelaide, Adelaide, South Australia)
Mrs. James
JAMES, Ralph Duncan, Prof.
(University of British Columbia, B. C., Canada)
Mrs. James
Mr. Alan James
Mr. Peter James
JAMES, Robert Clarke, Prof.
(University of California, Calif., U. S. A.)
JAMISON, Steven Lyle, Prof.
(Florida State University, Fla., U. S. A.)
Mrs. Jamison
JANET, Maurice, Prof.
(Université de Paris, Paris, France)
Mrs. Janet
JARDETZKY, Wencelas S., Dr.
(Columbia University; Manhattan College, N. Y., U. S. A.)
JASPER, Samuel Jacob, Prof.
(Kent State University, Ohio, U. S. A.)
Mrs. Jasper
JEFFREY, Ralph Lent, Prof.
(Queen's University, Ont., Canada)
JENKINS, Emerson D., Prof.
(Kent State University, Ohio, U. S. A.)
JENKINS, James Allister, Prof.
(Johns Hopkins University, Md., U. S. A.)
JENNER, William Elliott
(University of Michigan, Mich., U. S. A.)
JENNINGS, Stephen Arthur, Prof.
(University of British Columbia, B. C., Canada)
JENNINGS, Walter, Prof.
(United States Naval Postgraduate School, Md., U. S. A.)
JERISON, Meyer, Instr.
(University of Illinois, Ill., U. S. A.)
Mrs. Jerison
Mr. Michael Jerison
JESSEN, Børge Christian, Prof.
(University of Copenhagen, Copenhagen, Denmark)
JOHANSON, Ralph Nathaniel, Prof.
(Boston University, Mass., U. S. A.)
JOHN, Fritz, Prof.
(New York University, N. Y., U. S. A.)
JOHNS, Alfred Edward, Prof.
(McMaster University, Ont., Canada)
JOHNSON, Richard Edward, Prof.
(Smith College, Mass., U. S. A.)
JOHNSON, Roberta Frances, Prof.
(Wilson College, Pa., U. S. A.)
JOHNSTON, Francis Edgar, Prof.
(George Washington University, D. C., U. S. A.)
Mrs. Johnston
JONES, Alfred Welwood, Prof.
(Rensselaer Polytechnic Institute, N. Y., U. S. A.)
Mrs. Jones
JONES, Burton Wadsworth, Prof.
(University of Colorado, Colo., U. S. A.)
Mrs. Jones
Miss Louisa Jones
Mr. Christopher Jones
Miss Phyllis Jones
JONES, Floyd Burton, Prof.
(University of North Carolina, N. C., U. S. A.)
JONES, John, Jr., Prof.
(Mississippi Southern College, Miss., U. S. A.)
Mrs. Jones

JONES, Phillip Sanford, Prof.
(University of Michigan, Mich., U. S. A.)
JONES, Robert Clark, Mathematical Physicist
(Polaroid Corporation, Mass., U. S. A.)
JÓNSSON, Bjarni, Prof.
(Brown University, R. I., U. S. A.)
JULIA, Gaston Maurice, Prof.
(Institut de France, Paris, France)
JUNCOSA, Mario Leon, Instr.
(Johns Hopkins University, Md., U. S. A.)

K

KAC, Mark, Prof.
(Cornell University, N. Y., U. S. A.)
KAHAL, Robert, Prof.
(Washington University, Mo., U. S. A.)
KAKUTANI, Shizuo, Prof.
(Yale University, Conn., U. S. A.)
KALISCH, Gerhard K., Prof.
(University of Minnesota, Minn., U. S. A.
Mrs. Kalisch
KALISH, Aida, Instr.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
KAMKE, Erich, Prof.
(Universität Tübingen, Tübingen, Germany)
KAMPÉ DE FÉRIET, Joseph, Prof.
(Université de Lille, Lille, France)
KAPLAN, Edward Lynn
(Princeton University, N. J., U. S. A.)
KAPLAN, Wilfred, Prof.
(University of Michigan, Mich., U. S. A.)
KAPLANSKY, Irving, Prof.
(University of Chicago, Ill., U. S. A.)
KARAMATA, Jovan, Prof.
(University of Belgrade, Belgrade, Yugoslavia)
KARHUNEN, Kari Onni, Docent
(University of Helsinki, Helsinki, Finland)
KARLIN, Samuel, Prof.
(California Institute of Technology, Calif., U. S. A.)
KARUSH, William, Prof.
(University of Chicago, Ill.; National Bureau of Standards, Calif., U. S. A.)
KASNER, Edward, Prof. Emeritus
(Columbia University, N. Y., U. S. A.)
KATO, Chosaburo, Prof.
(Denison University, Ohio, U. S. A.)
Mrs. Kato
KATZ, Leo, Prof.
(Michigan State College, Mich., U. S. A.)
KAUFMAN, Hyman, Research Geophysicist
(Continental Oil Company, Okla., U. S. A.)
KAY, Alan Francis
(Harvard University, Mass., U. S. A.)
Mrs. Kay
KAYS, Samuel N., Dr.
(New York University, N. Y., U. S. A.)
KEEN, Walter Hughes, Jr., Commander, United States Navy
(Naval Air Development Center, Pa., U. S. A.)
KEHL, William Brunner, Instr.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Kehl
KELLER, Herbert Bishop, Research Scientist
(New York University, N. Y., U. S. A.)
KELLEY, John L., Prof.
(Tulane University of Louisiana, La., U. S. A.)
Mrs. Kelley
KELLINGTON, Myrtle R., Mathematician
(National Bureau of Standards, D. C., U. S. A.)
Elsie Lundt
KELLY, John Beckwith, Instr.
(University of Wisconsin, Wis., U. S. A.)
KETCHUM, Pierce W., Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Ketchum
Miss Laura Ketchum
KHAMIS, Salem Hanna, Statistician
(United Nations, N. Y., U. S. A.)
Mrs. Khamis
KIBBEY, Donald Eugene, Prof.
(Syracuse University, N. Y., U. S. A.)
KICHLINE, William L., Prof.
(University of New Hampshire, N. H., U. S. A.)
KIEFER, Jack C.
(Columbia University, N. Y., U. S. A.)
KIELY, Edmond R., Prof.
(Iona College, N. Y., U. S. A.)
KIMBALL, Spofford Harris, Prof.
(University of Maine, Me., U. S. A.)

KINCAID, Wilfred MacDonald, Instr.
(University of Michigan, Mich., U. S. A.)
KINNEY, Evelyn Kendrick, Prof.
(Ferrum Junior College, Va., U. S. A.)
KINSOLVING, Herbert Leigh, Prof.
(United States Naval Academy, Md., U. S. A.)
KIOKEMEISTER, Fred, Prof.
(Mount Holyoke College, Mass., U. S. A.)
KJELLBERG, Bo Erik Ambjörn, Docent
(Mathematiska Institutionen, Uppsala, Sweden)
KLAMKIN, Murray S., Instr.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
KLEE, Victor La Rue, Jr., Prof.
(University of Virginia, Va., U. S. A.)
Mrs. Klee
Miss Wendy Pamela Klee
Miss Barbara Christine Klee
KLEENE, Stephen Cole, Prof.
(University of Wisconsin, Wis., U. S. A.)
Mrs. Kleene
KLEIN, Joseph, Actuarial Assistant
(Department of Banking and Insurance, N. J., U. S. A.)
KLEINFELD, Erwin, Research Assistant
(University of Wisconsin, Wis., U. S. A.)
Mrs. Kleinfeld
KLINE, John Robert, Prof.
(University of Pennsylvania, Pa., U.S.A.)
KLINE, Morris, Prof.
(New York University, N. Y., U. S. A.)
KLOOSTERMAN, Hendrik Douwe, Prof.
(University of Leiden, Leiden, Netherlands)
Mrs. Kloosterman
KNEALE, Samuel George
(Harvard University, Mass., U. S. A.)
Mrs. Kneale
KNEBELMAN, Morris S., Prof.
(State College of Washington, Wash., U. S. A.)
KNEDLER, Paul Allen, Prof.
(State Teachers College, Pa., U. S. A.)
KNIGHT, Lyman Coleman, Jr., Prof.
(Muskingum College, Ohio, U. S. A.)
Mrs. Knight
Miss Judy Knight
Master Coleman Knight, III
KNOPP, Konrad, Prof.
(Universität Tübingen, Tübingen, Germany)
KODAIRA, Kunihiko, Dr.
(Institute for Advanced Study, N. J., U. S. A.; Tokyo University, Tokyo, Japan)
KODITSCHEK, Paul, Dr.
(Long Island University, N. Y., U. S. A.)
KOEHLER, Fulton, Prof.
(University of Minnesota, Minn., U. S. A.)
KOEHLER, Truman Lester, Prof.
(Muhlenberg College, Pa., U. S. A.)
KOGBETLIANTZ, Ervand George, Prof.
(New School for Social Research, N. Y., U. S. A.)
KOKOMOOR, Franklin Wesley, Prof.
(University of Florida, Fla., U. S. A.)
Mrs. Kokomoor
KOKSMA, Jurjen Ferdinand, Prof.
(Mathematisch Centrum, Amsterdam, Netherlands)
KOLCHIN, Ellis R., Prof.
(Columbia University, N. Y., U. S. A.)
Mrs. Kolchin
Mr. Peter Kolchin
KOOPMAN, Bernard Osgood, Prof.
(Columbia University, N. Y., U. S. A.)
KOPAL, Zdeněk, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
KOREVAAR, Jacob, Visiting Lecturer
(Purdue University, Ind., U. S. A.)
Mrs. Korevaar
KOSS, Walter Eddie, Prof.
(Texas Agricultural and Mechanical College, Tex., U. S. A.)
KÖTHE, Göttfried Maria, Prof.
(University of Mainz, Mainz, Germany)
KOTIK, Jack, Research Assistant
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Kotik
KOVACS, Elmer Charles, Instr.
(University of Pittsburgh, Pa., U. S. A.)
KOVITZ, Allan Theodore, Instr.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
KRAMER, Max, Prof.
(New Mexico College of Agriculture and Mechanic Arts, N. M., U. S. A.)
Mrs. Kramer
Miss Etel Thea Kramer
KRAMER, Mildred T., Instr.
(University of Pennsylvania, Pa., U. S. A.)

KRANTZ, John J., Prof.
(St. Bonaventure College, N. Y., U. S. A.)
KRANZER, Herbert Clyde
(New York University, N. Y., U. S. A.)
KRASNER, Marc, Dr.
(Centre National de la Recherche Scientifique, Paris, France)
KRAVETZ, Saul
(Harvard University, Mass., U. S. A.)
KRENTEL, William Delbert
(Oklahoma Agricultural and Mechanical College, Okla., U. S. A.)
KRIEGER, Cypra Cecilia, Prof.
(University of Toronto, Ont., Canada)
KRIKORIAN, Alice
(Columbia University, N. Y., U. S. A.)
KROCH, Aryeh
(Hebrew Institute of Technology, Haifa, Israel)
KRONSBEIN, John, Prof.
(Evansville College, Ind., U. S. A.)
KROTKOV, Valentina N., Instr.
(Queen's University, Ont., Canada)
KRUEGER, Raymond Leslie, Prof.
(Wittenberg College, Ohio, U. S. A.)
KRULL, Wolfgang, Prof.
(Universität Bonn, Bonn, Germany)
KRUSE, Arthur Herman
(University of Kansas, Kan., U. S. A.)
KUEBLER, Roy Raymond, Jr., Prof.
(Dickinson College, Pa., U. S. A.)
KUHN, Harold William, Instr.
(Princeton University, N. J., U. S. A.)
Mrs. Kuhn
VAN DER KULK, Wouter, Prof.
(Brown University, R. I., U. S. A.)
KULLBACK, Solomon, Lecturer
(George Washington University, D. C., U. S. A.)
KUNDERT, Essayas George, Dr.
(Washington, D. C., U. S. A.)
KUO, Ke-Chan, Dr.
(McKeesport, Pa., U. S. A.)
KUREPA, George, Prof.
(University of Zagreb, Zagreb, Yugoslavia)
KURTZ, Thomas Eugene
(Princeton University, N. J., U. S. A.)
KUTMAN, Helen, Instr.
(Hunter College, N. Y., U. S. A.)
KYLE, Roger Halliday
(Princeton University, N. J., U. S. A.)

L

LABARTHE, Pedro A., Dr.
(National School of Engineers, Lima, Peru)
Miss Lola Turchi
L'ABBÉ, Maurice Albert, Prof.
(Université de Montreal, Que., Canada)
LABRA, Manuel, Dr.
(Institute of Secondary Education of Matanzas, Matanzas, Cuba)
LAGERSTROM, Paco Axel, Prof.
(California Institute of Technology, Calif., U. S. A.)
LAGUARDIA, Rafael, Industrial Engineer
(Institute of Mathematics and Statistics, Montevideo, Uruguay)
LAHAYE, Edmond L., Prof.
(Université de Bruxelles, Brussels, Belgium)
LAMBEK, Joachim, Prof.
(McGill University, Que., Canada)
LAMPEN, Albert E., Prof.
(Hope College, Mich., U. S. A.)
LANCASTER, Otis Ewing, Dr.
(Bureau of Aeronautics, Navy Department, D. C., U. S. A.)
LANDERS, Aubrey Wilfred, Prof.
(Brooklyn College, N. Y., U. S. A.)
Mrs. Landers
Miss Patricia J. Landers
Master Richard B. Landers
Master Robert K. Landers
LANDIN, Joseph, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Landin
LANG, Serge
(Princeton University, N. J., U. S. A.)
LANGE, Lester H., Instr.
(Valparaiso University, Ind., U. S. A.)
LANGER, Rudolph E., Prof.
(University of Wisconsin, Wis., U. S. A.)
Mrs. Langer
LANING, J. Halcombe, Jr., Dr.
(Massachusetts Institute of Technology, Mass., U. S. A.)
LANZANO, Paolo, Instr.
(St. Louis University, Mo., U. S. A.)
LAPAZ, Lincoln, Prof.
(University of New Mexico, N. M., U. S. A.)
LAREW, Gillie Aldah, Prof.
(Randolph-Macon Woman's College, Va., U. S. A.)

LARGUIER, Rev. Everett Henry, Prof.
(Spring Hill College, Ala., U. S. A.)
LARKIN, James Richard, Research Assistant
(University of Kansas, Kan., U. S. A.)
LARSSON, Robert Dustin, Prof.
(Clarkson College, N. Y., U. S. A.)
LARUE, Alexandre, Prof.
(Laval University, Que., Canada)
LASALLE, Joseph Pierre, Prof.
(University of Notre Dame, Ind., U. S. A.)
Mrs. LaSalle
LASLEY, John Wayne, Jr., Prof.
(University of North Carolina, N. C., U. S. A.)
LASS, Harry, Prof.
(University of Illinois, Ill., U. S. A.)
LAUSH, George, Prof.
(University of Pittsburgh, Pa., U. S. A.)
LAWTON, Walter S., Prof.
(Temple University, Pa., U. S. A.)
Mrs. Lawton
Master Leslie Lawton
LEAH, Philip J.
(University of Toronto, Ont., Canada)
LEAVITT, William Grenfell, Prof.
(University of Nebraska, Neb., U. S. A.)
LE CORBEILLER, Philippe Emanuel, Prof.
(Harvard University, Mass., U. S. A.)
LEE, Erastus Henry, Prof.
(Brown University, R. I., U. S. A.)
LEE, Joseph Ross, Instr.
(University of Michigan, Mich., U. S. A.)
LEE, Mary Ann, Prof.
(Sweet Briar College, Va., U. S. A.)
LEFSCHETZ, Solomon, Prof.
(Princeton University, N. J., U. S. A.)
LEGER, George F., Jr.
(University of Illinois, Ill., U. S. A.)
LEHMER, D. H., Prof.
(University of California, Calif., U. S. A.)
Mrs. Lehmer
Miss Laura Lehmer
LEHNER, Joseph, Prof.
(University of Pennsylvania, Pa., U. S. A.)
Mrs. Lehner
Miss Janet Louise Lehner
LEHR, Marguerite, Prof.
(Bryn Mawr College, Pa., U. S. A.)
LEHTO, Olli Erkki, Dr.
(University of Helsinki, Helsinki, Finland)
LEIGHTON, Walter, Prof.
(Washington University, Mo., U. S. A.)
LEIMANIS, Eugene, Prof.
(University of British Columbia, B. C., Canada)
LEIMBACHER, Werner Rudolf, Research Assistant
(Swiss Federal Institute of Technology, Zürich, Switzerland)
LEIPNIK, Roy Bergh, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
LEISENRING, Kenneth Baylis, Instr.
(University of Michigan, Mich., U. S. A.)
LEONOR, Concepcion
(University of Santo Tomas, Philippines)
LEPAGE, Théophile H. J., Prof.
(Université de Bruxelles, Brussels, Belgium)
LEPSON, Benjamin, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
LERAY, Jean, Prof.
(Collège de France, Paris, France)
LESTER, Caroline Avery, Prof.
(New York State College for Teachers at Albany, N. Y., U. S. A.)
LEUTERT, Werner Walter, Prof.
(University of Maryland, Md., U. S. A.)
LEVENE, Howard, Instr.
(Columbia University, N. Y., U. S. A.)
LEVENSON, Arthur J., Research Analyst
(Armed Forces Security Agency, D. C., U. S. A.)
LEVEQUE, William Judson, Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. LeVeque
LEVI, Howard, Prof.
(Columbia University, N. Y., U. S. A.)
LEVINSON, Norman, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Levinson
Miss Joan E. Levinson
Miss Sylvia C. Levinson
LEVIT, Robert Jules, Prof.
(University of Georgia, Ga., U. S. A.)
LEVITZKI, Jakob, Dr.
(Hebrew University, Jerusalem, Israel)
Mrs. Levitzki
LEVY, Harry, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Levy

LÉVY, Paul Pierre, Prof.
(École Polytechnique, Paris, France)
Mrs. Lévy
LEWIS, Anne Louise, Prof.
(Woman's College of the University of North Carolina, N. C., U. S. A.)
LEWIS, Father Charles Joseph
(Brown University, R. I., U. S. A.)
LEWIS, Daniel Clark, Prof.
(Johns Hopkins University, Md., U. S. A.)
LEWIS, David R.
(University of Minnesota, Minn., U. S. A.)
LEWIS, Donald John, Instr.
(Ohio State University, Ohio, U. S. A.)
LEWIS, Brother George, Prof.
(LaSalle College, Pa., U. S. A.)
LEWIS, John Allen, Research Physicist
(Corning Glass Works, N. Y., U. S. A.)
Mrs. Lewis
LEWIS, Sister Mary Teresine, Instr.
(Fontbonne College, Mo., U. S. A.)
Sister M. Pachomia Lackey
LEWY, Hans, Prof.
(University of California, Calif., U. S. A.)
Mrs. Lewy
LI, Ta Chung-Heng, Prof.
(Drake University, Iowa, U. S. A.)
LICHNEROWICZ, André, Prof.
(Université de Paris, Paris, France)
LIEBLEIN, Julius, Mathematician
(National Bureau of Standards, D. C., U. S. A.)
Mrs. Lieblein
Mr. Jack B. Lieblein
LIN, Chia Chiao, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Lin
LINK, Richard F.
(University of Oregon, Ore., U. S. A.)
LIPSEY, Sally Irene, Lecturer
(Hunter College, N. Y., U. S. A.)
Mr. Robert E. Lipsey
LIPSICH, H David, Instr.
(University of Cincinnati, Ohio, U. S. A.)
LITZINGER, Marie, Prof.
(Mount Holyoke College, Mass., U. S. A.)
LJUNGGREN, Wilhelm, Prof.
(Universitetet i Bergen, Bergen, Norway)
LOCKE, Cynthia Anne
(Harvard University, Mass., U. S. A.)
LOCKE, Robert W. (Associate Member)
(McGraw-Hill Book Company, Inc., N. Y., U. S. A.)
LOCKHART, Brooks J., Prof.
(United States Naval Postgraduate School, Md., U. S. A.)
LODE, Tenny
(University of Minnesota, Minn., U. S. A.)
LOEWNER, Charles, Prof.
(Syracuse University, N. Y., U. S. A.)
Mrs. Loewner
Miss Marian Loewner
LOGSDON, Mayme I., Prof.
(University of Miami, Fla., U. S. A.)
LOH, Zung-Nyi, Prof.
(Wilson College, Pa., U. S. A.)
LOHWATER, Arthur J., Instr.
(University of Michigan, Mich., U. S. A.)
Mrs. Lohwater
LONGLEY, William Raymond, Prof.
(Yale University, Conn., U. S. A.)
Mrs. Longley
LOOMIS, Lynn H., Prof.
(Harvard University, Mass., U. S. A.)
Mrs. Loomis
Miss Susan Loomis
Miss Joan Loomis
LOONSTRA, Frans, Prof.
(Technische Hogeschool, Delft, Netherlands)
LORCH, Edgar Raymond, Prof.
(Columbia University, N. Y., U. S. A.)
LORCH, Lee, Prof.
(Fisk University, Tenn., U. S. A.)
LORENTZ, George, Prof.
(University of Toronto, Ont., Canada)
LORENZEN, Paul, Dozent
(Universität Bonn, Bonn, Germany)
LOUD, Warren Simms, Prof.
(University of Minnesota, Minn., U. S. A.)
LOVETT, Edgar Odell, President Emeritus
(Rice Institute, Tex., U. S. A.)
LOWAN, Arnold N., Dr.
(Naval Ordnance Laboratory, Md., U. S. A.)
LOWELL, Sherman C., Prof.
(Institute for Mathematics and Mechanics, New York University, N. Y., U. S. A.)
LOZADA, Faustino Roy
(University of the Philippines, Manila, Philippines)
LUBIN, Clarence Isaac, Prof.
(University of Cincinnati, Ohio, U. S. A.)

LUCE, Robert Duncan, Instr.
(Massachusetts Institute of Technology, Mass., U. S. A.)
LÜHRSEN, J. Conrad, Dr.
(National Board of Economics, Havana, Cuba)
Mrs. Lührsen
LUKACS, Eugene, Statistician
(National Bureau of Standards, D. C., U. S. A.)
Mrs. Lukacs
LUKENS, Jaywood
(Darien, Conn., U. S. A.)
LYNCH, Rev. John Joseph, Prof.
(Fordham University, N. Y., U. S. A.)
LYNDON, Roger C., Prof.
(Princeton University, N. J., U. S. A.)
DE LYRA, Carolos Benjamin
(University of São Paulo, São Paulo, Brazil)

M

MCARTNEY, June Marie, Instr.
(University of Buffalo, N. Y., U. S. A.)
MCBRIEN, Vincent O., Prof.
(College of the Holy Cross, Mass., U. S. A.)
MCCALLION, William James, Lecturer
(McMaster University, Ont., Canada)
MCCARTHY, John
(Princeton University, N. J., U. S. A.)
MCCLENON, Raymond Benedict, Prof.
(Grinnell College, Iowa, U. S. A.)
Mrs. McClenon
Mr. John R. McClenon
MCCONNELL, Albert Joseph, Prof.
(Trinity College, University of Dublin, Dublin, Ireland)
MCCONNELL, James R., Prof.
(St. Patrick's College, Maynooth, Ireland)
MCCONNELL, Robert Kinsey, Instr.
(New York University, N. Y., U. S. A.)
MCCONNON, Myles, Prof.
(Norwich University, Vt., U. S. A.)
Mrs. McConnon
MCCOY, Neal Henry, Prof.
(Smith College, Mass., U. S. A.)
MCCUSKEY, Sidney W., Prof.
(Case Institute of Technology, Ohio, U. S. A.)
MCDANIEL, Wilbur Charles, Prof.
(Southern Illinois University, Ill., U. S. A.)
MCDONALD, Janet, Prof.
(Vassar College, N. Y., U. S. A.)
MCDONALD, Sophia Levy, Prof.
(University of California, Calif., U. S. A.)
MCDOUGLE, Edith Augusta, Instr.
(University of Delaware, Del., U. S. A.)
MCEWEN, William Hayward, Prof.
(University of Manitoba, Man., Canada)
MCEWEN, William R., Prof.
(University of Minnesota, Duluth, Minn., U. S. A.)
MCFARLAN, Lee Horace, Prof.
(University of Washington, Wash., U. S. A.)
Mrs. McFarlan
MCKEACHIE, Duane D.
(General Motors Institute, Mich., U. S. A.)
MCKINSEY, J. C. C., Mathematician
(RAND Corporation, Calif., U. S. A.)
MCKNIGHT, Betty, Prof.
(Centenary College of Louisiana, La., U. S. A.)
MCLAUGHLIN, Isabel Cecilia, Instr.
(Hunter College, N. Y., U. S. A.)
MCMANAMA, Sister M. Thaddeus, Prof.
(Seton Hill College, Pa., U. S. A.)
MCMILLAN, Brockway, Dr.
(Bell Telephone Laboratories, N. J., U. S. A.)
Mrs. McMillan
Miss Sarah McMillan
Mr. Douglas McMillan
MCNEIL, Sister Marie Gertrude, Prof.
(Seton Hill College, Pa., U. S. A.)
MCSHANE, Edward James, Prof.
(University of Virginia, Va., U. S. A.)
MCVITTIE, George Cunliffe, Prof.
(Queen Mary College, University of London, London, England)
Mrs. McVittie
MCVOY, Kirk Warren, Jr.
(Carleton College, Minn., U. S. A.)
MACCOLL, LeRoy Archibald, Dr.
(Bell Telephone Laboratories, N. Y., U. S. A.)
MACDUFFEE, Cyrus Colton, Prof.
(University of Wisconsin, Wis., U. S. A.)
MACKEY, George Whitelaw, Prof.
(Harvard University, Mass., U. S. A.)

MacLane, Gerald Robinson, Prof.
(Rice Institute, Tex., U. S. A.)
Mrs. MacLane
MacLane, Saunders, Prof.
(University of Chicago, Ill., U. S. A.)
Mrs. MacLane
MacNeille, H. M., Executive Director
(American Mathematical Society, N. Y., U. S. A.)
MacNeish, Harris Franklin, Prof.
(University of Miami, Fla., U. S. A.)
Mrs. MacNeish
MacPhail, Moray St. John, Prof.
(Carleton College, Ont., Canada)
MacPhee, Hugh J.
(Loyola College, Que., Canada)
Madow, William Gregory, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Madow
Magee, Gordon R., Prof.
(University of Western Ontario, Ont., Canada)
Mrs. Magee
Mr. Peter Magee
Magnus, Arne
(Washington University, Mo., U. S. A.)
Magnus, Margaret Hope, Instr.
(Washington University, Mo., U. S. A.)
Magnus, Wilhelm, Dr.
(California Institute of Technology, Calif., U. S. A.)
Mahalanobis, Prasanta Chandra, Director
(Indian Statistical Institute, Calcutta, India)
Mahler, Kurt, Reader
(University of Manchester, Manchester, England)
Malin, Henry, Prof.
(City College of New York, N. Y., U. S. A.)
Mall, Adolph William, Instr.
(St. Lawrence University, N. Y., U. S. A.)
Mrs. Mall
Malti, Michel George, Prof.
(Cornell University, N. Y., U. S. A.)
Mr. George Malti
Malvern, Lawrence E., Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
Mandelbrojt, Szolem, Prof.
(Collège de France, Paris, France)
Mann, Henry Berthold, Prof.
(Ohio State University, Ohio, U. S. A.)
Mann, Irwin, Lecturer
(Columbia University, N. Y., U. S. A.)
Mann, W. Robert, Prof.
(University of North Carolina, N. C., U. S. A.)
Manneback, Charles, Prof.
(Université de Louvain, Louvain, Belgium)
Manning, Rhoda, Prof.
(Oregon State College, Ore., U. S. A.)
Manos, Eleutherios Theophanes
(Harvard University, Mass., U. S. A.)
Marcou, René J., Prof.
(Boston University, Mass., U. S. A.)
Marden, Morris, Prof.
(University of Wisconsin, Wis., U. S. A.)
Marshall, Andrew Walter, Statistician
(RAND Corporation, Calif., U. S. A.)
Marth, Ella Carolyn, Prof.
(Harris Teachers College, Mo., U. S. A.)
Martin, Antonio R.
(San Francisco, Calif., U. S. A.)
Martin, Monroe Harnish, Prof.
(University of Maryland, Md., U. S. A.)
Martin, William Ted, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Martin
Mr. Seelye Martin
Mr. James E. Martin
Mr. William G. Martin
Mason, Hazel Lolita
(Grand Prairie Junior High School, Tex., U. S. A.)
Mason, Sewell Luverne, Prof.
(University of North Dakota, N. D., U. S. A.)
Massera, José L., Industrial Engineer
(University of Uruguay, Montevideo, Uruguay)
Massey, William S., Dr.
(Princeton University, N. J., U. S. A.)
Matheson, John Douglas, Instr.
(United States Military Academy, N. Y., U. S. A.)
Mathews, Charles Willard, Prof.
(Washington University, Mo., U. S. A.)
Mautner, Frederich Ignaz, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Mautner
May, Kenneth O., Prof.
(Carleton College, Minn., U. S. A.)

MAY, Lida B., Prof.
(Texas Technological College, Tex., U. S. A.)
Mrs. Minnie L. Stewart
MAYBERRY, John Patterson
(Princeton University, N. J., U. S. A.)
MAYOR, John Roberts, Prof.
(University of Wisconsin, Wis., U. S. A.)
MEACHAM, Alan D.
(Remington Rand, Inc., N. Y., U. S. A.)
MEACHAM, Robert C., Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
MEAD, David Godfrey, Instr.
(Williams College, Mass., U. S. A.)
MEDER, Albert Eugene, Jr., Dean
(Rutgers University, N. J., U. S. A.)
Dr. Elsa M. Meder
MEHLENBACHER, Lyle E., Prof.
(University of Detroit, Mich., U. S. A.)
MEIER, Paul, Statistician
(Philadelphia Tuberculosis and Health Association, Pa., U. S. A.)
MENDEL, Clifford William, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Mendel
MENDELSOHN, Nathan S., Prof.
(University of Manitoba, Man., Canada)
MENGER, Karl, Prof.
(Illinois Institute of Technology, Ill., U. S. A.)
MERRIELL, David McCray, Instr.
(University of Chicago, Ill., U. S. A.)
MESERVE, Bruce Elwyn, Prof.
(University of Illinois, Ill., U. S. A.)
MEWBORN, A. Boyd, Prof.
(United States Naval Postgraduate School, Calif., U. S. A.)
MEYERS, Leroy Frederick, Instr.
(Syracuse University, N. Y., U. S. A.)
MICHAEL, Ernest A.
(University of Chicago, Ill., U. S. A.)
MICHALOPOULOS, Nikolaos George, Dr.
(University of Athens, Athens, Greece)
MICHALUP, Erich M., Prof.
(University of Caracas, Caracas, Venezuela)
Mrs. Michalup
Miss Eva Michalup
Mr. Freddy Michalup
MILES, Ernest P., Jr., Prof.
(Alabama Polytechnic Institute, Ala., U. S. A.)
Mrs. Miles
MILGRAM, Arthur Norton, Prof.
(Syracuse University, N. Y., U. S. A.)
MILGRAM, Martin
(Brooklyn, N. Y., U. S. A.)
MILLER, Alton Lombard, Dr.
(Charles N. Miller Co., Mass., U. S. A.)
MILLER, Don Dalzell, Prof.
(University of Tennessee, Tenn., U. S. A.)
MILLER, Jeffrey Charles Percy, Consultant
(National Bureau of Standards, D. C., U. S. A.)
MILLER, Jack Milton
(University of Michigan, Mich., U. S. A.)
MILLER, Kenneth Sielke, Instr.
(New York University, N. Y., U. S. A.)
MILLER, Norman, Prof.
(Queen's University, Ont., Canada)
Mrs. Miller
MILLOUX, Henri, Prof.
(Université de Bordeaux, Bordeaux, France)
Mrs. Milloux
MILLS, Vicente, Engineer
(Tuguegaras Branch Office, United States-Philippine War Damage Commission, Manila, Philippines)
MILLS, William H., Instr.
(Yale University, Conn., U. S. A.)
Mrs. Mills
MILLSAPS, Knox, Prof.
(Alabama Polytechnic Institute, Ala., U. S. A.)
MILNE, W. E., Prof.
(Oregon State College, Ore., U. S. A.)
Mrs. Milne
MILNOR, John Willard
(Princeton University, N. J., U. S. A.)
MILOSLAVSKY, George Constantinovich
(New York, N. Y., U. S. A.)
MINAKSHISUNDARAM, Subbaramiah, Prof.
(Andhra University, Waltair, S. India)
DE MIRANDA-NETTO, Antonio Garcia, Prof.
(University of Brazil, Rio de Janeiro, Brazil)
Mrs. de Miranda-Netto
MIRKIL, Hazelton
(University of Chicago, Ill., U. S. A.)
MISER, Hugh J., Operations Analyst
(Headquarters United States Air Force, D. C., U. S. A.)
MISER, Wilson Lee, Prof.
(Vanderbilt University, Tenn., U. S. A.)

von Mises, Richard, Prof.
(Harvard University, Mass., U. S. A.)
Mitchell, Josephine Margaret, Prof.
(University of Illinois, Ill., U. S. A.)
Mode, Elmer Beneken, Prof.
(Boston University, Mass., U. S. A.)
Mrs. Mode
Moise, Edwin Evariste, Dr.
(Tulane University of Louisiana, La., U. S. A.)
Molina, Edward Charles, Lecturer
(Newark College of Engineering, N. J., U. S. A.)
Monjallon, Albert Eugène, Prof.
(Lycée Henri IV, Paris, France)
Monro, Sutton
(University of North Carolina, N. C., U. S. A.)
Monroe, Robert E. (Associate Member)
(McGraw-Hill Book Company, Inc., N. Y., U. S. A.)
Montague, Harriet Frances, Prof.
(University of Buffalo, N. Y., U. S. A.)
Monteiro, A. A., Prof.
(University of Brazil, Rio de Janeiro, Brazil)
Montgomery, Deane, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
Montgomery, Mabel D., Instr.
(University of Buffalo, N. Y., U. S. A.)
Moody, Edward F., Jr.
(John Wiley and Sons, Inc., N. Y., U. S. A.)
Mooers, Calvin N., Director of Research
(Zator Company, Mass., U. S. A.)
Moore, Charles Napoleon, Prof.
(University of Cincinnati, Ohio, U. S. A.)
Mrs. Moore
Moore, John Coleman
(Belmont, Mass., U. S. A.)
Moore, Thomas W., Prof.
(United States Naval Academy, Md., U. S. A.)
Moore, Walter Lee, Prof.
(University of Louisville, Ky., U. S. A.)
Moore, Warren Keith
(University of Kansas, Kan., U. S. A.)
Mrs. Moore
Mr. David Moore
Moorman, Richard Herbert, Prof.
(Tennessee Polytechnic Institute, Tenn., U. S. A.)
del Moral, Luis Marti, Prof.
(Sancti Spiritus Institute, Las Villas, Yslade, Cuba)
Morawetz, Cathleen Synge
(New York University, N. Y., U. S. A.)
Mordell, Louis Joel, Prof.
(Cambridge University, Cambridge, England)
Mrs. Mordell
Morenus, Eugenie Maria, Prof.
(Sweet Briar College, Va., U. S. A.)
Morgan, Kathryn Anne, Instr.
(Syracuse University, N. Y., U. S. A.)
Moriguti, Sigeiti, Prof.
(Tokyo University, Tokyo, Japan)
Morikawa, George Kiyoshi, Aerodynamicist
(Howard Hughes Aircraft Company, Calif., U. S. A.)
Morley, Raymond Kurtz, Prof.
(Worcester Polytechnic Institute, Mass., U. S. A.)
Morrey, Charles Bradfield, Jr., Prof.
(University of California, Calif., U. S. A.)
Mrs. Morrey
Miss Carolyn G. Morrey
Mr. Robert A. Morrey
Master Walter T. Morrey
Morris, Frank R., Prof.
(Fresno State College, Calif., U. S. A.)
Mrs. Morris
Morse, David Sherman, Prof.
(Union College, N. Y., U. S. A.)
Mrs. Morse
Morse, Marston, Prof.
(Institute for Advanced Study, N. J., U. S. A.)
Mrs. Morse
Moser, William Oscar
(University of Minnesota, Minn., U. S. A.)
Mosteller, Frederick, Prof.
(Harvard University, Mass., U. S. A.)
Moston, Lester Turner, Dean
(Waynesburg College, Pa., U. S. A.)
Mostow, George Daniel, Prof.
(Syracuse University, N. Y., U. S. A.)
Mrs. Mostow
Motzkin, Theodore S., Prof.
(Hebrew University, Jerusalem, Israel)
Moulton, Elton James, Prof.
(Northwestern University, Ill., U. S. A.)
Mrs. Moulton

MOUSINHO, Maria Larua, Dr.
(University of Brazil, Rio de Janeiro, Brazil)
MOYLS, Benjamin Nelson, Prof.
(University of British Columbia, B. C., Canada)
Mrs. Moyls
MULLENDER, Pieter, Dr.
(Free University of Amsterdam, Amsterdam, Netherlands)
MUNDHJELD, Sigurd, Prof.
(Concordia College, Minn., U. S. A.)
MUNSHOWER, Carl Wallace, Prof.
(Colgate University, N. Y., U. S. A.)
Mrs. Munshower
Miss Ann Chalmers Munshower
MURPHY, Ray Bradford, Instr.
(Carnegie Institute of Technology, Pa., U. S. A.)
MURRAY, Forrest Hamilton
(Oak Ridge National Laboratory, Tenn., U. S. A.)
MUSSELMAN, John Rogers, Prof.
(Western Reserve University, Ohio, U. S. A.)
MYERS, Sumner B., Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Myers
Mr. David F. Myers
Miss Alison Myers

N

NACHBIN, Leopoldo, Prof.
(University of Brazil, Rio de Janeiro, Brazil)
NAKAYAMA, Tadasi, Prof.
(Nagoya University, Nagoya, Japan)
NANNI, Luis Fernando, Instr.
(Rutgers University, N. J., U. S. A.)
NÁPOLES-GÁNDARA, Alfonso, Dr.
(University of Mexico, Mexico, D. F., Mexico)
NASH, John Forbes, Jr.
(Princeton University, N. J., U. S. A.)
NATHAN, David S., Prof.
(City College of New York, N. Y., U. S. A.)
NEELLEY, John Haven, Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
NEHARI, Zeev, Prof.
(Washington University, Mo., U. S. A.)
Mrs. Nehari

NELSON, Cyril A., Prof.
(New Jersey College for Women, N. J., U. S. A.)
NELSON, David, Prof.
(George Washington University, D. C., U. S. A.)
NELSON, Thorman Archer, Instr.
(Muhlenberg College, Pa., U. S. A.)
NEMÉNYI, Paul F., Dr.
(Naval Research Laboratory, D. C.; University of Maryland, Md., U. S. A.)
NERING, Evar Dare, Prof.
(University of Minnesota, Minn., U. S. A.)
NESBEDA, Paul P., Instr.
(Catholic University of America, D. C., U. S. A.)
NETANYAHU, Elisha, Lecturer
(Hebrew Technical College, Haifa, Israel)
Mrs. Netanyahu
NEUGEBAUER, O., Prof.
(Brown University, R. I., U. S. A.)
VON NEUMANN, John, Prof.
(Institute for Advanced Study, N. J., U. S. A.)
Mrs. von Neumann
NEVANLINNA, Rolf Herman, Prof.
(University of Zürich, Zürich, Switzerland)
NEWBURGH, John David, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
NEWMAN, Maxwell H. A., Prof.
(University of Manchester, Manchester, England)
NEWSOM, Carroll Vincent, Dr.
(University of the State of New York, N. Y., U. S. A.)
Mrs. Newsom
Mr. Walter B. Newsom
Mr. Gerald H. Newsom
NEYMAN, Jerzy, Prof.
(University of California, Calif., U. S. A.)
NICKERSON, Helen K., Dr.
(Princeton, N. J., U. S. A.)
Master Kenneth Nickerson
NIELSEN, Kaj Leo, Head, Mathematics Division
(Naval Ordnance Plant, Ind., U. S. A.)
Mrs. Nielsen
NIKODÝM, Otton Martin, Prof.
(Kenyon College, Ohio, U. S. A.)
Mrs. Nikodým

NILSON, Edwin Norman, Prof.
(Trinity College, Conn., U. S. A.)
Mrs. Nilson

NIVEN, Ivan Morton, Prof.
(University of Oregon, Ore., U. S. A.)

NOEL MARIE, Sister, Prof.
(College of St. Rose, N. Y., U. S. A.)

NOMIZU, Katsumi, Instr.
(Nagoya University, Nagoya, Japan)

NORDEN, Monroe L., Mathematical Statistician
(Operations Research Office, D. C., U. S. A.)
Mrs. Norden

NORDGAARD, Martin R., Prof.
(Upsala College, N. J., U. S. A.)

NØRLUND, Niels Erik, Prof.
(University of Copenhagen, Copenhagen, Denmark)
Mrs. Nørlund

NORMAN, Philip Brown, Prof.
(Wagner College, N. Y., U. S. A.)

NORRIS, Michael Joseph, Prof.
(College of St. Thomas, Minn., U. S. A.)
Mrs. Norris

NORRIS, Nilan, Prof.
(Hunter College, N. Y., U. S. A.)

NORTHROP, Eugene P., Prof.
(University of Chicago, Ill., U. S. A.)

NORWOOD, Lawrence, Mathematician
(United States Signal Corps, N. J., U. S. A.)

NORWOOD, Virginia Tower
(Red Bank, N. J., U. S. A.)

NOVAK, Ilse Lisl, Research Fellow
(University of California, Calif., U. S. A.)

O

OAKLEY, Cletus O., Prof.
(Haverford College, Pa., U. S. A.)

OBI, Chike, Dr.
(Pembroke College, Cambridge University, Cambridge, England)

O'BRIEN, Katharine E., Dr.
(Deering High School, Me., U. S. A.)

O'CONNOR, R. Eric., Prof.
(Loyola College, Que., Canada)

ODLE, John William, Dr.
(Naval Ordnance Test Station, Calif., U. S. A.)
Mrs. Odle

O'DONNELL, George Anthony, Dean, Graduate School
(Boston College, Mass., U. S. A.)

O'DONNELL, Ruth Eileen, Prof.
(Duquesne University, Pa., U. S. A.)

ODQVIST, Folke Gustaf, Prof.
(Royal Institute of Technology, Stockholm, Sweden)
Mrs. Odqvist

OFFORD, Albert Cyril, Prof.
(Birkbeck College, University of London, London, England)

OGAWA, George, Instr.
(Illinois Institute of Technology, Ill., U. S. A.)

OGG, Frank Chappell, Prof.
(Bowling Green State University, Ohio, U. S. A.)
Mrs. Ogg
Mr. Frank C. Ogg, Jr.

OLDENBURGER, Rufus, Mathematician-engineer
(Woodward Governor Company, Ill., U. S. A.)
Mrs. Oldenburger

OLMSTED, John M. H., Prof.
(University of Minnesota, Minn., U. S. A.)

OLUM, Paul, Prof.
(Cornell University, N. Y., U. S. A.)

O'NEILL, Sister Anna Concilio, Prof.
(College of St. Elizabeth Convent, N. J., U. S. A.)
Sister Agnes Concilio Ryan

O'NEILL, Anne Frances, Prof.
(Smith College, Mass., U. S. A.)

OPPENHEIM, Alexander, Dr.
(University of Malaya, Singapore, Malaya)

O'SHEA, Rev. Edward Francis
(Weston College, Mass., U. S. A.)

OSSERMAN, Robert
(Harvard University, Mass., U. S. A.)

OTTER, Richard, Prof.
(University of Notre Dame, Ind., U. S. A.)

OVERHOLTZER, Gordon Klinzman, Prof.
(Purdue University, Ind., U. S. A.)

OVERN, Orlando Evan Anthony, Dr.
(State Teachers College, Wis., U. S. A.)

OWCHAR, Margaret, Prof.
(Southwest Missouri State College, Mo., U. S. A.)

OWENS, Frederick William, Prof.
(Pennsylvania State College, Pa., U. S. A.)
Mrs. Owens

Oxtoby, John C., Prof.
(Bryn Mawr College, Pa., U. S. A.)

P

Paine, Stephen Tyler, Mathematician
(National Bureau of Standards, Calif., U. S. A.)
Palais, Richard Sheldon
(Harvard University, Mass., U. S. A.)
Pall, Gordon, Prof.
(Illinois Institute of Technology, Ill., U. S. A.)
Mrs. Pall
Miss Gail Pall
Mr. Martin Pall
Palmer, Theodore Paine, Prof.
(Rose Polytechnic Institute, Ind., U. S. A.)
Mrs. Palmer
Papoulis, Athanasios, Instr.
(University of Pennsylvania, Pa., U. S. A.)
Papy, George Léopold, Dr.
(Université de Bruxelles, Brussels, Belgium)
Pararas, Toula L., Prof.
(University of Athens, Athens, Greece)
Pardee, Otway O'Meara, Instr.
(Syracuse University, N. Y., U. S. A.)
Parke, Nathan Grier, III, Dr.
(Concord, Mass., U. S. A.)
Parker, Sidney Thomas, Prof.
(Kansas State College, Kan., U. S. A.)
Pars, Leopold Alexander, University Lecturer
(Jesus College, Cambridge University, Cambridge, England)
Parvulescu, Antares, Prof.
(University of Witwatersrand, Johannesburg, South Africa)
Patterson, George W., Prof.
(University of Pennsylvania, Pa., U. S. A.)
Paxton, Earle Kerr, Prof.
(Virginia Military Institute, Va., U. S. A.)
Peabody, Mary Kent, Instr.
(Smith College, Mass., U. S. A.)
Pease, Donald Knight, Instr.
(University of Connecticut, Conn., U. S. A.)
Peck, John Edward Lancelot, Instr.
(Brown University, R. I., U. S. A.)
Peck, Leslie Gilbert, Dr.
(Los Alamos Scientific Laboratory, N.M., U. S. A.)
Pedoe, Daniel, Reader in Mathematics
(Westfield College, University of London, London, England)
Pedrick, George B.
(Oklahoma Agricultural and Mechanical College, Okla., U. S. A.)
Peixoto, Mauricio Matos, Prof.
(University of Brazil, Rio de Janeiro, Brazil)
Mrs. Peixoto
Pekeris, Chaim Leib, Prof.
(Weizmann Institute of Science, Rehovoth, Israel)
Mrs. Pekeris
Pellegrino, Franco, Coadiutore dell' Istituto Nazionale di Alta Matematica
(Università di Roma, Rome, Italy)
Pendry, Frances Minnie Anne
(John Howard School, London, England)
Penez, Jacqueline Lydie
(Brown University, R. I., U. S. A.)
Penico, Anthony Joseph, Instr.
(Tufts College, Mass., U. S. A.)
Pennell, Walter Otis
(Massachusetts Institute of Technology, Mass., U. S. A.)
Pennington, John Virgil, Director of Research
(Drilling Research, Inc., Tex., U. S. A.)
Pennington, William Barry
(Cambridge University, Cambridge, England)
Pepper, Paul Milton, Prof.
(Ohio State University, Ohio, U. S. A.)
Pepper, Roy Irvine, Prof.
(Winthrop College, S. C., U. S. A.)
Perkins, Fred William, Prof.
(Dartmouth College, N. H., U. S. A.)
Perry, Clay Lamont, Jr., Prof.
(University of Arkansas, Ark., U. S. A.)
Peters, Iland Dee, Prof.
(West Virginia University, W. Va., U. S. A.)
Mrs. Peters
Peters, Ruth M., Prof.
(St. Lawrence University, N. Y., U. S. A.)
Peterson, Homer Charles, Instr.
(University of Denver, Colo., U. S. A.)
Peterson, Thurman Stewart, Prof.
(University of Oregon, Ore., U. S. A.)

PETTIS, Billy James, Prof.
(Tulane University of Louisiana, La., U. S. A.)
PETTIT, Harvey P., Prof.
(Marquette University, Wis., U. S. A.)
PFLAUM, Charles Whaley
(Darrow School, N. Y., U. S. A.)
PFLUGER, Albert Joseph, Prof.
(Eidgenössische Technische Hochschule, Zürich, Switzerland)
Mrs. Pfluger
PHALEN, Harold Romaine, Prof.
(College of William and Mary, Va., U. S. A.)
PHILLIPS, Ralph Saul, Prof.
(University of Southern California, Calif., U. S. A.)
PICONE, Mauro, Prof.
(Università di Roma, Rome, Italy)
PIERCE, Joseph Alphonso, Prof.
(Texas State University, Tex., U. S. A.)
PIERCE, William Augustus, Dr.
(Harvard University, Mass., U. S. A.)
PIHLBLAD, Margaret M., Instr.
(University of Kansas, Kan., U. S. A.)
PILLAI, S. Sivasankaranarayana, Lecturer
(Calcutta University, Calcutta, India)
PINNEY, Edmund, Prof.
(University of California, Calif., U. S. A.)
PIPER, Robert Irving
(Pacific Telephone and Telegraph Company, Calif., U. S. A.)
PIPES, Louis Albert, Prof.
(University of California, Los Angeles, Calif., U. S. A.)
PIRANIAN, George, Prof.
(University of Michigan, Mich., U. S. A.)
PITCAIRN, Joel
(University of Pennsylvania, Pa. U. S. A.)
Mrs. Pitcairn
PITCHER, Arthur Everett, Prof.
(Lehigh University, Pa., U. S. A.)
Mrs. Pitcher
Miss Joan S. Pitcher
Miss Susan S. Pitcher
PITTS, Raymond Jackson, Prof.
(Fort Valley State College, Ga., U. S. A.)
PLITHIDES, Constantinos C.
(Columbia University, N. Y., U. S. A.)
PODMELE, Theresa Louise
(East High School, Buffalo, N. Y., U. S. A.)
Miss Lorraine T. Podmele

POHLHAUSEN, Karl, Dr.
(Office of Air Research, Wright-Patterson Air Force Base, Ohio, U. S. A.)
POLACHEK, Harry, Dr.
(Naval Ordnance Laboratory, Md., U. S. A.)
POLANSKY, Frank Joseph, Prof.
(University of Minnesota, Minn., U. S. A.)
POLCYN, Stephen Joseph, Instr.
(New York University, N. Y., U. S. A.)
POLLACZEK, Félix, Dr.
(Centre National de la Recherche Scientifique, Paris, France)
POLLAK, Henry Otto
(Harvard University, Mass., U. S. A.)
POLLARD, Harry, Prof.
(Cornell University, N. Y., U. S. A.)
Mrs. Pollard
POLLEY, Joseph Crawford, Prof.
(Wabash College, Ind., U. S. A.)
PÓLYA, George, Prof.
(Stanford University, Calif., U. S. A.)
PONDS, Johnny Webster, Prof.
(West Virginia State College, W. Va., U. S. A.)
POPKEN, Jan, Prof.
(Rijks-Universiteit, Utrecht, Netherlands)
PORITSKY, Hillel, Consulting Engineer
(General Electric Company, N. Y., U. S. A.)
Mrs. Poritsky
POSEY, Eldon Eugene, Instr.
(University of Tennessee, Tenn., U. S. A.)
POST, Emil Leon, Prof.
(City College of New York, N. Y., U. S. A.)
POSTON, Paul Lehman, Instr.
(Wayne University, Mich., U. S. A.)
POULIOT, Adrien, Dean of the Faculty of Science
(Laval University, Que., Canada)
POUNDER, Irvine Rudsdale, Prof.
(University of Toronto, Ont., Canada)
Mrs. Pounder
PRAGER, William, Prof.
(Brown University, R. I., U. S. A.)
PRAKASH BAMBAH, Ram
(St. John's College, Cambridge University, Cambridge, England)
PRASAD, Badri Nath, Prof.
(Science College, Patna University, Patna, India)

PRESTON, Glenn Wetherby, Physicist
(Goodyear Aircraft Corporation, Ohio, U. S. A.)
PRICE, Griffith Baley, Prof.
(University of Kansas, Kan., U. S. A.)
Mrs. Price
PRIEST, Richard Eugene
(University of Illinois, Ill., U. S. A.)
PRIM, Robert Clay, Dr.
(Bell Telephone Laboratories, N. J., U. S. A.)
PRODI, Giovanni
(Università di Milano, Milan, Italy)
PROSKAUER, Eric S., Editor in Chief
(Interscience Publishers, N. Y., U. S. A.)
PROTTER, M. H., Prof.
(Syracuse University, N. Y., U. S. A.)
Mrs. Protter
PULLIAM, Francis McConnell, Prof.
(United States Naval Postgraduate School, Md., U. S. A.)
PURCELL, Edwin J., Prof.
(University of Arizona, Ariz., U. S. A.)
PURDY, John F., Mathematician
(Goodyear Tire and Rubber Company, Ohio, U. S. A.)
PUTNAM, Alfred Lunt, Prof.
(University of Chicago, Ill., U. S. A.)
PYLE, H. Randolph, Prof.
(Whittier College, Calif., U. S. A.)

R

RABINOWITZ, Philip
(University of Pennsylvania, Pa., U.S.A.)
RACAH, Giulio, Prof.
(Hebrew University, Jerusalem, Israel)
RACINE, Charles, Prof.
(Loyola College, Madras, India)
RADEMACHER, Hans, Prof.
(University of Pennsylvania, Pa., U.S.A.)
Mrs. Rademacher
RADO, Richard, University Reader
(King's College, University of London, London, England)
RADÓ, Tibor, Prof.
(Ohio State University, Ohio, U. S. A.)
Mrs. Radó
RAEZER, Spencer Dorworth
(Franklin and Marshall College, Pa., U. S. A.)
RAFAEL, Henry, Prof.
(Catholic Institute of Arts and Industries)
RAINBOW, Henry
(Shell Oil Company, England)
Mrs. Rainbow
Mrs. Henry B. Rainbow
Miss Margaret W. Rainbow
Miss Virginia Rainbow
RAINICH, George Yuri, Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Rainich
RAINVILLE, Earl David, Prof.
(University of Michigan, Mich., U. S. A.)
RAISBECK, Gordon, Dr.
(Bell Telephone Laboratories, N. J., U. S. A.)
Mrs. Raisbeck
Mr. Michael Raisbeck
RAJCHMAN, Jan A., Dr.
(RCA Laboratories, Princeton, N. J., U. S. A.)
RAMANATHAN, K. G.
(Institute for Advanced Study, N. J., U. S. A.)
RAM BEHARI, Prof.
(University of Delhi, Delhi, India)
RAMLER, Otto Joseph, Prof.
(Catholic University of America, D. C., U. S. A.)
RANDOLPH, John F., Prof.
(University of Rochester, N. Y., U. S. A.)
RANEY, George Neal, Instr.
(Columbia University, N. Y., U. S. A.)
RANKIN, Robert Alexander, Dr.
(Clare College, Cambridge University, Cambridge, England)
RAO, Narasinga, Prof.
(Andhra University, Waltair, India)
RAPOPORT, Anatol, Prof.
(University of Chicago, Ill., U. S. A.)
Mrs. Rapoport
RAPP, Philip C.
(Bell Aircraft Corporation, N. Y., U.S.A.)
RATNER, Lawrence T., Prof.
(Vanderbilt University, Tenn., U. S. A.)
RATTRAY, Basil Andrew
(Princeton University, N. J., U. S. A.)
RAUCH, Harry Ernest, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
RAUCH, Lawrence Lee, Prof.
(University of Michigan, Mich., U. S. A.)
RAY, Daniel
(Cornell University, N. Y., U. S. A.)

RAYNOR, George Emil, Prof.
(Lehigh University, Pa., U. S. A.)
Mrs. Raynor
READE, Maxwell O., Prof.
(University of Michigan, Mich., U. S. A.)
RECILLAS-JUÁREZ, Félix, Investigador de Carrera
(University of Mexico, Mexico, D. F., Mexico)
Mrs. Recillas-Juárez
REDHEFFER, Raymond Moos, Instr.
(Harvard University, Mass., U. S. A.)
REES, Carl John, Prof.
(University of Delaware, Del., U. S. A.)
Mrs. Rees
REES, Mary Rebecca, Lecturer
(University of Reading, Reading, England)
REES, Mina S., Director, Mathematical Sciences Division
(Office of Naval Research, D. C., U. S. A.)
REGAN, Francis, Prof.
(St. Louis University, Mo., U. S. A.)
REID, William Hill
(University of California, Calif., U. S. A.)
REID, William Thomas, Prof.
(Northwestern University, Ill., U. S. A.)
REINER, Irving, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Reiner
REINGOLD, Haim, Prof.
(Illinois Institute of Technology, Ill., U. S. A.)
REISSNER, Eric, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Reissner
Miss Eva Reissner
Mr. John Reissner
RELLICH, Franz, Prof.
(University of Göttingen, Göttingen, Germany)
Mrs. Rellich
REMAGE, Russell, Jr., Prof.
(University of Delaware, Del., U. S. A.)
Mrs. Remage
RESCH, Daniel, Instr.
(University of Michigan, Mich., U. S. A.)
RESCHOVSKY, Helene, Instr.
(University of Connecticut, Conn., U. S. A.)
REY-PASTOR, Julio, Prof.
(University of Buenos Aires, Buenos Aires, Argentina)
REYNOLDS, Clarence Newton, Jr., Prof.
(West Virginia University, W. Va., U. S. A.)
REYNOLDS, Robert Raymond, Mathematician
(National Bureau of Standards, Calif., U. S. A.)
Mrs. Reynolds
DE RHAM, Georges William, Prof.
(University of Geneva, Geneva; University of Lausanne, Lausanne, Switzerland)
RHODES, Charles Everett, Prof.
(Alfred Univers'ty, N. Y., U. S. A.)
RHODES, Ida
(National Bureau of Standards, D. C., U. S. A.)
RIABOUCHINSKY, Dimitri P., Prof.
(Centre National de la Recherche Scientifique, Paris, France)
Mrs. Riabouchinsky
RIBEIRO, Hugo Baptista, Prof.
(University of Nebraska, Neb., U. S. A.)
Mrs. Ribeiro
RICCI, Giovanni, Prof.
(Università Statale di Milano, Milan, Italy)
RICE, Sister Mary Edmund, Instr.
(Our Lady of Cincinnati College, Ohio, U. S. A.)
RICE, R. B., Senior Physicist
(Phillips Petroleum Company, Okla., U. S. A.)
RICHARDSON, Davis Payne, Prof.
(University of Arkansas, Ark., U. S. A.)
RICHARDSON, Moses, Prof.
(Brooklyn College, N. Y., U. S. A.)
*RICHARDSON, Mrs. R. G. D. (Associate Member)
(Providence, R. I., U. S. A.)
RICHMOND, Donald E., Prof.
(Williams College, Mass., U. S. A.)
RICKART, Charles Earl, Prof.
(Yale University, Conn., U. S. A.)
RIDDER, Jacob, Prof.
(State University, Groningen, Netherlands)
RIDER, Paul Reece, Prof.
(Washington University, Mo., U. S. A.)
Mrs. Rider

---

* Special Guest of Organizing Committee.

RIES, Henry Francis, Principal Actuary
(State of Colorado Insurance Department, Colo., U. S. A.)
RIESS, John Karlem, Prof.
(Tulane University of Louisiana, La., U. S. A.)
RINEHART, Robert Fross, Prof.
(Case Institute of Technology, Ohio, U. S. A.)
RITT, Joseph Fels, Prof.
(Columbia University, N. Y., U. S. A.)
Mrs. Ritt
RITT, R. K., Instr.
(University of Michigan, Mich., U. S. A.)
RITTER, Eugene Kerfoot, Dr.
(University of Michigan, Mich., U. S. A.)
Mrs. Ritter
Mrs. E. Kerfoot Ritter, Jr.
Miss Martha L. Ritter
RIVERA, Nariño, Prof.
(University of Panama, Panama)
Mrs. Rivera
ROBB, Dean Leroy, Prof.
(Baldwin-Wallace College, Ohio, U. S. A.)
Mrs. Robb
Mr. Frazer Robb
ROBBINS, Herbert, Prof.
(University of North Carolina, N. C., U. S. A.)
ROBERTS, Helen Murray, Prof.
(University of Connecticut, Conn., U. S. A.)
ROBERTS, Robert Abram
(University of Michigan, Mich., U. S. A.)
ROBERTSON, Howard P., Prof.
(California Institute of Technology, Calif., U. S. A.)
ROBERTSON, Malcolm Slingsby, Prof.
(Rutgers University, N. J., U. S. A.)
ROBINSON, Abraham, Lecturer
(College of Aeronautics, Cranfield, Bletchley, England)
ROBINSON, Gilbert de B., Prof.
(University of Toronto, Ont., Canada)
ROBINSON, Joan Elizabeth, Instr.
(Wilson College, Pa., U. S. A.)
ROBINSON, Julia Bowman, Dr.
(Berkeley, Calif., U. S. A.)
ROBINSON, Lewis Bayard
(Baltimore, Md., U. S. A.)
ROBINSON, Raphael Mitchel, Prof.
(University of California, Calif., U. S. A.)
ROBINSON, Robin, Prof.
(Dartmouth College, N. H., U. S. A.)
Mrs. Robinson
Mr. Peter Robinson
Miss Julia A. Robinson
ROBINSON, Selby Lemley, Prof.
(City College of New York, N. Y., U. S. A.)
Mrs. Robinson
Mr. Richard Robinson
ROEVER, William Henry, Prof.
(Washington University, Mo., U. S. A.)
Mrs. Roever
ROFFIEL, Alfredo Gutierrez, Prof.
(Naval Academy; Merchant Nautical School, Vera Cruz, Mexico)
ROGERS, Claude Ambrose, Dr.
(University College, University of London, London, England; Institute for Advanced Study, N. J., U. S. A.)
ROGERS, Hartley, Jr.
(Princeton University, N. J., U. S. A.)
ROGERS, Paul Carney
(University of Maryland, Md., U. S. A.)
ROGOSINSKI, Werner Wolfgang, Prof.
(King's College, University of Durham, Newcastle-on-Tyne, England)
ROME, Canon Adolphe, Prof.
(Université de Louvain, Louvain, Belgium)
ROMERO JUAREZ, Antonio, Prof.
(Institute of Physics, Mexico, D. F., Mexico)
ROOM, Thomas Gerald, Prof.
(University of Sydney, Sydney, Australia)
ROSE, Gene F., Instr.
(University of Wisconsin, Wis., U. S. A.)
ROSE, Nicholas John, Instr.
(Stevens Institute of Technology, N. J., U. S. A.)
ROSEN, David, Instr.
(University of Pennsylvania, Pa., U. S. A.)
ROSEN, Saul, Prof.
(Drexel Institute of Technology, Pa., U. S. A.)
Mrs. Rosen
ROSEN, William G.
(University of Illinois, Ill., U. S. A.)
Mrs. Rosen
ROSENBAUM, Ira, Prof.
(University of Miami, Fla., U. S. A.)
ROSENBAUM, Robert A., Prof.
(Reed College, Ore., U. S. A.)
Mrs. Rosenbaum

ROSENBERG, Reinhardt M., Prof.
(University of Washington, Wash., U. S. A.)
ROSENBLATT, Murray, Dr.
(Cornell University, N. Y., U. S. A.)
Mrs. Rosenblatt
ROSENBLOOM, Joshua H., Physicist
(Naval Ordnance Laboratory, Md., U. S. A.)
ROSENBLOOM, Paul Charles, Prof.
(Syracuse University, N. Y., U. S. A.)
ROSENHEAD, Louis, Prof.
(University of Liverpool, Liverpool, England)
ROSENLICHT, Maxwell A., National Research Fellow
(University of Chicago, Ill., U. S. A.)
ROSENTHAL, Arthur, Prof.
(Purdue University, Ind., U. S. A.)
ROSS, Arnold Ephraim, Prof.
(University of Notre Dame, Ind., U. S. A.)
Mrs. Ross
ROSS, Shepley Littlefield
(Boston University, Mass., U. S. A.)
ROSSER, J. Barkley, Prof.
(National Bureau of Standards, Calif., U. S. A.)
ROSSING, Erling, Lektor
(Tönder Statsskole, Tönder, Denmark)
ROSSKOPF, Myron F., Prof.
(Syracuse University, N. Y., U. S. A.)
ROTHBERGER, Fritz, Prof.
(University of New Brunswick, N. B., Canada)
ROTHE, Erich Hans, Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Rothe
ROUBINE, Elie, Prof.
(École Supérieure d'Electricité; École Polytechnique, Paris, France)
ROWLAND, Sidney A., Prof.
(Ohio Wesleyan University, Ohio, U. S. A.)
ROY, Samarendra Nath, Prof.
(University of North Carolina, N. C., U. S. A.)
RUBIN, Herman, Prof.
(Stanford University, Calif., U. S. A.)
RUDERMAN, Harry Daniel
(Manhattan High School of Aviation Trades, N. Y., U. S. A.)
Mrs. Ruderman
RUDIN, Walter, Instr.
(Duke University, N. C., U. S. A.)
RUMBAUT TOLEDO, Alicia, Dr.
(Institute of Secondary Education, Cienfuegos, Cuba)
RUSE, Harold Stanley, Prof.
(University of Leeds, Leeds, England)
RUSK, Evelyn Carroll, Prof.
(Wells College, N. Y., U. S. A.)
Mr. William Sener Rusk
RUSSEK, Arnold, Research Physicist
(New York University, N. Y., U. S. A.)
RUSSELL, Bernard M., Prof.
(Gannon College, Pa., U. S. A.)
RUSSELL, Helen Gertrude, Prof.
(Wellesley College, Mass., U. S. A.)
RUSSELL, Jules P., Prof.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
RUTLEDGE, William Austin, Instr.
(University of Tennessee, Tenn., U. S. A.)
RUTT, Norman Eby, Prof.
(Louisiana State University, La., U. S. A.)
RYSER, Herbert John, Prof.
(Ohio State University, Ohio, U. S. A.)

S

SÁENZ, Albert William, Consultant, Theoretical Physics
(Naval Research Laboratory, D. C., U. S. A.)
SAIBEL, Edward, Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
SAKELLARIOU, Nilos Th., Prof.
(University of Athens, Athens, Greece)
Mrs. Sakellariou
Mr. Theodore Sakellariou
SALCEDO, Enrique
(National Bank of Cuba, Havana, Cuba)
Mrs. Salcedo
SALKIND, Charles T., Instr.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
SALZER, Herbert Ellis, Mathematician
(National Bureau of Standards, D. C., U. S. A.)
SAMELSON, Hans, Prof.
(University of Michigan, Mich., U. S. A.)
SAMUELSON, Paul Anthony, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)

SANCHEZ-DIAZ, Rafael, Prof.
(New Mexico School of Mines, N. M., U. S. A.)
SANGREN, Ward Conrad, Prof.
(Miami University, Ohio, U. S. A.)
Mrs. Sangren
Mr. Paul S. Sangren
Miss Karen R. Sangren
SAN JUAN LLOSÁ, Ricardo
(University of Madrid, Madrid, Spain)
SANSONE, Giovanni, Prof.
(Università di Firenze, Florence, Italy)
SARANTOPOULOS, Spyridion Basil, Prof.
(University of Athens, Athens, Greece)
SARD, Arthur, Prof.
(Queens College, N. Y., U. S. A.)
SARGENT, Winifred Lydia Crunden, Lecturer
(Bedford College, University of London, London, England)
SARGINSON, Kathleen, University Teacher
(Somerville College, Oxford University, Oxford, England)
SARIO, Leo Reino, Dr.
(Helsinki, Finland)
Mrs. Sario
SASLAW, Samuel Saul, Prof.
(United States Naval Academy, Md., U. S. A.)
SAUER, Robert, Prof.
(Technische Hochschule München, Munich, Germany)
SAVAGE, I. Richard
(Columbia University, N. Y., U. S. A.)
SAVAGE, Leonard Jimmie, Prof.
(University of Chicago, Ill., U. S. A.)
SAXER, Walter, Prof.
(Eidgenössische Technische Hochschule, Zürich, Switzerland)
Mrs. Saxer
SCHACKNOW, Aronld B., Instr.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
SCHAEFFER, Albert C., Prof.
(University of Wisconsin, Wis., U. S. A.)
SCHAFER, Richard Donald, Prof.
(University of Pennsylvania, Pa., U. S. A.)
Mrs. Schafer
SCHAFFER, Juan Jorge
(University of Uruguay, Montevideo, Uruguay)
SCHATTEN, Robert, Prof.
(University of Kansas, Kan., U. S. A.)
SCHATZ, Joseph Arthur
(Brown University, R. I., U. S. A.)
Mrs. Schatz
SCHECTER, Samuel, Instr.
(Syracuse University, N. Y., U. S. A.)
Mrs. Schecter
SCHEERER, Anne Elizabeth
(University of Pennsylvania, Pa., U. S. A.)
SCHEFFÉ, Henry, Prof.
(Columbia University, N. Y., U. S. A.)
SCHEIER, Rev. Maurice Andrew, Prof.
(St. Bonaventure College, N. Y., U. S. A.)
SCHELKUNOFF, Sergei A., Research Mathematician
(Bell Telephone Laboratories, N. J., U. S. A.)
VON SCHELLING, Hermann, Statistician
(Naval Medical Research Laboratory, Conn., U. S. A.)
Mrs. von Schelling
SCHENKMAN, Eugene Victor, Prof.
(Louisiana State University, La., U. S. A.)
SCHERBERG, Max G., Chief of Applied Mathematics
(Office of Air Research, Wright-Patterson Air Force Base, Ohio, U. S. A.)
SCHERK, Peter, Prof.
(University of Saskatchewan, Sask., Canada)
SCHIFFER, Menahem Max, Prof.
(Hebrew University, Jerusalem, Israel)
Mrs. Schiffer
SCHLESINGER, Ernest C.
(Harvard University, Mass., U. S. A.)
SCHMIDT, Herman Karl, Prof.
(Technische Hochschule Braunschweig, Braunschweig, Germany)
SCHMIED, Robert Whittaker, Instr.
(Tulane University of Louisiana, La., U. S. A.)
SCHMITT, Samuel A., Research Analyst
(Armed Forces Security Agency, D. C., U. S. A.)
SCHNECKENBURGER, Edith Ruth, Prof.
(University of Buffalo, N. Y., U. S. A.)
SCHNITTKE, Raymond Ira, Major
(Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
SCHOENBERG, Isaac Jacob, Prof.
(University of Pennsylvania, Pa., U. S. A.)

SCHOENFELD, Lowell, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Schoenfeld
SCHOLOMITI, Nicholas Christ, Instr.
(University of Illinois, Ill., U. S. A.)
Mrs. Scholomiti
SCHOUTEN, J. A., Prof.
(University of Amsterdam, Amsterdam, Netherlands)
SCHRADER, Dorothy Virginia, Prof.
(College of St. Teresa, Minn., U. S. A.)
SCHREMP, Edward Jay, Dr.
(Naval Research Laboratory, D. C., U. S. A.)
SCHUB, Pincus, Dr.
(University of Pennsylvania, Pa., U. S. A.)
SCHUTZBERGER, Henry, Division Supervisor
(Sandia Corporation, N. M., U. S. A.)
SCHWARTZ, Abraham, Dr.
(City College of New York, N. Y., U. S. A.)
SCHWARTZ, Laurent, Prof.
(Université de Nancy, Nancy, France)
SCHWEDER, William Herbert, Prof.
(Georgetown University, D. C., U. S. A.)
SCHWEIGERT, George E., Prof.
(University of Pennsylvania, Pa., U. S. A.)
Mrs. Schweigert
SCOTT, Carol Scott, Instr.
(St. Petersburg Junior College, Fla., U. S. A.)
SCOTT, Walter Tandy, Prof.
(Northwestern University, Ill., U. S. A.)
SCOTT, William Raymond, Prof.
(University of Kansas, Kan., U. S. A.)
Mrs. Scott
Miss Dorothy Scott
Miss Martha Scott
SEAL, Hilary Latham, Dr.
(Wickenden, Morss and Associates, N. Y., U. S. A.)
SEALANDER, Carl E., Research Engineer
(Battelle Memorial Institute, Ohio, U. S. A.)
SEDGEWICK, Charles Hill Wallace, Prof.
(University of Connecticut, Conn., U. S. A.)
SEEBER, Robert Rex, Jr., Senior Staff Member
(International Business Machines Corporation, N. Y., U. S. A.)
SEGAL, Irving E., Prof.
(University of Chicago, Ill., U. S. A.)
SEGRE, Beniamino, Prof.
(Università di Bologna, Bologna, Italy)
SEIDEN, Esther, Lecturer
(University of California, Calif., U. S. A.)
SEIDENBERG, Abraham, Prof.
(University of California, Calif., U. S. A.)
Mrs. Seidenberg
SELBERG, Atle, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
Mrs. Selberg
SELBERG, Sigmumd, Prof.
(Norges Tekniske Høgskole, Trondheim, Norway)
SEMIN, Ferruh, Prof.
(University of Istanbul, Istanbul, Turkey)
SEN, Nikhilranjan, Prof.
(University College of Science and Technology, Calcutta, India)
SERRELL, Robert
(RCA Laboratories, N. J., U. S. A.)
SETH, Bhoj Raj, Visiting Prof.
(Iowa State College, Iowa, U. S. A.)
Mrs. Seth
Miss Neena Seth
Mr. Arun Seth
Mr. Kiran Seth
SEVERI, Francesco, Prof.
(Università di Roma, Rome, Italy)
SEYBOLD, Mary Anice, Prof.
(North Central College, Ill., U. S. A.)
Miss Ethel Seybold
SHAFFER, Dorothy Browne
(Corning, N. Y., U. S. A.)
SHANKS, Daniel, Physicist
(Naval Ordnance Laboratory, Md., U. S. A.)
SHANKS, Eugene Baylis, Prof.
(Vanderbilt University, Tenn., U. S. A.)
SHANKS, Merrill Edward, Prof.
(Purdue University, Ind., U. S. A.)
SHANNON, Claude Elwood, Research Mathematician
(Bell Telephone Laboratories, N. J., U. S. A.)
Mrs. Shannon
SHAPIRO, David F., Electrical Engineer; Instr.
(RCA Institutes, Inc., N. Y., U. S. A.)
Mrs. Shapiro

SHAPIRO, George I., Actuary
(American Jewish Congress, N. Y., U. S. A.)
SHAPIRO, Harold N., Prof.
(New York University, N. Y., U. S. A.)
SHAPLEY, Lloyd Stowell
(RAND Corporation, Calif., U. S. A.)
SHARMA, H. G. S., Prof.
(Lingeraj College, Belgaum, India)
SHARP, William Tout
(Princeton University, N. J., U. S. A.)
SHAW, Robert F., Logical Design Engineer
(Eckert-Mauchly Computer Corporation, Pa., U. S. A.)
SHAW, Robert Parker, Mathematician
(New York University, N. Y., U. S. A.)
SHEFFER, Isador Mitchell, Prof.
(Pennsylvania State College, Pa., U. S. A.)
Mrs. Sheffer
SHELDON, Ernest Wilson, Prof.
(Acadia University, N. S., Canada)
SHERIDAN, Laurence Ward, Prof.
(College of St. Thomas, Minn., U. S. A.)
SHERMAN, Jack, Dr.
(Beacon Laboratories, Texas Company, N. Y., U. S. A.)
SHERMAN, Seymour, Dr.
(Lockheed Aircraft Corporation, Calif., U. S. A.)
SHEWHART, Walter Andrew, Research Engineer
(Bell Telephone Laboratories, N. J., U. S. A.)
SHIELDS, William Henry, Instr.
(Massachusetts Institute of Technology, Mass., U. S. A.)
SHIFFMAN, Max, Prof.
(Stanford University, Calif., U. S. A.)
Mrs. Shiffman
Mr. Bernard Shiffman
Mr. David Shiffman
SHILS, Lee Haliday, Civil Engineer
(American Car and Foundry Company, N. J., U. S. A.)
SHIVELY, Levi Stephen, Prof.
(Ball State Teachers College, Ind., U. S. A.)
Mrs. Shively
SHOWALTER, Alan B.
(University of Kansas, Kan., U. S. A.)
SHÜ, Shien Siu, Prof.
(Illinois Institute of Technology, Ill., U. S. A.)
SHUHANY, Elizabeth Anna, Instr.
(Boston University, Mass., U. S. A.)
SICHEL, Herbert Simon, Research Officer
(Council for Scientific and Industrial Research, Johannesburg, South Africa)
Mrs. Sichel
SIEGEL, Keeve Milton, Section Leader, High Altitude Studies
(Aeronautical Research Center, Mich., U. S. A.)
SIGNORINI, Antonio, Prof.
(Università di Roma, Rome, Italy)
SILVA, Joseph Andrew, Instr.
(Duke University, N. C., U. S. A.)
SILVERMAN, Edward, Prof.
(Kenyon College, Ohio, U. S. A.)
SILVERMAN, Robert J.
(University of Illinois, Ill, U. S. A.)
SIMON, Webster Godman, Prof.
(Western Reserve University, Ohio, U. S. A.)
SIMONART, Fernand, Prof.
(Université de Louvain, Louvain, Belgium)
SIMONS, William Haddock, Prof.
(University of British Columbia, B. C., Canada)
Mrs. Simons
Miss Christina Joy Simons
SIMPSON, T. M., Dean, Graduate School
(University of Florida, Fla., U. S. A.)
SIMPSON, William B., Assistant Director of Research
(Cowles Commission for Research in Economics, Ill., U. S. A.)
SINCLAIR, Annette, Instr.
(University of Tennessee, Tenn., U. S. A.)
SINCLAIR, Mary Emily, Prof. Emeritus
(Oberlin College, Ohio, U. S. A.)
SINGER, James, Prof.
(Brooklyn College, N. Y., U. S. A.)
SKOLEM, Thoralf Albert, Prof.
(University of Oslo, Oslo, Norway)
SLADE, James Jeremiah, Jr., Prof.
(Rutgers University, N. J., U. S. A.)
SLATER, John C., Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)

SLATER, Morton Lincoln, Research Associate
(Cowles Commission for Research in Economics, Ill., U. S. A.)
SLOTNICK, Morris Miller, Mathematical Geophysicist
(Humble Oil and Refining Company, Tex., U. S. A.)
Mr. Joseph Stanley Slotnick
SMAIL, Lloyd Leroy, Prof.
(Lehigh University, Pa., U. S. A.)
SMILEY, Malcolm Finlay, Prof.
(State University of Iowa, Iowa, U. S. A.)
Mrs. Smiley
SMITH, Edgar Clarence, Jr.
(Brown University, R. I., U. S. A.)
Mrs. Smith
SMITH, Franklin C., Prof.
(College of St. Thomas, Minn., U. S. A.)
SMITH, Guy Watson, Prof.
(University of Kansas, Kan., U. S. A.)
SMITH, James J., Electrical Engineer
(General Electric Company, N. Y., U. S. A.)
SMITH, Marianne Freundlich, Instr.
(University of California, Calif., U. S. A.)
Mr. Lloyd Smith
Master Peter Smith
Mr. Phil Merryman
SMITH, Paul A., Prof.
(Columbia University, N. Y., U. S. A.)
SMITH, Samuel S., Prof.
(University of Utah, Utah, U. S. A.)
SMITHIES, Frank, Lecturer
(Cambridge University, Cambridge, England)
SNAPPER, Ernst, Prof.
(University of Southern California, Calif., U. S. A.)
SNYDER, Bernhart Richard, Instr.
(Johns Hopkins University, Md., U. S. A.)
Mrs. Snyder
SNYDER, Walter Stephen, Prof.
(University of Tennessee; Oak Ridge National Laboratory, Tenn., U. S. A.)
SOBCZYK, Andrew, Prof.
(Boston University, Mass., U. S. A.)
SOHON, Rev. Frederick Wyatt, Prof.
(Georgetown University, D. C., U. S. A.)
SOKOLNIKOFF, Elizabeth Stafford, Prof.
(University of Wisconsin, Wis., U. S. A.)
SOLKY, Charles Gerard
(New York University, N. Y., U. S. A.)
SOLOMONOFF, Ray Joseph
(University of Chicago, Ill., U. S. A.)
SOPKA, John J., Dr.
(Applied Physics Laboratory, Johns Hopkins University, Md., U. S. A.)
SPANIER, Edwin Henry, Prof.
(University of Chicago, Ill. U. S. A.)
SPEAR, Joseph, Prof.
(Northeastern University, Mass., U. S. A.)
SPEARS, William Kebren, Research Analyst
(United States Army Signal Corps, D. C., U. S. A.)
SPECHT, Edward John, Prof.
(Emmanuel Missionary College, Mich., U. S. A.)
SPECKER, Ernst Paul, Dr.
(Eidgenössische Technische Hochschule, Zürich, Switzerland)
SPENCER, Domina Eberle, Prof.
(Brown University, R. I., U. S. A.)
SPENCER, Donald Clayton, Prof.
(Princeton University, N. J., U. S. A.)
SPENCER, Guilford Lawson, II
(University of Michigan, Mich., U. S. A.)
SPENCER, Vivian Eberle, Principal Statistician
(Bureau of the Census, D. C., U. S. A.)
SPRAGUE, Atherton H., Prof.
(Amherst College, Mass., U. S. A.)
SPRINGER, Charles Eugene, Prof.
(University of Oklahoma, Okla., U. S. A.)
SPRINGER, George, Instr.
(Massachusetts Institute of Technology, Mass., U. S. A.)
SPRINGER, Melvin Dale, Dr.
(Naval Ordnance Plant, Ind., U. S. A.)
Mrs. Springer
STAAL, Ralph Axel, Prof.
(University of New Brunswick, N. B., Canada)
STABLER, Edward Russell, Prof.
(Hofstra College, N. Y., U. S. A.)
STAKGOLD, Ivar, Instr.
(Harvard University, Mass., U. S. A.)
STANAITIS, Otonas Edmundas, Instr.
(St. Olaf College, Minn., U. S. A.)
STARK, Lloyd William, Prof.
(Atlantic Christian College, N. C., U. S. A.)
STARK, Marion Elizabeth, Prof.
(Wellesley College, Mass., U. S. A.)

STARKE, Emory Potter, Prof.
(Rutgers University, N. J., U. S. A.)
STAUFFER, Jacob Reiff Kob, Prof.
(Rhode Island State College, R. I., U. S. A.)
Mrs. Stauffer
Miss Carolyn Ruby Stauffer
Miss Mitzi Jane Stauffer
Miss Sara Ann Stauffer
STEELE, Rev. Domhnall Arthur, Prof.
(Fordham University, N. Y., U. S. A.)
STEELE, Sister Mary Philip, Prof.
(Rosary College, Ill., U. S. A.)
Sister Marie Stephen
STEEN, Frederick Henry, Prof.
(Allegheny College, Pa., U. S. A.)
STEENROD, Norman Earl, Prof.
(Princeton University, N. J., U. S. A.)
Mrs. Steenrod
STEIN, Charles Max, Prof.
(University of California, Calif., U. S. A.)
STEINBERG, Robert, Instr.
(University of California, Los Angeles, Calif., U. S. A.)
STEINHAUSER, Harry H., Jr., Prof.
(Rensselaer Polytechnic Institute, N. Y., U. S. A.)
STELSON, Hugh E., Prof.
(Michigan State College, Mich., U. S. A.)
STEPHENS, Rothwell, Prof.
(Knox College, Ill., U. S. A.)
Mrs. Stephens
Mr. Kirk Stephens
STEVENSON, Guy, Prof.
(University of Louisville, Ky., U. S. A.)
STEWART, Bonnie M., Prof.
(Michigan State College, Mich., U. S. A.)
STEWART, Frank Moore, Prof.
(Brown University, R. I., U. S. A.)
STEWART, James Collier, Prof.
(Lawrence College, Wis., U. S. A.)
STOCKMAN, John Franklin
(University of Colorado, Colo., U. S. A.)
STOKER, J. J., Prof.
(New York University, N. Y., U. S. A.)
STOKES, Ruth Wyckliffe, Prof.
(Syracuse University, N. Y., U. S. A.)
STOLL, Charles Buckner, Publisher
(John Wiley and Sons, Inc., N. Y., U. S. A.)
STOLL, Robert Roth, Prof.
(Lehigh University, Pa., U. S. A.)
STONE, Arthur Harold, Lecturer
(University of Manchester, Manchester, England)
STONE, Dorothy Maharam, Visiting Prof.
(Wellesley College, Mass., U. S. A.)
STONE, Marshall Harvey, Prof.
(University of Chicago, Ill., U. S. A.)
STONE, Ralph Bushnell, Prof.
(Purdue University, Ind., U. S. A.)
STONE, William Mathewson, Prof.
(Oregon State College, Ore., U. S. A.)
STOUFFER, Ellis B., Prof.
(University of Kansas, Kan., U. S. A.)
Mrs. Stouffer
STRACK, Sainsbury Lee
(University of New Zealand, Wellington, New Zealand)
STRAUS, Ernst Gabor, Instr.
(University of California, Los Angeles, Calif., U. S. A.)
STRUIK, Dirk Jan, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
STUBBE, John S., Prof.
(Clark University, Mass., U. S. A.)
STUBBS, Harold LeRoy, Instr.
(Northeastern University, Mass., U. S. A.)
STURLEY, Eric Avern, Prof.
(Allegheny College, Pa., U. S. A.)
SUETUNA, Zyoiti, Prof.
(Tokyo University, Tokyo, Japan)
SUFFA, Mary Clegg, Prof.
(Elmira College, N. Y., U. S. A.)
SUGAR, Alvin Charles, Prof.
(University of Southern California, Calif., U. S. A.)
SULLIVAN, Sister Mary Carmel, Instr.
(Albertus Magnus College, Conn., U. S. A.)
SULLIVAN, Mildred Marie, Prof.
(Queens College, N. Y., U. S. A.)
SUTER, Mary Frances, Prof.
(Roanoke College, Va., U. S. A.)
Mrs. D. I. Suter
SVERDRUP, Erling, Lecturer
(University of Oslo, Oslo, Norway)
SWAIN, R. L., Prof.
(Ohio State University, Ohio, U. S. A.)
SWANSON, Jessie E., Instr.
(Valparaiso University, Ind., U. S. A.)
SYDLER, Jean-Pierre, Dr.
(Eidgenössische Technische Hochschule, Zürich, Switzerland)

SYER, Henry William, Prof.
(Boston University, Mass., U. S. A.)
SZÁSZ, Otto, Prof.
(National Bureau of Standards, Calif.; University of Cincinnati, Ohio, U. S. A.)
Mrs. Szász
SZEGÖ, Gabor, Prof.
(Stanford University, Calif., U. S. A.)
SZMIELEW, Wanda, Lecturer
(University of California, Calif., U. S. A.)

T

TALBOT, Walter Richard, Prof.
(Lincoln University, Mo., U. S. A.)
TAMARI, Dov, Research Attaché
(Centre National de la Recherche Scientifique, Institut Henri Poincaré, Paris, France)
TARSKI, Alfred, Prof.
(University of California, Calif., U. S. A.)
Mrs. Tarski
TATE, Herbert, Prof.
(McGill University, Que., Canada)
Mrs. Tate
TATE, John Torrence, Jr., Instr.
(Princeton University, N. J., U. S. A.)
TAUB, Abraham H., Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Taub
TAYLOR, James Sturdevant, Prof.
(University of Pittsburgh, Pa., U. S. A.)
TAYLOR, John J.
(Westinghouse Electric Corporation, Pa., U. S. A.)
TAYLOR, Margaret Owsley, Mathematician
(Gulf Research Development Company, Pa., U. S. A.)
TAYLOR, Mildred Ellen, Prof.
(Mary Baldwin College, Va., U. S. A.)
TAYLOR, Robert Lawrence, Instr.
(Columbia University, N. Y., U. S. A.)
TAYLOR, William Clare, Prof.
(University of Cincinnati, Ohio; Ballistic Research Laboratories, Aberdeen Proving Ground, Md., U. S. A.)
Mrs. Taylor
Miss Patricia Taylor
TEAR, Richard Theodore, Instr.
(Rensselaer Polytechnic Institute, N. Y., U. S. A.)
TECERA-MARTINEZ, Alfredo, Civil Engineer
(Córdoba University, Córdoba, Argentina)
TEMPLE, George Frederick James, Prof.
(University of London, London, England)
Mrs. Temple
TERRACINI, Alessandro Marco, Prof.
(Università di Torino, Turin, Italy)
TERZIOĞLU, Nazim, Prof.
(University of Istanbul, Istanbul, Turkey)
TERZUOLI, Andrew J., Instr.
(Polytechnic Institute of Brooklyn, N. Y., U. S. A.)
THIELMAN, Henry P., Prof.
(Iowa State College, Iowa, U. S. A.)
Mrs. Thielman
Miss Pauline Thielman
Mr. John Thielman
THOMAS, George Brinton, Jr., Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Thomas
THOMAS, Joseph Miller, Prof.
(Duke University, N. C., U. S. A.)
THOMPSON, Gerald Luther
(University of Michigan, Mich., U. S. A.)
THORNHILL, Charles Kenneth
(Ministry of Supply, Fort Halstead, England)
THRALL, Robert McDowell, Prof.
(University of Michigan, Mich., U. S. A.)
THURSTON, Herbert Stanley, Prof.
(University of Alabama, Ala., U. S. A.)
Mrs. Thurston
Mrs. Lewis Smith
TINTNER, Gerhard, Prof.
(Iowa State College, Iowa, U. S. A.)
Mrs. Tintner
TITUS, Charles Joseph, Instr.
(University of Michigan, Mich., U. S. A.)
Mrs. Titus
TODD, John Arthur, Lecturer
(Cambridge University, Cambridge, England)
TODD, Olga Taussky, Dr.
(National Bureau of Standards, D. C., U. S. A.)
Mr. John Todd
TOLA PASQUEL, José, Dr.
(University of San Marcos, Lima, Peru)
Mrs. Tola
TOLSTED, Elmer, Prof.
(Pomona College, Calif., U. S. A.)

TOMBER, Marvin L., Instr.
(University of Pennsylvania, Pa., U. S. A.)
Mrs. Tomber
Mr. Robert Welber
TORALBALLA, Leopoldo V., Prof.
(Marquette University, Wis., U. S. A.)
TORANZOS, Fausto Ismael, Prof.
(University of Cuyo, Mendoza, Argentina)
TORNHEIM, Leonard, Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Tornheim
Miss Holly Tornheim
Mr. Keith Tornheim
TORRANCE, Charles Chapman, Prof.
(United States Naval Postgraduate School, Md., U. S. A.)
Mrs. Torrance
TORRES, Guillermo Diaz, Dr.
(University of Mexico, Mexico, D. F., Mexico)
TORREY, Marian Marsh, Prof.
(Goucher College, Md., U. S. A.)
TOWNSEND, Robert René, Instr.
(Pennsylvania State College, Pa., U. S. A.)
TRACEY, Joshua Irving, Prof.
(Yale University, Conn., U. S. A.)
Mrs. Tracey
TRANSUE, William Reagle, Prof.
(Kenyon College, Ohio, U. S. A.)
Mrs. Transue
TRANTER, Clement John, Prof.
(Military College of Science, Shrivenham, England)
TREMBLAY, Charles J., Instr.
(Bard College, N. Y., U. S. A.)
TRENT, Horace Maynard, Head, Applied Mathematics Branch
(Naval Research Laboratory, D. C., U. S. A.)
TREUENFELS, Peter Martin
(Institute for Mathematics and Mechanics, New York University, N. Y., U. S. A.)
TRICOMI, Francesco Giacomo, Prof.
(Università di Torino, Turin, Italy)
Mrs. Tricomi
TRJITZINSKY, Waldemar J., Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Trjitzinsky
TROST, Ernest Walter, Prof.
(Technikum Winterthur, Switzerland)
TRUESDELL, Clifford Ambrose, Prof.
(Naval Research Laboratory, D. C.; University of Maryland, Md., U. S. A.)
Mrs. Truesdell
TRYPANIS, Alexander Athanasius, Prof.
(National Technical University of Athens, Athens, Greece)
TUCKER, Albert William, Prof.
(Princeton University, N. J., U. S. A.)
TUCKERMAN, Bryant, Prof.
(Oberlin College, Ohio, U. S. A.)
TUKEY, John Wilder, Prof.
(Princeton University, N. J., U. S. A.)
Mrs. Tukey
TULLER, Annita, Prof.
(Hunter College, N. Y., U. S. A.)
TURNBULL, Herbert Westren, Prof.
(University of St. Andrews, Dundee, Scotland)
TURNER, Bird Margaret, Prof.
(West Virginia University, W. Va., U. S. A.)
TURNER, Nura Dorothea, Instr.
(New York State College for Teachers, N. Y., U. S. A.)
TURRIN, Gino, Prof.
(University of Cuyo, San Luis, Argentina)
TURRITTIN, Hugh Lonsdale, Prof.
(University of Minnesota, Minn., U. S. A.)
Mrs. Turrittin
Miss Rachel M. Turrittin
Mr. Anton H. Turrittin
TYLER, George William, Dr.
(Navy Electronics Laboratory, Calif., U. S. A.)

U

ULAM, Stanislaw Marcin, Dr.
(Los Alamos Scientific Laboratory, N. M., U. S. A.)
ULLMAN, Joseph Leonard, Instr.
(University of Michigan, Mich., U. S. A.)
ULMER, Gilbert, Prof.
(University of Kansas, Kan., U. S. A.)
ULRICH, Floyd Edward, Prof.
(Rice Institute, Tex., U. S. A.)
URBANO, Rocco H., Instr.
(Northeastern University, Mass., U. S. A.)

V

VAIDYANATHASWAY, R.
(University of Madras, Madras, India)

VALIRON, George Jean Marie, Prof.
(Institut Henri Poincaré, Paris, France)
Mrs. Valiron
VALLE FLORES, Enrique
(University of Mexico, Mexico, D. F., Mexico)
DE LA VALLÉE POUSSIN, Charles, Prof.
(Université de Louvain, Louvain, Belgium)
VANCE, Elbridge Putnam, Prof.
(Oberlin College, Ohio, U. S. A.)
VANDERSLICE, John Livezey, Prof.
(University of Maryland, Md., U. S. A.)
VANDIVER, Harry Shultz, Prof.
(University of Texas, Tex., U. S. A.)
VAN HOVE, Léon Charles
(Université Libre de Bruxelles, Brussels, Belgium)
VAN TUYL, A. H., Dr.
(Naval Ordnance Laboratory, Md., U. S. A.)
VATNSDAL, John Russell, Prof.
(State College of Washington, Wash., U. S. A.)
Mrs. Vatnsdal
Miss Mary Vatnsdal
VAUDREUIL, Sister Mary Felice, Prof.
(Mount Mary College, Wis., U. S. A.)
VAUGHAN, Herbert Edward, Prof.
(University of Illinois, Ill., U. S. A.)
Mrs. Vaughan
VAUGHT, Robert Lawson
(University of California, Calif., U. S. A.)
VÁZQUEZ-GARCÍA, Roberto, Prof.
(University of Mexico, Mexico, D. F., Mexico)
Mrs. Vázquez-García
VEBLEN, Oswald, Prof.
(Institute for Advanced Study, N. J., U. S. A.)
Mrs. Veblen
VEDOVA, George C., Prof.
(Newark College of Engineering, N. J., U. S. A.)
VEST, Marvin Lewis, Prof.
(West Virginia University, W. Va., U. S. A.)
VIAL, Gabriel, Prof.
(Collège Chartreux, Lyon, France)
VIDAV, Ivan, Prof.
(University of Ljubljana, Yugoslavia)
VIGNAUX, Juan Carlos, Prof.
(University of Buenos Aires, Buenos Aires, Argentina)
VINCENSINI, Paul Félix, Prof.
(Université de Marseilles, Marseilles, France)
VINOGRADE, Bernard, Prof.
(Iowa State College, Iowa, U. S. A.)
VINTER HANSEN, Julie Marie, Observator
(University Observatory, Copenhagen, Denmark)
VITALE, Renato L., Electrical Engineer
(Board of Transportation of the City of New York, N. Y., U. S. A.)
VOGELI, Bruce Ramon
(Mount Union College, Ohio, U. S. A.)
VOTAW, David F., Jr., Prof.
(Yale University, Conn., U. S. A.)
VROOMAN, Sumner I., Prof.
(Rensselaer Polytechnic Institute, N. Y., U. S. A.)
VYTHOULKAS, Dennis, Prof.
(National University of Engineering Science, Athens, Greece)

W

WADDELL, Mary E. G., Barrister at Law
(Osgood Halls, Ont., Canada)
WADE, L. I., Prof.
(Louisiana State University, La., U. S. A.)
Mrs. Wade
Miss Billie Jeanne Wade
Mr. Luther Wade
Master Johnny Wade
WADE, Thomas Leonard, Prof.
(Florida State University, Fla., U. S. A.)
Mrs. Wade
WAGNER, Daniel H.
(Brown University, R. I., U. S. A.)
WALD, Abraham, Prof.
(Columbia University, N. Y., U. S. A.)
WALKER, Gordon L., Prof.
(Purdue University, Ind., U. S. A.)
WALKER, Robert John, Prof.
(Cornell University, N. Y., U. S. A.)
WALKLEY, Stephen E.
(University of Illinois, Ill., U. S. A.)
WALLACE, Alexander Doniphan, Prof.
(Tulane University of Louisiana, La., U. S. A.)
Mrs. Wallace
Miss Catherine A. Wallace

WALLACE, Andrew Hugh, Lecturer
(University College, University of St. Andrews, Dundee, Scotland)
WALLACH, Sylvan, Dr.
(Westinghouse Electric Corporation, Pa., U. S. A.)
WALLMAN, Henry, Prof.
(Chalmers Tekniska Högskola, Gothenburg, Sweden)
Mrs. Wallman
Mr. Stephen Wallman
Miss Rhoda Beth Wallman
WALMSLEY, Charles, Prof.
(Dalhousie University, N. S., Canada)
Mrs. Walmsley
WALSH, John E., Mathematician
(RAND Corporation, Calif., U. S. A.)
WALSH, Joseph Leonard, Prof.
(Harvard University, Mass., U. S. A.)
Mrs. Walsh
Miss Betty Walsh
WALTER, Charlton Marshall
(Harvard University, Mass., U. S. A.)
WALTHER, Alwin, Prof.
(Technische Hochschule, Darmstadt, Germany)
WANG, Chi-Teh, Prof.
(New York University, N. Y., U. S. A.)
Mrs. Wang
Miss Jane Wang
WANG, Hsien-Chung, Dr.
(Louisiana State University, La., U. S. A.)
WARD, Morgan, Prof.
(California Institute of Technology, Calif., U. S. A.)
Mrs. Ward
Miss Audrey Ward
Mr. Eric Ward
Mr. Richard Ward
Mr. Samuel Ward
WARDWELL, James Fletcher, Prof.
(Colgate University, N. Y., U. S. A.)
WARSCHAWSKI, S. E., Prof.
(University of Minnesota, Minn., U. S. A.)
WASHNITZER, Gerard, Dr.
(Princeton University, N. J., U. S. A.)
WASOW, Wolfgang Richard, Mathematician
(National Bureau of Standards, Calif., U. S. A.)
WATERMAN, Daniel
(University of Chicago, Ill., U. S. A.)
WATSON, Neal T.
(Harvard University, Mass., U. S. A.)
WATSON, Richard Elvis, Dr.
(Leeds and Northrup Company, Pa., U. S. A.)
WATTS, Dorothy DeWitt, Instr.
(Iowa State Teachers College, Iowa, U. S. A.)
Mr. Alan W. Watts
WAYNE, Alan
(Brooklyn High School of Automotive Trades, N. Y., U. S. A.)
WEBBER, G. Cuthbert, Prof.
(University of Delaware, Del., U. S. A.)
WEBSTER, Merritt S., Prof.
(Purdue University, Ind., U. S. A.)
Mrs. Webster
Miss Anne W. Webster
Mr. John S. Webster
Miss Edith H. Webster
WEHAUSEN, John V., Executive Editor
(Mathematical Reviews, Brown University, R. I., U. S. A.)
WEIL, André, Prof.
(University of Chicago, Ill., U. S. A.)
Mrs. Weil
Miss S. Weil
Miss N. Weil
WEIL, Herschel, Dr.
(General Electric Company, N. Y., U. S. A.)
WEINBERGER, Hans F., Dr.
(Institute for Fluid Dynamics, University of Maryland, Md., U. S. A.)
WEINSTEIN, Alexander, Prof.
(University of Maryland, Md., U. S. A.)
Mrs. Weinstein
WEISNER, Louis, Prof.
(Hunter College, N. Y., U. S. A.)
WEISS, Eleanor Sylvia
(Harvard University, Mass., U. S. A.)
WEISS, Harry J., Instr.
(Carnegie Institute of Technology, Pa., U. S. A.)
WEISS, Marie, Prof.
(Tulane University of Louisiana, La., U. S. A.)
WELCH, Bernard Alfred, Prof.
(Manhattan College, N. Y., U. S. A.)
WELCHMAN, W. Gordon
(Massachusetts Institute of Technology, Mass., U. S. A.)
WELLS, Charles Prentiss, Prof.
(Michigan State College, Mich., U. S. A.)

WELMERS, Everett Thomas, Chief of Dynamics
(Bell Aircraft Corporation, N. Y., U. S. A.)
Mrs. Welmers
Miss Marina Jean Welmers
Mr. Thomas E. Welmers
WEND, David Van Vranken, Instr.
(Reed College, Ore., U. S. A.)
WENDEL, James G., Instr.
(Yale University, Conn., U. S. A.)
WENDROFF, Burton
(New York University, N. Y., U. S. A.)
WERMER, John
(Harvard University, Mass., U. S. A.)
WESTERN, Donald W., Prof.
(Franklin and Marshall College, Pa., U. S. A.)
WESTON, Jeffrey Dennis, Dr.
(King's College, University of Durham, Newcastle-on-Tyne, England)
WETHERELL, Elizabeth, Instr.
(Syracuse University, N. Y., U. S. A.)
WETZEL, Marion D., Prof.
(Denison University, Ohio, U. S. A.)
WEYL, F. Joachim, Dr.
(Office of Naval Research, D. C., U. S. A.)
WHEELER, Anna Pell., Prof. Emeritus
(Bryn Mawr College, Pa., U. S. A.)
WHITCOMBE, David W.
(Oak Ridge, Tenn., U. S. A.)
WHITE, Myron Edward, Tutor
(Queens College, N. Y., U. S. A.)
Mrs. White
WHITEHEAD, George W., Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Whitehead
WHITEHEAD, John Henry Constantine, Prof.
(Oxford University, Oxford, England)
WHITEMAN, Albert Leon, Prof.
(University of Southern California, Calif., U. S. A.)
Mrs. Whiteman
WHITMAN, Edwin A., Prof.
(Carnegie Institute of Technology, Pa., U. S. A.)
WHITMAN, Philip Martin, Dr.
(Johns Hopkins University, Md., U. S. A.)
WHITMORE, William Francis, Operations Analyst
(Operations Evaluation Group, D. C., U. S. A.)
WHITNEY, Anne Marie, Instr.
(University of Pennsylvania, Pa., U. S. A.)
WHITNEY, Hassler, Prof.
(Harvard University, Mass., U. S. A.)
WHYBURN, Gordon Thomas, Prof.
(University of Virginia, Va., U. S. A.)
WHYBURN, William Marvin, Prof.
(University of North Carolina, N. C., U. S. A.)
WICHT, Marion Cammack, Prof.
(North Georgia College, Ga., U. S. A.)
WIDDER, David Vernon, Prof.
(Harvard University, Mass., U. S. A.)
Mrs. Widder
Mr. David Charles Widder
WIELANDT, Helmut Wilhelm, Prof.
(Universität Mainz, Mainz, Germany)
WIENER, Norbert, Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
Mrs. Wiener
Miss Peggy Wiener
WIGGIN, Evelyn Prescott, Prof.
(Randolph-Macon Woman's College, Va., U. S. A.)
WILANSKY, Albert, Prof.
(Lehigh University, Pa., U. S. A.)
Mrs. Wilansky
Miss Eleanor Wilansky
WILCOX, Lee Roy, Prof.
(Illinois Institute of Technology, Ill., U. S. A.)
WILDER, Raymond Louis, Prof.
(University of Michigan, Mich., U. S. A.)
Mrs. Wilder
WILKES, Maurice Vincent, Director
(University Mathematical Laboratory, Cambridge, England)
WILKS, S. S., Prof.
(Princeton University, N. J., U. S. A.)
Mrs. Wilks
Mr. Stanley N. Wilks
WILLIAMS, Charles Wiley, Prof.
(Washington and Lee University, Va., U. S. A.)
Mrs. Williams
Miss Jane Carroll Williams
WILLIAMS, Lloyd B., Prof.
(Reed College, Ore., U. S. A.)
WILLIAMS, Mary Elizabeth., Prof.
(Skidmore College, N. Y., U. S. A.)

WILLIAMSON, Charles Owen, Prof.
(College of Wooster, Ohio, U. S. A.)
WILSON, Edwin Bidwell, Prof.
(Harvard University School of Public Health, Mass., U. S. A.)
WILSON, Levi Thomas, Prof.
(United States Naval Academy, Md., U. S. A.)
WILSON, Robert L., Prof.
(University of Tennessee, Tenn., U. S. A.)
WINGER, Roy Martin, Prof.
(University of Washington, Wash., U. S. A.)
Mrs. Winger
WINSOR, Charles P., Prof.
(Johns Hopkins University, Md., U. S. A.)
WINSTON, Clement, Business Economist
(Department of Commerce, D. C., U. S. A.)
Mrs. Winston
Miss Mollie B. Zion
WIRSZUP, Isaak, Instr.
(University of Chicago, Ill., U. S. A.)
WISHART, David M. G.
(University of St. Andrews, Dundee, Scotland)
WITMER, Enos Eby, Prof.
(University of Pennsylvania, Pa., U. S. A.)
WOLF, František, Prof.
(University of California, Calif., U. S. A.)
WOLFE, James H., Prof.
(University of Utah, Utah, U. S. A.)
WOLFOWITZ, Jacob, Prof.
(Columbia University, N. Y., U. S. A.)
Mrs. Wolfowitz
WOLONTIS, Vidar Michael, Prof.
(University of Kansas, Kan., U. S. A.)
Mrs. Wolontis
Miss Marjorie Annikki Wolontis
WONG, Yue-Kei
(Institute for Advanced Study, N. J., U. S. A.)
WOODBURY, Max Atkin, Dr.
(Institute for Advanced Study, N. J., U. S. A.)
WOOLSON, John Robert, Geophysicist
(United Geophysical Company, Inc., Calif., U. S. A.)
WREN, Frank Lynwood, Prof.
(Peabody College, Tenn., U. S. A.)
Mrs. Wren
WRENCH, John William, Jr., Dr.
(United States Navy Bureau of Ships, D. C., U. S. A.)
Mrs. Wrench
Miss Paula F. Wrench
WU, Ta-You, Prof.
(National Research Council of Canada, Ont., Canada)
WURSTER, Marie Anna, Prof.
(Temple University, Pa., U. S. A.)
WYLIE, Clarence R., Jr., Prof.
(University of Utah, Utah, U. S. A.)
WYLIE, Shaun, Lecturer
(Cambridge University, Cambridge, England)

Y

YACHTER, Morris
(M. W. Kellogg Company, N. J., U. S. A.)
YANO, Kentaro, Prof.
(Tokyo University, Tokyo, Japan)
YARNELLE, John E., Prof.
(Hanover College, Ind., U. S. A.)
YEARDLEY, Nelson Paul, Prof.
(Purdue University, Ind., U. S. A.)
YEVICK, George Johannus, Prof.
(Stevens Institute of Technology, N. J., U. S. A.)
YOOD, Bertram, Prof.
(Cornell University, N. Y., U. S. A.)
YOSIDA, Kôsaku, Prof.
(Nagoya University, Nagoya, Japan)
YOUNG, David M., Jr., Dr.
(Harvard University, Mass., U. S. A.)
Mrs. Young
YOUNG, Laurence Chisholm, Prof.
(University of Wisconsin, Wis., U. S. A.)
Mrs. Young
YOUNG, Mabel Minerva, Prof.
(Wellesley College, Mass., U. S. A.)
YOUNG, Rosalind Cecilia Hildegard, Lecturer
(Imperial College of Science and Technology, London, England)
YOUNGS, J. W. T., Prof.
(Indiana University, Ind., U. S. A.)

Z

ZALUAR-NUNES, Manuel, Prof.
(Sociedade Portuguesa de Matemática, Lisbon, Portugal)
ZARANTONELLO, Eduardo Hector, Dr.
(Harvard University, Mass., U. S. A.)
Mrs. Zarantonello

ZARISKI, Oscar, Prof.
(Harvard University, Mass., U. S. A.)
Mrs. Zariski
Miss Vera Laetitia Zariski
ZASSENHAUS, Hans Julius, Prof.
(McGill University, Que., Canada)
ZEICHNER, Arthur J., Instr.
(Oklahoma Agricultural and Mechanical College, Okla., U. S. A.)
ZELDIN, Samuel D., Prof.
(Massachusetts Institute of Technology, Mass., U. S. A.)
ZELINKA, Martha, Instr.
(Brookline High School, Mass., U. S. A.)
ZEMMER, Joseph Lawrence, Jr., Instr.
(University of Missouri, Mo., U. S. A.)
Mrs. Zemmer
ZERVOS, Panajiotis, Prof.
(Academy of Athens, Athens, Greece)
ZETTLER-SEIDEL, Philipp Wolfgang, Consultant
(Naval Ordnance Laboratory, Md., U. S. A.)
ZILBER, Joseph Abraham, Instr.
(Johns Hopkins University, Md., U. S. A.)
ZIPPIN, Leo, Prof.
(Queens College, N. Y., U. S. A.)
ZUBIETA-RUSSI, Francisco
(University of Mexico, Mexico, D. F., Mexico)
ZUCKERMAN, Herbert S., Prof.
(University of Washington, Wash., U. S. A.)
ZYGMUND, Antoni, Prof.
(University of Chicago, Ill., U. S. A.)

PROGRAM

WEDNESDAY, AUGUST 30

ADDRESSES

3:30 P. M.

Mallinckrodt MB9

A. Beurling, University of Uppsala.
On null-sets in harmonic analysis and function theory.
(*Address by invitation of the Organizing Committee*)

Fogg Large Room

H. Hopf, Swiss Federal School of Technology.
Die n-dimensionalen Sphären und projektiven Räume in der Topologie.
(*Address by invitation of the Organizing Committee*)

4:45 P. M.

Mallinckrodt MB9

H. Cartan, University of Paris.
Sur les fonctions analytiques de variables complexes.
(*Address by invitation of the Organizing Committee*)

Fogg Large Room

R. L. Wilder, University of Michigan.
The cultural basis of mathematics.
(*Address by invitation of the Organizing Committee*)

THURSDAY, AUGUST 31

ADDRESSES

9:00 A.M.

Mallinckrodt MB9

S. Bochner, Princeton University.
Laplace operator on manifolds.
(*Address by invitation of the Organizing Committee*)

Fogg Large Room

K. Gödel, Institute for Advanced Study.
Rotating universes in general relativity theory.
(*Address by invitation of the Organizing Committee*)

THURSDAY, AUGUST 31
10:15 A.M.
Emerson D

CONFERENCE IN ALGEBRA
ALGEBRAIC GEOMETRY

). ZARISKI, Harvard University.
The fundamental ideas of abstract algebraic geometry.
(*Address by invitation of the Organizing Committee*)

\. WEIL, University of Chicago.
Number-theory and algebraic geometry.
(*Address by invitation of the Organizing Committee*)

THURSDAY, AUGUST 31
10:15 A.M.
Harvard 1

SECTION II
ANALYSIS

I. A. RADEMACHER, University of Pennsylvania.
Remarks on the theory of partitions. (30 min.)

}. KJELLBERG, University of Uppsala.
On the growth of minimal positive harmonic functions in a plane region.

:. P. MILES, JR., Alabama Polytechnic Institute.
A minimal problem for harmonic functions in space.

}. CHOQUET, University of Paris.
Fonctions croissantes d'ensembles et capacités.

THURSDAY, AUGUST 31
10:15 A.M.
Harvard 5

SECTION II
ANALYSIS

|. AGMON, Rice Institute.
On the existence of summation functions for a class of Dirichlet series.

I. DELANGE, University of Clermont-Ferrand.
Sur les théorèmes taubériens pour les séries de Dirichlet.

:. G. STRAUS, University of California, Los Angeles.
On a class of integral-valued Dirichlet series.

[. CHANDRASEKHARAN, Tata Institute of Fundamental Research, Bombay.
On the summation of multiple Fourier series.

l. J. DUFFIN and A. C. SCHAEFFER, Carnegie Institute of Technology and University of Wisconsin.
A class of nonharmonic Fourier series.

. KARAMATA, University of Belgrade.
Sur une notion de continuité régulière avec application aux séries de Fourier.

). SZÁSZ, National Bureau of Standards.
Tauberian theorems for summability (R_1).

THURSDAY, AUGUST 31
10:15 A.M.
Harvard 4

SECTION III
GEOMETRY AND TOPOLOGY

L. A. Santaló, University of Rosario. (Read by C. B. Allendoerfer)
Integral geometry in general spaces. (30 min.)
H. W. Alexander, Adrian College.
The edge of regression of pseudo-spherical surfaces.
J. DeCicco, DePaul University.
Polygenic functions of several complex variables.
P. C. Hammer, Los Alamos Scientific Laboratory.
Convex bodies associated with a convex body.
A. Kroch, Hebrew Institute of Technology, Haifa.
Solids filling space.
J. Kronsbein, Evansville College.
A method of visualizing four-dimensional rotations.
L. A. MacColl, Bell Telephone Laboratories.
Geometrical properties of two-dimensional wave motion.
R. Rado, King's College, London.
Covering theorems for systems of similar sets of points.

THURSDAY, AUGUST 31
10:15 A.M.
Emerson 211

SECTION IV
PROBABILITY AND STATISTICS, ACTUARIAL SCIENCE, ECONOMICS

E. Lukacs and O. Szász, National Bureau of Standards and University of Cincinnati.
Some non-negative trigonometric polynomials connected with a problem in probability.
W. D. Baten, Michigan State College.
A history of probability in the United States of America to 1926.
E. Michalup, University of Caracas.
New developments in interpolation formulae.
H. E. Stelson, Michigan State College.
The accuracy of linear interpolation.
S. H. Khamis, Statistical Office, United Nations.
A note on the general Chebycheff inequality.

THURSDAY, AUGUST 31
2:15 P.M.
Emerson D

CONFERENCE IN ALGEBRA
GROUPS AND UNIVERSAL ALGEBRA

G. Birkhoff, Harvard University.
Problems in lattice theory.

S. MacLane, University of Chicago.
Cohomology theory of abelian groups.
R. Baer, University of Illinois.
Cohomology theory of a pair of groups.
C. Chevalley, Columbia University.
Determination of the Betti numbers of the exceptional Lie groups.

THURSDAY, AUGUST 31
2:15 P.M.
Sever 11

CONFERENCE IN ANALYSIS
ANALYSIS IN THE LARGE

L. Bers, Syracuse University.
Singularities of minimal surfaces.
S. Bergman, Harvard University.
Geometric methods in the theory of functions of several complex variables.
L. Cesari and T. Radó, University of Bologna and Ohio State University.
Applications of area theory in analysis. (Presented by Cesari)
C. B. Morrey, University of California.
Problem of Plateau on a Riemannian manifold.

THURSDAY, AUGUST 31
2:15 P.M.
Harvard 4

SECTION III
GEOMETRY AND TOPOLOGY

E. M. Bruins, University of Amsterdam.
The symbolical method in algebraic geometry.
P. Du Val, University of Georgia.
Regular surfaces of genus 2.
L. Godeaux, University of Liège.
Singularités des points de diramation isolés des surfaces multiples.
T. R. Hollcroft, Wells College.
Systems of singular primals in S_r.
C. C. Hsiung, Northwestern University.
A general theory of conjugate nets in projective hyperspace.
D. Pedoe, University of London.
The intersection of algebraic varieties.
T. G. Room, University of Sydney. (Read by F. Chong)
Quadrices associated with the Clifford matrices.
C. R. Wylie, University of Utah.
Linear line involutions without a complex of invariant lines

THURSDAY, AUGUST 31

2:15 P.M.

Emerson 211

SECTION IV

PROBABILITY AND STATISTICS, ACTUARIAL SCIENCE, ECONOMICS

R. Fortet, University of Caen.
Éléments aléatoires de nature quelconque.
M. Castellani, University of Kansas City.
Random functions on Divisia ensemble.
E. S. Cansado, Consejo Superior de Estadística, Madrid.
On the logarithmico-Pearson distributions.
F. I. Toranzos, National University of Cuyo, Argentina.
A frequency system that generalizes the Pearson system.
H. von Schelling, Naval Medical Research Laboratory.
Distribution for the ordinal number of simultaneous events which last during a finite time.
G. Tintner, Iowa State College.
Some formal relations in multivariate analysis.

THURSDAY, AUGUST 31

2:15 P.M.

Harvard 1

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

A. Charnes and E. Saibel, Carnegie Institute of Technology.
On some cavitational flows in lubrication.
T. M. Cherry, University of Melbourne.
Exact solutions for flow of a perfect gas in a two-dimensional Laval nozzle.
J. C. Cooke, University of Malaya.
Pohlhausen's method for three-dimensional boundary layers.
A. G. Hansen and M. H. Martin, University of Maryland.
Some geometrical properties of plane flows.
C. K. Thornhill, Armament Research Establishment, Fort Halstead, England.
The dispersion, under gravity, of a column of fluid supported on a rigid horizontal plane.
C. Truesdell, University of Maryland and Naval Research Laboratory.
A new vorticity theorem.
C. T. Wang, New York University.
The application of variational methods to the compressible flow problems.
E. H. Zarantonello, Harvard University.
A constructive theory for the equations of flows with free boundaries.

THURSDAY, AUGUST 31

2:15 P.M.

Harvard 5

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

C. G. Darwin, Cambridge, England.
The refractive index of an ionized gas. (30 min.)

P. G. Bergmann, Syracuse University.
Covariant quantization of nonlinear field theories.

A. J. Coleman, University of Toronto.
Gravitational shift in the solar spectrum.

P. Drumaux, University of Ghent. (Read by H. P. Robertson)
La récession des nébuleuses extra-galactiques.

O. E. Glenn, Lansdowne, Pennsylvania.
The mathematical nature of Lamarck's hypothesis that a biological species tends to increase in size.

K. Sarginson, Somerville College, Oxford, England.
An expansion of a four-dimensional plane wave in terms of eigenfunctions.

E. J. Schremp, Naval Research Laboratory.
On the interpretation of the parameters of the proper Lorentz group.

A. H. Taub, University of Illinois.
Empty space-times admitting three-parameter groups of motion.

E. E. Witmer, University of Pennsylvania.
Integral relationships between nuclear quantities.

FRIDAY, SEPTEMBER 1

ADDRESSES

9:00 A.M.

Mallinckrodt MB9

M. Morse, Institute for Advanced Study.
Recent advances in variational theory in the large.
(*Address by invitation of the Organizing Committee*)

Fogg Large Room

A. Rome, University of Louvain.
The calculation of an eclipse of the sun according to Theon of Alexandria.
(*Address by invitation of the Organizing Committee*)

FRIDAY, SEPTEMBER 1

10:15 A.M.

Emerson D

SECTION I

ALGEBRA AND THEORY OF NUMBERS

A. H. Clifford, The Johns Hopkins University.
A class of partially ordered abelian groups related to Ky Fan's characterizing subgroups.

D. Ellis, University of Florida.
On distance sets and distanciallity in naturally metrized groups.
F. Loonstra, Technische Hogeschool, Delft.
The classes of ordered groups.
H. Ribeiro, University of California.
On lattices of abelian groups with a finite basis.
R. M. Thrall, University of Michigan.
On a Galois connection between algebras of linear transformations and lattices of subspaces of a vector space.
B. H. Arnold, Oregon State College.
Distributive lattices with a third operation defined.
A. L. Foster, University of California.
Boolean-partition-vector extensions and (sub) direct-powers of rings and general operational algebras.
F. Haimo, Washington University.
Some limits of Boolean algebras.
F. Harary, University of Michigan.
On complete atomic proper relation algebras.

FRIDAY, SEPTEMBER 1
10:15 A.M.
Harvard 1

SECTION II
ANALYSIS

A. Erdélyi, California Institute of Technology.
The general form of hypergeometric series of two variables.
R. San Juan Llosá, Madrid, Spain. (Read by T. R. Bachiller)
Les fondements d'une théorie générale de séries divergentes.
G. G. Lorentz, University of Toronto.
Direct theorems on methods of summability.
J. C. P. Miller, National Bureau of Standards.
The determination of converging factors for the asymptotic expansions for the Weber parabolic cylinder functions.
W. Rudin, Duke University.
Uniqueness theory for Hermite series.
R. E. Graves, University of Minnesota.
A closure criterion for orthogonal functions.

FRIDAY, SEPTEMBER 1
10:15 A.M.
Harvard 5

SECTION II
ANALYSIS

E. F. Beckenbach and L. K. Jackson, University of California, Los Angeles, and University of Nebraska.
Subfunctions and elliptic partial differential equations.

P. P. GILLIS, University of Brussels.
Équations de Monge-Ampère, du type élliptique, et problèmes réguliers du calcul des variations.

F. JOHN, New York University.
On the fundamental solution of linear elliptic partial differential equations with analytic coefficients.

M. H. PROTTER, Syracuse University.
Boundary value problems for a partial differential equation of mixed type.

C. B. MORREY, JR., University of California.
Differentiability properties of the solutions of variational problems for multiple integrals.

M. SHIFFMAN, Stanford University.
On variational analysis in the large.

L. C. YOUNG, University of Wisconsin.
Generalized parametric surfaces.

FRIDAY, SEPTEMBER 1
10:15 A.M.
Sever 11

SECTION III
GEOMETRY AND TOPOLOGY

A. BERNHART, University of Oklahoma.
Irreducible rings in minimal five color maps.

A. ERRERA, University of Brussels.
Sur les conséquences, pour le problème de quatre couleurs, d'un théorème de M. Whitney.

B. GELBAUM, G. KALISCH, and J. M. H. OLMSTED, University of Minnesota.
On the embedding of topological semigroups and integral domains.

R. INZINGER, Technische Hochschule, Vienna.
A realization of the geometry of the Hilbert space in the plane.

M. JERISON, University of Illinois.
Characterizations of certain spaces of continuous functions.

N. J. FINE and G. E. SCHWEIGERT, University of Pennsylvania.
On the structure of the group of homeomorphisms of an arc.

C. N. REYNOLDS, University of West Virginia.
Applications of a calculus of finite differences to the 4-color problem.

H. C. WANG, Louisiana State University.
Metric space and its group of isometries.

FRIDAY, SEPTEMBER 1
10:15 A.M.
Emerson 211

SECTION VI
LOGIC AND PHILOSOPHY

T. SKOLEM, Matematiske Institut, Oslo.
Remarks on the foundation of set theory. **(30 min.)**

M. DOLCHER, University of Trieste.
On displacements of systems of data and structures.

J. B. Rosser, Institute for Numerical Analysis, National Bureau of Standards.
Transfinite cardinal arithmetic in Quine's New Foundations.
R. M. Robinson, University of California.
An essentially undecidable axiom system.
W. Szmielew and A. Tarski, University of California.
Mutual interpretability of some essentially undecidable theories.

FRIDAY, SEPTEMBER 1
10:15 A.M.
Harvard 4

SECTION VII
HISTORY AND EDUCATION

J. L. Coolidge, Harvard University.
The origin of polar coordinates.
C. B. Boyer, Brooklyn College.
The foremost textbook of modern times.
H. W. Turnbull, University of St. Andrews, Scotland.
The Scottish contribution to the early history of the calculus.
P. S. Jones, University of Michigan.
Brook Taylor and the mathematical theory of linear perspective, his contributions and influence.
M. Richardson, Brooklyn College.
Fundamentals in the teaching of undergraduate mathematics.
L. E. Boyer, Millersville State Teachers College, Pennsylvania.
A note on the teaching of general mathematics.
R. L. Swain, Ohio State University.
Condensed graphs.
W. Betz, Public School System of Rochester.
Mathematics for the millions, or for the few?
B. H. Gundlach, University of Arkansas.
Gestalt theory in the teaching of mathematics.

FRIDAY, SEPTEMBER 1
2:15 P.M.
Sever 11

CONFERENCE IN ANALYSIS
EXTREMAL METHODS AND GEOMETRIC THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

L. V. Ahlfors, Harvard University.
Introduction.
A. C. Schaeffer and D. C. Spencer, University of Wisconsin and Stanford University.
Coefficient regions for schlicht functions.
M. M. Schiffer, The Hebrew University, Jerusalem.
Variational methods in the theory of conformal mapping.
H. Grunsky, University of Tübingen.
Über Tchebycheffsche Probleme.

R. Nevanlinna, University of Zürich.
Surfaces de Riemann ouvertes.
G. Szegö, Stanford University.
On certain set-functions defined by extremum properties in the theory of functions and in mathematical physics.

FRIDAY, SEPTEMBER 1
2:15 P.M.
Emerson D

CONFERENCE IN TOPOLOGY
HOMOLOGY AND HOMOTOPY THEORY

W. Hurewicz, Massachusetts Institute of Technology.
Homology and homotopy.
(Address by invitation of the Organizing Committee)
S. Eilenberg, Columbia University.
Homotopy groups and algebraic homotopy theories.
J. H. C. Whitehead, Oxford University.
Algebraic homotopy theory.
G. W. Whitehead, Massachusetts Institute of Technology.
Homotopy groups of spheres.

FRIDAY, SEPTEMBER 1
2:15 P.M.
Emerson 211

SECTION I
ALGEBRA AND THEORY OF NUMBERS

G. Y. Rainich, University of Michigan.
Invariants of vectors with noncommutative components, and application to geometry.
G. L. Papy, University of Brussels.
Un théorème d'arithmétique en algèbre de Grassmann.
T. Evans, University of Manchester.
The word problem for abstract algebras.
A. T. Brauer, University of North Carolina.
On algebraic equations with all but one root in the interior of the unit circle.
D. P. Vythoulkas, National University of Engineering, Athens.
On the minimum modulus of a root of a polynomial.
W. Jacobs, George Washington University.
The effect on the inverse of a change in a matrix. Preliminary report.
W. D. Krentel, J. C. C. McKinsey, and W. V. Quine, Oklahoma Agricultural and Mechanical College, The RAND Corporation, and Harvard University.
Information patterns for games in extensive form.
M. Dresher, The RAND Corporation.
Solution of polynomial-like games.
J. Popken, University of Utrecht.
Two arithmetical theorems concerning linear differential-difference equations.

FRIDAY, SEPTEMBER 1

2:15 P.M.

Harvard 4

SECTION IV

PROBABILITY AND STATISTICS, ACTUARIAL SCIENCE, ECONOMICS

R. C. BOSE, University of North Carolina.
Mathematical theory of factorial designs. (30 min.)
B. DE FINETTI, University of Trieste.
La nozione di "beni indipendenti" in base ai nuovi concetti per la misura della "utilità."
A. WALD and J. WOLFOWITZ, Columbia University.
Two methods of randomization in statistics and the theory of games.
H. ROBBINS, University of North Carolina.
Asymptotically sub-minimax solutions of statistical decision problems.
C. N. MOOERS, Zator Company, Boston.
Information retrieval viewed as temporal signalling.
L. S. SHAPLEY, The RAND Corporation.
Information and the formal solution of many-moved games.
J. W. TUKEY, Princeton University.
Estimation in the alternative family of distributions.

FRIDAY, SEPTEMBER 1

2:15 P.M.

Harvard 1

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

F. RELLICH, University of Göttingen.
Störungstheorie der Spektralzerlegung. (30 min.)
R. V. CHURCHILL, University of Michigan.
A modified equation of diffusion.
S. GOLDSTEIN, The Technion, Haifa.
On diffusion by discontinuous movements, and on the telegraph equation.
M. S. KLAMKIN, Polytechnic Institute of Brooklyn.
A moving boundary filtration problem or "The Cigarette Problem."
A. J. McCONNELL, Trinity College, Dublin.
The hypercircle method of approximation to the solution of a general class of boundary-value problems.
H. PORITSKY and H. WEIL, General Electric Company.
Electromagnetic measurements of the flow velocity of a fluid in a pipe of elliptical cross section.
S. A. SCHELKUNOFF, Bell Telephone Laboratories.
Biconical antennas of arbitrary angle.
W. C. TAYLOR, University of Cincinnati and Aberdeen Proving Ground.
Formal solutions of an integro-differential equation for multiply scattered radiation.

FRIDAY, SEPTEMBER 1

7:00 P.M.

Harvard Union

S. BERGMAN, Harvard University.
    On visualization of domains in the theory of functions of two complex variables.

8:00 P.M.

Sever 11

CONFERENCE IN ANALYSIS

ANALYSIS AND GEOMETRY IN THE LARGE

J. LERAY, Collège de France.
    La théorie des points fixes et ses applications en analyse.
G. DE RHAM, University of Geneva and University of Lausanne.
    Harmonic integrals and the theory of intersections.
A. LICHNEROWICZ, University of Paris.
    Curvature and Betti numbers.

SATURDAY, SEPTEMBER 2

9:00 A.M.

Emerson D

CONFERENCE IN TOPOLOGY

FIBER BUNDLES AND OBSTRUCTIONS

P. OLUM, Institute for Advanced Study.
    Theory of obstructions.
W. S. MASSEY, Princeton University.
    Homotopy groups of triads.
G. HIRSCH, University of Brussels.
    Homology invariants and fibre spaces.
E. SPANIER, University of Chicago.
    Homology theory of fiber bundles.

SATURDAY, SEPTEMBER 2

9:00 A.M.

Emerson 211

SECTION I

ALGEBRA AND THEORY OF NUMBERS

K. MAHLER, University of Manchester.
    Farey sections in the fields of Gauss and Eisenstein. (30 min.)
K. IWASAWA, Tokyo University.
    A note on L-functions.
O. T. TODD, National Bureau of Standards.
    Classes of matrices and quadratic fields.
M. WARD, California Institute of Technology.
    Arithmetical properties of lemniscate polynomials.

M. Gut, University of Zürich.
  Die Bedeutung der Euler'schen Zahlen für den grossen Fermat'schen Satz und für die Klassenanzahl des Körpers der $4l$-ten Einheitswurzeln.

S. Chowla and A. B. Showalter, University of Kansas.
  On the solutions of $h(d) = 1$.

H. Davenport, University College, London.
  Binary cubic forms.

H. S. M. Coxeter, University of Toronto.
  Extreme forms.

L. Tornheim, University of Michigan.
  The extreme smoothed octagon.

H. Cohn, Wayne University.
  A periodic algorithm for cubic forms.

SATURDAY, SEPTEMBER 2

9:00 A.M.

Harvard 1

SECTION II

ANALYSIS

M. Brelot, University of Grenoble.
  Sur l'évolution du problème de Dirichlet.

A. Edrei, University of Saskatchewan.
  On mappings of a uniform space onto itself.

J. L. Kelley, Tulane University of Louisiana.
  On commutative self-adjoint operator algebras.

C. Goffman, University of Oklahoma.
  Lusin's theorem for one to one measurable transformations.

J. P. LaSalle, University of Notre Dame.
  Successive upper and lower approximations.

E. R. Lorch, Columbia University.
  Differentiable inequalities, convexity, and mixed volumes.

L. Nachbin, University of Brazil.
  On the continuity of positive linear transformations.

M. H. Stone, University of Chicago.
  The spectrum and the operational calculus for a family of operators.

J. L. Kelley and R. L. Vaught, Tulane University of Louisiana and University of California.
  A note on Banach algebras.

J. W. T. Youngs, Indiana University and The RAND Corporation.
  Surface area and homotopy.

A. Douglis, Institute for Mathematics and Mechanics, New York University.
  An extremum principle for solutions of a class of elliptic systems of differential equations with continuous coefficients.

SATURDAY, SEPTEMBER 2

9:00 A.M.

Harvard 5

SECTION II

ANALYSIS

L. Van Hove, University of Brussels.
A set of unitary representations of the group of contact transformations.

H. Wielandt, University of Mainz.
Über die Eigenwertaufgaben mit reellen diskreten Eigenwerten.

A. Wilansky, Lehigh University.
Summability matrices coincident with regular matrices, Banach space methods.

F. Wolf, University of California.
Perturbation of analytic operators.

R. H. Cameron and R. E. Graves, University of Minnesota.
Additive functionals on a space of continuous functions.

S. P. Diliberto and E. G. Straus, University of California and University of California, Los Angeles.
On approximating to functions of several variables by functions of fewer variables.

R. B. Leipnik, Institute for Advanced Study.
Axiomatic Perron inversion.

A. Papoulis, University of Pennsylvania.
On the strong differentiation of the indefinite integral.

A. Denjoy, University of Paris.
Les permutations clivées.

D. B. Goodner, Florida State University.
A note on separable normed linear spaces.

G. Kurepa, University of Zagreb.
Sur les ensembles partiellement ordonnés.

W. D. Berg and O. M. Nikodým, Kenyon College.
On convex sets in linear spaces.

SATURDAY, SEPTEMBER 2

9:00 A.M.

Sever 11

SECTION IV

PROBABILITY AND STATISTICS, ACTUARIAL SCIENCE, ECONOMICS

P. Lévy, École Polytechnique, Paris.
Processus à la fois stationnaires et markoviens pour les systèmes ayant une infinité dénombrable d'états possibles. (30 min.)

K. L. Chung, Cornell University.
An ergodic theorem for stationary Markov chains with a countable number of states.

K. Yosida, Nagoya University.
Stochastic processes built from flows.

M. Rosenblatt, Cornell University.
On a class of two-dimensional Markov processes.

H. Bergström, Chalmers University of Technology, Gothenburg.
On asymptotical expansions of probability functions.

W. R. WASOW, Institute for Numerical Analysis, National Bureau of Standards.
On the mean duration of random walks in n dimensions.

B. O. KOOPMAN, Columbia University.
Improbable events in general stationary-transition Markoff chains.

SATURDAY, SEPTEMBER 2

9:00 A.M.

Harvard 4

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

D. C. DRUCKER and W. PRAGER, Brown University, and H. J. GREENBERG, Carnegie Institute of Technology.
On the pressing of a rigid stamp into an elastic-plastic body in plane strain.

D. GRAFFI, University of Bologna.
Su alcuni questioni di elasticità ereditaria.

G. H. HANDLEMAN and A. E. HEINS, Carnegie Institute of Technology.
Remarks on the direct integration of the equations of elasticity.

P. G. HODGE, JR., University of California, Los Angeles.
The method of characteristics applied to problems of steady motion in plane plastic stress.

W. S. JARDETSKY, Columbia University and Manhattan College.
The problem of Atlantis.

E. H. LEE, Brown University.
The analysis of plastic flow in plane strain with large strains.

S. MORIGUTI, Tokyo University.
Some remarks on the method of solving two-dimensional elastic problems.

A. W. SÁENZ and P. F. NEMÉNYI, Naval Research Laboratory and University of Maryland.
On the geometry of two-dimensional elastic stress systems.

A. SIGNORINI, University of Rome.
A simple case of "incompatibility" between linear elasticity and the theory of finite deformations.

C. P. WELLS and R. A. BETH, Michigan State College and Western Reserve University.
A new approach to cantilever-strut problems.

SATURDAY, SEPTEMBER 2

2:15 P.M.

Emerson D

CONFERENCE IN ALGEBRA

STRUCTURE THEORY OF RINGS AND ALGEBRAS

A. A. ALBERT, University of Chicago.
Power-associative algebras.
(*Address by invitation of the Organizing Committee*)

R. BRAUER, University of Michigan.
On the representations of groups of finite order.

N. JACOBSON, Yale University.
Representation theory of Jordan rings.

J. DIEUDONNÉ, University of Nancy.
Minimal ideals.

T. Nakayama, Nagoya University.
Two topics in the structural theory of rings.

SATURDAY, SEPTEMBER 2

2:15 P.M.

Sever 11

CONFERENCE IN APPLIED MATHEMATICS

RANDOM PROCESSES IN PHYSICS AND COMMUNICATIONS

C. E. Shannon, Bell Telephone Laboratories.
Some topics in information theory.

S. M. Ulam, Los Alamos Scientific Laboratory.
Random processes and transformations.

SATURDAY, SEPTEMBER 2

2:15 P.M.

Harvard 1

SECTION II

ANALYSIS

H. Bohr, University of Copenhagen.
A survey of the different proofs of the main theorems in the theory of almost periodic functions. (30 min.)

W. Magnus, California Institute of Technology.
On a class of bounded matrices.

M. Owchar, Southwest Missouri State College.
Wiener integrals of multiple variations.

J. Chazy, University of Paris.
La solution du problème des trois corps par Sundman, et ses conséquences.

G. Fichera, University of Rome.
Methods for solving linear functional equations, developed by the Italian Institute for the Applications of Calculus.

R. N. Haskell, University of Texas.
Sub-biharmonic functions.

E. Hopf, Indiana University.
On the initial value problem for the Navier-Stokes equations.

M. Janet, University of Paris.
Équations semi-canoniques.

V. Wolontis, University of Kansas.
The change of resistance under circular symmetrization.

J. Elliott, Cornell University.
Some singular integral equations of the Cauchy type.

SATURDAY, SEPTEMBER 2

2:15 P.M.

Harvard 5

SECTION II

ANALYSIS

S. Mandelbrojt, Rice Institute and Collège de France.
Théorèmes d'unicité de la théorie des fonctions. (30 min.)

E. F. Collingwood and M. L. Cartwright, Alnwick, England and Cambridge University.
Boundary theorems for functions meromorphic in the unit circle.

H. Milloux, University of Bordeaux.
Fonctions méromorphes et dérivées.

G. Valiron, University of Paris.
Fonctions méromorphes d'ordre nul.

B. Epstein and J. Lehner, University of Pennsylvania.
On Ritt's representation of analytic functions as infinite products.

P. Erdös, F. Herzog, and G. Piranian, University of Aberdeen, Scotland, Michigan State College, and University of Michigan.
Schlicht gap series whose convergence on the unit circle is uniform but not absolute.

A. W. Goodman and M. S. Robertson, University of Kentucky and Rutgers University.
A class of multivalent functions.

O. Lehto, University of Helsinki.
On the boundary behavior of analytic functions.

G. R. MacLane, Rice Institute.
Riemann surfaces and asymptotic values associated with certain real entire functions.

Z. Nehari, Washington University.
Some extremal problems involving single-valued analytic functions.

J. L. Walsh, Harvard University.
On Rouché's theorem and the integral-square measure of approximation.

SATURDAY, SEPTEMBER 2

2:15 P.M.

Emerson 211

SECTION III

GEOMETRY AND TOPOLOGY

P. Hartman, The Johns Hopkins University.
On the uniqueness of geodesics.

V. Hlavatý, Indiana University.
Spinor space and line geometry.

S. B. Jackson, University of Maryland.
Angular measure and the Gauss-Bonnet formula.

A. Fialkow, Polytechnic Institute of Brooklyn.
A correspondence principle in conformal geometry.

C. C. MacDuffee, University of Wisconsin.
Curves in Minkowski space.

S. B. Myers, University of Michigan.
Curvature of closed hypersurfaces.

N. SAKELLARIOU, University of Athens.
Über Strahlensysteme deren abwickelbaren Flächen eine Fläche unter geodätischen Linien und ihren geodätischen Parallelen schneiden.

J. L. VANDERSLICE, University of Maryland.
Nonlinear displacements in affine-connected space.

P. VINCENSINI, University of Marseille.
Sur certains réseaux tracés sur une surface et leur rôle en géométrie différentielle.

K. YANO, Tokyo University.
Affine and projective geometries of systems of hypersurfaces.

SATURDAY, SEPTEMBER 2

2:15 P.M.

Harvard 4

SECTION VII

HISTORY AND EDUCATION

G. PÓLYA, Stanford University.
On plausible reasoning. **(30 min.)**

E. ROSSING, Tönder Statsskole, Denmark.
The teaching of mathematics in Denmark.

K. MAY and K. MCVOY, Carleton College.
Simplification of rigorous limit proofs.

O. E. OVERN, State Teachers College, Milwaukee.
Current trends in the teaching of plane trigonometry.

H. P. FAWCETT, Ohio State University.
Unifying concepts in mathematics.

M. S. KRAMER, New Mexico College of Agriculture and Mechanic Arts.
The introduction of applied problems for the enrichment of classroom instruction in the schools and colleges.

F. L. GRIFFIN, Reed College.
Further experience with undergraduate mathematical research.

MONDAY, SEPTEMBER 4

9:00 A.M.

Emerson D

SECTION I

ALGEBRA AND THEORY OF NUMBERS

A. SELBERG, Institute for Advanced Study.
The general sieve method and its place in prime number theory. **(30 min.)**

D. H. LEHMER, University of California.
Problems concerning Ramanujan's function.

D. SHANKS, Naval Ordnance Laboratory.
On the density of reducible integers and some sequences associated with them.

I. NIVEN, University of Oregon.
Sets of integers of density zero.

M. HALL, JR., and H. J. RYSER, Ohio State University.
Cyclic incidence matrices.

S. Chowla and A. L. Whiteman, University of Kansas and University of Southern California.
On exponential and character sums.
W. Ljunggren, University of Bergen.
On the integral solutions of the diophantine system $ax^2 - by^2 = c, a_1z^2 - b_1y^2 = c_1$.
L. V. Toralballa, Marquette University.
A generalization of finite integration.
N. C. Scholomiti, University of Illinois.
Expansion of discontinuous functions. Applications to the theory of numbers.
A. A. Trypanis, National Technical University, Athens.
On Fermat's last theorem.

MONDAY, SEPTEMBER 4
9:00 A.M.
Harvard 1

SECTION II
ANALYSIS

L. Carleson, University of Uppsala.
On a class of meromorphic functions.
G. Springer, Massachusetts Institute of Technology.
The coefficient problem for schlicht mappings of the exterior of the unit circle.
N. Terzioğlu, University of Istanbul.
Über den Verzerrungssatz von Koebe.
L. R. Ford, Illinois Institute of Technology.
Fundamental regions for discontinuous groups of linear transformations.
B. Lepson, Institute for Advanced Study.
On irregular points of normal convergence and M-convergence for series of analytic functions.
S. Rosen, Drexel Institute of Technology.
Modular transformation of certain series.
L. Sario, Harvard University.
On open Riemann surfaces.
O. Szász and N. Yeardley, University of Cincinnati and Purdue University.
Representation of an analytic function by general Laguerre series.
F. G. Tricomi, University of Turin.
On the incomplete gamma function.
S. E. Warschawski, University of Minnesota.
On the effective determination of the mapping function in conformal mapping.

MONDAY, SEPTEMBER 4
9:00 A.M.
Harvard 5

SECTION II
ANALYSIS

E. Hewitt and H. S. Zuckerman, University of Washington.
On convolution algebras.

G. RACAH, The Hebrew University, Jerusalem.
On the characterization of the rows and columns of the representations of the semi-simple Lie groups.
R. K. RITT, University of Michigan.
Algebraic functions in an abelian normed ring.
I. J. SCHOENBERG, University of Pennsylvania.
On the number of variations of signs in a sequence of linear forms.
S. SHERMAN, Lockheed Aircraft Corporation.
The second adjoint of a C^* algebra.
O. M. NIKODÝM, Kenyon College.
On extension of measure.
J. KAMPÉ DE FÉRIET, University of Lille.
Sur l'analyse harmonique des fonctions à carré moyen fini.
N. J. FINE, University of Pennsylvania.
On the asymptotic distribution of certain sums.
E. HILLE, Yale University.
"Explosive" solutions of Fokker-Planck's equation.
S. KAKUTANI, Yale University.
Brownian motion and duality of locally compact abelian groups.
H. RUBIN, Stanford University.
An elementary treatment of uniqueness for the Hamburger moment problem.
A. SARD, Queens College.
Least square error and variance.

MONDAY, SEPTEMBER 4

9:00 A.M.

Harvard 4

SECTION III

GEOMETRY AND TOPOLOGY

A. L. BLAKERS, Lehigh University.
A generalization of the Hurewicz isomorphism theorem.
R. E. CHAMBERLIN, University of Utah.
On the mappings of a 4-complex into certain simply connected spaces.
B. ECKMANN, Swiss Federal School of Technology.
Räume mit Mittelabbildungen.
S. T. HU, Tulane University of Louisiana.
The equivalence of fibre bundles.
J. LERAY, Collège de France.
L'emploi, en topologie algébrique, du formalisme du calcul différentiel extérieur.
E. PITCHER, Lehigh University and Institute for Advanced Study.
Homotopy groups of the space of curves with application to spheres.
S. S. CHERN and E. H. SPANIER, University of Chicago.
Transgression and the homology structure of fiber bundles.
N. E. STEENROD, Princeton University.
Reduced powers of a cocycle.
R. L. WILDER, University of Michigan.
A generalization of a theorem of Pontrjagin.

MONDAY, SEPTEMBER 4

9:00 A.M.

Sever 11

SECTION IV

PROBABILITY AND STATISTICS, ACTUARIAL SCIENCE, ECONOMICS

S. N. Roy, University of North Carolina.
On some aspects of statistical inference. (30 min.)

A. C. Cohen, Jr., University of Georgia.
Estimating parameters of logarithmic-normal distributions by the method of maximum likelihood.

P. R. Rider, Washington University.
The distribution of ranges in samples from a discrete rectangular population.

A. W. Marshall and J. E. Walsh, The RAND Corporation.
Some tests for comparing percentage points of two arbitrary continuous populations.

MONDAY, SEPTEMBER 4

9:00 A.M.

Hunt A

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

G. Borg, Mathematiska Institutionen, Uppsala.
An inversion formula.

P. Brock, Reeves Instrument Corporation, New York.
The nature of solutions of a Rayleigh type forced vibration equation with large coefficient of damping.

F. H. Brownell, University of Washington.
Asymptotically ergodic output under ergodic input of delay differential machines.

R. M. Foster, Polytechnic Institute of Brooklyn.
The number of series-parallel networks.

M. Goldberg, Bureau of Ordnance, Navy Department.
Rotors in spherical polygons.

R. Kahal, Washington University.
The realization of the transfer function of the finite, four-terminal network.

E. Leimanis, University of British Columbia.
Some new cases of integration of differential equations of exterior ballistics by quadratures.

A. Rapoport and R. Solomonoff, University of Chicago.
Structure of random nets.

R. M. Rosenberg, University of Washington.
A note on the response of systems with or without nonlinear elements.

A. W. Sáenz, Naval Research Laboratory.
On integrals of motion of the Runge type in classical and quantum mechanics.

J. J. Smith and P. L. Alger, General Electric Company.
The use of the null-unit function in generalized integration.

MONDAY, SEPTEMBER 4
9:00 A.M.
Emerson 211

SECTION VI
LOGIC AND PHILOSOPHY

A. Tarski, University of California.
Some notions and methods on the borderline of algebra and metamathematics.
(*Address by invitation of the Organizing Committee*) (30 min.)

A. Robinson, College of Aeronautics, Cranfield, England.
Applied symbolic logic. (30 min.)

J. Robinson, Berkeley, California.
Existential definability in arithmetic.

I. Rosenbaum, University of Miami.
A logistic proof of a theorem related to Landau's Theorem 4.

E. R. Stabler, Hofstra College.
Applied logic and modern problems.

MONDAY, SEPTEMBER 4
2:15 P.M.
Emerson D

CONFERENCE IN ANALYSIS
ALGEBRAIC TENDENCIES IN ANALYSIS

Report on Group Representations

***Panel*:**

H. Cartan, University of Paris.
A. M. Gleason, Harvard University.
R. Godement, University of Nancy.
G. W. Mackey, Harvard University.
F. I. Mautner, Massachusetts Institute of Technology.
L. Schwartz, University of Nancy.

***Spokesman*:**

R. Godement

Report on Topological Algebra

***Panel*:**

J. Dieudonné, University of Nancy.
I. Kaplansky, University of Chicago.
I. E. Segal, University of Chicago.

***Spokesman*:**

I. Kaplansky

Report on Measure Theory

***Panel*:**

J. Dieudonné, University of Nancy.
P. R. Halmos, University of Chicago.

J. C. OXTOBY, Bryn Mawr College.
D. M. STONE, University of Manchester.
S. M. ULAM, Los Alamos Scientific Laboratory.

Spokesman:
P. R. HALMOS

MONDAY, SEPTEMBER 4
2:15 P.M.
Sever 11

CONFERENCE IN APPLIED MATHEMATICS
PARTIAL DIFFERENTIAL EQUATIONS

J. VON NEUMANN, Institute for Advanced Study.
Shock interaction and its mathematical aspects.
(*Address by invitation of the Organizing Committee*)
R. COURANT, New York University.
Boundary value problems in modern fluid dynamics.
S. GOLDSTEIN, The Technion, Haifa.
Selected problems in gas dynamics.
W. HEISENBERG, Max Planck Institute for Physics, Göttingen.
Die Stabilitätsfragen der Flüssigkeitsdynamik im Zusammenhang mit der statistischen Turbulenztheorie.
W. PRAGER, Brown University.
Boundary value problems of plasticity.
J. J. STOKER, New York University.
Mathematical theory of water waves.

MONDAY, SEPTEMBER 4
2:15 P.M.
Emerson 211

SECTION I
ALGEBRA AND THEORY OF NUMBERS

W. GIVENS, University of Tennessee.
Some properties of the Dieudonné determinant.
R. D. SCHAFER, University of Pennsylvania.
A theorem on the derivations of Jordan algebras.
A. J. PENICO, Tufts College.
On the structure of standard algebras.
T. NAKAYAMA, Nagoya University.
On the theory of Galois algebras.
TH. H. J. LEPAGE, University of Brussels.
Idéaux homogènes de l'algèbre extérieure.
J. LEVITZKI, The Hebrew University, Jerusalem.
On the algebraic elements of a ring with operators.
R. H. BRUCK, University of Wisconsin.
On the associativity theorems for alternative rings and Moufang loops.
M. F. SMILEY, State University of Iowa.
Topological alternative rings.

. Bourne, Institute for Advanced Study.
The Jacobson radical of a semiring.
. Abellanas, University of Madrid.
Variété fondamentale par rapport d'une correspondance algébrique.

MONDAY, SEPTEMBER 4
2:15 P.M.
Harvard 1

SECTION II
ANALYSIS

. Gelbart, Syracuse University.
An extension of the Riemann mapping theorem associated with minimal surfaces.
. A. Barnett, University of Cincinnati.
Functional invariants of integro-differential equations.
. Pinney, University of California.
A system of functional equations.
[. G. Arsove, Brown University.
Functions representable as differences of subharmonic functions.
[. Fekete, The Hebrew University, Jerusalem.
On transfinite radius.
:. Schmidt, Technische Hochschule Braunschweig.
Über Nullgebilde analytischer Funktionen zweier Veränderlichen, die in singulären Punkten münden.

MONDAY, SEPTEMBER 4
2:15 P.M.
Harvard 4

SECTION III
GEOMETRY AND TOPOLOGY

. Segre, University of Bologna.
Arithmetical properties of algebraic varieties. (30 min.)
. Andreotti, University of Rome.
Sopra alcune superficie algebriche.
Barsotti, University of Pittsburgh.
Algebraic theory of intersections for cycles of an algebraic variety.
'. R. Hutcherson, University of Florida.
Imperfect point on invariant space curves.
. Kasner, Columbia University.
The converse of the theorem of Mehmke-Segre.
. S. Motzkin, The Hebrew University, Jerusalem.
Duality and neighbor points.
F. Nash, Jr., Princeton University.
Algebraic approximations to manifolds.
. J. Purcell, University of Arizona.
A series of non-involutorial Cremona transformations in [n].
. M. Terracini, University of Turin.
Direttrici congiunte e bicongiunte di una rigata.

TUESDAY, SEPTEMBER 5

ADDRESSES

9:00 A.M.

Mallinckrodt MB9

A. WALD, Columbia University.
Basic ideas of a general theory of statistical decision rules.
(*Address by invitation of the Organizing Committee*)

Fogg Large Room

H. WHITNEY, Harvard University.
r-dimensional integration in n-space.
(*Address by invitation of the Organizing Committee*)

10:15 A.M.

Mallinckrodt MB9

W. V. D. HODGE, Cambridge University.
Topological invariants of algebraic varieties.
(*Address by invitation of the Organizing Committee*)

Fogg Large Room

J. F. RITT, Columbia University.
Differential groups.
(*Address by invitation of the Organizing Committee*)

11:30 A.M.

Mallinckrodt MB9

H. DAVENPORT, University College, London.
Recent progress in the geometry of numbers.
(*Address by invitation of the Organizing Committee*)

Fogg Large Room

L. SCHWARTZ, University of Nancy.
Distributions and principal applications.
(*Address by invitation of the Organizing Committee*)

TUESDAY, SEPTEMBER 5

2:15 P.M.

Emerson D

CONFERENCE IN ALGEBRA

ARITHMETIC ALGEBRA

E. ARTIN, Princeton University.
Modern development of algebraic number theory and class field theory.
W. KRULL, University of Bonn.
Jacobsonsches Radikal, Hilbertscher Nullstellensatz, Dimensionstheorie.

M. Eichler, University of Münster. (Read by R. D. Schafer)
Arithmetics of orthogonal groups.
M. Krasner, National Center of Scientific Research, Paris.
Essai d'une théorie non-abélienne des corps de classes.

TUESDAY, SEPTEMBER 5

2:15 P.M.

Sever 11

CONFERENCE IN ANALYSIS

ALGEBRAIC TENDENCIES IN ANALYSIS

Report on Spectral Theory

***Panel*:**

W. Ambrose, Massachusetts Institute of Technology.
J. Dixmier, Faculté des Sciences, Dijon.
N. Dunford, Yale University.
F. J. Murray, Columbia University.
J. von Neumann, Institute for Advanced Study.
F. Rellich, University of Göttingen.
B. de Sz. Nagy, University of Szeged.
K. Yosida, Nagoya University.

***Spokesman*:**

N. Dunford

Report on Applied Functional Analysis

***Panel*:**

N. Aronszajn, Oklahoma Agricultural and Mechanical College.
S. Bergman, Harvard University.
J. W. Calkin, Rice Institute.
K. Friedrichs, New York University.
K. Kodaira, Tokyo University.
A. Weinstein, University of Maryland.

***Spokesman*:**

N. Aronszajn

Report on Ergodic Theory

***Panel*:**

N. Dunford, Yale University.
W. Eberlein, University of Wisconsin.
G. A. Hedlund, Yale University.
E. Hille, Yale University.
S. Kakutani, Yale University.
J. C. Oxtoby, Bryn Mawr College.

***Spokesman*:**

S. Kakutani
Ergodic theory.
(*Address by invitation of the Organizing Committee*)

TUESDAY, SEPTEMBER 5

2:15 P.M.

Emerson 211

CONFERENCE IN TOPOLOGY

DIFFERENTIABLE MANIFOLDS

S. S. CHERN, University of Chicago.
Differential geometry of fiber bundles.
(Address by invitation of the Organizing Committee)
C. EHRESMANN, University of Strasbourg.
Almost complex manifolds.
B. ECKMANN, Swiss Federal School of Technology.
Topologie der komplexen Mannigfaltigkeiten.
C. B. ALLENDOERFER, Haverford College.
Cohomology on real differentiable manifolds.

TUESDAY, SEPTEMBER 5

2:15 P.M.

Harvard 1

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

R. H. BATTIN, Massachusetts Institute of Technology.
On the stability of the boundary layer over a body of revolution.
R. BERKER, Technical University of Istanbul.
Sur l'écoulement d'un fluid visqueux autour d'un obstacle.
A. GHAFFARI, University of Tehran.
Simple waves in two-dimensional compressible flow.
C. C. LIN, Massachusetts Institute of Technology.
On the stability of zonal winds over a rotating spherical earth.
K. MILLSAPS and K. POHLHAUSEN, Alabama Polytechnic Institute and Office of Air Research, Wright-Patterson Air Force Base.
The kinetic structure of plane shock waves.
H. PORITSKY, General Electric Company.
The collapse or growth of a spherical bubble or cavity in a viscous fluid.
R. SAUER, Technische Hochschule München.
Linearisierte Überschallströmung um langsam schwingende Drehkörper.
B. R. SETH, Iowa State College and Hindu College, Delhi, India.
Synthetic method for nonlinear problems.
S. S. SHÜ, Illinois Institute of Technology.
On the solution in the large of a Cauchy problem (with special references to the compressible flow after a stationary shock).
K. M. SIEGEL, Aeronautical Research Center, University of Michigan.
An exact solution to the nonlinear differential equation describing the passage of plane waves of sound through air.

TUESDAY, SEPTEMBER 5

2:15 P.M.

Harvard 5

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

F. L. ALT, National Bureau of Standards.
Almost-triangular matrices.

G. R. BOULANGER, Polytechnic Faculty of Mons and University of Brussels.
New advance in the structural study of multiplane nomograms.

H. BÜCKNER, Minden, Germany.
Remarks on an algebraic method for numerically solving the Fredholm integral equation $y - \lambda \mathfrak{k} y = f$.

S. H. CRANDALL, Massachusetts Institute of Technology.
On a relaxation method for eigenvalue problems.

W. FELLER and G. E. FORSYTHE, Cornell University and National Bureau of Standards.
New matrix transformations for obtaining characteristic vectors.

L. FOX, National Physical Laboratory, Teddington, England.
The numerical solution of ordinary differential equations.

F. N. FRENKIEL and H. POLACHEK, Naval Ordnance Laboratory.
An algorithm for the construction of a polynomial representing a given tabular function.

M. R. HESTENES, University of California, Los Angeles, and National Bureau of Standards.
Iterative methods of obtaining solutions of boundary value problems.

M. R. HESTENES and W. KARUSH, University of California, Los Angeles, and National Bureau of Standards, and University of Chicago and National Bureau of Standards.
A method of gradients for the calculation of the characteristic roots and vectors of a real symmetric matrix.

R. F. SHAW, Eckert-Mauchly Computer Corporation, Philadelphia.
Determination of instruction codes for automatic computers.

H. WALLMAN, Massachusetts Institute of Technology and Chalmers Institute of Technology, Gothenburg.
Solution of partial differential equations by means of continuous-variable mathematical machines.

P. W. ZETTLER-SEIDEL, Naval Ordnance Laboratory.
Improved Adams method of numerical integration of ordinary differential equations.

WEDNESDAY, SEPTEMBER 6

9:00 A.M.

Emerson D

CONFERENCE IN TOPOLOGY

TOPOLOGICAL GROUPS

P. A. SMITH, Columbia University.
Some topological notions connected with a set of generators.

D. MONTGOMERY, Institute for Advanced Study.
Properties of finite-dimensional groups.

K. IWASAWA, Tokyo University.
Locally compact groups.

A. Gleason, Harvard University.
One-parameter subgroups and Hilbert's fifth problem.
R. H. Fox, Princeton University.
Recent development of knot theory at Princeton.

WEDNESDAY, SEPTEMBER 6

9:00 A.M.

Emerson 211

SECTION I

ALGEBRA AND THEORY OF NUMBERS

H. D. Kloosterman, University of Leiden.
The characters of binary modular congruence groups. (30 min.)
K. A. Hirsch, University of Durham.
A characteristic property of nilpotent groups.
K.-T. Chen, Columbia University.
Some commutator subgroups of a linkage group.
H. W. Kuhn, Princeton University.
Subgroup theorems for groups presented by generators and relations.
J. C. Abbott and T. J. Benac, United States Naval Academy.
Similarity and isotopy.
R. R. Stoll, Lehigh University.
Matroid semigroups.
M. Krasner, National Center of Scientific Research, Paris.
Généralisation abstraite de la théorie de Galois.
L. R. Wilcox, Illinois Institute of Technology.
On the generation of transitive relations.
P. Dubreil, University of Paris.
Sur une classe de relations d'équivalence.
L. C. Hutchinson, Polytechnic Institute of Brooklyn.
Incidence relations and canonical forms for alternating tensors.
P. Scherk, University of Saskatchewan.
On a theorem by Cartan.
E. Grosswald, University of Saskatchewan.
On the genus of the fundamental region of some subgroups of the modular group.

WEDNESDAY, SEPTEMBER 6

9:00 A.M.

Harvard 1

SECTION II

ANALYSIS

T. Bang, University of Copenhagen.
Metric spaces of infinitely differentiable functions.
D. G. Bourgin, University of Illinois.
Approximately multiplicative operators.

A. DVORETZKY, The Hebrew University, Jerusalem.
On Hausdorff measures.

F. A. FICKEN, New York University and University of Tennessee.
The continuation method for functional equations.

L. GÅRDING, University of Lund.
The asymptotic distribution of the eigenvalues and eigenfunctions of a general vibration problem.

H. L. HAMBURGER, University of Ankara.
On the reduction of a completely continuous linear transformation in Hilbert space.

R. C. JAMES, University of California.
Projections and bases in Banach spaces.

G. KÖTHE, University of Mainz.
Eine einfache Klasse lokalkonvexer linearer Räume.

M. M. PEIXOTO, University of Brazil.
Note on uniform continuity.

A. C. OFFORD, University of London.
Spaces of integral functions.

R. S. PHILLIPS, University of Southern California.
A general spectral theory.

P. C. ROSENBLOOM, Syracuse University.
The Cauchy-Kowalewski existence theorem.

WEDNESDAY, SEPTEMBER 6

9:00 A.M.

Harvard 5

SECTION II
ANALYSIS

C. BLANC, University of Lausanne.
Sur les équations différentielles linéaires à coefficients variables.

M. L. CARTWRIGHT and J. E. LITTLEWOOD, Cambridge University.
Some topological problems connected with forced oscillations.

S. KARLIN, California Institute of Technology.
Moment theory and orthogonal polynomials.

K. S. MILLER and M. M. SCHIFFER, New York University and The Hebrew University, Jerusalem.
On the Green's functions of ordinary differential systems.

W. C. SANGREN, Miami University.
Generalized Fourier integrals.

G. SANSONE, University of Florence.
Su una classe di equazioni di Liénard aventi una sola soluzione periodica.

F. SIMONART, University of Louvain.
Équation différentielle des systèmes isothermes.

H. PORITSKY and J. J. SLADE, General Electric Company and Rutgers University.
On the solution of certain linear differential equations.

I. VIDAV, University of Ljubljana.
Sur les théorèmes de Klein dans les équations différentielles linéaires.

S. B. SARANTOPOULOS, University of Athens.
Some nuclei of contour integrals which satisfy linear differential equations.

WEDNESDAY, SEPTEMBER 6

9:00 A.M.

Harvard 4

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

H. Lewy, University of California.
Developments at the confluence of analytic boundary conditions. (30 min.)

F. Bureau, University of Liège.
Le problème de Cauchy pour certains systèmes d'équations linéaires aux dérivées partielles totalement hyperboliques.

Y. W. Chen, Institute for Advanced Study.
On a quasi-linear system of hyperbolic differential equations with a parameter and a singularity.

J. B. Diaz, University of Maryland.
Concerning scalar products and certain minimum and maximum principles of mathematical physics.

B. Friedman, New York University.
Multiple representations for Green's functions of second order partial differential equations.

R. D. Gordon, University of Buffalo.
The general integration of a quasi-linear partial differential equation of second order composed of symmetric cartesian invariants.

M. Herzberger, Eastman Kodak Research Laboratory, Rochester, New York.
An optical model of physics.

C. L. Pekeris, Weizmann Institute, Rehovoth, Israel.
The correlation tensor of the electromagnetic field in cavity radiation.

D. E. Spencer, Brown University.
Coordinate systems permitting separation of the Laplace and Helmholtz equations.

WEDNESDAY, SEPTEMBER 6

2:15 P.M.

Emerson D

CONFERENCE IN APPLIED MATHEMATICS

STATISTICAL MECHANICS

N. Wiener, Massachusetts Institute of Technology.
Comprehensive view of prediction theory.
(*Address by invitation of the Organizing Committee*)

W. Feller, Princeton University.
Mathematical theory of diffusion processes.

WEDNESDAY, SEPTEMBER 6

2:15 P.M.

Harvard 1

SECTION II

ANALYSIS

E. Baiada, University of Pisa.
The uniqueness for the equation $p = f(x, y, z, q)$ with the Cauchy data.

L. M. COURT, Rutgers University.
A theorem on conditional extremes with an application to total differentials.

H. F. MACNEISH, University of Miami.
A uniform method for integrating (a) $\int (1/Q_1(Q_2)^{1/2})dx$ and (b) $\int (1/(Q_1Q_2)^{1/2})dx$, where Q_1 and Q_2 are distinct quadratic functions of x.

A. M. WHITNEY, University of Pennsylvania.
A criterion for total positivity of matrices.

A. P. CALDERÓN and A. ZYGMUND, University of Chicago.
On singular integrals in the theory of the potential.

WEDNESDAY, SEPTEMBER 6
2:15 P.M.
Harvard 5

SECTION II
ANALYSIS

M. AISSEN, I. J. SCHOENBERG, and A. WHITNEY, University of Pennsylvania.
Generating functions for totally positive sequences. Preliminary report.

V. F. COWLING, University of Kentucky.
On the partial sums of a Taylor series.

J. L. ULLMAN, University of Michigan.
Hankel determinants of sections of a Taylor's series.

S. CHOWLA, University of Kansas.
The asymptotic behaviour of solutions of difference equations.

H. P. THIELMAN, Iowa State College.
Note on a functional equation.

M. O. GONZÁLEZ, University of Havana.
An alternative approach to the theory of elliptic functions.

H. HELSON, Harvard University.
Spectral synthesis of bounded functions.

WEDNESDAY, SEPTEMBER 6
2:15 P.M.
Harvard 4

SECTION III
GEOMETRY AND TOPOLOGY

R. D. ANDERSON, University of Pennsylvania.
Continuous collections of continua.

R. ARENS, University of California, Los Angeles.
Operations induced in conjugate spaces.

C. ARF, Istanbul, Turkey.
Quasiconvex polyhedra.

R. H. BING, University of Wisconsin.
Higher dimensional hereditarily indecomposable continua.

O. H. HAMILTON, Oklahoma Agricultural and Mechanical College.
Fixed-point theorems for pseudo-arcs and certain other metric continua.

A. HELLER, Institute for Advanced Study.
On equivariant maps of spaces with operators.

V. L. Klee, Jr., University of Virginia.
A proof that Hilbert space is homeomorphic with its solid sphere.
M. J. Norris, College of St. Thomas.
Topological spaces having the same regular open sets.
R. Remage, Jr., University of Delaware.
Invariance and periodicity properties of nonalternating in the large transformations.
C. W. Williams, Washington and Lee University.
Incompressibility and periodicity.

WEDNESDAY, SEPTEMBER 6

2:15 P.M.

Emerson 211

SECTION VI

LOGIC AND PHILOSOPHY

S. C. Kleene, University of Wisconsin.
Recursive functions and intuitionistic mathematics. (30 min.)
W. Craig, Princeton University.
Incompletability, with respect to validity, in every finite nonempty domain, of first order functional calculus.
H. B. Curry, Pennsylvania State College.
The inferential theory of negation.
M. Davis, University of Illinois.
Relatively recursive functions and the extended Kleene hierarchy.
J. K. Feibleman, Tulane University of Louisiana.
Ontological positivism.
F. Fiala, University of Neuchâtel.
Sur les bases philosophiques de la formalisation.
P. Lorenzen, University of Bonn.
Konstruktive Begründung der klassischen Mathematik.
Z. Suetuna, Tokyo University.
On the mathematical existence.
G. C. Vedova, Newark College of Engineering.
An inquiry into the nature of knowledge.

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

CAMBRIDGE, MASSACHUSETTS

From August 30 to September 6, 1950, an International Congress of Mathematicians was held at Harvard University under the auspices of the American Mathematical Society. In addition to the principal host, Harvard University, the following acted as co-hosts: The American Academy of Arts and Sciences, Boston College, Boston University, The Massachusetts Institute of Technology, and Tufts College.

This was the first International Congress of Mathematicians held in the United States since that assembled in connection with the Chicago World's Fair in 1893. At that time an international gathering of mathematicians met at Northwestern University under the presidency of Professor William E. Story of Clark University. Professor Felix Klein of the University of Göttingen attended this Congress as the official representative of the German government and brought with him a number of papers by prominent foreign mathematicians. These papers formed an important part of the program of the meetings. The nations represented among the authors of papers submitted to the Congress were as follows: Austria, France, Germany, Italy, Russia, Switzerland, and the United States. There were twenty-five mathematicians in attendance. In the two weeks immediately following the Congress, Professor Klein, who had taken such a prominent part in the Congress, gave a series of Colloquium Lectures at Northwestern University. These twelve lectures were published under the title *The Evanston Colloquium Lectures on Mathematics.*

The 1950 Congress was the first International Congress of Mathematicians held on the North American continent since that in Toronto in 1924.

At the International Congress in Oslo in 1936, the American delegation invited the mathematicians of the world to hold their next general gathering in the United States in 1940. Plans had been practically completed for such a gathering at Harvard University when the outbreak of World War II necessitated the cancellation of the Congress. At the close of the war in 1945, the Council of the Society at once began discussion concerning the possibility of a gathering of the mathematicians of the world. Those guiding the policies of the American Mathematical Society were insistent that there should be no international congress until such a time that the gathering could be truly international in the sense that mathematicians could be invited irrespective of national or geographic origins. On December 10, 1947, the Council of the Society voted unanimously to accept the recommendation of the Emergency Committee for the International Congress of Mathematicians that steps be taken at once to elect officers and committees for the next congress.

Early in 1948, the following officers of the Congress were elected: President Designate: Oswald Veblen, Institute for Advanced Study; Secretary: J. R.

Kline, University of Pennsylvania; Associate Secretary: R. P. Boas, Jr., Mathematical Reviews.

Garrett Birkhoff was elected chairman of the Organizing Committee with W. T. Martin as vice chairman. The other members of this committee, as finally constituted, were : A. A. Albert, J. L. Doob, G. C. Evans, T. H. Hildebrandt, Einar Hille, J. R. Kline, Solomon Lefschetz, Saunders MacLane, Marston Morse, John von Neumann, Oswald Veblen, J. L. Walsh, Hassler Whitney, D. V. Widder, and R. L. Wilder.[1]

Invitations to send delegates to the Congress were sent to national academies and royal societies, universities and colleges, and to the mathematical societies throughout the world. In the case of all national academies and royal societies of countries with which the United States Government maintained diplomatic relations, these invitations were transmitted through the diplomatic mail pouch of the State Department. The various mathematical societies cooperated by distributing to their members invitations to and literature about the Congress which was furnished to them by the Secretariat. In attempting to maintain the non-political nature of the Congress, many serious difficulties had to be overcome. In the solution of these problems, officers of the Congress found the various officials of the Department of State most sympathetic and helpful. As a part of the effort to keep the Congress apolitical, they tried to secure a visa for every mathematician who notified them about any visa difficulties before cancelling his passage. As far as they know only one mathematician from any independent nation was prevented from attending the Congress because he failed to pass a political test and this man did not notify the officers of the Congress about his difficulties. Only two mathematicians from occupied countries failed to secure visas. Mathematicians from behind the Iron Curtain were uniformly prevented from attending the Congress by their own governments which generally refused to issue passports to them for the trip to the Congress. Their non-attendance was not due to any action of the United States Government.

Just before the opening of the Congress there was received from the President of the Soviet Academy of Sciences the following cablegram:

> "USSR Academy of Sciences appreciates receiving kind invitation for Soviet scientist take part in International Congress of Mathematicians to be held in Cambridge. Soviet Mathematicians being very much occupied with their regular work unable attend congress. Hope that impending congress will be significant event in mathematical science. Wish success in congress activities.
>
> S. Vavilov, President, USSR Academy of Sciences."

This cablegram was read at the opening plenary session of the Congress which

[1] R. G. D. Richardson and J. L. Synge were also originally members of this Committee. Dean Richardson died July 17, 1949, while Professor Synge resigned from the Committee when he left the United States to assume his position at the Institute for Advanced Studies in Dublin in August 1948.

was held on Wednesday afternoon, August 30, in the Sanders Theatre of Harvard University.

The Congress was officially opened by Professor Garrett Birkhoff as chairman of the Organizing Committee of the Congress. Professor Birkhoff spoke as follows:

"Fellow Mathematicians:

"It gives me great satisfaction to see you all assembled here. The organizing of a successful International Congress at such a time of political tensions, and after a gap of fourteen years, has had its anxious moments, as many of you know. Your presence here promises that our efforts will be crowned with success.

"It is needless to say that the organization of the Congress would have been impossible without the generous and loyal assistance of many people. I wish I could mention them individually, by name, but time does not permit this. Those who did the work are quite cognizant of the fact, I am sure, and do not need to be reminded of it.

"In fact, our scientific program is so full that there will not even be time to listen, in this first preliminary session, to the official greetings from the representatives of many countries and learned societies which have sent delegates here. Therefore, it will be necessary for these greetings to be conveyed informally; I am sure that they are appreciated nonetheless.

"To save time I shall say no more, but shall call immediately upon Mr. Skolem, as the delegate of President Størmer of the Oslo Congress in 1936, to nominate the President of this Congress."

Professor T. A. Skolem of the University of Oslo presented the name of Professor Oswald Veblen for the Presidency of the Congress. Professor Veblen was unanimously elected and then delivered his presidential address, which follows:

OPENING ADDRESS OF PROFESSOR OSWALD VEBLEN

In taking the chair today I feel that I am just acting as deputy for my friend, George Birkhoff, whose untimely death has kept him from performing this duty. It was he who could have best welcomed the mathematicians of the world both on behalf of his University and on behalf of the American Mathematical Society.

If this Congress could have been held, as originally planned, in 1940 it would have marked in rather a definite sense the coming of age of mathematics in the United States. At the time of the International Congress in Chicago, in 1893, there was no indigenous mathematical tradition in this country, but there were a few active mathematicians, some of whom were beginning to diverge a bit from the lines laid down by their European teachers. By the time of the Oslo Congress, which was so admirably conducted by our Norwegian colleagues, a notable growth and transformation had taken place. Important discoveries had been made by American mathematicians. New branches of mathematics were being cultivated and new tendencies in research were showing themselves. Some American universities were receiving students and research workers from overseas, and interchanges of all sorts tended to be more and more on terms of equality. The colonial period was ending. At the same time mathematics had attained a small but growing amount of recognition from the rest of the American community—enough, at least, to encourage us to invite the mathematicians of the world to a congress in this country in 1940.

Now, fourteen years have elapsed since the invitation was issued, and we are approaching the end of another epoch. I mean the period during which North America has absorbed so many powerful mathematicians from all over the world that the indigenous traditions and tendencies of mathematical thought have been radically changed as well as enriched. These American gains have seemed to be at the cost of great losses to European mathematics. But there are so many signs of vitality in Europe that it is now possible to hope that the losses will be only temporary while the American gains will be permanent.

We are holding the Congress in the shadow of another crisis, perhaps even more menacing than that of 1940, but one which at least does allow the attendance of representatives from a large part of the mathematical world. It is true that many of our most valued colleagues have been kept away by political obstacles and that it has taken valiant efforts by the Organizing Committee to make it possible for others to come. Nevertheless, we who are gathered here do represent a very large part of the mathematical world. I will also venture the much more hazardous statement that we represent most of the currents of mathematical thought that are discernible in the world today. I hope that this remark will be dissected and, if possible, pulverized in the private conversations that are so valuable a part of any scientific meeting.

I have referred to the political difficulties which have harassed this Congress, but think that if there are to be future international congresses, an even more serious difficulty will be the vast number of people who have a formal, and even an actual, reason for attending. This makes all meetings, even for very specialized purposes, altogether too large and unwieldy to accomplish their purposes.

Mathematics is terribly individual. Any mathematical act, whether of creation or apprehension, takes place in the deepest recesses of the individual mind. Mathematical thoughts must nevertheless be communicated to other individuals and assimilated into the body of general knowledge. Otherwise they can hardly be said to exist. But the ideal communication is to a very few other individuals. By the time it becomes necessary to raise one's voice in a large hall some of the best mathematicians I know are simply horrified and remain silent.

The Organizing Committee of the present Congress has tried to meet this problem by means of a series of conferences, more informal than the regular program, but even in the conferences the problem of numbers will remain. It is to be hoped that our colleagues who have been meeting in New York to consider organizing an International Mathematical Union will have something to say to us on this and other problems before this Congress adjourns.

The solution will not be to give up international mathematical meetings and organizations altogether, for there is a deep human instinct that brings them about. Every human being feels the need of belonging to some sort of a group of people with whom he has common interests. Otherwise he becomes lonely, irresolute, and ineffective. The more one is a mathematician the more one tends to be unfit or unwilling to play a part in normal social groups. In most cases that I have observed, this is a necessary, though definitely not a sufficient, condition for doing mathematics. But it has made it necessary for mathematicians to group themselves together *as* mathematicians. The resultant organizations of various kinds have accomplished many important things known to us all. Of these accomplishments I am sure that the most important is the maintenance of a set of standards and traditions which enable us to preserve that coherent and growing something which we call Mathematics.

To our non-mathematical friends we can say that this sort of a meeting, which cuts across all sorts of political, racial, and social differences and focuses on a universal human interest will be an influence for conciliation and peace. But the Congress is, after all, just a meeting of mathematicians. Let us get about our business.

Harvard University
Cambridge, Massachusetts
August 30, 1950

Immediately after the address by Professor Veblen, Professor Harald Bohr of the University of Copenhagen spoke on behalf of the Committee to award the Fields Medals.[2] The medals were awarded to Professor Laurent Schwartz of the University of Nancy and to Professor Atle Selberg of the Institute for Advanced Study. Professor Bohr gave an excellent résumé of the work of Schwartz on distributions and of the work of Selberg on the Riemann zeta function and his elementary proof of the celebrated prime number theorem. Professor Bohr's address follows:

[2] At the International Congress at Toronto in 1924 it was decided that at each international mathematical congress two gold medals should be awarded. Professor J. C. Fields, the Secretary of the 1924 Congress, presented a fund to subsidize these medals. They were first awarded in Oslo in 1936. The Committee to select the winners of the 1950 medals was: Professor Harald Bohr (Chairman), Professors L. V. Ahlfors, Karol Borsuk, Maurice Fréchet, W. V. D. Hodge, A. N. Kolmogoroff, D. Kosambi, and Marston Morse.

ADDRESS OF PROFESSOR HARALD BOHR

At a meeting of the organizing committee of the International Congress held in 1924 at Toronto the resolution was adopted that at each international mathematical congress two gold medals should be awarded, and in a memorandum the donor of the fund for the founding of the medals, the late Professor J. C. Fields, expressed the wish that the awards should be open to the whole world and added that, while the awards should be a recognition of work already done, it was at the same time intended to be an encouragement for further mathematical achievements. The funds for the Fields' medals were finally accepted by the International Congress in Zürich in 1932, and two Fields medals were for the first time awarded at the Congress in Oslo 1936 to Professor Ahlfors and Professor Douglas. And now, after a long period of fourteen years, the mathematicians meet again at an international congress, here in Harvard.

In the fall of 1948 Professor Oswald Veblen, as nominee of the American Mathematical Society for the presidency of the Congress in Harvard, together with the chairman of the organizing committee, and with the secretary of the Congress, appointed an international committee to select the two recipients of the Fields medals to be awarded at the Congress in Harvard, the committee consisting of Professors Ahlfors, Borsuk, Fréchet, Hodge, Kolmogoroff, Kosambi, Morse, and myself. With the exception of Professor Kolmogoroff, whose valuable help we were sorry to miss, all the members of the committee have taken an active part in the discussions. As chairman of the committee I now have the honor to inform the Congress of our decisions and to present the gold medals together with an honorarium of $1,500 to each of the two mathematicians selected by the committee.

The members of the committee were, unanimously, of the opinion that the medals, as on the occasion of the first awards in Oslo, should be given to two really young mathematicians, without exactly specifying, however, the notion of being "young." But even with this principal limitation the task was not an easy one, and it was felt to be very encouraging for the expectations we may entertain of the future development of our science that we had to choose among so many young and very talented mathematicians, each of whom should certainly have been worthy of an official appreciation of his work. Our choice fell on Professor *Atle Selberg* and Professor *Laurent Schwartz*, and I feel sure that all members of the Congress will agree with the committee that these two young mathematicians not only are most promising as to their future work but have already given contributions of the uttermost importance and originality to our science; indeed they have already written their names in the history of mathematics of our century.

Before having the honor of presenting the medals to Professor Selberg and Professor Schwartz, I shall try briefly, and in a very general way, to emphasize some of the most important results obtained by the two recipients and those which have especially attracted the admiration of the committee.

Atle Selberg who studied and took his doctor's degree in his native country Norway, with its great mathematical traditions since the days of Abel, some years ago followed a call to the modern center of mathematics, the Institute for Advanced Study in Princeton. His scientific production is very extensive, his interests centering on the theory of numbers, including the theory of those functions which dominate the analytical theory of numbers. Of great importance is his generalization of the method of his very original and ingenious countryman Viggo Brun, the Eratostenes sieve method; I shall not, however, enter into any details of this part of Selberg's work, preferring to dwell on two other fundamental achievements of Selberg which may be easier to explain in general terms. The first deals with the Riemann Zeta function $\zeta(s) = \zeta(\sigma + it)$ and allied functions. The study of the Zeta function, introduced by Riemann in his classical paper on the distribution of the prime numbers, has, as is well-known, been most inspiring and valuable not only for the development of the theory of the primes but also for the general theory of analytic functions of a complex variable. However, it still presents an open problem of fundamental character, namely the problem concerning the truth of the famous Riemann hypothesis that all the zeros of $\zeta(s)$ in the critical strip $0 < \sigma < 1$ lie on the line $\sigma = 1/2$. The existence of infinitely many roots in this strip having been established by Hadamard by means of his general theory of integral functions of finite order, von Mangoldt succeeded in proving—what was already conjectured by Riemann—that the number $N(\tau)$ of roots in the critical strip with ordinates between 0 and τ is asymptotically equal to $(1/2\pi)\tau \log \tau$. Some progress in the direction of the Riemann hypothesis was made at the beginning of this century; in fact, it could be shown, on the one hand that the overwhelming number of the roots in the critical strip were lying in the infinite neighborhood of the line $\sigma = 1/2$, and on the other hand that there were actually infinitely many zeros situated on this line itself. The results in both these directions were greatly improved by Selberg in his doctor's dissertation. I shall limit myself to speaking about the last problem, the question concerning the zeros on the line $\sigma = 1/2$. After the first proof of Hardy of the very existence of infinitely many roots on this line, Hardy and Littlewood were able to prove that there were in fact very many roots on the line; indeed, denoting by $N_0(t)$ the number of roots on $\sigma = 1/2$ with ordinates between 0 and t, they showed that $N_0(t)$ was asymptotically larger than a constant times t. But Selberg was the first who succeeded in filling the remaining gap between t and $t \log t$ by proving that $N_0(t)$ was even larger than a constant times $t \log t$, i.e., that the number of roots on the line $\sigma = 1/2$ has at any rate the same order of magnitude as the total number of roots in the critical strip. Although it does not seem possible in this way to solve the Riemann hypothesis, this achievement of Selberg, surpassing the results obtained by two of the strongest analysts of our times, certainly gave evidence of his great ingenuity and his extraordinary penetrating power. Of quite singular importance, and indeed of a sensational character, is Selberg's work on the distribution of the primes, namely his discovery in 1948 of an elementary proof of the celebrated prime number theorem. I may recall briefly the very interesting history of this fundamental theorem which states that the number $\pi(x)$ of primes smaller than x is asymptotically equal to $x/\log x$. This theorem, already conjectured by Gauss as a young boy and in a letter of Abel characterized as perhaps the most remarkable theorem in all mathematics, through a long period withstood all attempts of proof. Tschebyscheff was the first who by very ingenious elementary methods succeeded in making substantial progress; indeed he showed that the function $\pi(x)$ has at any rate the order of magnitude $x/\log x$, but his method could not lead to a proof of the prime number theorem itself. As is well-known, such a proof was first obtained simultaneously by Hadamard and de la Vallée Poussin; the proofs of both were of an analytical character, based on the Hadamard theory of integral functions applied to the Riemann zeta function. Later on several greatly simplified proofs, avoiding the Hadamard theory, were given, in particular by Landau and Wiener, but all of them were based on the Riemann zeta function, a decisive element in all the proofs being the fact that the zeta function has no zeros on the line $\sigma = 1$, the right-hand boundary of the critical strip. Naturally, much thought has been given to the problem whether one could arrive at the theorem by more elementary means. In this connection, I may quote a passage from a lecture of Hardy delivered to the Mathematical Society of Copenhagen in 1921: "No elementary proof of the prime number theorem is known, and one may ask whether it is reasonable to expect one. Now we know that the theorem is roughly equivalent to a theorem about an analytic function, the theorem that Riemann's zeta function has no roots on a certain line. A proof of such a theorem, not fundamentally dependent upon the ideas of the theory of functions, seems to me extraordinarily unlikely. It is rash to assert that a mathematical theorem *cannot* be proved in a particular way; but one thing seems quite clear. We have certain views about the logic of the theory; we think that some theorems, as we say 'lie deep' and others nearer to the surface. If anyone produces an elementary proof of the prime number theorem, he will show that these views are wrong, that the subject does not hang together in the way we have supposed, and that it is time for the books to be cast aside and for the theory to be rewritten." And now Atle Selberg has produced such an elementary proof! About this most ingenious proof, which is certainly quite elementary—in the sense that it uses practically no analysis except the properties of the logarithm—but by no means is an easy one, I shall only say that it is based on a relatively simple asymptotic formula discovered by Selberg

$$\sum_{p \leq x} \log^2 p + \sum_{pq \leq x} \log p \log q = 2x \log x + O(x)$$

containing summations partly over the primes p themselves and partly over the product of two primes p and q. From this formula there are, as was pointed out by Selberg, several ways to deduce the prime number theorem.

To the proof in its original version an important contribution was given by another brilliant young mathematician, Paul Erdös. Recently Selberg has succeeded in generalizing his elementary proof of the prime number theorem to cover also the corresponding theorem for the prime numbers in an arbitrary arithmetic progression.

And now I turn to the work of the other, slightly older, of the two recipients, the French mathematician *Laurent Schwartz*. Having passed through the old celebrated institution École Normale Supérieure in Paris, he is now Professor at the University of Nancy. He belongs to the group of most promising and closely collaborating young French mathematicians who secure for French mathematics in the years to come a position worthy of its illustrious traditions. Like Selberg, Schwartz can look back on an extensive and varied production, but when comparing the work of these two young mathematicians one gets a strong impression of the richness and variety of the mathematical science and of its many different aspects. While Selberg's work dealt with clear cut problems concerning notions which, as the primes, are, so to say, given a priori, one of the greatest merits of Schwartz's work consists on the contrary in his creation of new and most fruitful notions adapted to the general problems the study of which he has undertaken. While these problems themselves are of classical nature, in fact dealing with the very foundation of the old calculus, his way of looking at the problem is intimately connected with the typical modern development of our science with its highly general and often very abstract character. Thus once more we see in Schwartz's work a confirmation of the words of Felix Klein that great progress in our science is often obtained when new methods are applied to old problems. In the short time at my disposal I think I may give the clearest impression of Schwartz's achievements by limiting myself to speak of the very central and most important part of his work, his theory of "distributions." The first publication of his new ideas was given in a paper in the Annales de l'Université de Grenoble, 1948, with the title *Généralisation de la notion de fonction, de dérivation, de transformation de Fourier et applications mathématiques et physiques*, a paper which certainly will stand as one of the classical mathematical papers of our times. As the title indicates it deals with a generalization of the very notion of a function better adapted to the process of differentiation than the ordinary classical one. In trying to explain briefly the new notions of Schwartz and their importance I think I can do no better than start by considering the same example as that used by Schwartz himself, namely, the simple function $f(x)$ which is equal to 0 for $x \leqq 0$ and equal to 1 for $x > 0$. This function has a derivative $f'(x) = 0$ for every $x \neq 0$, but this fact evidently does not tell us anything about the magnitude of the jump of $f(x)$ at the point $x = 0$. In order to overcome this inconvenience the physicist and the technicians had accustomed themselves to say that the function $f(x)$ has as derivative the "Dirac-function" $f'(x) = \delta(x)$ which is 0 for $x \neq 0$ and equal to $+\infty$ for $x = 0$ and moreover has the property that its integral over any interval containing the point $x = 0$ shall be equal to 1. But this is of course not a legitimate way of speaking; from a mathematical point of view—using the idea of a Stieltjes' integral—we naturally would think of the derivative of our function $f(x)$ not as a function but as a mass-distribution, in this case of the particularly simple type with the whole mass 1 placed at the origin $x = 0$. Now, according to a classical theorem of F. Riesz, there is a most intimate connection, in fact a one-to-one correspondence, between an arbitrary mass-distribution μ on the x-axis and a linear continuous functional $\mu(\phi)$ defined in the space of all continuous functions $\phi(x)$, vanishing outside a finite interval, where the topology of the ϕ-space is fixed by the simple claim that convergence of a sequence ϕ_n shall mean that the functions $\phi_n(x)$ are all zero outside a fixed finite interval and that the sequence $\phi_n(x)$ shall be uniformly convergent. This correspondence between the mass-distributions μ and the functionals $\mu(\phi)$ is given simply by the relation

$$\mu(\phi) = \int_{-\infty}^{+\infty} \phi(x)\, d\mu.$$

In the Schwartz theory of distributions the new notion, generalizing, or rather replacing, that of a function is nothing else than just such a linear continuous functional, but of a kind essentially different from that above, the underlying ϕ-space and its topology being of a quite different nature. The new notion—once invented—is so easy to explain that I cannot resist the temptation, notwithstanding the general solemn nature of this opening meeting, to go into some detail. Let us consider, then, with Schwartz a quite arbitrary function $f(x)$, assumed only to be integrable in the sense of Lebesgue over any finite interval, and let us try to characterize the function $f(x)$, not as in the classical Dirichlet way by the values it takes for the different values of x, but by what we may call its effect when operating on an arbitrary auxiliary function $\phi(x)$ of which, for the moment, we suppose only, as above, that $\phi(x)$ is continuous and equal to zero outside a finite interval; by the effect of the given function $f(x)$ on the auxiliary function $\phi(x)$ we here mean simply the value of the integral

$$\int_{-\infty}^{+\infty} f(x)\phi(x)\, dx.$$

This integral, obviously, is a linear functional, associated with the function $f(x)$, and we shall denote it $f(\phi)$, using the same letter f, as we wish, so to speak, to identify it with the function $f(x)$ itself. In the special case where the given function $f(x)$ has a continuous derivative $f'(x)$ we may of course, starting with $f'(x)$ instead of with $f(x)$, in the same way build a functional associated with $f'(x)$, i.e., the functional

$$f'(\phi) = \int_{-\infty}^{+\infty} f'(x)\phi(x)\, dx.$$

If now—and this is an essential point—we assume also the auxiliary function $\phi(x)$ to have a continuous derivative $\phi'(x)$, we immediately find through partial integration

$$\int_{-\infty}^{+\infty} f'(x)\phi(x)\,dx = -\int_{-\infty}^{+\infty} f(x)\phi'(x)\,dx,$$

i.e., the simple relation

$$f'(\phi) = -f(\phi').$$

In order that the derivative of any function $\phi(x)$ of our space shall also belong to the space, we must obviously assume, with Schwartz, that the auxiliary functions $\phi(x)$ to be considered shall possess derivatives not only of the first order but of arbitrarily high order. In the space consisting of all such functions $\phi(x)$, i.e., of all functions $\phi(x)$ zero outside a finite interval and with derivatives of any order, and topologized by the definition that convergence of a sequence ϕ_n shall mean not only as above that the functions $\phi_n(x)$ shall all be zero outside a fixed finite interval and that $\phi_n(x)$ shall converge uniformly, but moreover that all the derivated sequences $\phi'_n(x)$, $\phi''_n(x)$ $\cdots$ shall converge uniformly, Schwartz now takes into consideration all continuous linear functionals $J(\phi)$. These functionals $J(\phi)$ are just what Schwartz denotes as "distributions." Among them are in particular the distributions $f(\phi)$ derived in the manner above from an ordinary function $f(x)$, and more generally we have distributions, which we denote by $\mu(\phi)$, which are associated with a mass-distribution μ—evidently the word distribution has been chosen to remind us vaguely of these mass-distributions—but the whole class of Schwartz distributions is far from being exhausted by the special distributions of the type $f(\phi)$ or $\mu(\phi)$. Now—and this is the decisive point in the theory—Schwartz assigns to every one of his distributions $J(\phi)$ another distribution $J'(\phi)$ as the derivative of $J(\phi)$, namely, immediately suggested by the consideration above, the distribution defined by

$$J'(\phi) = -J(\phi').$$

In the special case where the distribution $J(\phi)$ is of the type $f(\phi)$ and moreover is derived from a function $f(x)$ with a continuous derivative, the derivated distribution $J'(\phi) = -J(\phi')$ is, evidently, nothing else than the distribution $f'(\phi)$ associated with the function $f'(x)$. But generally, if $f(x)$ is an arbitrary function with no derivative, the corresponding functional $f(\phi)$ still has a derivative $f'(\phi)$ which, however, is no longer associated with any function, neither, generally, with any mass-distribution, but is just some Schwartz distribution.

And now, one will naturally ask, what has been gained by Schwartz's generalization of a function $f(x)$ to that of a distribution $J(\phi)$. Naturally, the aim of any such generalization of basic notions—as, for instance, the generalization of the notion of a real number to that of a complex number—is, in principle, the same and of a double kind; on the one hand, and this is the primary purpose, one aims at getting simplifications in the treatment of problems concerning the old notions through the greater freedom in carrying out operations, provided by the new notions, and on the other hand, one may hope to meet with new

:uitful problems concerning these new notions themselves. In both these re- pects the theory of Schwartz may be said to be a great success. I think that very reader of his cited paper, like myself, will have felt a considerable amount f pleasant excitement, on seeing the wonderful harmony of the whole structure f the calculus to which the theory leads and on understanding how essential n advance its application may mean to many parts of higher analysis, such as pectral theory, potential theory, and indeed the whole theory of linear partial ifferential equations, where, for instance, the important notion of the "finite art" of a divergent integral, introduced by Hadamard, presents itself in a most atural way when the distributions and not the functions are taken as basic lements. And as to the harmony brought about I shall mention only one single, ery simple, but most satisfactory result. Not only has, as we have seen, every istribution $J(\phi)$ a derivative $J'(\phi)$, and hence derivatives of every order, but onversely it also holds that every distribution possesses a primitive distribu- ion, i.e., is the derivative of another distribution which is uniquely determined part from an additive constant (i.e., of course, a distribution associated vith a constant). The simplification obtained, and not least the easy jus- ification of different "symbolic" operations often used in an illegitimate way y the technicians, is of such striking nature that it seems more than a utopian hought that elements of the theory of the Schwartz distributions may find their lace even in the more elementary courses of the calculus in universities and echnical schools.

Schwartz is now preparing a larger general treatise on the theory of distribu- ions, the first, very rich, volume of which has already appeared. In his intro- uction to this treatise he emphasizes the fact that ideas similar to those under- ying his theory have earlier been applied by different mathematicians to various ubjects—here only to mention the methods introduced by Bochner in his tudies on Fourier integrals—and that the theory of distributions is far from eing a "nouveauté revolutionaire." Modestly he characterizes his theory as une synthèse et une simplification." However, as in the case of earlier ad- ances of a general kind—to take only one of the great historic examples, that f Descartes' development of the analytic geometry which, as is well-known, vas preceded by several analytic treatments by other mathematicians of special eometric problems—the main merit is justly due to the man who has clearly een, and been able to shape, the new ideas in their purity and generality.

No wonder that the work of Schwartz has met with very great interest in nathematical circles throughout the world, and that a number of younger nathematicians have taken up investigations in the wide field he has opened or new researches.

And now I am at the end of my short review of some of the most important arts of the work of the two elected young mathematicians, and I have the honor o call upon Professor Selberg and Professor Schwartz to present to them the olden medals and the honorarium.

In the name of the committee, I think I dare say of the whole Congress, I congratulate you most heartily on the awards of the Fields medals. Repeatin the wish of Fields himself I may finally express the hope that the great admira tion of your achievements of which the medals are a token may also mean a encouragement to you in your future work.

Harvard University
Cambridge, Massachusetts
August 30, 1950

There were in attendance at the Congress 2,302 persons. These people were distributed as follows:

United States and Canadian	1,410
United States and Canadian, associate members	540
Members outside United States and Canada	290
Associate members outside the United States and Canada	76
Total members in attendance	2,316

In addition there were a number of mathematicians who became members of the Congress but did not attend the gathering at Cambridge. Some of these persons joined fully expecting to be present, but were prevented from attending either by illness or other unavoidable circumstances, while others assumed membership to support an enterprise which they considered most worthy. The statistics in this connection are as follows:

United States and Canadian members unable to attend	160
United States and Canadian associate members unable to attend	35
Members outside United States and Canada unable to attend	57
Associate members outside United States and Canada unable to attend	3
Total members unable to attend	255
Total Congress membership	2,571

The following countries outside the United States and Canada were represented:

Argentina, Australia, Austria, Belgium, Brazil, Burma, Chile, China, Colombia, Cuba, Denmark, Egypt, England, Finland, France, Germany, Greece, India, Iran, Ireland, Israel, Italy, Japan, Mexico, Netherlands, Nigeria, Norway, Panama, Peru, Philippines, Scotland, South Africa, Spain, Sweden, Switzerland, Turkey, Uruguay, Venezuela, and Yugoslavia.

Every state in the United States and the District of Columbia were represented, with the exception of South Dakota.

THE SCIENTIFIC PROGRAM

In addition to the contributed ten-minute papers and the stated addresses, which were delivered on invitation of the Organizing Committee, the scientific program of the Congress included four conferences on the following topics: Algebra, Analysis, Applied Mathematics, and Topology. The chairman of each section for contributed papers was given the privilege of inviting not more than three persons to deliver thirty minute addresses in his section.

Stated Addresses

The following stated addresses were delivered by invitation of the Organizing Committee:

A. A. Albert: *Power-associative algebras.*

Arne Beurling: *On null-sets in harmonic analysis and function theory.*

S. Bochner: *Laplace operator on manifolds.*

H. Cartan: *Sur les fonctions analytiques de variables complexes.*

S. S. Chern: *Differential geometry of fiber bundles.*

H. Davenport: *Recent work in the geometry of numbers.*

Kurt Gödel: *Rotating universes in general relativity theory.*

W. V. D. Hodge: *Topological invariants of algebraic varieties.*

Heinz Hopf: *Die n-dimensionalen Sphären und projektiven Räume in der Topologie.*

Witold Hurewicz: *Homology and homotopy theory.*

S. Kakutani: *Ergodic theory.*

M. Morse: *Recent advances in variational theory in the large.*

J. von Neumann: *Shock interaction and its mathematical aspects.*

J. F. Ritt: *Differential groups.*

A. Rome: *The calculation of an eclipse of the sun according to Theon of Alexandria.*

L. Schwartz: *Distributions and principal applications.*

A. Wald: *Basic ideas of a general theory of statistical decision rules.*

A. Weil: *Number-theory and algebraic geometry.*

H. Whitney: *r-dimensional integration in n-space.*

N. Wiener: *Comprehensive view of prediction theory.*

R. L. Wilder: *The cultural basis of mathematics.*

O. Zariski: *The fundamental ideas of abstract algebraic geometry.*

Conferences

In recent years mathematicians have been much impressed by the success of the conference method of presenting research in various fields in which vigorous advances have been made or are in progress. In such conferences there is a well-coordinated program of formal lectures and open informal discussion. At the International Congress the Organizing Committee decided to hold four conferences in the following fields: Algebra, Analysis, Applied Mathematics, and Topology. A number of the Stated Addresses listed above were integrated into the work of the Conferences.

A. *Conference in Algebra.* The Conference in Algebra was under the Chairmanship of Professor A. A. Albert. There were four sessions, devoted to:

1. Algebraic geometry. In this session Professors A. Weil and O. Zariski gave their Stated Addresses.

2. Groups and universal algebra. The speakers were Professors G. Birkhoff, S. MacLane, R. Baer, and C. Chevalley.

3. Structure theory of rings and algebras. The Stated Address of Professor

A. A. Albert was part of this session. Other speakers in this conference were Professors R. Brauer, N. Jacobson, J. Dieudonné, and T. Nakayama.

4. Arithmetic algebra. Principal speakers in this session were Professors E. Artin, W. Krull, M. Eichler, and M. Krasner.

B. *Conference in Analysis.* This Conference was under the Chairmanship of Professor Marston Morse. Professor Arnaud Denjoy of the Faculté des Sciences, Paris, acted as Honorary Chairman of this Conference. There were four sessions devoted solely to analysis with an additional section in which the participants of the Topology Conference were joined. This was held on Friday evening, September 1. The sessions were devoted to:

1. Analysis in the large. The speakers were Professors L. Bers, S. Bergman, L. Cesari and T. Radó, and C. B. Morrey.

2. Extremal methods and geometric theory of functions of a complex variable. The speakers were Professors L. V. Ahlfors, A. C. Schaeffer and D. C. Spencer, M. M. Schiffer, H. Grunsky, R. Nevanlinna, and G. Szegö.

3. Algebraic tendencies in analysis. This subconference consisted of reports of various aspects of the main topic of algebraic tendencies in analysis. A panel of experts surveyed the field and then prepared a report. This report of the panel was presented to the Congress by the spokesman of the group. Two sessions, one on Monday afternoon and the other on Tuesday, were devoted to this subconference. The topics treated were:

a. Group Representations. Spokesman, R. Godement.
b. Topological Algebra. Spokesman, I. Kaplansky.
c. Measure Theory. Spokesman, P. R. Halmos.
d. Spectral Theory. Spokesman, N. Dunford.
e. Applied Functional Analysis. Spokesman, N. Aronszajn.
f. Ergodic Theory. Spokesman, S. Kakutani.

4. The Joint Session with the Topology Conference was devoted to analysis and geometry in the large. The speakers were Professors J. Leray, G. de Rham, and A. Lichnerowicz.

C. *Conference in Applied Mathematics.* This conference was under the Chairmanship of Professor J. von Neumann. There were three sessions devoted to:

1. Random processes in physics and communications. The speakers were Doctors C. E. Shannon and S. M. Ulam.

2. Partial differential equations. The speakers were Professors J. von Neumann, R. Courant, S. Goldstein, W. Heisenberg, W. Prager, J. J. Stoker. Professor von Neumann's address was the one which he presented by invitation of the Organizing Committee.

3. Statistical Mechanics. In this session there was the Stated Address of Professor N. Wiener. In addition Professor W. Feller gave a 45 minute address.

D. *Conference in Topology.* This conference was under the Chairmanship of Professor Hassler Whitney and was divided into four sessions.

1. Homology and homotopy theory. The speakers were Professors W.

Hurewicz, S. Eilenberg, J. H. C. Whitehead, and G. W. Whitehead. Professor Hurewicz's address was given by invitation of the Organizing Committee.

2. Fiber bundles and obstructions. The speakers were Professors P. Olum, W. S. Massey, G. Hirsch, and E. Spanier.

3. Differentiable manifolds. The speakers were Professors S. Chern, C. Ehresmann, B. Eckmann, and C. B. Allendoerfer. Professor Chern's address was the Stated Address which he was invited to deliver by the Organizing Committee.

4. Topological groups. The speakers were Professors P. A. Smith, Deane Montgomery, K. Iwasawa, A. Gleason, and R. H. Fox.

Contributed Papers

There were the usual sessions for the presentation of contributed papers. Each member of the Congress was given the privilege of submitting one paper for presentation in person. Presentation by title was not permitted. The contributed papers were distributed among seven sections, listed below. In addition, the Chairman of each section was permitted to invite not more than three persons to deliver thirty minute addresses in his section.[3] In all, 374 contributed papers were presented at the Congress.

Section I, Algebra and theory of numbers. Professor H. A. Rademacher was the Chairman of this Section, and Professor L. J. Mordell of Cambridge University, England, acted as Honorary Chairman of the Section. Fifty-eight contributed papers were presented. The following thirty minute addresses were given by invitation of the Chairman:

K. Mahler: *Farey sections in the fields of Gauss and Eisenstein.*

A. Selberg: *The general sieve method and its place in prime number theory.*

H. D. Kloosterman: *The characters of binary modular congruence groups.*

Section II, Analysis. Professor G. C. Evans was the Chairman of this Section in which 127 contributed papers were presented. The following invited addresses were given by invitation of the Chairman:

H. A. Rademacher: *Remarks on the theory of partitions.*

H. Bohr: *A survey of the different proofs of the main theorems in the field of almost periodic functions.*

S. Mandelbrojt: *Théorèmes d'unicité de la théorie des fonctions.*

In addition, Doctor Stefan Bergman gave an address, illustrated with models: *On visualization of domains in the theory of functions of two complex variables.*

Section III, Geometry and topology. Professor S. Eilenberg was the Chairman of this Section and Professor Francesco Severi of the University of Rome acted as Honorary Chairman of the Section. Fifty-eight contributed papers were pre-

[3] An exception was later permitted in the section on Logic and Philosophy, where four persons presented longer papers.

sented. The following invited addresses were given by invitation of the Chairman:

L. A. SANTALÓ: *Integral geometry in general spaces* (read by C. B. ALLENDOERFER).

B. SEGRE: *Arithmetical properties of algebraic varieties.*

Section IV, Probability and statistics, actuarial science, economics. Professor J. L. Doob was the Chairman of this Section in which twenty-seven contributed papers were presented. The following invited addresses were given by invitation of the Section Chairman:

R. C. BOSE: *Mathematical theory of factorial designs.*

P. LÉVY: *Processus lapelaciens et équations différentielles stochastiques.*

S. N. ROY: *On some aspects of statistical inference.*

Section V, Mathematical physics and applied mathematics. Professor R. Courant was the Chairman of this Section in which seventy-four contributed papers were presented. The following invited addresses were given by invitation of the Chairman:

C. G. DARWIN: *The refractive index of an ionized gas.*

F. RELLICH: *Störungstheorie der Spektralzerlegung.*

H. LEWY: *Developments at the confluence of analytic boundary conditions.*

Section VI, Logic and philosophy. Professor Alfred Tarski was the Chairman and Professor L. E. J. Brouwer of the University of Amsterdam, Netherlands, was selected as Honorary Chairman of this Section. Sixteen contributed papers were presented. The following invited addresses were given by invitation of the Chairman:

T. SKOLEM: *Remarks on the foundation of set theory.*

A. TARSKI: *Some notions and methods on the borderline of algebra and metamathematics.*[4]

A. ROBINSON: *Applied symbolic logic.*

S. C. KLEENE: *Recursive functions and intuitionistic mathematics.*

Section VII, History and education. Dr. C. V. Newsom was the Chairman of this Section and Professor H. W. Turnbull of the University of St. Andrews, Scotland, acted as Honorary Chairman of the Section. Fifteen contributed papers were presented. The following invited address was given by invitation of the Chairman:

G. PÓLYA: *On plausible reasoning.*

THE SOCIAL PROGRAM

The entertainment of the members of the Congress was in the hands of an Entertainment Committee, whose Chairman was L. H. Loomis. The other members of the Committee were C. R. Adams, Mrs. L. V. Ahlfors, Mrs. G. D. Birkhoff, J. A. Clarkson, Mrs. W. C. Graustein, F. B. Hildebrand, Evelyn M.

[4] This address was given by special invitation of the Organizing Committee.

Hull, J. R. Kline, E. R. Lorch, Mrs. W. T. Martin, E. B. Mode, G. A. O'Donnell, Mrs. H. B. Phillips, Helen G. Russell, J. H. Van Vleck, Mrs. J. L. Walsh, Mrs. D. V. Widder, and Mrs. Norbert Wiener.

The members and associate members of the Congress were housed in the dormitories of Harvard University. Excellent meals were served in the Harvard Union. The local members of the Entertainment Committee arranged a fine exhibition of mathematical models and a book exhibit at the Harvard Cooperative Society. In addition to tours of Harvard University and the Massachusetts Institute of Technology there were trips to historic and modern Boston as well as to Lexington and to Concord.

On the opening night there was a reception for all members of the Congress at the Fogg Art Museum of Harvard University. On the following evening, August 31, the Busch String Quartet gave a concert in the Sanders Theatre. At the same time Professor Howard Aiken, the designer of the Harvard Computing Machine, gave an interesting lecture on computing machines.

On Friday afternoon, September 1, the mathematicians and their friends were the guests of Wellesley College at a tea. At the same time another portion of the members of the Congress were entertained by the Harvard Observatory at Oak Ridge. On Saturday evening, September 2, there was an informal dance in Lowell House, while other members of the Congress attended a beer party in Memorial Hall.

There were no sessions of the Congress on Sunday, September 3. In the late afternoon, Boston University received the members of the Congress, and an organ recital was provided in the Daniel L. Marsh Chapel of the University. In the evening Richard Dyer-Bennet gave a concert in the Sanders Theatre of ballads of various nations.

On Monday evening, September 4, the committee arranged for a concert in Symphony Hall by Miss Helen Traubel. The concert was enthusiastically received by the audience and Miss Traubel received a tremendous ovation at the end of the performance.

The Congress banquet was held on Tuesday evening, September 5, in the Sever Quadrangle. Professor Marston Morse was the Toastmaster and Dr. Detlev Bronk, President of the National Academy of Sciences, gave the principal address. Other speakers were Edward Crane, the Mayor of Cambridge; Dr. James R. Killian, the President of Massachusetts Institute of Technology; Professor W. V. D. Hodge of Cambridge University; and Professor Percy Bridgeman of Harvard University. Professor Hodge expressed the sincere appreciation of our foreign members of the Congress to their American hosts for their splendid hospitality. Dr. Bronk's address follows:

ADDRESS OF
DR. DETLEV BRONK
PRESIDENT, NATIONAL ACADEMY OF SCIENCES

Professor Morse, Professor Veblen, and colleagues of the Congress. As a mere biologist, I accepted the invitation to come here this evening with great trepidation; for I was reminded of the first time that I met Lord Rutherford in company with my good friend, E. B. Hill, who had been a senior wrangler, I believe, at Cambridge. To me, Lord Rutherford said, "It is interesting to find another physicist and mathematician who has ventured into biology." Indeed, when I was cruising in the fogbound waters of Maine during these past few weeks, I had one consolation in the difficulties of navigation. For I felt that could I not escape from my fogbound visitations, I would escape the duties that now confront me. But I regret to say that although I never reached the peaks of mathematical ability, I could do the simple arithmetic of dead reckoning that brought me back in time.

It is my pleasant privilege to bring to you the greetings of the National Academy of Sciences of the United States of America. It is especially appropriate that we your colleagues, who work in other fields of science, do so, for we are keenly conscious of our debt to you for intellectual tools we all require. In these days of increasing specialization, mathematics is a unifying focus. We recognize in your deliberations which we may not comprehend, the genesis of thoughts and concepts which will increase the acuity of our own investigations. We perceive in this great Congress the furtherance of intellectual powers that will facilitate increased understanding of natural forces without regard of the boundaries of scientific discipline. To you from other nations I bring the cordial welcome of my colleagues. It is a welcome filled with gratitude from the scientists of a youthful nation. At a time when men and women who have come from your several countries were absorbed in pioneers' practical duties on geographic frontiers, we benefited from the endeavors of your scientific ancestors. Now that our geographical frontiers are passed, we may join with you in exploring the endless frontiers of knowledge. We would thus pay back to you the debt for the basic facts and concepts which enabled us to satisfy our needs in technology and the applications of science. Having benefited thus, we clearly recognize science as a universal heritage of all men and women in all nations. That you have thus gathered with your distant colleagues without regard for race or tongue or nation is reaffirmation of your faith in the international amity of science. To be an isolationist in science is to handicap one's own achievements. The course of new discoveries starts from the territory of established knowledge. The genesis of new ideas is catalyzed by the work and thought of others. Recognizing this, scientists were among the first to realize the practical dependence of their own work on the efforts of those in distant lands. Together with the trades in rare goods, they have sought intellectual products and new discoveries wherever they were to be found. Out of this desire for the advantages which can be gained from the work of others has come that admirable phrase "my foreign colleague," so frequently heard in scientific circles and so seldom heard in others. It is worthy of emphasis that this desire for international cooperation derives from no unique nobility of spirit but comes rather from the simple realization of the advantages that derive from a free exchange of ideas. If scientists are better prepared for the acceptance of the principles of world unity, it is because we have long ago realized the benefits that come from such cooperation. The desire for world-wide dissemination of thoughts on science and of scientific discoveries motivated those who shaped the earliest association of scientific workers.

The Academia de Scientia of Rome, first of the academies of sciences, in 1609 laid plans for the establishment of common scientific nonclerical monasteries, not only in Rome, but in the four quarters of the globe. In each house, every observation, every discovery, was to be communicated without delay to the head house and to all the sister houses. A similar purpose was subsequently found in Bacon's proposal for the creation of the House of Solomon wherein there were to be twelve fellows who were to sail into foreign countries and return with the books and abstracts and the patterns of experiments made in other nations. These we call, significantly said Bacon, merchants of might. This international idea was an essential motive in the early activities of the Royal Society of London. That may be judged from the lines of William Glanville, written in 1660, while the Society was forming at Gresham College, that "Gresham College shall hereafter be the whole world's university." This was not an unfulfilled aspiration. Oldenburg, the first secretary of the Royal Society, lists among his duties, "I write all letters abroad and answer received thereto, entertaining correspondence with at least some thirty persons." Because the reading of these communications from foreign scientists became so important a part of the meetings of the Society, one of their fellows got the sum of five hundred pounds for the support of a fellow to carry on a foreign correspondence. Science has facilitated the movements of scientists by means of swift transportation. The exchange of information is speeded by new methods of communication. But two recent wars and the present conflicts of national purpose show how insecure is our privilege to exercise that freedom upon which depends the furtherance of science. As scientists we have a precious stake in the preservation of personal liberties that do not infringe the liberties of others. But our freedom of thought, of opinion, and of debate will be guaranteed only by a social system which guarantees to all such freedoms. In these times when stress, bewilderment, and fear encourage few to gain control of many, we must couple with our scientific efforts vigorous defenses of freedom, of undistorted science uncontrolled except by experimental test and reason. These are times that challenge us to double effort. These are times that challenge our loyalty to those ideals that made possible the intellectual adventures of the past which led to scientific progress.

In these times when national conflicts threaten human welfare, the scientist will not forget that the social value of his accomplishments makes him a citizen of all free nations. A new chemical agent, or the treatment of disease, is of potential benefit to all men. The laws of electromagnetic induction, discovered by Faraday, the Englishman, relieve the labors of the citizens of many lands The observations of Galileo, Copernicus, and Newton have increased the intellectual horizons of no one national group. Scientific research conducted in a spirit of freedom and published without restriction increases the welfare and the resources of all nations. To further scientific investigations is a common responsibility and a common advantage for all countries. As Francis Bacon did foresee, science enlarges the bounds of human empire and the effecting of all things possible from a knowledge of the causes and secret motions of things. Science gives to those who would limit the bounds of human empire awful power over others. Because of this, the progress of science is endangered by others who would use scientists and science to achieve their selfish ends. To do so, they would restrict the free statement of ideas and information. But science cannot flourish if the discoveries and thoughts of scientists are the secret knowledge of the few. Science cannot increase the understanding and improve the welfare of all men unless free access to knowledge is recognized as a fundamental human right. To deserve that right, the peoples of the world must restore regard for truth and for the democratic determination of individual and national action. The spirit of science will not long survive in a world half free to think and speak, to investigate and question, half slave to prejudice and dictation. So long as wars are waged to gain advantage and control, science will be used to implement aggression, and to fortify the defense of freedom. Thus will the proper purpose of scientists be deflected, inquiry to satisfy curiosity and increase understanding will be subordinated. The mere application of scientific facts, freedom for discussion, will be curtailed, and truth will bow to propaganda.

In these days, we as scientists are challenged to adhere to our traditional ideals of intellectual freedom, to align ourselves with those who would guarantee the freedom of peoples everywhere. Great though our temporary sacrifice may be, our future right to inquiry demands it. The use of modern science gives a nation tremendous power and material advantage. Since science is developed more in some than in other countries, there will be a further imbalance in the intellectual and material welfare of different peoples. There lies a grave threat to peace and wholesome progress. This accents the responsibility of the scientists of more favored nations to share their knowledge and their methods with all people, and especially with those who are victims of poverty and disease and ignorance. For scientists have an important role in shaping world cultures suitable for these times.

In these days of international tension, American scientists like to recall that one of our greatest statesmen was also one of our first and greatest scientists. Benjamin Franklin was fitted for his tasks in foreign capitals by many qualifications; but not the least of these was his eminence as a man of science. Because of this, we know that he was heard as one who contributed to the welfare of all peoples while seeking as a patriot to improve the material and political circumstances of his own countrymen. Of especial importance was the fact that he carried through his tasks an instinct for internationalism, which had been developed through his scientific career. It is not unreasonable to assume that this gave him a tolerance and a breadth of outlook that favored the course for which he pled and gave it reasonableness.

It was this quality which prompted him in March of 1779 to address to all captains and commanders of armed ships, acting by permission from the Congress of the United States of America, then at war with Great Britain, this directive: "Gentlemen, A ship having been fitted out from England before the commencement of this war to make discoveries in unknown seas, within the conduct of that most celebrated discoverer and navigator Captain Cook, which is an undertaking laudable in itself as the increase in geographical knowledge facilitates the communication between distant nations, and sciences of other kinds are increased which have benefit for mankind in general; this, then, is to recommend to you that in case the said ship should fall into your hands, you should not consider her as an enemy, nor permit any plunder of her effects, nor obstruct her return to England." Would that we had more statesmen that possessed this attitude toward the international values of science. Would that we had more scientists who would participate in the international affairs of nations. As you bring to a close your distinguished Congress, I would, on behalf of the Academy I represent, express our gratitude for your visit to our country and to our colleagues, and to pledge anew our devotion to the fraternity of free scholars.

Harvard University
Cambridge, Massachusetts
September 5, 1950

On Wednesday evening, September 6, the mathematicians were the guests of the Director and Board of Trustees of Gardner Museum at a farewell party.

On Wednesday morning, September 6, there was held a final plenary session of the Congress in Sanders Theatre. Professor M. H. Stone gave a report on the conference which had been held in New York City immediately preceding the International Congress for the purpose of considering the formation of an International Mathematical Union. He reported that Statutes and By-Laws had been adopted and that these would be submitted to the proper scientific groups in the various national or geographic areas in which there was significant mathematical activity. When a specified number of groups have signified their acceptance of these Statutes and By-Laws, the Union will be declared in existence and a meeting of the General Assembly arranged.

Professor van der Corput, on behalf of the delegation from the Netherlands, presented a cordial invitation to the International Congress to hold its next meeting in the Netherlands in 1954. The Congress unanimously voted to accept the gracious invitation of our Dutch colleagues. After addresses, there was an address of appreciation by President Harald Cramér of the University of Stockholm. After Professor A. A. Albert had presented a resolution of thanks to Harvard University and to the various committees of the Congress, which resolution was unanimously adopted, the Congress adjourned.

The Congress was undoubtedly the largest gathering of persons ever assembled in the history of the world for the discussion of mathematical research. However, the real measure of its success lies not in the large number of persons present, but in the excellence of its scientific program and in the contributions which it made to the cause of closer cooperation among scientists and to the cause of international good will.

J. R. Kline, *Secretary,*
International Congress of Mathematicians

STATED ADDRESSES

POWER-ASSOCIATIVE ALGEBRAS

A. A. ALBERT

This address was given as part of the Conference in Algebra, see Volume 2, ›age 25.

ON NULL-SETS IN HARMONIC ANALYSIS AND FUNCTION THEORY

ARNE BEURLING

No manuscript of this address has been received by the editors.

LAPLACE OPERATOR ON MANIFOLDS

S. Bochner

This address appears in the report of the Conference in Analysis, see Volume 2, page 189.

PROBLÈMES GLOBAUX DANS LA THÉORIE DES FONCTIONS ANALYTIQUES DE PLUSIEURS VARIABLES COMPLEXES[1]

HENRI CARTAN

Ayant l'honneur de parler ici de la théorie des fonctions analytiques de plusieurs variables complexes, je me propose non pas de vous donner un aperçu complet de la théorie dans son état actuel, mais de passer en revue quelques problèmes typiques de son développement récent; il s'agira surtout de problèmes *globaux* ("in the large"). La théorie des fonctions automorphes de plusieurs variables pourrait certes rentrer dans mon sujet; cependant je n'y ferai que quelques allusions, laissant à d'autres, plus qualifiés que moi, le soin de vous en entretenir éventuellement.

1. Variétés à structure analytique complexe. Une variété analytique-complexe, c'est, par définition, une variété de dimension paire $2n$ (c'est-à-dire un espace topologique dont chaque point possède un voisinage ouvert homéomorphe à l'espace euclidien de dimension $2n$), munie en outre de la donnée, en chaque point, d'un ou plusieurs systèmes de "coordonnées complexes locales": un système de coordonnées locales, en un point P, est un système de n fonctions à valeurs complexes $z_1, \cdots, z_n$, définies dans un voisinage ouvert V de P, et qui appliquent biunivoquement et bicontinûment V sur un ensemble ouvert de l'espace C^n de n variables complexes. Au sujet des systèmes de coordonnées locales, on fait les hypothèses suivantes: 1°. Tout système de coordonnées locales pour un point P est aussi un système de coordonnées locales pour tout point P' suffisamment voisin de P; 2°. Etant donnés deux systèmes de coordonnées locales en un point P, on passe de l'un à l'autre par une transformation *analytique-complexe* au voisinage de P. L'entier n se nomme la *dimension* (complexe) de la variété analytique-complexe.

Par exemple, l'espace projectif complexe (de dimension quelconque n) est une variété analytique-complexe. Voici un autre exemple: soit D un sous-ensemble ouvert de l'espace C^n; soit Γ un groupe discontinu d'automorphismes de D (automorphisme = transformation analytique-complexe, biunivoque, de D sur D); l'espace quotient D/Γ est muni d'une structure analytique-complexe, obtenue par passage au quotient à partir de la structure analytique-complexe naturelle de C^n.

Etant donnée une variété analytique-complexe B, on a la notion de *fonction analytique* (ou *holomorphe*) dans B: c'est une fonction définie dans B, à valeurs complexes, et qui, au voisinage de chaque point P, s'exprime comme fonction analytique des n coordonnées locales d'un système attaché à P. Par exemple, une fonction analytique dans D/Γ n'est autre chose qu'une fonction analytique dans

[1] Cette communication était mentionnée sur le programme imprimé sous le titre *Sur les fonctions analytiques de variables complexes.*

) et invariante par le groupe Γ (fonction *automorphe*). Plus généralement, on a la otion de fonction holomorphe dans un ensemble ouvert de B (un tel ensemble uvert étant lui-même muni d'une structure de variété analytique-complexe, ıduite par la structure de B). Précisons encore la notion de fonction holomorphe ans un sous-ensemble *compact* K de B: c'est, par définition, une fonction définie t holomorphe dans un voisinage ouvert de K; on convient d'identifier deux ɔnctions quand elles coïncident dans un voisinage de K. Avec cette définition, les ɔnctions holomorphes dans K forment un *anneau*; en particulier, on a l'anneau es fonctions holomorphes *en un point* de la variété B.

L'étude générale des variétés analytiques, et des fonctions holomorphes sur ces ariétés, est encore très peu avancée. Un des premiers problèmes qui se posent est ɜ suivant: existe-t-il une fonction holomorphe dans B et non constante? Il n'en xiste certainement pas si B est une variété *connexe* et *compacte,* parce qu'une ɔnction holomorphe ne peut admettre de maximum, même au sens large, en un oint P de B sans être constante au voisinage de P. Mais on peut alors poser la uestion de savoir s'il existe des fonctions *méromorphes* non constantes sur une ariété compacte connexe B (une fonction méromorphe est une fonction qui, au oisinage de chaque point, peut s'écrire comme quotient de deux fonctions olomorphes). Par exemple: supposons que B soit l'espace quotient de l'espace $'^n$ (considéré comme groupe additif) par un sous-groupe discret Γ engendré par n éléments linéairement indépendants; B est alors homéomorphe à un tore à $2n$ imensions réelles, muni d'une structure analytique-complexe; les fonctions ıéromorphes dans B ne sont autres que les "fonctions abéliennes", et on sait que ɜur existence n'est assurée que s'il y a certaines relations entre les périodes.[2] ſous reviendrons plus loin sur les variétés analytiques compactes; nous allons uparavant nous occuper de la "réalisation" d'une variété analytique dans espace C^n de n variables complexes.

Une "réalisation" de B, c'est une application analytique de B dans l'espace C^n, 'est-à-dire un système de n fonctions holomorphes $f_1, \cdots, f_n$. Si B est compacte onnexe, de telles fonctions sont nécessairement constantes, et il n'existe donc ue des réalisations triviales. On ne s'occupera donc que de la réalisation des ariétés (connexes) non compactes. Plus précisément, nous allons imposer aux ɔnctions $f_1, \cdots, f_n$ d'avoir leur déterminant fonctionnel $\neq 0$ en tout point de B; 'est la condition nécessaire et suffisante pour que l'application f de B dans C^n ɔit *localement biunivoque*; cela exprime aussi que $f_1, \cdots, f_n$ constituent un ystème de coordonnées locales en tout point de B. Une réalisation de ce type appellera un *domaine étalé* dans C^n; en particulier, ce domaine étalé est *univalent* 'schlicht") si l'application f est une application *biunivoque* de B sur un sous-ısemble (ouvert) de C^n. Bien entendu, une même variété peut être susceptible de lusieurs réalisations; par exémple, elle peut avoir des réalisations *bornées* et 'autres réalisation non bornées (ainsi: le cercle $|z| < 1$ et le demi-plan $\mathrm{Re}(z) > 0$).

[2] A ce sujet, consulter le cours récent de C. L. Siegel, *Analytic functions of several com-lex variables*, Institute for Advanced Study, Princeton, 1950, et un exposé de A. Weil au éminaire Bourbaki de mai 1949.

2. Domaines d'holomorphie. Il n'existe actuellement une théorie des *domaines d'holomorphie* que pour les variétés susceptibles d'une réalisation comme "domaine étalé".

Pour $n = 1$, tout domaine est le domaine total d'existence d'une fonction holomorphe convenable; mais on sait qu'il n'en est plus de même pour $n \geqq 2$. Rappelons l'exemple classique de Hartogs (1906): dans l'espace des 2 variables z_1 et z_2, considérons la réunion A des 2 ensembles compacts

$$| z_1 | \leqq 1, \qquad | z_2 | = 1,$$

et

$$z_1 = 0, \qquad | z_2 | \leqq 1;$$

toute fonction holomorphe dans A se laisse prolonger en une fonction holomorphe dans le polycylindre compact $| z_1 | \leqq 1$, $| z_2 | \leqq 1$.

E. E. Levi a montré, peu d'années plus tard, qu'on en peut dire autant pour les fonctions méromorphes; il a aussi prouvé que toute fonction holomorphe (ou méromorphe) au voisinage de la sphère $| z_1 |^2 + | z_2 |^2 = 1$ se laisse prolonger en une fonction holomorphe (ou méromorphe) dans la boule $| z_1 |^2 + | z_2 |^2 \leqq 1$, résultat qui s'étend au cas de n variables complexes ($n \geqq 2$), et qui a été généralisé par Severi au cas où certaines variables sont réelles (l'une au moins étant complexe). On trouve, dans le livre récent de Bochner et Martin, une généralisation intéressante de ces résultats.

Il m'est impossible de retracer ici l'historique du développement de la théorie des domaines d'holomorphie. Retenons-en seulement deux choses: d'abord, la propriété, pour une variété B, de posséder une fonction holomorphe dans B et non prolongeable au-delà de B, est une propriété indépendante de la réalisation de B comme domaine étalé; autrement dit, le fait, pour B, d'être un domaine d'holomorphie, est une propriété de la variété analytique-complexe B, dès que B admet une réalisation spatiale comme domaine étalé.

La seconde chose que nous voulons mentionner, c'est la caractérisation des domaines d'holomorphie par une propriété interne de *convexité*[3] qui s'avère essentielle dans beaucoup de problèmes. Soit B une réalisation spatiale d'un domaine d'holomorphie, et soit K un sous-ensemble compact de B; soit r la distance de K à la frontière de B; alors, pour tout point z de B dont la distance à la frontière de B est $<r$, il existe une fonction f holomorphe dans B et telle que $| f(z) | > \sup_K | f |$ (théorème de Thullen). Réciproquement, si une réalisation spatiale B jouit de cette propriété (ou même d'une propriété affaiblie que nous ne précisons pas ici), B est un domaine d'holomorphie. Or l'ensemble K_r des points de B dont la distance à la frontière de B est $\leqq r$ est *compact*, tout au moins si B est univalent, ou, plus généralement, s'il existe un entier N tel que tout point de l'espace C^n soit couvert au plus N fois par B (domaine à un nombre borné de "feuillets"). Nous nous limiterons désormais à ce cas; alors le résultat précédent entraîne celui-ci: si B est un domaine d'holomorphie à un nombre

[3] Voir H. Cartan et P. Thullen, Math. Ann. t. 106 (1932) pp. 617–647.

borné de feuillets, B est réunion d'une suite croissante d'ensembles *compacts* P_k, dont chacun est défini par un nombre fini d'inégalités de la forme $|f_{kj}(z)| \leqq 1$ $(j = 1, 2, \cdots)$, les f_{kj} étant holomorphes dans B; plus exactement, P_k est une composante connexe, supposée compacte, d'un ensemble ainsi défini. Un ensemble compact du type de P_k sera appelé un *polyèdre analytique*; ainsi, tout domaine d'holomorphie à un nombre borné de feuillets est réunion d'une suite croissante de polyèdres analytiques; j'ignore si ce théorème est encore exact quand le nombre des feuillets de B n'est pas borné. La réciproque est vraie: si un domaine étalé B est réunion d'une suite croissante de polyèdres analytiques, c'est un domaine d'holomorphie; cela résulte d'un théorème de Behnke et Stein,[4] qui dit que la réunion d'une suite croissante de domaines d'holomorphie est un domaine d'holomorphie.

Les polyèdres analytiques ont été considérés explicitement pour la première fois par André Weil en 1935; *l'intégrale de* Weil,[5] qui généralise l'intégrale de Cauchy, exprime une fonction holomorphe dans un polyèdre analytique par une intégrale portant sur les valeurs de cette fonction sur les "arêtes" à n dimensions (réelles) du polyèdre. Mais la construction du noyau de cette intégrale soulève, dans le cas général, des difficultés qui ne peuvent être surmontées que grâce à la théorie des idéaux de fonctions holomorphes, dont il sera parlé plus loin.

A quoi reconnaît-on qu'un domaine donné B est un domaine d'holomorphie? Dès 1911, E. E. Levi avait indiqué des conditions nécessaires auxquelles doit satisfaire la frontière d'une réalisation spatiale de B, lorsque cette frontière est suffisamment différentiable. Il s'agissait alors de savoir, étant donné un point frontière z_0 de B, s'il existe, dans l'intersection de B et d'un voisinage de z_0, une fonction holomorphe (ou méromorphe) qui admette z_0 comme point singulier essentiel. Il s'agissait donc d'un critère de nature *locale*, qui d'ailleurs s'exprimait en écrivant qu'une certaine forme hermitienne était définie positive au point z_0 considéré. Pendant longtemps la question est restée posée de savoir si de telles conditions locales, supposées vérifiées on chaque point frontière d'une réalisation spatiale de B, étaient suffisantes pour que B soit, *globalement*, un domaine d'holomorphie. Il était réservé à Oka de résoudre ce problème par l'affirmative en 1942, tout au moins dans le cas des domaines univalents.[6] La solution de Oka (qu'il a exposée pour $n = 2$) est l'aboutissement d'une suite de recherches difficiles et met en oeuvre toute une technique spéciale, d'ailleurs liée, elle aussi, à la théorie des idéaux de fonctions holomorphes. Le lemme de Oka qui donne la clef du problème est le suivant: soit z l'une des variables complexes de l'espace ambiant C^n; soient B_1 et B_2 les intersections du domaine B avec $\mathrm{Re}(z) < \epsilon$ et $\mathrm{Re}(z) > -\epsilon$ respectivement $(\epsilon > 0)$; alors, si B_1 et B_2 sont des domaines d'holomorphie, B est aussi un domaine d'holomorphie. On reconnaît ici un énoncé du type de ceux qui permettent d'effectuer, de proche en proche, le passage d'une propriété locale à une propriété globale.

[4] Math. Ann. t. 116 (1938) pp. 204–216.
[5] Math. Ann. t. 111 (1935) pp. 178–182.
[6] Tôhoku Math. J. t. 49 (1942) pp. 15–52.

3. Etude globale des idéaux de fonctions holomorphes. On peut y être conduit en analysant la notion de *sous-variété analytique*. Soit B une variété à structure analytique-complexe, de dimension (complexe) n; un sous-ensemble M de points de B sera une sous-variété analytique dans B (éventuellement décomposable) si c'est un sous-ensemble *fermé* de B, et si chaque point z de M possède, dans B, un voisinage ouvert $V(z)$ tel que $M \cap V(z)$ soit exactement l'ensemble des zéros communs à une famille de fonctions holomorphes dans $V(z)$; autrement dit, si, au voisinage de z, M peut être définie par des équations analytiques, qu'on peut d'ailleurs supposer en nombre fini. Il faut prendre garde que, malgré la terminologie de "sous-variété", M, comme espace topologique, n'est pas en général une variété: M peut posséder des points singuliers, dits *non essentiels*, analogues aux points singuliers d'une sous-variété algébrique de l'espace; l'ensemble de ces points singuliers forme, à son tour, une sous-variété analytique; au voisinage de tout point non singulier, M est doué d'une structure de variété analytique (au sens du §1), avec des systèmes de coordonnées locales.

La définition d'une sous-variété analytique M de B a ainsi un caractère *local*. Problème: est-il possible de définir M, *globalement* dans B, en égalant à zéro une famille (finie ou infinie) de fonctions holomorphes dans B tout entier? Ce problème peut avoir une réponse négative: par exemple, si B est une variété connexe compacte, et si M est non vide, toute fonction holomorphe qui s'annule sur M est identiquement nulle. Voici un autre contre-exemple: soit, comme dans l'exemple de Hartogs, B la réunion de

$$|z_1| < 1 + \epsilon, \qquad 1 - \epsilon < |z_2| < 1 + \epsilon,$$

et de

$$|z_1| < \epsilon, \qquad |z_2| < 1 + \epsilon \qquad (0 < \epsilon < 1/2).$$

Considérons, dans B, l'ensemble M des points tels que $z_1 = z_2$, $|z_1| < \epsilon$; M est bien une sous-variété analytique de B. Toute fonction f, holomorphe dans B, et qui s'annule sur M, est holomorphe dans le polycylindre $|z_1| < 1 + \epsilon$, $|z_2| < 1 + \epsilon$ et s'annule aux points tels que $z_1 = z_2$ (car la trace de f sur la variété $z_1 = z_2$ est une fonction holomorphe de z_1, nulle au voisinage de $z_1 = 0$, donc identiquement nulle pour $|z_1| < 1 + \epsilon$). Or, parmi les points de B tels que $z_1 = z_2$, il en est qui n'appartiennent pas à M; donc M ne peut pas être défini comme l'ensemble des zéros d'une famille de fonctions holomorphes dans B.

Ce dernier exemple montre qu'il est raisonnable de se borner d'abord au cas où B est un domaine d'holomorphie. Dans ce cas, nous allons voir que le problème posé est toujours résoluble. Donnons quelques précisions: considérons la sous-variété M de B, qui nous est donnée; si un point z de B appartient à M, les fonctions holomorphes au point z et qui s'annulent sur M (c'est-à-dire en tout point de M suffisamment voisin de z) forment un *idéal* I_z dans l'anneau des fonctions holomorphes au point z; si $z \notin M$, I_z désignera l'idéal-unité (idéal de toutes les fonctions holomorphes en z). Posons-nous la question suivante: existe-t-il un ensemble de fonctions holomorphes dans B tout entier, et qui, en

:haque point z de B, engendre l'idéal I_z attaché à ce point? On peut évidemment e borner à chercher un ensemble de fonctions qui soit un *idéal* I de l'anneau des onctions holomorphes dans B. Voici maintenant une réponse à la question)osée: si B est un *domaine d'holomorphie* (plus exactement: la réunion d'une uite croissante de polyèdres analytiques), un tel idéal existe toujours; si en)utre on le suppose *fermé* (ce qui signifie que toute fonction holomorphe dans B [ui est, sur tout compact, limite uniforme de fonctions de l'idéal I, appartient ı I), alors un tel idéal I est *unique*: c'est l'idéal de toutes les fonctions, holo-norphes dans B, qui s'annulent en tout point de M; enfin, sur tout compact K :ontenu dans B, l'idéal I peut être engendré par un nombre *fini* d'éléments, et)ar suite la sous-variété M peut être définie par un nombre fini d'équations lans le voisinage d'un ensemble compact K arbitraire.

Ces résultats rentrent dans le cadre d'une théorie générale des idéaux de onctions holomorphes, théorie développée à une époque récente, parallèlement,)ar Oka et H. Cartan.[7] Supposons qu'à chaque point z de B on ait attaché un déal I_z de l'anneau des fonctions holomorphes au point z; cherchons s'il existe, lans B, un idéal I qui engendre I_z en chaque point z de B. Or il y a une condition ·videmment nécessaire: c'est que tout point z de B possède un voisinage ouvert lans lequel existe un idéal engendrant $I_{z'}$ en tout point z' assez voisin de z. ;i cette condition est remplie, nous dirons que le système des idéaux I_z est *ohérent*. Par exemple, le système des idéaux qu'une sous-variété analytique M)ermet d'attacher aux divers points de B est un système cohérent; c'est là un héorème de nature locale, qui est d'ailleurs assez difficile à prouver; une fois ce héorème démontré, on peut aborder l'étude globale des sous-variétés analytiques, ·tude dont nous avons déjà indiqué les principaux résultats.

Cela dit, supposons, d'une manière générale, qu'on nous ait donné un système :ohérent d'idéaux I_z dans B; alors on peut démontrer ceci: sur tout polyèdre ,nalytique P contenu dans B, il existe un idéal *et un seul* qui engendre I_z en out point z de P, et cet idéal a un nombre *fini* de générateurs; de plus, tout déal, dans un polyèdre P, est *fermé*. Si en outre on suppose que B est un domaine l'holomorphie, alors il existe dans B un idéal fermé et un seul qui engendre le ystème cohérent donné: c'est l'idéal des fonctions qui, en chaque point z de B, ,ppartiennent à l'idéal I_z attaché à ce point. D'ailleurs, dans un domaine d'holo-norphie, tout idéal engendré par un nombre fini d'éléments est fermé. A titre l'application des résultats précédents, signalons aussi le théorème suivant: si p onctions f_i, holomorphes dans un domaine d'holomorphie B, n'ont pas de zéro ommun, elles sont liées par une relation $\sum_i c_i f_i = 1$ à coefficients c_i holomorphes lans B. Ceci permet notamment de lever les difficultés qu'on rencontre dans la onstruction du noyau de l'intégrale de Weil.

La théorie des idéaux dont nous venons d'esquisser rapidement quelques ésultats, sans pouvoir donner le principe des démonstrations, permet de prouver

[7] H. Cartan, Journal de Math. sér. 9 t. 19 (1940) pp. 1–26; H. Cartan, Ann. Ecole Norm. . 61 (1944) pp. 149–197; K. Oka, Bull Soc. Math. France t. 78 (1950) pp. 1–27; H. Cartan, 3ull. Soc. Math. France t. 78 (1950) pp. 29–64.

une ancienne conjecture d'André Weil: soit M une sous-variété analytique dans un domaine d'holomorphie B; et soit f une fonction holomorphe dans M, c'est-à-dire holomorphe dans un voisinage de M; alors il existe une fonction g, holomorphe dans tout B, et égale à f en tout point de M. Ce théorème peut servir, notamment, à montrer que, dans un polyèdre analytique défini par des inégalités $|\varphi_k(z)| \leqq 1$, toute fonction holomorphe est limite uniforme de *polynomes* par rapport aux variables z_j et aux fonctions $\varphi_k(z)$.

4. Prolongement analytique des sous-variétés. Le problème est le suivant: soit M une sous-variété analytique dans un sous-ensemble ouvert D de B; existe-t-il, dans B, une sous-variété analytique M' telle que $M' \cap D = M$? S'il en existe une, elle n'est pas nécessairement unique, mais il existe alors une sous-variété M' *minimale*, qui est contenue dans toutes les autres; car toute intersection de sous-variétés analytiques dans B est une sous-variété analytique dans B.

Sur ce problème, nous n'avons aujourd'hui que des résultats fragmentaires. Le plus ancien est dû à Thullen:[8] soit, dans la variété analytique-complexe B de dimension n, une sous-variété analytique M_0 de dimension $n - 1$, indécomposable; soit D un sous-ensemble ouvert de B, contenant l'ensemble complémentaire de M_0 et au moins un point de M_0 ; si M est une sous-variété analytique dans D, de dimension $n - 1$, alors M se prolonge en une sous-variété analytique dans B tout entier. En d'autres termes: si une sous-variété analytique M, de dimension $n - 1$, n'a pas de singularité essentielle en dehors d'une sous-variété indécomposable M_0 de dimension $n - 1$, deux cas seulement sont possibles: ou bien tous les points de M_0 sont effectivement des points singuliers essentiels de M, ou bien aucun point de M_0 n'est singulier essentiel pour M. En particulier, une sous-variété M de dimension $n - 1$ ne peut avoir de singularité essentielle *isolée*, si $n \geqq 2$. Cet intéressant théorème de Thullen peut, comme Stein me l'a signalé récemment, se généraliser aux sous-variétés de dimension quelconque: il suffit, dans l'énoncé précédent, de supposer que M_0 et M sont toutes deux de même dimension $k < n$; alors: ou bien tous les points de M_0 sont des points singuliers essentiels de M, ou bien aucun point de M_0 n'est singulier essentiel pour M. En particulier, une sous-variété M de dimension $k \geqq 1$ n'a jamais de point singulier essentiel isolé. Ceci donne notamment une nouvelle démonstration du théorème de Chow[9] suivant lequel toute sous-variété analytique de l'espace projectif complexe est nécessairement une sous-variété *algébrique*. En effet, en prenant des coordonnées homogènes dans l'espace projectif, on est ramené à considérer, dans l'espace C^{n+1}, un cône qui est analytique au voisinage de chacun de ses points sauf peut-être en son sommet; comme il n'a pas de singularité essentielle isolée, il est aussi analytique au voisinage de son sommet, ce qui implique aussitôt qu'il est algébrique.

Le prolongement analytique des sous-variétés n'obéit pas aux mêmes lois que le

[8] Math. Ann. t. 111 (1935) pp. 137–157.

[9] Amer. J. Math. t. 71 (1949) pp. 893–914.

prolongement analytique des fonctions. Par exemple, reprenons, avec Hartogs, la réunion A des 2 ensembles compacts $|z_1| \leqq 1$, $|z_2| = 1$, et $z_1 = 0$, $|z_2| \leqq 1$; soit $f(z_1)$ une fonction holomorphe pour $|z_1| < 1$, admettant la circonférence $|z_1| = 1$ comme coupure essentielle, et telle que $|f(z_1)| < 1/2$. Soit, dans un voisinage de A, la sous-variété analytique définie par $|z_1| < 1$, $z_2 = f(z_1)$; elle ne peut pas se prolonger en une sous-variété analytique dans un voisinage du polycylindre B: $|z_1| \leqq 1$, $|z_2| \leqq 1$, bien que toute fonction holomorphe dans A se prolonge en une fonction holomorphe dans B.

Cependant, il semble, à en croire un mémoire de Rothstein qui vient de paraître aux Math. Ann. (1950), qu'on doive s'attendre à des théorèmes généraux concernant le prolongement analytique des sous-variétés, théorèmes qui, quoique différents de ceux qui concernent le prolongement des fonctions, leur ressemblent tout de même un peu. Par exemple, Rothstein démontre ceci: soit, dans l'espace de 3 variables complexes, une sous-variété analytique M de dimension 2 au voisinage de la sphère $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$; alors M se prolonge en une sous-variété analytique dans toute la boule $|z_1|^2 + |z_2|^2 + |z_3|^2 \leqq 1$ (la proposition analogue, pour 2 variables et une sous-variété de dimension 1, est *fausse*). Voici un autre théorème de Rothstein, qui ressemble au théorème de Hartogs: si M, de dimension 2, est analytique au voisinage de $|z_1| \leqq 1$, $|z_2|^2 + |z_3|^2 = 1$, et au voisinage de $z_1 = 0$, $|z_2|^2 + |z_3|^2 \leqq 1$, M se prolonge en une sous-variété analytique au voisinage de $|z_1| \leqq 1$, $|z_2|^2 + |z_3|^2 \leqq 1$. Il y a là les prémices d'une théorie pleine d'intérêt, dans laquelle on verra plus clair quand on aura des énoncés valables pour des sous-variétés de dimension p dans l'espace de dimension n.

5. Relations avec la topologie. On sait que, sur une variété différentiable, une forme différentielle qui est localement une différentielle exacte peut ne pas l'être globalement. Considérons en particulier, sur une variété analytique-complexe B, une forme différentielle du premier degré ω qui, localement, soit la différentielle totale d'une fonction holomorphe; utilisant la terminologie de la géométrie algébrique, nous l'appellerons une *différentielle de première espèce*. La primitive d'une telle forme différentielle ω est une fonction holomorphe *multiforme*, qui se reproduit augmentée d'une constante lorsqu'on parcourt un lacet dans la variété B; elle définit donc un homomorphisme du premier groupe d'homologie $H_1(B)$ dans le groupe additif C des nombres complexes (il s'agit d'homologie à coefficients entiers). Cet homomorphisme est évidemment nul sur le sous-groupe $H_1'(B)$ des éléments d'ordre fini de $H_1(B)$, d'où un homomorphisme du groupe de Betti $H_1(B)/H_1'(B) = H_1''(B)$ dans C. Lorsque B est une variété *compacte*, $H_1''(B)$ est un groupe libre à un nombre fini r de générateurs, dont chacun peut être défini par un circuit fermé de B. L'homomorphisme de $H_1''(B)$ dans C est alors déterminé quand on connait l'intégrale de ω le long de chacun de ces r circuits: ce sont les "périodes" de ω. Quand B est une variété algébrique plongée sans singularité dans l'espace projectif complexe, l'entier r est *pair*, et il existe une différentielle de première espèce ω et une seule dont les périodes aient des *parties* *réelles* données. La démonstration de Hodge[10] vaut, plus généralement, lorsque B est une variété analytique *compacte* (connexe) susceptible d'être munie d'une *métrique kählérienne*, c'est-à-dire d'une forme différentielle quadratique hermitienne $\sum_\nu \omega_\nu \bar{\omega}_\nu$, définie positive, telle que la forme extérieure associée $\Omega = \sum_\nu \omega_\nu \cap \bar{\omega}_\nu$ satisfasse à $d\Omega = 0$.[11]

Le cas où B est un *domaine d'holomorphie* (étalé dans l'espace C^n) donne lieu à des résultats tout différents. Par les méthodes de Oka, on peut montrer[12] qu'il existe toujours une différentielle de première espèce dont les périodes soient des *nombres complexes* arbitrairement données; d'une façon précise, étant donné arbitrairement un homomorphisme du groupe d'homologie $H_1(B)$ dans C, il existe une différentielle de première espèce qui donne naissance à cet homomorphisme.

Nous allons aussi être amenés à des considérations topologiques en examinant un problème très simple concernant les idéaux de fonctions. Considérons, dans B, un système cohérent d'idéaux I_z tel que chaque idéal I_z soit *principal*, c'est-à-dire engendré par une seule fonction holomorphe au point z. Alors I_z définit, au voisinage de z, un nombre fini de sous-variétés analytiques de dimension $n - 1$, dont chacune est affectée d'un ordre de multiplicité (entier > 0). La donnée du système cohérent des I_z est équivalente à la donnée, dans B, d'une famille de sous-variétés analytiques de dimension $n - 1$, affectées d'ordres de multiplicité (cette famille pouvant être infinie, pourvu que chaque sous-ensemble compact de B n'en rencontre qu'un nombre fini). C'est ce qu'on appelle une "donnée de Cousin" dans B. Supposons que B soit un domaine d'holomorphie; alors, d'après les résultats généraux de la théorie des idéaux, il existe dans B un idéal fermé I, et un seul, qui engendre I_z en chaque point z de B. Mais le "problème de Cousin" consiste à chercher une *fonction unique*, holomorphe dans B, et qui, en chaque point z, engendre l'idéal I_z (c'est-à-dire s'annule sur chaque variété avec l'ordre de multiplicité voulu, et pas ailleurs). En d'autres termes, on exige que l'idéal I soit un idéal *principal*. Or cette nouvelle exigence ne peut pas toujours être satisfaite, même si B est un domaine d'holomorphie: comme nous allons le voir, elle pose des conditions de nature topologique.

Avant d'en parler, rappelons que le problème précédent avait été résolu par l'affirmative par Cousin, dès 1895,[13] dans le cas où B est un polycylindre, produit de domaines $z_k \in B_k$ supposés tous simplement connexes sauf un au plus. Cousin avait étudié ce problème à la suite de Poincaré qui s'était préoccupé de mettre une fonction méromorphe dans l'espace C^n sous la forme du quotient de 2 fonctions entières, premières entre elles.

Pour étudier l'aspect topologique du problème, plaçons-nous d'abord dans le cas général où B est une variété quelconque, à structure analytique-complexe.

[10] *The theory and applications of harmonic integrals*, Cambridge, 1941.

[11] Voir A. Weil, Comment. Math. Helv. t. 20 (1947) pp. 110–116.

[12] Résultat de H. Cartan, non encore publié; des cas particuliers en étaient connus auparavant, notamment lorsque B est une surface de Riemann étalée dans le plan d'une variable complexe.

[13] Acta Math. t. 19 (1895) pp. 1–62.

Une donnée de Cousin dans B définit un nouvel espace topologique E que voici: un point de E sera, par définition, un couple (z, f) formé d'un point z de B et d'un élément générateur f de l'idéal principal I_z attaché au point z; on identifiera les couples (z, f) et (z', f') si $z = z'$ et si le quotient f/f' (qui est holomorphe et $\neq 0$ au point z) est *égal à un* au point z. Faisons opérer, dans cet espace E, le groupe multiplicatif C^* des nombres complexes $\neq 0$, comme suit: un nombre complexe $\alpha \neq 0$ transforme (z, f) en $(z, \alpha f)$. Le groupe C^*, en opérant ainsi dans E, définit une relation d'équivalence; les classes d'équivalence, ou *fibres*, sont isomorphes à C^*, et l'espace quotient de E par la relation d'équivalence n'est autre que l'espace B. Dans le langage de la topologie moderne, E est un *espace fibré principal*, de groupe C^*, ayant B pour espace de base. L'hypothèse suivant laquelle les idéaux I_z forment un système cohérent exprime que chaque fibre de E possède un voisinage isomorphe au produit $U \times C^*$ d'un ensemble ouvert U de B par la fibre C^*; ceci permet de définir, sur E, une structure de variété analytique-complexe.

Ainsi, une donnée de Cousin, sur une variété analytique B de dimension n, définit une variété analytique E de dimension $n + 1$, qui est un espace fibré principal de base B et de groupe C^*. On voit aussitôt qu'une solution du problème de Cousin définit une *section analytique* de cet espace fibré, et réciproquement (une section analytique est une application analytique de B dans E, qui transforme chaque point z de B en un point de la fibre correspondant à z). Ainsi: pour que le problème de Cousin ait une solution, il faut et il suffit que l'espace fibré E ait une section analytique, ou, ce qui revient au même, qu'il soit isomorphe au produit $B \times C^*$ (il s'agit d'isomorphisme au sens analytique-complexe). Dans le langage de la théorie des espaces fibrés, notre espace fibré E doit être *trivial*; mais non pas trivial au sens topologique, ni même au sens de la structure différentiable, mais au sens analytique-complexe.

Les remarques qui précèdent sont dues à André Weil, qui attira récemment mon attention sur cette intervention de la notion d'espace fibré dans ce problème bien connu de la théorie des fonctions analytiques. Ainsi, on peut appliquer au problème de Cousin les résultats donnés par la théorie topologique des espaces fibrés: pour que l'espace fibré E défini plus haut soit topologiquement trivial, il faut et il suffit que la classe caractéristique de cet espace fibré, qui est un élément u du deuxième groupe de cohomologie $H^2(B)$ à coefficients entiers, soit nulle. C'est donc là une condition *nécessaire* pour que l'espace soit analytiquement trivial, mais peut-être pas suffisante.

Or Oka, dès 1939,[14] a montré que si B est un domaine d'holomorphie, et si le problème de Cousin peut être résolu dans le champ des fonctions continues, alors il admet aussi une solution dans le champ des fonctions analytiques. Cela revient à dire que si l'espace fibré E défini par la donnée de Cousin est topologiquement trivial, il est analytiquement trivial. Par conséquent, si B est un domaine d'holomorphie, la nullité de l'élément $u \in H^2(B)$ défini par la donnée de Cousin est nécessaire et suffisante pour que le problème de Cousin ait une solution.

[14] Journal of Science of the Hirosima University (1939) pp. 7–19.

En fait, dès 1941, Stein[15] avait explicité des conditions de nature homologique pour la résolubilité du problème de Cousin, sans faire appel à la théorie des espaces fibrés.

Indépendamment de tout problème de Cousin, on peut se demander si tout espace fibré analytique E, de groupe C^*, qui est topologiquement trivial, est analytiquement trivial. Or il en est bien ainsi quand l'espace de base B est un domaine d'holomorphie. Ce dernier résultat peut être utilisé pour le problème de Cousin généralisé, dans lequel on se donne un système cohérent de fonctions f_z, non plus holomorphes, mais méromorphes; cela revient à se donner, dans B, une famille de sous-variétés analytiques M_j, de dimension $n - 1$, affectées d'ordres de multiplicité p_j qui sont des entiers de signe quelconque; dans le langage de la géométrie algébrique, on se donne un "diviseur", combinaison linéaire, à coefficients entiers p_j, de sous-variétés analytiques de dimension $n - 1$. Une donnée de Cousin généralisée définit encore un espace fibré principal de base B et de groupe C^*; sa classe caractéristique $u \in H^2(B)$ est facile à interpréter à l'aide du cycle de dimension réelle $2n - 2$ défini par le diviseur. Lorsque B est un domaine d'holomorphie, la nullité de la classe d'homologie définie par le "diviseur" est nécessaire et suffisante pour que le problème de Cousin généralisé soit résoluble.

Le problème de Cousin dont nous venons de parler est appelé, par les spécialistes, "deuxième problème de Cousin". Le *premier* problème de Cousin consiste à chercher une fonction méromorphe dans B, dont la partie principale est donnée en chaque point de B. On voit facilement que la donnée de ces parties principales définit un espace fibré de base B, dont le groupe est cette fois le groupe *additif* C des nombres complexes. Un tel espace est toujours topologiquement trivial. La démonstration qu'a donnée Oka du fait que le premier problème de Cousin a toujours une solution quand B est un domaine d'holomorphie, prouve en réalité le théorème suivant: lorsque B est un domaine d'holomorphie, tout espace fibré de base B et de groupe C est analytiquement trivial.

Nous voudrions maintenant dire quelques mots des problèmes précédents dans le *cas où B est une variété compacte, kählérienne.* Alors, il n'est plus vrai qu'un espace fibré de base B et de groupe C soit toujours analytiquement trivial: un tel espace possède un invariant (de sa structure fibrée analytique-complexe), qui est un élément du premier groupe de cohomologie de B à coefficients réels; la nullité de cet invariant est nécessaire et suffisante pour que l'espace soit analytiquement trivial. Interprétons cet invariant lorsque la structure fibrée de groupe C provient de la donnée des parties principales d'une fonction méromorphe inconnue (premier problème de Cousin): il n'existe pas, en général, de fonction méromorphe f admettant ces parties principales; mais si on tolère une fonction f *multiforme*, on peut lui imposer de se reproduire augmentée d'une constante *réelle* par tout lacet dans B, et alors le problème a toujours une solution et une seule. Les "périodes" réelles de cette solution définissent l'invariant homologique cherché.

[15] Math. Ann. t. 117 (1941) pp. 727–657

Revenons au deuxième problème de Cousin généralisé, toujours dans le cas où B est une variété compacte kählérienne. Si on se donne un "diviseur", la nullité le l'élément $u \in H^2(B)$ qu'il définit assure seulement l'existence d'une fonction néromorphe *multiforme* admettant ce diviseur; on peut imposer à cette fonction l'être multipliée par une constante > 0 par tout lacet dans B. Si on tolère les multiplicateurs qui soient des constantes complexes, il n'est même plus nécessaire que u soit nul: dans ce cas, il faut et il suffit que l'intersection du 'diviseur" avec tout cycle à 2 dimensions réelles soit nulle; et ce résultat vaut aussi bien lorsque B est une variété compacte kählérienne (A. Weil, Kodaira[16]) que lorsque B est un domaine d'holomorphie. Dans un cas comme dans l'autre, on peut astreindre les multiplicateurs à être des nombres complexes de valeur absolue égale à un; si B est kählérienne compacte, cette restriction entraîne 'unicité de la solution, à un facteur constant près.

Il resterait à parler du deuxième problème de Cousin dans le cas général où il n'existe même pas de solution non uniforme admettant des multiplicateurs constants. C'est le problème que l'on rencontre dans la théorie classique des *onctions thêta* de n variables: B est alors le quotient de C^n par un sous-groupe Γ engendré par $2n$ éléments indépendants ("périodes"); étant donnée une fonction néromorphe dans $C^n = \tilde{B}$ et admettant les $2n$ périodes, les *pôles* d'une telle fonction définissent un "diviseur"; il est certain que l'élément $u \in H^2(B)$ défini par ce diviseur n'est pas nul (en vertu d'un théorème connu,[17] une sous-variété analytique d'une variété kählérienne compacte n'est jamais homologue à zéro). Mais il existe toujours une "fonction thêta" qui admette un diviseur arbitrairement donné: c'est une fonction holomorphe dans $\tilde{B}$ et qui, par tout lacet dans B, est multipliée par $e^{\varphi(z)}$, où φ est une fonction primitive d'une différentielle de première espèce de B (dans le cas présent, ceci implique que la fonction $\varphi(z)$ est linéaire dans l'espace C^n). Enoncé sous cette forme, ce résultat a été généralisé par Kodaira[16] à toutes les variétés compactes kählériennes, de la manière suivante: si l'élément du deuxième groupe de cohomologie *réel* défini par le "diviseur" est une somme de produits d'éléments du premier groupe de cohomologie, alors ce diviseur est celui d'une "fonction thêta généralisée".

Or il est remarquable que des fonctions analogues aient été considérées par Stein en 1941[18] dans le cas, fort différent, où B est un polycylindre de la forme

$$z_1 \in B_1, \cdots, z_n \in B_n.$$

L'invariant $u \in H^2(B)$ défini par une donnée de Cousin est alors caractérisé par la loi d'intersection du diviseur avec les produits $\gamma_j \times \gamma_k$ d'un cycle $\gamma_j \in H_1(B_j)$ et d'un cycle $\gamma_k \in H_1(B_k)$. Le résultat de Stein, que j'énonce pour simplifier dans le cas $n = 2$, est le suivant: le problème de Cousin possède une solution holomorphe

[16] A. Weil, *loc. cit.* en (11); K. Kodaira, Chapter V du cours de G. De Rham sur les intégrales harmoniques, Institute for Advanced Study, Princeton, 1950.

[17] Voir B. Eckmann et H. Guggenheimer, C. R. Acad. Sci. Paris t. 229 (4) (1949) pp. 577–579.

[18] Voir §4 du mémoire cité en 15.

$f(z_1, z_2)$, uniforme par rapport à z_1 (z_2 étant fixé), et qui, pour z_1 fixé, est multipliée par un facteur $f_\gamma(z_1)$ (holomorphe, uniforme, et $\neq 0$) quand z_2 décrit un cycle γ. De plus, Stein montre qu'il existe toujours une donnée de Cousin dont l'invariant $u \in H^2(B)$ soit un élément arbitrairement donné de $H^2(B)$, contrairement à ce qui se passe pour les fonctions thêta: dans le cas d'une fonction thêta, la loi d'intersection du diviseur qu'elle définit, avec les produits $\gamma_j \times \gamma_k$, donne naissance à une forme bilinéaire alternée à $2n$ variables réelles qui n'est pas quelconque, car elle doit être la partie imaginaire d'une forme quadratique hermitienne positive à n variables complexes.

Dans cet ordre d'idées, il se pose de nombreux problèmes que je ne puis même pas mentionner. Je serai heureux si j'ai réussi à vous montrer que de nouveaux domaines s'ouvrent aujourd'hui à la théorie des fonctions analytiques de plusieurs variables. Les résultats déjà obtenus sont encourageants, mais encore assez fragmentaires pour exciter notre curiosité. Dans les recherches qu'ils ne manqueront pas de susciter, l'algèbre moderne aussi bien que la topologie auront leur rôle à jouer. Ainsi s'affirmera, une fois de plus, l'unité de la mathématique.

University of Paris,
Paris, France.

DIFFERENTIAL GEOMETRY OF FIBER BUNDLES

S. S. Chern

This address was given as part of the Conference in Topology, see Volume 2, page 397.

RECENT PROGRESS IN THE GEOMETRY OF NUMBERS

H. DAVENPORT

In this address I shall endeavour to give an account of some of the work which has been done on the geometry of numbers since the time of the last Congress. At that Congress, Professor Mordell gave an address[1] on *Minkowski's theorems and hypotheses concerning linear forms,* in which he discussed two unproved conjectures of Minkowski; and it may be appropriate if I begin today by mentioning the progress that has since been made in connection with these two conjectures.

The first conjecture concerns what we should now call *the critical lattices of an n-dimensional cube*. A lattice in n dimensional space consists of all points $(x_1, \cdots, x_n)$ given by n linear forms in n variables $u_1, \cdots, u_n$ which take all integral values. In other words, a lattice is an affine transform of the set of all points with integral coordinates. Given a bounded region K in n-dimensional space, which contains the origin 0 as an inner point, we consider all those lattices which have no point except 0 in the interior of K. The lower bound of their determinants is called the *critical determinant* of K, denoted by $\Delta(K)$, and the lattices for which this lower bound is attained are called the *critical lattices* of K. In the case when K is the unit cube defined by

$$| x_1 | \leqq 1, \cdots, | x_n | \leqq 1,$$

Minkowski proved that $\Delta(K) = 1$, and he made a conjecture about the nature of the critical lattices of K. The conjecture was that these lattices are given essentially by linear forms with triangular matrices and with unit elements in the principal diagonal.

There is an alternative way of formulating the conjecture, which is perhaps more readily grasped. A lattice is a critical lattice for the cube K if and only if it provides a space-filling for the cube $K/2$, so that when the cube $K/2$ is translated to have its centre at each lattice point, the resulting cubes exactly fill up space. The alternative form of the conjecture is that such a space-filling must necessarily be built up in layers, parallel to one of the coordinate planes. The cubes in any one layer constitute an $(n - 1)$-dimensional space-filling, and the successive layers are obtained by applying repeatedly a certain translation. In two or three dimensions, it is easy to see that this must be so, and proofs were given by Minkowski himself. The general conjecture, after resisting the efforts of all who had worked on it from the time of Minkowski, was eventually proved by Hajós in 1940.[2] His proof is based on an interpretation of the space-filling situation in terms of group-algebra.

[1] L. J. Mordell, Comptes Rendus du Congrès International des Mathématiciens, Oslo 1936, vol. 1, pp. 226–238.

[2] G. Hajós, Math. Zeit. vol. 47 (1941) pp. 427–467. Some simplifications in the proof have been made by L. Rédei, Acta Univ. Szeged. vol. 13 (1949) pp. 21–35, or Comment. Math. Helv. vol. 23 (1949) pp. 272–281, and T. Szele, Publicationes Math. (Debrecen) vol. 1 (1949) pp. 56–62.

The second of Minkowski's conjectures relates to quite a different question. Suppose we have any lattice of determinant 1 in n-dimensional space. The conjecture is that for any point $(c_1, \cdots, c_n)$ in the space there is a lattice point $(x_1, \cdots, x_n)$ such that

$$(1) \qquad | (x_1 - c_1) \cdots (x_n - c_n) | \leqq 1/2^n.$$

In other words, the product of n nonhomogeneous linear forms of determinant 1 always assumes a value $\leqq 1/2^n$, for integral values of the variables.

Minkowski himself proved this in the case $n = 2$, and many other proofs for that case are known. The case $n = 3$ was settled by Remak[3] in 1923, and the case $n = 4$ by Dyson[4] in 1946. In these cases, rather more is proved than is actually asserted in Minkowski's conjecture. What is proved is that it is possible to distort the lattice, by a transformation of the form

$$(2) \qquad y_1 = \lambda_1 x_1, \cdots, y_n = \lambda_n x_n \qquad (\lambda_1 \cdots \lambda_n = 1),$$

so that the new lattice in y space has its points distributed throughout the space with a certain uniformity. The precise meaning of this is that for any point $(d_1, \cdots, d_n)$ there is a lattice point $(y_1, \cdots, y_n)$ such that

$$(3) \qquad (y_1 - d_1)^2 + \cdots + (y_n - d_n)^2 \leqq \tfrac{1}{4}n.$$

Taking $d_1 = \lambda_1 c_1, \cdots, d_n = \lambda_n c_n$, it follows by the inequality of the arithmetic and geometric means that (1) holds. The method of choosing the multipliers $\lambda_1, \cdots, \lambda_n$ is to select them so that the quadratic form

$$(4) \qquad (\lambda_1 x_1)^2 + \cdots + (\lambda_n x_n)^2$$

attains its minimum value at n independent lattice points in x-space. In other words, the distorted lattice has n independent lattice points at the same minimal distance from the origin. Thus the proofs when $n = 3$ and $n = 4$ fall into two stages, the first stage being the proof that multipliers $\lambda_1, \cdots, \lambda_n$ with this property exist, the second stage being the proof that when they have been so chosen, the lattice in y-space has the property formulated in (3). Both stages become very difficult when $n = 4$, but more especially the first stage. Dyson establishes the existence of the λ's by topological arguments in the three-dimensional space of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, treated as homogeneous coordinates. These arguments are subtle and novel, at least in this connection.

The question whether the situation in the n-dimensional space is similar, if one is content with less exact constants, was stressed by Mordell in his Oslo lecture, although he expressed it in a slightly different form. This question was answered affirmatively by Siegel[5] in 1937. He proved that it is possible to choose the multipliers $\lambda_1, \cdots, \lambda_n$ so that the successive minima of the quadratic form (4), though not necessarily equal, have their ratios bounded by constants

[3] R. Remak, Math. Zeit. vol. 17 (1923) pp. 1–34; vol. 18 (1923) pp. 173–200. A simpler proof was given by H. Davenport, J. London Math. Soc. vol. 14 (1939) pp. 47–51.

[4] F. J. Dyson, Ann. of Math. vol. 49 (1948) pp. 75–81, 82–109.

[5] See H. Davenport, Acta Arithmetica vol. 2 (1937) pp. 262–265.

depending only on n. If the λ's are chosen in this way, results analogous to (3) and (1) follow, but with large constants, depending on n, on the right-hand sides.

As early as 1934, Tschebotareff[6] had made an important contribution to Minkowski's problem, using quite a different method. His work, however, did not receive adequate publicity until several years later. What he proved was that Minkowski's conjecture, expressed by (1), is true if instead of $1/2^n$ on the right, there stands any number greater than $1/(2^{1/2})^n$. The proof is simple and ingenious, though it does not establish the possibility of distorting the lattice so that it has a property analogous to that expressed by (3). The method underlying Tschebotareff's work has recently been generalized by Macbeath,[7] and expressed by him in the form of a general principle, applicable to a variety of problems.

In spite of all the work I have mentioned, Minkowski's conjecture remains unproved, and is an outstanding challenge to all who are interested in the subject.

There are particular cases which invite detailed study. The case when the linear forms $x_1, \cdots, x_n$ have rational coefficients is almost trivial, since then one can transform them into diagonal form, and satisfy (1) by choosing the integral variables $u_1, \cdots, u_n$ one at a time. A case which gives rise to interesting considerations is that in which the linear forms $x_1, \cdots, x_n$ correspond to a totally real algebraic number-field of degree n. We take x_1 to be the linear form which represents the general algebraic integer of the field, and $x_2, \cdots, x_n$ to be the algebraically conjugate linear forms. The problem then is: how small can one make $|(x_1 - c_1) \cdots (x_n - c_n)|$ for arbitrary real numbers $c_1, \cdots, c_n$? Allowing for the fact that the determinant of the linear forms is no longer 1 but $d^{1/2}$, where d is the discriminant of the field, Minkowski's conjecture (known to be true for $n \leqq 4$) would imply that we can make the product less than $2^{-n}d^{1/2}$. For any particular number-field F there will be a best possible estimate $\mathfrak{M}(F)$ for the product, and this constant has been determined for several quadratic and cubic fields.[8] The problem is one which, if appropriately formulated, retains its significance when F is not totally real, although Minkowski's conjecture is no longer relevant and $\mathfrak{M}(F)$ is no longer of the order $d^{1/2}$.

The problem just mentioned (that of determining, or estimating, the number $\mathfrak{M}(F)$) has a bearing on the question of the validity or invalidity of Euclid's algorithm in the field F. If $\mathfrak{M}(F) > 1$ for a particular field F, then there exist

[6] N. Tschebotareff, Scientific Notes of Kazan University vol. 94 Part 7 (1934) pp. 4–16; also in Vierteljahresschrift der Naturforschenden Gesellschaft in Zürich vol. 85 (1940) pp. 27–30. The proof is also accessible in Hardy and Wright, *Introduction to the theory of numbers*, 2d ed., 1945, p. 396.

[7] A. M. Macbeath, in his Princeton Ph.D. dissertation, not yet published.

[8] The literature for quadratic fields is somewhat extensive. For cubic fields, see H. Davenport, Proc. Cambridge Philos. Soc. vol. 43 (1947) pp. 137–152; A. V. Prasad, Neder. Akad. Wetensch. vol. 52 (1949) pp. 240–250, 338–350; L. E. Clarke, Quart. J. Math. Oxford Ser. (in course of publication).

numbers $c_1, \cdots, c_n$ (all real if F is totally real, and otherwise real or complex in correspondence with F and its conjugate fields) such that

$$| (x_1 - c_1) \cdots (x_n - c_n) | > 1$$

for all integers x_1 of F. If these numbers can be so chosen that c_1 is a nonintegral number of F, and $c_2, \cdots, c_n$ are the algebraic conjugates of c_1, the product becomes the norm of $x_1 - c_1$, and it follows that Euclid's algorithm does not hold in the field F. For real quadratic fields, I proved[9] in 1948 that

$$\mathfrak{M}(F) > \frac{1}{128} d^{1/2},$$

and further that this remains true if c_1, c_2 are restricted in the way described above. It follows that Euclid's algorithm cannot hold in a real quadratic field if $d > (128)^2$. This result led to the final enumeration[10] of all the real quadratic fields in which the algorithm holds. I have since proved a similar inequality for $\mathfrak{M}(F)$ for two other types of field, namely (a) cubic fields of negative discriminant, and (b) totally complex quartic fields; and have proved in this way that there are only a finite number of such fields for which Euclid's algorithm holds.[11]

We now turn from Minkowski's two conjectures, and questions connected with them, to the geometry of numbers as a whole. There have been so many developments in the subject in the last fourteen years that we can only mention briefly a selection of them. The central problem of the subject is that of finding the critical determinant $\Delta(K)$ of a given region K, which we suppose for the present to be bounded. It is convenient to consider the ratio $\Delta(K)/V(K)$, where $V(K)$ is the volume of K, since the ratio is invariant under affine transformations. Two general theorems concerning this ratio are known. The first is the fundamental theorem of Minkowski, which states that

$$\frac{\Delta(K)}{V(K)} \geqq \frac{1}{2^n}$$

provided K is convex and symmetrical about 0. The reason for the theorem lies in the equivalence (for convex bodies) between the problem of the critical determinant and the problem of closest packing; and the inequality simply expresses the fact that the density of closest packing cannot exceed 1.

The second theorem is an inequality in the opposite direction, which was

[9] H. Davenport, Proc. London Math. Soc. (in course of publication). See also H. Davenport, Quart. J. Math. Oxford Ser. (2) vol. 1 (1950) pp. 54–62.

[10] See H. Chatland and H. Davenport, Canadian Journal of Mathematics vol. 2 (1950) pp. 289–296, and K. Inkeri, Annales Academiae Scientiarum Fennicae. Ser. A vol. 41 (1948) pp. 5–34.

[11] H. Davenport, Acta Math. vol. 84 (1950) pp. 159–179 and Trans. Amer. Math. Soc. vol. 68 (1950) pp. 508–532.

stated by Minkowski in his last paper on the geometry of numbers, but without proof. It is that

$$\frac{\Delta(K)}{V(K)} \leqq \frac{1}{2\zeta(n)},$$

where $\zeta(n) = 1 + 2^{-n} + 3^{-n} + \cdots$, provided that K is a symmetrical star body relative to 0. The first general proof was given by Hlawka[12] in 1943, and simpler proofs were given later, first by Siegel[13] and then by Rogers.[14] The result is essentially an existence theorem: it asserts, in effect, the existence of a lattice having no point except 0 in K, and having a determinant which is bounded in terms of $V(K)$. All the proofs depend on "averaging" arguments, applied to lattices of a given determinant.

We have, then, these two general estimates for $\Delta(K)/V(K)$. Each of them can be improved by imposing further restrictions on K. In the case of an n-dimensional sphere, the lower estimate was greatly improved by Blichfeldt in two famous papers,[15] but even in this case there is a wide gap between the best known estimates from above and from below.

Minkowski developed a systematic procedure for determining $\Delta(K)$ when K is a two-dimensional or three-dimensional region which is convex and symmetrical about 0. Mahler[16] has adapted this procedure so that it applies to bounded star regions in the plane, not necessarily convex, and detailed results have been worked out by him and by his pupils for many particular regions, both convex and nonconvex.

Let us now turn to unbounded regions, which are in various ways more interesting than bounded regions. All the regions we shall consider will be star bodies relative to 0 and symmetrical about 0. We again define the critical determinant of a region K to be the lower bound of the determinants of all lattices which have no point except 0 in the interior of K. It may be, however, that *every* lattice has a point other than 0 in the interior of K; in this case we make the natural convention that $\Delta(K) = \infty$. One of the most interesting problems of the subject is to find criteria which will decide whether $\Delta(K)$ is finite or infinite. One obvious fact is that $\Delta(K)$ will be infinite if K contains convex regions, symmetrical about 0, of arbitrarily large volume, but this does not say much.

There are a few unbounded regions for which $\Delta(K)$ is known already from classical results concerning the corresponding Diophantine inequalities. These are the regions which correspond to indefinite quadratic forms in two, three, or

[12] E. Hlawka, Math. Zeit. vol. 49 (1943) pp. 285–312.

[13] C. L. Siegel, Ann. of Math. vol. 46 (1945) pp. 340–347.

[14] C. A. Rogers, Ann. of Math. vol. 48 (1947) pp. 996–1002.

[15] H. F. Blichfeldt, Trans. Amer. Math. Soc. vol. 15 (1914) pp. 227–235, and Math. Ann. vol. 101 (1929) pp. 605–608.

[16] K. Mahler, J. London Math. Soc. vol. 17 (1942) pp. 130–133, and other papers.

four variables; namely

$$| xy | \leqq 1 \qquad \text{in two dimensions,}$$

$$| x^2 + y^2 - z^2 | \leqq 1 \qquad \text{in three dimensions,}$$

$$\left.\begin{array}{l} | x^2 + y^2 + z^2 - w^2 | \leqq 1 \\ | x^2 + y^2 - z^2 - w^2 | \leqq 1 \end{array}\right\} \text{in four dimensions.}$$

The critical determinants of all these regions follow at once from classical work of Korkine and Zolotareff, and Markoff, on the minima of indefinite quadratic forms.[17] When we come to the corresponding regions in five or more dimensions, we meet precisely the problem mentioned above. It is conjectured that *every* lattice, no matter how large its determinant may be, has a point other than 0 in the n-dimensional region

$$| \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_n^2 | \leqq 1,$$

where the signs are fixed but not all the same, and $n \geqq 5$. In the cases $n \leqq 4$, the critical lattices of the regions are lattices for which the corresponding quadratic form (xy or $x^2 + y^2 - z^2$, etc.) becomes a certain special quadratic form with integral coefficients in the u variables, namely the form of least determinant which does not represent zero. Now there is a theorem of Meyer which states that any indefinite form with integral coefficients in five or more variables necessarily represents zero, and this is the main ground for conjecturing that the critical determinant of the n-dimensional region is infinite when $n \geqq 5$. Expressed arithmetically, the conjecture is that any indefinite quadratic form in five of more variables assumes values which are arbitrarily small numerically, for integral values of the variables, not all zero. This conjecture presents one of the most interesting unsolved problems in the subject.

In 1937 I found the critical determinant of another unbounded region, namely the three-dimensional region defined by $| xyz | \leqq 1$; and this proved to be the starting point for a good deal of new work. The value of the critical determinant is 7, and the critical lattices are closely related (as indeed was expected) to a particular cubic field. This is the cubic field of least positive discriminant, 49, and is generated by the equation $\theta^3 + \theta^2 - 2\theta - 1 = 0$. If ξ is the linear form which represents the general algebraic integer of this field, the critical lattices for the region in question are given by

$$x = \lambda\xi, \qquad y = \mu\xi', \qquad z = \nu\xi'',$$

where accents denote algebraic conjugates, and λ, μ, ν are any constants with $\lambda\mu\nu = \pm 1$. The linear forms x, y, z have determinant 7, and xyz is a nonzero

[17] For the results concerning quadratic forms, see P. Bachmann, *Die Arithmetik der quadratischen Formen* II, and L. E. Dickson, *Studies in the theory of numbers*. For a simple proof of Markoff's results on binary forms, see J. W. S. Cassels, Ann. of Math. vol. 50 (1949) pp. 676–685. New results on ternary forms have been given by A. Venkov, Bull. Acad. Sci. URSS. Sér. Math. vol. 9 no. 6 (1945).

integer for all values of the integral variables, not all zero. My original proof[18] of these results was complicated, but later I found a much simpler proof[19] on the same general lines.

In 1938 I found the critical determinant of the three-dimensional region defined by

$$| x(y^2 + z^2) | \leqq 1,$$

its value being $(23)^{1/2}/2$. The proof[20] was complicated, and even now no simple direct proof is known. The critical lattices are related in a similar way to a particular cubic field, namely the field of numerically least negative discriminant, -23. This last fact is rather difficult to prove, and has only recently been established[21] by using methods due to Mahler.

The corresponding problems in four dimensions are still unsolved.

In 1940, Mordell found a new method of approach to the three-dimensional regions just discussed. He showed[22] that the results could be deduced from similar results for certain two-dimensional regions. The two-dimensional region required for $| xyz | \leqq 1$ was that given by

$$| xy(x + y) | \leqq 1,$$

which has the critical determinant $7^{1/3}$. That required for $| x(y^2 + z^2) | \leqq 1$ was given by

$$| x(x^2 + y^2) | \leqq 1,$$

which has the critical determinant $((23)^{1/2}/2)^{1/3}$. Two-dimensional regions are naturally easier to deal with than three-dimensional regions, although in the present case the gain in simplicity is partially compensated for by the fact that the two-dimensional regions do not possess the continuous infinity of automorphisms which is the main feature of the three-dimensional regions.

Mordell gave several proofs[23] of his two-dimensional results, and the methods which he evolved for the purpose proved to be applicable to other regions in the plane, some of them being regions of a certain degree of generality. It is a striking fact that it is often easier to find the critical determinant of a nonconvex region than of a convex region whose definition is superficially similar. For example, Mordell found the critical determinant of the nonconvex region $| x |^p + | y |^p \leqq 1$

[18] H. Davenport, Proc. London Math. Soc. (2) vol. 44 (1938) pp. 412–431.

[19] H. Davenport, J. London Math. Soc. vol. 16 (1941) pp. 98–101. See also Proc. Cambridge Philos. Soc. vol. 39 (1943) pp. 1–21.

[20] H. Davenport, Proc. London Math. Soc. (2) vol. 45 (1939) pp. 98–125.

[21] See H. Davenport and C. A. Rogers, Phil. Trans. Roy. Soc. London Ser. A vol. 242 (1950) pp. 311–344.

[22] L. J. Mordell, J. London Math. Soc. vol. 16 (1941) pp. 83–85; vol. 17 (1942) pp. 107–115.

[23] L. J. Mordell, Proc. London Math. Soc. (2) vol. 48 (1943) pp. 198–228, 339–390; J. London Math. Soc. vol. 18 (1943) pp. 201–217, vol. 19 (1944) pp. 92–99.

for $0.33 < p < 1$, whereas our knowledge about the convex region defined by the same inequality when $p > 1$ is still only fragmentary. The explanation seems to be that the critical lattices of a nonconvex region often bear a simple relation to the shape of the region.

In 1946, Mahler[24] developed a general theory of star bodies in n-dimensional space, in which he clarified much that was previously obscure, and proved many general theorems, some of a positive and some of a negative character. He also introduced much of the terminology which we have been using throughout the discussion. One of his basic results is that if $\Delta(K)$ is finite, then certainly there exists at least one critical lattice for K. A critical lattice need not have any point on the boundary of K, but must of course have a point in the region $(1 + \epsilon)K$ for any $\epsilon > 0$. Mahler also proved that each of the special unbounded bodies we have mentioned (arising from quadratic forms or from products of coordinates) is *boundedly reducible*, that is, contains a bounded star body with the same critical determinant. Thus, for example, the body defined by

$$| xyz | \leqq 1, \qquad x^2 + y^2 + z^2 \leqq R^2$$

has the critical determinant 7, if R is sufficiently large. The proofs that these various special bodies are boundedly reducible all depend, however, on information already available about the critical determinant and critical lattices of the body. A general method for proving that a body is boundedly reducible would be of great interest. Mahler proposes many problems in his paper, and although some of these have since been solved, an ample number remain for the attention of future investigators.

The work of Mahler has been carried further in some directions by Rogers and myself.[25] We proved, for various unbounded star bodies, that any lattice of determinant less than $\Delta(K)$ must have not only one point but an infinity of points in the interior of K. We have also[26] drawn attention to some facts which are implicit in Mahler's work, concerning bodies such as $| xyz | \leqq 1$. Instead of reducing this to a bounded body, there are other ways in which it can be modified without changing the value of the critical determinant: for example, the region defined by

$$| x | (| x | + | y |) (| x | + | y | + | z |) \leqq 1$$

has the same critical determinant, namely 7, and the same critical lattices as the region defined by $| xyz | \leqq 1$. Mahler had already shown[24] that this was true for the region defined by

$$x^2y^2(x^2 + y^2 + z^2) \leqq 1,$$

[24] K. Mahler, Proc. Roy. Soc. London. Ser. A vol. 187 (1946) pp. 151–187; and Neder. Akad. Wetensch. vol. 49 (1946) pp. 331–343, 444–454, 524–532, 622–631.

[25] H. Davenport and C. A. Rogers, Phil. Trans. Roy. Soc. London. Ser. A vol. 242 (1950) pp. 311–344.

[26] H. Davenport and C. A. Rogers, J. London Math. Soc. vol. 24 (1949) pp. 271–280.

and had given this as a simple example of a region whose critical lattices have no points on the boundary. The basic reason for the validity of these results, which seem at first sight curious, lies in the simple form of the automorphisms of the region $|xyz| \leqq 1$:

$$x = \lambda x', \qquad y = \mu y', \qquad z = \nu z' \qquad\qquad (\lambda\mu\nu = \pm 1).$$

University College, University of London,
London, England.

ROTATING UNIVERSES IN GENERAL RELATIVITY THEORY

KURT GÖDEL

In this lecture I am setting forth the main results (for the most part without proofs) to which my investigations on rotating universes have led me so far.

1. Definition of the type of rotatory solutions to be considered. I am starting from the relativistic field equations:[1]

$$R_{ik} - \frac{1}{2}g_{ik}R = T_{ik} - \lambda g_{ik} \tag{1}$$

and am assuming that:

1) the relative velocity of masses (i.e. galactic systems) close to each other is small compared with c.

2) no other forces except gravitation come into play.

Under these assumptions T_{ik} takes on the form:

$$T_{ik} = \rho v_i v_k \tag{2}$$

where:

$$\rho > 0, \tag{3}$$

$$g^{ik} v_i v_k = -1, \tag{4}$$

and, of course:

(5) The signature of g_{ik} is $+2$.

The local angular velocity of matter relative to the compass of inertia can be represented by the following vector ω (which is always orthogonal on v):

$$\omega^i = \frac{\epsilon^{iklm}}{12(-g)^{1/2}} a_{klm} \tag{6}$$

where the skew-symmetric tensor a_{klm} is defined by:

$$a_{klm} = v_k\left(\frac{\partial v_l}{\partial x_m} - \frac{\partial v_m}{\partial x_l}\right) + v_l\left(\frac{\partial v_m}{\partial x_k} - \frac{\partial v_k}{\partial x_m}\right) + v_m\left(\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k}\right). \tag{7}$$

That ω represents the angular velocity relative to the compass of inertia is seen as follows: In a coordinate system which, in its origin, is geodesic and normal, and in whose origin matter is at rest (i.e. for which in O: $\partial g_{ik}/\partial x_l = 0$, $g_{ik} = \eta_{ik}$, $v^4 = 1$, $v^i = 0$ for $i \neq 4$),[2] one obtains for ω^i in O:

$$\omega^1 = \frac{1}{2}\left(\frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}\left(\frac{v^3}{v^4}\right) - \frac{\partial}{\partial x_3}\left(\frac{v^2}{v^4}\right)\right), \quad \text{etc.} \tag{8}$$

$$\omega^4 = 0. \tag{9}$$

[1] I am supposing that such measuring units are introduced as make $c = 1$, $8\pi\kappa/c^2 = 1$.

[2] A coordinate system satisfying the first two conditions may fittingly be called a "local inertial system".

In such a coordinate system, however, since parallel-displacement (in its origin) means constancy of the components, the angular velocity relative to the compass of inertia, in O, is given by the same expressions as in Newtonean physics, i.e. the right-hand sides of (8) are its components. *Evidently ω is the only vector the first 3 components of which, in the particular coordinate systems defined, coincide with the angular velocity computed as in Newtonean physics and the 4th component is 0.*

Any Riemann 4-space with some ρ, v_i defined in it, which everywhere satisfies the conditions (1)–(5) and permits of no extension free from singularities, and for which, moreover, ω is continuous and $\neq 0$ in every point, represents a rotating universe. However, in the sequel I am chiefly concerned with solutions satisfying the following three further postulates (suggested both by observation and theory):

I. The solution is to be homogeneous in space (i.e. for any two world lines of matter l, m there is to exist a transformation of the solution into itself which carries l into m).

II. Space is to be finite (i.e. the topological space whose points are the world lines of matter is to be closed, i.e. compact).

III. ρ is not to be a constant.

Postulate III is indispensable also for rotating universes, since it can be proved that a *red-shift which, for small distances, increases linearly with the distance implies an expansion, no matter whether the universe rotates or not.*[3]

As to the question of the existence of rotating solutions satisfying the postulates I, II, III, cf. §5.

2. Some general properties of these solutions. In view of III the equation ρ = const. defines a one-parameter system of 3-spaces. *In rotating universes these 3-spaces of constant density cannot be orthogonal on the world lines of matter.* This follows immediately from the fact that $a_{klm} = 0$ is the necessary and sufficient condition for the existence of *any* system of 3-spaces orthogonal on a vector field v.

The inclination of the world lines of matter toward the spaces of constant density yields *a directly observable necessary and sufficient criterion for the rotation of an expanding spatially homogeneous and finite universe:* namely, *for sufficiently great distances, there must be more galaxies in one half of the sky than in the other half.*

In the first approximation, i.e., for solutions differing little from one spatially isotropic, the magnitude of this effect is given by the following theorem: *If N_1, N_2 are the numbers of galaxies in the two hemispheres into which a spatial sphere*[4] *of radius r (small compared with the world radius R) is decomposed by a*

[3] Provided, of course, that the atomic constants do not vary in time and space, or, to be more exact, provided that the dimensionless numbers definable in terms of the constants of nature (such as e^2/hc) are the same everywhere.

[4] I.e., one situated in a 3-space orthogonal on v at the point under consideration.

plane orthogonal on ω, *then:*

$$\frac{|N_1 - N_2|}{N_1 + N_2} = \frac{9}{8} \cdot \frac{|\omega| rRh}{c^2} \tag{10}$$

where h *is Hubble's constant* $(=\dot{R}/R)$.

For plausible values of the constants (where ω is estimated from the velocity of rotation of the galaxies[5]) this effect is extremely small. But the uncertainty in the knowledge of the constants is too great for drawing any definitive conclusions.

The group of transformations existing owing to I evidently carries each of the spaces ρ = const. into itself, and therefore (the case of isotropy being excluded) can only have 3 or 4 parameters.[6] The number 4 (i.e., the case of rotational symmetry) cannot occur either. *There exist no rotationally symmetric rotating universes satisfying the conditions stated in* §1.[7] The only symmetry around one point which can occur is that of one rotation by π. This case will be referred to as the symmetric one.

In any case the group of transformations must be 3-parameter. Since moreover, owing to II, it must be compact, and since (as can easily be shown) it cannot be commutative in rotating universes,[8] it follows that *the group of transformations of any rotating solution of the type characterized in* §1 *must be isomorphic (as a group of transformations) with the right (or the left) translations of a* 3*-space of constant positive curvature, or with these translations plus certain rotations by an angle* π. Hence also the topological connectivity of space must be that of a spherical or elliptical 3-space.

The metric g_{ik} can be decomposed (relative to the world lines of matter) into a space-metric $\overline{g_{ik}}$ and a time-metric $\overline{\overline{g_{ik}}}$, by defining the spatial distance of two neighbouring points P_1, P_2 to be the orthogonal distance of the two world lines of matter passing through P_1, P_2, and the temporal distance to be the orthogonal projection of P_1P_2 on one of these two lines. This decomposition evidently is exactly that which (in the small) holds for the observers moving along the world lines of matter. It has the following properties:

$$\begin{gathered} \overline{\overline{g_{ik}}} = -v_iv_k, \qquad \overline{g_{ik}} = g_{ik} + v_iv_k, \\ \operatorname{Det}(\overline{g_{ik}}) = \operatorname{Det}(\overline{\overline{g_{ik}}}) = 0. \end{gathered} \tag{11}$$

If the coordinate system is so chosen that the x_4-lines are the world lines of matter and the x_4-coordinate measures the length of these lines, $\overline{g_{ik}}$ takes on the form:

$$\overline{g_{ik}} = \begin{Vmatrix} h_{ik} & 0 \\ 0 & 0 \end{Vmatrix} \tag{12}$$

[5] Cf. my paper in Reviews of Modern Physics vol. 21 (1949) p. 450.

[6] There exists, in every space ρ = const., a positive definite metric which is carried into itself, namely the metric h_{ik} defined below.

[7] This even is true irrespective of postulate II (the finiteness of space).

[8] The reason is that the curl of a vector field invariant under a transitive commutative group vanishes identically.

(where h_{ik} is positive definite) and the Hubble-constant in the space-direction dx^i (orthogonal on v), as measured by an observer moving along with matter, becomes equal to:

$$\frac{1}{2}\frac{\dot{h}_{ik}\, dx^i\, dx^k}{h_{ik}\, dx^i\, dx^k}, \quad \text{where} \quad \dot{h}_{ik} = \frac{\partial h_{ik}}{\partial x_4}.$$

The surface $\dot{h}_{ik}x_i x_k = 1$ in the 3-dimensional subspace, orthogonal on v, of the tangent space, may be called the ellipsoid of expansion or, more generally, the quadric of expansion.

The theorem about the nonexistence of rotationally symmetric solutions,[9] *under the additional hypothesis that the universe contains no closed time-like lines* (cf. §3), can be strengthened to the statement that *the quadric of expansion, at no moment of time, can be rotationally symmetric around* ω. In particular it can never be a sphere, i.e., the expansion is necessarily coupled with a deformation. This even is true for *all* solutions satisfying I–III and gives another directly observable property of the rotating universes of this type.

Moreover the assymmetry of the expansion around ω opens up a possibility for the explanation of the spiral structure of the galaxies. For, if under these circumstances a condensation is formed, the chances are that it will become an oblong body rotating around one of its smaller axes; and such a body, because its outer parts will rotate more slowly, will, in the course of time, be bent into a spiral. It remains to be seen whether a quantitative elaboration of this theory of the formation of spirals will lead to agreement with observation.

3. Rotation and time-metric. The formulae (6), (7), (11) show that it is, in the first place, the time-metric (relative to the observers moving along with matter) which determines the behaviour of the compass of inertia. In fact *a necessary and sufficient condition for a spatially homogeneous universe to rotate is that the local simultaneity of the observers moving along with matter be not integrable* (i.e., do not define a simultaneity in the large). This property of the time-metric in rotating universes is closely connected with the possibility of closed time-like lines.

The latter anomaly, however, occurs only if the angular velocity surpasses a certain limit. This limit, roughly speaking, is that value of $|\omega|$ for which the maximum linear velocity caused by the rotation becomes equal to c; i.e., it is approximately c/R if, at the moment considered, the space-metric in the 3-space ρ = const. does not differ too much from a space of the constant curvature $1/R^2$. *The precise necessary and sufficient condition for the nonexistence of closed time-like lines* (provided that the one-parameter manifold of the spaces ρ =

[9] This theorem makes it very likely that there exist no rotating spatially homogeneous and expanding solutions whatsoever in which the ellipsoid of expansion is *permanently* rotationally symmetric around ω.

const. is not closed) *is that the metric in the spaces of constant density be space-like.*[10] This holds for solutions satisfying all conditions stated in §1.

For these solutions, also, *the nonexistence of closed time-like lines is equivalent with the existence of a "world-time"*, where by a world-time we mean an assignment of a real number t to every space-time point, which has the property that t always increases if one moves along a time-like line in its positive direction.[11] If in addition any two 3-spaces of simultaneity are equidistant and the difference of t is their distance, one may call it a *metric world-time.* If the spaces of constant density are space-like, a metric world-time can be defined by taking these 3-spaces as spaces of simultaneity. Evidently (up to transformations $\bar{t} = f(t)$) this is the only world-time invariant under the group of transformations of the solution.

4. Behaviour of the angular velocity in the course of the expansion. No matter whether postulates I–III are satisfied or not, the temporal change of ω is described by the following theorem: *In a coordinate system in which the x_4-lines are the world lines of matter, $g_{44} = -1$ everywhere, and moreover $g_{i4} = 0$ (for $i \neq 4$) on the X_4-axis, one has along the whole X_4-axis:*

$$\omega^i(-g)^{1/2} = \omega^i h^{1/2} = \text{const.} \qquad (i = 1, 2, 3). \tag{13}$$

The proof can be given in a few lines: Evidently $v^4 = 1$, $v^i = 0$ (for $i \neq 4$) everywhere; hence: $v_i = g_{i4}$. Substituting these values of v_i in (7), one obtains on X_4:

$$a_{4ik} = \frac{\partial g_{4k}}{\partial x_i} - \frac{\partial g_{4i}}{\partial x_k}, \qquad a_{123} = 0. \tag{14}$$

But $\partial g_{4i}/\partial x_4 = 0$ (because the x_4-lines are geodesics and $g_{44} = -1$). Hence by (14), $\partial a_{klm}/\partial x_4 = 0$ on X_4. Hence by (6) also, $\partial(\omega^i(-g)^{1/2})/\partial x_4 = 0$ on X_4.

The equation (13) means two things:

A. that the vector ω (or, to be more exact, the lines l_ω whose tangent everywhere has the direction ω) permanently connect the same particles with each other,

B. that the absolute value $|\omega|$ increases or decreases in proportion to the contraction or expansion of matter orthogonal on ω, where this contraction or expansion is measured by the area of the intersection of an infinitesimal spatial cylinder[4] around l_ω (permanently including the same particles) with a surface orthogonal on l_ω.

Since in the proof of (13) nothing was used except the fact that the world

[10] This condition, too, means that at the border separating the two cases the linear velocity caused by the rotation becomes equal to c, if by this linear velocity is understood the velocity of matter relative to the orthogonals on the spaces of constant density.

[11] A time-like vector is positive if it is contained in the same half of the light-cone as the vector v.

lines of matter are geodesics (and in particular the homogeneity of space was not used), (13), and therefore A, B, also describe the behaviour of the angular velocity, if condensations are formed under the influence of gravitation;[12] i.e., $|\omega|$, under these circumstances, increases by the same law as in Newtonean mechanics.

The direction of ω, even in a homogeneous universe, need not be displaced parallel to itself along the world lines of matter. *The necessary and sufficient condition for it to be displaced parallel at a certain moment is that it coincide with one of the principal axes of the quadric of expansion.* For, if P, Q are two neighbouring particles connected by ω, then, only under the condition just formulated, the direction PQ at the given moment, will be at rest relative to the compass of inertia (in order to see this one only has to introduce the local inertial system defined in §1 (cf. footnote 2) and then argue exactly as in Newtonean physics). Since however (because of A) the direction of ω coincides *permanently* with the direction of PQ, the same condition applies for the direction of ω. This condition however, in general, is not satisfied (only in the symmetric case it is always satisfied).

The fact that the direction of ω need not be displaced parallel to itself might be the reason for the irregular distribution of the directions of the axes of rotation of the galaxies (which at first sight seems to contradict an explanation of the rotation of the galaxies from a rotation of the universe). For, if the axis of rotation of the universe is not displaced parallel, the direction of the angular momentum of a galaxy will depend on the moment of time at which it was formed.

5. Existence theorems. It can be shown that, *for any value of* λ (*including* 0), *there exist* ∞^8 *rotating solutions satisfying all conditions stated in* §1. *The same is true if in addition it is required that a world-time should exist* (*or should not exist*). The value of the angular velocity is quite arbitrary, even if ρ and the mean world radius (at the moment under consideration) are given. In particular, there exist rotating solutions with $\lambda = 0$ which differ arbitrarily little from the spatially isotropic solution with $\lambda = 0$.

Thus the problem arises of distinguishing, by properties of symmetry or simplicity, certain solutions in this vast manifold of solutions. E.g., one might try to require that the universe should expand from one point and contract to one point.

6. Method of proof. The method of proof by which the results given above were obtained is based on postulate I of §1. This postulate implies that all world lines of matter (and all orthogonals on the spaces of constant density) are equivalent with each other. It is, therefore, sufficient to confine the consideration to one

[12] Of course, only as long as the gas and radiation pressure remain small enough to be neglected.

uch world line (or one such orthogonal). This reduces the problem to a system of *rdinary* differential equations.

Moreover, this system of differential equations can be derived from a Hamil-onean principle, i.e., it is a problem of analytical mechanics with a finite number f degrees of freedom. The equations of relativity theory, however, assign definite alues to the integrals of energy and momentum, so that the relativistic problem ; a little more special than the corresponding one of analytical mechanics.

The symmetric case, by means of the integrals of momentum, can be reduced o a problem with three degrees of freedom (g_1, g_2, g_3), whose Lagrangean function eads as follows:

$$15) \quad \left\{\sum_{i<k} \frac{\dot{g}_i \dot{g}_k}{g_i g_k} + \frac{1}{g}\left[2\sum_i g_i^2 - \left(\sum_i g_i\right)^2\right] + \frac{V^2}{g_1(g_2 - g_3)^2}\right\} g^{1/2} + 2\left(1 + \frac{V^2}{g_1}\right)^{1/2}$$

'here $g = g_1 g_2 g_3$ and V is a constant which determines the velocity of rotation. 'he general case can be reduced to a system of differential equations of the th order.

7. Stationary rotating solutions. It might be suspected that the desired par-icular solutions (cf. §5 above) will have a close relationship to the stationary omogeneous solutions, and it is therefore of interest to investigate these, too. ;y a stationary homogeneous solution we mean one whose group, for any two oints P, Q of the whole 4-space, contains transformations carrying P into Q.

These solutions can all be determined and expressed by elementary functions. 'ne thus obtains the following results:

1. *There exist no stationary homogeneous solutions with* $\lambda = 0$.

2. *There exist rotating stationary homogeneous solutions with finite space, no 'osed time-like lines, and* $\lambda > 0$; in particular also such as differ arbitrarily ttle from Einstein's static universe.

The world lines of matter in these solutions, however, are not equidistant: eighbouring particles of matter, relative to the compass of inertia, rotate round each other, not in circles, but in ellipses (or, to be more exact, in rotating llipses).

Institute for Advanced Study,
Princeton, N. J., U. S. A.

THE TOPOLOGICAL INVARIANTS OF ALGEBRAIC VARIETIES

W. V. D. Hodge

1. Since the origins of the birational geometry of algebraic varieties can be traced back to Riemann's theory of algebraic functions, it is not surprising that topological considerations have played a considerable role in the theory of algebraic varieties defined over the field of complex numbers. For instance in the theory of algebraic functions of two complex variables, due mainly to Picard, topological considerations are used extensively, and in the more geometrical theory of surfaces, due to the Italian school of geometers, many arguments are based essentially on topological methods. But in all this work the topology is subservient to function-theoretic or geometrical ends, and when Lefschetz's memoir [9][1] of 1921 appeared, followed, in 1924, by his Borel Tract, *L'analyse situs et la géometrie algébrique*, a new chapter in the theory of algebraic varieties was opened in which the topological properties of varieties are given equal status with the geometrical and function-theoretic properties. The researches thus initiated have now taken their place in the general theory of complex manifolds and have made important contributions to the problem of classifying these manifolds. In this lecture, I propose to give some account of the present state of our knowledge of the topological properties of algebraic varieties and of the contributions of this theory to the theory of complex manifolds.

What I have to say about algebraic varieties is valid, strictly speaking, only for algebraic varieties in complex projective space which are without singular points. A few of the results may be generalised to apply to varieties with singular points of a sufficiently simple character, but in what follows we shall not consider such generalisations, and we shall henceforth assume that all the varieties considered are non-singular.

2. The investigations described in Lefschetz's Borel Tract fall into two classes; in the first, there is a straightforward study of an algebraic variety V_m, of m dimensions, as a topological space, and in the second, the variety is investigated as a complex differentiable manifold. In the investigation of the purely topological properties of V_m, the fact that there exist systems of algebraic subvarieties on V_m is fundamental, and the whole of Lefschetz's investigation depends on the selection of a suitable system. Broadly speaking, any system $| V_{m-1} |$ of varieties on V_m which correspond to the hyperplane sections of a variety in bi-regular birational correspondence with V_m will serve, though this is, in fact, a more severe restriction than is necessary. We shall refer to the system selected as the selected "allowable system". By means of an allowable system $| V_{m-1} |$ it is possible to determine the topological properties of V_m from a knowledge of the geometrical and topological properties of $| V_{m-1} |$. In this way, Lefschetz is able to establish a

[1] Numbers in brackets refer to the references at the end of the paper.

number of general theorems on the topology of algebraic varieties. Not only is it possible to compute the homology groups of V_m and to establish such general theorems as

(a) the Betti numbers R_p of even dimension of an algebraic variety are always positive;

(b) the Betti numbers of odd dimension are always even;

(c) for $p \leqq m$, $R_p \geqq R_{p-2}$;

but it is possible to classify the p-cycles of V_m into subsets in a significant way. This classification is not an absolute classification, but is relative to the selected allowable system. We denote by V_r the intersection of $m - r$ general varieties V_{m-1} of the selected allowable system, or the Riemannian manifold of $2r$ real dimensions corresponding to it. The results obtained by Lefschetz, or directly deducible from these, are as follows.

I. If $p \leqq m$ and Γ_q is any q-cycle of V_m $(q \leqq p)$, there is a q-cycle in V_p homologous to Γ_q.

II. Every homology between the q-cycles of V_m lying in V_p $(p > q)$ which holds in V_m also holds in V_p.

III. If $p \leqq m$, a basis for the p-cycles of V_m can be chosen as follows:

(0) $R_p - R_{p-2}$ independent p-cycles lying in V_p, no combination of which is homologous to a cycle in V_{p-1}. Cycles homologous to any combination of these p-cycles are called effective p-cycles;

(i) $R_{p-2} - R_{p-4}$ independent p-cycles lying in V_{p-1}, no combination of which is homologous to a cycle in V_{p-2}. Cycles homologous to any combination of these p-cycles are said to be *ineffective* of rank 1;

. . . .

(q) $q = [p/2]$, R_{p-2q} independent p-cycles lying in V_{p-q}. Cycles homologous to any combination of these p-cycles are said to be *ineffective* of rank q.

IV. If Γ_p is an ineffective p-cycle of rank q, the intersection $\Gamma_p \cdot V_{m-r}$ is an ineffective $(p - 2r)$-cycle of rank $q - r$ if $r \leqq q$, and is homologous to zero if $r > q$.

V. Corresponding to each effective p-cycle Γ_p there is a $(2m - p)$-cycle Γ_{2m-p} whose homology class is uniquely defined, with the property $\Gamma_{2m-p} \cdot V_p \sim \Gamma_p$. These dual cycles have the property that if $\Gamma_{2m-p+2r}$ is the dual of an effective $(p - 2r)$-cycle, $\Gamma_{2m-p+2r} \cdot V_{p-r}$ is an ineffective p-cycle of rank r. Further, if $\Gamma_{2m-p+2r}$ and $\Gamma_{2m-p+2s}$ are the duals of effective $(p - 2r)$- and $(p - 2s)$-cycles,

$$\Gamma_{2m-p+2r} \cdot \Gamma_{2m-p+2s} \cdot V_{p-r-s}$$

is homologous to zero if $r \neq s$.

We shall return to these results later. We now consider the properties of V_m as a differentiable manifold. In order to describe Lefschetz's contribution to this theory, we recall that any unmixed algebraic subvariety of dimension d of V_m defines a cycle on V_m of dimensions $2d$. The main results of Lefschetz which I quote refer to the case $m = 2$, that is, algebraic surfaces. The first tells us that if U_1 and U_1' are two curves on the surface V_2, a necessary and sufficient condition that they belong to the same algebraic system of cuves on V_2 is that they be homologous as 2-cycles; this extends a result due to Severi [12] which proves that if U_1 and U_1' are homologous as cycles there must exist a non-zero integer λ such that λU_1 and $\lambda U_1'$ are algebraically equivalent. Using another result due to Severi [12], it is possible to show that there exists a finite subset of 2-cycles, every one of which corresponds to a curve on V_2, which has the property that any curve on the surface is algebraically equivalent to some integral combination of the set—in fact, we get the theory of the base for surfaces. The generalisation of this to subvarieties of dimension d of a variety of m dimensions is not yet complete in the case $d < m - 1$.

The second result of Lefschetz tells us that a necessary and sufficient condition that a 2-cycle Γ_2 in V_2 be *algebraic*, that is, homologous to a cycle defined by a curve on V_2, is that the algebraic double integrals of the first kind attached to V_2 should all have period zero on Γ_2. This result has many geometrical applications; as an example, the theorem, taken with the theory of the base, enables us to develop very easily the Hurwitz theory of correspondences between algebraic curves [10].

It is clearly a matter of great importance to extend Lefschetz's condition for a 2-cycle to be algebraic. The general problem is as follows. It follows from the general topological properties of an algebraic variety, described above, that a p-cycle on V_m is always homologous to a cycle lying in an algebraic subvariety U_q of V_m where $q \leqq p$. If it is homologous to a cycle lying in some subvariety U_{p-k}, we say that it is of rank k. Clearly $k \leqq [p/2]$. If p is even and $k = p/2$, the cycle is homologous to some U_k, that is, it is *algebraic*. A necessary condition that a p-cycle Γ_p be of rank k is

$$\int_{\Gamma_p} A^{(p-h,h)} = 0,$$

for every exact p-form which can be written in the form

$$A^{(p-h,h)} = \sum_{\alpha_1,\cdots,\alpha_{p-h}=1}^{m} \sum_{\beta_1,\cdots,\beta_h=1}^{m} A_{\alpha_1\cdots\alpha_{p-h}\beta_1\cdots\beta_h}\, dz^{\alpha_1} \cdots dz^{\alpha_{p-h}}\, d\bar{z}^{\beta_1} \cdots d\bar{z}^{\beta_h},$$

which is finite everywhere on V_m, and for which $h < k$. Lefschetz's theorem tells us that this condition is sufficient for the case $m = p = 2$, and another of his results shows that it is sufficient for the case $p = 2m - 2$, any m. Subsequently, I [7] was able to show that if the condition is satisfied for $p = 2$, any m, there exists a non-zero integer λ such that $\lambda\Gamma_2$ is algebraic, and recently [8] I have proved the sufficiency of the condition in the case $m = p = 3$ when the hyperplane sections of V_3 have geometric genus equal to zero. Beyond this, the problem is an unsolved one; I believe, however, that if we can solve it for the case $p = m$ (any m) and $k = 1$, a general solution may be deduced.

3. These results due to Lefschetz were the starting point of an extensive investigation of an algebraic variety as a complex differentiable manifold. A complex manifold of one dimension is always the Riemann surface of an algebraic curve, uniquely defined to within a birational transformation, but a similar result does not hold for complex manifolds of more than one dimension; there are complex differentiable manifolds of dimension m ($m > 1$) which are not algebraic. Let us, for the moment, turn our attention to a general complex differentiable manifold M_m of m dimensions. On this, we can construct multiple integrals, and for these integrals we have the theorems of de Rham [3]: if Γ_{2m-p} is any $(2m - p)$-cycle on M_m, there exists an exterior p-form A with coefficients which are functions of $z_1, \cdots, z_m, \bar{z}_1, \cdots, \bar{z}_m$ regular in the domain of the parameters $z_1, \cdots, z_m$, which is exact, and which satisfies the condition

$$\int_{\Gamma_p} A = I(\Gamma_{2m-p}\cdot\Gamma_p)$$

for every p-cycle Γ_p, where $I(\Gamma_{2m-p}\cdot\Gamma_p)$ is the intersection number of Γ_{2m-p} and Γ_p. For brevity, we say that A is homologous to Γ_{2m-p}, writing $A \sim \Gamma_{2m-p}$. Conversely, given any exact p-form A which is regular on M_m, there exists a $(2m - p)$-cycle Γ_{2m-p}, defined to within its homology class, which is homologous to A, the cycles being considered as elements of a vector space over the field of complex numbers. $\Gamma_{2m-p} \sim 0$ is a necessary and sufficient condition that A should be the derived form dB of a $(p - 1)$-form B.

It is always possible to attach a Hermitian metric of class r to a complex manifold. A complex manifold to which a Hermitian metric is attached is called a Hermitian manifold. Topologically, it has the same degree of generality as the general complex manifold, but the existence of the metric allows us to introduce two new concepts, harmonic integrals and characteristic classes.

An exterior p-form A is harmonic if it has the following property. We define the derivative of A in a p-fold direction at a point O as

$$\lim_{\delta\to 0} \frac{1}{\delta}\int_{E_p} A,$$

where E_p is a p-cell containing O, of measure δ, which is tangent to the given direction at O, and whose diameter tends to zero with δ. It is easy to show that there exists a unique $(2m - p)$-fold form $*A$ whose derivative at every point O in any $(2m - p)$-fold direction is equal to the derivative of A in the absolutely perpendicular direction. A is harmonic if it is exact and if the dual form $*A$ is also exact; $*A$ is then also harmonic. The fundamental theorem is that there is one and only one harmonic form A homologous to any given $(2m - p)$-cycle Γ_{2m-p}.

The characteristic classes on a Hermitian manifold H_m are defined by Chern [2] as follows. Let K_{2m} be complex covering H_m, and let K_s denote its s-dimensional skeleton. Consider the skeleton $K_{2m-2r+1}$. It is possible to define, in a continuous manner, an ordered set of r mutually orthogonal unit complex vectors in H_m at all points of $K_{2m-2r+1}$. Let $E_{2m-2r+2}$ be any $(2m - 2r + 2)$-cell of K_{2m}. Its boundary $\partial E_{2m-2r+2}$ is a $(2m - 2r + 1)$-sphere, and the vectors given on $K_{2m-2r+1}$ map this boundary on the space $U_{m,r}$ of all ordered sets of r mutually orthogonal unit vectors of m-dimensional complex vector space. Now the homotopy groups of $U_{m,r}$ of dimension less than $2m - 2r + 1$ are all zero, while the homotopy group of dimension $2m - 2r + 1$ is an infinite cyclic group. We choose a generator γ of this group once and for all. If $\partial E_{2m-2r+2}$ is mapped on $a\gamma$, we attach the integer a to $E_{2m-2r+2}$. The $(2m - 2r + 2)$-cells of K_{2m}, with coefficients attached in this way, form a cocycle, and the cohomology class to which this cocycle belongs is independent of the field of r vectors given on $K_{2m-2r+1}$, and also of the Hermitian metric chosen. It is called the rth characteristic class of H_m, and is an invariant of the complex manifold. A $(2r - 2)$-cycle dual to the cocycles of the rth characteristic class is called a characteristic cycle; it is sometimes more convenient to deal with characteristic cycles than with characteristic cohomology classes.

4. In order that an algebraic variety can be treated as a Hermitian manifold we have to specify a metric on it. While this can be done arbitrarily, it is desirable that the metric chosen should have some intrinsic relationship with the structure of the variety. Such a metric can be found in this way. Let V_{m-1} be any algebraic variety belonging to an allowable system on V_m. Then it can be shown that there exists an exact 2-form

$$\omega = \sum a_{\alpha\beta} dz^\alpha d\bar{z}^\beta$$

homologous to $(-1)^{1/2}$ times V_{m-1}, such that the quadratic differential form

$$\sum a_{\alpha\beta}(dz^\alpha d\bar{z}^\beta)$$

defines a positive definite Hermitian metric on V_m. We use this metric, which bears a special relationship to the selected allowable system $| V_{m-1} |$.

The chosen metric has the property

$$\frac{\partial a_{\alpha\gamma}}{\partial z_\beta} = \frac{\partial a_{\beta\gamma}}{\partial z_\alpha}, \qquad \frac{\partial a_{\alpha\beta}}{\partial \bar{z}_\gamma} = \frac{\partial a_{\alpha\gamma}}{\partial \bar{z}_\beta},$$

which is equivalent to the condition that the *fundamental* 2-form ω associated with the metric is exact. Such a metric is called a Kähler metric. Kähler metrics appear in so many places in mathematics that it is worth while studying Kähler manifolds, that is, Hermitian manifolds with Kähler metrics, in some detail. On a general Kähler manifold $\mathcal{K}_m$ the $(2m - 2)$-cycle Δ homologous to the fundamental 2-form ω is not restricted to be a multiple of a cycle whose cells have integral coefficients, as is the case when we consider algebraic varieties.

Not all complex manifolds carry Kähler metrics. A property possessed by all Kähler manifolds, but not by more general complex manifolds, is easily established. The forms

$$\omega = \omega_1, \qquad \omega_2 = \omega_1 \times \omega, \cdots, \qquad \omega_r = \omega_{r-1} \times \omega$$

are all exact. If ω_r were derived, so would ω_s be, for $s > r$. But $\omega_m = m!$ times the element of volume of $\mathcal{K}_m$, and cannot be a derived form since its integral over $\mathcal{K}_m$ is not zero. Hence no ω_r is derived, and the $2(m - r)$-cycle homologous to ω_r is not bounding, for $r = 1, \cdots, m$. It follows that the Betti numbers R_{2s} of even dimension are positive for $\mathcal{K}_m$.

5. Kähler manifolds have, however, many other special properties. Some of these have recently been described by Eckmann and Guggenheim [6]. The properties of the metric implied by the Kähler condition lead to a classification of the harmonic integrals into subclasses. It can be shown that, if $p \leqq m$, there are exactly $R_p - R_{p-2}$ independent p-forms P which are harmonic and satisfy the condition $P \times \omega_{m-p+1} = 0$. The forms satisfying these conditions are called the effective p-forms. An important fact concerning an effective p-form is that the equation $P \times \omega_{m-p} = 0$ implies that P is zero.

If $p \geqq 2$, there are $R_{p-2} - R_{p-4}$ independent effective $(p - 2)$-forms; the product of any effective $(p - 2)$-form by ω_1 is a harmonic p-form, which we say is ineffective of rank one. Similarly, if $p \geqq 4$, we have $R_{p-4} - R_{p-6}$ independent effective $(p - 4)$-forms, and the product of an effective $(p - 4)$-form by ω_2 is a harmonic p-form, ineffective of rank 2. And so on. The $(R_p - R_{p-2}) + (R_{p-2} + R_{p-4}) + \cdots = R_p$ harmonic forms so constructed are linearly independent, and form a base for the set of harmonic p-forms.

We call attention to the similarity between this classification of harmonic forms and Lefschetz's classification of cycles, described earlier. This similarity is not accidental. Let Δ be a $(2m - 2)$-cycle homologous to the fundamental form ω, and let $\Gamma_{2m-p+2r}$ be a cycle homologous to an effective $(p - 2r)$-form P. Since P is effective, $P \times \omega_{m-p+r}$ is a non-zero harmonic $(2m - p)$-form. Also [3]

$$P \times \omega_{m-p+r} \sim \Gamma_{2m-p+2r} \cdot (\Delta)^{m-p+r},$$

hence $\Gamma_{2m-p+2r} \cdot (\Delta)^{m-p+r}$ is a p-cycle which is not homologous to zero. Corresponding to the $R_{p-2r} - R_{p-2r-2}$ ineffective p-forms $P \times \omega_r$ of rank r, we obtain $R_{p-2r} - R_{p-2r-2}$ p-cycles $\Gamma_{2m-p+2r} \cdot (\Delta)^{m-p+r}$, which we call ineffective cycles of rank r. In the case of an algebraic variety, in which Δ is, save for a factor, a cycle representing a variety of the selected allowable system, it is easy to show that these ineffective p-cycles of rank r are just the ineffective p-cycles of rank r of the Lefschetz classification, and most of the results obtained by Lefschetz can then be read off from the corresponding results for the harmonic forms. Thus nearly all of the results of Lefschetz follow directly from the fact that an algebraic variety carries a Kähler metric associated with the selected allowable system. It can further be shown that the ineffective p-forms of rank r have zero periods on the ineffective p-cycles of rank s, if $s \neq r$, and from this it follows that the essential properties of harmonic p-forms can easily be deduced once one knows the properties of the effective p-forms, and it is necessary to study only the latter. This applies to all values of p not exceeding m; the theory for values of p exceeding m is easily deduced by duality.

The effective p-forms on a Kähler manifold can be further resolved. We call a p-form of the type

$$\sum_{\alpha_1,\cdots,\alpha_h=1}^{m} \sum_{\beta_1,\cdots,\beta_k=1}^{m} A_{\alpha_1\cdots\alpha_h\beta_1\cdots\beta_k}\, dz^{\alpha_1} \cdots dz^{\alpha_h}\, d\bar{z}^{\beta_1} \cdots d\bar{z}^{\beta_k}$$

$(h + k = p)$ a form of type (h, k). Any p-form A can be written uniquely as a sum of p-forms, one of type $(p, 0)$, one of type $(p - 1, 1)$, $\cdots$, one of type $(0, p)$. If A is effective, it is a consequence of the Kählerian condition that each of these forms is effective. We can resolve the vector space of dimension $R_p - R_{p-2}$ of effective p-forms into the sum of a space of dimension $\sigma^{p,0}$ formed by the effective forms of type $(p, 0)$, a space of dimension $\sigma^{p-1,1}$ formed by the effective forms of type $(p - 1, 1)$, and so on. If A is an effective form of type (h, k) homologous to the $(2m - p)$-cycle Γ_{2m-p}, its complex conjugate $\bar{A}$ is an effective form of type (k, h), homologous to the complex conjugate cycle $\bar{\Gamma}_{2m-p}$. In particular, $\sigma^{h,k} = \sigma^{k,h}$. From this it follows that, if p is odd, R_p is even. The numbers $\sigma^{h,k}$, for all h, k such that $h + k \leqq m$, can be shown to be invariants of the differential manifold which carries the Kähler metric. In the case of an algebraic variety, $\sigma^{p,0}$ is the number of linearly independent exact algebraic p-fold integrals of the first kind attached to the variety, an important invariant in the classical theory of algebraic varieties.

This further subdivision of the effective p-forms leads at once to a further classification of the effective p-cycles. If $\Gamma^i_{h,k}$ $(i = 1, \cdots, \sigma^{h,k})$ are the $(2m - p)$-cycles corresponding to a basis for the effective p-forms of type (h, k), it can be shown that the intersection matrix

$$\| I(\Gamma^i_{h,k} \cdot \bar{\Gamma}^j_{h',k'} \cdot (\Delta)^{m-p}) \| \qquad (h + k = p = h' + k')$$

is zero unless $h = h'$, $k = k'$, and is, save for a scalar multiplier, a positive definite Hermitian matrix when $h = h'$, $k = k'$. These properties have no counterpart in the Lefschetz theory. However, if we impose on our Kähler manifold a *further condition, that the* $(2m - 2)$-*cycle* Δ *be a scalar multiple of a cycle* Γ *whose cells have integral coefficients,* we are able to obtain new theorems about the "integral" topology of $\mathcal{K}_m$. This condition is fulfilled for algebraic varieties.

When this new condition is satisfied, it is easy to show that the classification of the p-cycles of $\mathcal{K}_m$ into effective cycles, etc., can be carried out rationally, that is, that a basis for the ineffective p-cycles of rank r can be found consisting of $R_{p-2r} - R_{p-2r-2}$ integral cycles $(r = 0, 1, 2, \cdots)$. The further classification, corresponding to the classification of forms according to their type (h, k), is not rational, but from the results stated above for this classification we can deduce that if **I** is the intersection matrix of effective p-cycles (regarded as cycles in $(\Gamma)^{m-p}$), **I** is symmetric when p is even and has the signature[2]

[2] If **M** is any real symmetric matrix, we can find a real non-singular matrix **T** such that **TMT**$'$ has $+1$ in the first α places on the principal diagonal, -1 in the next β places on the principal diagonal, and zero everywhere else. Whatever matrix **T** is used to reduce **M** to this diagonal form, we always obtain the same numbers α, β. (α, β) is called the signature of **M** [4].

$$(\sigma^{p,0} + \sigma^{p-2,2} + \cdots + \sigma^{0,p},\ \sigma^{p-1,1} + \sigma^{p-3,3} + \cdots + \sigma^{1,p-1}).$$

In the case $p = 2$, this gives $\sigma^{2,0}$ as a pure topological invariant of $\mathcal{K}_m$. Another result of considerable interest is that when $\Delta = k\Gamma$, Γ being an integral cycle, the period matrix of the $\sigma^{1,0}$ independent everywhere finite simple integrals $\int A_i dz^i$ is a Riemann matrix, and hence, if $R_1 > 0$, it is possible, by the use of Θ-functions, to define one-valued functions on $\mathcal{K}_m$ which are algebroid at all points.

6. The results which we have described show that many of the classical invariants of an algebraic variety V_m appear as invariants of the complex manifold associated with it, and that some are pure topological invariants. Let us consider the case $m = 2$, that is, of an algebraic surface. The number $\sigma^{1,0} = R_1/2$ is equal to the number of linearly independent simple algebraic integrals of the first kind attached to V_2, and this is known to be equal to the irregularity. Hence we have

$$R_1 = 2(p_g - p_n).$$

Stated in a slightly different way, another result described above tells us that if the intersection matrix $\mathbf{I}$ of the 2-cycles of V_2 is reduced to diagonal form by a transformation $\mathbf{TIT}'$, the number of positive terms on the diagonal is $2\sigma^{2,0} + 1 = 2p_g + 1$. Thus both p_g and p_n can be determined as pure topological characters. It has also been shown by Alexander [1] that the Euler-Poincaré invariant of V_2 is equal to $I + 4$, when I is the Zeuthen-Segre invariant of the surface. Since the linear genus ν of the surface is connected with I and p_n by the formula

$$I + \nu = 12p_n + 9,$$

we conclude that the four principal numerical invariants of a surface can all be expressed in topological terms.

7. So far, we have dealt only with topological properties of a variety which can be deduced by means of the theory of harmonic integrals. Let us now consider the characteristic classes of the Kähler manifold defined by an algebraic variety V_m and a selected allowable system on it. The determination of the dual characteristic cycles for an algebraic variety V_m, in the case in which V_m carries a sufficient number of algebraically independent simple integrals of the first kind, can be deduced from results of Severi [13], Todd [14], and Eger [5]. For, any simple integral $\int A_i dz^i$ defines a covariant vector-field A_i, and when we have r of these which are algebraically independent, we can use them to define a field of r independent vectors on the skeleton $K_{2m-2r+1}$ of a covering complex K_{2m}. From these, by a standard process, we can deduce an ordered set of r mutually orthogonal unit vectors. It is then not difficult to show that to define the coefficient associated with any $(2m - 2r + 2)$-cell of the covering complex in the corresponding characteristic cocycle we have only to take the intersection number of this cell with the locus, of $r - 1$ complex dimensions, of points of V_m at which the Jacobian matrix of the simple integrals has rank less than r. It then follows that the cycles of the rth characteristic homology class are homologous to the variety of points of V_m at which the Jacobian of the simple integrals has rank less than r. This locus has been determined, in the case $m = 2$, $r = 1$, by Severi; in the case $r = 1$ (any m) by Todd; and generally by Eger. It is a variety of the canonical system $X_{r-1}(V_m)$, as defined by Severi [13], Segre [11], Todd [15, 16], and Eger [5].

The cycles of the rth characteristic homology class, as determined above differ in sign from those defined by Chern [2], since our computation has made use of covariant vectors, while Chern uses contravariant vectors. According to Chern's definition, the cycles of the rth characteristic class are homologous to $(-1)^{m-r+1} X_{r-1}(V_m)$.

This determination of the characteristic classes on an algebraic variety is valid, however, only in the case of a variety carrying a sufficiently large number of algebraically independent simple integrals. Nor does Todd's geometrical theory of canonical systems lead directly to a proof of the desired result in the general case, and to prove it we have to proceed along other lines.

Let ψ be a rational function on V_m, U its locus of zeros, and U' its locus of poles. We construct a covering complex K_{2m} so that U is covered by a subcomplex of K_{2m}, and U' by a subcomplex of the dual complex. The vector $\partial\psi/\partial z_i = \psi_i$ is defined on all cells of K_{2m} except those which meet U', but it is possible to modify ψ_i in the neighbourhood of U' so that we have a vector ζ_i defined on the skeleton $K_{2m-2r-1}$. Let $\zeta_i^{(\alpha)}$ ($\alpha = 1, \cdots, r$) be r vectors defined on $K_{2m-2r-1}$, independent of ζ_i. We can use the vectors $\zeta_i, \zeta_i^{(1)}, \cdots, \zeta_i^{(r)}$ to determine a cocycle of the $(r+1)$ th characteristic class on V_m, and the orthogonal projections of $\zeta_i^{(1)}, \cdots, \zeta_i^{(r)}$ on U to determine a cocycle of the rth characteristic class on U. It is then possible to determine a relation between these. If we define characteristic classes by means of covariant vectors, and denote by $X_{r-1}(V_m)$ a cycle of the rth characteristic homology class on V_m, we obtain the following homology:

$$X_{r-1}(U) \sim [X_r(U) + X_r(V_m)] \cdot U.$$

Now Chern has shown that, for a projective space S_m, $X_{r-1}(S_m)$ is homologous to a cycle representing a variety of the $(r-1)$-dimensional canonical system. From this, the adjunction formula just obtained enables us to establish the identity of the characteristic cycles and canonical cycles on any algebraic variety.

This result raises a number of other questions which remain to be investigated. For instance, it is known that the stationary points of an effective integral of type $(m, 0)$ form a variety of the $(m-1)$-dimensional canonical system, but what do we know about the stationary points of integrals of type (h, k)? I mention only one preliminary result which I have found: If $m = 2$, the stationary points of an effective integral of type $(1, 1)$ form a locus of two real dimensions, in general, which is a cycle of the second characteristic class.

8. While it would be easy to draw rash conclusions, it is surely not without significance that so many of the basic concepts of classical algebraic geometry have a counterpart in the theory of complex manifolds, and the subject calls for much further investigation. I conclude with some remarks of a very general nature.

The results which I have described fall into two classes. In the one, we use only the fact that an algebraic variety is a complex manifold; although we make use of a Hermitian metric on it, this is only incidental, for a Hermitian metric can be attached to any complex variety, and the results to which I refer—for example, those dealing with characteristic classes—do not depend on our choice of metric. In the other, the fact that the manifold carries a Kähler metric is fundamental, and we have seen that this imposes a considerable restriction on the topology of the manifold. Yet many of the results do not depend on the actual Kähler metric selected, while others depend on the fact that the fundamental 2-form ω is homologous to a scalar multiple of an integral cycle, but beyond this do not depend on the choice of metric. We are therefore led to ask whether it is possible to characterise, in any reasonable way, the complex manifolds which can carry a Kähler metric, and in particular those on which the Kähler metric can be chosen so that the associated 2-form ω is homologous to a multiple of an integral cycle. It is known that not all Kähler manifolds are algebraic; for instance, the generalised complex torus can be made into a Kähler manifold by a suitable choice of metric, but it is not algebraic unless its period matrix is Riemannian. On the other hand, I know of no example of a Kähler manifold whose fundamental 2-form ω is homologous to a multiple of an integral cycle, except the algebraic manifolds. There is surely a problem of great interest and importance here, which is a challenge to all interested in the theory of complex manifolds.

References

1. J. W. Alexander, Atti della Accademia Nazionale dei Lincei. Rend. V. (2) vol. 23 (1914) p. 55.
2. S. S. Chern, Ann. of Math. (2) vol. 37 (1946) p. 85.
3. G. de Rham, Journal de Mathématiques (9) vol. 10 (1931) p. 115.
4. L. E. Dickson, *Modern algebraic theories*, Chicago, 1926, Chapter IV.
5. M. Eger, C. R. Acad. Sci. Paris vol. 204 (1937) pp. 92, 217.
6. B. Eckmann and H. Guggenheim, C. R. Acad. Sci. Paris. vol. 229 (1949) pp. 464, 489, 503, 577.
7. W. V. D. Hodge, Proc. Cambridge Philos. Soc. vol. 31 (1935) p. 18.
8. ———, Annali di Matematici (4) vol. 29 (1949) p. 115.
9. S. Lefschetz, Trans. Amer. Math. Soc. vol. 22 (1921) p. 327.
10. ———, Ann. of Math. (2) vol. 28 (1927) p. 342.
11. B. Segre, Memoire della Accademia d'Italia vol. 5 (1934) p. 479.
12. F. Severi, Math. Ann. vol. 62 (1906) p. 194.
13. ———, Comment. Math. Helv. vol. 4 (1932) p. 268.
14. J. A. Todd, Proc. London Math. Soc. (2) vol. 43 (1937) p. 139.

15. ———, Proc. London Math. Soc. (2) vol. 43 (1937) p. 127.
16. ———, Proc. London Math. Soc. (2) vol. 45 (1939) p. 410.

The proofs of new results referred to in this lecture will be published in two papers in Proc. London Math. Soc. (3) vol. 1 (1951).

General References

W. V. D. Hodge, *The theory and applications of harmonic integrals*, Cambridge, 1941.
S. Lefschetz, *L'analyse situs et la géométrie algébrique*, Paris, 1924.

Cambridge University,
Cambridge, England.

[E n-DIMENSIONALEN SPHÄREN UND PROJEKTIVEN RÄUME IN DER TOPOLOGIE

HEINZ HOPF

Dieser Vortrag ist nicht für Topologen bestimmt; ihnen werde ich nichts aues sagen, und die Probleme, die ich hier nur oberflächlich streife, werden el gründlicher in der Topologie-Konferenz behandelt werden. Ich wende ich vielmehr ausdrücklich an die Mathematiker, denen die Topologie ferner gt, und möchte versuchen, sie über einen Teil dessen zu informieren, was uns der Topologie heute beschäftigt und bewegt; ich werde versuchen, dies durch zu tun, dass ich über einige aktuelle Probleme spreche—aktuell in gendem Sinne: sie sind ungelöst, aber es wird an ihnen gearbeitet, und man t den Eindruck, dass sie ihrer Lösung näher gebracht werden.

Glücklicherweise ist eine ganze Reihe aktueller topologischer Probleme so ıfach zu formulieren, dass es leicht ist, sie auch den fernerstehenden Mathe-ıtikern auseinanderzusetzen. Insbesondere werden im Folgenden keine anderen pologischen Räume vorkommen als, neben den euklidischen Räumen R^n, e n-dimensionalen Sphären S^n und die n-dimensionalen projektiven Räume '. Der Raum R^n ist der Koordinatenraum der reellen n-Tupel $(x_1, x_2, \cdots, x_n)$; e Sphäre S^n ist im R^{n+1} durch die Gleichung $\sum_1^{n+1} x_j^2 = 1$ definiert, und der ojektive Raum P^n ist der Raum aller Verhältnisse $(x_1 : x_2 : \cdots : x_{n+1})$, wobei $:0: \cdots :0)$ ausgeschlossen ist. Man kann den Raum P^n auch als den Raum er Geraden interpretieren, die durch den Nullpunkt des R^{n+1} gehen; betrachtet ın gleichzeitig die S^n, so sieht man, dass zwischen S^n und P^n folgender Zusam-menhang besteht: durch Identifikation je zweier antipodischer Punkte der S^n tsteht der P^n.

1. Ich beginne mit einer alten Geschichte, nämlich mit dem berühmten tz von H. Poincaré (1885), dass es unmöglich ist, die gewöhnliche Kugelfläche mit einer einfachen Kurvenschar ohne Singularität zu überdecken (etwa e sie auf der Torusfläche existiert). Denken wir dabei nur an stetig differen-rbare Kurven; dann ist der Satz im wesentlichen mit dem folgenden identisch: ıf der S^2 gibt es kein stetiges Feld tangentialer Richtungen ohne Singularität.

E. J. Brouwer hat (1910) gezeigt, dass dieser Poincarésche Satz für alle hären gerader Dimension seine Gültigkeit behält: Auf der S^{2m} gibt es kein tiges Feld tangentialer Richtungen. Dagegen existieren auf den Sphären gerader Dimension S^{2m+1} derartige Felder: man bringe in dem Punkt der $^{n+1}$, dessen Koordinaten $c_0, c_1, \cdots, c_{2m+1}$ sind, den Vektor mit den Kom-nenten

$$-c_1, c_0, -c_3, c_2, \cdots, -c_{2m+1}, c_{2m}$$

Man kann diesen Unterschied zwischen den geraden und den ungerad
Dimensionszahlen auch so formulieren: wir betrachten die lineare Gleichung

$$c_0 z_0 + c_1 z_1 + \cdots + c_n z_n = 0 \tag{1}$$

und suchen für alle Koeffizienten c_j mit $\sum c_j^2 = 1$ Lösungen $(z_0, z_1, \cdots, z_n$ welche nirgends trivial sind, d.h., für welche $\sum z_j^2 \neq 0$ ist und welche stet von den c_j abhängen (c ist der Punkt auf der S^n, z der Vektor, der die Ta gentialrichtung bestimmt). Der Brouwersche Satz lehrt, dass bei geradem die Gleichung (1) keine Lösung der gewünschten Art besitzt; für ungerad $n = 2m + 1$ dagegen ist

$$z_0 = -c_1, \quad z_1 = c_0, \cdots, \quad z_{2m} = -c_{2m+1}, \quad z_{2m+1} = c_{2m} \tag{2}$$

eine solche Lösung. Hat man einmal dem Satz diese Formulierung im Rahm der "stetigen linearen Algebra" gegeben, wobei die Koeffizienten c_j und d Unbekannten z_j reelle Zahlen sind, so erhebt sich in natürlicher Weise die Frag ob der analoge algebraische Satz gilt, wenn man statt der reellen Zahlen komple zulässt, wobei wir natürlich $\sum c_j^2$ und $\sum z_j^2$ durch $\sum |c_j|^2$ und $\sum |z_j|^2$ z ersetzen haben. Die Tatsache, dass bei ungeradem n die Lösung (2) existie bleibt unverändert bestehen; bleibt, als Analogon des Brouwerschen Satz auch die Behauptung richtig, dass bei geradem $n = 2m$ die Gleichung (1) f $\sum |c_j|^2 = 1$ keine stetige und nirgends triviale Lösung besitzt? Setzen wi um diese Frage zu untersuchen,

$$c_j = a_j + ib_j, \quad z_j = x_j + iy_j$$

mit reellen a_j, b_j, x_j, y_j, so ist (1) gleichbedeutend mit dem Gleichungssyste

$$a_0 x_0 - b_0 y_0 + \cdots + a_n x_n - b_n y_n = 0$$

$$a_0 y_0 + b_0 x_0 + \cdots + a_n y_n + b_n x_n = 0;$$

deuten wir $a_0, b_0, \cdots, a_n, b_n$ als Koordinaten von Punkten c im R^{2n+2}, liefert jede Lösung dieses Systems zwei Vektoren

$$x_0, -y_0, \cdots, x_n, -y_n$$

$$y_0, x_0, \cdots, y_n, x_n,$$

die im Punkte c tangential an die Sphäre S^{2n+1} und überdies, falls $\sum |z_j|^2 \neq$ ist, linear unabhängig voneinander sind. Die Vermutung, dass unsere ob formulierte Frage (bezüglich der Nicht-Existenz einer stetigen, nirgends t vialen Lösung im Komplexen für $n = 2m$) zu bejahen ist, führt daher zu d Vermutung, dass, in Analogie zu dem alten Brouwerschen Satz über die Ric tungsfelder auf den Sphären S^{2m}, der folgende Satz gilt: Auf den Sphären d Dimensionen $4m + 1$ existieren nicht zwei stetige tangentiale Richtungsfeld die frei von Singularitäten und überall linear unabhängig voneinander sind.

Dieser Satz ist von B. Eckmann und von G. Whitehead bewiesen word (1941). Man weiss also: auf der S^n gibt es, wenn n gerade ist, kein Richtungsfe (tangential, stetig, ohne Singularität); wenn n ungerade ist, so gibt es ein solches Feld; wenn $n = 4m + 1$ ist, so gibt es zwar ein Feld, aber nicht zwei Felder, die überall unabhängig voneinander sind. Es ist weiter leicht zu verifizieren—durch Formeln, die zu (2) analog sind,— dass es für $n = 4m + 3$ auf der S^n wenigstens 3 unabhängige Felder gibt; auf welchen Sphären gibt es 4 unabhängige Felder? Diese Frage gehört zu dem folgenden allgemeineren Problem, das eines der aktuellen Probleme ist, auf die ich hier hinweisen wollte: Wie gross ist, bei gegebenem n, die Maximalzahl linear unabhängiger Felder auf der Sphäre S^n?

2. Die analoge Frage ist sinnvoll nicht nur für die Sphären, sondern für beliebige differenzierbare n-dimensionale Mannigfaltigkeiten, und sie ist in dieser Allgemeinheit von E. Stiefel in einer wichtigen Arbeit behandelt worden (1936). Es würde mein Programm überschreiten, über diese Arbeit zu berichten, und ich werde nur von einer Anwendung sprechen, die Stiefel von seiner allgemeinen Theorie auf die projektiven Räume gemacht hat (1940). Die Sphären selbst entziehen sich dieser Theorie aus Gründen, auf die ich ebenfalls hier nicht eingehe; ich möchte aber darauf hinweisen, dass Richtungsfelder im P^n als spezielle Richtungsfelder auf der S^n gedeutet werden können: da der P^n aus der S^n entsteht, wenn man auf dieser je zwei antipodische Punkte miteinander identifiziert, ist ein Feld im P^n nichts anderes als ein Feld auf der S^n, welches "ungerade" in dem Sinne ist, dass die in zwei antipodischen Punkten angebrachten Richtungen immer einander entgegengesetzt sind; daher ist es auch verständlich, dass wir über die Richtungsfelder in den Räumen P^n mehr wissen als über die Felder auf den Sphären. Der Satz von Stiefel, den ich anführen will, lautet: Man schreibe die Zahl $n + 1$ in der Form $n + 1 = 2^r u$ mit ungeradem u; dann ist die Maximalzahl linear unabhängiger Richtungsfelder im P^n kleiner als 2^r. Für $r = 0$ ist dies der Brouwersche Satz, dass es bei geradem n kein stetiges Richtungsfeld im P^n gibt; für $r = 1$ ist es der Eckmann-Whiteheadsche Satz für die projektiven Räume.

3. Eine besonders interessante Konsequenz des Stiefelschen Satzes ist die folgende: Im n-dimensionalen projektiven Räume können n unabhängige Richtungsfelder höchstens dann existieren, wenn $n + 1$ eine Potenz von 2 ist. Ob diese notwendige Bedingung auch hinreichend ist, ist fraglich; die Existenz von n unabhängigen Feldern im P^n ist bekannt nur für $n = 1, 3, 7$. Ebenso verhält es sich für die Sphären: auf denen der Dimensionen 1, 3, 7, und nur auf diesen, kennt man Systeme von n unabhängigen Feldern, und man weiss nicht, ob noch andere Sphären die analoge Eigenschaft haben.

Diese Frage nach der Existenz von n unabhängigen Richtungsfeldern auf der S^n oder im P^n (oder auch in einer anderen n-dimensionalen Mannigfaltigkeit) ist aus verschiedenen Gründen interessant. Erstens: wenn man n solche Felder hat, so kann man in jedem Punkt n linear unabhängige Vektoren der Länge 1 auszeichnen, die stetig mit dem Punkte variieren; diese Vektoren kann man als Basisvektoren in den lokalen tangentialen Vektorräumen benutzen und nun zwischen den Tangentialvektoren in verschiedenen Punkten den folgenden natürlichen Begriff des Parallelismus erklären: zwei Vektoren sind parallel, wenn sich ihre Komponenten in bezug auf die genannten lokalen Basen nur um einen positiven Faktor unterscheiden. Umgekehrt folgt, wie man leicht sieht, aus der Möglichkeit, einen Parallelismus einzuführen, der einige plausible Bedingungen erfüllt, die Existenz von n unabhängigen Richtungsfeldern. Diese Existenz ist also gleichbedeutend mit der Möglichkeit, in der betrachteten Mannigfaltigkeit, in unserem Fall in der S^n oder im P^n, einen "stetigen Fernparallelismus" zu definieren oder, wie man auch sagt, die Mannigfaltigkeit zu "parallelisieren". Dass die Frage, welche Mannigfaltigkeiten in diesem Sinne "parallelisierbar" sind, an und für sich geometrisches Interesse verdient, ist wohl klar.

4. Dasselbe Problem ist aber auch aus anderen Grunden interessant, welche mehr algebraischer Natur sind und mit dem Problemkreis der kontinuierlichen oder topologischen Gruppen zu tun haben. Nehmen wir an, dass wir in der S^n oder im P^n n unabhängige Richtungsfelder haben; durch eine kanonische Orthogonalisierung kann man erreichen, dass sie sogar orthogonal zueinander sind. Dann existiert in der Gruppe der Drehungen der S^n, bezw. der elliptischen Bewegungen des P^n, eine einfach transitive Schar in folgendem Sinne: ein Punkt e sei willkürlich ausgezeichnet, und für jeden Punkt a sei T_a diejenige Drehung, bezw. elliptische Bewegung, welche e in a und die n in e ausgezeichneten Feldrichtungen unter Erhaltung ihrer Reihenfolge in die analogen Richtungen im Punkte a befördert; die Schar dieser T_a ist einfach transitiv in bezug auf den Punkt e. Diese Schar kann man nun benutzen, um zwischen den Punkten der S^n, bezw. des P^n, eine Multiplikation zu definieren: man setze $T_a(x) = ax$ (für beliebige Punkte a, x); da $T_a(e) = a$ ist, ist $ae = a$; da T_e die Identität ist, ist $ea = a$. Das Produkt hängt natürlich stetig von den beiden Faktoren ab. Wenn also die Sphäre S^n parallelisierbar ist, so lässt sich auf ihr eine "algebraische Struktur" mit den soeben genannten Eigenschaften einführen: eine stetige Multiplikation mit einem Einselement; und das Analoge gilt für die projektiven Räume P^n. In diesem Fall kann S^n, bezw. P^n, als verallgemeinerter Gruppenraum aufgefasst werden; das assoziative Gesetz wird allerdings im allgemeinen nicht gelten. Gewöhnliche, assoziative Gruppenräume sind S^n und P^n bekanntlich nur für $n = 1$ und $n = 3$: $S^1 = P^1$ ist die multiplikative Gruppe der komplexen Zahlen vom Betrage 1, S^3 die Gruppe der Quaternionen vom Betrage 1, P^3 die Gruppe der projektiv gemachten (d.h., nur bis auf einen reellen, von 0 verschiedenen Faktor bestimmten) Quaternionen (oder auch die Gruppe der Drehungen der S^2). Neben den komplexen Zahlen und Quaternionen gibt es noch ein interessantes hyperkomplexes System mit 8 Einheiten über dem Körper der reellen Zahlen: die Cayleyschen Oktaven, deren Eigenschaften denen der Quaternionen verwandt sind (insbesondere gibt es keine Nullteiler), für welche aber das assoziative Gesetz nicht gilt; mit ihrer Hilfe kann man in der S^7 und im P^7

wohl eine stetige Multiplikation mit Einselement als auch einen Parallelismus nführen. Auf der Existenz der komplexen Zahlen, der Quaternionen und der ayleyschen Oktaven beruht die früher erwähnte Ausnahmestellung der Dimenonszahlen 1, 3, 7 im Rahmen dessen, was wir über die Parallelisierbarkeit der hären und der projektiven Räume wissen. Aus dem zitierten Stiefelschen tz über die Parallelisierbarkeit der projektiven Räume folgt leicht: der Grad nes hyperkomplexen Systems über dem Körper der reellen Zahlen, das keine ullteiler besitzt, aber nicht assoziativ zu sein braucht, ist notwendigerweise ne Potenz von 2; für diesen, auf topologischem Wege entdeckten Satz hat origens F. Behrend einen rein algebraischen Beweis angegeben (1939); ob er ein solches System mit einem Grade $2^r > 8$ existiert, das konnte bis heute eder mit topologischen, noch mit algebraischen Methoden entschieden werden. Das Problem, auf das ich hier hinweisen wollte, ist das, ob sich in der S^n ler im P^n gewisse algebraische Strukturen, von der oben besprochenen oder ich von allgemeinerer oder von speziellerer Art, einführen lassen.

5. Ich will darauf aber hier nicht weiter eingehen, sondern kurz von dem oblem der Existenz einer speziellen *analytischen* Struktur auf einer S^n, nämlich ner "komplex-analytischen" Struktur sprechen. Die Fragestellung hat nur i geradem $n = 2m$ einen Sinn; dass die S^{2m}—oder allgemeiner eine $2m$-dimenonale Mannigfaltigkeit—eine komplex-analytische Struktur besitzt, oder irz: dass sie eine "komplexe Mannigfaltigkeit" ist, soll bedeuten, dass sie istande ist, die Rolle einer (mehrdimensionalen) abstrakten Riemannschen äche zu spielen, dass also auf ihr Begriffe wie "analytische Funktion von m omplexen Variablen", "komplex-analytisches Differential," usw. sinnvoll nd. Dies ist dann und nur dann der Fall, wenn man sie mit lokalen Systemen on m komplexen Parametern $(z_1, \cdots, z_m)$, $(w_1, \cdots, w_m)$, $\cdots$ so überdecken inn, dass die Parametertransformationen, die dort entstehen, wo mehrere eser Systeme übereinandergreifen, analytisch sind. Die Frage ist also: auf elchen Sphären S^{2m} ist eine solche komplex-analytische Parametrisierung öglich? Der klassische Fall ist der der S^2; das ist die Riemannsche Zahlkugel der wöhnlichen Funktionentheorie. Gibt es für eine Sphäre S^{2m} mit $m > 1$ etwas hnliches? Man weiss es nicht, und dies zu entscheiden, ist auch eines der tuellen Probleme, mit denen sich heute Topologen beschäftigen. Das am eitesten gehende Resultat (bei Beschränkung auf Sphären) ist der folgende, hr einfach beweisbare Satz von A. Kirchhoff (1947): Wenn die S^n eine komex-analytische Struktur besitzt, so ist die S^{n+1} parallelisierbar. Hieraus und is dem früher besprochenen Satz von Eckmann-Whitehead folgt, dass die hären S^{4k} keine komplex-analytische Struktur besitzen. Im Lichte des Kirchffschen Satzes hängt die komplexe Struktur der S^2 mit der Parallelisierbarkeit r S^3 zusammen, die ihrerseits etwas mit den Quaternionen zu tun hat; aber auch die S^7 parallelisierbar ist, kann der Kirchhoffsche Satz keinen Aufschluss irüber geben, ob die S^6 eine komplex-analytische Struktur zulässt. Gerade i der S^6 haben bisher alle Versuche, die Nicht-Existenz einer solchen Struktur zu beweisen, versagt; in der Tat besitzt die S^6 eine sogenannte "fast-komplexe Struktur—ich kann darauf hier aber nicht eingehen und brauche es auch un soweniger zu tun, als Herr Ehresmann in der Topologie-Konferenz ja ausführlic über diesen Fragenkreis sprechen wird, ohne sich übrigens auf Sphären zu b schränken.

6. Über die *Methoden*, mit denen man die Probleme behandelt, von dene ich gesprochen habe, kann ich natürlich in diesem kurzen Vortrag nicht vi sagen. Ich möchte nur an einem Spezialfall erläutern, welcher Art oft der Ker der Schwierigkeit ist, die man zu überwinden hat. Nehmen wir an, wir wolle untersuchen, ob es, bei gegebenem n und k, auf der S^n k linear unabhängig oder, was auf dasselbe hinauskommt, k zueinander orthogonale Richtungsfelde gibt. Es ist leicht, ein solches System von k Feldern zu konstruieren, das i einem einzigen Punkte singulär wird: im R^n ist die Existenz eines solchen System ohne Singularität trivial; man projiziere den R^n samt diesem System stere graphisch auf die S^n; dann entsteht auf dieser ein System der gewünschten A mit dem Projektionszentrum o als einziger Singularität. Diese Singularit versuchen wir zu beseitigen. Wir betrachten eine Umgebung U von o, die w als Vollkugel in einem n-dimensionalen euklidischen Raum auffassen könne und tilgen die im Inneren von U schon definierten Feldrichtungen; dann hande es sich darum, die folgende Randwertaufgabe zu lösen: die auf dem Rande S^{n-} der Vollkugel U gegebenen k orthogonalen Richtungsfelder sollen stetig a U erweitert werden, sodass sie auch dort orthogonal sind. Diese Aufgabe läss sich so deuten: wir betrachten die Mannigfaltigkeit V aller geordneten orth gonalen k-Tupel in einem Punkt des R^n; dann definieren die auf dem Rand S^{n-} gegebenen k-Tupel in natürlicher Weise (vermittels der Parallelität im R' eine stetige Abbildung f von S^{n-1} in V, und unsere Randwertaufgabe ist offenba äquivalent mit der folgenden: f soll zu einer stetigen Abbildung der Vollkug U in V erweitert werden. Diese Aufgabe aber ist ihrerseits, wie man sehr leich sieht, gleichbedeutend mit der Aufgabe, das Sphärenbild $f(S^{n-1})$ in der Mar nigfaltigkeit V stetig auf einen Punkt zusammenzuziehen. Die entscheidend Frage ist also schliesslich die, ob diese Zusammenziehung möglich, also, um di übliche Terminologie zu benutzen: ob $f(S^{n-1})$ in V homotop 0 ist.

Damit habe ich schon angedeutet, in welches Gebiet man bei der Behandlun unserer Probleme gerät: in die Homotopietheorie, durch deren Begründun (1935) W. Hurewicz der ganzen Topologie einen so grossartigen neuen Impul gegeben hat.

Bei der Frage, auf die wir soeben geführt worden sind, beachte man, das die Abbildung f der Sphäre S^{n-1} eine ganz spezielle, wohldefinierte Abbildun ist, die man, wenn man will, durch explizite Formeln beschreiben kann; vo dieser speziellen Abbildung also soll man entscheiden, ob sie homotop 0 is oder nicht. Ich betone das deswegen, weil gerade in dieser Hinsicht eine irrtüm liche Meinung darüber weit verbreitet ist, was die Topologen eigentlich tu und von welcher Art ihre Schwierigkeiten sind: es handelt sich in der Topologi gar nicht ausschliesslich um Theorien von sehr hoher Allgemeinheit, sondern gerade unter unseren aktuellsten Probleme gibt es solche von ausgesprochen speziellem Charakter.

7. Ich will die spezielle Frage, von der wir hier gesprochen haben, aber verlassen; oft treten ähnliche Fragen auf, bei denen es sich darum handelt, ob eine Abbildung einer Sphäre in einen Raum V (der nicht der vorhin betrachtete zu sein braucht) homotop 0 ist. Ein besonders einfacher Fall, dessen Untersuchung übrigens auch für die Behandlung vieler anderer Falle von ausschlaggebender Bedeutung ist, ist natürlich der, in dem V selbst eine Sphäre ist. Von den Abbildungen von Sphären auf Sphären also—sagen wir: einer S^N auf eine S^n—will ich jetzt sprechen. Wie kann man einen Überblick über sie gewinnen, oder besser: nicht so sehr über die Abbildungen selbst, als über die Abbildungsklassen? Dabei werden zwei Abbildungen zu einer Klasse gerechnet, wenn man sie stetig ineinander überführen kann. Die "unwesentlichen" oder null-homotopen Abbildungen, d.h., diejenigen, die man stetig in Abbildungen auf einen einzigen Punkt überführen kann, bilden eine Klasse, die "Nullklasse".

Die erste und wichtigste Frage ist natürlich: Gibt es, bei gegebenem N und n, auch "wesentliche" Abbildungen, mit anderen Worten: ist die Anzahl der Klassen grösser als 1? Es ist trivial, dass es für $N < n$ nur unwesentliche Abbildungen gibt, und es ist leicht beweisbar, dass es sich für $N > n = 1$ ebenso verhält. Aber man weiss nicht, ob dies die einzigen Fälle ohne wesentliche Abbildungen sind. Hier liegt ein besonders wichtiges ungelöstes Problem vor.

Längst erledigt ist der Fall $N = n$: die Brouwersche Theorie des Abbildungsgrades (1911) lehrt, dass es unendlich viele Abbildungsklassen gibt. Unendlich viele Klassen gibt es auch für alle Paare $N = 4m - 1$, $n = 2m$ (H. Hopf 1935). Andere Fälle mit unendlich vielen Klassen sind nicht bekannt. Auch hier haben wir ein ungelöstes Problem.

Auf die allgemeine Frage nach der Anzahl $A(N, n)$ der Abbildungsklassen $S^N \to S^n$, die nach dem Gesagten nur für $N > n > 1$ interessant ist, kennen wir nur Bruchstücke von Antworten; dieses Problem ist heute wohl eines der merkwürdigsten in der Topologie. Die einzige Aussage von allgemeinem Charakter, die man bis heute über die Abhängigkeit der Zahl A von den Argumenten N und n machen konnte, ist in einem schönen Satz von H. Freudenthal (1937) enthalten, der besagt, dass $A(N, n)$, wenn n im Verhältnis zu N nicht zu klein ist, nur von der Differenz $N - n$ abhängt; der Satz lautet: Bei festem k haben in der Folge $A(n + k, n)$ mit $n = 1, 2, \cdots$ alle Zahlen A von $n = k + 1$ an, und bei geradem k sogar schon von $n = k$ an, denselben Wert. Es ist nun natürlich wichtig, diesen Wert, den wir $A'(k)$ nennen wollen, zu bestimmen. Ausser für den Fall $k = 0$, in dem $A' = \infty$ ist, ist A' bisher nur für $k = 1$ und $k = 2$ bestimmt worden: in beiden Fallen ist $A'(k) = 2$ (H. Freudenthal 1937 für $k = 1$; L. Pontrjagin 1950 für $k = 2$, womit er seine frühere Behauptung (1936), es sei $A'(2) = 1$, korrigiert hat). Ferner hat Freudenthal noch bewiesen, dass

$A'(3) > 1$ und $A'(7) > 1$ ist. Das ist, soviel ich sehe, alles, was man über $A'(k)$ weiss.

Noch weniger wissen wir, wenn n im Verhältnis zu N klein ist. Eine spezielle bekannte Tatsache (W. Hurewicz 1935) ist die, dass $A(N, 2) = A(N, 3)$ für alle $N > 2$ ist. Ich nenne noch einige weitere spezielle Tatsachen, die in dem bisher Gesagten nicht enthalten sind: $A(6, 2) = A(6, 3) > 1$; $A(N, 4) > 1$ für $N = 8, 10, 14$; dagegen weiss man—soviel mir bekannt ist—z.B. nicht, ob sich eine Sphäre einer Dimension $N > 6$ wesentlich auf die S^2 abbilden lässt (Literatur: G. Whitehead, Ann. of Math. 1950).

Die gegenwärtige Situation in diesem Gebiet ist sehr unübersichtlich, und es ist verständlich und erfreulich, dass sich viele Geometer bemühen, hier Klarheit zu schaffen und ein Gesetz zu erkennen.

Ich habe hier nur von den *Anzahlen* $A(N, n)$ der Klassen gesprochen; sie sind die Ordnungen der N-ten Hurewiczschen Homotopiegruppen der n-dimensionalen Sphären, und die Aufgabe, die A zu bestimmen, ist ein Teil der schärferen Aufgabe, die Strukturen dieser Homotopiegruppen zu ermitteln; andererseits wird auch die Bestimmung der Anzahlen A selbst erleichtert und geklärt, wenn man die Homotopiegruppen heranzieht. Ich möchte aber auf die Homotopiegruppen hier nicht eingehen.

8. Aber ich will noch eine Bemerkung anderer Art zu dem Problemkreis der Abbildungen von Sphären auf Sphären machen. Auf der S^N, die im R^{N+1} liegt, seien $n + 1$ stetige Funktionen $f_0, f_1, \cdots, f_n$ gegeben, die keine gemeinsame Nullstelle haben. Man kann sie natürlich stetig in die ganze, von der S^N berandete Vollkugel U hinein fortsetzen; wir fragen: ist dies so möglich, dass auch im Inneren keine gemeinsamen Nullstellen entstehen? Der einfachste Fall ist der, in dem $N = n = 0$ ist: dann ist U ein Intervall, sein Rand S^0 besteht aus zwei Punkten, und in diesen sind zwei Werte gegeben, die nicht 0 sind; diese beiden Werte sind dann und nur dann die Randwerte einer in U stetigen und von 0 verschiedenen Funktion, wenn sie das gleiche Vorzeichen haben. Das ist das klassische Prinzip von Bolzano, und unsere obige Frage kann aufgefasst werden als die Frage, ob das Bolzanosche Prinzip eine Verallgemeinerung auf Systeme von $n + 1$ Funktionen von $N + 1$ Variablen besitzt. Kehren wir also zu beliebigen positiven N und n zurück. Ausser der S^N betrachten wir eine Sphäre S^n in einem R^{n+1} mit Koordinaten $y_0, y_1, \cdots, y_n$. Die auf S^N gegebenen Funktionen $f_0, f_1, \cdots, f_n$, die keine gemeinsame Nullstelle haben, vermitteln eine Abbildung f von S^N auf S^n, die durch die Formeln

$$y_j = f_j \cdot \left(\sum f_i^2\right)^{-1/2}, \qquad j = 0, 1, \cdots, n,$$

gegeben ist. Eine nullstellenfreie Fortsetzung der Funktionen f_j in U hinein ist gleichbedeutend mit einer Fortsetzung der Abbildung f zu einer Abbildung $U \to S^n$, und die Existenz einer solchen Abbildung ist, wie man sehr leicht sieht, gleichbedeutend mit der Deformierbarkeit der Abbildung f von S^N in eine Abbildung auf einen einzigen Punkt, also mit der Unwesentlichkeit von f.

Wir sehen also: wenn die durch die Randwerte f_j bestimmte Abbildung f der S^N auf die S^n wesentlich ist, so existieren für jede Fortsetzung des Randwertsystems ins Innere von U gemeinsame Nullstellen der Funktionen f_j; man hat also eine, durch die Randwerte ausgedrückte hinreichende Bedingung für die Existenz einer Lösung des Gleichungssystems

$$f_j(x_0, x_1, \cdots, x_N) = 0, \qquad j = 0, 1, \cdots, n,$$

in U. Randwerte, welche diese Bedingung erfüllen, gibt es aber dann und nur dann, wenn—in unserer früheren Terminologie—$A(N, n) > 1$ ist.

Diese Formulierung dürfte zeigen, dass unser Problem der Abbildungen von Sphären auf Sphären recht natürlich ist und etwas mit den Grundtatsachen der reellen Analysis zu tun hat. Vielleicht wäre es auch für die Behandlung unseres Problems ganz nützlich, über die hier angedeuteten Zusammenhänge mit der Analysis etwas nachzudenken.

9. Ich habe jetzt einige Zeit von den Sphären gesprochen und möchte nun noch kurz auf ein Problem hinweisen, das für die Sphären trivial, aber für andere Mannigfaltigkeiten, besonders auch für die projektiven Räume interessant ist. Der projektive Raum P^n ist nicht als Punktmenge in einem euklidischen Raum erklärt, wie z.B. die Sphäre, sondern auf eine etwas abstraktere Weise. Da aber die Teilmengen der euklidischen Räume als besonders "konkret" oder "anschaulich" gelten, entsteht die Aufgabe, den P^n in einen euklidischen Raum R^k einzubetten, d.h., ihn eineindeutig und stetig in den R^k hinein abzubilden; das Bild ist dann ein Modell des P^n. Diese Einbettung ist leicht vorzunehmen, wenn k hinreichend gross ist. Interessant ist aber die Aufgabe, bei gegebenem n das kleinste k zu finden, für welches ein Modell des P^n im R^k existiert; wir wollen dieses minimale k mit $k(n)$ bezeichnen. Die projektive Gerade P^1 ist eine einfach geschlossene Linie und besitzt daher ein Modell in der Ebene R^2; es ist also $k(1) = 2$. Die projektive Ebene P^2 ist eine nicht-orientierbare geschlossene Fläche und besitzt daher bekanntlich kein Modell im Raum R^3; das Analoge gilt nicht nur für alle P^n mit geraden n, die ebenfalls nicht-orientierbar sind, sondern auch für die orientierbaren P^n mit ungeraden $n > 1$; es ist somit $k(n) \geqq n + 2$ für alle $n > 1$ (H. Hopf 1940). Für $n = 2$ und $n = 3$ verifiziert man leicht, dass $k(n) = n + 2$ ist, d.h., dass man P^2 in R^4 und P^3 in R^5 einbetten kann. Für gewisse andere n ist bekannt, dass $k(n) \geqq n + 3$ ist (S. S. Chern 1947). Die Aufgabe, $k(n)$ allgemein zu bestimmen, ist ungelöst und scheint schwierig zu sein.

Wenn der P^n in den R^k eingebettet ist, so sind die Koordinaten $y_1, \cdots, y_k$ des R^k stetige Funktionen in P^n; da P^n aus S^n durch Identifikation antipodischer Punkte entsteht, können die y_j als Funktionen auf der S^n aufgefasst werden, welche "gerade" sind, d.h., in antipodischen Punkten immer gleiche Werte haben; die Tatsache, dass die Abbildung des P^n in den R^k eineindeutig ist, äussert sich in den y_j auf der S^n in naheliegender Weise. Das genannte Einbettungsproblem der projektiven Räume ist somit gleichbedeutend mit der Frage nach der Existenz gewisser Funktionensysteme auf den Sphären. Wenn man verlangt, dass die Einbettung differenzierbar sei (was wohl keine wesentliche Einschränkung bedeutet), so kommt man zu interessanten differentialgeometrischen Fragen. Ferner ist es vom algebraischen Standpunkt interessant, das Problem unter der zusätzlichen Forderung zu diskutieren, dass die y_j homogene Formen (geraden Grades) der Koordinaten $x_0, x_1, \cdots, x_n$ des R^{n+1} sind, in dem die S^n liegt.

Ich kann aber hierauf nicht mehr eingehen, und auch nicht auf einige andere Punkte, die eigentlich noch auf meinem Programm standen (komplexe projektive Räume; Faserungen von Sphären), denn die Zeit, die für diesen Vortrag zur Verfügung steht, ist abgelaufen.

SWISS FEDERAL SCHOOL OF TECHNOLOGY,
ZÜRICH, SWITZERLAND.

HOMOLOGY AND HOMOTOPY

W. Hurewicz

This address was given as part of the Conference in Topology, see Volume 2, page 344.

ERGODIC THEORY

S. Kakutani

This address was given as part of the Conference in Analysis, see Volume 2, page 128.

RECENT ADVANCES IN VARIATIONAL THEORY IN THE LARGE

M. Morse

This address appears in the report of the Conference in Analysis, see Volume 2, page 143.

SHOCK INTERACTION AND ITS MATHEMATICAL ASPECTS

J. von Neumann

This address was given as part of the Conference in Applied Mathematics, but no manuscript has been received by the editors.

DIFFERENTIAL GROUPS

J. F. RITT

We shall introduce the notion of differential group by means of an example. The operation of substituting one function of x into another is associative, and confers to an extent, upon the functions of x, the status of a group. As the identical element is x, we write functions in the form $x + f(x)$. The result of substituting $x + u(x)$ into $x + v(x)$ is

$$x + u(x) + v(x + u(x)). \tag{1}$$

If we denote the function in (1) by $x + z(x)$, we have

$$z = u + v(x + u). \tag{2}$$

The second member of (2) furnishes an operation which produces a function z out of two functions u and v. This operation is associative and makes the functions of x a group with 0 for identical element.

We expand $v(x + u)$ in powers of u by Taylor's theorem and write

$$z = u + v + \sum_{n=1}^{\infty} \frac{v^{(n)}u^n}{n!} \tag{3}$$

with $v^{(n)}$ the nth derivative of $v(x)$. The second member of (3), viewed as a formal series, furnishes, as we shall now explain, an associative operation of a formal character. Suppose that, in (3), we replace v by

$$v + w + \sum_{n=1}^{\infty} \frac{w^{(n)}v^n}{n!}. \tag{4}$$

In this we understand that each $v^{(j)}$ in (3) is replaced by the jth derivative of (4), calculated formally. Let the result of substitution be arranged as a series P of homogeneous polynomials in u, the $v^{(j)}$ and $w^{(j)}$.[1] Now let u be replaced in (3) by (3) itself and let each $v^{(j)}$ be replaced by $w^{(j)}$. After a rearrangement of terms, we secure again the series P.

The second member of (3) thus exemplifies a formal generalization of the notion of Lie group. To formulate this generalization completely, we need some definitions.

A differential field is a field, defined as in algebra, in which an operation of differentiation is performable. The formulas for the derivatives of sums and products are as in analysis. We employ a differential field $\mathfrak{F}$ of characteristic zero. A differential indeterminate is a letter endowed with certain of the formal properties of an arbitrary analytic function. A differential indeterminate u has associated with it, for every positive integer j, a letter $u^{(j)}$ called the jth derivative of u. Given a finite number of indeterminates, one forms polynomials in them and

[1] We understand, for instance, that $v^{(0)} = v$.

their derivatives, with coefficients in $\mathfrak{F}$. Such polynomials are called differential polynomials. Differentiation is performed on them as in analysis.

We use differential indeterminates $u_1, \cdots, u_n; v_1, \cdots, v_n$. Consider a formal infinite series

$$\sum_{j=1}^{\infty} P_j \tag{5}$$

in which P_j is a differential polynomial which is either zero or else homogeneous, and of degree j, in the u, v and their derivatives. We call (5) a differential series in the u and v.

Let there be given now n differential series[2] in the u and v

$$z_i = A_i(u_1, \cdots, u_n; v_1, \cdots, v_n), \qquad i = 1, \cdots, n, \tag{6}$$

which satisfy the following two conditions:

(a) A_i reduces to u_i when the v are replaced by zero[3] and to v_i when the u are so replaced.

(b) With $w_1, \cdots, w_n$ new indeterminates, one has, for $i = 1, \cdots, n$,

$$A_i(A(u, v), w) = A_i(u, A(v, w)).$$

We call the set of n series A_i a *differential group*.

The author, in four papers,[4] has developed foundations of a theory of differential groups. There are obtained counterparts, for differential groups, of Lie's basic results on infinitesimal transformations and structure constants. The relation between infinitesimal transformations and subgroups is examined. The differential groups of orders one and two are examined. For order unity there is, in addition to the group (3), the ordinary Lie group $z = u + v$. For this formulation of results, a suitable relation of equivalence of groups must be employed. For order two, there are thirteen types, some of which involve parameters. In three of these types, one meets direct products of two groups of order unity. The thirteen types form two categories, the finite and the substitutional. The finite types involve a finite number of derivatives of u and v, and, in a sense, are Lie groups which can be written as differential groups. In the ten substitutional types, one has

$$z_2 = u_2 + v_2(x + u_2);$$

as one ascends through the types, z_1, which is $u_1 + v_1$ for the first type, increases in complication.

COLUMBIA UNIVERSITY,
NEW YORK, N. Y., U. S. A.

[2] We write, briefly, $z_i = A_i(u, v)$.

[3] Each $v_j^{(k)}$ is made zero.

[4] *Associative differential operations*, Ann. of Math. vol. 51 (1950) pp. 756–765. In the same journal, there will soon appear a paper *Differential groups and formal Lie theory* for an infinite number of parameters, and a paper *Differential groups of order two*. A paper *Subgroups of differential groups* is now in the hands of the editors of Ann. of Math.

THE CALCULATION OF AN ECLIPSE OF THE SUN ACCORDING TO THEON OF ALEXANDRIA

A. Rome

In English-speaking countries, Theon of Alexandria, and still more his laughter, Hypatia, are well known through a novel by Charles Kingsley. Whether or not there are still many people who actually have read it, I don't now. Novels grow old quickly, and that particular one is nearly a centenarian, naving been published in 1853. It is also rather difficult to understand for anyone who is not acquainted at the same time with ancient Greek literature and history, he history of Greek science, and ancient church history, as well as the religious iistory of England in the Victorian Era, Wiseman, Pusey, Newman, and the ther Tractarians. But I often remarked in England that the name Hypatia apealed to many people there, and I should not be surprised if it were the same here.

Now, I happen to be engaged in the editing of the *Commentary of Theon of lexandria on the Almagest of Ptolemy*. In the course of that work, I found that Iypatia had made a revision of the third book of her father's big work, which vas hitherto unknown, and which is now the only text of Hypatia actually ublished. By the way, those 150 pages show that Kingsley, in his chapter ntitled Nephelococcugia (an allusion to "The Birds" of Aristophanes), was erhaps not altogether well inspired when he imagined Hypatia speaking like roclus and making mystical speculations on mathematics. The only text we an read of hers is rather elementary, but quite sound, mathematics and asronomy. It is without all the mystical nonsense some people of the second ophistic period were dreaming à propos of still more elementary mathematics.

My editorial work has just reached the Sixth Book of the *Commentary on the lmagest*, which explains the theory of eclipses of the sun and moon. As an xample of the calculation of an eclipse of the sun, Theon has taken the eclipse f June 16, 364 A.D. We shall rapidly examine it. The Greek text of it is not et completely "established" as we classical philologists say. Before going to he printer, it will undergo a few minor changes. But on the whole, I think it s already sufficiently secure to enable me to speak about it.[1]

[1] The members of the Congress present at the address were given an offset copy of the hole calculation of Theon, which he made twice: first according to the *Almagest*, and a econd time according to the *Handy tables* of Ptolemy. I took those calculations from differnt places: the first from the Sixth Book of Theon's *Commentary on the Almagest*; the second om the *Small commentary* of Theon on the *Handy tables* (ed. Halma); the third from the reat commentary of Theon on the *Handy tables*, which is unpublished. As my text of the ixth Book is not yet established, I shall not publish the full calculation here, for it surely ould make the present article longer than it ought to be. I have changed here the text f my address in such a way as to make it comprehensible for a reader who does not have ne full calculation under his eyes. I note here only that the offset copy corrects a false xplanation I had given in my edition of Pappus' *Commentary on the Almagest* in 1931: . lxix, instead of $c + d' + x' + a$, it should have been $d' + x' + a$.

Theon says that he has observed the eclipse of the sun of June 16, 364, and he gives the times he has recorded. On the other hand, at several places in his Sixth Book, he calculates the different elements of the same eclipse, to show how it must be done, and also to prove that the results of the *Handy tables* are tolerably concordant with the results of the *Almagest.* He explains also the trigonometrical proofs of the *Almagest,* and occasionally adds some proofs of his own. I shall show you one of these trigonometrical computations, selecting a very short one, for the greatest defect of ancient trigonometry is that it is very cumbrous to handle, and generally frightfully long.

Both in the *Almagest* and in the *Handy tables,* the time of the eclipse is determined by seeking first the time of mean conjunction, then the time of true conjunction, and finally, taking the sun's and the moon's parallax into account, the time of apparent conjunction for the given place, which is here Alexandria. Having thus found the elements of the eclipse, he proceeds to compute the circumstances, the magnitude, and the time table, of which he gives two successive approximations. Finally, there is a calculation of the "prosneusis", an azimuthal direction that enables one to find the place on the sun's disc where the first and last contacts can be observed.

Having thus taken a bird's eye view of the calculation, we can now examine some details.

The date is given according to two systems: the "vague" year of 365 days, and the "Alexandrian" one of 365 days with a leap year every fourth year: the Julian calendar is based on Alexandrian astronomy. In the corresponding explanations, Theon gives a model of the calculations needed to pass from one system to the other. Those models enable us to fix with absolute certainty, if there were any doubt about it, when the leap years were started at Alexandria. The names of the months might seem more puzzling. Every month has 30 days, and after 12 months there are 5 or 6 complementary days. Thoth, Payni, and so on, are the original Egyptian names, which were used by the Alexandrian Greeks. Alexandria was a bilingual town. By the way, you know that the French revolution at the end of the 18th century tried to introduce a new calendar. In fact, it was simply the Alexandrian one, with new names, starting with the autumn equinox, because the "Alexandrian" year started about that time, August 29 or 30. How the French came to that idea is simple to find when one knows that the decimal system of weights and measures was in great part organized by Delambre, the man who made the triangulation of the arc of meridian Dunkirk-Paris-Barcelona, which was originally the base of the decimal meter. Now, that same man was also an historian of astronomy; his history of Greek astronomy is not yet completely superseded, after 150 years. So it is easy to guess where he found the original version of Floréal and Germinal and Messidor.

Theon says he has observed the eclipse. For the first contact he says he is quite sure of the time, and for the last, he says it is approximate. Now, Delambre finds that this result is too beautiful to be true. But Delambre has a knack of asting a doubt on the reality of all sorts of observations made by Greek as-ronomers. Strangely enough, people do not read Delambre any more (and in his, I think they are wrong), and still, all those hypercritical doubts are linger-ng about in general histories of astronomy.

First, note that Theon does not give exact figures for the minutes: he observed he first contact on the 24th of Thoth, year 1112 of the era of Nabonassar, $2\frac{1}{2}\frac{1}{3}$ seasonal hours P.M. The last contact, approximately $4\frac{1}{2}$ seasonal hours P.M.[2]

Then if we compare the calculated time tables given by Theon, we see clearly what he means: He finds the first contact according to the *Almagest*, at $3\frac{1}{3}$ equi-noctial hours P.M. which he reduces to $2\frac{1}{2}\frac{1}{3}$ seasonal hours P.M., but according o the *Handy tables*, he finds $3\frac{1}{4}$ equinoctial hours, and he reduces it again to $2\frac{1}{2}\frac{1}{3}$ easonal ones. The maximum phase, according to the *Almagest*, at $4\frac{1}{3}$ equinoctial ours, is taken to be equivalent to $3\frac{1}{2}\frac{1}{4}\frac{1}{20}$ seasonal ones; according to the *Handy ables* it is at $4\frac{1}{4}$ equinoctial hours or $3\frac{1}{2}\frac{1}{6}$ seasonal hours. The last contact, accord-ng to the *Almagest*, is at $5\frac{1}{4}$ equinoctial hours, or $4\frac{1}{2}$ seasonal; according to the *Handy tables* it is at $5\frac{1}{6}$ equinoctial or $4\frac{1}{2}$ seasonal. Maybe some of those figures will be corrected when my text is completely "established", but it is quite clear hat the objection of Delambre is not justified. In fact, I am under the impres-ion that whenever Theon or Ptolemy uses Egyptian fractions instead of sexa-gesimal ones, he intends only to give a rougher approximation.

Can we get an idea of their approximations? Ptolemy (*Almagest*, ed. Heiberg, st vol., p. 505, 24) speaking of eclipses, considers $\frac{1}{16}$ of an equinoctial hour as egligible. Theon considers as small an error as one that can be reckoned to 10 minutes.

It is interesting to compute our eclipse with modern tables. The concordance s fairly good. With the tables of P. V. Neugebauer, which are Schram's tables orrected to bring them into accord with Schoch's, I found for Alexandria: irst contact, $15^h\ 12^m$ true time; maximum phase $16^h\ 25^m$; last contact $17^h\ 20^m$; magnitude 5.5. (Theon finds with the *Almagest*, 4; 39, 18 digits, and with the *Handy tables* 4; 58 digits.) This compares fairly well with the above results in quinoctial hours. If we use Oppolzer, *Kanon der Finsternisse*, we get the first ontact at $15^h\ 28^m$ true time at Alexandria; maximum phase, $16^h\,20.88^m$; last ontact, $17^h\ 19^m$; magnitude 4.3. Ginzel, *Spezieller Kanon*, speaks of our eclipse, . 213. I am afraid he has not been able to read the "Egyptian" fractions in the ld Basel edition of 1538, and his figures are all wrong.

But, writing this, I see already some of the readers preparing an objection:

[2] This somewhat puzzling way of writing the fractions as a sum of elementary fractions ach having a numerator equal to unity (with the exception of 2/3 which had a sign of s own), was taken by the Alexandrian Greeks from the Egyptians. An alternative way of riting fractions, which gave any degree of accuracy that might be wanted, was the sex-gesimal system.

The "seasonal" (day) hour is the calculated 12th of the time between sunrise and sun-et, neglecting refraction. The "equinoctial" hour is practically the same as our true solar me.

can we use a modern table to control the accuracy, to a few minutes, of an observation by Theon? I think we may not. At any rate, we may not always. The astronomers who make tables of eclipses choose the coëfficients of their formulae to fit observations that they deem interesting. Oppolzer took a minimum of them, 7 from antiquity, 3 from the middle ages, with the result that his *Kanon der Finsternisse* does not fit exactly the observations from 1650 to our time. Ginzel gave the maximum weight to the eclipses of +71 and +1385. For the rest, he considered only 3 eclipses of antiquity (all after Plutarch, 2nd century A.D.) and all the medieval ones; but he, too, overlooked all the modern observations.

Schoch tried to fit in the modern observations, too. P. V. Neugebauer has shown, for instance, how Schoch's tables give an excellent description of the sun eclipse of April 17, +1912: For the point chosen, Senlis in France, the curve of centrality calculated with Schoch's tables is not farther from what most probably happened, according to observation, than the curves given by the current almanacs, which are all slightly different from one another. But in addition to this, Schoch took into account all the sun eclipses of antiquity he could know, and all the eclipses of the Middle Ages. For Greek antiquity, he depends on Fotheringham's *A solution of ancient eclipses of the sun.*[3] Now Fotheringham has examined also the eclipse of June 16, 364. You see that if we try to control Theon by means of Schoch's tables, we are arguing in a circle.

I would make some remarks about that article of Fotheringham.

He calls his number 8, *The eclipse of Hipparchus.* When he wrote his article, he had to rely upon a partial edition of a few pages of Pappus, by Hultsch. Now the full text of Pappus' Fifth Book is printed. But there is much hesitation about the date of that eclipse, and I dare not say anything about it.

Fotheringham's number 11 is just our eclipse of June 16, 364. He took all the particulars about it from the only edition available, the Basel edition of 1538. There, the observed times of Theon are recorded distinctly as being equinoctial hours. The consequence of this was that the time supposed to be recorded by Theon could not be brought into line with the other eclipses recorded, and with the theory. Finally, Fotheringham admitted that "The time of Theon was 37, 32, and 38 minutes slow at the three observed phases of the eclipse."

First, I showed you that Theon did not give the time to the minute. Then, letting alone the question of night determination of time, which is of course irrelevant here, I am afraid it is impossible to admit that Theon had such wretched sundials at his disposal. But there is another thing: although my text of the Sixth Book is not yet definitively established, I am quite sure that the original text of Theon did not speak of equinoctial hours; in the best manuscript, the Medicaeus 28, 18, the word equinoctial is absent. Besides, Theon says that the calculated hours conform to the observed ones. But the conformity exists only with the calculated seasonal hours, and of course they cannot be mistaken for equinoctial ones, as the whole calculation is given. If the hours recorded by

[3] Schoch has been the assistant of Fotheringham at Oxford. Reciprocally, in his article, Fotheringham has used a first redaction of Schoch's tables.

heon are seasonal, they fit fairly well with the hour angles called "set *A*" by otheringham, which are obtained by combining Newcomb's tables with rown's movements of the moon and Fotheringham's obliquity of ecliptic.

In consequence, if tables of syzygy were to be calculated again on the prinples of Schoch, there would be some change in two or three of the equations.[4]

One might ask how they proceeded to observe an eclipse of the sun. Up to now, don't think a text has ever been discovered describing the method of observing , when the sun is not quite close to the horizon. Probably the ancient astronners found that detail too trivial to mention. Many systems have been sugested, but I venture to add that there is a possibility which all historians always em to overlook or to discard. When making glass, the trouble is not in getting coloured, but in getting it white. If we now consult the archaeologists, we shall e that the glassmakers were able to produce glass (and transparent glass, too) practically all shades, up to black. Alexandria was especially renowned for s blue glass, from dark to light blue. On the other hand, if we look at the *Optics* Ptolemy, we shall find that he could cut and polish glass to plane or cylinrical forms. The blue Alexandrian glass was exported as a half-finished product the form of cubes, which were used by glassmakers very far away, up to the hine. I think it was not at all difficult for an astronomer at Alexandria to find tolerably plane screen that would do nicely to observe the sun. And it is not travagant to suppose that they had noticed it was possible to look at the sun rough that screen. Think also of the famous emerald of Nero.

This suggestion that they might have resorted to glass screens is of course a ere hypothesis, but I think that hypothesis is not worse than the others genally advanced in that connection.

Ginzel points out that screens seem to have been used only since the 17th ntury. This does not prove that the ancient Greeks did not know them. On e other hand, it would be a grave exaggeration to say that, as they had coloured ass and could not possibly not see that it was an excellent screen, they surely ed it. When you have in hand all the elements of the solution of a problem, u are not yet saved if you don't realize that you have found the solution. Of is I shall give you a clear example when I show you a trigonometrical solution.

Next we may perhaps find some interesting things, if we look at the calculaons of mean conjunction according to the *Almagest* and the *Handy tables*.

Ptolemy says distinctly in the *Almagest* that the mean conjunction might be mputed simply by using the tables of the sun's and moon's mean motion. In ct, in the *Handy tables*, there is no table of mean syzygies. As the moon epicle's center at the time of mean syzygy must be on the apogee of the eccentric, find the time of mean conjunction the only thing we have to do, when we have e moon's position for the beginning of the month we are inquiring about, is to ek in the tables for the epicycle's center quantities that will complete the 360 grees of a circle exactly. The days and hours corresponding to those arcs, are

[4] I say three because I think that, upon closer examination, the famous eclipse of Archilhos should receive a weight of "nil".

the sought days and hours of the mean conjunction, and the only thing we nee to do is to fill in the other columns for the same days and hours to get all th required information.

In the *Small commentary on the Handy tables,* we find an alternative wa which is interesting, because it is exactly the method that is followed to dete mine the date of Easter in the church calendar. Even the word "epact" is use by Theon. I say Theon, although I cannot yet guarantee that this place is no interpolated. But it seems prima facie to be authentic. The Ptolemaic astronom has surely been used by the experts who tried to solve the vexed question of th Easter date. The council of Nicea took place in 325, just about the time whe Pappus wrote his *Commentary*. I think the date of the equinox was fixed on th 21st of March, because the tables of the sun in the *Almagest* point to that dat for 325 A.D. Tannery supposed that the council relied upon an observatio of the equinox, but as the council took place somewhere in May, it was to late to make such an observation. It was much easier and more in accord wit the habits of the period, too, to have it computed from the *Almagest* or fro the *Handy tables*. So the system of epacts might also derive from the Ptolema School.

In the *Almagest,* Ptolemy has found it useful to make special tables of syzygie one for new moons and another for full moons. He starts from the 1st of Thot of year 1 in the era of Nabonassar. From the elongation at this moment, an the table of moon's mean elongation, he can find the time of the first new moo of that year. From there he starts his cycles of 25 years.

Now those cycles of 25 years have been quite recently studied by Professo Neugebauer and van der Waerden. Egyptian texts published by Professo Neugebauer use just the same lunar cycles of 25 years, and suppose that 2 "Egyptian" years are exactly as long as 309 synodic months. Ptolemy knows more accurate equivalent, viz, 309 months minus 0; 2, 47, 5 days.

Did Ptolemy know Egyptian texts of that type? In more than one place i the *Almagest,* he assumes a distinctly polemical tone, to say that people thinkin they can make tables valid for all times to come cannot be considered as seekin the truth. It is tempting to suppose that the Egyptian texts in question ar just based upon those "eternal tables."

On the other hand, the *Commentary of Theon* on the Sixth Book makes anothe hypothesis which can be interesting, too. The table is supposed to have 4 lines. That number of 45 lines plays a great role in the *Almagest.* A whole lo of tables are expressedly mentioned as having 45 lines. Such a mention gave very quick way of controlling whether the copy was accurately done. Ther are many places in the *Almagest* and its commentators where the reader i given practical hints on how to detect errors in his manuscript. That was c course the great danger to avoid when using tables. 45 lines is nearly the highes number of lines found on Greek literary papyri.[5] Publishers were consequentl bound to use the largest usual size of papyrus rolls when they copied the *Almo*

[5] Cf. Kenyon, *Books and readers in ancient Greece and Rome*, Oxford, 1932, p. 56.

st. Now, with 45 lines, and periods of 18 years, as in the other tables of the *lmagest*, we come only to 810 of the era of Nabonassar (63 A.D.). Hipparchus orked about 625 Nabonassar. But the time of Ptolemy is about 885 Nabonas-ır. Ptolemy has obviously kept the disposition of Hipparchus' tables for the ıean motions of sun, moon, and planets. When using those tables after 810 'abonassar, it is quite simple to put down first the data for 810 and then add a ıfficient number of 18 year periods to reach the given dates. But that does not ork with the table of syzygy. Accordingly, he took cycles of 25 years, which rought him up to 354 A.D., quite enough for him, but not for Theon. But tolemy was not a believer in "eternal tables." His successors, disregarding his arnings, because they were no more able to do original research work, simply ıt appropriate prolongations in their manuscripts. There are even manuscripts here one can see by different hands the successive prolongations of the Byzan-ne period, many centuries after the period of validity of the tables.

In the *Handy tables*, 25 year cycles are used everywhere, which, further, start om the first year of Philippus, not of Nabonassar.

The table of syzygies is also an exception in the *Almagest* for a second reason: ıere the cycles are tabulated under the ordinal number of the year, and not as ʼerywhere else under the number of years elapsed since the beginning of the 'a. This last way of tabulating is the cause of many distractions. I have had ›veral opportunities of experimenting with it. In the *Handy tables*, the cycle 'guments are given as in the table of syzygies.

In fact, you see that Ptolemy has probably kept in the *Almagest* the disposi-on of Hipparchus' tables. He says expressly how he corrected them. But in ıe table of syzygies, which probably did not exist in Hipparchus' works, he ied another disposition, which he adopted completely in the *Handy tables*. hat table of syzygies marks thus a transition between his two great astronomi-ıl works.

We might now have a look at a trigonometrical calculation. As I told you, I ıall take the shortest possible case, because ancient trigonometry is always ɜry simple (all problems of spherical trigonometry are solved by application ' one theorem, always the same; and all problems of plane trigonometry are ›lved as I shall show you); but it is nearly always very long. That is the reason hy I am sure that Ptolemy had a whole regiment of calculators, perhaps slaves, , his disposal. Otherwise, he could not possibly have made all the tables that 'e published under his name: I don't think it was possible for a calculator orking very rapidly and quite used to the Greek methods, to find one single ıtum of the fourth column in the *Handy table* of parallaxes in less than two hours. he practical way of doing it is explained by Theon. I had to use it, with the ɜlp of a machine, to find a missing figure. It took me twice as much time as ıat. The existence of those calculators can even be detected, I think, in the table trigonometrical ratios.

We shall take as an example the table of eclipses in the *Almagest*. There is table for the maximum distance of the moon, and another for the minimum distance; and finally, a table of corrections for intermediate positions. Tha table of corrections is made on the assumption that the digits of the eclips and the length of the moon's path during the eclipse vary proportionally to th distance of the moon from the earth. The same table is used also for parallaxes Accordingly, the calculation I shall show you is not to be found in the Sixtl Book of the *Almagest*, but in the fifth (ed. Heiberg, p. 433), where Ptolemy explains how he has computed the seventh column of his table of parallaxes

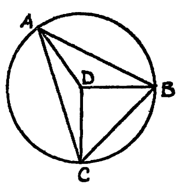


FIGURE 1

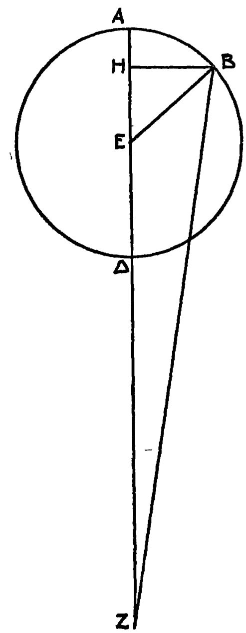


FIGURE 2

The solution is based on the following principle: if we have inscribed triangl *ABC*, and if we have made a table of chords of all the arcs of a circle, takin the radius as unity, it is clear that $CB = \text{crd}\ 2\,CAB \times DB$.

You see at once that all the triangles can be solved, with a little patience You see also that the solution is general and applies to any triangle. Well, th Greeks never saw it. They applied it only to rectangular triangles, and for th others always took the way around of cutting them into two rectangular tri angles. This is the example which I promised to show you of the fact that whe all the elements of a solution are known, there may still remain the need t realize that the solution is found.

If we now call Z the center of earth, E the center of the epicycle, with A a apogee and Δ as perigee, let B be any position of the moon on its epicycle. Th value tabulated is $ZA - ZB$ for $A\Delta = 60$, that is to say, taking $A\Delta$ as a unit, sinc we are in the sexagesimal system. For want of a symbolic notation like ours, th

ncients were bound to give their general solutions under the form of a numeri-
al example. Let us take AB = 60 degrees.

We shall now proceed to solve the triangle HEB.

If we circumscribe a circle around the triangle HEB, the diameter will be B, the chord of 2HEB = HB = crd 120 degrees, and HE will be the chord of HBE = crd 60 degrees.

The Greeks used to express that by saying: having angle HEB = 60 parts of hich there are 360 in a circle, we write down the same angle in parts of which iere are 360 in a half circle, that is, angle HEB = 120 degrees.

That way of speaking blurs somewhat the logical principle of the solution, hich I explained a moment ago.

Anyhow, we get now, taking the value of chords in the table of the first book ι the *Almagest*,

$$\begin{aligned} BH &= \text{crd } 120 \text{ degrees} = 103;\ 55 \\ EH &= \text{crd } \ 60 \text{ degrees} = \ 60;\ 0 \\ EB &= 120, \end{aligned}$$

B being the diameter, and the radius of the circle being 1. If Ptolemy had sed the sexagesimal notation consistently, he would have written something ke 2, 0; 0 = EB. But he does not use sexagesimal notation for the entire part his numbers,[6] and he writes EB = 120, just as he writes BH = 103; 55 and ot 1, 43; 55.

Now, as EB is the radius of the epicycle, and ZE the distance of the center the epicycle from the earth at the time of the eclipse, he has calculated else-here that ZE/EB = 60/5; 15. Reducing proportionally the above equalities, e have:

$$\begin{aligned} EB &= 5;\ 15 \\ BH &= 4;\ 33 \\ EH &= 2;\ 38 \\ EH + EZ &= 62;\ 38 \\ ZH^2 + HB^2 &= ZB^2 \\ ZB &= 62;\ 48. \end{aligned}$$

Now

$$\begin{aligned} ZA &= 65;\ 15 \\ A\Delta &= 10;\ 30 \\ ZA - ZB &= 2;\ 27 \qquad \text{if } A\Delta = 10;\ 30 \\ ZA - ZB &= 14 \qquad\quad \text{if } A\Delta = 60. \end{aligned}$$

onsequently, the value tabulated for the point B will be 14.

The same table of corrections is used, as I told you, in the table of parallaxes here it is the seventh column. It is easy to see that the data for AB = 12 degrees, degrees, 36 degrees ··· have been directly calculated, and that in our table

[6] Must I add that he does not use Arabic ciphers nor signs equivalent to our symbolical tation?

of corrections, one value of ZA − ZB has been interpolated between each pa
of calculated data, while in the table of parallaxes, two such values have bee
interpolated. The fact is immediately detected by forming all the difference
between all the data of both tables.

It happens often that tables of the collection contained in the *Handy tabl*
have been obtained by interpolation. These interpolations are simply propo
tional, but whenever there are remainders to distribute, that distribution is n
made at random. Ptolemy had instinctively, if not explicitly, the sensation th
it was better to reproduce more or less in the interpolated terms the variatior
of the calculated terms. Theon, in his *Great commentary on the Handy table*
calls the attention of his readers to that point. Another curiosity of the *Hand*
tables is that, in order to save labour, Ptolemy sometimes uses a table that w
not originally meant for the problem he is solving, but which gives a numeric
answer to a sufficient degree of approximation. We met a curious example
that process in the calculation of the moon's speed according to the *Hand*
tables: take in the Προκανόνιον the number corresponding to the datum calle
"moon's center," divide by 10 and add 0 degrees 30′: you have the moon
speed with an approximation of to within 1 minute.

We might finish our very incomplete exploration of the Sixth Book of Theo
by having a look at the calculation of prosneusis. As it is always difficult t
see the first and last contacts of an eclipse, it may be useful to know beforehan
in which part of the sun's disc it will happen. Our modern astronomical almana
always mention it. The astronomers of the Ptolemaic school used to calculat
the point where a great circle passing through the sun and moon's centers shoul
cut the horizon. If they put one of the circles of a spherical astrolabe at the sam
time on the sun's center, and on the azimuth so defined, that circle was also o
the point of the sun's disc at which to look. That was, I think, the use of th
prosneusis computation. I don't think it had any astrological significance. A
any rate, I could not find, either in the *Tetrabiblos*, the great bible of astrolog
nor in the catalogue of Cumont, any text showing an astrological significan
such as would, for instance, point out to what country the omen would appl
If there is no astrological significance in the prosneusis, I don't see any possib
use for the prosneusis except the one I mentioned.

Delambre pretends it was not used for that purpose, because he thinks th
ancient astronomers did not care to observe the first contact accurately. I hav
spoken already of this hypercritical opinion.

It would have been possible to compute the prosneusis for each case ver
accurately. But, as Greek trigonometry was very cumbrous to handle, th
computation would have been very long. Accuracy on that point was not r
quired. Ptolemy has, consequently, given in the *Almagest* and in the *Hand*
tables approximate methods that are said by Theon to give concordant result
In fact, if we try them both on the eclipse of June 16, 364, we get with th
Almagest a prosneusis of 16°51′ from W in the northern direction, and with th
Handy tables, a prosneusis of 21°41′ from W in the northern direction. If th

ircle of the astrolabe were put exactly on the center of the sun, and on the points f the horizon indicated by those calculations, this shifting of 4° on the horizon 'ould of course have an influence on the location of the point where the first ontact was to be looked for. In our particular case, the circle of the astrolabe 'ould indicate two points situated at 4° from one another on the circumference f the sun. But as the diameter of the sun is seen under an angle of 1/2°, that rc of 4° on the circumference of the sun would have been seen under an angle f less than 0°1′, which could not be appreciated with naked eyes; in both cases, ıe circle would point exactly to the same spot of the sun's disc. Thus it can be xplained why Theon calls both results equivalent.

This conclusion is at variance with the conclusion of the article that I pubshed in 1948 on the prosneusis. And I am not sure that I shall not change my ıind again before I publish the fourth volume of the commentaries on the *lmagest,* where those notes will be printed. In fact, some of the Greek texts 'e have been exploring now together are not quite ready for publication, and ɔme of them are not even ready at all. But I thought you would prefer an xploration of this kind, even with its hesitations and mistakes, to looking at ıings which are better settled but already published everywhere.[7]

UNIVERSITY OF LOUVAIN,
LOUVAIN, BELGIUM.

[7] I express my thanks to Professor P. S. Jones of University of Michigan, who has kindly vised my text, and Professor Cohen, Harvard University, who assisted me in the correcon of proofs. At the same time, as this address was delivered within a few yards of the 'idener Library, I take the occasion to thank *Isis* and its founder, Professor G. Sarton, r the help which is constantly received from that quarter by anyone wishing to study the story of science.

THÉORIE DES NOYAUX[1]

LAURENT SCHWARTZ

Nous supposerons connus les fondements de la théorie des distributions. La question des notations est fondamentale dans cet article. Si $\varphi(x) \in (\mathfrak{D})_x$ $T_x \in (\mathfrak{D}')_x$, le produit scalaire $T(\varphi) = T \,.\, \varphi$ sera *toujours* noté comme intégrale $\int_{R^n} \varphi T$ du produit φT, et même, pour qu'aucune confusion ne soit possible sur les variables, sous la forme

(1) $$T \,.\, \varphi = \int_{R^n} T_x \varphi(x)\, dx.$$

Lorsque T_x est une fonction $g(x)$, on retrouve l'intégrale classique $\int_{R^n} g(x)\varphi(x)\, dx$ Cette notation est assez lourde mais évite toute confusion.

1. Opérateurs intégraux. Soient X^m, Y^n deux espaces vectoriels réels isomorphes respectivement à R^m, R^n; x sera un point de X^m, y un point de Y^n Un noyau $K(x, y)$, c'est-à-dire une fonction (à valeurs complexes) localement sommable sur $X^m \times Y^n$, définit un opérateur intégral $f \to g = K \cdot f$, qui à toute fonction continue à support compact $f(y)$ sur Y^n fait correspondre une fonction $g(x)$ localement sommable sur X^m:

(2) $$\begin{cases} g = K \cdot f \\ g(x) = \displaystyle\int_{Y^n} K(x, y) f(y)\, dy. \end{cases}$$

Il est bien connu que toutes les opérations linéaires ne peuvent pas être représentées par de tels noyaux, comme on le voit pour l'opérateur identique ou les opérateurs de multiplication ou de dérivation sur R^n. Mais Dirac,[4] dans le mémoire où il introduit sa célèbre "fonction" δ, montre que cette "fonction" permet de représenter les opérations précédentes à l'aide de noyaux:

(3) $$\begin{cases} f(x) = \displaystyle\int_{R^n} \delta(x - \xi) f(\xi)\, d\xi \\ \alpha(x) f(x) = \displaystyle\int_{R^n} [\alpha(x)\delta(x - \xi)]\, f(\xi)\, d\xi \\ \dfrac{\partial f}{\partial x_1} = \displaystyle\int_{R^n} \left[\frac{\partial}{\partial x_1}\, \delta(x - \xi)\right] f(\xi)\, d\xi. \end{cases}$$

[1] Cette communication était mentionnée sur le programme imprimé sous le titre *Distributions and principal applications*.

[2] Le lecteur trouvera dans notre livre (*Théorie des distributions*, Paris, Hermann, 1950 tous les renseignements nécessaires sur les termes employés ici, notamment en consultant l'index terminologique et l'index des notations situés à la fin du volume. Nous noterons cet ouvrage par (TD).

[3] Voir (TD), formule (V, 1; 5).

[4] Dirac, *The physical interpretation of the quantum dynamics*, Proc. Roy. Soc. London Ser. A t. 113 (1926–1927) pp. 621–641.

e but de cet article est de montrer comment la théorie des distributions permet ɜ réaliser correctement une telle représentation, pour toutes les opérations néaires rencontrées dans la pratique, et d'étudier les noyaux en relation avec s opérations linéaires. Nous donnerons dans un autre article les démonstraons des théorèmes, et les applications de cette théorie.

2. Noyaux-distributions. Nous appellerons noyau-distribution ou simplement ɔyau une distribution $K_{x,y}$ sur $X^m \times Y^n$; $K_{x,y} \in (\mathfrak{D}')_{x,y}$ est donc une forme néaire continue sur $(\mathfrak{D})_{x,y}$.

Un tel noyau définit en outre:

1°. Une forme bilinéaire $B_K(u, v)$ sur $(\mathfrak{D})_x \times (\mathfrak{D})_y$:

$$B_K(u, v) = \iint_{X^m \times Y^n} K_{x,y}\, u(x)v(y)\, dx\, dy.^5 \tag{}$$

Lorsque l'une des fonctions u, v est fixée, B est continue par rapport à l'autre t uniformément lorsque la première reste bornée); de plus, elle est continue ır rapport à l'ensemble des deux arguments u, v lorsque les supports de u et restent dans des compacts fixes de X^m et Y^n.

2°. Une transformation linéaire $v \to \mathfrak{L}_K(v)$ de $(\mathfrak{D})_y$ dans $(\mathfrak{D}')_x$, définie comme iit pour $u(x) \in (\mathfrak{D})_x$:

$$\int_{X^m} [\mathfrak{L}_K(v)]_x u(x)\, dx = B_K(u, v) = \iint_{X^m \times Y^n} K_{x,y}\, u(x)v(y)\, dx\, dy.$$

C'est cette transformation qui, si l'on remplace v par f et $\mathfrak{L}_K(v)$ par g, et si est une fonction $K(x, y)$, est définie par la formule (2): nous pouvons donc mplacer $\mathfrak{L}_K(f)$ par $K \cdot f$, et adopter la notation suivante, pour $f(y) \in (\mathfrak{D})_y$, $x) \in (\mathfrak{D})_x$, $K \cdot f = [K \cdot f]_x \in (\mathfrak{D}')_x$:

$$\int_{X^m} [K \cdot f\,]_x\, \varphi(x)\, dx = \iint_{X^m \times Y^n} K_{x,y}\, \varphi(x)f(y)\, dx\, dy.$$

$\mathfrak{L}_K$ est une opération continue de $(\mathfrak{D})_y$, muni de la topologie forte, dans $\mathfrak{D}')_x$, muni de la topologie faible ou forte. On a ainsi:

THÉORÈME I. *Tout noyau-distribution K définit une transformation linéaire $\to K \cdot f$ continue de $(\mathfrak{D})$ dans $(\mathfrak{D}')$.*

3°. Une transformation linéaire continue $u \to \mathfrak{L}'_K(u)$ de $(\mathfrak{D})_x$ dans $(\mathfrak{D}')_y$, finie comme suit pour $v(y) \in (\mathfrak{D})_y$:

$$\int_{Y^n} [\mathfrak{L}'_K(u)]_y v(y)\, dy = B_K(u, v) = \iint_{X^m \times Y^n} K_{x,y}\, u(x)v(y)\, dx\, dy.$$

ɜtte transformation $\mathfrak{L}'_K$ est évidemment la transposée de la précédente $\mathfrak{L}_K$. ɜmarquons que l'isomorphisme canonique $(x, y) \to (y, x)$ de $X^m \times Y^n$ sur

[5] Il s'agit toujours de la notation définie à la formule (1), mais nous emploierons le mbole d'intégrale double quand ce sera sur l'espace produit $X^m \times Y^n$, le symbole d'ingrale simple quand ce sera sur l'espace X^m ou Y^n.

$Y^n \times X^m$ associe à la distribution $K_{x,y} \in (\mathfrak{D}')_{x,y}$ une distribution ${}^*K_{y,x} \in (\mathfrak{D}')_{y,x}$ et on a bien évidemment $\mathfrak{L}'_K = \mathfrak{L}_{(*K)}$, et $\mathfrak{L}'_K(f) = {}^*K\cdot f$.

Si les espaces X^m, Y^n sont confondus avec un même espace R^n, il y a lieu d[e] changer les notations. Un noyau sera une distribution $K_{x,\xi}$ sur $R^n \times R^n$. L'opéra[-] tion $\mathfrak{L}_K$ est celle qui fera correspondre à une fonction $f(x) \in (\mathfrak{D})_x$ une distribu[-] tion $[\mathfrak{L}_K(f)]_x = [K\cdot f]_x \in (\mathfrak{D}')_x$, définie par

$$\int_{R^n} [K\cdot f]_x \varphi(x)\, dx = \iint_{R^n\times R^n} K_{x,\xi}\, \varphi(x) f(\xi)\, dx\, d\xi, \tag{8}$$

tandis que l'opération $\mathfrak{L}'_K$ de $(\mathfrak{D})_x$ dans $(\mathfrak{D}')_x$ fera correspondre à une fonctio[n] $f(x) \in (\mathfrak{D})_x$ une distribution $[\mathfrak{L}'_K(f)]_x \in (\mathfrak{D}')_x$, définie par

$$\int_{R^n} [\mathfrak{L}'_K(f)]_x \varphi(x)\, dx = \iint_{R^n\times R^n} K_{x,\xi}\, \varphi(\xi) f(x)\, dx\, d\xi. \tag{9}$$

On voit que dans ce cas il est logique d'introduire le noyau ${}^sK$, "symétrique" de K (associé à K par la "symétrie", isomorphisme canonique $(x, \xi) \to (\xi, x)$ d[e] $R^n \times R^n$ sur lui-même), défini par

$$\iint_{R^n\times R^n} ({}^sK)_{x,\xi}\, \varphi(x, \xi)\, dx\, d\xi = \iint_{R^n\times R^n} K_{x,\xi}\, \varphi(\xi, x)\, dx\, d\xi. \tag{10}$$

Alors l'opération $\mathfrak{L}'_K$, transposée de $\mathfrak{L}_K$, est l'opération $\mathfrak{L}_{({}^sK)}$ définie par l[e] symétrique de K, et peut être notée $f \to \mathfrak{L}'_K(f) = {}^sK\cdot f$.

3. Exemples. Dans tous les exemples qui suivent, X^m et Y^n seront identique[s] à l'espace R^n.

Exemple 1.

$$\begin{cases} \text{Noyau } I_{x,\xi} \text{ défini par } \displaystyle\iint_{R^n\times R^n} I_{x,\xi}\, \varphi(x, \xi)\, dx\, d\xi = \int_{R^n} \varphi(t, t)\, dt; \\ I = \text{mesure, portée par la diagonale } x = \xi; \\ I\cdot f = f \text{ (opérateur identique); } {}^sI = I. \end{cases} \tag{11}$$

Nous avons là une expression correcte de la première formule (3).

Exemple 2. Soit $S_x \in (\mathfrak{D}')_x$.

$$\begin{cases} \text{Noyau } K \text{ défini par } \displaystyle\iint_{R^n\times R^n} K_{x,\xi}\, \varphi(x, \xi)\, dx\, d\xi = \int_{R^n} S_t\, \varphi(t, t)\, dt; \\ K\cdot f = Sf \text{ (multiplication par } S\text{); } {}^sK = K; \\ \text{Si } S_x = \alpha(x) \text{ est une fonction continue, } K_{x,\xi} = \alpha(x) I_{x,\xi}. \end{cases} \tag{12}$$

Nous avons là l'expression correcte de la deuxième formule (3).

Exemple 3. Soit $S_x \in (\mathfrak{D}')_x$.

$$
(3) \begin{cases} \text{Noyau } K \text{ défini par } K_{x,\xi} = S_{x-\xi}, \text{ ou} \\ \displaystyle\iint_{R^n \times R^n} S_{x-\xi}\varphi(x, \xi)\, dx\, d\xi = \int_{R^n} S_x \left[\int_{R^n} \varphi(x+t, t)\, dt\right] dx; \\ K \cdot f = S * f \text{ (convolution avec } S);^{6} \\ {}^{s}K \text{ s'obtient en remplaçant } S \text{ par } \check{S}. \end{cases}
$$

Avec ces notations, le noyau $I_{x,\xi}$ de l'exemple 1 peut s'écrire $\delta_{x-\xi}$.

Exemple 4. Soit D un polynome de dérivation sur R^n. L'opération de dériva-on $f \to Df$ définie par D correspond au noyau $(D\delta)_{x-\xi}$ (exemple 3) qui est aussi noyau $D_x\, I_{x,\xi}$ (obtenu en appliquant à $I_{x-\xi}$ la dérivation partielle D_x). Cette ernière forme est aussi valable si D est un opérateur différentiel à coefficients ıdéfiniment dérivables. Nous avons là l'expression correcte de la troisième forıule (3). L'opérateur différentiel adjoint (au sens classique) D' sera défini par noyau symétrique, qui n'est autre que $D_\xi I_{x,\xi}$. Ainsi l'adjoint s'obtient tou-ıurs par la symétrie $K \to {}^{s}K$, qu'il s'agisse d'un opérateur intégral ou différentiel.

4. Noyaux et opérations linéaires. Ces exemples suggèrent, comme l'a in-iqué Dirac, que des classes très larges d'opérations linéaires peuvent être éfinies par des noyaux. On peut effectivement démontrer la réciproque du héorème 1, qui sera fondamentale dans la suite:

THÉORÈME II. *Toute transformation linéaire continue de* $(\mathfrak{D})_y$ *(muni de la pologie forte) dans* $(\mathfrak{D}')_x$ *(muni de la topologie faible) peut être définie par* $f \to K \cdot f$, *ù* K *est un noyau déterminé d'une manière unique.*

L'unicité est évidente;[7] l'existence de K fait au contraire appel à des propriétés ssez profondes. Remarquons qu'en prenant la topologie forte sur $(\mathfrak{D})$ et la pologie faible sur $(\mathfrak{D}')$, nous faisons les hypothèses les moins restrictives ossible. Mais elles entraînent alors d'après le Théorème 1, la continuité de D) fort dans $(\mathfrak{D}')$ fort.

Une forme équivalente du théorème est la suivante:

Toute forme bilinéaire $B(u, v)$ *sur* $(\mathfrak{D})_x \times (\mathfrak{D})_y$, *continue par rapport à chacun s arguments quand l'autre est fixé, est de la forme* $B_K(u, v)$ *(formule* 4), *où* K *t un noyau déterminé d'une manière unique.*

5. Prolongement de l'application $\mathfrak{L}_K$. Supposons choisi une fois pour toutes n ensemble $\mathfrak{F}$ d'espaces vectoriels localement convexes de distributions (es-aces fonctionnels généralisés) $\mathfrak{A}$ ayant les propriétés suivantes:

[6] Nous désignerons ici par convolution ce que nous avons appelé dans (TD) composition, ıur éviter la confusion avec la composition de Volterra, qui en est une généralisation, et ra indiquée plus loin. Pour $\check{S}$, voir (TD) formule (VI, 4; 7).

[7] Voir (TD), Chapitre IV, Théorème III.

1°. $(\mathfrak{D}) \subset \mathfrak{A} \subset (\mathfrak{D}')$; les applications identiques de $(\mathfrak{D})$ fort dans $\mathfrak{A}$ et de (dans $(\mathfrak{D}')$ faible sont continues;

2°. Si $\mathfrak{A}$ est dans $\mathfrak{F}$, le dual faible $\mathfrak{A}'$ de $\mathfrak{A}$ est aussi dans $\mathfrak{F}$;

3°. Dans l'intersection $\mathfrak{A} \cap \mathfrak{B}$ de deux espaces appartenant à $\mathfrak{F}$, munie d la topologie borne supérieure des topologies définies par $\mathfrak{A}$ et $\mathfrak{B}$, $(\mathfrak{D})$ est dense Ou encore: Si $T \in (\mathfrak{A} \cap \mathfrak{B})$, il existe des fonctions $\varphi_j \in (\mathfrak{D})$, qui convergent ver T à la fois pour les topologies $\mathfrak{A}$ et $\mathfrak{B}$.

Pratiquement, $\mathfrak{F}$ pourra contenir tous les espaces vectoriels usuels de l'analys fonctionnelle,[8] à condition de les munir éventuellement de leur topologie faibl (dans L^∞ ou dans l'espace $(\mathfrak{C})$ des mesures, $(\mathfrak{D})$ ne serait pas dense pour la topol ogie forte). [Pour chacun de ces espaces $\mathfrak{A}$, si l'on choisit *une suite* de régularis antes $\rho_j \in (\mathfrak{D})$, telles que $\rho_j \geqq 0$, $\int_{R^n}\rho_j(x)\, dx = +1$, et que pour $j \to \infty$, leur supports convergent vers l'origine de R^n, les régularisées $T * \rho_j$ de $T \in \mathfrak{A}$ conver gent pour $j \to \infty$ vers T dans $\mathfrak{A}$. D'autre part si l'on choisit une suite de multi plicateurs $\alpha_j \in (\mathfrak{D})$, tels que $0 \leqq \alpha_j \leqq 1$, et que les fonctions $1 - \alpha_j$ convergen vers 0 uniformément sur tout compact en restant bornées sur R^n, ainsi qu chacune de leurs dérivées,[9] les $\alpha_j S$, pour $S \in \mathfrak{A}$, convergent pour $j \to \infty$ vers S dans $\mathfrak{A}$. De plus, la suite des $\alpha_j(T * \rho_j) \in (\mathfrak{D})$ et la suite des $(\alpha_j T) * \rho_j \in (\mathfrak{D})$ convergeront pour $j \to \infty$ vers T dans $\mathfrak{A}$. Ces suites étant associées à T indé pendamment de l'espace $\mathfrak{A}$ auquel elle appartient, la condition 3° sera assurée On peut d'ailleurs appeler *espace permis* tout espace $\mathfrak{A}$ vérifiant la condition 1° et tel que pour toute $T \in \mathfrak{A}$, les suites $\alpha_j(T * \rho_j)$ et $(\alpha_j T) * \rho_j$ convergen vers T dans $\mathfrak{A}$ pour $j \to \infty$, et $\mathfrak{F}$ sera *l'ensemble de tous les espaces permis*.]

On supposera définie la famille $\mathfrak{F}$ pour les différents espaces numérique X^m, Y^n, etc. qui interviendront.

Soit alors $\mathfrak{L}$ une transformation linéaire continue de $\mathfrak{A} \in \mathfrak{F}$ dans $\mathfrak{B} \in \mathfrak{F}$ D'après 1° elle est continue de $(\mathfrak{D})$ fort dans $(\mathfrak{D}')$ faible, donc il existe un noyau K unique tel que $\mathfrak{L} = \mathfrak{L}_K$ sur $(\mathfrak{D})$; mais alors la connaissance de K détermine $\mathfrak{L}$ sur $(\mathfrak{D})$ donc sur $\mathfrak{A}$ puisque $(\mathfrak{D})$ est dense dans $\mathfrak{A}$ (3°). Donc *toute application linéaire continue d'un espace de distributions dans un autre, si tous deux appartien nent à $\mathfrak{F}$, est entièrement définie par un noyau K unique*. Ainsi les noyaux définissent toutes les opérations linéaires qu'on peut rencontrer dans la pratique. Même dans le cas des espaces fonctionnels les plus simples l'introduction des distribu tions est inévitable. Ainsi la transformation de Hilbert $(X^m = Y^n = R^1)$, $f(x) \to g(x) = vp \int_{-\infty}^{+\infty} f(\xi)d\xi/(x - \xi)$ de L^2 dans L^2 est définie par la distribution $K_{x,\xi} = vp/(x - \xi)$.

Réciproquement soit K un noyau tel que, pour $f \in (\mathfrak{D})$, $K \cdot f$ soit dans $\mathfrak{B}$ et qu'en outre $\mathfrak{L}_K$ soit une application continue de $(\mathfrak{D})$ dans $\mathfrak{B}$ lorsqu'on munit $(\mathfrak{D})$ de la topologie induite par $\mathfrak{A}$. Alors $\mathfrak{L}_K$ peut se prolonger d'une manière unique en une application linéaire continue de $\mathfrak{A}$ dans $\mathfrak{B}$. Pour $T \in \mathfrak{A}$, nous

[8] Nous entendons par là les L^p, et espaces apparentés, et tous les espaces de distributions introduits dans (TD), etc.

[9] Nous avons couramment utilisé ces régularisations et ces multiplications dans (TD) t. 2.

ɔterons toujours par $\mathcal{L}_K(T)$ ou $K \cdot T$ l'image de T par cette application prongée. Nous dirons que K est un noyau $\mathcal{A} \to \mathcal{B}$. Remarquons qu'alors $\mathcal{A}'$ et $\mathcal{B}'$, ı moins lorsqu'ils sont munis des topologies faibles, sont dans $\mathcal{F}$ (condition 2°) ; que $^*K$ (et ^sK si $X^m = Y^n = R^n$) est un noyau $\mathcal{B}' \to \mathcal{A}'$.

On voit d'autre part aisément que si $\mathcal{B}$ est un espace permis, *pour que K ·it un noyau $(\mathcal{D}) \to \mathcal{B}$, il suffit que $\mathcal{L}_K(\mathcal{D}) \subset \mathcal{B}$.*

Naturellement un même noyau K est $\mathcal{A} \to \mathcal{B}$ pour une infinité de choix fférents de $\mathcal{A}$ et $\mathcal{B}$. Ainsi le noyau $I_{x,\xi}$ (§3, exemple 1) est $\mathcal{A} \to \mathcal{A}$, quel que ıit $\mathcal{A}$; le noyau de Fourier exp $(-2i\pi x \cdot \xi)$ égal à son symétrique est $(\mathcal{S}) \to (\mathcal{S})$, ;') $\to (\mathcal{S}')$, $(\mathcal{O}_M) \to (\mathcal{O}'_C)$, $(\mathcal{O}'_C) \to (\mathcal{O}_M)$, $L^2 \to L^2$, etc.

Si K est $\mathcal{A}_1 \to \mathcal{B}_1$ et $\mathcal{A}_2 \to \mathcal{B}_2$, et si $T \in (\mathcal{A}_1 \cap \mathcal{A}_2)$, la valeur de $K \cdot T$ est la .ême (et située dans $\mathcal{B}_1 \cap \mathcal{B}_2$) dans le premier et dans le deuxième prolongement ı vertu de la propriété 3° de l'ensemble $\mathcal{F}$:[10] car $K \cdot T = \lim_{j \to \infty} K \cdot \varphi_j$. On ɔurra donc dire que $K \cdot T$ a un sens, s'il existe un espace $\mathcal{A} \in \mathcal{F}$ tel que $T \in \mathcal{A}$, · que K soit $\mathcal{A} \to (\mathcal{D}')$.

La méthode de prolongement de $\mathcal{L}_K$ indiquée ici est celle que nous avons nployée en théorie des distributions pour étendre à des espaces variés de stributions les opérations de multiplication ou de composition, ou la transrmation de Fourier.

6. Problèmes de support; noyaux compacts.

Théorème III. *Pour que $K_{x,y}$ soit un noyau* $(\mathcal{E}) \to (\mathcal{D}')$ (resp. $(\mathcal{D}) \to (\mathcal{E}')$), *faut et il suffit que l'image réciproque, par la projection* $(x, y) \to x$ (resp. $(x, y) \to$, *de tout compact de* X^m (resp. Y^n), *coupe le support de $K_{x,y}$ dans $X^m \times Y^n$:ivant un compact, ce qui revient à dire que la projection* $(x, y) \to x$ (resp. $(x, y) \to$ *est "régulière à l'infini" sur le support de K.*

Théorème IV. *Pour que $K_{x,y}$ soit un noyau $(\mathcal{E}) \to (\mathcal{E}')$, il faut et il suffit que le ·pport de K dans $X^m \times Y^n$ soit compact, et alors le support de $K \cdot f$ est toujours ntenu dans un compact fixe.*

Un noyau qui est à la fois $(\mathcal{D}) \to (\mathcal{E}')$ et $(\mathcal{E}) \to (\mathcal{D}')$ sera dit *noyau compact*; ›ur un tel noyau, il n'y a jamais de difficultés dues au comportement à l'infini. ɑ noyau $(\mathcal{E}) \to (\mathcal{E}')$ sera dit *compactifiant*, puisque l'image $K \cdot f$ est toujours à pport contenu dans un compact fixe. Le transposé $^*K$ (et, pour $X^m = Y^n = R^n$, symétrique ^sK) d'un noyau compact ou compactifiant a la même propriété. ;s paramétrix des opérateurs différentiels elliptiques sont des noyaux com- ,cts.

[10] On rencontre parfois des espaces de distributions non permis, par exemple l'espace de .lbert $\mathcal{H}$ des distributions d'énergie finie en théorie du potentiel (voir J. Deny, *Les po- ıtiels d'énergie finie*, Acta Math. t. 82). Si T est dans $\mathcal{H}$, on ne sait pas si les α, T ɑt dans $\mathcal{H}$. Mais $(\mathcal{D})$ est dense dans $\mathcal{H}$. Alors toute application linéaire continue de $\mathcal{H}$ ns $(\mathcal{D}')$ sera définie d'une manière unique par un noyau K. Mais si T est à la fois dans ` et dans un espace fonctionnel classique $\mathcal{A}$, rien ne prouve que $K \cdot T$ ne puisse pas avoir ux sens différents lorsque l'on considère T comme dans $\mathcal{H}$ ou comme dans $\mathcal{A}$.

Les noyaux des exemples 1, 2, 4 du §3 sont compacts; le noyau de la multi plication (exemple 2) est compactifiant si S est à support compact. Le noyau de la convolution (exemple 3) est compact si S est à support compact.

Naturellement d'autres extensions relatives aux supports sont possibles comme dans la théorie du produit de convolution. Par exemple, $K \cdot f$ *a toujours un sens si f est indéfiniment dérivable et si le produit de tout compact de X^m par le support de f dans Y^n coupe le support de K dans $X^m \times Y^n$ suivant un compact* (formule 6). Cette extension a un grand intérêt dans l'étude des noyaux liés aux opérateurs différentiels hyperboliques.

En particulier soit K un noyau $(\mathcal{E}) \to (\mathcal{D}')$. Alors $K \cdot 1$ a un sens, et peut être noté sous la forme

$$K \cdot 1 = \int_{Y^n} K_{x,y}\, dy \tag{14}$$

par analogie avec ce qui se passe si K est une fonction $K(x, y)$. Si maintenant K est un noyau quelconque, et si $f \in (\mathcal{D})$, le produit multiplicatif $K_{x,y} f(y)$ est un noyau $(\mathcal{E}) \to (\mathcal{D}')$, et on voit que l'on peut écrire

$$K \cdot f = [K_{x,y} f(y)] \cdot 1 = \int_{Y^n} K_{x,y} f(y)\, dy \tag{15}$$

ce qui, si K est une fonction $K(x, y)$, redonne la formule (2).

Si K est un noyau à support compact, on a la formule

$$\iint_{X^m \times Y^n} K_{x,y}\, dx\, dy = \int_{X^m} \left[\int_{Y^n} K_{x,y}\, dy \right] dx; \tag{16}$$

alors si K est un noyau quelconque, et si $\varphi(x, y) \in (\mathcal{D})_{x,y}$, on aura

$$\iint_{X^m \times Y^n} K_{x,y}\, \varphi(x, y)\, dx\, dy = \int_{X^m} dx \left[\int_{Y^n} K_{x,y}\, \varphi(x, y)\, dy \right], \tag{16$^{\text{bis}}$}$$

l'ordre des intégrations pouvant être interverti.

7. Problèmes de régularité locale. Noyaux réguliers. Soit $f \to \mathcal{L}(f)$ une opération linéaire continue de $(\mathcal{D})_y$ fort dans $(\mathcal{D}')_x$ faible, telle que $\mathcal{L}(f)$ soit toujours une fonction m fois continûment différentiable de x, soit $\mathcal{L}(f)(x)$ C'est alors une opération continue de $(\mathcal{D})_y$ dans $(\mathcal{E}^m)_x$. Pour x fixé, $f \to \mathcal{L}(f)(x)$ est une forme linéaire continue sur $(\mathcal{D})_y$, donc une distribution S_y, qu'on devra noter $S_y(x)$, puisqu'elle dépend de x; l'application $x \to S_y(x)$ est alors une fonction vectorielle (à valeurs dans $(\mathcal{D}')_y$) de x, m fois continûment différentiable Réciproquement une telle fonction vectorielle $S_y(x)$ définit l'application linéaire continue de $(\mathcal{D})_y$ dans $(\mathcal{E}^m)_x$

$$f \to \int_{Y^n} S_y(x) f(y)\, dy, \tag{17}$$

l'intégrale (au sens de la formule (1)) étant calculable pour toute valeur individuelle de x. Le Théorème 2 est ici trivial, car $\mathcal{L}(f) = K\cdot f$, le noyau K étant défini par

$$\iint_{X^m \times Y^n} K_{x,y}\varphi(x, y)\, dx\, dy = \int_{X^m} dx \left[\int_{Y^n} S_y(x)\varphi(x, y)\, dy\right], \tag{18}$$

le crochet étant calculable pour toute valeur de x, et fonction numérique de x m fois continûment différentiable à support compact. Nous pouvons donc énoncer:

Théorème V. *Pour que $f \to \mathcal{L}(f)$ soit une opération linéaire continue de $(\mathcal{D})_y$ dans $(\mathcal{E}^m)_x$, il faut et il suffit que $\mathcal{L}(f) = K\cdot f$, K étant un noyau défini par la formule* (18), *où $x \to S_y(x)$ est une application m fois continûment différentiable* de X^m *dans* $(\mathcal{D}')_y$.

Soit maintenant $T \to \mathcal{L}(T)$ une application linéaire faiblement continue de $(\mathcal{E}'^m)_y$ dans $(\mathcal{D}')_x$. Elle est alors transposée d'une application linéaire continue de $(\mathcal{D})_x$ dans $(\mathcal{E}^m)_y$, dont nous connaissons la forme. Il existe donc une application m fois continûment différentiable $y \to \Phi(y) = \Sigma_x(y)$ de Y^n dans $(\mathcal{D}')_x$, telle que $\mathcal{L}(f) = K\cdot f$, le noyau K étant défini par

$$\iint_{X^m \times Y^n} K_{x,y}\,\varphi(x, y)\, dx\, dy = \int_{Y^n} dy \left[\int_{X^m} \Sigma_x\,(y)\varphi(x, y)\, dx\right] \tag{19}$$

et réciproquement.

Remarquons alors que la formule

$$\int_{X^m} (K\cdot f)_x\varphi(x)\, dx = \int_{Y^n} f(y)\, dy \left[\int_{X^m} \Sigma_x(y)\varphi(x)\, dx\right] \tag{20}$$

montre que $K\cdot f$ peut s'écrire comme intégrale (faible ou forte) de la fonction continue vectorielle $\Phi(y) = \Sigma_x(y)$ (à valeurs dans $(\mathcal{D}')_x$):[11]

$$K\cdot f = \int_{Y^n} \Phi(y)f(y)\, dy. \tag{21}$$

Mais comme $\Phi(y)$ est fonction m fois continûment différentiable de y, on peut dans cette formule remplacer $f(y)$ par une distribution $T_y \in (\mathcal{E}'^m)_y$, et écrire

$$K\cdot T = \int_{Y^n} \Phi(y)T_y\, dy. \tag{22}$$

On a d'ailleurs les formules suivantes, qui auraient pu être considérées directement et servir à l'étude de ce cas indépendamment de celui qui précède:[12]

$$K\cdot\delta_{y;(\lambda)} = \Phi(\lambda) \tag{23}$$

[11] Voir (TD), t. 1, p. 30.

[12] Rappelons que $\delta_{y;(\lambda)}$ désigne la masse $+1$ au point λ dans l'espace Y^n de la variable y.

(et $\Phi(\lambda)$ doit être une fonction m fois continûment différentiable de λ, puisque $\delta_{y;(\lambda)}$ est une fonction (à valeurs dans $(\mathcal{E}'^m)_y$) m fois continûment différentiable de λ);

$$(24)\qquad K\cdot T_y = K\cdot\left[\int_{Y^n} T_\lambda\,\delta_{y;(\lambda)}\,d\lambda\right] = \int_{Y^n} T_\lambda\,d\lambda[K\cdot\delta_{y;(\lambda)}] = \int_{Y^n}\Phi(\lambda)T_\lambda\,d\lambda.$$

Et nous pouvons énoncer:

THÉORÈME VI. *Pour que $T \to \mathcal{L}(T)$ soit une opération linéaire faiblement continue de $(\mathcal{E}'^m)_y$ dans $(\mathcal{D}')_x$, il faut et il suffit que $\mathcal{L}(T) = K\cdot T$, K étant un noyau défini par la formule* (19), *et $K\cdot T$ par la formule* (22), *où $\Sigma_x(y)$ est une application m fois continûment différentiable de Y^n dans $(\mathcal{D}')_x$.*

Un noyau $K_{x,y}$ défini par $\Sigma_x(y)$ pourra être écrit $K_x(y)$, pour indiquer que c'est une distribution en x, y, qui est une fonction en y; de même un noyau $K_{x,y}$ défini par $S_y(x)$ sera noté $[K(x)]_y$ (plutôt que $K_y(x)$, de façon à conserver l'ordre des 2 variables x, y).

Nous appellerons *noyau régulier* un noyau qui est à la fois $(\mathcal{D}) \to (\mathcal{E})$ et $(\mathcal{E}') \to (\mathcal{D}')$. Il sera alors à la fois de la forme $K_x(y) = \Sigma_x(y)$ (application indéfiniment différentiable de Y^n dans $(\mathcal{D}')_x$) et $[K(x)]_y$ (application indéfiniment différentiable de X^m dans $(\mathcal{D}')_y$). Mais il n'y a aucune raison pour qu'un tel noyau puisse être écrit comme $K(x, y)$, fonction de x et y; c'est là un aspect d'un *principe d'incertitude* très général dont nous reparlerons ailleurs. Si K est régulier, $^*K$ (et ^sK si $X^m = Y^n = R^n$) est aussi régulier.

Les noyaux des exemples 1, 3, 4, du §3 sont réguliers; le noyau de la multiplication (exemple 2) n'est régulier que si $S_x = \alpha(x) \in (\mathcal{E})_x$. Les noyaux intervenant dans la théorie des équations aux dérivées partielles hyperboliques ou elliptiques (noyau élémentaire, paramétrix, etc.) sont toujours réguliers.

Considérons par exemple le noyau $S_{x-\xi}$ de l'exemple 3 du §3. Il peut être écrit sous les deux formes suivantes:

$$(25)\qquad \begin{cases} [K(x)]_\xi = \tau_x\check{S}_\xi \text{ (translaté } \tau_x \text{ du noyau } \check{S}_\xi \text{ dans l'espace de la variable } \xi); \\ K_x(\xi) = \tau_\xi S_x \text{ (translaté } \tau_\xi \text{ du noyau } S_x \text{ dans l'espace de la variable } x); \end{cases}$$

mais il est impossible de mettre un tel noyau sous la forme d'une fonction $K(x, \xi)$ de x et ξ, si la distribution S n'est pas une fonction.

THÉORÈME VII. *Si K est un noyau régulier compact, il est* $(\mathcal{D}) \to (\mathcal{D})$, $(\mathcal{E}) \to (\mathcal{E})$, $(\mathcal{E}') \to (\mathcal{E}')$, $(\mathcal{D}') \to (\mathcal{D}')$.

Les paramétrix des opérateurs différentiels elliptiques sont réguliers compacts.

THÉORÈME VIII. *Pour que K soit un noyau* $(\mathcal{E}') \to (\mathcal{E})$, *il faut et il suffit que K soit une fonction $K(x, y)$, indéfiniment dérivable sur $X^m \times Y^n$.*

Un tel noyau sera appelé *régularisant*, puisqu'il transforme une distribution n fonction indéfiniment dérivable. Les noyaux de convolution (exemple 3 du 3) sont régularisants lorsque $S \in (\mathcal{E})$. Alors $S_{x-\xi} = S(x - \xi)$. Si K est un oyau régularisant, l'opération $T \to K \cdot T$ sera une *régularisation*.

Naturellement ces problèmes de régularité locale peuvent être étudiés de ien plus près, et donnent lieu à l'introduction de nombreuses catégories in-ēressantes de noyaux.

8. Produit de composition de Volterra. Si $H(x, y)$ est un noyau-fonction ır $X^m \times Y^n$, $K(y, z)$ un noyau-fonction sur $Y^n \times Z^p$, le produit de composition e Volterra est défini usuellement par

$$
L(x, z) = H(x, y) \circ K(y, z) = \int_{Y^n} H(x, y)K(y, z)\, dy. \tag{26}
$$

'est un noyau sur $X^m \times Z^p$, quand il est défini.

Théorème IX. *Si $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, sont 3 espaces de distributions appartenant à $\mathfrak{F}$, $H_{x,y}$ est un noyau $\mathfrak{B} \to \mathfrak{C}$, $K_{y,z}$ un noyau $\mathfrak{A} \to \mathfrak{B}$, on peut définir un produit de ımposition de Volterra $H \circ K$, qui est un noyau $\mathfrak{A} \to \mathfrak{C}$; le noyau obtenu est in-ēpendant de la séquence $\mathfrak{A} \to \mathfrak{B} \to \mathfrak{C}$ utilisée.*

En effet K définit l'opération linéaire continue $\mathfrak{L}_K$ de $\mathfrak{A}$ dans $\mathfrak{B}$, et H l'opération néaire continue $\mathfrak{L}_H$ de $\mathfrak{B}$ dans $\mathfrak{C}$; alors $\mathfrak{L}_H \circ \mathfrak{L}_K$ est une opération linéaire ɔntinue de $\mathfrak{A}$ dans $\mathfrak{C}$, il lui correspond donc un noyau, qui sera par défini-on $H \circ K$. Ce produit est indépendant de la séquence $\mathfrak{A} \to \mathfrak{B} \to \mathfrak{C}$ utilisée. upposons en effet que $\mathfrak{A}_1 \to \mathfrak{B}_1 \to \mathfrak{C}_1$ et $\mathfrak{A}_2 \to \mathfrak{B}_2 \to \mathfrak{C}_2$ soient deux séquences ɔssibles pour les mêmes noyaux H et K. Pour $f \in (\mathfrak{D})$, $T = K \cdot f$ est dans $\mathfrak{B}_1 \cap$ $_2$, et sa définition est indépendante de tout prolongement. Mais alors d'après s résultats du §5 (axiome 3°), $U = H \cdot T \in (\mathfrak{D}')$ est la même distribution (né-ɔssairement dans $\mathfrak{C}_1 \cap \mathfrak{C}_2$), que l'on considère H comme $\mathfrak{B}_1 \to \mathfrak{C}_1$ ou comme $\mathfrak{B}_2$ $\to \mathfrak{C}_2$; donc $U = (H \circ K) \cdot f$ est indépendante de la séquence utilisée, et par iite aussi le noyau $H \circ K$. Cette indépendance nous permet d'écrire $H \circ K$, uand il existe, sans spécifier la séquence $\mathfrak{A} \to \mathfrak{B} \to \mathfrak{C}$ ayant servi à le définir.

Dans le cas de noyaux-fonctions, cette définition redonne bien le produit de ımposition usuel de Volterra.

On définit de même le produit de composition de plusieurs noyaux $K_1 \circ K_2 \circ$ $_3 \circ \cdots \circ K_l$, à partir d'une séquence d'espaces de distributions $\mathfrak{A}_{l+1} \to \mathfrak{A}_l \to$ $\cdot \to \mathfrak{A}_3 \to \mathfrak{A}_2 \to \mathfrak{A}_1$ appartenant à $\mathfrak{F}$; un tel produit, défini d'emblée, peut re défini de proche en proche, et ce produit est alors associatif, et indépendant ı procédé de définition.[13]

La composition de noyaux définissant des multiplications (exemple 2 du §3)

[13] Il peut arriver que $K_1 \circ K_2$, $(K_1 \circ K_2) \circ K_3$, $K_2 \circ K_3$, $K_1 \circ (K_2 \circ K_3)$, aient un sens, que $(K_1 \circ K_2) \circ K_3 \neq K_1 \circ (K_2 \circ K_3)$, lorsque on ne peut pas définir d'emblée le produit $_1 \circ K_2 \circ K_3$. Voir (TD), t. 1, p. 119 et t. 2, p. 27.

donne lieu à un produit multiplicatif des distributions S correspondantes; l
composition (de Volterra) de noyaux associés à des "convolutions" (exempl
3 du §3) donne lieu à un produit de convolution des distributions S corres
pondante.

On dira que deux noyaux sont composables si leur produit de composition d
Volterra a un sens (cette notion dépendra de l'ensemble $\mathcal{F}$ d'espaces vectoriel
utilisés; pratiquement on prendra les "espaces permis", voir §5). Comme deu
noyaux arbitraires ne sont pas composables, il est intéressant d'avoir des critères

THÉORÈME X. *Le produit de composition de plusieurs noyaux a un sens, s tous, sauf un au plus, sont compacts, et tous, sauf un au plus, réguliers.*

Raisonnons dans le cas de deux noyaux. On aura alors les cas suivants:
H compact et K régulier: séquence $(\mathcal{D}) \to (\mathcal{E}) \to (\mathcal{D}')$.
H régulier et K compact: séquence $(\mathcal{D}) \to (\mathcal{E}') \to (\mathcal{D}')$.
K régulier compact: séquence $(\mathcal{D}) \to (\mathcal{D}) \to (\mathcal{D}')$.
H régulier compact: séquence $(\mathcal{D}) \to (\mathcal{D}') \to (\mathcal{D}')$.

THÉORÈME XI. *Dans les conditions du Théorème* X, *si tous les noyaux facteur sont compacts* (*resp. réguliers*), *il en est de même du produit*; *si tous les noyau facteurs sont compacts* (*resp. réguliers*) *et l'un au moins compactifiant* (*resp. régu larisant*), *le produit est compactifiant* (*resp. régularisant*); *si le premier et le dernie noyaux facteurs sont compactifiants* (*resp. régularisants*), *il en est de même d produit.*

Il est intéressant de pouvoir représenter $H \circ K$ sous une forme rappelant (26) Supposons par example que H soit compact et K régulier, on aura

$$\iint_{X^m \times Z^p} (H \circ K)_{x,z}\, \varphi(x, z)\, dx\, dz = \iint_{X^m \times Y^n} H_{x,y}\, dx\, dy \left[\int_{Z^n} [K(y)]_z\, \varphi(x, z)\, dz\right] \tag{27}$$

le crochet étant calculable pour toute valeur individuelle de x et y, et fonctio numérique indéfiniment dérivable de x et y. Cette relation peut s'écrire sous l forme symbolique suivante:

$$(H \circ K)_{x,z} = \int_{Y^n} H_{x,y}[K(y)]_z\, dy, \tag{28}$$

l'intégrale étant une généralisation de celle de la formule (15), correspondan au cas où l'on remplace $f(y)$ par $F(y) = [K(y)]_z$, fonction vectorielle indéfini ment dérivable de y à valeurs dans $(\mathcal{D}')_z$.

Cette théorie des noyaux a des applications à la théorie des opérateurs dif férentiels elliptiques et hyperboliques, que nous développerons ailleurs.

UNIVERSITY OF NANCY,
NANCY, FRANCE.

BASIC IDEAS OF A GENERAL THEORY OF STATISTICAL DECISION RULES

ABRAHAM WALD

1. Introduction. The theory of statistics has had an extraordinarily rapid growth during the last 30 years. The line of development has been largely set by two great schools, the school of R. A. Fisher and that of Neyman and Pearson. A basic feature of the theories represented by these schools is the development of various criteria for the best possible use of the observations for purposes of statistical test and estimation procedures. In this connection I would like to mention the basic notions of efficiency and sufficiency introduced by Fisher, and that of the power of a test introduced by Neyman and Pearson. It is unnecessary to dwell on the importance of these notions, since they are well known to all statisticians.

Until about 10 years ago, the available statistical theories, except for a few scattered results, were restricted in two important respects: (1) experimentation was assumed to consist of a single stage, i.e., the number of observations to be made was assumed to be fixed in advance of experimentation; (2) the decision problems treated were restricted to special types known in the literature under the names of testing a hypothesis, point and interval estimation. In the last few years a general decision theory has been developed (see, for example, [1][1]) that is free of both of these restrictions. It allows for multi-stage experimentation and it includes the general multi-decision problem.

I would like to outline the principles of this general theory and some of the results that have been obtained.

A statistical decision problem is defined with reference to a sequence $X = \{X_i\}$ ($i = 1, 2, \cdots$, ad inf.) of random variables. For any sequence $x = \{x_i\}$ ($i = 1, 2, \cdots$, ad inf.) of real values, let $F(x)$ denote the probability that $X_i < x_i$ holds for all positive integral values i. The function $F(x)$ is called the distribution function of X. A characteristic feature of any statistical decision problem is that F is unknown. It is merely assumed to be known that F is a member of a given class Ω of distribution functions. The class Ω is to be regarded as a datum of the decision problem. Another datum of the decision problem is a space D, called decision space, whose elements d represent the possible decisions that can be made by the statistician in the problem under consideration.

For the sake of simplicity, we shall assume for the purposes of the present discussion that (1) each element F of Ω is absolutely continuous, i.e., it admits a probability density function; (2) the space D consists of a finite number of elements, $d_1, \cdots, d_k$ (say); (3) experimentation is carried out sequentially, i.e., the first stage of the experiment consists of observing the value of X_1. After the value of X_1 has been observed, the statistician may decide either to terminate

[1] Numbers in brackets refer to the references at the end of the paper.

experimentation with some final decision d, or to observe the value of X_2. In the latter case, after X_1 and X_2 have been observed, the statistician may again decide to terminate experimentation with some final decision d, or to observe the value of X_3, and so on.

The above assumptions on the spaces Ω and D and the method of experimentation are replaced by considerably weaker ones in the general theory given in [1]. These assumptions are made here merely for the purpose of simplifying the discussion.

A decision rule δ, that is, a rule for carrying out experimentation and making a final decision d, can be given in terms of a sequence of real-valued and Borel measurable functions $\delta_{im}(x_1, \cdots, x_m)$ $(i = 0, 1, \cdots, k; m = 0, 1, 2, \cdots, \text{ad inf.})$ where $x_1, x_2, \cdots$, etc. are real variables and the functions δ_{im} are subject to the following two conditions:

$$(1.1) \qquad \delta_{im} \geqq 0; \qquad \sum_{i=0}^{k} \delta_{im} = 1 \ (i = 0, 1, \cdots, k; m = 0, 1, 2, \cdots, \text{ad inf.}).$$

The decision rule δ is defined in terms of the function δ_{im} as follows: Let x_i denote the observed value of X_i. At each stage of the experiment (after the mth observation, for each integral value m) we compute the values $\delta_{0m}(x_1, \cdots, x_m), \delta_{1m}(x_1, \cdots, x_m), \cdots, \delta_{km}(x_1, \cdots, x_m)$ and then perform an independent random experiment with the possible outcomes $0, 1, 2, \cdots, k$ constructed so that the probability of the outcome i is δ_{im}. If the outcome is a number $i > 0$, we terminate experimentation with the decision d_i. If the outcome is 0, we make an additional observation (we observe the value of X_{m+1}) and repeat the process with the newly computed values $\delta_{0,m+1}, \delta_{1,m+1}, \cdots, \delta_{k,m+1}$, and so on.

The above described decision rule may be called a randomized decision rule, since at each stage of the experiment an independent chance mechanism is used to decide whether experimentation be terminated with some final decision d or whether an additional observation be made. The special case when the functions δ_{im} can take only the values 0 and 1 is of particular interest, since in this case the decision to be made at each stage of the experiment is based entirely on the observed values obtained and one can dispense with the use of an independent chance mechanism. We shall call a decision rule $\delta = \{\delta_{im}\}$ nonrandomized if the functions δ_{im} can take only the values 0 and 1. The question whether it is sufficient to consider only nonrandomized decision rules for the purposes of statistical decision making is of considerable interest. We shall return to this question later.

2. Loss, cost, and risk functions. A basic problem in statistical decision theory is the problem of a proper choice of a decision rule δ. In order to judge the relative merits of the various possible decision rules, it is necessary to state the cost of experimentation and the relative degree of preference we would have for the various possible final decisions d if some element F of Ω were known to us to be he true distribution. The latter may be described by a non-negative funcion $W(F, d)$, called loss function, which expresses the loss suffered by the statstician when the decision d is made and F happens to be the true distribution f X. In most decision problems each element d_i of D can be interpreted as the lecision to accept the hypothesis that the unknown distribution F is an element f a given subclass ω_i of Ω. In such a case we put $W(F, d_i) = 0$ when $F \in \omega_i$ and >0 when $F \notin \omega_i$. The cost of experimentation can be described by a sequence $c_m(x_1, \cdots, x_m)\}$ $(m = 1, 2, \cdots$, ad inf.) of non-negative functions where $_m(x_1, \cdots, x_m)$ denotes the cost of experimentation if the experiment consists of $\imath$ observations and x_i is the observed value of X_i $(i = 1, \cdots, m)$. The loss and ost functions are to be regarded as data of the decision problem. The cost unction $c_m(x_1, \cdots, x_m)$ is, of course, assumed to be Borel measurable.

Let $p(m, d_i \mid \delta, x_1, \cdots, x_m)$ denote the conditional probability that experiaentation will consist of m observations and the decision d_i will be made when is the decision rule adopted and x_j is the observed value of X_j $(j = 1, \cdots, m)$. Clearly,

$$
(2.1) \quad p(m, d_i \mid \delta, x_1, \cdots, x_m) = \delta_{00}\delta_{01}(x_1) \cdots \delta_{0,m-1}(x_1, \cdots, x_{m-1})\delta_{im}(x_1, \cdots, x_m).
$$

For any positive integral value m, let $f(x_1, \cdots, x_m \mid F)$ denote the joint ensity function of $X_1, \cdots, X_m$ when F is the true distribution of X. The exected loss, i.e., the expected value of $W(F, d)$ depends only on the true disribution F and the decision rule δ adopted. It is given by the expression

$$
(2.2) \quad \begin{aligned} r_1(F, \delta) &= \sum_{i=1}^{k} W(F, d_i)\delta_{0i} \\ &+ \sum_{m=1}^{\infty} \sum_{i=1}^{k} \int_{R_m} W(F, d_i)p(m, d_i \mid \delta, x_1, \cdots, x_m)f(x_1, \cdots, x_m \mid F)\, dx_1, \cdots, dx_m \end{aligned}
$$

'here R_m denotes the space of all m-tuples $(x_1, \cdots, x_m)$.

The expected cost of experimentation depends only on the true distribution F nd the decision rule δ adopted. It is given by

$$
(2.3) \quad r_2(F, \delta) = \sum_{m=1}^{\infty} \sum_{i=1}^{k} \int_{R_m} c_m(x_1, \cdots, x_m)p(m, d_i \mid \delta, x_1, \cdots, x_m)f(x_1, \cdots, x_m \mid F)\, dx_1, \cdots, dx_m .
$$

Let

$$
(2.4) \qquad r(F, \delta) = r_1(F, \delta) + r_2(F, \delta).
$$

The quantity $r(F, \delta)$ is called the risk when F is true and the decision rule δ adopted. For any fixed decision rule δ^0, the risk is a function of F only. We iall call $r(F, \delta^0)$ the risk function associated with the decision rule δ^0.

It is perhaps not unreasonable to judge the merit of any particular decision ıle entirely on the basis of the risk function associated with it. We shall say that the decision rule δ^1 is *uniformly better* than the decision rule δ^2 if $r(F, \delta^1) \leqq r(F, \delta^2)$ for all F and $r(F, \delta^1) < r(F, \delta^2)$ for at least one member F of Ω. A decision rule δ will be said to be *admissible* if there exists no uniformly better decision rule. Two decision rules δ^1 and δ^2 will be said to be *equivalent* if they have identical risk functions, i.e., if $r(F, \delta^1) = r(F, \delta^2)$ for all F in Ω. For any $\epsilon > 0$, two decision rules δ^1 and δ^2 will be said to be ϵ-equivalent if $| r(F, \delta^1) - r(F, \delta^2) | \leqq \epsilon$ for all F in Ω.

3. Elimination of randomization when Ω is finite. It was proved recently by Dvoretzky, Wolfowitz, and the author [2] that if Ω is finite, then for every decision rule δ there exists an equivalent nonrandomized decision rule δ^*. Thus, in this case one can dispense with randomization and it is sufficient to consider only nonrandomized decision rules. A similar result for a somewhat more special type of randomized decision rule than the one described here was obtained independently by Blackwell [3]. The proof is based on an extension of a theorem by Liapounoff [4] concerning the range of a vector measure. The continuity of the distribution of X_1 (implied by our assumption of absolute continuity of F) is essential for the above stated result. In case of discontinuous distributions there may exist randomized decision rules with risk functions having some desirable properties that cannot be achieved by any nonrandomized decision rule.

The finiteness of the space Ω is a very restrictive condition which is seldom fulfilled in statistical decision problems. However, it was shown in [2] that for any decision rule δ and for any $\epsilon > 0$ there exists an ϵ-equivalent and nonrandomized decision rule δ^* under very general conditions which are usually fulfilled in decision problems arising in applications.

An interesting result on the possible elimination of randomization, but of a somewhat different nature, was found recently by Hodges and Lehmann [5]. They proved that if the decision problem is a point estimation problem, if D is a Euclidean space, and if the loss $W(F, d)$ is a convex function of d for every F, then for any randomized decision rule δ (with bounded risk function) there exists a nonrandomized decision rule δ^* such that $r(F, \delta^*) \leqq r(F, \delta)$ for all F in Ω. It is remarkable that neither the finiteness of Ω nor the continuity of its elements F are needed for the validity of this result.

4. A convergence definition in the space of decision rules and some continuity theorems. A natural convergence definition in the space of decision rules would seem to be the following one: $\lim_{j=\infty} \delta^j = \delta^0$ if $\lim_{j=\infty} p(m, d_1 \mid \delta^j, x_1, \cdots, x_m) = p(m, d_i \mid \delta^0, x_1, \cdots, x_m)$ for all m, all $i > 0$, and for all $x_1, \cdots, x_m$. This convergence definition is, however, too strong for our purposes. Instead, we shall adopt the following weaker one: We shall say that

$$\lim_{j=\infty} \delta^j = \delta^0 \tag{4.1}$$

if

$$\lim_{j=\infty} \delta^j_{i0} = \delta^0_{i0} \qquad (i = 0, 1, \cdots, k) \tag{4.2}$$

and

$$\lim_{j=\infty} \int_{S_m} p(m, d_i \mid \delta^j, x_1, \cdots, x_m)\, dx_1 \cdots dx_m = \int_{S_m} p(m, d_i \mid \delta^0, x_1, \cdots, x_m)\, dx_1 \cdots dx_m \tag{4.3}$$

$$(i = 0, 1, \cdots, k; m = 1, 2, \cdots, \text{ad inf.})$$

holds for every measurable subset S_m of the space of all m-tuples $(x_1, \cdots, x_m)$.

It was shown (see, for example, Theorem 3.1 in [1]) that adopting the above convergence definition the following theorem holds.

THEOREM 4.1. *The space of all decision rules is compact, i.e., every sequence* $\{\delta^j\}$ $(j = 1, 2, \cdots$, ad inf.) *of decision rules admits a convergent subsequence.*

The above theorem is a simple consequence of known theorems on the "weak" compactness of a set of functions (see, for example, Theorem 17b (p. 33) of [6]).[2]

Before stating certain continuity theorems, we shall formulate two conditions concerning the loss and cost functions.

Condition I. *$W(F, d_i)$ is a bounded function of F for $i = 1, 2, \cdots, k$.*

Condition II. *The cost function has the following properties*: (i) $c_m(x_1, \cdots, x_m) \geqq 0$; (ii) $c_{m+1}(x_1, \cdots, x_{m+1}) \geqq c_m(x_1, \cdots, x_m)$; (iii) $c_m(x_1, \cdots, x_m)$ *is a bounded function of* $x_1, \cdots, x_m$ *for every fixed* m; (iv) $\lim_{m=\infty} c_m(x_1, \cdots, x_m) = \infty$ *uniformly in* $x_1, \cdots, x_m$.

The following continuity theorems have been proved in [1]:

THEOREM 4.2. *Let* $\{\delta^j\}$ $(j = 0, 1, 2, \cdots$, ad inf.) *be a sequence of decision rules such that* $\lim_{j=\infty} \delta^j = \delta^0$ *and such that*

$$\delta^j_{00}\delta^j_{01}(x_1)\delta^j_{02}(x_1, x_2) \cdots \delta^j_{0N}(x_1, \cdots, x_N) = 0 \qquad (j = 0, 1, 2, \cdots, \text{ad inf.})$$

identically in $x_1, \cdots, x_N$ *for some positive integer* N. *Then, if Conditions* I *and* II *hold, we have* $\lim_{j=\infty} r(F, \delta^j) = r(F, \delta^0)$ *for all* F.

THEOREM 4.3. *If* $\lim_{j=\infty} \delta^j = \delta^0$ *and if Conditions* I *and* II *hold, then* $\liminf_{j=\infty} r(F, \delta^j) \geqq r(F, \delta^0)$ *for all* F.

5. Bayes and minimax solutions of the decision problem. In this section we shall discuss the notions of Bayes and minimax solutions and some of their properties. These solutions are not only of intrinsic interest, but they play an important role in the construction and characterization of complete classes of

[2] Theorem 3.1 in [1] is actually much stronger and more difficult to prove, since D is not assumed there to be finite.

decision rules discussed in the next section. We shall start out with some definitions.

By an a priori probability distribution ξ in Ω we shall mean a non-negative and countably additive set function ξ defined over a properly chosen Borel field of subsets of Ω for which $\xi(\Omega) = 1$. The Borel field is chosen such that $r(F, \delta)$ is a measurable function of F for every fixed δ.

For any a priori probability distribution ξ, let

$$r^*(\xi, \delta) = \int_\Omega r(F, \delta)\, d\xi. \tag{5.1}$$

A decision rule δ^0 is said to be a Bayes solution relative to the a priori distribution ξ if

$$r^*(\xi, \delta^0) = \operatorname*{Min}_\delta r^*(\xi, \delta). \tag{5.2}$$

A decision rule δ^0 is said to be a Bayes solution in the strict sense if there exists an a priori distribution ξ such that δ^0 is a Bayes solution relative to ξ.

A rule δ^0 is said to be a Bayes solution relative to the sequence $\{\xi_i\}$ ($i = 1, 2, \cdots$, ad inf.) of a priori distributions if

$$\lim_{i=\infty} [r^*(\xi_i, \delta^0) - \operatorname*{Inf}_\delta r^*(\xi_i, \delta)] = 0 \tag{5.3}$$

where the symbol Inf_δ stands for infimum with respect to δ.

We shall say that a decision rule δ^0 is a Bayes solution in the wide sense if there exists a sequence $\{\xi_i\}$ of a priori distributions such that δ^0 is a Bayes solution relative to $\{\xi_i\}$.

A decision rule δ^0 is said to be a *minimax solution* if

$$\operatorname*{Sup}_F r(F, \delta^0) \leqq \operatorname*{Sup}_F r(F, \delta) \text{ for all } \delta, \tag{5.4}$$

where the symbol Sup_F stands for supremum with respect to F.

An a priori distribution ξ_0 is said to be *least favorable* if the following relation is satisfied:

$$\operatorname*{Inf}_\delta r^*(\xi_0, \delta) \geqq \operatorname*{Inf}_\delta r^*(\xi, \delta) \text{ for all } \xi. \tag{5.5}$$

The reason that an a priori distribution ξ_0 satisfying the above relation is called least favorable is this: If an a priori distribution ξ actually exists and is known to the statistician, a satisfactory solution of the decision problem is to use a Bayes solution δ relative to ξ since δ minimizes the average risk (averaged in accordance with the a priori distribution ξ). The minimum average risk that can be achieved will generally be different for different a priori distributions and an a priori distribution ξ may be regarded the less favorable from the point of view of the statistician the greater the minimum average risk associated with ξ. Thus, an a priori distribution satisfying (5.5) will be least favorable from the point of view of the statistician.

We shall state some of the results obtained concerning Bayes and minimax olutions.[3]

THEOREM 5.1. *If Conditions* I *and* II *hold, for any a priori distribution* ξ, *there xists a decision rule* δ *such that* δ *is a Bayes solution relative to* ξ.

THEOREM 5.2. *If Conditions* I *and* II *hold, there exists a minimax solution.*

The above existence theorems can easily be derived from the theorems stated ı §4. With the help of these theorems we can even prove the slightly stronger esult that admissible Bayes and admissible minimax solutions always exist.

THEOREM 5.3. *If Conditions* I *and* II *hold, then a minimax solution is always a ayes solution in the wide sense.*

THEOREM 5.4. *If* δ^0 *is a minimax solution and* ξ_0 *is a least favorable a priori istribution, then, if Conditions* I *and* II *hold,* δ^0 *is a Bayes solution relative to* $_0$ *and the set* ω *of all members* F *of* Ω *for which* $r(F, \delta^0) = \operatorname{Sup}_F r(F, \delta^0)$ *has the robability measure* 1 *according to* ξ_0.

The last part of Theorem 5.4 implies that the risk function of a minimax soluion has a constant value over a subset ω of Ω whose probability measure is ne according to every least favorable a priori distribution ξ. In many decision roblems the risk function of a minimax solution is constant over the whole pace Ω.

Some additional results can be stated if the validity of the following additional ondition is postulated:

Condition III. *The space* Ω *is compact and the loss function* $W(F, d)$ *is continuus in* F *in the sense of the following convergence definition in* Ω: *We shall say at* $\lim_{i=\infty} F_i = F_0$ *if for every positive integer* m *we have*

$$\lim_{i=\infty} \int_{S_m} f(x_1, \cdots, x_m \mid F_i)\, dx_1 \cdots dx_m = \int_{S_m} f(x_1, \cdots, x_m \mid F_0)\, dx_1 \cdots dx_m$$

niformly in all measurable subsets S_m *of the space of all m-tuples* $(x_1, \cdots, x_m)$.

THEOREM 5.5. *If Conditions* I, II, *and* III *hold, a least favorable a priori distriution exists.*

The proof of this theorem is based on the fact that the space of all probability ıeasures ξ on a compact space Ω is compact in the sense of the following conergence definition: $\lim_{i=\infty} \xi_i = \xi_0$ if $\lim \xi_i(\omega) = \xi_0(\omega)$ for any open subset ω of ıe space Ω whose boundary has probability measure zero according to ξ_0. 'his result was proved in [1, Theorem 2.15, p. 50]. A closely related result was

[3] For a detailed discussion and proofs, see §3.5 in [1].

obtained by Kryloff and Bogoliouboff [7]. Their convergence definition in the space of the probability measures ξ is somewhat different from the one used here.

THEOREM 5.6. *If Conditions* I, II, *and* III *hold, a minimax solution is always a Bayes solution in the strict sense.*

The above theorem is an immediate consequence of Theorems 5.4 and 5.5.

6. Complete classes of decision rules. A class C of decision rules δ is said to be *complete* if for any rule δ not in C there exists a rule δ^* in C such that δ^* is uniformly better. We shall say that a class C of decision rules is *essentially complete* if for any rule δ not in C there exists a rule δ^* in C such that $r(F, \delta^*) \leqq r(F, \delta)$ for all F in Ω.

Clearly, if C is a complete or at least an essentially complete class of decision rules, we can disregard all decision rules outside C and the problem of choice is reduced to the problem of choosing a particular element of C. Thus, the construction of complete or essentially complete classes of decision rules is of great importance in any statistical decision problem.

The first result concerning complete classes of decision rules is due to Lehmann [8] who constructed such a class in a special case. Soon after the publication of Lehmann's paper, results of great generality were obtained. To state some of these results, let Δ denote the set of all decision rules δ with bounded risk functions. We shall say that a class C of decision rules δ is complete, or essentially complete, relative to Δ if the corresponding condition is fulfilled for every δ in Δ. Among others, the following results have been proved in [1].

THEOREM 6.1. *If Conditions* I *and* II *hold, then the class of all Bayes solutions in the wide sense is complete relative to* Δ.

THEOREM 6.2. *If Conditions* I *and* II *hold, then the closure of the class of all Bayes solutions in the strict sense is essentially complete relative to* Δ.

THEOREM 6.3. *If Conditions* I, II, *and* III *hold, then the class of all Bayes solutions in the strict sense is complete relative to* Δ.

To avoid any possibility of a misunderstanding, it may be pointed out that the notions of Bayes solutions and a priori distributions are used here merely as mathematical tools to express some results concerning complete classes of decision rules, and in no way is the actual existence of an a priori distribution in Ω postulated here.

7. Relation to von Neumann's theory of games. The statistical decision theory as outlined here, is intimately connected with von Neumann's theory of zero sum two person games [9]. The normalized form of a zero sum two person game is given by von Neumann as follows: There are two players and there is given a bounded and real-valued function $K(u, v)$ of two variables u and v where u may be any point of a space U and v may be any point of a space V. Player 1 chooses a point u in U and player 2 chooses a point v in V, each choice being made in ignorance of the other. Player 1 then gets the amount $K(u, v)$ and player 2 the amount $-K(u, v)$.

Any statistical decision problem may be viewed as a zero sum two person game. Player 1 is the agency, say Nature, who selects an element F of Ω to be the true distribution of X, and player 2 is the statistician who chooses a decision rule δ. The outcome is then given by the risk $r(F, \delta)$ which depends on both the choice F of Nature and the choice δ of the statistician. The theory of zero sum two person games was developed by von Neumann for finite spaces U and V. In statistical decision problems, however, the number of strategies available to Nature (number of elements of Ω) and the number of strategies (number of decision rules) available to the statistician are usually infinite. Many of the results in statistical decision theory were obtained by extending von Neumann's theory to the case of infinite spaces of strategies. In particular, it has been shown in [1] that if Conditions I and II hold, the statistical decision problem, viewed as a zero sum two person game, is strictly determined in the sense of von Neumann's theory, i.e.,

$$\operatorname*{Sup}_{\xi} \operatorname*{Inf}_{\delta} r^*(\xi, \delta) = \operatorname*{Inf}_{\delta} \operatorname*{Sup}_{\xi} r^*(\xi, \delta). \tag{7.1}$$

The above relation plays a fundamental role in the theory of zero sum two person games. In statistical decision theory, the above relation is basic in deriving the results concerning complete classes of decision rules, but otherwise it is of no particular intrinsic interest.

8. Discussion of some special cases. I would like to discuss briefly application of the general theory to a few special cases.

Suppose that Ω consists of two elements F_1 and F_2. According to F_i the random variables $X_1, X_2, \cdots$, ad inf. are independently distributed with the common density function $f_i(t)$ $(i = 1, 2)$. The decision space D consists of two elements d_1 and d_2 where d_i denotes the decision to accept the hypothesis that F_i is the true distribution $(i = 1, 2)$. Let the loss $W(F_i, d_j) = W_{ij} > 0$ when $i \neq j$ and $=0$ when $i = j$. The cost of experimentation is assumed to be proportional to the number of observations, i.e., $c_m(x_1, \cdots, x_m) = cm$ where c denotes the cost of a single observation.

An a priori distribution is given by a set of two non-negative numbers (ξ_1, ξ_2) such that $\xi_1 + \xi_2 = 1$. The quantity ξ_i denotes the a priori probability that F_i is true. It was shown by Wolfowitz and the author [10] that any Bayes solution must be a decision rule of the following type: Let x_j denote the observed value of X_j and let

$$z_j = \log \frac{f_2(x_j)}{f_1(x_j)}. \tag{8.1}$$

We choose two constants a and b ($b < a$) and at each stage of the experiment (after the mth observation for each integral value m) we compute the cumulative sum $Z_m = z_1 + \cdots + z_m$. At the first time when $b < Z_m < a$ does not hold, we terminate experimentation.[4] We make the decision d_1 (accept the hypothesis that F_1 is true) if $Z_m \leqq b$, and the decision d_2 (accept the hypothesis that F_2 is true) if $Z_m \geqq a$. A decision rule of the above type is called a sequential probability ratio test.

Applying the complete class theorem to this case, we arrive at the following result: The class of all sequential probability ratio tests corrseponding to all possible values of the constants a and b is a complete class. This means that if δ is any decision rule that is not a sequential probability ratio test, then there exist two constants a and b such that the sequential probability ratio test corresponding to the constants a and b is uniformly better than δ.[5]

Due to the completeness of the class of all sequential probability ratio tests, the problem of choosing a decision rule is reduced to the problem of choosing the values of the constants a and b. A method for determining the constants a and b such that the resulting sequential probability ratio test is a minimax solution, or a Bayes solution relative to a given a priori distribution, is discussed by Arrow, Blackwell, and Girshick [11].

The properties of the sequential probability ratio tests have been studied rather extensively. The recently developed sequential analysis (see, for example, [12] and [13]) is centered on the sequential probability ratio test. It may be of interest to mention that the stochastic process represented by the sequential probability ratio test is identical with the one-dimensional random walk process that plays an important role in molecular physics.

We shall now consider the case when Ω contains more than two but a finite number of elements. It will be sufficient to discuss the case when Ω consists of 3 elements F_1, F_2, and F_3, since the extension to any finite number >3 is straightforward. As before, the random variables $X_1, X_2, \cdots$, are independently distributed with the common density function $f_i(t)$ when F_i is true ($i = 1, 2, 3$). The decision space D consists of 3 elements d_1, d_2, and d_3 where d_i denotes the decision to accept the hypothesis that F_i is true. Let $W(F_i, d_j) = W_{ij} = 0$ for $i = j$, and >0 when $i \neq j$. The cost of experimentation is again assumed to be proportional to the number of observations, and let c denote the cost of a single observation. Any a priori distribution $\xi = (\xi^1, \xi^2, \xi^3)$ can be represented by a point with the coordinates ξ^1, ξ^2, and ξ^3. The totality of all possible a priori distributions ξ will fill out the triangle T with the vertices V_1, V_2, V_3 where V_i represents the a priori distribution whose ith component ξ^i is equal to 1 (see fig. 1).

[4] If $Z_m = a$ or $= b$, the statistician may use any chance mechanism to decide whether to terminate experimentation or to take an additional observation.

[5] This result follows also from an optimum property of the sequential probability ratio test proved in [10].

In order to construct a complete class of decision rules for this problem, it is necessary to determine the Bayes solution relative to any given a priori distribution $\xi_0 = (\xi_0^1, \xi_0^2, \xi_3^0)$. Let x_i denote the observed value of X_i. After m observations have been made the a posteriori probability distribution $\xi_m = (\xi_m^1, \xi_m^2, \xi_m^3)$ is given by the following expression:

$$\xi_m^i = \frac{\xi_0^i f_i(x_1) f_i(x_2) \cdots f_i(x_m)}{\sum_{j=1}^{3} \xi_0^j f_j(x_1) f_j(x_2) \cdots f_j(x_m)}. \tag{8.2}$$

At each stage of the experiment, the a posteriori probability distribution ξ_m is represented by a point of the triangle T.

It was shown by Wolfowitz and the author [14] that there exist three fixed (independent of the a priori distribution ξ_0), closed and convex subsets S_1, S_2, and S_3 of the triangle T such that the Bayes solution relative to ξ_0 is given by the following decision rule: At each stage of the experiment (after the mth

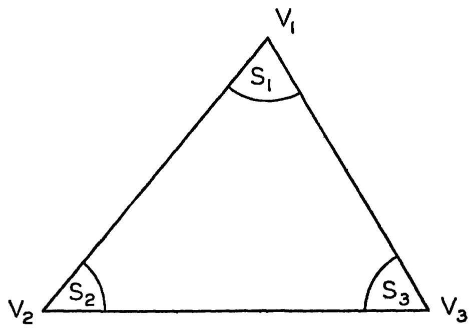


FIGURE 1

observation, for $m = 0, 1, 2, \cdots$) compute the point ξ_m in T. Continue taking additional observations as long as ξ_m does not lie in the union of S_1, S_2, and S_3. If ξ_m lies in the interior of S_i, stop experimentation with the final decision d_i ($i = 1, 2, 3$). If ξ_m lies on the boundary of S_i, an independent chance mechanism may be used to decide whether experimentation be terminated with the final decision d_i or whether an additional observation be made.

The convex sets S_1, S_2, and S_3 depend only on the constants W_{ij} and c. So far no method is available for the explicit computation of the sets S_1, S_2, and S_3 for given values of W_{ij} and c.[6] The development of a method for the explicit determination of the sets S_1, S_2, and S_3 would be of great value, since it would probably indicate a way of dealing with similar difficulties in many other sequential decision problems.

More general results concerning the nature of the Bayes solution for the decision problem described above, admitting also nonlinear cost functions, were obtained by Arrow, Blackwell, and Girshick [11].

[6] The boundary points of the sets S_1, S_2, and S_3 on the periphery of the triangle T and the tangents at these points have been determined in [14].

As a last example, consider the following decision problem: It is known that $X_1, X_2, \cdots$ are independently and identically distributed and the common distribution is known to be rectangular with unit range. Thus, the midpoint θ of the range is the only unknown parameter. The common density function of the chance variables $X_1, X_2, \cdots$ is given by

$$f(t, \theta) = 1 \text{ when } |t - \theta| \leqq 1/2 \\ = 0 \text{ otherwise.} \tag{8.3}$$

For any real value θ^* let d_{θ^*} denote the decision to estimate θ by the value θ^*. The decision space D consists of the elements d_{θ^*} corresponding to all real values θ^*. Let the loss be given by $(\theta - \theta^*)^2$ when θ is the true value of the midpoint of the range and the decision d_{θ^*} is made. The cost of experimentation is assumed to be proportional to the number of observations made. Let c denote the cost of a single observation.

It was shown in [1] that a minimax solution for this problem is given by the following decision rule: Take at least one observation. At each stage of the experiment (after the mth observation, for each positive integral value m) compute the quantity

$$l_m = 1 + \text{Min}(x_1, \cdots, x_m) - \text{Max}(x_1, \cdots, x_m). \tag{8.4}$$

Continue experimentation as long as $l_m > (24c)^{1/3}$. At the first time when $l_m \leqq (24c)^{1/3}$ stop experimentation and estimate θ by the value

$$\theta^* = \frac{\text{Min}(x_1, \cdots, x_m) + \text{Max}(x_1, \cdots, x_m)}{2}. \tag{8.5}$$

The risk function associated with this minimax solution is constant over the whole space Ω. The admissibility of the above minimax solution was proved by C. Blyth.

9. Concluding remark. While the general decision theory has been developed to a considerable extent and many results of great generality are available, explicit solutions have been worked out so far only in a relatively small number of special cases. The mathematical difficulties in obtaining explicit solutions, particularly in the sequential case, are still great, but it is hoped that future research will lessen these difficulties and explicit solutions will be worked out in a great variety of problems.

REFERENCES

1. A. WALD, *Statistical decision functions*, New York, John Wiley & Sons, 1950.
2. A. DVORETZKY, A. WALD, and J. WOLFOWITZ, *Elimination of randomization in certain statistical decision procedures and zero-sum two person games*, Ann. Math. Statist. vol. 22 (1951).
3. D. BLACKWELL, *On a theorem of Liapounoff*, Ann. Math. Statist. vol. 22 (1951).
4. A. LIAPOUNOFF, *Sur les fonctions-vecteurs complètement additives*, Bull. Acad. Sci. URSS. Ser. Math. vol. 4 (1940).

5. J. L. HODGES and E. L. LEHMANN, *Problems in minimax point estimation*, Ann. Math. Statist. vol. 21 (1950).
6. DAVID VERNON WIDDER, *The Laplace transform*, Princeton, Princeton University Press, 1946.
7. N. KRYLOFF and N. BOGOLIOUBOFF, *La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non-linéaire*, Ann. of Math. vol. 38 (1937).
8. E. L. LEHMANN, *On families of admissible tests*, Ann. Math. Statist. vol. 18 (1947).
9. J. VON NEUMANN and O. MORGENSTERN, *Theory of games and economic behaviour*, Princeton, Princeton University Press, 1944.
10. A. WALD and J. WOLFOWITZ, *Optimum character of the sequential probability ratio test*, Ann. Math. Statist. vol. 19 (1948).
11. K. J. ARROW, D. BLACKWELL, and M. A. GIRSHICK, *Bayes and minimax solutions of sequential decision problems*, Econometrica vol. 17 (1949).
12. A. WALD, *Sequential analysis*, New York, John Wiley & Sons, 1947.
13. Statistical Research Group, Columbia University, *Sequential analysis of statistical data: applications*, New York, Columbia University Press, 1945.
14. A. WALD and J. WOLFOWITZ, *Bayes solution of sequential decision problems*, Ann. Math. Statist. vol. 21 (1950).

COLUMBIA UNIVERSITY,
NEW YORK, N. Y., U. S. A.

NUMBER-THEORY AND ALGEBRAIC GEOMETRY

A. WEIL

This address was given as part of the Conference in Algebra, see Volume 2, page 90.

r-DIMENSIONAL INTEGRATION IN n-SPACE

HASSLER WHITNEY

1. Introduction. There are various elementary and fundamental questions in integration theory as applied to geometry and physics that are not covered by the modern theory of the Lebesgue integral. In particular, basic problems concerning integrals over domains of dimension less than that of the containing space, as functions of the domain, are largely untouched. We shall present here a general approach to this type of problem.[1]

For an example from physics, consider the flux through a surface S in Euclidean 3-space E^3. Cut S into small pieces $\sigma_1, \sigma_2, \cdots$; we find the flux through each σ_i, and add. Take a typical small piece σ, in the form of a parallelogram with vectors v_1, v_2 along two of its sides, and containing the point p. The flux $X(\sigma)$ through σ depends on p, the area of σ, and the direction of σ. If we change the direction of σ, $X(\sigma)$ varies; $X(\sigma)$ is approximately proportional (for small σ) to the cosine of the angle between the normal $v_1 \times v_2$ to σ and the direction of the flux through p. In particular, for a rotation through π, $X(\sigma)$ is replaced by its negative. Thus σ must be taken as oriented, say by ordering the vectors (v_1, v_2).

We may represent the flux by a vector function $\omega(p)$; then for small σ as above, $X(\sigma)$ is approximately $\omega(p)\cdot(v_1 \times v_2)$, and thus is linear and skew-symmetric in the vectors v_1, v_2 determining σ. More generally, for surface integrals in E^n, an integral $\int_S$ may be determined by naming a function $\omega(p)$ which, for each p, is a linear skew-symmetric function of v_1, v_2; that is, ω is a differential form, or a "2-form" for short. An r-dimensional integral may be determined similarly by an r-form.

Given $\omega(p)$, say with $r = 2$, the integral $\int_S \omega$ may be defined as follows. Cut S into small triangles (approximately) $\sigma_1, \cdots, \sigma_m$. S being oriented, let v_{i1}, v_{i2} be the vectors of two of the sides of σ_i, so that (v_{i1}, v_{i2}) orient σ_i like S. Let p_i be a point of σ_i. Then

$$\int_S \omega = \lim \frac{1}{2} \sum_i \omega(p_i; v_{i1}, v_{i2}),$$

taking the limit as the mesh of the subdivision of S approaches 0. The factor 1/2 is due to the use of triangles instead of parallelograms. The usual Riemann type theory may be set up in this fashion, essentially without use of coordinate systems.[2]

[1] Some of the theory given here has appeared in notes by the author, *Algebraic topology and integration theory*, Proc. Nat. Acad. Sci. vol. 33 (1947) pp. 1–6, and *La topologie algébrique et la théorie de l'intégration*, in the volume *Topologie algébrique*, Colloques Internationaux du Centre National de la Recherche Scientifique, Paris, 1949.

[2] This was carried out by P. Olum in a Senior Thesis at Harvard, 1940. See also H. Federer, *An introduction to differential geometry*, mimeographed notes, Stenographic Bureau, Brown University, Providence, R. I. (see Math. Reviews vol. 10 (1949) p. 264), and A. Lichnerowicz, *Algèbre et analyse linéaires*, Masson, Paris, 1947.

2. The general problem. We now ask what a general theory of integration should look like. For a given "integrand" X, the integral is a function of the domain A; we shall write it as $X \cdot A$. The basic property in the usual theory is additivity:

$$\int_{P \cup Q} = \int_P + \int_Q \quad \text{if } P \cap Q = 0.$$

In the general case, we may wish to integrate over domains with self-intersections. For instance, we may desire the line integral over a curve C consisting, in succession, of the oriented arcs C_1, C_2, C_3, C_2, C_4. We clearly should have

$$\int_C = \int_{C_1} + 2\int_{C_2} + \int_{C_3} + \int_{C_4}.$$

Thus a general r-dimensional domain is an "r-chain" $A = \sum a_i \sigma_i^r$, with oriented r-cells σ_i^r, each having a coefficient a_i (which we may take as any real number). Now $\int_A = \sum a_i \int_{\sigma_i^r}$, i.e.,

$$X \cdot \sum a_i \sigma_i^r = \sum a_i (X \cdot \sigma_i^r), \tag{2.1}$$

and the integral is a linear function of r-chains, generalizing the additivity above.

Recall the Theorem of Stokes: an r-form ω with differentiable coefficients has an exterior derivative $\delta\omega$; given an $(r + 1)$-chain A, with boundary ∂A,

$$\int_A \delta\omega = \int_{\partial A} \omega. \tag{2.2}$$

If X is an integral (abstractly defined) over r-chains, we may *define* an integral δX over $(r + 1)$-chains by the corresponding formula

$$\delta X \cdot A = X \cdot \partial A. \tag{2.3}$$

If we wish an integral $X \cdot A$ to be expressible in terms of A and local properties of X, such as is the case if X is defined by a differential form, we cannot let A be too general. For instance, for $r = 1$, there is a curve C and a differentiable function f in the plane, such that the partial derivatives of f vanish at all points of C, and yet f is not constant in C.[3] The natural assumption here is that C should be rectifiable. A similar assumption is in order for $r > 1$; see §8 below.

When we have chosen what domains should be allowable for setting up r-chains, we must decide how general the allowable linear functions X should be. Without some restrictions, little theory could be obtained. For instance, for $r = 0$, a typical domain is a point p; then $X \cdot p = \phi(p)$ would be an arbitrary function. In the usual Lebesgue theory, bounded measurable functions as integrands form a special class of summable functions; yet they illustrate much of the basic theory. We might invent some kind of corresponding bound, which we

[3] See H. Whitney, *A function not constant on a connected set of critical points*, Duke Math. J. vol. 1 (1935) pp. 514–517.

call the "mass" $|X|$, of X. It turns out that assuming both $|X|$ and $|\delta X|$ are bounded leads to a very satisfactory theory, to be described below; see (4.2). In particular, any X can be represented by a differential form (§6). An important problem is to obtain general results under weaker hypotheses than the boundedness of $|X|$ and $|\delta X|$.

3. Polyhedral chains. We introduce integration theory in a space R as follows. First choose a set of elements, "r-chains", forming a linear space; introduce a norm in this space; when completed, this gives a Banach space $\mathbf{C}^r$. Then the space of "r-cochains" X is the conjugate space $\bar{\mathbf{C}}^r$ of $\mathbf{C}^r$. The function $X(A) = X \cdot A$ is the "integral" of X over A. By making the norm in $\mathbf{C}^r$ small, we enlarge the set of elements which can be in $\mathbf{C}^r$ (i.e., for which the norm exists); the corresponding norm in $\bar{\mathbf{C}}^r$ is then large, which restricts the elements occurring in $\bar{\mathbf{C}}^r$.

We shall remain mostly in Euclidean space E^n; later we consider briefly the case of manifolds and more general spaces. The norms in $\mathbf{C}^r$ and $\bar{\mathbf{C}}^r$ will depend on the metric of E^n; but the sets of elements in these spaces are independent of the metric.

Among r-chains in E^n we must certainly include polyhedral r-chains $\sum a_i \sigma_i^r$, each σ_i^r and its boundary cells being flat. Define the *mass* by

$$|\textstyle\sum a_i \sigma_i^r| = \sum |a_i| |\sigma_i^r| \tag{3.1}$$

(the σ_i^r non-overlapping), $|\sigma_i^r|$ denoting the r-dimensional volume of σ_i^r.

The mass is too large a norm for our purposes. For instance, for $r = 0$, using $|p| = 1$ shows that any bounded function would define an element of $\bar{\mathbf{C}}^0$. Consider the case $r = n$ for a moment. If we take a domain P and translate it by a small vector v, giving $Q = T_v P$, then $\int_P$ and $\int_Q$ are nearly the same. For a general r, we might require $\int_{\sigma^r}$ and $\int_{T_v \sigma^r}$ to be nearly the same, even though v need not lie in the r-plane of σ^r. In terms of chains, we may require the norm $|A|^\#$ to satisfy

$$|T_v \sigma^r - \sigma^r|^\# \leqq |v| |\sigma^r|/(r + 1). \tag{3.2}$$

This does not relate r-cells in different r-planes. To take care of this, assume that for any $(r + 1)$-cell σ^{r+1}, whose boundary is an r-chain $\partial\sigma^{r+1}$, we have $|\partial\sigma^{r+1}|^\# \leqq N|\sigma^{r+1}|$ for some fixed N. There is a uniquely defined largest norm satisfying these conditions, and also $|A|^\# \leqq |A|$. We call it the *tight norm*. It is independent of N for N large.

With this norm the conjugate space corresponds exactly to what we formerly called the set of "tensor cochains" in E^n. (See §5 below.) Though these cochains X are easy to work with, they are not general enough for some purposes. For instance, f^*X, defined in §9, is not tightly Lipschitz.

The most important norm for our purposes is the *Lipschitz norm* $|A|^*$, intermediate in size between $|A|$ and $|A|^\#$. It is the largest norm satisfying the conditions

(3.3) $$|\sigma^r|^* \leqq |\sigma^r|, \qquad |\partial\sigma^{r+1}|^* \leqq |\sigma^{r+1}|.$$

It is easy to find an explicit expression for $|A|^*$, as follows. Considering all polyhedral $(r + 1)$-chains D, $|A|^*$ is the greatest lower bound

(3.4) $$|A|^* = \operatorname*{GLB}_D (|A - \partial D| + |D|).$$

For example, let σ be a segment of length a, and let σ' be $T_v\sigma$, $|v| = b$, b small. Let D be the parallelogram formed by carrying σ into σ', oriented so that $\partial D = \sigma' - \sigma + C$, where C is composed of the remaining short sides of D. Then $|D| \leqq ab$, $|C| = 2b$, and hence

$$|\sigma' - \sigma|^* \leqq |C| + |D| \leqq (a + 2)b.$$

Of course $|\sigma' - \sigma| = 2a$; also $|\sigma' - \sigma|^\# = ab/2$ for b small.

Given A, taking $D = 0$ in (3.4) shows that

(3.5) $$|A|^* \leqq |A|.$$

If $A^r = \partial B^{r+1}$, choosing D_2^{r+2} so that

$$|B - \partial D_2| + |D_2| < |B|^* + \epsilon$$

and setting $D_1 = B - \partial D_2$ gives

$$|A - \partial D_1| + |D_1| = |D_1| < |B|^* + \epsilon;$$

this proves

(3.6) $$|\partial B|^* \leqq |B|^* \leqq |B|.$$

In particular, (3.3) is proved.

For some special dimensions, considering $aq - bp$ as a 0-chain and $q - p$ as a vector, we have

$$r = 0\colon\ |1q - 1p|^* = |1q - 1p|^\# \leqq |q - p|,$$

with the sign $=$ if $|q - p| \leqq 2$, and

$$r = n\colon\ |A^n|^* = |A^n|,$$

since there are no nontrivial $(n + 1)$-chains in E^n.

If $A \neq 0$, then $|A|$, $|A|^*$, and $|A|^\#$ are all >0.

For any σ^r, $|\sigma^r| = |\sigma^r|^* = |\sigma^r|^\#$. In fact, $|A| = |A|^* = |A|^\#$ if $A = \sum a_i\sigma_i^r$, the σ_i^r are parallel and similarly oriented, and the a_i are $\geqq 0$.

4. Lipschitz cochains. A Lipschitz r-cochain in E^n (or in a subset R of E^n) is an element of the conjugate space $\overline{\mathbf{C}}^r$ of $\mathbf{C}^r$ (or of $\mathbf{C}'^r(R)$, using chains in R only). The Lipschitz norm $|X|^*$ is the norm of X in $\overline{\mathbf{C}}^r$; the mass $|X|$, corresponding to $|A|$, is defined. The definitions are

(4.1) $$|X|^* = \underset{|A|^*=1}{\mathrm{LUB}} |X\cdot A|, \qquad |X| = \underset{|A|=1}{\mathrm{LUB}} |X\cdot A|.$$

'hough $\mathbf{C}^r$ is separable, $\bar{\mathbf{C}}^r$ is not; hence $\mathbf{C}^r$ is not reflexive.

Note that $\delta\delta X = 0$; for $\delta\delta X \cdot A^{r+2} = \delta X \cdot \partial A = X\cdot\partial\partial A = 0$.

The following relation shows that $X \in \bar{\mathbf{C}}^r$ if and only if $|X|$ and $|\delta X|$ are nite:

(4.2) $$|X|^* = \max(|X|, |\delta X|).$$

'o prove this, note first that $|X| \leqq |X|^*$, because of (3.5) and (4.1). Next, 2.3), (4.1), and (3.6) give, using any B^{r+1},

$$|\delta X\cdot B| = |X\cdot\partial B| \leqq |X|^* |\partial B|^* \leqq |X|^* |B|^*;$$

ence δX is a Lipschitz $(r+1)$-cochain, and $|\delta X| \leqq |X|^*$. This proves $\geqq$ ı (4.2). To prove the reverse inequality, we need merely show that for any olyhedral r-chain A,

(4.3) $$|X\cdot A| \leqq \max(|X|, |\delta X|)\,|A|^*.$$

iven $\epsilon > 0$, choose $D = D^{r+1}$ so that

$$|A - \partial D| + |D| < |A|^* + \epsilon.$$

'hen if $C = A - \partial D$,

$$|X\cdot A| \leqq |X\cdot C| + |X\cdot\partial D| \leqq |X|\,|C| + |\delta X|\,|D|$$
$$\leqq \max(|X|, |\delta X|)\,(|A|^* + \epsilon),$$

hich gives (4.3).

Consider the case $r = 0$. For any Lipschitz 0-cochain X, $w(p) = X\cdot p$ is a al-valued function. The relations

$$|w(p)| = |X\cdot p| \leqq |X|\,|p| = |X|,$$
$$|w(q) - w(p)| = |X\cdot(1q - 1p)| = |\delta X\cdot(pq)| \leqq |\delta X|\,|q-p|$$

ıow that w is bounded and satisfies a Lipschitz condition. Conversely, any ıch w defines an X.

Now take $r = n$. With E^n oriented, $X\cdot Q$ is defined for polyhedral regions Q iented like E^n. Since $|X\cdot Q| \leqq |X|\,|Q|$, X is extendable to be an additive t function over measurable sets, satisfying the same inequality. Define the ullness" of a set Q by

(4.4) $$\Theta_n(Q) = |Q|/[\mathrm{diam}\, Q]^n.$$

y standard Lebesgue theory, there is a measurable function $D_X(p)$, $|D_X(p)| \leqq |X|$, such that

(4.5) $$X\cdot Q = \int_Q D_X(p)\,dp \qquad (Q \text{ oriented like } E^n).$$

Moreover, using sequences of cells $\sigma_1, \sigma_2, \cdots$ containing p and with

$$\Theta_n(\sigma_i) \geqq \eta > 0$$

for all i,

$$D_X(p) = \lim_{i\to\infty} X\cdot\sigma_i/|\sigma_i| \qquad \text{a. e. in } E^n. \tag{4.6}$$

5. The λ-norms. Suppose, instead of (3.3), we require

$$|\sigma^r|_\lambda^* \leqq |\sigma^r|, \qquad |\partial\sigma^{r+1}|_\lambda^* \leqq |\sigma^{r+1}|/\lambda. \tag{5.1}$$

We then obtain the *Lipschitz λ-norm*, with the property

$$|A|_\lambda^* = \underset{D}{\mathrm{GLB}}\,(|A - \partial D| + |D|/\lambda). \tag{5.2}$$

The corresponding λ-norm for cochains satisfies

$$|X|_\lambda^* = \max\,(|X|, \lambda\,|\delta X|). \tag{5.3}$$

We obtain $|A|_\lambda^\#$ similarly, using $|v|\,|\sigma^r|/(r+1)\lambda$ in (3.2). Define the *Lipschitz constant* of X by

$$\mathfrak{L}(X) = \mathrm{LUB}\,\frac{|X\cdot(T_v\sigma - \sigma)|}{|\sigma|\,|v|}; \tag{5.4}$$

call X *tightly Lipschitz* (a "tensor cochain" in[1]) if this is finite. If this holds for the Lipschitz r-cochain X, then

$$|\delta X| \leqq (r+1)\mathfrak{L}(X), \tag{5.5}$$

$$|X|_\lambda^\# = \max\,[|X|, (r+1)\lambda\mathfrak{L}(X)]. \tag{5.6}$$

The sets of elements in the spaces $\overline{\mathbf{C}}_\lambda^r$ for various λ are the same; only the norms differ; similarly for $\overline{\mathbf{C}}_\lambda^{r\#}$.

Given σ^r, it is not hard to construct a tightly Lipschitz X vanishing outside an arbitrary neighborhood of σ, such that $|X| = 1$, $X\cdot\sigma = |\sigma|$. Then for λ small enough, $|X|_\lambda^\# = 1$. Using this, we may prove, for polyhedral A,

$$\lim_{\lambda\to 0} |A|_\lambda^\# = \lim_{\lambda\to 0} |A|_\lambda^* = |A|. \tag{5.7}$$

From this we may prove that $|A|$ is a lower semicontinuous function of polyhedral chains A in $\mathbf{C}^r$ (in fact, in any $\mathbf{C}_\lambda^r$ or $\mathbf{C}_\lambda^{r\#}$). For a general A in $\mathbf{C}^r$, *define* $|A|$ as the lower bound of $\liminf |A_i|$ for sequences of polyhedral chains A_i with $|A_i - A|^* \to 0$; it may be infinite. $|A|$ is still lower semicontinuous.

For $r = 0$, $|A|_\lambda^\# = |A|_\lambda^*$.

6. Lipschitz cochains and differential forms. We present here the theorem of Wolfe,[4] that the Lipschitz r-cochains in an open subset R of E^n correspond to

[4] J. H. Wolfe, *Tensor fields associated with Lipschitz cochains*, Harvard Thesis, 1948.

he differential forms in R satisfying certain conditions. Suppose X is given. .t any point p we wish to find a corresponding linear skew-symmetric function $(p;\, v_1, \cdots, v_r)$, or equivalently, a linear function $\omega(p)\cdot\alpha$ of contravariant ·vectors α. The natural definition is the following. The set of points

$$p + \sum_{i=1}^{r} t_i v_i \qquad (0 \leqq t_i \leqq t,\, i = 1, \cdots, r)$$

›rms a parallelopiped σ_t, oriented by the ordered set $(v_1, \cdots, v_r)$. Set

$$\omega(p;\, v_1, \cdots, v_r) = \lim_{t\to 0} X\cdot\sigma_t/t^r, \tag{3.1}$$

this exists. (Cf. (4.6).) Then for almost all p in R, this exists and defines a near function $\omega(p)$; for fixed α, $\omega(p)\cdot\alpha$ is measurable. Also $|\,\omega(p;\, v_1, \cdots, v_r)\,| \leqq$ $X \mid |\, v_1 \,| \cdots |\, v_r\,|$. The same is true for $\delta\omega$, defined from δX. Moreover, for ll σ,

$$X\cdot\sigma = \int_\sigma \omega(p;\, e_1, \cdots, e_r)\, dp, \tag{3.2}$$

$_1, \cdots, e_r)$ being an orthonormal set in the plane of σ, oriented like σ.

The theorem is proved by first smoothing X by an averaging process (giving tightly Lipschitz cochain), in which case the corresponding form is more easily ›und, and then passing to the limit. The reason $\omega(p)$ turns out to be linear may e illustrated for the case $r = 1$ as follows. Given vectors u, v, set

$$q_t = p + tu, \qquad q_t' = q_t + tv = p + t(u + v);$$

›t $\sigma_t = pq_tq_t'$. For small t, $\omega(p)\cdot tu = \int_{pq_t}\omega$ approximately, etc.; thus

$$\begin{aligned} |\,\omega(p)\cdot tu + \omega(p)\cdot tv - \omega(p)\cdot t(u+v)\,| &=_{ap} |\, X\cdot\partial\sigma_t\,| \\ &= |\,\delta X\cdot\sigma_t\,| \leqq t^2\,|\,\delta X\,|\,|\,\sigma_1\,|\,; \end{aligned}$$

ividing by t and letting $t \to 0$ gives the result.

Conversely, suppose $\omega(p)$ is defined and linear a. e. in R, with $\omega(p)\cdot\alpha$ measrable for each α; suppose $|\,\omega(p)\,|$ is bounded as above, and $|\int_{\partial\sigma}\omega\,| \leqq N\,|\,\sigma\,|$ ›r each $(r + 1)$-cell σ for which the integral is defined. Such a form we call a *ipschitz r-form* in R. (The conditions can be weakened; for instance, we need ıerely continuity in α of $\omega(p)\cdot\alpha$ for simple unit r-vectors α, not linearity.) Then ıere is a corresponding Lipschitz r-cochain X in R, for which (6.2) holds whenver the integral is defined.

Say ω_1, ω_2 are *equivalent* if for each α, $\omega_1(p)\cdot\alpha = \omega_2(p)\cdot\alpha$ a. e. in R. Then *ıe Lipschitz r-cochains in R correspond exactly to the classes of equivalent Lip-:hitz r-forms in R.*

Note that for any Lipschitz r-form ω in R, we may find the corresponding ipschitz r-cochain X, take δX, and find the corresponding $\delta\omega$, even if the comonents of ω are not differentiable; whenever they are, this definition of $\delta\omega$ agrees ith the analytic one.

A "simple" contravariant r-vector is one expressible as a product of r vectors. Define the "simple norm" $|\xi|_s$ of a covariant r-vector ξ by

$$(6.3) \qquad |\xi|_s = \text{LUB}\,|\xi\cdot\alpha|, \qquad \alpha \text{ simple}, \qquad |\alpha| = 1.$$

Then for any X and corresponding ω,

$$(6.4) \qquad |X| = \underset{p}{\text{essential LUB}}\,|\omega(p)|_s.$$

7. General r-chains. Recall that $\mathbf{C}^r$ was the completion of the space of polyhedral r-chains; the new elements of $\mathbf{C}^r$ we call "general Lipschitz r-chains".

Consider first the case $r = 0$. Let R be a locally compact separable metric space. An additive set function Φ in R assigns to each Borel set Q a number $\Phi(Q)$, such that $\Phi(\sum Q_i) = \sum \Phi(Q_i)$ if the Q_i are disjoint. Any such set function defines uniquely a general Lipschitz 0-chain A_Φ, as follows. Let $R = R_1 \cup \cdots \cup R_m$ be a partition of R into disjoint Borel sets; choose $p_i \in R_i$, and set

$$(7.1) \qquad B = \sum \Phi(R_i)p_i;$$

this is an approximation to A_Φ by a polyhedral 0-chain, if the R_i are "small" enough (i.e. cut up Φ finely enough).

The mass $|A_\Phi|$ is the total variation of Φ. The expression for $|A_\Phi|^*$ is more complicated. Not all 0-chains are expressible in this manner; but the 0-chains described are dense in $\mathbf{C}^0$. For any Lipschitz 0-cochain X, corresponding to the function $\omega(p)$,

$$(7.2) \qquad X\cdot A_\Phi = \int_R \omega(p)\, d\Phi(p),$$

using the Lebesgue Stieltjes integral. The integral is defined for more general functions ω; on the other hand, $X\cdot A$ is defined for more general A.

Of course $\Phi(Q)$ is defined without regard to orientation properties. This corresponds to the fact that a 0-cell, i.e., a point, has a natural orientation. We may consider the theory of additive set functions and the Lebesgue Stieltjes integral as 0-dimensional integration; r-dimensional integration for $r > 0$ requires orientation properties.

Consider next n-chains in E^n. Recall the L^1-norm for summable functions F:

$$(7.3) \qquad |F| = \int_{E^n} |F(p)|\, dp.$$

If F is cellwise constant for some subdivision of E^n, it defines an n-chain A with $|A|^* = |A| = |F|$; this shows that summable functions define elements of $\mathbf{C}^n$; in fact, they give the whole of $\mathbf{C}^n$.

Take the case $n = 1$. If F is not only summable but differentiable, it is easy to show that $-dF/dx$ defines a 0-chain which is exactly the boundary (see below) of A_F. On the other hand, suppose

$$F_a(x) = \log |x|, \quad x < 0; \qquad F_a(x) = \log x + a, \quad x > 0;$$

hen $-dF_a(x)/dx = -1/x$, all $x \neq 0$. For each a, F_a defines a 1-chain A_a, nd the same function $-1/x$ (which is not summable) "corresponds" to ∂A_a.

The boundary ∂A of a general chain A may always be defined. For if A is lefined by the sequence $A_1, A_2, \cdots$ of polyhedral chains A_i, then

$$| \partial A_j - \partial A_i |^* \leqq | A_j - A_i |^*$$

y (3.6), and hence $\partial A_1, \partial A_2, \cdots$ is a Cauchy sequence, and defines ∂A.

Now take $0 < r < n$, in E^n. Let $\alpha(p)$ be any field of contravariant r-vectors, he components being summable functions. It may be shown that α defines niquely a general Lipschitz r-chain A_α, with the property that for any X, orresponding to ω,

$$X \cdot A_\alpha = \int_{E^n} \omega(p) \cdot \alpha(p)\, dp. \tag{7.4}$$

uch A_α we call "spread out" r-chains. They are dense in $\mathbf{C}^r$; this holds even we require the components to be analytic functions. (This holds in open subets of E^n.)

If the components $\alpha^{\lambda_1 \cdots \lambda_r}$ of α are differentiable, and

$$\beta^{\lambda_1 \cdots \lambda_{r-1}} = \sum_k \partial \alpha^{\lambda_1 \cdots \lambda_{r-1} k} / \partial x_k, \tag{7.5}$$

hen

$$\partial A_\alpha = (-1)^r A_\beta. \tag{7.6}$$

Define the "mass" of a contravariant r-vector α by

$$| \alpha |_m = \text{GLB} \sum | \alpha_i |, \tag{7.7}$$

r expressions $\alpha = \sum \alpha_i$ of α as a sum of simple r-vectors. (Then the norms $\alpha |_m$ and $| \xi |_s$ of (6.3) correspond, considering the spaces of covariant and conavariant r-vectors as conjugate spaces of each other.) Then

$$| A_\alpha | = \int | \alpha(p) |_m\, dp. \tag{7.8}$$

here seems to be no simple expression for $| A_\alpha |^*$.

8. Lipschitz r-chains. In the applications, one integrates commonly over omains which are not polyhedral (curved surfaces, etc.). The domains are enerally expressible as images of flat domains, under mappings f which are ifferentiable or piecewise differentiable. We shall assume merely that f satisfies Lipschitz condition: If ρ denotes distance,

$$\mathfrak{L}_f = \text{LUB}\, \rho(f(p), f(q))/\rho(p, q) \text{ is finite.} \tag{8.1}$$

If f is a Lipschitz mapping of an r-dimensional polyhedron P, in which an chain A is given, into a metric space R, we call (A, f), or fA for short, a *Lip-*

schitz r-chain in R. We may as well take the cells of P as disjoint cells in E^r. With E^r oriented, A may be replaced by a summable function φ (see §7); φ defines a general r-chain $\tilde{\varphi}$ in E^r; we call $(\tilde{\varphi}, f) = f\tilde{\varphi}$ a *Lipschitz-Lebesgue r-chain* in R.

If R is a metric space satisfying certain conditions, Lipschitz chains in R can be used to set up integration theory. The mass $|A|$ and norm $|A|^*$ can be defined; see below. Particularly important is the case that R is a smooth manifold M; if a metric is not given, one may be introduced, and the spaces of chains and cochains (though not the norms) are independent of the metric (at least in compact subsets of M).

A Lipschitz-Lebesgue r-chain $f\tilde{\varphi}$ in E^n is a general Lipschitz chain, given by approximating φ by a cellwise constant function φ' and f by a simplicial mapping f' (thus defining an approximating polyhedral chain). The following continuity theorem holds: Given $f\tilde{\varphi}$ and numbers L and $\epsilon > 0$, there is a $\zeta > 0$ with the following property. For any φ' with $|\tilde{\varphi} - \tilde{\varphi}'|^* \leqq \zeta$ and for any f' with $\mathfrak{L}_{f'} \leqq L$ and $|f'(p) - f(p)| \leqq \zeta$ (all p), we have $|f'\tilde{\varphi}' - f\tilde{\varphi}|^* < \epsilon$. For given polyhedral $A = \tilde{\varphi}$, and L, keeping $\varphi' = \varphi$, we may take $\zeta = c\epsilon$ for some c $(c = L^r|A| + L^{r-1}|\partial A|)$; but this is not possible in the general case.

A Lipschitz chain A, being a general chain, has a boundary ∂A; a Lipschitz $(r-1)$-form ω defines X and hence δX. General theory now gives Stokes' Theorem:

$$\int_{\partial A} \omega = X \cdot \partial A = \delta X \cdot A = \int_A \delta\omega.$$

It is always possible to represent a Lipschitz-Lebesgue chain in E^n as $f\tilde{\varphi}$ with f one-one in the carrier $\mathrm{Car}(\varphi)$ of φ, i.e., the set of points p where $\varphi(p) \neq 0$. For such a representation, the mass is given by the formula

$$|f\tilde{\varphi}| = \int |\varphi(p)|\,|J_f(p)|\,dp, \tag{8.2}$$

$J_f(p)$ being the Jacobian, which exists a.e. since f is Lipschitz (Rademacher's Theorem). For any representation, (8.2) holds with $\leqq$. In the proof, we make use of the existence of a tightly Lipschitz cochain X such that

$$|X| = 1, \qquad |f\tilde{\varphi}| - \epsilon < X \cdot f\tilde{\varphi} \leqq |f\tilde{\varphi}|, \tag{8.3}$$

for arbitrary $\epsilon > 0$.

Let $S_1, S_2, \cdots$ be a sequence of subdivisions of E^r with mesh $\to 0$. Then given $f\tilde{\varphi}$, if $\tilde{\varphi}_{ki}$ is the part of $\tilde{\varphi}$ in the cell σ_{ki} of S_k, we have $\sum_i \tilde{\varphi}_{ki} = \tilde{\varphi}$, and

$$\lim_{k\to\infty} \sum_i |f\tilde{\varphi}_{ki}|^* = |f\tilde{\varphi}|. \tag{8.4}$$

Given $f\tilde{\varphi}$ in E^n, with f one-one in $\mathrm{Car}(\varphi)$, and given $\eta > 0$ and $\epsilon > 0$, there is a $\zeta > 0$ with the following property. Let S be any simplicial subdivision of P of mesh $< \zeta$, whose simplexes σ_i satisfy $\Theta_r(\sigma_i) \geqq \eta$. Let φ' in σ_i be the verage of φ in σ_i, and let f' be the simplicial mapping which coincides with f at he vertices of S. Then

$$(8.5) \qquad \left| \, | f\tilde{\varphi} | - | f'\tilde{\varphi}' | \, \right| < \epsilon.$$

Given a Lipschitz mapping f of a subset Q of E^r into a metric space R, define he "reduced Lipschitz constant"

$$(8.6) \qquad \bar{\mathfrak{L}}_{f,Q} = \underset{\varphi}{\mathrm{GLB}}\, \mathfrak{L}_{f\varphi^{-1}(Q)},$$

vhere φ is an affine volume-preserving mapping of E^r into itself, using f in Q nd φ in $\varphi^{-1}(Q)$. Define the "local constant" $\mathfrak{L}_f^*(p)$ of f at p as follows. Let $\rho_r a^r$ e the volume of the set $x_1^2 + \cdots + x_r^2 \leqq a$. Let $U_\zeta(p)$ denote the ζ-neighborhood f p. Given ζ and η such that $\eta < \rho_r \zeta^r$, set

$$(8.7) \qquad \mathfrak{L}_f^*(p, \zeta, \eta) = \underset{Q}{\mathrm{GLB}}\, \bar{\mathfrak{L}}_{f,Q}, \qquad Q \subset U_\zeta(p), \qquad |Q| > \eta,$$

$$(8.8) \qquad \mathfrak{L}_f^*(p) = \lim_{\mu \to 0} \; \liminf_{\zeta \to 0} \; \mathfrak{L}_f^*[p, \zeta, (1 - \mu)\rho_r\zeta^r].$$

If f is a Lipschitz mapping of a measurable subset Q of E^r into E^n, then

$$(8.9) \qquad | J_f(p) | = [\mathfrak{L}_f^*(p)]^r \text{ a. e. in } Q.$$

Now the set of $(\tilde{\varphi}, f)$, with certain equivalence relations, forms the space of ipschitz r-chains A in R. For any such A, considering the various expressions $\tilde{\varphi}, f)$ of A, define

$$(8.10) \qquad | A | = \mathrm{GLB} \int | \varphi(p) | \, [\mathfrak{L}_f^*(p)]^r \, dp.$$

ctually (with some restrictions on R) we may choose $(\tilde{\varphi}, f)$ so that f is one-one 1 $\mathrm{Car}(\varphi)$; then $| A |$ equals the integral above. In particular, taking $\varphi(p) = 1$ ives the "r-volume" of the "rectifiable" manifold $f(\mathrm{Car}(\varphi))$ in R. The definitions gree with the previous ones if R is an open subset of E^n.

The norm $| A |^*$ is definable from $| A |$; then cochains may be introduced as efore.

9. Properties of Lipschitz mappings. Let f be a Lipschitz mapping of E^n into E^m. Then each polyhedral r-chain A in E^n is carried into a Lipschitz r-chain fA in E^n. The definition may be extended to general Lipschitz r-chains in E^n. 'he usual properties hold. In particular,

$$(9.1) \qquad |fA| \leqq \mathfrak{L}_f^r |A|, \quad |fA|^* \leqq \max(\mathfrak{L}_f^r, \mathfrak{L}_f^{r+1}) \, |A|^*.$$

Given X in E^m, f^*X in E^n is defined by $f^*X \cdot A = X \cdot fA$. Then $|f^*X| \leqq \mathfrak{L}_f^r |X|$, tc.

If ω is a Lipschitz r-form in E^m, it defines a Lipschitz r-cochain X, this defines f^*X, and this gives ω^* in E^n; call this $f^*\omega$. Recall that X defines δX and hence $\delta\omega$; also $\delta f^*X = f^*\delta X$; hence $\delta f^*\omega = f^*\delta\omega$. The usual analytic theory requires the differentiability of both f and ω for any parts of this last formula even to be defined.

Nevertheless, with the "Lipschitz" hypotheses used here, an analytic treatment of $f^*\omega$ can be given. We first note a fundamental difficulty: ω may fail to be defined in a set of measure 0 in E^m, and f might even map the whole of E^n into this set. Thus the first necessity is to enlarge or improve the definition of ω. This is done by determining the corresponding X, and finding a new ω' from X; then $\omega' = \omega$ a.e. (see §6). Then the usual analytic formula defines $f^*\omega'$ a.e. in E^n.

The proof requires the following approximation theorem. Let X be a Lipschitz r-cochain in E^m, and let p be any point at which the corresponding $\omega(p)$ exists and is linear (see §6). Then for any $\eta > 0$ and $\epsilon > 0$ there is a $\zeta > 0$ with the following property. Let σ be any oriented r-simplex with

$$\sigma \subset U_\zeta(p), \qquad \Theta_r((p) \cup \sigma) \geqq \eta; \tag{9.2}$$

if $\{\sigma\}$ denotes the r-vector defined by σ, then

$$| X\cdot\sigma - \omega(p)\cdot\{\sigma\} | \leqq \epsilon | \sigma |. \tag{9.3}$$

As a particular case, suppose $r = 1$, and X is the coboundary δY of a Lipschitz 0-cochain Y, Y corresponding to the real-valued function $w(p)$. Then the theorem gives the Rademacher Theorem on the total differentiability a.e. of the Lipschitz function w.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., U. S. A.

COMPREHENSIVE VIEW OF PREDICTION THEORY

N. Wiener

This address was given as part of the Conference in Applied Mathematics, Volume 2, page 308.

THE CULTURAL BASIS OF MATHEMATICS[1]

R. L. WILDER

I presume that it is not inappropriate, on the occasion of an Internation Mathematical Congress which comes at the half-century mark, to devote little time to a consideration of mathematics as a whole. The addresses aı papers to be given in the various conferences and sectional meetings will in ge eral be concerned with special fields or branches of mathematics. It is the ai of the present remarks to get outside mathematics, as it were, in the hope attaining a new perspective. Mathematics has been studied extensively fro the abstract philosophical viewpoint, and some benefits have accrued to math matics from such studies—although generally the working mathematician inclined to look upon philosophical speculation with suspicion. A growing nuı ber of mathematicians have been devoting thought to the Foundations Mathematics, many of them men whose contributions to mathematics ha won them respect. The varying degrees of dogmatism with which some of the have come to regard their theories, as well as the sometimes acrimonious d bates which have occurred between holders of conflicting theories, makes oı wonder if there is not some vantage point from which one can view such matte more dispassionately.

It has become commonplace today to say that mankind is in its prese "deplorable" state because it has devoted so much of its energy to technic skills and so little to the study of man itself. Early in his civilized career, mı studied astronomy and the other physical sciences, along with the mathemati these subjects suggested; but in regard to such subjects as anatomy, for examp it was not easy for him to be objective. Man himself, it seemed, should be co sidered untouchable so far as his private person was concerned. It is virtual only within our own era that the study of the even more personal subjects, su as psychology, has become moderately respectable! But in the study of t behavior of man *en masse*, we have made little progress. This is evidently d to a variety of reasons such as (1) inability to distinguish between group b havior and individual behavior, and (2) the fact that although the avera person may grudgingly give in to being cut open by a surgeon, or analyzed by psychiatrist, those group institutions which determine his system of valu such as nation, church, clubs, etc., are still considered untouchable.

Fortunately, just as the body of the executed criminal ultimately becar available to the anatomist, so the "primitive" tribes of Australia, the Paci Islands, Africa, and the United States, became available to the anthropologi Using methods that have now become so impersonal and objective as to me

[1] I am indebted to my colleague, Professor L. A. White, of the Anthropology Departme University of Michigan, and to Betty Ann Dillingham, for reading this paper in manuscri and offering most helpful criticism and advice. However, responsibility for errors a opinions expressed herein is entirely my own, of course.

its being classed among the natural sciences rather than with such social studies as history, anthropology has made great advances within the past 50 years in the study of the group behavior of mankind. Its development of the culture concept and investigation of cultural forces will, perhaps, rank among the greatest achievements of the human mind, and despite opposition, application of the concept has made strides in recent years. Not only are psychologists, psychiatrists, and sociologists applying it, but governments that seek to extend their control over alien peoples have recognized it. Manifold human suffering has resulted from ignorance of the concept, both in the treatment of colonial peoples, and in the handling of the American Indian, for example.

Now I am not going to offer the culture concept as an antidote for all the ills that beset mathematics. But I do believe that only by recognition of the cultural basis of mathematics will a better understanding of its nature be achieved; moreover, light can be thrown on various problems, particularly those of the Foundations of Mathematics. I don't mean that it can solve these problems, but that it can point the *way* to solutions as well as show the *kinds* of solutions that may be expected. In addition, many things that we have believed, and attributed to some kind of vague "intuition," acquire a real validity on the cultural basis.

For the sake of completeness, I shall begin with a rough explanation of the concept. (For a more adequate exposition, see [10; Chap. 7] and [18].)[2] Obviously it has nothing to do with culture spelled with a "K", or with degrees from the best universities or inclusion in the "best" social circles. A culture is the collection of customs, rituals, beliefs, tools, mores, etc., which we may call *cultural elements*, possessed by a group of people, such as a primitive tribe or the people of North America. Generally it is not a fixed thing but changing with the course of time, forming what can be called a "culture stream." It is handed down from one generation to another, constituting a seemingly living body of tradition often more dictatorial in its hold than Hitler was over Nazi Germany; in some primitive tribes virtually every act, even such ordinary ones as eating and dressing, are governed by ritual. Many anthropologists have thought of a culture as a super-organic entity, having laws of development all its own, and most anthropologists seem in practice to treat a culture as a thing in itself, without necessarily referring (except for certain purposes) to the group or individuals possessing it.

We "civilized" people rarely think of how much we are dominated by our cultures—we take so much of our behavior as "natural." But if you were to propose to the average American male that he should wear earrings, you might, as you picked yourself off the ground, reflect on the reason for the blow that you have just sustained. Was it because he decided at some previous date that every time someone suggested wearing earrings to him he would respond with a

[2] References in brackets are to the bibliography at the end of the paper. The first number in a bracket refers to the corresponding number in the bibliography, the second number to pages, chapter, or volume of the work indicated.

punch to the nose? Of course not. It was decided for him and imposed on him by the American culture, so that what he did was, he would say, the "natural thing to do." However, there are societies such as Navajo, Pueblo, and certain Amazon tribes, for instance, in which the wearing of earrings by the males is the "natural thing to do." What we call "human nature" is virtually nothing but a collection of such culture traits. What is "human nature" for a Navajo is distinctly different from what is "human nature" for a Hottentot.

As mathematicians, we share a certain portion of our cultures which is called "mathematical." We are influenced by it, and in turn we influence it. As individuals we assimilate parts of it, our contacts with it being through teachers, journals, books, meetings such as this, and our colleagues. We contribute to its growth the results of our individual syntheses of the portions that we have assimilated.

Now to look at mathematics as a cultural element is not new. Anthropologists have done so, but as their knowledge of mathematics is generally very limited, their reactions have ordinarily consisted of scattered remarks concerning the types of arithmetic found in primitive cultures. An exception is an article [17] which appeared about three years ago, by the anthropologist L. A. White, entitled *The locus of mathematical reality*, which was inspired by the seemingly conflicting notions of the nature of mathematics as expressed by various mathematicians and philosophers. Thus, there is the belief expressed by G. H. Hardy [8; pp. 63–64] that "mathematical reality lies outside us, and that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations' are simply our notes of our observations." On the other hand there is the point of view expressed by P. W. Bridgman [3; p. 60] that "it is the merest truism, evident at once to unsophisticated observation, that mathematics is a human invention." Although these statements seem irreconcilable, such is not the case when they are suitably interpreted. For insofar as our mathematics is a part of our culture, it is, as Hardy says, "outside us." And insofar as a culture cannot exist except as the product of human minds, mathematics is, as Bridgman states, a "human invention."

As a body of knowledge, mathematics is not something I know, you know, or any individual knows: It is a part of our culture, our *collective* possession. We may even forget, with the passing of time, some of our own individual contributions to it, but these may remain, despite our forgetfulness, in the culture stream. As in the case of many other cultural elements, we are taught mathematics from the time when we are able to speak, and from the first we are impressed with what we call its "absolute truth." It comes to have the same significance and type of reality, perhaps, as their system of gods and rituals has for a primitive people. Such would seem to be the case of Hermite, for example, who according to Hadamard [7; p. xii] said, "We are rather servants than masters in Mathematics;" and who said [6; p. 449] in a letter to Königsberger, "—these otions of analysis have their existence apart from us,—they constitute a whole f which only a part is revealed to us, incontestably although mysteriously assoiated with that other totality of things which we perceive by way of the senses." Evidently Hermite sensed the impelling influence of the culture stream to which e contributed so much!

In his famous work *Der Untergang des Abendlandes* [15], O. Spengler discussed t considerable length the nature of mathematics and its importance in his rganic theory of cultures. And under the influence of this work, C. J. Keyser ublished [9] some views concerning *Mathematics as a Culture Clue*, constituting n exposition and defense of the thesis that "The type of mathematics found in ny major Culture is a clue, or key, to the distinctive character of the Culture iken as a whole." Insofar as mathematics is a part of and is influenced by the ulture in which it is found, one may expect to find some sort of relationship etween the two. As to how good a "key" it furnishes to a culture, however, I hall express no opinion; this is really a question for an anthropologist to answer. ince the culture dominates its elements, and in particular its mathematics, it ould appear that for mathematicians it would be more fruitful to study the elationship from this point of view.

Let us look for a few minutes at the history of mathematics. I confess I know ery little about it, since I am not a historian. I should think, however, that in riting a history of mathematics the historian would be constantly faced with e question of *what sort of material to include*. In order to make a clearer case, t us suppose that a hypothetical person, A, sets out to write a *complete* history, esiring to include all available material on the "history of mathematics." bviously, he will have to accept some material and reject other material. It eems clear that his criterion for choice must be based on knowledge of what onstitutes mathematics! If by this we mean a *definition* of mathematics, of ourse his task is hopeless. Many definitions have been given, but none has een chosen; judging by their number, it used to be expected of every selfespecting mathematician that he would leave a definition of mathematics to osterity! Consequently our hypothetical mathematician A will be guided, I nagine, by what is *called* "mathematics" in his culture, both in existing (preiously written) histories and in works *called* "mathematical," as well as by hat sort of thing people who are *called* "mathematicians" publish. He will, en, recognize what we have already stated, that mathematics is a certain art of his culture, and will be guided thereby.

For example, suppose A were a Chinese historian living about the year 1200 00 or 1500 would do as well). He would include a great deal about computing ith numbers and solving equations; but there wouldn't be any geometry as e Greek understood it in his history, simply because it had never been integrated with the mathematics of his culture. On the other hand, if A were a reek of 200 A.D., his history of mathematics would be replete with geometry, ut there would be little of algebra or even of computing with numbers as the

Chinese practiced it. But if A were one of our contemporaries, he would includ both geometry and algebra because both are part of what we call mathematics I wonder what he would do about logic, however?

Here is a subject which, despite the dependence of the Greeks on logica deduction, and despite the fact that mathematicians, such as Leibnitz an Pascal, have devoted considerable time to it on its own merits, has been give very little space in histories of mathematics. As an experiment, I looked in tw histories that have been popular in this country; Ball's [1] and Cajori's [5], bot written shortly before 1900. In the index of Ball's first edition (1888) there i no mention of "logic;" but in the fourth edition (1908) "symbolic an mathematical logic" is mentioned with a single citation, which proved to be a refer ence to an incidental remark about George Boole to the effect that he "was on of the creators of symbolic or mathematical logic." Thus symbolic logic barel squeezed under the line because Boole was a mathematician! The index c Cajori's first edition (1893) contains four citations under "logic," all referrin to incidental remarks in the text. None of these citations is repeated in th second edition (1919), whose index has only three citations under "logic" (tw of which also constitute the sole citations under "symbolic logic"), again referrin only to brief remarks in the text. Inspection of the text, however, reveals nearl four pages (407–410) of material under the title "Mathematical logic," al though there is no citation to this subject in the index nor is it cited unde "logic" or "symbolic logic." (It is as though the subject had, by 1919, achieve enough importance for inclusion as textual material in a history of mathematic although not for citation in the index!)

I doubt if a like situation could prevail in a history of mathematics whic covers the past 50 years! The only such history that covers this period, that am acquainted with, is Bell's *Development of Mathematics* [2]. Turning to th index of this book, I found so many citations to "logic" that I did not car to count them. In particular, Bell devotes at least 25 pages to the developmen of what he calls "mathematical logic." Can there be any possible doubt tha this subject, not considered part of mathematics in our culture in 1900, despit the pioneering work of Peano and his colleagues, is now in such "good stand ing" that any impartial definition of mathematics must be broad enough t include it?

Despite the tendency to approach the history of mathematics from the bio graphical standpoint, there has usually existed some awareness of the impact o cultural forces. For example, in commencing his chapter on Renaissance mathe matics, Ball points out the influence of the introduction of the printing press In the latest histories, namely the work of Bell already cited, and Struik's ex cellent little two volume work [16], the evidence is especially strong. For ex ample, in his introduction, Struik expresses regret that space limitation prevented sufficient "reference to the general cultural and sociological atmos phere in which the mathematics of a period matured—or was stifled." And h goes on to say "Mathematics has been influenced by agriculture, commerc and manufacture, by warfare, engineering and philosophy, by physics and by astronomy. The influence of hydrodynamics on function theory, of Kantianism and of surveying on geometry, of electromagnetism on differential equations, of Cartesianism on mechanics, and of scholasticism on the calculus could only be indicated [in his book];—yet an understanding of the course and content of mathematics can be reached only if all these determining factors are taken into consideration." In his third chapter Struik gives a revealing account of the rise of Hellenistic mathematics, relating it to the cultural conditions then prevailing. I hope that future histories of mathematics will similarly give more attention to mathematics as a cultural element, placing greater emphasis on its relations to the cultures in which it is imbedded.

In discussing the general culture concept, I did not mention the two major processes of cultural change, *evolution* and *diffusion*. By diffusion is meant the transmission of a cultural trait from one culture to another, as a result of some kind of contact between groups of people; for example, the diffusion of French language and customs into the Anglo-Saxon culture following the Norman conquest. As to how much of what we call cultural progress is due to evolution and how much to diffusion, or to a combination of both, is usually difficult to determine, since the two processes tend so much to merge. Consider, for example, the counting process. This is what the anthropologist calls a universal trait—what I would prefer to call, in talking to mathematicians, a cultural invariant—it is found in every culture in at least a rudimentary form. The "base" may be 10, 12, 20, 25, 60—all of these are common, and are evidently determined by other (variable) culture elements—but the counting process in its essence, as the Intuitionist speaks of it, is invariant. If we consider more advanced cultures, the notion of a zero element sometimes appears. As pointed out by the anthropologist A. L. Kroeber, who in his *Anthropology* calls it a "milestone of civilization," a symbol for zero evolved in the cultures of at least three peoples; the Neo-Babylonian (who used a sexagesimal system), the Mayan (who used a vigesimal system), and the Hindu (from whom our decimal system is derived) [10; pp. 468–472]. Attempts by the extreme "diffusionists" to relate these have not yet been successful, and until they are, we can surmise that the concept of zero might ultimately evolve in any culture.

The Chinese-Japanese mathematics is of interest here. Evidently, as pointed out by Mikami [13] and others, the Chinese borrowed the zero concept from the Hindus, with whom they established contact at least as early as the first century, A.D. Here we have an example of its introduction by diffusion, but without such contacts, the zero would probably have evolved in Chinese mathematics, especially since calculators of the rod type were employed. The Chinese mathematics is also interesting from another standpoint in that its development seems to have been so much due to evolution within its own culture and so little affected by diffusion. Through the centuries it developed along slender arithmetic and algebraic lines, with no hint of geometry as the Greeks developed it. Those who feel that without the benefit of diffusion a culture will eventually stagnate find some evidence perhaps in the delight with which Japanese mathematicians of the 17th and 18th centuries, to whom the Chinese mathematics had come by the diffusion process, solved equations of degrees as high as 3000 or 4000. One is tempted to speculate what might have happened if the Babylonian zero and method of position had been integrated with the *Greek* mathematics—would it have meant that Greek mathematics might have taken an algebraic turn? Its introduction into the Chinese mathematics certainly was not productive, other than in the slight impetus it gave an already computational tendency.

That the Greek mathematics was a natural concomitant of the other elements in Greek culture, as well as a natural result of the evolution and diffusion processes that had produced this culture in the Asia Minor area, has been generally recognized. Not only was the Greek culture conducive to the type of mathematics that evolved in Greece, but it is probable that it resisted integration with the Babylonian method of enumeration. For if the latter became known to certain Greek scholars, as some seem to think, its value could not have been apparent to the Greeks.

We are familiar with the manner in which the Hindu-Arabic mathematical cultures diffused via Africa to Spain and then into the Western European cultures. What had become stagnant came to life—analytic geometry appeared, calculus—and the flood was on. The mathematical cultural development of these times would be a fascinating study, and awaits the cultural historian who will undertake it. The easy explanation that a number of "supermen" suddenly appeared on the scene has been abandoned by virtually all anthropologists. A *necessary* condition for the emergence of the "great man" is the presence of suitable cultural environment, including opportunity, incentive, and materials. Who can doubt that potentially great algebraists lived in Greece? But in Greece, although the opportunity and incentive may have been present, the cultural materials did not contain the proper symbolic apparatus. The anthropologist Ralph Linton remarked [12; p. 319] "The mathematical genius can only carry on from the point which mathematical knowledge within his culture has already reached. Thus if Einstein had been born into a primitive tribe which was unable to count beyond three, life-long application to mathematics probably would not have carried him beyond the development of a decimal system based on fingers and toes." Furthermore, the evidence points strongly to the *sufficiency* of the conditions stated: That is, suitable cultural environment is sufficient for the emergence of the great man. If your philosophy depends on the assumption of free will, you can probably adjust to this. For certainly your will is no freer than the opportunity to express it; you may will a trip to the moon this evening, but you won't make it. There may be potentially great blancophrenologists sitting right in this room; but if so they are destined to go unnoticed and undeveloped because blancophrenology is not yet one of our cultural elements.

Spengler states it this way [15tr; vol. II, p. 507]: "We have not the freedom to reach to this or to that, but the freedom to do the necessary or to do nothing. And a task that historic necessity has set *will* be accomplished with the individual or against him." As a matter of fact, when a culture or cultural element has developed to the point where it is ready for an important innovation, the latter is likely to emerge in more than one spot. A classical example is that of the theory of biological evolution, which had been anticipated by Spencer and, had it not been announced by Darwin, was ready to be announced by Wallace and soon thereafter by others. And as in this case, so in most other cases,—and you can recall many such in mathematics—one can after the fact usually go back and map out the evolution of the theory by its traces in the writings of men in the field.

Why are so many giving their lives to mathematics today; why have the past 50 years been so productive mathematically? The mathematical groundwork laid by our predecessors, the universities, societies, foundations, libraries, etc., have furnished unusual opportunity, incentive, and cultural material. In addition, the processes of evolution and diffusion have greatly accelerated. Of the two, the latter seems to have played the greater role in the recent activity. For during the past 50 years there has been an exceptional amount of fusion of different branches of mathematics, as you well know. A most unusual cultural factor affecting the development of mathematics has been the emigration of eminent mathematicians from Germany, Poland, and other countries to the United States during the past 30 years. Men whose interests had been in different branches of mathematics were thrown together and discovered how to merge these branches to their mutual benefit, and frequently new branches grew out of such meetings. The cultural history of mathematics during the past 50 years, taken in conjunction with that of mathematics in ancient Greece, China, and Western Europe, furnishes convincing evidence that *no branch of mathematics can pursue its course in isolation indefinitely, without ultimately reaching a static condition.*

Of the *instruments* for diffusion in mathematics, none is more important, probably, than the journals. Without sufficient outlet for the results of research, and proper distribution of the same, the progress of mathematics will be severely hampered. And any move that retards international contacts through the medium of journals, such as restriction to languages not widely read, is distinctly an anti-mathematical act. For it has become a truism that today mathematics is international.

This brings us to a consideration of symbols. For the so-called "international character" of mathematics is due in large measure to the standardization of symbols that it has achieved, thereby stimulating diffusion. Without a symbolic apparatus to convey our ideas to one another, and to pass on our results to future generations, there wouldn't be any such thing as mathematics—indeed, there would be essentially no culture at all, since, with the possible exception of a few simple tools, culture is based on the use of symbols. A good case can be made for the thesis that man is to be distinguished from other animals by the way in which he uses symbols [18; II]. Man possesses what we might call *symbolic initiative*; that is, he assigns symbols to stand for objects or ideas, sets up relationships between them, and operates with them as though they were physical objects. So far as we can tell, no other animal has this faculty, although many animals do exhibit what we might call *symbolic reflex* behavior. Thus, a dog can be taught to lie down at the command "Lie down," and of course to Pavlov's dogs, the bells signified food. In a recent issue of a certain popular magazine a psychologist is portrayed teaching pigeons to procure food by pressing certain combinations of colored buttons. All of these are examples of symbolic reflex behavior—the animals do not create the symbols.

As an aspect of our culture that depends so exclusively on symbols, as well as the investigation of certain relationships between them, mathematics is probably the furthest from comprehension by the non-human animal. However, much of our mathematical behavior that was originally of the *symbolic initiative* type drops to the *symbolic reflex* level. This is apparently a kind of labor-saving device set up by our neural systems. It is largely due to this, I believe, that a considerable amount of what passes for "good" teaching in mathematics is of the symbolic reflex type, involving no use of symbolic initiative. I refer of course to the drill type of teaching which may enable stupid John to get a required credit in mathematics but bores the creative minded William to the extent that he comes to loathe the subject! What essential difference is there between teaching a human animal to take the square root of 2 and teaching a pigeon to punch certain combinations of colored buttons? Undoubtedly the symbolic reflex type of teaching is justified when the pupil is very young—closer to the so-called "animal" stage of his development, as we say. But as he approaches maturity, more emphasis should be placed on his symbolic initiative. I am reminded here of a certain mathematician who seems to have an uncanny skill for discovering mathematical talent among the undergraduates at his university. But there is nothing mysterious about this; he simply encourages them to use their symbolic initiative. Let me recall parenthetically here what I said about the perennial presence of potential "great men;" there is no reason to believe that this teacher's success is due to a preference for his university by the possessors of mathematical talent, for they usually have no intention of becoming mathematicians when they matriculate. It moves one to wonder how many potentially great mathematicians are being constantly lost to mathematics because of "symbolic reflex" types of teaching.

I want to come now to a consideration of the Foundations of Mathematics. We have witnessed, during the past 50 years, what we might call the most thorough soul-searching in the history of mathematics. By 1900, the Burali-Forti contradiction had been found and the Russell and other antinomies were soon to appear. The sequel is well known: Best known are the attempt of Russell and Whitehead in their monumental *Principia Mathematica* to show that mathematics can be founded, in a manner free of contradiction, on the symbolically expressed principles and methods of what were at the time considered universally valid logical concepts; the formulation, chiefly at the hands of Brouwer and his collaborators, of the tenets of Intuitionism, which although furnishing a theory evidently free of contradiction, introduces a highly complicated set theory and a mathematics radically restricted as compared with the mathematics developed during the 19th century; and the formalization of mathematics by Hilbert and his collaborators, together with the development of a metamathematical proof theory which it was hoped would lead to proofs of freedom from contradiction for a satisfactory portion, at least, of the classical mathematics. None of these "foundations" has met with complete success. Russell and Whitehead's theory of types had to be bolstered with an axiom which they had to admit, in the second edition of *Principia Mathematica*, has only pragmatic justification, and subsequent attempts by Chwistek, Wittgenstein, and Ramsey to eliminate or modify the use of this axiom generally led to new objections. The restricted mathematics known as Intuitionism has won only a small following, although some of its methods, such as those of a finite constructive character, seem to parallel the methods underlying the treatment of formal systems in symbolic logic, and some of its tenets, especially regarding constructive existence proofs, have found considerable favor. The possibility of carrying out the Hilbert program seems highly doubtful, in view of the investigations of Gödel and others.

Now the cultural point of view is not advanced as a substitute for such theories. In my title I have used the word "basis" instead of "foundations" in order to emphasize this point. But it seems probable that the recognition of the cultural basis of mathematics would clear the air in Foundation theories of most of the mystical and vague philosophical arguments which are offered in their defense, as well as furnish a guide and motive for further research. The points of view underlying various attempts at Foundations of Mathematics are often hard to comprehend. In most cases it would seem that the proponents have decided in their own minds just what mathematics is, and that all they have to do is formulate it accordingly—overlooking entirely the fact that because of its cultural basis, mathematics as they know it may be not at all what it will be a century hence. If the thought underlying their endeavors is that they will succeed in trapping the elusive beast and confining it within bounds which it will never break, they are exceedingly optimistic. If the culture concept tells us anything, it should teach us that the first rule for setting up any Foundation theory is that it should only attempt to encompass specific portions of the field as it is known in our culture. At most, a Foundation theory should be considered as a kind of constitution with provision for future amendments. And in view of the situation as regards such principles as the choice axiom, for instance, it looks at present as though no such constitution could be adopted by a unanimous vote!

I mentioned "mysticism and vague philosophical arguments" and their elimination on the cultural basis. Consider, for example, the insistence of Intuitionism that all mathematics should be founded on the natural numbers or the counting process, and that the latter are "intuitively given." There are plausible arguments to support the thesis that the natural numbers should form the starting point for mathematics, but it is hard to understand just what "intuitively given" means, or why the classical conception of the continuum, which the Intuitionist refuses to accept, should not be considered as "intuitively given." It makes one feel that the Intuitionist has taken Kronecker's much-quoted dictum that "The integers were made by God, but all else is the work of man" and substituted "Intuition" for "God." However, if he would substitute for this vague psychological notion of "intuition" the viewpoint that inasmuch as the counting process is a cultural invariant, it follows that the natural numbers form for every culture the most basic part of what has been universally called "mathematics," and should therefore serve as the starting point for every Foundations theory; then I think he would have a much sounder argument. I confess that I have not studied the question as to whether he can find further cultural support to meet all the objections of opponents of Intuitionism. It would seem, however, that he would have to drop his insistence that in construction of sets (to quote Brouwer [4; p. 86]) "neither ordinary language nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary," since no cultural trait on the abstract level of mathematics can be constructed other than by the use of symbols. Furthermore, and this is a serious objection, it appears to ignore *the influence that our language habits have on our modes of thought.*

Or consider the thesis that all mathematics is derivable from what some seem to regard as primitive or universal logical principles and methods. Whence comes this "primitive" or "universal" character? If by these terms it is meant to imply that these principles have a culturally invariant basis like that of the counting process, then it should be pointed out that cultures exist in which they do not have any validity, even in their qualitative non-symbolic form. For example, in cultures which contain magical elements (and such elements form an extremely important part of some primitive cultures), the law of contradiction usually fails. Moreover, the belief that our forms of thought are culturally invariant is no longer held. As eminent a philosopher as John Stuart Mill stated, [14; p. 11], "The principles and rules of grammar are the means by which the forms of language are made to correspond with the universal forms of thought." If Mill had been acquainted with other than the Indo-European language group, he could not have made such an error. The Trobriand Islanders, for example, lack a cause-and-effect pattern of thought; their language embodies no mechanism for expressing a relationship between events. As Malinowski pointed out [11; p. 360], these people have no conception of one event leading up to another, and chronological sequence is unimportant. (Followers of Kant should note that they can count, however.) But I hardly need to belabor the point. As Lukasciewicz and others have observed, not even Aristotle gave to the law of the excluded middle the homage that later logicians paid it! All I want to do in this connection is to indicate that on the cultural basis we find affirmation of what is already finding universal acceptance among mathematical logicians, I believe; namely, that the significance and validity of such material as that in *Principia Mathematica* is only the same as that of other purely formal systems.

It is probably fair to say that the Foundations of Mathematics as conceived and currently investigated by the mathematical logicians finds greatest support on the cultural basis. For inasmuch as there can exist, and have existed, different cultures, different forms of thought, and hence different mathematics, it seems impossible to consider mathematics, as I have already indicated, other than man-made and having no more of the character of necessity or truth than other cultural traits. Problems of mathematical existence, for example, can never be settled by appeal to any mathematical dogma. Indeed, they have no validity except as related to special foundations theories. The question as to the existence of choice sets, for instance, is not the same for an Intuitionist as for a Formalist. The Intuitionist can justifiably assert that "there is no such problem as the continuum problem" provided he adds the words "for an Intuitionist"—otherwise he is talking nonsense. *Because of its cultural basis, there is no such thing as the absolute in mathematics; there is only the relative.*

But we must not be misled by these considerations and jump to the conclusion that what constitutes mathematics in our culture is purely arbitrary; that, for instance, it can be defined as the "science of p implies q", or the science of axiomatic systems. Although the individual person in the cultural group may have some degree of variability allowed him, he is at the same time subject to the dominance of his culture. The individual mathematician can play with postulational systems as he will, but unless and until they are related to the existing state of mathematics in his culture they will only be regarded as idiosyncrasies. Similar ties, not so obvious however, join mathematics to other cultural elements. And these bonds, together with those that tie each and every one of us to our separate mathematical interests, cannot be ignored even if we will to do so. They may exert their influence quite openly, as in the case of those mathematicians who have recently been devoting their time to high speed computers, or to developing other new and unforeseen mathematics induced by the recent wartime demands of our culture. Or their influence may be hidden, as in the case of certain mathematical habits which were culturally induced and have reached the symbolic reflex level in our reactions. Thus, although the postulational method may turn out to be the most generally accepted mode of founding a theory, it must be used with discretion; otherwise the theories produced will not be mathematics in the sense that they will be a part of the mathematical component of our culture.

But it is time that I closed these remarks. It would be interesting to study evidence in mathematics of *styles* and of *cultural patterns*; these would probably be interesting subjects of investigation for either the mathematician or the anthropologist, and could conceivably throw some light on the probable future course of the field. I shall have to pass on, however, to a brief conclusion:

In man's various cultures are found certain elements which are called *mathe-*

matical. In the earlier days of civilization, they varied greatly from one culture to another so much so that what was called "mathematics" in one culture would hardly be recognized as such in certain others. With the increase in diffusion due, first, to exploration and invention, and, secondly, to the increase in the use of suitable symbols and their subsequent standardization and dissemination in journals, the mathematical elements of the most advanced cultures gradually merged until, except for minor cultural differences like the emphasis on geometry in Italy, or on function theory in France, there has resulted essentially one element, common to all civilized cultures, known as mathematics. This is not a fixed entity, however, but is subject to constant change. Not all of the change represents accretion of new material; some of it is a shedding of material no longer, due to influential cultural variations, considered mathematics. Some so-called "borderline" work, for example, it is difficult to place either in mathematics or outside mathematics.

From the extension of the notion of number to the transfinite, during the latter half of the 19th century, there evolved certain contradictions around the turn of the century, and as a consequence the study of Foundations questions, accompanied by a great development of mathematical logic, has increased during the last 50 years. Insofar as the search for satisfactory Foundation theories aims at any absolute criterion for truth in mathematics or fixation of mathematical method, it appears doomed to failure, since recognition of the cultural basis of mathematics compels the realization of its variable and growing character. Like other culture traits, however, mathematics is not a thoroughly arbitrary construction of the individual mathematician, since the latter is restricted in his seemingly free creations by the state of mathematics and its directions of growth during his lifetime, it being the latter that determines what is considered "important" at the given time.

In turn, the state and directions of growth of mathematics are determined by the general complex of cultural forces both within and without mathematics. Conspicuous among the forces operating from without during the past 50 years have been the crises through which the cultures chiefly concerned have been passing; these have brought about a large exodus of mathematicians from Western Europe to the United States, thereby setting up new contacts with resulting diffusion and interaction of mathematical ideas, as well as in the institution of new directions or acceleration of directions already under way, such as in certain branches of applied mathematics.

What the next 50 years will bring, I am not competent to predict. In his *Decline of the West*, Spengler concluded [15tr; pp. 89–90] that in the notion of *group*, Western "mathematic" had achieved its "last and conclusive creation," and he closed his second chapter, entitled "The meaning of numbers," with the words: "—the time of the *great* mathematicians is past. Our tasks today are those of preserving, rounding off, refining, selection—in place of big dynamic creation, the same clever detail-work which characterized the Alexandrian mathematic of late Hellenism." This was published in 1918—32 years ago—and

I leave it to your judgment whether he was right or not. It seems unlikely that the threatened division into two opposing camps of those nations in which mathematical activity is chiefly centered at present will be of long enough duration to set up two distinct mathematical cultures—although in other fields, such as botany, such a division appears to be under way. Nevertheless, as individual mathematicians we are just as susceptible to cultural forces as are botanists, economists, or farmers, and long separation in differing cultures can result in variations of personality that cannot fail to be reflected in our mathematical behavior. Let us hope that at the turn of the century 50 years hence, mathematics will be as active and unique a cultural force as it is now, with that free dissemination of ideas which is the chief determinant of growth and vitality.

Bibliography

1. W. W. R. Ball, *A short account of the history of mathematics*, London, Macmillan, 1888; 4th ed., 1908.
2. E. T. Bell, *The development of mathematics*, New York, McGraw-Hill, 2d ed., 1945.
3. P. W. Bridgman, *The logic of modern physics*, New York, Macmillan, 1927.
4. L. E. J. Brouwer, *Intuitionism and formalism*, Bull. Amer. Math. Soc. vol. 20 (1913–1914) pp. 81–96 (tr. by A. Dresden).
5. F. Cajori, *A history of mathematics*, New York, Macmillan, 1893; 2d ed., 1919.
6. A. Dresden, *Some philosophical aspects of mathematics*, Bull. Amer. Math. Soc. vol. 34 (1928) pp. 438–452.
7. J. Hadamard, *The psychology of invention in the mathematical field*, Princeton, Princeton University Press, 1945.
8. G. H. Hardy, *A mathematician's apology*, Cambridge, England, Cambridge University Press, 1941.
9. C. J. Keyser, *Mathematics as a culture clue*, Scripta Mathematica vol. 1 (1932–1933) pp. 185–203; reprinted in a volume of essays having same title, New York, Scripta Mathematica, 1947.
10. A. L. Kroeber, *Anthropology*, rev. ed, New York, Harcourt, Brace, 1948.
11. D. D. Lee, *A primitive system of values*, Philosophy of Science vol. 7 (1940) pp. 355–378.
12. R. Linton, *The study of man*, New York, Appleton-Century, 1936.
13. Y. Mikami, *The development of mathematics in China and Japan*, Leipzig, Drugulin 1913.
14. J. S. Mill, *Inaugural address*, delivered to the University of St. Andrews, Feb. 1, 1867, Boston, Littell and Gay.
15. O. Spengler, *Der Untergang des Abendlandes*, München, C. H. Beck, vol. I, 1918, (2d ed., 1923), vol. II, 1922. [15tr] English translation of [15] by C. F. Atkinson, under the title *The decline of the West*, New York, Knopf, vol. I, 1926, vol. II, 1928.
16. D. J. Struik, *A concise history of mathematics*, 2 vols., New York, Dover, 1948.
17. L. A. White, *The locus of mathematical reality*, Philosophy of Science vol. 14 (1947) pp. 289–303; republished in somewhat altered form as Chapter 10 of [18].
18. ———, *The science of culture*, New York, Farrar, Straus, 1949.

University of Michigan,
Ann Arbor, Mich., U.S.A.

THE FUNDAMENTAL IDEAS OF ABSTRACT ALGEBRAIC GEOMETRY

O. Zariski

This address was given as part of the Conference in Algebra, see Volume 2, page 77.

ADDRESSES AND COMMUNICATIONS IN SECTIONS

SECTION I. ALGEBRA AND THEORY OF NUMBERS

THE CHARACTERS OF BINARY MODULARY CONGRUENCE GROUPS

H. D. Kloosterman

Several special methods for the actual calculation of the group characters of a finite group have already been given by Frobenius [2].[1] Amongst the groups for which he solved this problem is also the binary modulary congruence group modulo p, consisting of all matrices of the form

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{1}$$

the elements of which belong to the prime field of prime characteristic p and with determinant $\alpha\delta - \beta\gamma = 1$. He tabulated the values of the characters belonging to the $p + 4$ irreducible representations, but he did not, however, give the representations themselves.

It has become apparent from several publications by *Hecke* [3] that the characters as well as the representations of the binary modulary congruence groups modulo N, consisting of all matrices (1), where now α, β, γ, δ are classes of residues modulo the positive integer N such that $\alpha\delta - \beta\gamma \equiv 1 \pmod N$ play an important role in the general theory of modular functions. He therefore raised the question of determining these characters and representations. By well-known methods the problem can be reduced to the case $N = p^\lambda$, where p is a prime number. The first contributions to this problem were published almost at the same time by *H. Rohrbach* [7] and *H. W. Praetorius* [6] who both gave the solution, as far as the determination of the characters is concerned, for the special case $N = p^2$ ($\lambda = 2$). The general problem was attacked by me in two papers in Ann. of Math. [5] (to be cited as I and II) by means of the transformation formulas under modular substitutions of certain multiple theta series. By using binary theta series only, I succeeded in determining explicitly the greater part of the characters and irreducible representations and I conjectured that the remaining irreducible representations might be determined by considering ternary or higher theta series. Since 1946, however, it has appeared to me that all characters can be found by considering binary theta series only.

Before giving a survey of the methods by means of which the irreducible representations of the binary modulary congruence group modulo N can be found, it is necessary to show how the representations of this group (which will be denoted by $G(N)$) enter into the theory of entire modular forms of level ("Stufe") N.

The (homogeneous) *modular group* $\Gamma = \Gamma(1)$ is the group of all matrices (1) such that α, β, γ, δ are rational integers, satisfying the equation $\alpha\delta - \beta\gamma = 1$.

[1] The numbers in brackets refer to the bibliography at the end of the paper.

For any positive integer N the *principal congruence subgroup* $\Gamma(N)$ *of level* N is the group of those matrices of Γ, for which

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}.$$

It is a normal subgroup in $\Gamma(1)$ with the index

$$g = N^3 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right)$$

and $G(N)$ is the quotient group $\Gamma(1)/\Gamma(N)$. An *entire modular form of level N and dimension* $-k$ is an analytic function $F(\tau)$ of a complex variable τ, such that

$$(\gamma\tau + \delta)^{-k} F\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = F(\tau)$$

for all

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(N),$$

which is regular in the upper half plane $\mathcal{I}(\tau) > 0$ and such that for any matrix of Γ the function $(\gamma\tau + \delta)^k F(\tau)$ has in the neighbourhood of the rational point $\tau = -\delta/\gamma$ on the real axis a power series development

$$\sum_{m=0}^{\infty} a_m \exp\left(2\pi i \frac{m}{N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \tag{2}$$

in the "uniformising variable" $\exp((2\pi i/N)((\alpha\tau + \beta)/(\gamma\tau + \delta)))$ for this rational point. It follows from the theory of algebraic functions that the number of linearly independent entire modular forms of type $(-k, N)$ is finite. They therefore define a finite-dimensional vector space R, the exact dimensionality of which can be determined by an application of the Riemann-Roch theorem for the algebraic manifold determined by the fundamental domain of $\Gamma(N)$.

The matrices of Γ can be made operators on the space R: we define the symbol FA by

$$FA = (\gamma\tau + \delta)^{-k} F\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right)$$

for any entire modular form F of type $(-k, N)$ (vector of the vector space R) and any matrix A of Γ. From the fact that $\Gamma(N)$ is a normal subgroup of Γ and since

$$(FA_1)A_2 = F(A_1A_2)$$

for any $A_1 \in \Gamma$, $A_2 \in \Gamma$, it follows that FA is again an entire modular form of type $(-k, N)$. If therefore $f_1, f_2, \cdots, f_m$ constitute a basis for the vector space R, then there exist relations

$$f_i A = \sum_{j=1}^{m} a_{ij} f_j \qquad (i = 1, 2, \cdots, m)$$

where (a_{ij}) is a matrix of complex numbers. The mapping

$$A \to (a_{ij}) \tag{3}$$

is a representation of Γ and since the elements of $\Gamma(N)$ are mapped upon the unit matrix, it even is a representation of the quotient group $G(N) = \Gamma/\Gamma(N)$. The representation (3) is not irreducible. An invariant subspace of the representation space (invariant under all elements $A \in \Gamma$) is for instance the space of the so-called cusp-forms (those entire modular forms for which the number a_0 in (2) is 0 in all rational points $\tau = -\delta/\gamma$ on the real axis. The problem arises of splitting the representation (3) up into its irreducible constituents. It can be solved if the characters of the irreducible representations of $G(N)$ are known. In this way, using the results obtained by Frobenius, Rohrbach, and Praetorius, the problem has been solved by *H. Feldmann* [1] and *H. Spies* [8] in the special cases $N = p$ and $N = p^2$.

The idea used by me [5] for the determination of the irreducible representations and characters of $G(N)$ is to consider in R the invariant subspaces determined by a certain set of theta functions. Their behaviour under modular substitutions can be determined by a method of *Hermite* [4] and in this way certain representations of $G(N)$ are obtained. In the case of binary theta series it appears to be possible to effect explicitly the reduction of their representation spaces into irreducible invariant subspaces and in this way not only the irreducible characters but also a complete set of inequivalent irreducible representations of $G(N)$ can be obtained.

The theta functions used for this purpose (in I) are defined in the following way. Let $\mathfrak{Q}$ be an integral (i.e., with rational integral elements) symmetric matrix with n rows and columns (n is a positive integer). If x is a variable column vector (matrix with n rows and one column) and x' its transposed (a row vector or matrix with one row and n columns), then $x'\mathfrak{Q}x$ (also denoted by $\mathfrak{Q}[x]$) is a quadratic form, which is supposed to be positive definite. The determinant Δ of $\mathfrak{Q}$ is therefore a positive integer. An integral column vector will be termed *special* if $\mathfrak{Q}a \equiv 0 \pmod{\Delta}$. Now let g, h, and a be special vectors, z a column vector with complex components, and τ a complex variable restricted to the upper half plane $\mathfrak{J}\tau > 0$. We write

$$\vartheta_{gh}(z \mid \tau; a, N) = \sum_{m \equiv a(N\Delta)} \exp\left\{\pi i \frac{h'\mathfrak{Q}(m-a)}{N\Delta^2} \right.$$
$$\left. + \frac{\pi i \tau}{N\Delta^2}\mathfrak{Q}[m + \tfrac{1}{2}g] + \frac{2\pi i}{\Delta}(m + \tfrac{1}{2}g)'\mathfrak{Q}z\right\},$$

where the summation is extended over all integral vectors m satisfying the condition $m \equiv a \pmod{N\Delta}$. It is easily seen to be a direct generalization in somewhat modified form of the usual definition of multiple theta series by the introduction of a congruence restriction on the summation, the modulus $N\Delta$ of this congruence involving the level N. The numbers $h'\mathfrak{Q}(m - a)/N\Delta^2$ are integers. The denominators $N\Delta^2$ and Δ in the second and third term in the exponential are introduced in order to secure that the linear combinations of the theta series with variable g, h, and a (and complex coefficients) form a closed set under the application of modular substitutions on the variable τ. Somewhat more precisely, the method of Hermite mentioned above yields the result that for any modular substitution

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$$

the function

$$\vartheta_{gh}\left(\frac{z}{\gamma\tau + \delta} \,\middle|\, \frac{\alpha\tau + \beta}{\gamma\tau + \delta}; a, N\right) \tag{4}$$

is a linear combination of the series $\vartheta_{g_1h_1}(z \mid \tau; b, N)$, where 1) b runs through a complete set of incongruent special vectors mod $N\Delta$; 2) $g_1 = \alpha g + \gamma h + \alpha\gamma N v$, $h_1 = \beta g + \delta h + \beta\delta N v$, where $v = \Delta\mathfrak{Q}^{-1}t$ and t is the vector whose components are the diagonal elements of the matrix $\mathfrak{Q}$; 3) the coefficients are of the form $W\varphi(a, b)$, where

$$W = \frac{1}{\Delta^{1/2}}\left(\left(\frac{-i(\gamma\tau + \delta)\operatorname{sgn}\gamma}{|\gamma| N}\right)^{1/2}\right)^n \exp\frac{N\gamma\pi i\mathfrak{Q}[z]}{\gamma\tau + \delta}$$

and where $\varphi(a, b)$ is a generalized Gaussian sum (depending on the matrix A)

In order to obtain the desired representation of $G(N)$ it is still necessary to perform 1) a specialization and 2) a normalization. In the case that N is a power of an odd prime, the specialization consists in taking $g = h \equiv v \pmod 2$. Then the function (4) (with $g = h$) becomes a linear combination of the functions $\vartheta_{gg}(z \mid \tau; b, N)$ (with the same $g = h$). By the normalization, some undesirable factors are removed. We write

$$x_a(z \mid \tau) = \frac{\vartheta_{gg}(z \mid \tau; a, N)}{\vartheta_{gg}^N(z \mid \tau; 0, 1)}.$$

Now defining the matrices A of Γ as operators on functions $F(z \mid \tau)$ by the definition

$$F(z \mid \tau)A = (\gamma\tau + \delta)^{n(N-1)/2} F\left(\frac{z}{\gamma\tau + \delta} \,\middle|\, \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right),$$

we can prove for matrices A of the group $\Gamma(\Delta)$ relations of the form

$$x_a(z \mid \tau)A = \sum_b C(a, b)x_b(z \mid \tau),$$

where both a and b run through a complete set of incongruent special vectors mod $N\Delta$ and where the coefficients $C(a, b)$ are either 0 or roots of unity. The mapping

$$A \rightarrow (C(a, b)) \tag{6}$$

s a representation of $\Gamma(\Delta)$ and since it can be proved that the matrices of the ubgroup $\Gamma(N\Delta)$ of $\Gamma(\Delta)$ are mapped on the unit matrix, the representation (6) ndeed even is a representation of the quotient group $\Gamma(\Delta)/\Gamma(N\Delta)$.

Two cases must now be considered separately. If N is a power p^λ of the odd orime p, we consider first the case $\Delta \not\equiv 0 \pmod p$ (this is the only case considered n II). In this case we have the isomorphism

$$\Gamma(\Delta)/\Gamma(N\Delta) \cong \Gamma(1)/\Gamma(N) = G(N)$$

ınd the representation (6) of the quotient group $\Gamma(\Delta)/\Gamma(N\Delta)$ is also a representa-ion of $G(p^\lambda)$. We denote it by $R(p^\lambda)$. In the binary case ($n = 2$) the splitting up of $R(p^\lambda)$ into irreducible representations is principally based on the following wo fundamental facts:

1) the representation $R(p^\lambda)$ contains the representation $R(p^{\lambda-2})$ ($\lambda \geqq 2$);

2) if V is an automorph modulo N of the quadratic form $\mathfrak{Q}$ (i.e., a matrix V uch that $V'\mathfrak{Q}V \equiv \mathfrak{Q} \pmod N$, where V' is the transposed of V), then the numbers C have the property

$$C(Va, Vb) = C(a, b).$$

n the case $n = 2$ the automorphs V ("units") constitute a simple dihedral group. The two mentioned facts suggest in the representation space the introduction of a new basis by means of which the splitting up into irreducible representations s accomplished. Though I found in this way in II the greater part of the ir-educible representations of $G(p^\lambda)$, I did not however find them all. I had over-ooked the fact that the remaining irreducible representations can be found by onsidering binary quadratic forms with a discriminant Δ which is divisible by p. Supposing $\Delta \equiv 0 \pmod p$, $\Delta \not\equiv 0 \pmod{p^2}$, $N = p^\lambda$, we now have the iso-norphism

$$\Gamma(\Delta)/\Gamma(N\Delta) \cong \Gamma(p)/\Gamma(p^{\lambda+1}),$$

nd the representation (6) does not give immediately a representation of the roup $G(p^\lambda)$, but only a representation of a subgroup of $G(p^{\lambda+1})$, namely the ubgroup of those elements of $G(p^{\lambda+1})$ which are

$$\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod p.$$

But this representation induces a representation of the whole group $G(p^{\lambda+1})$ nd this induced representation can be split up into its irreducible constituents y means of the same devices as used in the first case $\Delta \not\equiv 0 \pmod p$. It appears hat all irreducible representations of $G(p^\lambda)$ occur either in the first case $\Delta \not\equiv 0$ (mod p) or in the second case $\Delta \equiv 0 \pmod p$.

The detailed proofs of these statements will be published elsewhere.

REFERENCES

1. H. FELDMANN, *Ueber das Verhalten der Modulfunktionen von Primzahlstufe bei beliebigen Modulsubstitutionen*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1931) pp. 323–341.
2. G. FROBENIUS, *Ueber Gruppencharactere*, Preuss. Akad. Wiss. Sitzungsber. (1896) pp. 985–1021.
3. E. HECKE,
   (a) *Ueber ein Fundamentalproblem aus der Theorie der elliptischen Modulfunktionen*, Abh. Math. Sem. Hamburgischen Univ. vol. 6 (1928) pp. 235–257.
   (b) *Ueber das Verhalten der Integrale erster Gattung bei Abbildungen, insbesondere in der Theorie der elliptischen Modulfunktionen*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1931) pp. 271–281.
   (c) *Grundlagen einer Theorie der Integralgruppen und der Integralperioden bei den Normalteilern der Modulgruppe*, Math. Ann. vol. 116 (1939) pp. 469–510.
   (d) *Analytische Arithmetik der positiven quadratischen Formen*, Det. Kgl. Danske Vidensk. Selskab, Math.-fys. Medd. XVII, 12 (1940).
4. CH. HERMITE, *Sur quelques formules relatives à la transformation des fonctions elliptiques*, Journal de Liouville (2) vol. 3 (1858) p. 26.
5. H. D. KLOOSTERMAN, *The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups*, I, II, Ann. of Math. vol. 47 (1946) pp. 317–375, 376–447.
6. H. W. PRAETORIUS, *Die Charactere der Modulargruppen der Stufe q^2*, Abh. Math. Sem. Hamburgischen Univ. vol. 9 (1933) pp. 365–394.
7. H. ROHRBACH, *Die Charactere der binären Kongruenzgruppen* mod p^2, Schriften d. math. Sem. d. Univ. Berlin. vol 1 (1932) pp. 33–94.
8. H. SPIES, *Die Darstellung der inhomogenen Modulargruppe* mod q^n *durch die ganzen Modulformen gerader Dimension*, Math. Ann. vol. 111 (1935) pp. 329–354.

UNIVERSITY OF LEIDEN,
LEIDEN, NETHERLANDS.

FAREY SECTIONS IN THE FIELDS OF GAUSS AND EISENSTEIN

K. MAHLER

In the study of the approximation of irrational numbers by rational ones, one is led to consider certain special sets of rational numbers, the so-called Farey sections. If N is any positive integer, then the Nth Farey section $\mathfrak{H}_N$ consists of all *different* fractions $x = a/b$ for which

$$|a| \leqq N, \qquad |b| \leqq N.$$

We allow a or b to vanish, but exclude the case that both are zero. In particular, $\mathfrak{H}_N$ contains the improper element

$$\frac{a}{0} = \infty \qquad (a \neq 0),$$

where the sign of a is immaterial.

With $\mathfrak{H}_N$, we associate a subdivision of the infinite real axis into a finite number of subsets, as follows.

If a/b is any element of $\mathfrak{H}_N$, where without loss of generality $(a, b) = 1$, then the generalized distance $|x; a/b|$ of an arbitrary real point x from a/b is defined by

$$\left|x; \frac{a}{b}\right| = |bx - a| = |b| \left|x - \frac{a}{b}\right|.$$

Next $\mathfrak{R}(a/b)$ is the set of all real numbers x for which $|x; a/b| \leqq |x; a'/b'|$ for all $a'/b' \in \mathfrak{H}_N$, thus which are nearest to a/b with respect to its distance function. There are thus as many sets $\mathfrak{R}(a/b)$ as there are different elements a/b of $\mathfrak{H}_N$.

We can arrange the elements of $\mathfrak{H}_N$ according to increasing size and may then speak of consecutive elements of $\mathfrak{H}_N$.

The following theorems are all well known:

A. *If a'/b', a/b, a''/b'', where $b' \geqq 0$, $b > 0$, $b'' \geqq 0$, are consecutive elements of $\mathfrak{H}_N$, then $\mathfrak{R}_N$ is the interval*

$$\frac{a + a'}{b + b'} \leqq x \leqq \frac{a + a''}{b + b''};$$

on the other hand, $\mathfrak{R}(1/0)$ consists of the points satisfying

$$x \leqq -(N + 1) \text{ or } x \geqq (N + 1).$$

B. *If $b > 0$, $b' > 0$, then a/b and a'/b' are consecutive elements of $\mathfrak{H}_N$ if and only if*

$$ab' - a'b = \mp 1; \tag{1}$$

(2) *the median* $(a + a')/(b + b')$ *is not in* $\mathfrak{H}_N$.

C. *All terms of* $\mathfrak{H}_{N+1}$ *are either in* $\mathfrak{H}_N$ *or are medians of consecutive terms of* $\mathfrak{H}_N$.

D. *If* a/b *is in* $\mathfrak{H}_N$, *and* $|a/b| \leqq 1$, *then* $|x; a/b| \leqq 1/N$ *for* $x \in \mathfrak{R}(a/b)$.

From D, one easily deduces Minkowski's theorem on two linear forms.

I shall not speak here on this classical theory, but give you instead some information about a very similar theory in the fields $k(i)$ of Gauss and $k(\rho)$ of Eisenstein. The results in Gauss's field are rather more difficult, so that I shall go more into detail in this case.

First I make some remarks on the history of the problem. In 1940 I obtained empirically all the results which I am going to discuss, and I also had a good guess as to how to obtain proofs. But other work kept me from occupation with this question, and I took it up again only in 1948 and 1949, being greatly helped by two of my Manchester colleagues, W. Ledermann and I. W. S. Cassels. These two finally obtained complete proofs for all my guesses, and it was Cassels who put our paper into its final form.[1]

Let, say, $k(i)$ be the Gaussian field, and let N be a positive integer; we might assume, for simplicity, that N is the norm of an integral element of $k(i)$, but shall not do so. Denote by $\mathfrak{H}_N$ the set of all simplified different fractions a/b where a, b are integers in $k(i)$ not both zero of norm not greater than N. We include in $\mathfrak{H}_N$ the improper point $1/0 = \infty$ and call $\mathfrak{H}_N$ the Farey section of order N; it consists thus of a finite number of points in the complex plane, including the point at infinity.

Let now a/b be any element of $\mathfrak{H}_N$, so that $(a, b) = 1$, that is, a and b are relatively prime. We associate with the point a/b a distance function

$$|z; a/b| = |bz - a| = |b|\cdot|z - a/b|,$$

giving the distance of an arbitrary complex point z from a/b; in particular

$$|z; 1/0| = 1 \text{ for all } z.$$

We next define $\mathfrak{R}(a, b)$ as the set of all complex points z for which

$$\left|z; \frac{a}{b}\right| \leqq \left|z; \frac{a'}{b'}\right| \quad \text{for all} \quad \frac{a'}{b'} \in \mathfrak{H}_N.$$

Then $\mathfrak{H}_N$ and the corresponding system of sets $\mathfrak{R}(a/b)$, where $a/b \in \mathfrak{H}_N$, are invariant under the mappings

$$z \to i^k z, \qquad z \to i^k/z, \qquad z \to \bar{z}$$

of the complex plane.

The points in the complex plane are not ordered; therefore in order to study the sets $\mathfrak{R}(a/b)$ we must use different methods than in the real case, essentially

[1] Our joint paper will soon appear in the Transactions of the Royal Society.

a combination of elementary topology with simple arithmetical properties of the Gaussian field.

To this end we must study the boundaries and inner points of the sets $\mathfrak{R}(a/b)$ in detail. It is clear from the definition that no two such sets have inner points in common; they do, however, have possibly common boundary points, and apart from these cover the whole plane without overlapping.

In every boundary point z of $\mathfrak{R}(a/b)$, we must evidently be equidistant from a/b and a second point a'/b' in $\mathfrak{H}_N$:

$$\left| z; \frac{a}{b} \right| = \left| z; \frac{a'}{b'} \right|,$$

that is,

$$| bz - a | = | b'z - a' |.$$

Hence the boundary of $\mathfrak{R}(a/b)$ consists of arcs of circles or lines, and evidently of only a finite number of these.

A rather lengthy study of $\mathfrak{R}(a/b)$ leads now to the following important result:

A. $\mathfrak{R}(a, b)$ *is, in the closed complex plane, a simply connected region, and is even a star domain with respect to the point* a/b *if* $N \geqq N_0$.

(This is probably true for all N, as the figures suggest and as we have proved for all regions $\mathfrak{R}(a/b)$ inside the unit circle.)

We call a/b the centre of $\mathfrak{R}(a/b)$. We further say that two regions $\mathfrak{R}(a/b)$ and $\mathfrak{R}(a'/b')$, and their centres a/b and a'/b', are adjacent if $\mathfrak{R}(a/b)$ and $\mathfrak{R}(a'/b')$ have boundary points (but of course no inner points) in common. The following necessary and sufficient condition holds then:

B. *The reduced points* a/b *and* a'/b' *in* $\mathfrak{H}_N$ *are adjacent if and only if*

(1) $\qquad ab' - a'b = i^k$ *or* $= i^k(1 + i)$;

(2) $\qquad$ *not all four medians* $(a + i^k a')/(b + i^k b')$ *are in* $\mathfrak{H}_N$.

In addition, the following result holds:

C. *Every point of* $\mathfrak{H}_{N+1}$ *is either an element of* $\mathfrak{H}_N$ *or can be written as the median of two adjacent elements of* $\mathfrak{H}_N$.

We next obtain the following analogue to the real case:

D. *If* a/b *lies in the interior of the unit circle and if* $z \in \mathfrak{R}(a/b)$, *then*

$$\left| z; \frac{a}{b} \right| = | bz - a | \leqq \frac{\kappa}{N}$$

where

$$\kappa = \frac{2^{1/2}}{3 - 3^{1/2}} \qquad (\kappa > 1),$$

and this is best possible.

From this theorem, we obtain a very short and simple proof of the following well-known result of Minkowski (also proved by E. Hlawka):

E. *Let* α, β, γ, δ *be four complex numbers of determinant* $\alpha\delta - \beta\gamma = 1$. *Then there exist two Gaussian integers* x, y *not both zero satisfying*

$$| \alpha x + \beta y | \leqq \kappa,$$

$$| \gamma x + \delta y | \leqq \kappa,$$

and here $\kappa = 2^{1/2}/(3 - 3^{1/2})$ *is the best possible constant.*

It was the search for a simpler proof of this theorem which led me originally to a study of the Farey sections in $k(i)$.

We continue now with the study of the regions $\mathfrak{R}(a/b)$. As already mentioned, their boundary is formed by arcs of circles or lines. At every boundary point of $\mathfrak{R}(a/b)$ the set touches at least one similar set $\mathfrak{R}(a/b)$. We call now a *node* any point in the complex plane where at least three sets $\mathfrak{R}(a/b)$ meet. The result as follows can then be proved:

F. *At a node, either three or four regions* $\mathfrak{R}(a/b)$ *meet. If four regions meet, they subtend equal angles* $\pi/2$. *If only three regions meet, then they either subtend angles* $\pi/2$, $3\pi/4$, $3\pi/4$, *or they subtend angles* $2\pi/3$, $2\pi/3$, $2\pi/3$. *In the first two cases, the node is an element of* $k(i)$ *and, in fact, a median of points in* $\mathfrak{H}_N$; *in the last case, the node does not lie in* $k(i)$, *but in the biquadratic field* $k(i, (-3)^{1/2})$.

The general region $\mathfrak{R}(a/b)$ is found to be a polygon bounded by a finite number of arcs of circles or lines. The number of these sides can be arbitrarily large, depending on N.

What we have found for $k(i)$ applies with very little change also for Farey sections in Eisenstein's field $k(\rho)$ where

$$\rho = \frac{-1 + (-3)^{1/2}}{2},$$

but the existence of six units $\mp\rho^k$ leads to some simplification. Theorem A is unchanged. In Theorem B, the conditions are now

(1) $\qquad ab' - a'b = \mp\rho^k$ or $= \mp\rho^k(1 + \rho)$;

(2) $\qquad$ not all medians $(a + \epsilon\rho^k a')/(b + \epsilon\rho^k b')$, $\epsilon = \mp 1$, are in $\mathfrak{H}_N$.

Theorem C is unchanged. In Theorems D and E, κ may be taken equal to 1. In Theorem F, all nodes lie in $k(\rho)$; either four or three regions come together, and the angles are in the first case

$$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3},$$

while in the second case they are

$$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3} \quad \text{or} \quad \frac{\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{6}.$$

The methods used by us are simple and general and there seems little doubt that analogous, although more complicated, results hold for all imaginary quadratic fields. I also think that a similar theory can be developed for the quaternion ring.

Another possible generalization deals with the simultaneous approximations of two real numbers x, y by means of fractions a/c, b/c of the same denominator. We may take, say,

$$\left| x, y; \frac{a}{c}, \frac{b}{c} \right| = c \max\left(\left| x - \frac{a}{c} \right|, \left| y - \frac{b}{c} \right| \right)$$

as the distance of (x, y) from $(a/c, b/c)$. But the theory becomes then very difficult, and the regions $\mathfrak{R}(a/c, b/c)$ are possibly not simply connected. A much simpler theory arises if

$$\left| x, y; \frac{a}{c}, \frac{b}{c} \right| = c\left(\left(x - \frac{a}{c} \right)^2 + \left(y - \frac{b}{c} \right)^2 \right)^{1/2}.$$

I conclude my lecture. You will agree that the generalization of Farey sections to complex fields leads to pretty figures and also to results of interest both in geometry and in the theory of numbers. Moreover there seems little doubt that there is scope for much further work, even in connection with Hermite's troublesome problem of the famous approximations of several real numbers.

UNIVERSITY OF MANCHESTER,
MANCHESTER, ENGLAND.

THE GENERAL SIEVE-METHOD AND ITS PLACE IN PRIME NUMBER THEORY

Atle Selberg

Ever since Viggo Brun introduced his ingenious sieve-method, it has been a very important tool in connection with problems in the theory of primes. This is especially due to the extreme generality of the method, which makes it yield results where the finer analytic tools will not work. But it is also characteristic of the sieve method that it leads only to partial and incomplete results. Also it was a disadvantage that to obtain good results with the method required quite extensive numerical computations due to the complexity of the method. This complexity actually increased as various technical improvements were made by several mathematicians, among them Rademacher, Estermann, Ricci, and Buchstab.

I shall in this lecture speak of a more general sieve-method, which I have been studying during the last years.

This method includes Brun's method and the improvements of it as a special case. It leads to a clearer and simpler formulation of the main problems connected with the sieve-method, which makes it possible to get information about the limitations of the method. It should also be mentioned that it leads to better results than Brun's method.

1. The problem that the sieve method is concerned with can be formulated as follows:

We have a set of integers n the total number of which we denote by N, and a certain set of primes p_i, $i = 1, 2, \cdots, r$, and we want to estimate the number of n's that are not divisible by any of the given primes p_i.[1]

Often this is stated in an only apparently more general form, where we are counting the n's that are not congruent to a certain set of residues modulo each p_i.

If we denote the number of n's that are not divisible by any p_i by $N(p_1, p_2, \cdots, p_r)$, we have, if d in the following denotes the positive integers composed only by primes p_i and $\mu(d)$ is the Möbius-function, that

$$N(p_1, \cdots, p_r) = \sum_n \sum_{d \mid n} \mu(d) = \sum_d \mu(d) \sum_{d \mid n} 1. \tag{1}$$

We now suppose that we have an approximate expression for the number of n's divisible by d, of the form

[1] One may generalize this to ask for the number of n's that contain not more than k prime factors from the given set of p_i, and counting them with weights depending on these prime factors. This is often advantageous if we want to prove that there exist numbers of a given type with a small number of prime factors. For instance, one can prove in this way that every large even number can be written as a sum of two positive numbers, of which one contains at most 2 the other at most 3 prime factors.

$$\sum_{d|n} 1 = \frac{N}{f(d)} + R_d, \tag{2}$$

here $f(d)$ is a multiplicative function,[2] and R_d is a remainder term about which e know nothing more than an upper bound for $| R_d |$. From (1) we shall then et,

$$\begin{aligned} N(p_1, \cdots, p_r) &= N \sum_d \frac{\mu(d)}{f(d)} + \theta \sum_d | R_d | \\ &= N \prod_{i=1}^{r} \left(1 - \frac{1}{f(p_i)}\right) + \theta \sum_d | R_d |, \end{aligned} \tag{3}$$

here $-1 \leqq \theta \leqq 1$. The disadvantage of (3), however, is that, except in the al- ost trivial case when r is very small compared to N, the remainder term will e much larger than the main term, so that (3) is of no use.

The sieve-method is devised to take care of this difficulty. It is based upon e following consideration. Instead of determining $N(p_1, p_2, \cdots, p_r)$ directly, e shall try to find upper and lower bounds for it by replacing the expression $\sum_{d|n} \mu(d)$ occurring in (1) with a similarly built expression which respectively ajorizes or minorizes it, and at the same time reduces the remainder term to a asonable size.

Thus, to find an upper bound for $N(p_1, \cdots, p_r)$, we take a set of real numbers with $\rho_1 = 1$, and such that for any integer n,

$$\sum_{d|n} \rho_d \geqq \sum_{d|n} \mu(d); \tag{4}$$

en we get an upper bound

$$N(p_1, \cdots, p_r) \leqq N \sum_d \frac{\rho_d}{f(d)} + \sum_d | \rho_d | \, | R_d |. \tag{5}$$

hat remains is then the problem of determining ρ's which satisfy (4) and make e right-hand side of (5) as small as possible.

Similarly, if we have a set of ρ_d's with $\rho_1 = 1$, and such that for every integer n

$$\sum_{d|n} \rho_d \leqq \sum_{d|n} \mu(d), \tag{6}$$

e get a lower bound

$$N(p_1, \cdots, p_r) \geqq N \sum_d \frac{\rho_d}{f(d)} + \sum_d | \rho_d | \, | R_d |, \tag{7}$$

d the problem is to choose the ρ's which make the right-hand side a maximum der the conditions (6).

The problems of finding an upper and a lower bound for $N(p_1, \cdots, p_r)$ are us reduced to two extremal problems, which unfortunately seem to be very

[2] This is the form that usually occurs in applications. One may also consider more gen- al forms of the leading term in (2).

difficult to solve. Therefore, we shall have to be satisfied, if by introducing more restrictions on the ρ's we can get a new extremal problem which is such that we can solve it. This, of course, will imply that our result will most likely not be the best possible. However, in some cases, we can actually obtain best possible results.

2. It would take too long to try to go into details about the principles we may use advantageously in order to get an extremal problem that we can solve, and which give a good result, so that only a few hints will have to do.

First, if we look at the right-hand side of (5) or (7), we see that in order that we should have a good result, the second term $\sum_d |\rho_d| |R_d|$ must not be too large. Since, in the cases that are of most interest, the $|R_d|$ are not very large this can be obtained by restricting the size of $\sum_d |\rho_d|$ in a suitable way. This one may try to obtain by requiring that, except for a certain number of them the ρ's should be zero, for instance, by taking $\rho_d = 0$ for $d > z$, where z is suitably chosen, we may expect that $\sum_d |\rho_d|$ is not essentially greater than z in order of magnitude.[3]

Then comes the problem of satisfying the inequalities (4) or (6). For (4) this may be done in a very simple manner by taking a set of real numbers λ_d with $\lambda_1 = 1$ and putting

$$\rho_d = \sum_{d_1 d_2/\kappa = d} \lambda_{d_1} \lambda_{d_2}, \qquad \kappa = (d_1, d_2) \tag{8}$$

Then

$$\sum_{d|n} \rho_d = \{\sum_{d|n} \lambda_d\}^2 \geqq \sum_{d|n} \mu(d),$$

so that (4) is satisfied. In order to make $\rho_d = 0$ for $d > z$, we require that $\lambda_d = 0$ for $d > z^{1/2}$. It then remains to make the first term on the right-hand side of (5) as small as possible. This term takes the form

$$N \sum_{d_1, d_2 \leqq z^{1/2}} \frac{\lambda_{d_1} \lambda_{d_2}}{f(d_1) f(d_2)} f(\kappa), \tag{9}$$

and our problem is now simply to find the minimum of this expression under the condition $\lambda_1 = 1$. This can easily be done, the minimum being

$$\frac{N}{\displaystyle\sum_{d \leqq z^{1/2}} \frac{\mu^2(d)}{f'(d)}}, \tag{10}$$

where

$$f'(d) = f(d) \prod_{p|d} \left(1 - \frac{1}{f(p)}\right).$$

[3] This can actually be proved to be the case if $\sum_d \rho_d/f(d)$ is small, and the ρ's satisfy either (4) or (6).

If z then is chosen suitably, the term $\sum_d |\rho_d| |R_d|$ becomes small enough in comparison with (10) to give us a good upper bound (5).

The case of the lower bound is essentially more complicated since it is not quite so simple to satisfy the inequalities (6) as the inequalities (4).

One way of doing this is, for instance, to write

$$\rho_d = - \sum_{d_1 d_2 = \kappa d,\, p|\mu} \lambda_{d_1} \lambda_{d_2},$$

where p is the largest prime dividing d, and further take $\lambda_p = 1$ for all p. Then (6) is automatically satisfied. Further, if we prescribe that $\lambda_d = 0$ for $d > (zp)^{1/2}$ where p is the largest prime dividing d, we have $\rho_d = 0$ for $d > z$. We have then to determine the maximum of the first term on the right-hand side of (7). This can actually be done, but is considerably more complicated than in the preceding case. There are also some alternative methods, which however all involve quite extensive computations if one tries to get a good result.

Finally, it should be mentioned that there are certain principles, by which in many cases one can step by step improve the results obtained for the upper or lower bound by the previously mentioned methods. Unfortunately this procedure also requires quite extensive computations in most cases.

The results that one obtains by these methods are better than those obtained by the classical sieve-method, but in most cases certainly do not represent the best possible result since we have subjected our ρ's to rather severe restrictions in order to get an extremal-problem that we could solve.

3. As long as we cannot solve the extremal problems connected with the sieve-method in their general form, it is of interest to have bounds for these extremal values. Such bounds in one direction are of course given by the results obtained for the restricted extremal problems. But for the cases of most interest for number-theory, for instance, when the numbers n are the values taken by an integral-valued polynomial $P(x)$ without fixed prime divisors as the argument x runs over N consecutive integers, and the set of primes $p_1, \cdots, p_r$ is the set of all primes less than a certain number ξ, we can prove interesting results in the other direction. That is, we can give bounds, which no result obtained by the sieve-method can be better than. In particular these results show that for some problems which have been attacked repeatedly by means of the sieve-method, a solution in this way is certainly not possible. They also show, what is more surprising, that in some special cases the results obtained by the methods explained in §2 are actually the best possible results.

The reason that the sieve-method cannot give "too good" results can be said to be that it is not very sensitive to the order of magnitude of the remainder terms R_d in a certain sense, for instance, if we have $n = P(x)$ as x runs over N successive integers and $P(x)$ is an integral-valued polynomial without fixed prime divisors. In this case if $u(d)$ denotes the number of solutions of the congruence $P(x) = 0 \pmod d$, we have $1/f(d) = u(d)/d$, and for R_d we have the result

$| R_d | \leqq u(d)$. However, we get essentially the same results by the sieve-method, if we suppose only that

$$R_d = O\left(\frac{u(d)N}{d(\log (N/d))^k}\right),$$

for a sufficiently high exponent k which depends on the number of irreducible factors into which $P(x)$ can be factored. This remark makes it possible to replace the original problem by one which is essentially equivalent with respect to the sieve-method, but for which we try to make the corresponding $N(p_1, \cdots, p_r)$ as large or as small as possible, and thus get results of the type we want.

Let us consider for instance the simple example that the n's are N consecutive integers, and we try to estimate an upper bound of the number of these integers that have no prime factor less than $\xi = N^\alpha$ where $0 < \alpha < 1$. We write for brevity $N(\xi)$ instead of $N(p_1, \cdots, p_r)$. In this case we have $f(d) = d$ and $| R_d | \leqq 1$, so that (5) takes the form

$$N(\xi) \leqq N \sum_d \frac{\rho_d}{d} + \sum_d | \rho_d |. \tag{11}$$

Now the method explained in §2 gives very easily an upper bound of the form $O(N/\log N)$, so that we may limit ourselves to consider the case when:

$$\sum_d \frac{\rho_d}{d} = O\left(\frac{1}{\log N}\right), \qquad \sum_d | \rho_d | = O\left(\frac{N}{\log N}\right). \tag{12}$$

From this and (4) one can deduce

$$\sum_d \frac{| \rho_d |}{d} = O(\log N). \tag{12'}$$

Now if we consider the set of all positive integers $n' \leqq N$, with an odd number of prime-factors, we have

$$\sum_{d | n'} 1 = \frac{N}{2d} + O\left(\frac{N}{d}\, e^{-(\log(N/d))^{1/2}}\right),$$

by virtue of a well-known result from analytic number-theory. Thus if we apply our sieve on this set of numbers n', and denote by $N'(\xi)$ the number of them with no prime factor $\leqq \xi$, we get

$$2N'(\xi) \leqq N \sum_d \frac{\rho_d}{d} + O\left(N \sum_d \frac{| \rho_d |}{d}\, e^{-(\log(N/d))^{1/2}}\right). \tag{13}$$

However, we cannot here be sure that the remainder term is small enough in comparison to the main term, so that we cannot draw any immediate conclusions from (13). This changes if we replace the N here by a somewhat larger number, e.g., by $N_1 = N^{1+\epsilon}$ where $\epsilon > 0$ tends to zero in a suitable way as N tends to infinity, then (12) and (12′) suffice to make the remainder term small enough. From this we can draw conclusions which give us a lower bound for $\sum_d \rho_d/d$, ıd thus a lower bound for the upper bound for $N(\xi)$ we can obtain from (11). he result is that we cannot obtain an upper bound which is smaller than

$$2N'(\xi) - O\left(N\left(\frac{\log\log N}{\log N}\right)^2\right). \tag{4)}$$

ι a similar way we can prove that a lower bound obtained by the sieve-method ınnot be larger than

$$2N''(\xi) + O\left(N\left(\frac{\log\log N}{\log N}\right)^2\right). \tag{4')}$$

ere $N''(\xi)$ denotes the number of positive integers $\leqq N$ with an even number prime factors and no prime factor $\leqq \xi$. If we take $\alpha = 1/2$; $\xi = N^{1/2}$, we have

$$N'(\xi) = \pi(N) - \pi(N^{1/2}) = \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right),$$

ıd

$$N''(\xi) = 1.$$

hus (14) gives

$$\frac{2N}{\log N} - O\left(N\left(\frac{\log\log N}{\log N}\right)^2\right)$$

a lower bound for the upper bound. With the method described in §2 we can tually obtain the upper bound

$$\frac{2N}{\log N} + O\left(\frac{N}{\log^2 N}\right),$$

ıich thus is essentially a best possible result. (14′) becomes

$$O\left(N\left(\frac{\log\log N}{\log N}\right)^2\right).$$

ıus the sieve-method cannot give the right order of magnitude for the number primes $\leqq N$. For $1/2 < \alpha < 1$, we can even prove that the lower bound bemes negative for large N.[4] If we let α decrease from 1/2 to 0 we see that $N'(\xi)$ ıd $N''(\xi)$ differ less and less. This agrees with the fact that the sieve-method ɔrks better for small exponents α.

We cannot prove that the bounds given by (14) and (14′) essentially represent e true limits for what can be obtained by the sieve-method. However, by the ethods explained in §2 one can get very close to them. Special interest is atched to the value of the exponent α, when the lower bound stops being positive ıd becomes negative as α increases. We may call this the sieving limit of the

[4] This is really due to the fact that we required ρ_1 to be 1. If we ask only that $\rho_1 \leqq 1$, e lower bound can never become negative, since we may take all $\rho_d = 0$. However, if the ʋer bound is positive, it will be reached with a set of ρ's with $\rho_1 = 1$.

problem. From the previous we have that the sieving limit in this case is $\leq 1/$. By the methods in §2, we can very easily show that it is also >0.465, a boun that can be improved step by step by further computations. Whether this pr cedure would actually converge to 1/2 I do not know.

This analysis can easily be extended to the case that the n's are the values an integral-valued irreducible polynomial $P(x)$ without fixed prime divisor as x runs over N successive integers, with essentially the same results.

For the case of reducible polynomials we can prove similar results, but he there is a greater gap between these results and the bounds actually obtaine by the methods of §2. A case of special interest is the polynomial composed two irreducible factors, which includes two well-known problems which hav been attacked repeatedly by means of the sieve-method, namely the problem the infinitude of prime twins and the Goldbach problem.

The first one corresponds to taking the polynomial $x(x + 2)$ for $x = 1, 2, \cdots$ N, and $\xi = (N + 2)^{1/2}$. If we then could get a positive lower bound, the proble would be solved. However, we can prove that for $\xi = N^{\alpha}$, the lower boun actually is negative for large N if $\alpha > 1/(1 + e^{3/4})$,[5] thus the sieve-method cann solve this problem. The corresponding result holds for Goldbach's problem an in general for the case of a polynomial with only two irreducible factors. W can also show that there exist similar limitations for the upper bounds that ca be obtained by the sieve-method, but in this case the discrepancy between the and the results we can actually obtain by the methods of §2 is greater. We ca show, for instance, that the best upper bound that can be obtained for the numb of prime twins $\leq N$ is for large N more than 4 times[6] larger than what is generall assumed to be the true asymptotic value.[7] By the method described in §2 w actually can obtain a bound which is 8 times too large. I am inclined to co jecture that this in reality is the best possible.

4. In view of this it seems that the sieve-method will be of little value for th further progress of these problems in prime number theory which it was orig inally designed to deal with. But it remains as an extremely general and versati tool for establishing, for instance, upper bounds, and may perhaps, when i some way combined with an analytic approach, still play an important part i the future of these problems.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

[5] This number can actually be replaced by a slightly smaller one.
[6] The factor 4 may be replaced by a slightly larger one.
[7] Namely $T(N) \sim kN/\log^2 N$ with $k = 2\Pi_{p \geq 3}(1 - 1/(p - 1)^2) = 1.320\cdots$.

THEORY OF NUMBERS AND FORMS

ON THE SOLUTIONS OF $h(d) = 1$

S. Chowla and A. B. Showalter

Karl Schaffstein (Math. Ann. vol. 98 (1928) pp. 745–748) listed a large number of positive discriminants d with the property $h(d) = 1$. Here $h(d)$ is the number of primitive classes of binary quadratic forms whose discriminant is d. He did not describe the method by which he obtained the table. Our purpose in this paper was to devise a rapid method (based on simple continued fractions) by which we can find the values of d for which $h(d) = 1$. According to a famous unproved conjecture of Gauss, there are infinitely many positive discriminants d with this property.

Note added March 6, 1951. In an unpublished paper Ankeny, Artin, and Chowla proved that $2(hu/t) \equiv (a + b)/p \pmod p$. Here h is the class-number of the real quadratic field $R(p^{1/2})$ whose fundamental unit is $(t + u \cdot p^{1/2})/2$, a denotes the product of the quadratic residues of p, while b denotes the product of the non-residues of p.

University of Kansas,
Lawrence, Kan., U. S. A.

ON EXPONENTAL AND CHARACTER SUMS

Sarvadaman Chowla and Albert Leon Whiteman

The authors investigate the sums $S(f(x), n) = \sum_{x=1}^{p} e^{2\pi i n f(x)/p}$ and $T(f(x), n) = \sum_{x=1}^{p} ((f(x) + n)/p)$, where p is a prime, $f(x)$ is a polynomial of degree k, and (n/p) is the Legendre symbol. Typical results are the following. There exists an integer n such that $| S(f(x), n) | < k^{1/2}p^{1/2}$. A similar result holds also for $T(f(x), n)$. If k is any integer, then $T(x^k, 1) = O(p^{1/2})$. For a prime p such that $(p - 1)/4$ is an odd square, the value of $| S(x^4, n) |^2$ is the same for every $n \not\equiv 0 \pmod p$, and further is equal to a rational number. A connection between the theory of the sums $S(x^4, n)$ and the theory of "difference sets" as developed by Marshall Hall in his paper, *Cyclic projective planes* (Duke Math. J. vol. 14 (1947) pp. 1079–1090) is also obtained.

University of Kansas,
Lawrence, Kan., U. S. A.
University of Southern California,
Los Angeles, Calif., U. S. A.

A PERIODIC ALGORITHM FOR CUBIC FORMS

HARVEY COHN

A lattice in three-dimensional space will be called *reduced* if its basis vectors $x_1^{(i)}$, $x_2^{(i)}$, $x_3^{(i)}$ together with $x_4^{(i)} = -\sum x_j^{(i)} (j = 1, 2, 3)$, lie one in each octant of space or its negative octant. It may be seen that every lattice has a reduced sublattice of universally bounded index (and the examination of a handful of cases may even reveal this index to be unity). This type of algorithm is very similar to one by Minkowski in which the basis vectors were three shortest independent vectors under certain deformations, and two bases were called neighbors if they had two basis vectors (out of three) in common, making three neighbors in general. In the author's algorithm the bases are "not far" from Minkowski's bases, and the process of finding neighbors involves agreement of two (out of four) basis vectors. Here there are three to six neighbors in general, making for more complicated patterns.

Similarly, reduced quaternary forms $\prod\sum n_i x_j^{(i)}$ $(j = 1, 2, 3, 4;\ i = 1, 2, 3)$ can be defined. If this form is the norm in a totally real cubic module, it can be shown there are only a finite number of reduced forms under change of basis; and the neighbor concept will lead to finite (periodic) multi-dimensional structures. Finding the structures by this algorithm is very simple since it involves only patterns of signs. For the integers in the cubic field for which $D = 49$, the structure of the norm is 8 forms connected like the vertices of a cube. For $D = 81$ and $D = 148$, the structures have 16 and 28 forms respectively. Structures with certain symmetries turn out to correspond to abelian fields, and the periodicity of the structures, of course, leads to unities of the field.

WAYNE UNIVERSITY,
DETROIT, MICH., U. S. A.

EXTREME FORMS

H. S. M. COXETER

E. Stiefel (Comment. Math. Helv. vol. 14 (1942) pp. 350–380) used a discrete group generated by reflections to represent a family of locally isomorphic compact simple Lie groups, and distinguished the individual members of each family by considering the point-lattices that are invariant under the discrete group. It is thus possible to make a complete list of such Lie groups, in a natural generalization of Cartan's notation, as follows: A_n, A_n^r $(n \geqq 1, r \mid n + 1)$, C_n, C_n^2 $(n \geqq 2)$, B_n, B_n^2 $(n \geqq 3)$, D_n, ${}^2D_n$, D_n^4 $(n \geqq 4)$, D_n^2 (n even, $\geqq 6$), E_6, E_6^3, E_7, E_7^2, E_8, F_4, G_2. One lattice may arise from several different discrete groups; e.g., A_8^3, D_8^2, and E_8 all have the same lattice. Using the same notation for conven ent representatives of the classes of positive definite quadratic forms which have these for their Gauss lattices, we find

$$
\begin{aligned}
A_n &= x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + \cdots - x_{n-1}x_n + x_n^2\,, \\
A_n^r &= A_{n-1} - x_q x_n + q(1 - r^{-1})x_n^2/2 \qquad (n = qr - 1 > 1, r > 1), \\
C_n &= 2A_n - x_n^2 = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2, \\
C_n^2 &= x_1^2 - x_1x_n + x_2^2 - x_2x_n + \cdots - x_{n-1}x_n + nx_n^2/4, \\
D_n &= A_{n-1} - x_{n-2}x_n + x_n^2\,, \\
D_n^2 &= D_{n-1} - x_{n-1}x_n + nx_n^2/8 \qquad (n \text{ even}), \\
E_6 &= A_5 - x_3x_6 + x_6^2\,, \\
E_6^3 &= A_6 - x_5^2/3.
\end{aligned}
$$

It happens that all these forms, except A_n^{n+1} $(n > 2)$, A_3^2, A_5^2, C_n, C_n^2 $(n \neq 4)$, and D_6^2, are *extreme* in the sense of Korkine and Zolotareff (Math. Ann. vol. 6 (1873) pp. 366–389). In particular, the extreme binary, ternary, quaternary, quinary, and senary forms are A_2, A_3, A_4, D_4, A_5, A_5^3, D_5, A_6, D_6, E_6, E_6^3. The senary form E_6^3 was sought unsuccessfully by Hofreiter (Monatshefte für Mathematik und Physik vol. 40 (1933) pp. 129–152). Its 27 pairs of minimal vectors correspond to the lines on the general cubic surface. Similarly, the 28 pairs of minimal vectors for A_7^4 (or E_7^2) correspond to the bitangents of the general plane quartic curve.

University of Toronto,
Toronto, Ont., Canada.

BINARY CUBIC FORMS

Harold Davenport

Let $h(D)$ denote the number of classes of properly equivalent irreducible binary cubic forms with integral coefficients, of given discriminant D. It is proved that

$$\sum_{D=1}^{X} h(D) = KX + O(X^{1-\delta})$$

as $X \to \infty$, where K and δ are positive absolute constants.

University of London,
London, England.

DIE BEDEUTUNG DER EULER'SCHEN ZAHLEN FÜR DEN GROSSEN FERMAT'SCHEN SATZ UND FÜR DIE KLASSENANZAHL DES KÖRPERS DER 4l-TEN EINHEITSWURZELN

MAX GUT

Bedeutet l durchwegs eine beliebige ungerade Primzahl, und setzt man voraus, dass die Gleichung $X^{2l} + Y^{2l} = Z^{2l}$ eine Lösung in ganzen rationalen zu l teilerfremden Zahlen X, Y, Z hat, so lässt sich vermöge des Reziprozitätsgesetzes der l-ten Potenzreste im Körper der $4l$-ten Einheitswurzeln zeigen, dass ausser den Kongruenzen, in denen die Bernoulli'schen Zahlen und die Kummer'schen Polynome

$$(X^2 + Y^2)^w \cdot \left[\frac{d^w \log (X^2 + Y^2e^v)}{dv^w}\right]_{v=0}$$

bezw. die Bernoulli'schen Zahlen und die Mirimanoff'schen Polynome:

$$\varphi_w(t) = \sum_{m=1}^{l-1} (-1)^{m-1} m^{w-1} t^m, \qquad t = \frac{Y^2}{X^2},$$

bezw. die Mirimanoff'schen Polynome allein auftreten, weiter ganz analoge Kongruenzen gelten, in welchen die Euler'schen Zahlen und die Polynome

$$(X^2 + Y^2)^w \cdot \frac{Y}{X}\left[\frac{d^w \operatorname{arctg} \dfrac{XY(e^v - 1)}{X^2 + Y^2e^v}}{dv^w}\right]_{v=0},$$

bezw. die Euler'schen Zahlen und die Polynome

$$\varphi_w^*(t) = \sum_{m=1}^{l} (-1)^{m-1}(2m - 1)^{w-1} t^m,$$

bezw. diese Polynome allein auftreten. In den vorangehenden Formeln soll dabei w die Werte $2, 3, \cdots, l - 1$ annehmen dürfen. Insbesondere ergibt sich der Satz: Die Gleichung $X^{2l} + Y^{2l} = Z^{2l}$ hat keine Lösung in ganzen rationalen zu l teilerfremden Zahlen X, Y, Z, wenn wenigstens eine der Euler'schen Zahlen E_{l-3}, E_{l-5}, E_{l-7}, E_{l-9}, E_{l-11} nicht durch l teilbar ist. [Beweise in meiner Arbeit in den Comment. Math. Helv. vol. 24 (1950) p. 73.]

Wendet man die Formeln für die Takagi'schen Exponenten im Körper der $4l$-ten Einheitswurzeln auf die Klassenzahlformel dieses Körpers an, so ergibt sich der Satz: Die Klassenzahl des Körpers der $4l$-ten Einheitswurzeln ist dann und nur dann zu l teilerfremd, wenn die Zähler der ersten $l - 1$ Bernoulli'schen Zahlen von geradem Index und die ersten $l - 1$ Euler'schen Zahlen von geradem Index alle zu l teilerfremd sind. [Die den ausführlichen Beweis enthaltende Arbeit habe ich am 13. April 1950 der Redaktion der Comment. Math. Helv. eingereicht.]

UNIVERSITY OF ZÜRICH,
ZÜRICH, SWITZERLAND.

PROBLEMS CONCERNING RAMANUJAN'S FUNCTION

D. H. LEHMER

Ramanujan's function $\tau(n)$, which is the coefficient of x^n in the expansion of the product $x(1 - x)^{24}(1 - x^2)^{24} \cdots$ is generated by a classical invariant of Weierstrass' theory of elliptic functions. It is only in recent years however that its properties have been noticed and investigated. The values of $\tau(n)$ depend ultimately on the values of $\tau(p)$ for primes p. The many congruence properties of $\tau(n)$ which have been discovered recently are reducible to statements of the form:

(A) *$\tau(p)$ is congruent to a polynomial in p modulo m.* Such statements have already been proved for $m = 2^{10}, 3^5, 5^3$, and 691. The statement (A) is not strictly true for $m = 23$, although $\tau(p)$ has a special congruence property (mod 23).

In the present paper the statement (A) is proved for $m = 2^{11}, 3^7$, and 7^2. In the latter case the statement holds for p a nonresidue of 7 and is specifically

$$\tau(p) \equiv 3p(p^3 + 1) \pmod{49}, \qquad (p \equiv 3, 5, 6 \pmod 7)$$

It is further proved that (A) is false for $m = 5^4$ and 7^3, and for every prime p. for which $11 \leqq p < 1250$, except $p = 691$. Whether (A) holds for $m = 2^{12}$ or 3^7 are open questions.

Another unsolved problem is concerned with the question of whether $\tau(n) = 0$ for some n. By the methods of a previous paper [Duke Math. J. vol. 14 (1947) pp. 429–433] it is shown that $\tau(n) \neq 0$ for $n < 1586\ 17336\ 31999$.

A final problem is the proof or disproof of the so-called Ramanujan hypothesis: $|\tau(p)| < 2p^{11/2}$ for every prime p. This hypothesis is shown to be true for all primes $p < 2500$. This result is based on an extension of the table of $\tau(n)$ as far as $n = 2500$.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF., U. S. A.

ON THE INTEGRAL SOLUTIONS OF THE DIOPHANTINE SYSTEM $ax^2 - by^2 = c, \quad a_1z^2 - b_1y^2 = c_1$

WILHELM LJUNGGREN

Let a, b, c and a_1, b_1, c_1 denote rational integers, where a, b, a_1, b_1 are positive. In this note we shall show that it is possible to find an upper bound for the number of solutions in rational integers x, y, z of the Diophantine system

$$ax^2 - by^2 = c, \qquad a_1z^2 - b_1y^2 = c_1 \tag{1}$$

for given values of the constants. It is assumed that these equations are solvable separately, each of them with infinitely many solutions.

It is easy to prove that there is no restriction in assuming $c > 0$, and $c_1 < 0$. Obviously it is further sufficient to consider the case in which $(a, c) = (b, c) =$

$(a_1, c_1) = (b_1, c_1) = (x, y) = (y, z) = 1$, and a, a_1 being without any squared factors greater than 1.

From (1) we deduce

$$by + (-bc)^{1/2} = \frac{\alpha}{p^2}\mu^2, \qquad b_1 y + (-b_1 c_1)^{1/2} = \frac{\beta}{q^2}\nu^2,$$

where α, μ and β, ν are numbers in the rings $R((-bc)^{1/2})$ and $R_1((-b_1c_1))^{1/2}$ respectively, while p and q are ordinary integers. Furthermore α, β, p, q are finite in number. It can now be proved that

$$\delta = \left(\frac{\mu}{p}(b_1\alpha)^{1/2} + \frac{\nu}{q}(b\beta)^{1/2}\right)^2$$

is an integer of the field $K((bb_1\alpha\beta)^{1/2})$ with the relative norm $\kappa = (b_1(-bc)^{1/2} - b(-b_1c_1)^{1/2})^2$ in the subfield $k((-bc)^{1/2},(-b_1c_1)^{1/2})$.

If the product $-b_1c_1$ is not a perfect square, there exist two primitive units λ_1 and λ_2 in K with norm ± 1 relative to k, such that

$$\delta = \tau\lambda_1^m\lambda_2^n,$$

where τ, finite in number, is an integer in K with relative norm κ in k. This will give us two exponential equations to determine the ordinary integers m, n. If $-b_1c_1$ is a perfect square, we have to make $\lambda_2 = 1$, getting one equation for the determination of m. Our previously stated result now follows in consequence of a theorem due to Th. Skolem, *Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen*, Eighth Scandinavian Congress of Mathematicians, Stockholm, 1934.

Proofs and further results will be published later.

University of Bergen,
Bergen, Norway.

SETS OF INTEGERS OF DENSITY ZERO

Ivan Niven

Let A be a set of positive integers, and let $A\{p\}$ denote those elements of A which are divisible by p but not by p^2. If there exists a set of primes p_i such that $\sum p_i^{-1}$ diverges and such that each set $A\{p_i\}$ has density zero, then A has density zero, density being defined as a strict limit, not lower limit. This result furnishes a simple test for zero density in many cases. If density is defined in the sense of lower limit, the result is false.

University of Oregon,
Eugene, Ore., U. S. A.

EXPANSION OF A SINGLE-VALUED FUNCTION OF SEVERAL VARIABLES WHOSE RANGE IS A DENUMERABLE SET OF REAL NUMBERS AND WHOSE DOMAIN OF DEFINITION IS A SET OF REAL NUMBERS

NICHOLAS CHRIST SCHOLOMITI

Let $y = f(x_1, x_2, \cdots, x_n)$ and let the denumerable set G' of real numbers be the range of y: $G' = (\psi_1, \psi_2, \cdots)$ so that when the x's assume some particular set of values, y will take on as corresponding value some number ψ_i of G'. The expansion in which y is represented in terms of the x's is of the form: (1) $y = \sum_{j=1}^{\mu} A_j \cdot \psi_j \cdots$, where the A's are functions of the x's: $A_j = F(j, x_1, \cdots, x_n)$ and where $A_i = 0$ for every i except when $i = k$ in which case $A_k = 1$. Then (1) reduces to: $y = \psi_k$. k and μ are functions of the x's, and μ is completely characterized by the inequality: $\mu \geqq k$. We write: $\mu = \varphi(x_1, x_2, \cdots, x_n)$.

To determine (1) when f and G' are given we must first determine F and φ. As an illustration we can obtain an expansion for the least positive residue y of the non-negative integer x_1 modulo the posivite integer x_2. We get:

$$y = \sum_{j=0}^{x_2-1} \left(\tfrac{1}{2} - \tfrac{1}{2}(-1)^{2^{P(j)}}\right) \cdot j,$$

where $P(j) = \prod_{u=0}^{x_1} (x_1 - j - ux_2)^2$.

A more general expansion can be obtained by constructing the functions A_i so that not only one of them is unity but several. The development now takes the form: $y = A_1 \cdot \psi_1 + \sum_{u=2}^{\mu} A_u \cdot \psi_u \cdot \prod_{z=1}^{u-1} (1 - A_z)$. Using this result we can construct a function of an integral variable x which represents all the positive primes; thus if $P(x)$ is the xth prime in the series: 2, 3, 5, 7, 11, $\cdots$, we find:

$$P(x) = \sum_{w=3}^{2^x} w \cdot f(w) \cdot \left[\tfrac{1}{2} - \tfrac{1}{2}(-1)^{2^{(x-1-\sum_{u=3}^{w} f(u))^2}}\right]$$

where $f(w) = \frac{1}{2} + \frac{1}{2}(-1)^{2^{G(w)}}$; $G(w) = \prod_{z=2}^{w-1} F(z)$; $F(z) = \prod_{v=2}^{w} (w - vz)^2$, $x \geqq 2$.

UNIVERSITY OF ILLINOIS,
CHICAGO, ILL., U. S. A.

ON THE DENSITY OF REDUCIBLE INTEGERS AND SOME SEQUENCES ASSOCIATED WITH THEM

DANIEL SHANKS

A positive integer, r, is "reducible" if $\tan^{-1} r = \sum_{i=1}^{r-1} a_i \tan^{-1} i$ for some $a_i = 0, \pm 1, \pm 2, \cdots$ (see John Todd, *A problem on arctangent relations*, Amer. Math. Monthly vol. 56 (1949) pp. 517–528). Todd has conjectured that the density of the set R of reducible integers is $1 - \log 2$. Divide R into mutually exclusive subsets, R_j, $(j = 0, 1, 2, \cdots)$ such that $r \in R_j$ if $r^2 + 1$ has j and only j different factors $> r$ and $< 2r$. (Of the 143 reducible r's < 500, 15 are in R_0, 92 are in R_1, 34 are in R_2, and 2 are in R_3.) In distinction to the set R which contains each r in each R_j once, this paper considers a new set E which contains each r in each R_j, j times. Then the density, δ_E, of the set E satisfies the inequality $7/8\pi < \delta_E < 10/8\pi$. This follows from either of two distinct algorithms for obtaining every $e \in E$. Three vector sequences, u_n, v_n, and w_n, are defined by $u_1 = (0, 1)$, $v_1 = (1, 1)$, $u_{2n} = u_n$, $u_{2n+1} = v_n$, and $v_{2n} = v_{2n+1} = w_n$ where $w_n = u_n + v_n$. Then the sequence e_n is the scalar product $v_n \cdot w_n$ and the five other products, $u_n \cdot u_n$, $u_n \cdot v_n$, etc. give five other associated sequences. Alternatively, the six integer sequences may be determined by a seemingly independent simple addition algorithm without reference to the vectors. The nine sequences thus defined are intimately related to some interesting one-to-one correspondences between the set of positive integers and (a) the set of rational numbers > 0 and < 1, (b) the set of all finite sequences of positive integers, (c) the set of all regular, terminated, continued fractions, (d) the set of positive integers each taken $\varphi(n)$ times, (e) the set of all factors of all $n^2 + 1$, and (f) the set of every 2×2 matrix, M, with integer elements such that $M^2 = -I$.

NAVAL ORDNANCE LABORATORY,
SILVER SPRING, MD., U. S. A.

A GENERALIZATION OF FINITE INTEGRATION

L. V. TORALBALLA

An ordered set $x_1, x_2, \cdots, x_n$ in which each x_i is an integer $\geqq 1$ and $\sum_1^n x_i = A$ a given positive integer is called a permutational partition of A. We consider the problem of finding the sum of the values of a function $f(x_1, x_2, \cdots, x_s)$, $s \leqq n$, over the set of all the permutational partitions of A.

The principal results are:

1) $\sum_{A,n} x_1^{m_1} x_2^{m_2} \cdots x_s^{m_s} = \sum_{r=0}^{m-1} (P - 1)^{s\oplus r}]_{m_1,m_2,\cdots,m_s} A^{(n+r)}/(n + r)!$

where $A^{(n+r)} = A \times (A - 1) \times \cdots \times (A - r + 1)$;

$(P - 1)^{s\oplus r} = \sum_{m=0}^{n-s} (-1)^m_{s+r} C_m P^{s+r-m}$; $P^{s+t}]_{m_1,m_2,\cdots,m_s} = \sum_{s+r,s} x_1^{m_1} x_2^{m_2} \cdots x_s^{m_s}$;

$$m = \sum_1^s m_i.$$

2) $\sum_{A,n} x_1^{(m_1)} x_2^{(m_2)} \cdots x_s^{(m_s)}$

$= (m_1 ! \, m_2 ! \cdots m_s !) \sum_{p=0}^{s-1} ((s - 1)^{(p)}/p!) \, A^{(n-s+p+m)}/(n - s + p + m)!$.

MARQUETTE UNIVERSITY,
MILWAUKEE, WIS., U. S. A.

THE EXTREME SMOOTHED OCTAGON

LEONARD TORNHEIM

A new proof, of a geometric nature, is given of Mahler's result that every central symmetric octagon with all four circumscribed hexagons having the same area is affinely equivalent to the area common to two equal concentric squares. It is known that these are the only octagons which on rounding the corners with appropriate hyperbolic arcs become irreducible figures; i.e., their determinant, which is the greatest lower bound of the determinants of lattices with only the origin in the interior of the figure, will be diminished whenever symmetric portions of the figure are deleted. It is shown that of these "smoothed octagons" the smoothed regular octagon has the smallest value for the ratio of its area to its determinant. Since this ratio is proportional to that fraction of the area covered by a densest lattice packing of the figure, this result supports the conjecture of Reinhardt and Mahler that among symmetric convex figures the smoothed regular octagon is extreme, i.e., has the smallest ratio. The extreme convex figure is shown to be a "smoothed" polygon.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

ON FERMAT'S LAST THEOREM

ALEXANDER A. TRYPANIS

Criteria are established, applying to the case:

$$\text{(A)} \qquad x^p + y^p + z^p = 0, \qquad xyz \not\equiv 0 \pmod p,$$

where p is a (positive) odd prime ($p \neq 1$).

1. If (A) can be satisfied by rational integral values of x, y, z, then a rational integer a must exist which satisfies the conditions:

$$\text{(B)} \qquad \begin{cases} (a+1)^{p^2} - a^{p^2} - 1 \equiv 0 \pmod{p^4}, \\ 0 \not\equiv a \not\equiv -1 \pmod p, \qquad a^2 + a + 1 \not\equiv 0 \pmod p; \end{cases}$$

or the conditions equivalent to (B):

$$\begin{cases} (a^{p^3} + 1)^{p-1} - 1 \equiv 0 \pmod{p^4}, \\ a \not\equiv 0 \pmod p, \qquad a^2 + a + 1 \not\equiv 0 \pmod p. \end{cases}$$

2. If $(a+1)^p - a^p - 1 \equiv 0 \pmod{p^3}$, or if $(a+1)^{p^2} - a^{p^2} - 1 \equiv 0 \pmod{p^3}$, then:

$$(a+1)^{p^{v+1}} - a^{p^{v+1}} - 1$$
$$\equiv -\frac{[(a+1)^{p-1} - 1][a^{p-1} - 1]}{2} p^v + (a+1)^{p^v} - a^{p^v} - 1 \pmod{p^{v+3}},$$

for every positive rational integral value of v.

3. If a rational integer a, such that

$$-1 \not\equiv a \not\equiv 0 \pmod{p},$$

satisfies the congruence:

$$(a+1)^{p^v} - a^{p^v} - 1 \equiv 0 \pmod{p^{v+2}},$$

or the congruence:

$$(a^{p^v}+1)^{p-1} - 1 \equiv 0 \pmod{p^{v+1}},$$

for every positive rational integral value of v, then

$$a^2 + a + 1 \equiv 0 \pmod{p}.$$

For relative investigations by Carmichael, Meissner, et al., vide Dickson: *History of the theory of numbers*, vol. II, chap. 26.

National Technical University,
Athens, Greece.

GROUPS AND UNIVERSAL ALGEBRA

SIMILARITY AND ISOTOPY

J. C. Abbott and T. J. Benac

In 1939 Baer (R. Baer, *Nets and groups*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 110–141) introduced a concept of similarity for those coset multiplication systems which are loops. In this note this concept is extended to an arbitrary right quasigroup G with left unit by using the representation of G as a quotient set of its permutation group modulo the inner mapping congruence (Abbott and Benac, *Right congruences on groupoids*, Bull. Amer. Math. Soc. vol. 55 (1949) p. 1042). The concept of isotopy as defined by Albert (A. A. Albert, *Quasigroups* I, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 507–519) is then shown to be equivalent to similarity for this case, that is, whenever a representation exists.

U. S. Naval Academy,
Annapolis, Md., U. S. A.

DISTRIBUTIVE LATTICES WITH A THIRD OPERATION DEFINED

B. H. Arnold

If L is a distributive lattice closed under a binary operation $*$ which is idempotent, commutative, associative, and distributive with $*$, $\cup$, $\cap$ in all possible ways, then L is a subdirect union of two distributive lattices $A = \{a\}$ and $B = \{b\}$ in such a way that $(a, b) * (a', b') = (a \cup a', b \cap b')$. For L to be the entire direct union, it is sufficient that L contain an identity element for the operation $*$. A necessary and sufficient condition is also given.

Oregon State College,
Corvallis, Ore., U. S. A.

SOME COMMUTATOR SUBGROUPS OF A LINKAGE GROUP

Kuo-Tsai Chen

Let G be a group with a finite number of generators and relations. Using a method of integration in free groups, the author has given (Bull. Amer. Math. Soc. vol. 56 (1950) p. 156) a method for computing the (commutative) factor groups $G_d = {}^{d-1}G \cdot G^{(2)} / {}^{d}G \cdot G^{(2)}$ where ${}^{d}G$ is the dth lower central commutator subgroup, and $G^{(2)} = [[G, G], [G, G]]$. The method is applied to the case when G

is the fundamental group of the complement of a linkage consisting of n disjoint, polygonal orientated closed curves $L_1, L_2, \cdots, L_n$ in a Euclidean 3-space. $G_1 = G/[G, G]$ is then a free abelian group of rank n. Let a_{ij} be the looping coefficient of L_i and L_j ($i \neq j$). Assume $n > 2$ and consider the matrix M of the system of n linear equations $\sum_j a_{ij}x_{ij} = 0$ in the $n(n-1)/2$ indeterminates x_{ij} ($i, j = 1, 2, \cdots, n$, $x_{ij} = -x_{ji}$). Let $k_1, \cdots, k_n$ be the elementary divisors of M. Then $G_2 = [G, G]/[[G, G], G] \cong Z_1 \times \cdots \times Z_n \times H$ where Z_i is cyclic of order k_i, and H is free abelian of rank $n(n-3)/2$. For $n = 2$, G_2 is cyclic of order $|a_{12}|$. For $n = 1$, $G_2 = \{1\}$.

PRINCETON UNIVERSITY,
PRINCETON, N. J., U. S. A.

A CLASS OF PARTIALLY ORDERED ABELIAN GROUPS RELATED TO KY FAN'S CHARACTERIZING SUBGROUPS

A. H. CLIFFORD

In a recent paper (Ann. of Math. vol. 51 (1950) pp. 409–427) Ky Fan gives (Theorem 1) three simple conditions (I, II, III) on a partially ordered abelian group which are necessary and sufficient that it be isomorphic with a subgroup G of the group $C(\Omega)$ of all continuous real-valued functions defined on a compact Hausdorff space Ω, with the properties: (i) G contains all the constant functions defined on Ω, and (ii) if x_1, x_2 are distinct points of Ω, there exists f in G such that $f(x_1) \neq f(x_2)$. A subgroup G of $C(\Omega)$ is called by Ky Fan a "characterizing subgroup of $C(\Omega)$" if, in addition to these two properties, it has the following: (iii) for each point x_0 of Ω, the subset of G consisting of all f in G with $f(x_0) = 0$ is a compositive subgroup of G. Ky Fan gives (Theorem 4) a fourth condition IV which, together with I, II, III, is sufficient that G be isomorphic with a characterizing subgroup of some $C(\Omega)$ with compact Ω. He also shows that a characterizing subgroup G of $C(\Omega)$ satisfies the condition IV′: every maximal singular subgroup of G is also maximal convex. (A singular subgroup of G is one that is compositive and contains no archimedean element = strong unit of G.) In the present paper examples are given showing that the converse of neither of these theorems is true.

In his Theorem 1, Ky Fan takes for Ω the set Λ of maximal convex subgroups of G, and in Theorem 4 he takes for Ω the set Σ of maximal singular subgroups of G, showing that Σ is a closed subset of Λ. He introduces a topology into Λ in a manner similar to that used by I. Gelfand on the set of maximal ideals of a normed ring (Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 3–24). In the present paper we show that, for a group G satisfying I, II, III, IV′, the topology may be introduced in Σ in a manner similar to the second one used by I. Gelfand and G. Šilov (Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 25–38). Necessary and sufficient conditions are given for an additive group G of real functions on a set S (topologized or not) in order that I, II, III, IV′ hold for G.

Johns Hopkins University,
Baltimore, Md., U. S. A.

SUR UNE CLASSE DE RELATIONS D'ÉQUIVALENCE

Paul J. Dubreil

1. Soit D un demi-groupe *abélien*. Si G est un groupe homomorphe à D et si $\mathcal{R}$ désigne l'équivalence d'homomorphisme, la classe-unité U est un sous-demi-groupe *net* [*Contribution à la théorie des demi-groupes*, Mémoires de l'Académie des Sciences, Paris, vol. 63 (1941)] et $a \equiv a'(\mathcal{R}) \rightleftarrows \exists u,\ u' \in U$ tels que $ua = u'a'$. U est de plus *unitaire*. Si inversement S est un sous-demi-groupe *net* et si Σ_S est l'équivalence définie par $a \equiv a'(\Sigma_S) \rightleftarrows sa = s'a'(s,\ s' \in S)$, Σ_S est régulière et D/Σ_S est un groupe, dont l'élément-unité U est un sous-demi-groupe net et unitaire contenant S.

2. La recherche des groupes homomorphes à un demi-groupe D *non abélien* est résolue aussi [loc. cit.], mais par d'autres équivalences, dites *principales*, définies de la façon suivante. Désignons par $H:a$ $(0 \subset H \subsetneq D)$ l'ensemble des $x \in D$ vérifiant $ax \in H$, par $\mathcal{R}_H$ l'équivalence: $a\mathcal{R}_H b \rightleftarrows H:a = H:b$. H est *fort* si cette égalité a lieu dès que $(H:a) \cap (H:b) \neq 0$. Les groupes homomorphes à D correspondent biunivoquement aux sous-demi-groupes forts nets, symétriques et unitaires; l'équivalence d'homomorphisme est l'équivalence principale correspondante. L'étude générale de ces équivalences est liée à celle des idéaux de D [*Contribution à la théorie des demi-groupes*, II, Séminaire Mathématique, Rome (1950)].

3. Pour généraliser les relations intervenant dans 1, considérons ue sous-demi-groupe S de D (non abélien), la relation σ_S définie par $a\ \sigma_S\ a' \rightleftarrows sa = s'a'$ $(s,\ s' \in S)$ et sa fermeture transitive Σ_S. On a [ibid.] $sa \equiv a(\Sigma_S)$ et Σ_S est la plus fine des équivalences ayant cette propriété. En général, S n'est ni saturé, ni indivisible mod Σ_S. Pour que S soit saturé, il faut et il suffit qu'il soit unitaire à gauche; pour qu'il soit indivisible, il faut et il suffit qu'il soit lié à droite, ou qu'il n'existe aucun demi-groupe homomorphe à S dont tout élément soit neutre à gauche et qui comprenne plus d'un élément. Si S est fort et net à droite, on a $\Sigma_S = \mathcal{R}_S$ si et seulement si S est lié à droite et Σ_S simplifiable à droite.

The Sorbonne,
Paris, France.

ON DISTANCE SETS AND DISTANCIALLITY IN NATURALLY METRIZED GROUPS

DAVID ELLIS

Let G be an additive Abelian group and $| G |$ be the set of *unordered* pairs $(a, -a)$, $a \in G$. Set $(a, -a) = | a |$, $| 0 | = 0$. G is *naturally metrized* by the distance $d(a, b) = | a - b |$. Let $A \subset G, D(A) = \{\forall d \in G \mid \exists a, b \in A, {}_3 d(a, b = |d|\}$. Define $D^n(A) = D(D^{n-1}(A))$ for $n > 1$. Let A^* be the group closure of A and $A \oplus B$ be the direct union of $A, B \subset G$. $A, B \subset G$ are *congruent* if there is a biuniform mapping of one onto the other which preserves distance. A *motion* of G is a congruent mapping of G *onto* itself. $A, B \subset G$ are *distancial* if $D(A) = D(B)$. The equivalence classes into which the non-null subsets of G are divided by distanciallity form the *distance semigroup* of G under direct union of representatives. The principal results of this study are:

1) *If* $A \subset G$, $A \neq 0$, *then* $A = D(A)$ *if and only if* $A = A^*$;

2) *No two distinct subgroups of* G *are congruent*;

3) *If* A, B *are subgroups of* G *then* $\{\forall d \in G \mid \exists a \in A, b \in B, {}_3 d(a, b) = | d | \} = A \oplus B$;

4) *The positive integral powers of* D *on* $A \neq 0$ *form a monotone nondecreasing sequence of subsets of* G, *bounded above by* A^*, *which is finite if and only if every (finite) linear form in elements of* A *with integral coefficients is expressable as a linear form in not more than* v *elements of* A *with coefficients* ± 1 *for some fixed* v;

5) *The distance semigroup of* G *is commutative and possesses a unit element. The group of motions of* G *leaves fixed the elements of the distance semigroup of* G *but, in general, permutes subsets of* G *within these elements. No element of the distance semigroup of* G *contains more than one subgroup of* G *and each element contains with each proper subset of* G *a distinct subset of* G *with the same cardinal number*;

6) *If* G *and* G' *are isomorphic naturally metrized groups, their distance semigroups are isomorphic.*

The writer conjectures that the converse of proposition 6) is valid.

UNIVERSITY OF FLORIDA,
GAINESVILLE, FLA., U. S. A.

THE WORD PROBLEM FOR ABSTRACT ALGEBRAS

TREVOR EVANS

Let $\mathfrak{A}$ be a class of abstract algebras defined by a finite number of finitary operations $f_1, f_2, \cdots$ and a finite set of axioms $\phi_i = \phi_i'$. An incomplete $\mathfrak{A}$-algebra is a set of elements $a_1, a_2, \cdots$ such that $f_k(a_i, a_j, \cdots)$ is either assigned a value in the set or else left undefined, and such that (i) if both sides of an axiom can be assigned values for some substitution of $a_1, a_2, \cdots$ for the variables, then the axiom is satisfied, (ii) it is not possible by the use of the axioms to assign a value to any $f_k(a_i, a_j, \cdots)$ which is not already defined. An $\mathfrak{A}$-algebra generated by $g_1, g_2, \cdots$ is said to have a closed set of defining relations if (i) every relation is of the form $f_k(g_i, g_j, \cdots) = g_t$, (ii) the set $g_1, g_2, \cdots$ with $f_k(g_i, g_j, \cdots)$ assigned a value g_t corresponding to every defining relation $f_k(g_i, g_j, \cdots) = g_t$ satisfies the conditions for an incomplete $\mathfrak{A}$-algebra.

If, for any incomplete $\mathfrak{A}$-algebra, we can find an $\mathfrak{A}$-algebra containing it as a subset, then the class $\mathfrak{A}$ is said to have the embedding property. We show that for any class $\mathfrak{A}$ which has the embedding property there exists a finite algorithm for solving the word problem in any finitely related $\mathfrak{A}$-algebra. The proof depends on the following lemmas.

LEMMA I. *If $\mathfrak{A}$ has the embedding property, then in any $\mathfrak{A}$-algebra defined by a closed set of relations no two distinct generators are equivalent.*

LEMMA II. *Let A be an $\mathfrak{A}$-algebra defined by a finite set of relations and let u, v be any two words in the generators of A. Then we can find an $\mathfrak{A}$-algebra A' defined by a closed set of relations, which is isomorphic to A and such that in this isomorphism u, v correspond to generators of A'.*

Since loops and other related nonassociative systems can be defined equationally and satisfy the embedding condition, the above method gives a positive solution of the word problem for these classes of algebras. In addition, lattices also satisfy the embedding condition and so this gives a solution of the word problem for lattices. Some limitations of our method are shown by the fact that while neither semigroups nor abelian groups satisfy the embedding condition, the word problem can be solved for abelian groups but not for semigroups.

UNIVERSITY OF MANCHESTER,
MANCHESTER, ENGLAND.

BOOLEAN-PARTITION-VECTOR EXTENSIONS AND (SUB)DIRECT-POWERS OF RINGS AND GENERAL OPERATIONAL ALGEBRAS

ALFRED L. FOSTER

In one phase of the development of the *K-ality* theory completed to date (for a partial bibliography see Foster, *On n-ality theories in rings and their logical algebras, etc.*, Amer. J. Math. (1950)) it was shown that the intimate interdefinable bond between Boolean rings and their corresponding Boolean algebras may be defined on a much more general level, leading to a class of *ring-logics* (mod N). In exhibiting these higher ring-logics, new results on the structure of p-rings ($a^p = a$, $pa = 0$) were required, results which were obtained by means of a certain generalized (and nonlinear) hypercomplex representation of such rings.

The present paper, while still rooted in the K-ality theory, is more immediately concerned with extending the above (generalized) hypercomplex methods and also the main structure results to (a) a fairly comprehensive class of rings, among which may be mentioned the new class of p^k-rings ($a^{p^k} = a$, $pa = 0$, plus minor conditions), and, in part, even to (b) arbitrary operational algebras.

The (generalized) hypercomplex characterization of p-rings is widely dissimilar to the formulationally simpler but structurally less penetrating earlier direct-power characterization of McCoy and Montgomery, namely: the class of p-rings is coextensive (up to isomorphisms) with the class of subdirect-powers of F_p = field of residues, mod p. This equivalence is here shown to be an instance of a similar situation which holds on the general operational level.

University of California,
Berkeley, Calif., U. S. A.

ON THE GENUS OF THE FUNDAMENTAL REGION OF SOME SUBGROUPS OF THE MODULAR GROUP

Emil Grosswald

Let R be a fundamental region for a function, automorphic under a group G. Let $2n$ be the number of sides of R, l the number of cycles, q the number of cycles corresponding to corners of R which are fixed points of transformations of G, and let g be the genus of R. The general relation $g = (n - l + 1)/2$ (see L. R. Ford, *Automorphic functions*, p. 239) can be reduced, in cases studied by R. Fricke and F. Klein (see *Vorlesungen über die Theorie der autom. Funktionen* vol. 1 p. 319, vol. 2 p. 303) to $g = (n - q)/2$. Furthermore, if G is generated by m independent generators satisfying ν independent defining relations and if ρ of the generators are parabolic, then also $n = m$ and $q = \nu + \rho$ hold. Above relations and results of H. Rademacher (*Ueber die Erzeugenden von Kongruenzuntergruppen der Modulgruppe*, Abh. Math. Sem. Hamburgischen Univ. vol. 7 (1929) pp. 134–148) and H. Frasch (*Die Erzeugenden der Hauptkongruenzgruppen...*, Math. Ann. vol. 108 (1933) pp. 229–252) are used by the author in order to determine the genus of the fundamental regions for some subgroups of the modular group Γ. Γ having as elements the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$, the following subgroups are defined by additional conditions: $\Gamma_0(p)$ by $c \equiv 0 \pmod p$; $\Gamma_0^0(p)$ by $b \equiv c \equiv 0 \pmod p$; and $\Gamma(p)$ by $b \equiv c \equiv 0$ and $a \equiv d \equiv 1 \pmod p$. For $\Gamma_0(p)$ H. Rademacher showed that $m = 2[p/12] + 3$, $p > 3$, and $\nu = 2 + (-1/p) + (-3/p)$. It is shown that $\rho = 1$ for all p and it follows that $g = 0$ for $p = 2$ or 3 and $g = [p/12] + r(r^2 - 25)/24$, $r \equiv p$ (mod 12), $|r| \leqq 5$, for $p > 3$. For $\Gamma_0^0(p)$ the author has previously proved (*On the structure of some subgroups* $\cdots$, Amer. J. Math.) that $m = 2[(p + 2)(p - 1)/12] + 3$, $\nu = 2 + (-1/p) + (-3/p)$ and $\rho = 3$, so that $g = 0$ for $p = 2$ or 3

and $g = [(p + 2)(p - 1)/12] + r(r^2 - 25)/24 - 1$ for $p > 3$. For $\Gamma(p)$, H. Frasch showed that it can be generated by

$$S^p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

and other transformations, (λ, μ, ν), depending on 3 parameters. Then $\Gamma(p)$ are free groups, $\nu = 0$, and, for $p > 3$, $m = p(p^2 - 1)/12 + 1$. It follows that $l = (p^2 - 1)/2$ so that $g = (p^2 - 1)(p - 6)/24 + 1$.

UNIVERSITY OF SASKATCHEWAN,
SASKATOON, SASK., CANADA.

SOME LIMITS OF BOOLEAN ALGEBRAS

FRANKLIN HAIMO

Inverse and direct limits of sets of Boolean algebras are constructed. Ideals in the direct limit are studied, and decompositions of the direct limit are discussed.

WASHINGTON UNIVERSITY,
ST. LOUIS, MO., U. S. A.

ON COMPLETE ATOMIC PROPER RELATION ALGEBRAS

FRANK HARARY

By definition, a proper relation algebra is a relation algebra whose elements are relations; and a relation algebra is complete atomic if its Boolean algebra is complete atomic. An equivalence relation is found over the set of all atoms of any complete atomic proper relation algebra such that the number of atoms in any equivalence class is a perfect square cardinal number. It is then shown that (i) the number of isomorphism types of proper finite relation algebras of 2^m elements is equal to the number of partitions of the positive integer m into summands which are positive perfect squares, (ii) the isomorphism type of a complete atomic proper relation algebra is completely determined by the set of perfect square cardinals, each of which is the number of atoms in an equivalence class.

The statements made above are valid only for those complete atomic proper relation algebras in which each atom is a singleton, i.e., a relation consisting of exactly one ordered couple. The isomorphism types of *all* complete atomic proper relation algebras are more difficult, but have been found and will be presented elsewhere.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

A CHARACTERISTIC PROPERTY OF NILPOTENT GROUPS

KURT A. HIRSCH

It is well known that the following three properties are equivalent in a finite group:

I. The group is swept out by its ascending central series.

II. Every proper subgroup is different from its normaliser.

III. All maximal subgroups are normal.

A finite group with these properties is a direct product of p-groups.

For infinite groups the equivalence of I and II has so far been proved under the additional assumptions that

(i) the group is soluble (R. Baer),

(ii) the group satisfies the minimal condition for subgroups (S. N. Černikov and O. Schmidt).

We now prove the equivalence of I and II assuming that

(iii) the group satisfies the maximal condition for subgroups.

The equivalence of I and III need not hold for infinite groups, trivially, because the group need not possess maximal subgroups. The following example shows that a group may be soluble, possess maximal subgroups, and satisfy III, but not I. Let $G = \{Q, P_1, P_2, \cdots\}$ with the relations $P_{i+1}^2 = P_i$, $Q^2 = 1$, $(QP_i)^2 = 1$. The only maximal subgroup is $\{P_1, P_2, \cdots\}$ which is normal. But the centre of G is 1. It is known that I and III are equivalent, if the group is soluble and satisfies the maximal condition (K. A. Hirsch, Proc. London Math. Soc. (2) vol. 49 (1946) pp. 184–194). Whether the condition of solubility can be omitted, or whether the maximal condition can be weakened to the condition that every subgroup can be embedded in a maximal subgroup, are open problems.

An account will appear shortly in Mathematische Nachrichten.

KING'S COLLEGE, UNIVERSITY OF DURHAM,
NEWCASTLE-UPON-TYNE, ENGLAND.

INCIDENCE RELATIONS AND CANONICAL FORMS FOR ALTERNATING TENSORS

L. C. HUTCHINSON

In a linear associative algebra of rank n, represented by a projective point space S^n or an affine direction space E^{n-1}, any set of n linearly independent elements can serve as a base, or coordinate system, for the space. However, this is no longer true for the simple alternating tensors of valence m ($2 \leqq m \leqq n - 2$), represented by the linear subspaces S^m of S^n. (Although it is of course trivially true in the corresponding Grassmannian S^N, $N = C_m^n$.) The conditions that a given set of linearly independent S^m can become coördinate S^m is closely related to the problem of the classification of the general alternating tensor, or n-vector. This paper studies this general question. For example, the problem of the classification of the trivector in the field of real numbers becomes, with certain conditions, simply that of the different possible ways of forming combinations of n objects three at a time, such that no two have more than one common element. In the complex domain, the canonical forms are a subset of those obtained by the above process. The same approach should at least delimit the possible forms for the general n-vector.

POLYTECHNIC INSTITUTE OF BROOKLYN,
BROOKLYN, N. Y., U. S. A.

SUBGROUP THEOREMS FOR GROUPS PRESENTED BY GENERATORS AND RELATIONS

HAROLD W. KUHN

The theorem of Reidemeister and Schreier (Abh. Math. Sem. Hamburgischen Univ. vol. 5 (1926)) gives a construction for generators and relations of a subgroup of any group in terms of certain knowledge of its left cosets. In this paper, a new solution is given to this problem which permits direct application to the determination of the structure of subgroups of groups which are free products or free products with identified subgroups. In particular, the new construction provides an immediate proof of the Kurosh subgroup theorem, which states that every subgroup of a free product G is itself a free product, in which the factors are a free group and subgroups of the conjugates of the factors of G. As proved here, the statement of Kurosh's theorem includes the subsequent improvements made by Baer and Levi, Takahasi, and H. Neumann, admitting in addition an explicit construction for the generators and relations of the subgroup. Further applications are made to the commutator subgroups of free products and of free products of abelian groups with one identified subgroup. The former are shown to be determined up to isomorphism by the commutator subgroups of the factors with their indices in these factors, while the latter are free groups of computable rank. The methods used throughout the paper have a topological background relating to the well-known fact that the subgroups of a group G are precisely the fundamental groups of the unbranched covering complexes of a complex which has G as its fundamental group.

PRINCETON UNIVERSITY,
PRINCETON, N. J., U. S. A.

THE CLASSES OF ORDERED GROUPS

F. Loonstra

A set G is said to form an ordered group if the conditions I, II are satisfied: I. G is a group; II. G is simply ordered; moreover $a < b$ implies $ac < bc$ and $ca < cb$.

An ordered group G is either discretely ordered or densely ordered. If for two elements a and b $(e < a < b)$ a is archimedian with regard to b, then a and b are said to have the same "rank." If $e < a < b$ and $a^n < b$ for all n, a is said to be of lower rank than b; in symbols, $a \ll b$ For $a < e$, $b > e$, we define $a \ll b$ if $a^{-1} \ll b$. For $a < e$, $b < e$, we define $a \ll b$ if $a^{-1} \ll b^{-1}$. This enables us to divide G into classes of elements: one class contains those and only those elements that are of the same rank with respect to each other. The classes $A, B, \cdots$ of an ordered group define an ordered class-set Λ, by defining $A < B$, if and only if $a \ll b$ $(a \in A; b \in B)$. By Λ we mean the class-set without the identity-class E consisting of e. The class-set of a discretely ordered group has a first class A; A together with e forms a cyclic group.

We answer the question: What is the character of the ordering of the classes?

Theorem I. *In a discretely ordered group two successors a and b belong to the same class.*

Suppose $e < a < b$, then $a^{-1}b$ is the immediate successor of e. If $a^{-1}b = a < b$, then $b = a^2 < a^3$; if $a^{-1}b < a$, then $b < a^2$. In both cases, a and b belong to one class.

Theorem II. *If one element a of a class C has a successor b, then each element of C has a successor.*

If $e < a < b$ and $a^{-1}b = c$, then $a, ac, ac^2, \cdots, ac^{-1}, ac^{-2}, \cdots$ (and no other elements) belong to C.

Between two elements ac^k and ac^{k+1} there cannot be an element of C. If there were an element d $(e < d)$ with $c^n < d$ for all n, then $c \ll d$. Thus we conclude:

Theorem III. *A class of an ordered group is discretely or densely ordered.*

Theorem IV. *If one class is discretely ordered, all classes are.* Hence: *if one class is densely ordered, all classes are.*

If one class C consists of $\cdots ac^{-1}, a, ac, ac^2, \cdots$, then c is the successor of e. If there were a densely ordered class C', there would be an element d of C' such that $b < d < bc$; but then we would have $e < b^{-1}d < c$, contrary to the property of c.

Theorem V. *If a group is discretely ordered, all classes are discretely ordered.* Indeed the class of lowest rank is a cyclic group, which is discretely ordered; whence all classes are discretely ordered.

Thus we obtain:

Theorem VI. *If a group is densely ordered, all classes are densely ordered.*

We summarize:

THEOREM VII. *An ordered group is discretely ordered (densely ordered) if and only if all classes are discretely ordered (densely ordered).*

TECHNISCHE HOGESCHOOL,
DELFT, NETHERLANDS.

ON A THEOREM BY CARTAN

PETER SCHERK

Given a field $\mathfrak{F}$ whose characteristic is different from 2, let $\mathfrak{R}_n$ denote the space of all column vectors over $\mathfrak{F}$ with n components. The small letters a, b, x, y stand for vectors in $\mathfrak{R}_n$. Capital letters denote n-rowed squared matrices over $\mathfrak{F}$. A prime indicates transposition.

Let G be a fixed regular symmetric matrix. Thus $G = G'$; $|G| \neq O$. We call two vectors a, b perpendicular if $a'Gb = O$. An isotropic vector a is perpendicular to itself: $a'Ga = O$. A matrix T is called orthogonal if it leaves the scalar product $x'Gy$ invariant. This is equivalent to $T'GT = G$. If, in addition, rank $(T - I) = I$ [I = unit matrix], T is called a symmetry. A symmetry can be characterized more geometrically by the fact that it maps some nonisotropic vector a on $-a$ and every vector perpendicular to a onto itself. Obviously, the symmetries are involutions.

Cartan proved that every orthogonality T can be decomposed into a product of n or less than n symmetries. Let $m_0 = m_0(T)$ denote the minimum number of symmetries into which the orthogonality T can be decomposed. We prove that in general m_0 is equal to

$$(1) \qquad m = \text{rank } (T - I).$$

An exception occurs if and only if $G(T - I)$ is skew-symmetric. In that case m_0 is equal to $m + 2$. The proof does not make use of Cartan's theorem.

The fact that $m_0 \geqq m$ is a corollary of the almost obvious observation that for any two matrices A and B

$$(2) \qquad \text{rank } (AB - I) \leqq \text{rank } (A - I) + \text{rank } (B - I).$$

The induction proof that $m_0 \leqq m$ in the general case is constructive. Suppose $m > 1$. Let $\mathfrak{R}_{n-m}$ be the eigenspace belonging to the eigenvalue 1 of T. Let $b \not\subset \mathfrak{R}_{n-m}$, $a = (T - I)b$. Choose b such that a is nonisotropic. Thus a determines a symmetry A. Put $U = AT$. Thus $T = AU$, and the eigenspace of U belonging to the eigenvalue 1 contains both b and $\mathfrak{R}_{n-m}$. Hence rank $(U - I) \leqq m - 1$, and from (2)

$$(3) \qquad \text{rank } (U - I) = m - 1.$$

The main difficulty is the construction of a vector b such that $G(U - I)$ is sure not to be skew-symmetric.

The exceptional case that $G(T - I)$ is skew-symmetric can readily be discussed in detail.

UNIVERSITY OF SASKATCHEWAN,
SASKATOON, SASK., CANADA.

MATROID SEMIGROUPS

ROBERT R. STOLL

A matroid semigroup S is a semigroup which satisfies the following postulates: (a) every element is either k-regular or nilpotent and elements of both types occur, (b) S contains no properly nilpotent elements, (c) S contains a primitive idempotent. Here an element x is k-regular if every positive integral power x^i, $i \geqq k$, has a two-sided unit and relative to such a unit, a two-sided inverse. The terms nilpotent, etc. are defined as in ring theory.

THEOREM. *The set $M_n(F)$ of all n by n matrices over the field F form a matroid semigroup under mulitplication.*

In the set I of idempotents of S the following relations are defined: $e\mathcal{R}_1 f$ if $ef = f$, $e\mathcal{R}_2 f$ if $fe = f$, $e < f$ if $f\mathcal{R}_i e$ $(i = 1, 2)$, and $e\mathcal{R}f$ if there exists a sequence of idempotents $g, \cdots, h, k, \cdots, l$ such that $e\mathcal{R}_{i_1}g\mathcal{R}_{i_2} \cdots h\mathcal{R}_{i_p}f\mathcal{R}_{i_{p+1}}k \cdots l\mathcal{R}_{i_m}e$ where $i_j = 1$ or 2.

THEOREM. *The $<$ relation partially orders I and the $\mathcal{R}$ relation is an equivalence relation over I. Moreover these relations are noncircular so that it is possible to define a diagram of the idempotents of S where equivalent idempotents occur at the same level.*

THEOREM. *In the partially ordered set I of idempotents of $M_n(F)$, $e\mathcal{R}f$ if and only if $r(e) = r(f)$ and $e < f$ implies $r(e) < r(f)$ where $r(e)$ is the rank of e.*

For the matroid semigroups with unit element and with I of finite dimension (e. g., $M_n(F)$) it is found that the minimal two-sided ideals have the form SeS, where e is a primitive idempotent, and contain one or more classes of equivalent idempotents.

If, besides the above assumptions, a matroid semigroup contains a single minimal two-sided ideal, it is called *primitive.* Known methods for combining semigroups are applied to construct matroid semigroups from primitive ones. Matrix representations of primitive semigroups are investigated. In particular, numerical invariants for $M_n(F)$ where $F = GF(p^k)$ are found in order to decide when a finite semigroup is some matrix semigroup over a Galois field.

LEHIGH UNIVERSITY,
BETHLEHEM, PA., U. S. A.

ON THE GENERATION OF TRANSITIVE RELATIONS

L. R. WILCOX

Let M be a set, and let, for every binary relation X on $M \times M$, X_t be the smallest transitive relation containing X. If R is a given transitive relation, it is desired to obtain all *generators* X of R, that is, all relations X such that $X_t = R$. This problem is reduced to the cases where R is either a partial ordering or an equivalence relation in the following manner.

Denote the transpose of any relation Y by Y^* and the identity relation by E. Let R be any given transitive relation, and let $\mathbf{M}$ be the set of equivalence classes in M under the equivalence relation $P = (R + E)\cdot(R + E)^*$. Then R is expressible uniquely in the form $P + U - E_0$, where $E_0 \subset E$ and U is a partial ordering of M defined by the condition $(a, b) \in U$ if and only if $(a, b) \in R - P$. The partial ordering U defines in a natural way a partial ordering $\mathbf{U}$ of $\mathbf{M}$. Every generator of R is of the form $Y + Z - E_0$, where Y is a generator of P and Z is a relation associated with a generator $\mathbf{Z}$ of $\mathbf{U}$. In fact, all generators $\mathbf{Z}$ of $\mathbf{U}$ arise in this way, and the relations Z associated with $\mathbf{Z}$ are those which contain exactly one pair of representative elements chosen from each pair of equivalence classes in the relation $\mathbf{Z}$.

The following results are obtained toward the characterization of generators of partial orderings. Let $<$ be a partial ordering of M satisfying the descending chain condition. Let for each non-minimal $a \in M$ a set $A(a)$ be given such that (1) $x \in A(a)$ implies $x < a$; (2) $b < a$ implies the existence of $x \in A(a)$ such that $b \leqq x$. (In particular, $A(a)$ must contain all immediate predecessors of a.) Then a relation X, consisting of all pairs (x, a) with $x \in A(a)$ and $a \in M$ non-minimal, is a generator of $<$; conversely every generator is so obtained. If $<$ is a well-ordering, the sets $A(a)$ admit the following characterization: If a is not a limit element, $A(a)$ is any set of elements $x < a$ containing the immediate predecessor of a; otherwise $A(a)$ is any set such that $\sup A(a) = a$.

ILLINOIS INSTITUTE OF TECHNOLOGY,
CHICAGO, ILL., U. S. A.

RINGS AND ALGEBRAS

THE JACOBSON RADICAL OF A SEMIRING

SAMUEL BOURNE

The concept of the Jacobson radical of a ring is generalized to semirings. A semiring is a system consisting of a set S together with two binary operations, called addition and multiplication, which forms a semigroup relative to addition, a semigroup relative to multiplication, and the right and left distributive laws hold. The additive semigroup of S is assumed to be commutative. The right ideal I of a semiring S is said to be right semiregular if for every pair of elements i_1, i_2 in I there exist elements j_1 and j_2 in I such that $i_1 + j_1 + i_1j_1 + i_2j_2 = i_2 + j_2 + i_1j_2 + i_2j_1$. The Jacobson radical R of a semiring S is the sum of all the right semiregular ideals of S. It is also the sum of all the left semiregular ideals of S. It is shown that the Jacobson radical of a semiring S has the two following important properties: (i) If R is the Jacobson radical of the semiring S, then the difference semiring $S - R$ is semisimple. (ii) The Jacobson radical of a semiring S is a radical semiring. Thus the structure of an arbitrary semiring is reduced to the study of the structure of semisimple semirings and radical semirings. The paper concludes with a consideration of the Jacobson radical of a matrix semiring S_n. In the case S is a ring, the theory reduces to the Jacobson theory for arbitrary rings.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

ON THE ASSOCIATIVITY THEOREMS FOR ALTERNATIVE RINGS AND MOUFANG LOOPS

R. H. BRUCK

A ring R is alternative if $xx \cdot y = x \cdot xy$, $yx \cdot x = y \cdot xx$ for all x, y of R. The associator (x, y, z) is defined by $(x, y, z) = xy \cdot z - x \cdot yz$. If A_1, A_2, A_3 are subsets of R, (A_1, A_2, A_3) is the subset consisting of the elements (a_1, a_2, a_3) with a_i in A_i. Artin's theorem states: If A, B are subsets of the alternative ring R such that $(A, A, R) = (B, B, R) = 0$, then A and B together generate an associative subring. Zorn's inductive proof (M. Zorn, *Theorie der alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 9 (1933) pp. 395–402) is incomplete and the methods of Moufang (R. Moufang, *Zur Struktur von Alternativkörpern*, Math. Ann. vol. 110 (1935) pp. 416–430) involve complicated inductions. The present paper supplies a simple non-inductive proof. For any subset A of R, let K be the largest subset of R such that $(A, K, R) = 0$ and let A^* be the largest subset of R such that $(A^*, K, R) = 0$. Then A^* is a subring containing A, and

$(A, A, R) = 0$ implies $(A^*, A^*, R) = 0$. With A, B as in the theorem, define C, M, S as follows: C is the largest subset of R such that $(A^*, B^*, C) = 0$; M is the largest subset of C such that $MC \subset C$, $CM \subset C$ and $(A^* \cup B^*, M, C) = 0$; S is the largest subset of M such that $(S, M, C) = 0$. Then S is an associative subring containing A^* and B^*. An additive Moufang loop is a system with an addition such that: (i) in the equation $x + y = z$, any two of x, y, z uniquely determine the third; (ii) R has a zero element; (iii) $(x + y) + (z + x) = x + ((y + z) + x)$ for all x, y, z of R. The associator (x, y, z) is defined by $(x + y) + z = (x + (y + z)) + (x, y, z)$. The above form of Artin's theorem carries over to Moufang loops. The paper also considers the three-element associativity theorem of Moufang.

University of Wisconsin,
Madison, Wis., U. S. A.

IDÉAUX HOMOGÈNES DE L'ALGÈBRE EXTÉRIEURE

Th. H. J. LePage

1. Considérons l'algèbre extérieure $A(n, K)$ de degré n définie sur un corps commutatif quelconque K. Un idéal $\mathfrak{J}$ engendré par des formes de degré positif sera dit *homogène*; il est bilatère. Désignons par $\mathfrak{J}'$ son *annulateur*, c'est-à-dire, l'ensemble des éléments de $A(n, K)$ dont le produit extérieur pour tout élément de $\mathfrak{J}$ est nul; cet idéal est homogène donc bilatère et $(\mathfrak{J}')' = \mathfrak{J}$. L'ensemble des éléments d'un même degré appartenant à $\mathfrak{J}'$ sera dit *complètement décomposable* sur K^*, prolongement algébrique de K, s'il possède une base de formes simples (de degré égal au rang). Si chaque module de $\mathfrak{J}'$ est complètement décomposable, nous dirons que $\mathfrak{J}$ est *complet*. Les *zéros* de $\mathfrak{J}$ seront les éléments simples de $\mathfrak{J}'$.

Tout élément de $A(n, K)$ possèdant les mêmes zéros que $\mathfrak{J}$ appartiendra à $\mathfrak{J}$ si et seulement si $\mathfrak{J}$ est complet.

2. Soit $n = 2m$, $m > 1$, et soit $\mathfrak{A}$ l'idéal engendré par une forme quadratique de rang $2m$. Si la caractéristique de K ne divise pas m, $\mathfrak{A}$ sera complet sur K. Il en résulte notamment les propositions:

Si A, B sont deux matrices $m \times n$, $m \leqq n$, telles que la matrice (A, B) soit de rang r, il existe une matrice symétrique Z d'ordre n, d'éléments $+1$, -1, 0, telle que $A + BZ$ soit de rang r.

Toute matrice symplectique

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

est le produit de trois matrices

$$\begin{pmatrix} E & 0 \\ Z_1 & E \end{pmatrix} \begin{pmatrix} M & MZ_2 \\ 0 & M'^{-1} \end{pmatrix} \begin{pmatrix} E & 0 \\ Z & E \end{pmatrix},$$

Z_1, Z_2 matrices symétriques, M régulière, Z matrice symétrique régulière d'éléments $+1$, -1, 0.

UNIVERSITY OF BRUSSELS,
BRUSSELS, BELGIUM.

ON THE ALGEBRAIC ELEMENTS OF A RING WITH OPERATORS

JAKOB LEVITZKI

Let S be a ring with a domain Φ of operators (in short: S is a Φ-ring) satisfying the following conditions: (1) The domain Φ is a subring of the ring of endomorphisms of the additive group of S. (2) The ring Φ has an identity which is the identical automorphism of the additive group of S. (3) If $\alpha \in \Phi$ and $a, b \in S$, then $(ab)\alpha = a(b\alpha) = (a\alpha)b$.

A polynomial $f(x) = \sum_{i=0}^{n} x^i \alpha_i$ with $\alpha_i \in \Phi$ is called a Φ-polynomial. If $a \in S$ and for some monic Φ-polynomial (i.e., such that $\alpha_n = 1$) we have $f(a) = 0$, then a is called *algebraic over* Φ, or in short, *algebraic*.

THEOREM. *A Φ-ring satisfying* (1)-(3) *contains a maximal two-sided algebraic Φ-ideal $K(S)$, the algebraic kernel of S. $K(S)$ contains the upper radical of S. The algebraic kernel of $S/K(S)$ is* $= 0$.

A subring T of a Φ-ring is called *locally finite* if each Φ-subring generated by any finite set of elements belonging to T is a finite Φ-module. If in addition to (1)-(3) we assume that (4) each Φ-submodule of a finite Φ-module is also finite, then we have the following:

THEOREM. *Each Φ-ring satisfying* (1)–(4) *contains a maximal two-sided locally finite Φ-ideal $F(S)$, the locally finite kernel of S. $F(S)$ contains also all locally finite one-sided ideals and in particular all semi-nilpotent one-sided ideals of S. The locally finite kernel of $S/F(S)$ is* $= 0$.

Remark. Evidently $K(S) \supseteq F(S)$. It is not known whether cases exist where $K(S) \supset F(S)$.

Conditions (1)–(4) are satisfied for example in case S is an algebra over a field Φ. If $S = K(S)$, S is algebraic over Φ. It has been shown by Kaplansky with the aid of topological methods (*Topological representations of algebras* II, to appear in the Trans. Amer. Math. Soc.) that if $S = K(S)$ and S satisfies a polynomial identity, then $S = F(S)$. Applying the concept of the locally finite kernel we derive a purely algebraic proof of this result.

HEBREW UNIVERSITY,
JERUSALEM, ISRAEL.

ON THE THEORY OF GALOIS ALGEBRAS

TADASI NAKAYAMA

If an algebra $\mathfrak{A}$ over a field Ω possesses G, a finite group, as a group of automorphisms and is isomorphic as a G-Ω-module to the group ring $G(\Omega)$, then $\mathfrak{A}$ is called a Galois algebra with Galois group G. Its theory of construction and invariant characterization, which specializes to the celebrated theory of Kummer fields in case of abelian G, was first established by Hasse under the assumption that absolutely irreducible representations of G lie in Ω and the characteristic of Ω does not divide the order of G, and has recently been freed from the restriction on the characteristic by the author; the main tool for the extension being the theory of modular regular representations of Brauer's school, in particular the theorem of Brauer-Nesbitt-Osima on the Kronecker products of components of regular representation and the orthogonality relation of Nesbitt-Nakayama. By virtue of our modular extension we may treat the Artin-Schreier-Albert-Witt theory of p-fields from the view point of our theory. The extension is useful for arithmetic application of the theory, too.

The notion of Galois algebra can be generalized so that the G-Ω-module $\mathfrak{A}$ need not be isomorphic to $G(\Omega)$, but defines a certain given representation M of G. The theory of such Galois algebras, in the generalized sense, can be given by intertwining matrices. A matrix A which intertwines $M \times M$ with M defines multiplication in a Galois algebra belonging to M. Derivation from a fixed Galois algebra and isotopy, in Albert's sense, are discussed. Conditions for the associativity, commutativity, and absolute semisimplicity of $\mathfrak{A}$ may be given in terms of A.

Of special interest is the case where M is a quasi-regular representation. On taking a certain normalized form of the quasi-regular representation we can analize the structure of the matrix A for the case. The analysis provides a second approach, representation-theoretical rather than structural, to the theory of Galois algebras in the proper sense.

Another case of interest is the one where M is a permutation representation, which yields a duality theorem of Tannaka's type; it suggests a certain supplement as well as an extension of Tannaka's theorem itself, in which subgroups are taken care of.

NAGOYA UNIVERSITY,
NAGOYA, JAPAN.

ON THE STRUCTURE OF STANDARD ALGEBRAS

ANTHONY J. PENICO

A finite-dimensional algebra A is said to be a *standard* algebra if it satisfies the identities

$$[x, y, z] + [z, x, y] - [x, z, y] = 0,$$

$$[w, x, yz] - [wy, x, z] - [wz, x, y] = 0$$

for all w, x, y, z in A, where $[x, y, z] = (xy)z - x(yz)$, by definition. These algebras have been defined and studied by A. A. Albert (*Power-associative rings*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593). Jordan algebras and associative algebras are special cases of standard algebras. In fact, every standard simple algebra is either a Jordan algebra or is associative (ibid., p. 593). The radical of a standard algebra A is defined to be the maximal solvable ideal of A. Some useful identities are derived as consequences of the defining identities given above. These are used in obtaining a decomposition of a standard algebra A relative to a set of pairwise orthogonal idempotents in A. This decomposition reduces to a well-known decomposition in case A is a Jordan algebra or an associative algebra. For B an ideal of the standard algebra A, it is shown that the subalgebras $B_0 = B$, $B_{n+1} = AB_n^2 + B_n^2$, $n = 0, 1, 2, \cdots$, are all ideals in A, and that there is an integer k such that $B_k = B^2$. This implies that, if $N \neq 0$ is the radical of the standard algebra A, the subalgebra $AN^2 + N^2$ is an ideal of A properly contained in N. This latter result then permits the reduction, by the usual inductive proof, of the following theorem to the case $N^2 = 0$: If A is a standard algebra over a field F of characteristic zero, and if N is the radical of A, then there exists a subalgebra S of A such that $S = A - N$, and A is the direct sum $A = S + N$. The case $N^2 = 0$ is then established when $A - N$ is a Jordan algebra or an associative algebra. The proof makes use of the Wedderburn principal theorem for Jordan algebras, established in the author's dissertation, and the fact that a standard algebra is Jordan-admissible.

TUFTS COLLEGE,
MEDFORD, MASS., U. S. A.

A THEOREM ON THE DERIVATIONS OF JORDAN ALGEBRAS

RICHARD D. SCHAFER

G. Hochschild has proved that a Lie algebra [associative algebra] over a field F of characteristic 0 is semisimple if and only if its derivation algebra is semisimple [semisimple or $\{0\}$]. The analogue for Jordan algebras of these theorems is: a Jordan algebra A over F is semisimple with each simple component of dimension $\neq 3$ over its center if and only if the derivation algebra D of A is semisimple or $\{0\}$. The dimensional restriction arises from the fact that the central simple Jordan algebra of all 2×2 symmetric matrices has for its derivation algebra the abelian Lie algebra of dimension 1. The "only if" part of the theorem is an immediate consequence of the determination of the derivation algebras of all nonexceptional simple Jordan algebras by N. Jacobson and of the exceptional simple Jordan algebra by C. Chevalley and the present author. The proof of the converse depends upon the Wedderburn principal theorem for Jordan algebras, proved recently by A. Penico, and also the following lemma: if A is semisimple, then A (as a vector space) is the direct sum of its center C and the subspace P of A spanned by the associators $(ab)c - a(bc)$ of elements a, b, c in A.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA., U. S. A.

TOPOLOGICAL ALTERNATIVE RINGS

M. F. SMILEY

If the associative law is replaced by $a(ab) = a^2b$ and $(ab)b = ab^2$ in the definition of a topological ring, the definition of a topological *alternative* ring is obtained. Such systems are studied by the methods employed by Kaplansky (*Topological rings*, Amer. J. Math. vol. 69(1947) pp. 153–183). Let R be the Perlis-Jacobson radical of a topological alternative ring A (see the author's paper, *The radical of an alternative ring*, Ann. of Math. vol. 49(1948) pp. 702–709) and let $\mathfrak{R}$ denote the set of elements a of A such that $a + x + ax = 0$ for some x in A. Preliminary results are that $\mathfrak{R}$ is open if $\mathfrak{R}$ contains a neighborhood of zero, that R is closed whenever $\mathfrak{R}$ is either open or closed, and that every modular maximal right ideal M of A is closed whenever $\mathfrak{R}$ is open. (A right ideal I is *modular* in case there is an element e in A such that $ex - x$ is in I for every x in A. The concept is due to I. Segal, the term to N. Jacobson.) The proof of these results rests on lemmas which verify the desired associativity as in the previous paper cited. A basic definition, due to Shafarevich (*On the normalizability of topological fields*, C. R. (Doklady) Acad. Sci. URSS. vol. 40 (1943) pp. 133–135), is generalized to read: "An alternative ring A is *right bounded* in case to every neighborhood U of zero there corresponds a neighborhood V such that $V\rho \leqq U$ for every finite product ρ of right multiplications of A". The principal result is that *a compact and bounded alternative ring with zero Perlis-Jacobson radical is isomorphic and homeomorphic to a Cartesian direct sum of finite simple alternative rings.* The question of whether a compact alternative ring is necessarily bounded is left open.

STATE UNIVERSITY OF IOWA,
IOWA CITY, IOWA, U. S. A.

ARITHMETIC ALGEBRA

A NOTE ON FUNCTIONS

Kenkichi Iwasawa

Let K be a finite algebraic number-field, J the idèle group of K and P the group of principal idèles of K. If we introduce a *natural topology* in J, it becomes a locally compact group and P a discrete subgroup of J, and the zeta-function of K with its Γ- and other factors can then be written as an integral over J:

$$\xi(s) = \int_J \varphi(\mathfrak{a}) V(\mathfrak{a})^s \, d\mu(\mathfrak{a}),$$

where $\varphi(\mathfrak{a})$ is a suitable continuous function on J, $V(\mathfrak{a})$ the volume of $\mathfrak{a}$ in the sense of Artin-Whaples, and $\mu(\mathfrak{a})$ a Haar measure on J. The same integral can also be transformed into an integral over $\bar{J} = J/P$:

$$\xi(s) = \int_{\bar{J}} \bar{\varphi}(\bar{\mathfrak{a}}) V(\bar{\mathfrak{a}})^s d\mu(\bar{\mathfrak{a}}), \qquad \bar{\varphi}(\bar{\mathfrak{a}}) = \textstyle\sum_{\alpha \in P} \varphi(\mathfrak{a}\alpha),$$

and the functional equation $\xi(s) = \xi(1 - s)$ follows then immediately from the theta-formula for $\Theta(\bar{\mathfrak{a}}) = 1 + \bar{\varphi}(\bar{\mathfrak{a}})$ and from the invariance of $\mu(\bar{\mathfrak{a}})$. In carrying out this calculation, we obtain Dirichlet's unit theorem for K from the compactness of a subgroup of $\bar{J}$. The above method is also applicable for Hecke's L-functions "mit Grössencharakteren", for these characters form essentially a particular class of continuous characters of $\bar{J}$. (According to a letter of Professor A. Weil, he and Professor E. Artin also obtained these results some time ago.)

More generally, we can define L-functions for a division algebra A over the field of rational numbers by the help of the idèle group of A, and then prove in a similar way the functional equations for them.

Tokyo University,
Tokyo, Japan.

UN THÉORÈME D'ARITHMÉTIQUE EN ALGÈBRE DE GRASSMANN

G. L. Papy

Dans les algèbres de Grassmann sur un corps de caractéristique nulle, à 2n indéterminées $x_1, x_2, \cdots, x_n, y_1, \cdots, y_n$, toute forme de degré $n + k$ est divisible par la k-ième puissance (extérieure) de toute forme quadratique de rang $2n$. Il suffit d'établir la divisibilité par la k-ième puissance de la forme $H =$

$x_1y_1 + \cdots + x_ny_n$, puisque toute forme quadratique de rang $2n$ est équivalente à H.

Cette proposition cesse d'être valable en général dans les algèbres de Grassmann sur un anneau d'intégrité quelconque ou sur un corps dont la caractéristique ne soit pas nulle. En effet, comme la k-ième puissance de H est toujours divisible par $k!$, la proposition ci-dessus impliquerait que toute forme quadratique de degré $n + k$ soit divisible par $k!$, ce qui n'est pas vrai, en général, lorsque le domaine des coefficients est un anneau d'intégrité quelconque ou un corps de caractéristique non nulle.

Afin de nous débarrasser de l'influence perturbatrice des facteurs factorielles, nous avons introduit les *puissances réduites* des formes de degré pair non nul. La k-ième puissance réduite d'une forme de degré pair non nul est toujours définie et coïncide avec "la k-ième puissance extérieure divisée par $k!$" chaque fois que cette dernière expression possède un sens. Dans tous les cas, la k-ième puissance extérieure est égale à la k-ième puissance réduite multipliée par $k!$. Ainsi, le problème de la divisibilité des formes de degré supérieur à n par des puissances extérieures de formes quadratiques se ramène au problème plus général de la divisibilité par des puissances réduites de formes quadratiques.

Nous avons établi que dans les algèbres de Grassmann sur un anneau d'intégrité quelconque, à $2n$ indéterminées $x_1, \cdots, x_n, y_1, \cdots, y_n$, la forme F de degré d au moins égal à $n + k$ est divisible par la k-ième puissance réduite de H si et seulement si, pour tout entier naturel i non supérieur à $n - d/2$, le produit $F \cdot H^i$ (où H^i désigne la i-ième puissance réduite de H) est divisible par $C_{k+i,k}$.

Si l'anneau des coefficients est principal, la proposition restera valable si l'on substitue à H toute forme quadratique dont la n-ième puissance réduite ne soit divisible par aucun élément non inversible du domaine des coefficients.

La démonstration se fait en plusieurs étapes.

1. On montre qu'il suffit d'établir la proposition pour des formes F en les produits $x_1 \cdot y_1, x_2 \cdot y_2, \cdots, x_n \cdot y_n$.

2. Par la considération de systèmes d'équations diophantiennes remarquables, on établit la proposition pour les formes en les $x_1 \cdot y_1, \cdots, x_n \cdot y_n$ et de degré $d = n + k$, lorsque la caractéristique est nulle.

3. Par un homomorphisme de module, on étend cette dernière proposition au cas où la caractéristique n'est pas nulle.

4. En procédant par récurrence, on montre que la proposition obtenue est encore valable lorsque d est supérieur à $n + k$.

On trouvera la démonstration d'une généralisation du présent théorème ainsi que la bibliographie du sujet dans un mémoire de l'auteur à paraître en 1951 dans les Mémoires de l'Académie Royale de Belgique, Classe des Sciences, et intitulé *Sur l'arithmétique dans les algèbres de Grassmann.*

UNIVERSITY OF BRUSSELS,
BRUSSELS, BELGIUM.

TWO ARITHMETICAL THEOREMS CONCERNING LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

J. POPKEN

It is a matter of common knowledge that analysis often plays an important part in deriving purely arithmetical results. However by combining analytic methods with ideas from the theory of numbers one is often led to theorems of mixed arithmetical and analytic character.

I give here some new results of this type concerning integral transcendental functions

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!}, \qquad c_h = O(q^h), \tag{1}$$

with algebraic coefficients $c_0, c_1, c_2, \cdots$ as solutions of a linear differential-difference equation

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} y^{(\mu)}(z + \omega_\nu) = 0. \tag{2}$$

Here q denotes an arbitrary positive number; $A_{\mu\nu}, \omega_\nu (\mu = 0, 1, \cdots, m; \nu = 0, 1, \cdots, n)$ represent constants, the $A_{\mu\nu}$ not vanishing simultaneously and $\omega_0, \omega_1, \cdots, \omega_n$ being different.

1. *If in* (2) *all constants $A_{\mu\nu}$, ω_ν are algebraic and if the $n + 1$ equations*

$$\sum_{\mu=0}^{m} A_{\mu\nu} t^\mu = 0 \qquad (\nu = 0, 1, \cdots, n)$$

have no common root, then there exists no integral transcendental function of the form (1) *with algebraic coefficients $c_0, c_1, c_2, \cdots$ satisfying the equation* (2).

2. *Let the integral transcendental function* (1) *with algebraic coefficients $c_0, c_1, c_2, \cdots$ satisfy the equation* (2). *Then $y(z)$ is a transcendental number for every algebraic value of z with the exception of a finite number of values for z.*

The last theorem is a very special case of a similar theorem concerning linear differential equations of infinite order with constant coefficients, too long to reproduce here.

UNIVERSITY OF UTRECHT,
UTRECHT, NETHERLANDS.

VECTOR SPACES AND MATRICES

VARIÉTÉ FONDAMENTALE PAR RAPPORT D'UNE CORRESPONDANCE ALGÉBRIQUE

PEDRO ABELLANAS

Hypotèses et notations. Soit k un corps des constantes avec une infinité d'éléments; $\{x_0, \cdots, x_n; y_0, \cdots, y_m\}$ deux séries d'indeterminées; $A = k[x_0, \cdots, x_n; y_0, \cdots, y_m]$; $I = A(F_1(x; y), \cdots, F_\alpha(x; y))$ un idéal premier et bihomogène de A; T la correspondance algébrique déterminée par I; (ξ) et (η) éléments de A/I tels que $x_i \equiv \xi_i(I)$, $y_j \equiv \eta_j\,(I)$, $i = 0, \cdots, n$, $j = 0, \cdots, m$; V et V' les variétés originale et imâge, respectivement, avec les points généreaux $(\xi_0, \cdots, \xi_n)$ et $(\eta_0, \cdots, \eta_m)$, respectivement; $r + 1$, $s + 1$, $a + 1$, et $b + 1$, les dégrés de transcendance de Σ et Σ' sur k et de Ω sur Σ et Σ' respectivement. Soient $\lambda_{i,j}$, $i, j = 0, \cdots, m$, $(m + 1)^2$ indéterminées sur Ω et $y_i = \sum_{j=0}^{m} \lambda_{ij} y_j^*$; $K = k(\lambda_{ij})$; $A^* = K[x_0, \cdots, x_n; y_0^*, \cdots, y_m^*]$, $I^* = A^*(F_1(x; \Sigma\lambda y^*), \cdots, F_\alpha(x; \Sigma\lambda y^*))$; $\bar{A} = K[\xi_0, \cdots, \xi_n; y_0^*, \cdots, y_m^*]$, $\bar{I} = \bar{A}(F_1(\xi; \Sigma\lambda y^*), \cdots, F_\alpha(\xi; \Sigma\lambda y^*))$, $\tilde{A} = K(\xi)[y_0^*, \cdots, y_m^*]$; $\tilde{I} = \tilde{A}\,\bar{I}$; $y_0^*, \cdots, y_a^*$ tous les (y^*) indépendants par rapport à $\tilde{I}$; $\hat{A} = K(\xi)[y_0^*, \cdots, y_{a+1}^*]$; $J = \tilde{I} \cap \hat{A}$; $\hat{J} = K(\xi; y_0^*, \cdots, y_a^*)[y_{a+1}^*]J$.

Resultats. THÉORÈME 1. *Si la sous-variété W de V est fondamentale* (pour la définition de ce concept voir ma mémoire: *Théorie arithmétique des correspondances algébriques*, Revista Matemática Hispano-Americana (1949)) *par rapport à W' et ceci n'ést pas fondamentale par rapport à la compossante de sa transformée dans T^{-1} qui contient à W, on a*

$$\dim (W') > \dim (W) + a - b.$$

THÉORÈME 2. *Si la correspondance algébrique T est telle que $b = 0$ et que la sous-variété fondamentale dans V', par rapport à T^{-1} est vide, on a que la condition necessaire et suffisante pour qu'une sous-variété, W, de V soit fondamentale par rapport à la composante W' de sa transformée dans T est que*

$$\dim (W') > \dim (W) + a.$$

L'idéal $\hat{J}$ est premier et principal: $\hat{J} = (\Psi)$, où

$$\Psi = H(\xi; \lambda) y_{a+1}^{*\rho} + a_1(\xi; \lambda; y_0^*, \cdots, y_a^*) y_{a+1}^{*\rho-1} + \cdots + a_\rho(\xi; \lambda; y_0^*, \cdots, y_a^*);$$

ce polynôme est bihomogène par rapport des (ξ) et (y^*) et il s'évanouit pour la substitution $y^* \to \eta^*$, où $\eta_i = \sum_{j=0}^{m} \alpha_{ij} \eta_j^*$. L'équation qu'on obtient par ce substitution on peut simplifier, s'il est nécessaire, et écrire dans la forme suivante

$$H_1(\xi; \lambda)\eta_{a+1}^{*\rho} + c_1(\xi; \lambda; \eta_0^*, \cdots, \eta_a^*)\eta_{a+1}^{*\rho-1} + \cdots + c_\rho(\xi; \lambda; \eta_0^*, \cdots, \eta_a^*) = 0$$

tel que si on représente par $\mathfrak{F}$ l'idéal qu'on obtient quand on exclut du radical de $(H_1(\xi; \lambda))$ toutes les composantes premières dont les bases appartiennent à P, on a le suivant

Théorème 3. *La condition nécessaire et suffisante pour que la sous-variété déterminée par l'idéal premier et homogène β soit fondamentale est que*

$$\mathfrak{F} \equiv 0 \; (P[\lambda]\beta).$$

Corollaire. *La dimension de la variété fondamentale de V n'est pas supérieure à dim $(V)-2$*

University of Madrid,
Madrid, Spain.

SOME PROPERTIES OF THE DIEUDONNÉ DETERMINANT

Wallace Givens

Let K be a division ring (= skew field), K^* the multiplicative group of its nonzero elements, and let C be the commutator subgroup of K^*. Then Dieudonné has shown [Bull. Soc. Math. France vol. 71 (1943) pp. 27–45] that a square matrix A of order m with elements in K determines a unique coset, $\Delta(A)$, of C in K^*, or determines the zero element of K if A^{-1} does not exist. It is convenient to regard the determinant $\Delta(A)$ as a "determinantal class" of elements in K rather than as an element of the quotient group K^*/C (augmented with a "zero" element).

Defining $\Delta(A)\cdot\Delta(B)$ and $\Delta(A) + \Delta(B)$ as the class of all products and sums, respectively, of an element of $\Delta(A)$ and one of $\Delta(B)$, the former is a single determinantal class and the latter is a union of such classes. Let $C = A \otimes B$ be the matrix of order mn with elements $c_{(i,\alpha)(j,\beta)} = a_{ij}b_{\alpha\beta}$ where (i, α) and (j, β) have the range $(1, 1)$ to (m, n) in some order. Then $\Delta(A \otimes B) = [\Delta(A)]^n\cdot [\Delta(B)]^m$ holds for the direct product just as in the commutative case.

If A, B, and C are of the same order and have identical elements except in the ith row (or column), where they have row (column) vectors u_1, u_2, and $u_1 + u_2$, respectively, it can be proved that $\Delta(C) \subset \Delta(A) + \Delta(B)$. The inclusion relation implies equality in the commutative case since the determinantal classes then contain only one element.

The determinantal class of A can also be defined in terms of the (always possible) factorization of A into a product of factors of the form $I_n + M$, where M is of rank one: $\Delta(A)$ = the class of all values of $\prod_{i=1}^{N}(1 + v_iu_i)$, where $A = \prod_{i=1}^{N}(I_n + u_iv_i)$, the u_i being matrices of one column and the v_i matrices of one row. This suggests an extension of the definition of determinant to matrices over a principal ideal domain.

University of Tennessee,
Knoxville, Tenn., U. S. A.

CYCLIC INCIDENCE MATRICES

M. HALL, JR., AND H. J. RYSER

Let v elements be arranged into v sets such that each set contains exactly k elements and such that every pair of sets has exactly λ elements in common $(0 < \lambda < k < v)$. This combinatorial problem has been investigated in the recent papers of Bruck, Chowla, and Ryser (Canadian Journal of Mathematics vol. 1 (1949) pp. 88–93 and vol. 2 (1950) pp. 93–99). In the present paper the cyclic solutions are studied. Let v be odd and let t be an arbitrary positive divisor of v. Then a cyclic solution of the v, k, λ problem implies that the Diophantine equation $x^2 = (k - \lambda)y^2 + (-1)^{(t-1)/2}tz^2$ has a solution in integers not all zero. The techniques of M. Hall employed in the study of cyclic projective planes (Duke Math. J. vol. 14 (1947) pp. 1079–1090) are applied to the case $\lambda > 1$. In particular, the concept of a multiplier is introduced, and the existence of multipliers is studied. Their effectiveness in the analysis of the cyclic solutions of the v, k, λ problem is illustrated by a variety of examples.

OHIO STATE UNIVERSITY,
COLUMBUS, OHIO, U. S. A.

THE EFFECT ON THE INVERSE OF A CHANGE IN A MATRIX. PRELIMINARY REPORT

WALTER JACOBS

Let $A = (a_{ij})$ be a real, nonsingular $n \times n$ matrix, and Z be a real $n \times n$ matrix satisfying $N(Z - A) < L(A)$. Here $N(A) = [\sum_{i,j} a_{ij}^2]^{1/2}$, while $L(A)$, $U(A)$ designate the lower and upper bounds of A respectively, i.e., the minimum and maximum lengths of Ax for all unit vectors x. It is proved that

$$\frac{1}{U(A)} - \frac{1}{U(A) + N(Z - A)} \leqq N(Z^{-1} - A^{-1}) \leqq \frac{1}{L(A) - N(Z - A)} - \frac{1}{L(A)}.$$

The proof makes use of some new inequalities for $N(A \pm B)$ and $N(AB)$ in terms of the characteristic roots of $A'A$ and $B'B$. These inequalities are derived with the help of the following:

LEMMA. Let A be a real $n \times n$ matrix satisfying, when $\alpha, \beta \geqq 0$,

$$a_{ij} - a_{i+\alpha,j} - a_{i,j+\beta} + a_{i+\alpha,j+\beta} \geqq 0,$$

for all $i, j = 1, 2, \cdots, n$. Let X be a real $n \times n$ matrix whose elements are non-negative and all of whose rows and columns sum to 1. Then

$$\sum_{i=1}^{n} a_{i,n+1-i} \leqq \sum_{i,j=1}^{n} a_{ij}x_{ij} \leqq \sum_{i=1}^{n} a_{ii}.$$

GEORGE WASHINGTON UNIVERSITY,
WASHINGTON, D. C., U. S. A.

INVARIANTS OF VECTORS WITH NONCOMMUTATIVE COMPONENTS, AND APPLICATION TO GEOMETRY

GEORGE Y. RAINICH

A space V is considered whose vectors are linear combinations with coefficients in a noncommutative ring R of basis vectors of a metric vector space of r dimensions. It is found that a single such vector under transformations of the basis possesses invariants of two types which are elements of R. One invariant is a determinant of r equal rows each row consisting of the components of the given vector. The determinant is defined by an expression analogous to that of determinants with commutative elements in which the order of factors takes place of belonging to different rows. Other invariants are sums of squares of the minors of k rows of the above determinant. For k equal to one this reduces to the ordinary invariant of the vector in the commutative case—the square of the vector. The other invariants of this type vanish in the commutative case. Invariants of another type reduce in the commutative case to powers of the sum of the squares of the components.—As a particular case the elements of R are quadratic forms in another metric space S, multiplication being defined in terms of products of matrices whose elements are the coefficients of the quadratic forms. The invariants introduced above appear then as semi-invariants subjected to transformations of the space S. Expressions invariant under transformations in both V and S are obtained by considering the invariants and simultaneous invariants of the elements of R.—In geometry the normal flat space of a curved space of dimension n plays the part of the space V, the elements of R being quadratic forms in the tangent flat space which are generalizations of the second differential form of the theory of surfaces. When r and n are both two, the invariants are equivalent to those found by other authors.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

ON A GALOIS CONNECTION BETWEEN ALGEBRAS OF LINEAR TRANSFORMATIONS AND LATTICES OF SUBSPACES OF A VECTOR SPACE

ROBERT M. THRALL

Let V be a vector space of dimension n over a sfield k and let N be the lattice of all subspaces of V. A sublattice L of N is said to have the *relative imbedding property* if there exists a complemented modular lattice M with (a) $L \subseteq M \subseteq N$, (b) $\dim L = \dim M$, (c) any quotients of L which are projective in M are already projective in L. Let σ be the linear transformation of the residue class space $S - R$ onto $T - R$ defined by a projectivity of S/R and T/R in L, and let $b \in k$. Then the set Q_b of all vectors lying in any one of the cosets $Q_b(s) =$

$(s + R) + \sigma(s + R)b$ for $s \in S$ is an element of N which we say is *projectively related* to L. L is said to be *projectively closed* in N if L contains every space projectively related to it. With any sublattice L of N we associate the algebra $A = L^+$ of all linear transformations on V under which each element of L is invariant, to each algebra A we associate the lattice $L = A^*$ consisting of all A-subspaces of V. The mappings "$+$" and "$*$" constitute a Galois connection between the sets of lattices L and algebras A on V. A lattice L is said to be *closed* if $L^{+*} = L$ or equivalently if there exists A such that $L = A^*$. The main results of the present paper are (1) closure implies projective closure; (2) if A is cleft, then A^* has the relative imbedding property; (3) every distributive lattice is closed; (4) a complemented lattice L is closed if and only if it is projectively closed. It is not yet known whether projective closure implies the relative imbedding property or whether both of these properties together are sufficient for closure. For both of these questions the answer may depend on the nature of k. The motivation for all of this study is the possibility of applications to some unsolved problems in the theory of representations of algebras.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

CLASSES OF MATRICES AND QUADRATIC FIELDS

OLGA TAUSSKY TODD

A correspondence between ideal classes and matrix classes is applied to the study of the ideal class group in algebraic number fields by rational methods. Quadratic fields in particular and their 2-class group are investigated from this aspect. The classes of order 2 are subdivided into such classes for which the corresponding matrix class contains a symmetric matrix and those for which it does not. The significance of the sign of the norm of the fundamental unit in real quadratic fields is investigated from this point of view.

NATIONAL BUREAU OF STANDARDS,
WASHINGTON, D. C., U. S. A.

THEORY OF FIELDS AND EQUATIONS

ON ALGEBRAIC EQUATIONS WITH ALL BUT ONE ROOT IN THE INTERIOR OF THE UNIT CIRCLE

Alfred T. Brauer

Let C be the class of algebraic equations having all but one root in the interior of the unit circle. Let K be the field of rational numbers or an imaginary quadratic field. Perron [J. Reine Angew. Math. vol. 132 (1907) pp. 288–307] proved that every equation $f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0$ of C with integral rational coefficients and $a_n \neq 0$ is irreducible in K. This holds, in particular, if $|a_1| > 1 + |a_2| + \cdots + |a_n|$. That $f(x)$ belongs to C if this condition is satisfied was already proved by Mayer [Nouveaux Annales de Mathématiques (3) vol. 10 (1891) pp. 111–124]. Berwald [Math. Zeit. vol. 37 (1933) pp. 61–76] improved Mayer's theorem by showing that $f(x)$ belongs to C if $|a_1| > |1 + a_2| + |a_3| + \cdots + |a_n|$. Lipka [Acta Univ. Szeged vol. 5 (1931) pp. 78–82] considered for real a_ν the polygon determined by the points with the Cartesian coordinates $(-1, 0)$, $(0, 1)$, $(1, a_1)$, $(2, a_2)$, $\cdots$, (n, a_n) and proved that $f(x)$ belongs to C if this polygon is concave from below at $(1, a_1)$ and convex at the other vertices. In a later paper [Math. Ann. vol. 118 (1941–1943) pp. 235–245], Lipka remarked that this condition and also the inequality of Berwald are sufficient for irreducibility in K if $a_n \neq 0$. There exist two other criteria for a polynomial to belong to C. But it seems that the corresponding criteria for irreducibility have never been formulated. The polynomial $f(x)$ belongs to C if $a_1 < a_2 < \cdots < a_n < 0$ or if $a_1 - 1 > a_2 > a_3 > \cdots > a_n > 0$. This was proved by Hurwitz [Tôhoku Math. J. vol. 4 (1913–1914) pp. 89–93] and by Berwald in the mentioned paper respectively.

In this paper it will be proved that $f(x) = x^n + \epsilon(a_1 x^{n-1} + a_2x^{n-2} + \cdots + a_n)$ is irreducible in K for $\epsilon = \pm 1$ and rational integers a_ν if $a_1 > a_2 > \cdots > a_n > 0$ except for the case $f(x) = x^2 + a_1x + a_1 - 1$. This result contains the criteria obtained from the theorems of Lipka, Hurwitz, and Berwald as special cases. For $\epsilon = -1$, the weaker condition $a_1 \geqq a_2 \geqq \cdots \geqq a_n$ is sufficient for irreducibility. Some similar theorems will be proved. For instance, $f(x) = x^{2m+1} \pm (a_1x^{2m} + a_2x^{2m-1} + \cdots + a_{2m+1})$ is irreducible in K if $a_1 > a_3 > \cdots > a_{2m+1} > 0$ and $a_2 = a_4 = \cdots = a_{2m} = 0$.

University of North Carolina,
Chapel Hill, N. C., U. S. A.

GÉNÉRALISATION ABSTRAITE DE LA THÉORIE DE GALOIS

MARC KRASNER

Soient E un ensemble et U un ensemble auxiliaire, fixé une fois pour toutes, de puissance ${}_pU \geqq {}_pE$. Les applications $P = \{u \to P\cdot u\}_{u \in U}$ de U dans E seront dits *points* de E, et toute subdivision r de leur ensemble E^U en deux classes $D(r)$ et $\bar{D}(r)$ sera dite une *relation dans* E. $P \in E^U$ sera dit *satisfaire* ou *ne pas satisfaire* à r suivant que $P \in D(r)$ ou $P \in \bar{D}(r)$.

On introduira, pour ces relations, certaines opérations, dites *fondamentales*: 1°. *opposition* $r \to \bar{r}$: $\bar{r}$ est la relation telle que $D(\bar{r}) = \bar{D}(r)$; 2°. *V-projections*: si $V \subseteq U$, on appelle la V-projection de r la relation r_V telle que $P \in D(r_V)$ si et seulement s'il existe un $P' \in D(r)$, dont la restriction à V coïncide avec celle de P; 3°. *changements généralisés de noms de variables*: soit $V \subseteq U$ et soit C une relation d'équivalence dans V; r est dite *compatible* avec C si $r_V = r$ et si, pour tout $P \in D(r)$, $P\cdot u$ $(u \in V)$ ne dépend que de la classe $\zeta(u)$ de u suivant C [$P\cdot u$ sera noté $P\cdot\zeta(u)$]. V_1, V_2 étant $\subseteq U$, et C_1, C_2 étant des équivalences dans les V_1, V_2 telles que V_1/C_1 et V_2/C_2 aient une même puissance, soient ϵ une application biunivoque de V_1/C_1 sur V_2/C_2 et r une relation compatible avec C_1. Soit $r^{(\epsilon)}$ la relation compatible avec C_2 telle que $P \in D(r^{(\epsilon)})$ si et seulement s'il existe un $P' \in D(r)$ tel que, pour tout $\zeta \in V_1/C_1$, on ait $P\cdot(\epsilon\cdot\zeta) = P'\cdot\zeta$. $r \to r^{(\epsilon)}$ est dit le changement généralisé ϵ de noms de variables; 4°. R étant un ensemble de relations, *la prise du p.g.c.d.* $[R]$ de R:$[R]$ est la relation telle que $D([R]) = \bigcap_{r\in R} D(r)$.

Un ensemble R de relations est dit *logiquement fermé* s'il est fermé par rapport aux opérations fondamentales, appliquées à ses éléments ($\bar{r}$, r_V, $r^{(\epsilon)}$) ou sous-ensembles ($[R']$). R étant quelconque, l'intersection R_f de ses surensembles logiquement fermés l'est aussi et est dite la *fermeture logique de* R.

On appelle *structure* $S = (E, R)$ *dans* E l'ensemble E organisé par un ensemble R de relations. S et $S' = (E, R')$ sont dites *équivalentes* si et seulement si $R_f = R'_f$.

σ étant une permutation de E, soient $\sigma\cdot P$ le point tel que $(\sigma\cdot P)\cdot u = \sigma\cdot(P\cdot u)$ et $\sigma\cdot r$ la relation telle que $D(\sigma\cdot r) = \sigma\cdot D(r)$. Si $\sigma\cdot r = r$, on dit que σ *conserve* r. $S = (E, R)$ étant une structure, l'ensemble $G_{E/S}$ (qui est un groupe) des permutations de E, qui conservent toute $r \in R$, est dit le *groupe de Galois de* E/S. J'ai démontré:

LOI D'EXISTENCE. *Quelque soit le groupe G de permutations de E, il existe une structure S dans E telle que $G_{E/S} = G$;*

LOI D'EQUIVALENCE. *Deux structures S et S' dans E sont équivalentes si et seulement si $G_{E/S} = G_{E/S'}$.*

On appelle *corps abstrait* $K(S)$, défini par une structure S, la classe des structures équivalentes à S. L'ensemble des corps abstraits de E ne dépend pas de

U, et $K(S) \to G_{E/S}$ est une correspondance galoisienne de cet ensemble avec celui des groupes de permutations de E.

NATIONAL CENTER OF SCIENTIFIC RESEARCH,
PARIS, FRANCE.

ARITHMETICAL PROPERTIES OF LEMNISCATE POLYNOMIALS

MORGAN WARD

The arithmetical properties of the polynomials associated with the complex multiplication of elliptic functions are developed for the lemniscate case of multiplication by i. Most of the results of Lucas and Sylvester on the cyclotomic polynomials associated with the multiplication of the trigonometric functions extend to this case, and more generally, to any complex multiplication. A new feature is a double numerical periodicity of the polynomials over the ring of Gaussian integers if the basic elliptic function is replaced by a Gaussian integer.

CALIFORNIA INSTITUTE OF TECHNOLOGY,
PASADENA, CALIF., U. S. A.

ON THE MINIMUM MODULUS OF A ROOT OF A POLYNOMIAL

DENNIS P. VYTHOULKAS

MM. Féjer and P. Montel have shown that "the polynomial $P_k(z) = 1 + z + \alpha_1 z^{n_1} + \alpha_2 z^{n_2} + \cdots + \alpha_{k-1} z^{k-1} = 0$ has at least one root whose modulus does not exceed the number

$$n_1/(n_1 - 1) \cdot n_2/(n_2 - 1) \cdots n_{k-1}/(n_{k-1} - 1) = \lambda \leqq k."$$

We can reduce the surface within which there is a minimum modulus of a root of the above polynomial proving that "the polynomial $P_k(z) = 0$ has at least one root within or on the circumference of a circle with centre $-\lambda/2$ and radius $\lambda/2 \leqq k/2$" or "the polynomial $P_k(z) = 0$ of $k + 1$ terms has at least one root within or on the circumference of a circle of centre $-k/2$ and radius $k/2$".

The proof requires the following proposition "For any point z_0 on the plane, there is at least one root of the polynomial $f(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n$ within or on the circle $| z - z_0 | \leqq | f(z_0) |^{1/n}$ as there is one other root outside or on the circumference of the same circle". Indeed if $f(z) = (z - z_1) \cdot (z - z_2) \cdots (z - z_n)$, we find $f(z_0) = (z_0 - z_1)(z_0 - z_2) \cdots (z_0 - z_n)$. Hence for at least one root z_k we have $|z_0 - z_k| \geqq |f(z_0)|^{1/n}$. To prove the theorem and simplify operations let us consider the equation

$$f(z) = 1 + z + \alpha z^m + \beta z^n + \gamma z^p = 0 \qquad (2 \leqq m < n < p).$$

Putting $z = 1/\delta$ we have $\varphi(\delta) = \delta^p + \delta^{p-1} + \alpha\delta^{p-m} + \beta\delta^{p-n} + \gamma = 0$. The derivative is put into the form $\varphi'(\delta) = \delta^{p-n-1}\cdot\varphi_1(\delta)$ where

$$\varphi_1(\delta) = p\delta^n + (p-1)\delta^{n-1} + \alpha(p-m)\delta^{n-m} + \beta(p-n).$$

Similarly we write $\varphi_1'(\delta) = \delta^{n-m-1}\cdot\varphi_2(\delta)$ and $\varphi_2'(\delta) = p\cdot m\cdot n\cdot\delta^{m-2}\cdot(\delta + \theta)$ where $\theta = (p-1)/p\cdot(n-1)/n\cdot(m-1)/m$ putting $\pi(\delta) = \delta + \theta$. Let δ_0 be any point on the x-axis, then by the proposition there is at least one root of $\pi(\delta) = 0$ outside or on the circumference of centre δ_0 and radius $r = |\pi(\delta_0)| = |\delta_0 + \theta|$. By Lucas' theorem and following Féjer's proof we conclude that $\varphi(\delta) = 0$ has at least one root outside or on the circumference with centre δ_0 and radius $r = |\delta_0 + \theta|$. If the origin of the axes is within this circle, then by the substitution $z = 1/\delta$ we conclude that $f(z) = 0$ has at least one root within or on the corresponding circle of centre $\delta_0/(\delta_0^2 - r^2)$ and radius $R = r/(r^2 - \delta_0^2)$. Consequently the choice of the parameter δ_0 must be made in such a manner that (1) the circle of radius r includes the origin, (2) R is a minimum. For (1) $\delta_0 > 0 > -\theta/2$ suffices, and therefore we can put $R = (\delta_0 + \theta)/(\theta^2 + 2\delta_0\theta)$ and seek a positive value for δ_0 for which R is a minimum.

The minimum R is obtained for $\delta_0 \to +\infty$ when

$$R = 1/2\theta = (1/2)\cdot p/(p-1)\cdot n/(n-1)\cdot m/(m-1)$$

and the abscissa of the centre of the circle for $\delta_0 \to \infty$ is $\lim_{\delta_0 \to \infty} \delta_0/(\delta_0^2 - r^2) = -1/2\theta$. Q.E.D.

National University of Engineering,
Athens, Greece.

THEORY OF GAMES

SOLUTION OF POLYNOMIAL-LIKE GAMES

M. Dresher

The theory of infinite games with polynomial payoff function, $M(x, y) = \sum_{i,j=1}^{m,n} a_{ij}x^i y^j$, where x, y are pure strategies of the two players, can be generalized to a much larger class of games whose payoff is given by $M(x, y) = \sum_{i,j=1}^{m,n} a_{ij}r_i(x)s_j(y)$ where the functions r_i and s_j are S-integrable with respect to distribution functions on the interval [0, 1]. Define

$$\alpha_i = \int_0^1 r_i(x)\, dF(x), \qquad \beta_j = \int_0^1 s_j(y)\, dG(y)$$

where F and G are distribution functions. Then the expectation with mixed strategies reduces to the bilinear game

$$\phi(\alpha, \beta) = \sum_{i,j=1}^{m,n} a_{ij}\alpha_i\beta_j$$

where α and β are points in convex sets A and B. The minimax theorem for bilinear forms over convex sets guarantees that

$$\max_{\alpha \epsilon A} \min_{\beta \epsilon B} \phi(\alpha, \beta) = \min_{\beta \epsilon B} \max_{\alpha \epsilon A} \phi(\alpha, \beta).$$

The set A is the convex extension of the curve C in m dimensions defined by $\alpha_i = r_i(t)$; $0 \leqq t \leqq 1$, $i = 1, 2, \cdots, m$. Similarly for set B. Points on C in A are generated by distribution functions having single steps. Points in A correspond to step-functions with at most $m + 1$ steps (m steps if C is connected). From the properties of the solution of a game, it follows that each player has an optimal mixed strategy with at most m steps, or m pure strategies.

Suppose the players are restricted in their choice of mixed strategies—i.e., player I may pick only those F's for which $\int_0^1 h(x)\, dF(x) = 0$. Then we consider the curve in $m + 1$ dimensions defined by

$$\alpha_i = r_i(t), \qquad \alpha_{m+1} = h(t), \qquad 0 \leqq t \leqq 1, i = 1, \cdots, m.$$

The space of strategies for player I are those points in the convex extension of this curve whose $(m + 1)$th coordinate is zero. If this game has a solution, i.e., if the constraint can be satisfied, then some optimal strategy consists of at most m pure strategies. The solution can be obtained by solving a system of algebraic equations, most of them linear.

Using this type of analysis, we can construct a method of computing

$$\max_F \min_G \sum_{k=1}^{p} \int_0^{a_k} \int_0^{b_k} \sum_{i,j=1}^{m,n} a_{ij} r_i(x) s_j(y)\, dF(x)\, dG(y)$$

where $0 \leqq a_k \leqq 1$, $0 \leqq b_k \leqq 1$, $k = 1, \cdots, p$.

THE RAND CORPORATION,
SANTA MONICA, CALIF., U. S. A.

INFORMATION PATTERNS FOR GAMES IN EXTENSIVE FORM

W. D. KRENTEL, J. C. C. MCKINSEY, AND W. V. QUINE

An *information function* for a zero-sum two-person game in extensive form is, intuitively speaking, a function which, for each i, indicates what previous moves are known to the player making the ith move. An *information pattern* is an information function together with a specification which moves are made by which player. Two information patterns α and β are *equivalent* if every two games which differ only in that one has information pattern α, and the other has information pattern β, have the same value. Purely formal (as opposed to intuitive) definitions of these notions are given, and necessary and sufficient conditions are found for equivalence. The conditions found make it possible to construct, for a given game in extensive form, an equivalent game having a minimum number of (pure) strategies.

OKLAHOMA AGRICULTURAL AND MECHANICAL COLLEGE,
STILLWATER, OKLA., U. S. A.
HARVARD UNIVERSITY,
CAMBRIDGE, MASS., U. S. A.

Section II

ANALYSIS

SECTION II. ANALYSIS

A SURVEY OF THE DIFFERENT PROOFS OF THE MAIN THEOREMS IN THE THEORY OF ALMOST PERIODIC FUNCTIONS

HARALD BOHR

1. The theory of almost periodic functions, like any other mathematical theory of a somewhat general character, has several connections with different mathematical disciplines. Thus it has important applications to the functions occurring in the analytic theory of numbers, as the Riemann zeta function, the study of which was the very origin of the development of the theory of almost periodicity; and through the work of Weyl, von Neumann, and others it has given rise to a new chapter in the general theory of groups. In my lecture today, in which I shall try to give you a short survey of the different ways in which some of the main theorems of the theory can be established, several such connections with other mathematical fields will appear indirectly. In the following I shall limit myself to considering functions of a real variable x, $f(x) = u(x) + iv(x)$, and moreover only functions continuous for all x; hence, naturally, the notion of uniform convergence, which preserves continuity, will play an essential role. Besides uniform convergence, another predominant notion will be convergence in the mean.

2. It may be convenient first to say a few words about continuous pure periodic functions $w = f(x)$. Among the periodic movements described by such functions the simplest one is certainly a uniform movement on a circle, as given by a so-called pure oscillation $w = ae^{i\lambda x}$ where λ is a real and a a complex number. In the theory of periodic functions one considers only pure oscillations with a given period p, say $p = 2\pi$, i.e., the oscillations $a_n e^{inx}$ $(n = 0, \pm 1, \pm 2, \cdots)$. By superposition of a finite number of such oscillations we get the exponential polynomials

$$s(x) = a_0 + a_1 e^{ix} + a_{-1} e^{-ix} + \cdots + a_n e^{inx} + a_{-n} e^{-inx}.$$

Rounding off the set of all such polynomials $s(x)$ with the help of uniform convergence, we get the closure $\mathrm{Cl}\{s(x)\}$, the elements of which, evidently, are again continuous periodic functions with period 2π. The famous theorem of Weierstrass states that this set $\mathrm{Cl}\{s(x)\}$ contains all continuous periodic functions $P(x)$ with period 2π. Among the different proofs of this theorem, those based on the theory of Fourier series are of main interest for our purpose. Starting from the fact that the pure oscillations e^{inx}, considered in an interval of length 2π, form a normal orthogonal set, i.e.,

$$M\{e^{in_1x} \cdot e^{-in_2x}\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in_1x} \cdot e^{-in_2x}\, dx = \begin{cases} 0 \text{ for } n_1 \neq n_2 \\ 1 \text{ for } n_1 = n_2 \end{cases},$$

one is led to associate with every one of our functions $P(x)$ an infinite series

$$P(x) \sim \sum_{-\infty}^{\infty} A_n e^{inx}$$

as its Fourier series, namely the series with the coefficients $A_n = M\{P(x)e^{-inx}\}$. Further, one sees, with the help of Bessel's inequality, that the series $\sum |A_n|^2$ is convergent to a sum $\leqq M\{|P(x)|^2\}$. A fundamental theorem, the Parseval theorem, states that we always have the sign of equality, i.e., that

$$\sum |A_n|^2 = M\{|P(x)|^2\}$$

which, on account of Bessel's formula, is equivalent with the mean convergence of the Fourier series towards $P(x)$, i.e.,

$$M\left\{\left|P(x) - \sum_{-N}^{N} A_n e^{inx}\right|^2\right\} \to 0 \quad \text{as} \quad N \to \infty.$$

The usual way of proving this mean convergence, or the Parseval equation, is first to prove the much stronger Weierstrass theorem concerning uniform approximation, and this may be done by one or another suitable summation method applied to the—generally divergent—Fourier series of $P(x)$. In particular, as shown by Fejér, the application of the simple kernel

$$K_n(t) = \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n}\right) e^{i\nu t}$$

will lead to exponential polynomials

$$s_n(x) = \underset{t}{M}\{P(x + t)K_n(t)\} = \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n}\right) a_\nu e^{i\nu x}$$

which converge uniformly to $P(x)$.

3. We now pass to the main subject, the a.p. functions. As above we start from the pure oscillations $ae^{i\lambda x}$, but now we consider all of them, and not only a denumerable subset having a given number as a common period. Again we consider superpositions, i.e., exponential polynomials

$$s(x) = a_1 e^{i\lambda_1 x} + a_2 e^{i\lambda_2 x} + \cdots + a_n e^{i\lambda_n x},$$

but this time with arbitrary real exponents—so that the functions are no longer periodic—, and again we form the closure $\mathrm{Cl}\{s(x)\}$ of the set of all these exponential polynomials with respect to uniform convergence for all x. A main problem then presents itself: to characterize the functions of this closure $\mathrm{Cl}\{s(x)\}$ by means of structural properties, in analogy with and as a generalization of the classical Weierstrass theorem for functions with a given period. Let me at once remind you of the answer, which is that the function shall be what I have called "almost periodic", the exact definition being the following: A continuous function $f(x)$ is called an a.p. function if for every given $\epsilon > 0$ there exists an infinity of translation numbers or almost periods belonging to ϵ, i.e., numbers $\tau = \tau(\epsilon)$ such that

$$|f(x + \tau) - f(x)| \leqq \epsilon \qquad \text{for } -\infty < x < \infty,$$

and, moreover, if for any fixed ϵ the set of these translation numbers $\{\tau(\epsilon)\}$ is relatively dense in the sense that each sufficiently large interval on the real axis contains at least one of these numbers.

That every function in the closure $\mathrm{Cl}\{s(x)\}$ is an a.p. function is fairly easy to prove. The difficulty lies in proving the converse, namely that every a.p. function can really be approximated uniformly by exponential polynomials $s(x)$. Here—in contrast to the pure periodic case where the exponents are given beforehand—the problem arises how to force the given a.p. function to deliver its characteristic exponents, i.e., the exponents to be used in the approximating polynomials. This is done by attaching to every a.p. function a certain infinite series as its Fourier series. In this general case also, the starting point is that the set of our pure oscillations, i.e., here the non-denumerable set $\{e^{i\lambda x}\}$ form a normal orthogonal set, but now of course with the mean value taken over the whole infinite interval, i.e.,

$$M\{e^{i\lambda_1 x}\cdot e^{-i\lambda_2 x}\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i\lambda_1 x}\cdot e^{-i\lambda_2 x}\,dx = \begin{cases} 0 \text{ for } \lambda_1 \neq \lambda_2 \\ 1 \text{ for } \lambda_1 = \lambda_2. \end{cases}$$

For a given a.p. function $f(x)$, and an arbitrary real λ, we then naturally ask whether the pure oscillation $e^{i\lambda x}$ is resonant with $f(x)$, i.e., we form the mean value

$$a(\lambda) = M\{f(x)e^{-i\lambda x}\}$$

which may be shown to exist for every real λ. And rather easily we find the important result that this mean value differs from 0 for at most a denumerable set of values $\lambda = \Lambda_1, \Lambda_2, \cdots, \Lambda_n, \cdots$ (which may very well be everywhere dense on the real axis). With these numbers Λ_n as exponents and the corresponding values $a(\Lambda_n) = A_n$ as coefficients we form the infinite series, the Fourier series of $f(x)$,

$$f(x) \sim \sum A_n e^{i\Lambda_n x}.$$

4. Now, in analogy with the pure periodic case, we naturally ask whether the Fourier series of an a.p. function $f(x)$ also converges in mean to the function $f(x)$, i.e., whether

$$M\left\{\left|f(x) - \sum_1^N A_n e^{i\Lambda_n x}\right|^2\right\} \to 0 \quad \text{for} \quad N \to \infty,$$

and here again this question is equivalent to asking whether the infinite series $\sum |A_n|^2$, which is easily seen to converge to a sum $\leqq M\{|f(x)|^2\}$, is always equal to $M\{|f(x)|^2\}$, i.e., whether the Parseval equation

$$\sum |A_n|^2 = M\{|f(x)|^2\}$$

holds true for every a.p. function. That this is really the case is the clue to the whole theory, and constitutes the so-called fundamental theorem. Briefly I shall try to indicate the several proofs of this theorem given by various authors. Let me add that the Parseval equation immediately implies another fundamental result, the unicity theorem, stating that two different a.p. functions always have two different Fourier series, or, that the only a.p. function whose Fourier series is empty is the function $f(x)$ identically 0. In fact, in case of an empty Fourier series, the Parseval equation states that $M\{|f(x)|^2\} = 0$ and, luckily, on account of the almost periodicity, this implies that $f(x)$ must be identically 0.

5. In the original proof of the Parseval equation for an a.p. function the underlying idea was a very simple one, whereas the details were rather complicated. It consisted in reducing the general case to the special pure periodic case by considering the periodic function $f_T(x)$ with the period T, which coincides with the given a.p. function $f(x)$ in the period interval $0 < x < T$, next applying the Parseval equation to this periodic function $f_T(x)$, and finally—what was of course the difficult part of the proof—carrying out the limit process $T \to \infty$. An essentially simplified proof along these lines (i.e., to start by approximating the given function $f(x)$ by the periodic function $f_T(x)$) was given later on by de la Vallée Poussin. Both my original proof and the proof of de la Vallée Poussin have lately been further simplified by Jessen who, instead of working with the periodic function $f_T(x)$, uses the function $g_T(x)$ which like $f_T(x)$ is equal to $f(x)$ in the interval $0 < x < T$ but vanishes outside this interval. To this function $g_T(x)$ he applies the theory of Fourier integrals, which turns out to be better adapted for the purpose than the Fourier series of the periodic function $f_T(x)$.

6. The greater simplicity of de la Vallée Poussin's proof, as compared with my own, was due mainly to his application of the important process of convolution of two a.p. functions, a notion implicitly applied by Norbert Wiener in his proof of the Parseval equation, to which I shall return later on. This notion also is the basis of the proof of Weyl, of which I shall speak in a moment. In the special but particularly important case of the convolution of an a.p. function $f(x) \sim \sum A_n e^{i\Lambda_n x}$ with the a.p. function $\bar{f}(-x) \sim \sum \bar{A}_n e^{i\Lambda_n x}$ we form the mean value

$$F(x) = \underset{t}{M}\{f(x+t)\bar{f}(t)\},$$

to be denoted briefly as the "convolution of $f(x)$", which is easily seen again to be a.p. and to have the absolutely convergent Fourier series with positive coefficients

$$F(x) \sim \sum |A_n|^2 e^{i\Lambda_n x}.$$

By means of this convolution process it is easily seen that the unicity theorem is not only a consequence of, but in fact equivalent with, the Parseval equation. Further, we find that the proof of either of these two theorems is again equivalent to proving that in the above relation the sign $\sim$ may simply be replaced by the sign $=$, i.e., that the (absolutely convergent) Fourier series of $F(x)$ has the function $F(x)$ itself as its sum.

7. Now we turn to Weyl's proof of the fundamental theorem, say of the unicity theorem. The basis of this proof is an important connection between the classical theory of ordinary Fourier series for pure periodic functions on the one hand and the theory of integral equations on the other. This connection could easily be extended to the general case of a.p. functions, the integral equations in question being simply replaced by mean value equations of a quite similar type. His starting point was the following: If in the equation

$$a(\lambda) = M\{f(x)e^{-i\lambda x}\}$$

serving to determine the terms of the Fourier series of the a.p. function $f(x)$, we replace x by $x - y$, taking y as the variable and x as a constant, we get immediately, on account of the functional equation $e^{-i\lambda(x-y)} = e^{-i\lambda x}e^{i\lambda y}$, the mean value equation

$$a(\lambda)e^{i\lambda x} = \underset{y}{M}\{f(x - y)e^{i\lambda y}\}$$

or, if we replace $a(\lambda)$ by γ and $e^{i\lambda x}$ by $\varphi(x)$, the mean value equation

$$\gamma\varphi(x) = \underset{y}{M}\ \{f(x - y)\varphi(y)\}.$$

Hence we see that it is the same thing to say that $A_n e^{i\Lambda_n x}$ is a term of the Fourier series of $f(x)$ as to say that the mean value equation with the kernel $K(x, y) = f(x - y)$ has $\gamma = A_n$ as a characteristic value and $\varphi(x) = e^{i\Lambda_n x}$ as a corresponding characteristic function. Thus the content of the unicity theorem is equivalent with saying that our mean value equation, in case of an a.p. function $f(x)$ not identically 0, has a characteristic value $\neq 0$, and a pure oscillation as a corresponding characteristic function. As could be expected, rather than working with $f(x)$ itself, it is more convenient to work with its convolution $F(x)$, since in case of a non-empty Fourier series, the Fourier coefficients $|A_n|^2$ of $F(x)$ are all real and positive. Thus we consider the equation

$$\gamma\varphi(x) = \underset{y}{M}\{F(x - y)\varphi(y)\}.$$

On account of the almost periodicity of the kernel, this mean value equation may be treated practically as if it were an ordinary integral equation (where the mean value is to be taken over a finite interval). In fact the classical method of Erhard Schmidt could be applied almost unchanged to the mean value equation in question and led to the result that if $f(x)$ (and hence $F(x)$) is not identically 0, there always exists a positive characteristic value $\gamma = \gamma_0$ and a corresponding characteristic a.p. function $\varphi(x) = \varphi_0(x)$ not identically 0. However, this result is not yet the unicity theorem, as it would have been if the characteristic function

$\varphi_0(x)$ were known to be a pure oscillation and not merely an a.p. function. In order to complete the proof we naturally have to use the fact that the kernel $K(x, y) = F(x - y)$ depends only on the difference $x - y$; this immediately gives that together with $\varphi_0(x)$ every translated function $\varphi_0(x + h)$, where h is an arbitrary real number, is again a characteristic function belonging to the characteristic value γ_0. If now the space of all the characteristic functions belonging to γ_0 has the dimension 1, everything is easy; we then have the simple relation $\varphi_0(x + h) = c(h)\varphi_0(x)$ for all h, from which it follows that $\varphi_0(x)$ must be a pure oscillation. If, however, the space of our characteristic functions belonging to γ_0 has a dimension m which, though certainly finite, is >1 (this will really happen if in the Fourier series of $F(x)$ there is more than one term for which the coefficients $| A_n |^2$ is equal to γ_0), a certain difficulty appears. This is due to the fact that if $\varphi_1(x), \cdots, \varphi_m(x)$ is a normal orthogonal basis for the functional space in question, we cannot conclude that, for each ν, $\varphi_\nu(x + h) = c_\nu(h)\varphi_\nu(x)$, but only that $\varphi_\nu(x + h)$ is a linear combination $\varphi_\nu(x + h) = c_{\nu 1}\varphi_1(x) + \cdots + c_{\nu m}\varphi_m(x)$ of the m functions $\varphi_1(x), \cdots, \varphi_m(x)$. This difficulty was overcome by Weyl by the following simple group-theoretical consideration. Since, together with $\varphi_1(x), \cdots, \varphi_m(x)$, the translated system $\varphi_1(x + h), \cdots, \varphi_m(x + h)$ is, evidently, again a normal orthogonal basis for our functional space, the corresponding matrix $M = \{c_{\nu\mu}\}$ is, for each h, a unitary matrix, and further as these matrices $M = M(h)$ $(-\infty < h < \infty)$ form a commutative group (commutative simply because $h_1 + h_2 = h_2 + h_1$) they may, by a classical theorem of Frobenius, be transformed, simultaneously for all h, into matrices of the diagonal form with mere zeros outside the main diagonal. This means, however, that we may choose the coordinate system in our m-dimensional space, i.e., the basic orthogonal system $\varphi_1(x), \cdots, \varphi_m(x)$, in such a way that for each $\nu = 1, \cdots, m$ we have, just as in the one-dimensional case, a simple relation of the form $\varphi_\nu(x + h) = c_\nu(h)\varphi_\nu(x)$ and thus may conclude that the functions $\varphi_\nu(x)$ are really pure oscillations. It may be added that an interesting variant of Weyl's proof was given by Hammerstein who, in analogy with classical methods in the theory of ordinary integral equations, treated the mean value equation in question by means of the direct methods from the calculus of variations. Further, it may be mentioned that several interesting artifices have been indicated, from various sides, to avoid the explicit use of group-theoretical considerations in the last step of the proof.

8. We now proceed to the interesting proof of the Parseval equation due to Norbert Wiener—the first proof of the fundamental theorem to follow the original one—as a part of his important general theory of harmonic analysis. I shall give a short account of this proof in the simplified and very beautiful form given by Bochner. As in Weyl's proof it is essential to treat not the a.p. function $f(x)$ itself, but its convolution

$$F(x) = \underset{t}{M}\{f(x + t)\bar{f}(t)\}.$$

As emphasized above, the task is to prove that the convergent Fourier series $\sum | A_n |^2 e^{i\Lambda_n x}$ of $F(x)$, with positive coefficients, has the function $F(x)$ as its sum, and this is easily seen to be equivalent with showing that the function $F(x)$ may be expressed by a Stieltjes integral

$$F(x) = \int_{\alpha=-\infty}^{\alpha=\infty} e^{i\alpha x} \, dD(\alpha)$$

where $D(\alpha)$ is a bounded monotonically increasing function which is completely discontinuous, i.e., increases only through a denumerable number of jumps (namely the jumps $| A_n |^2$ in the points Λ_n). Now, more generally, we consider functions $G(x)$ which do not possess (as we shall prove about $F(x)$) only a discontinuous spectrum, but the spectrum of which may be partly discontinuous and partly continuous, i.e., functions $G(x)$ which may be represented as a Stieltjes integral

$$G(x) = \int_{-\infty}^{\infty} e^{i\alpha x} \, dV(\alpha)$$

where $V(\alpha)$ is a quite arbitrary bounded increasing function of α. Splitting $V(\alpha)$, in the usual way, into two monotonic components, $V(\alpha) = D(\alpha) + C(\alpha)$ where $D(\alpha)$ is completely discontinuous and $C(\alpha)$ continuous for all α, we get the function $G(x)$ split into two functions

$$G(x) = \int_{-\infty}^{\infty} e^{i\alpha x} \, dD(\alpha) + \int_{-\infty}^{\infty} e^{i\alpha x} \, dC(\alpha) = G_D(x) + G_C(x).$$

Here $G_D(x)$ has a pure discontinuous spectrum and hence has a strong structural property, namely that of almost periodicity, whereas $G_C(x)$ has a pure continuous spectrum and hence behaves in quite another way, namely has a tendency to tend to 0 for x tending to $\pm\infty$, or, more precisely, satisfies the mean value equation

$$M\{| G_C(x) |^2\} = 0.$$

For an arbitrary function $G(x)$ of the above type it is immediately shown (simply with the help of the functional equation of the exponential function) that it is a so-called positive definite function, a notion first introduced by Mathias. By this is meant that $G(x)$ is a continuous bounded function, of Hermitian character, $\bar{G}(-x) = G(x)$, which satisfies the condition that the quadratic form

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{m} G(x_\mu - x_\nu) \rho_\mu \bar{\rho}_\nu$$

is $\geqq 0$ for arbitrary real numbers $x_1, \cdots, x_m$ and complex numbers $\rho_1, \cdots, \rho_m$. Now Bochner shows, by means of the theory of Fourier transforms, that the property just mentioned is characteristic of the functions $G(x)$ in question, i.e., that conversely, every positive definite function may be expressed by a Stieltjes integral of the form in question. The proof then proceeds in the following manner. For a rather large class of functions $g(x)$, including the a.p. functions $f(x)$ as a special case, it is easily shown by direct considerations that their convolutions

$$G(x) = \underset{t}{M}\{g(x + t)\bar{g}(t)\}$$

possess all the properties demanded by the definition of a positive definite function. Hence each of these convolutions $G(x)$ is a function of the kind considered above and therefore may be written as a sum of two functions

$$G(x) = G_D(x) + G_C(x)$$

where $G_D(x)$ is almost periodic while $G_C(x)$ has a purely continuous spectrum. The remainder of the proof follows automatically. In fact, if the function $g(x)$ from which we started is just an a.p. function $f(x)$, we get for its convolution $F(x)$ the expression

$$F(x) = F_D(x) + F_C(x),$$

where the first term $F_D(x)$ is the almost periodic part of $F(x)$. Our task is to show that the second term $F_C(x)$ is identically 0, i.e., that the spectrum of $F(x)$ contains no continuous part. But this is evident, as on the one hand, as mentioned above, $M\{|F_C(x)|^2\}$ is $= 0$ while on the other hand $F_C(x)$, as the difference between the two a.p. functions $F(x)$ and $F_D(x)$, is itself an a.p. function, and an a.p. function for which the mean value of its numerical square is 0 must necessarily, as emphasized above, be identically 0.

9. Although the fundamental theorem dealing with the mean convergence of the Fourier series may be said to be the clue to the whole theory, it is not really the main theorem—at any rate not as long as we consider only the ordinary continuous a.p. functions. As the main theorem must naturally be considered the approximation theorem dealing with uniform convergence since it gives the exact characterization of our class of functions. Before finishing my lecture with a short review of one more interesting proof of the fundamental theorem, due to Bogolioùboff, I should like to say some few words concerning the various proofs of the approximation theorem. The starting point of my original proof was a consideration of the Fourier exponents of the function from an arithmetical point of view, namely, their representation by means of a so-called basis, i.e., as linear combinations of rationally independent numbers, generally infinitely many, and generally with rational (and not just integral) coefficients. Led by the Kronecker theorem on Diophantine approximation, and in generalization of ideas already used by Bohl in the special case of a finite integral basis, and by the lecturer in his studies of the value-distribution of Dirichlet series, a function of more variables was introduced, the "spatial extension" of $f(x)$, which is generally a so-called limit periodic function $L(x_1, x_2, \cdots)$ of infinitely many variables $x_1, x_2, \cdots$, and from which the given function $f(x)$ itself could again be obtained by considering the function $L(x_1, x_2, \cdots)$ on the main diagonal $x_1 = x_2 = \cdots = x$ of the infinite-dimensional space. For these limit periodic functions $L(x_1, x_2, \cdots)$ a theory of Fourier series was then established, and in particular an approximation theorem, based on Fejér kernels in several variables, was established, leading to exponential polynomials $s(x_1, x_2, \cdots)$ of more and more variables which tended uniformly to the function $L(x_1, x_2, \cdots)$; finally, exponential polynomials $s(x)$ in one variable approximating the given a.p. function $f(x)$ were then obtained by considering the approximating polynomials $s(x_1, x_2, \cdots)$ on the main diagonal of the space. A very interesting turn and a substantial simplification of this proof of the approximation theorem on arithmetic lines was given by Bochner who realized that one could arrive at the approximating polynomials $s(x)$ in question without the spatial extension of the function $f(x)$, and thus completely avoiding the use of functions of infinitely many variables which are, however, indispensable for the treatment of other problems in the theory of almost periodic functions. Bochner constructed composed kernels simply through multiplication of suitably chosen Fejér kernels, and then in the usual way formed the convolution $s(x) = M\{f(x + t)K(t)\}$. Another proof of the approximation theorem, on quite other lines, was given by Weyl who constructed kernels not, as Bochner, with the help of the exponents of the Fourier series, but by means of the translation numbers of the function. Another very beautiful proof on these lines was later given by Wiener who, by the construction of his kernels, made use of the so-called translation function of the given a.p. function, a function implicitly introduced already in the original proof of the fundamental theorem and studied in detail by Bochner. As to the two purest proofs of the approximation theorem, the Bochner proof and the Wiener proof, they may be considered as being of rather opposite character—and each of them having its particular interest—so far as the first is a general summation method, the same for all a.p. functions with the same set of Fourier exponents, while the latter is of a more individual character, especially adjusted to every single a.p. function.

10. And now, finally, some few words about Bogolioùboff's proof of the fundamental theorem, the latest one to appear and in some respect perhaps the most elementary one. It consists in directly showing that the translation numbers of an a.p. function possess some kind of arithmetical properties (connected with the Fourier exponents of the function) which hitherto could be deduced only by use of the fundamental theorem but of which it was early recognized that their establishment would be sufficient for the proof of the latter theorem. In Bogolioùboff's proof of these properties of the translation numbers the following very interesting and rather surprising general lemma, belonging to the additive theory of numbers, is the main tool. Let E be a quite arbitrary relatively dense set of integers and let E^* be the set of all integers of the form $n_1 + n_2 - n_3 - n_4$ where the n's belong to E; then E^* will always contain a "Diophantine set", i.e., a set consisting of all integral solutions of a finite number of Diophantine inequalities of the form

$$|\lambda_\nu x| < \delta \pmod 1 \qquad (\nu = 1, \cdots, N).$$

It may be interesting to observe that Bogolioùboff's proof of this beautiful lemma—which has such an important application to the theory of a.p. functions and their Fourier series—is itself based on the theory of Fourier series, but in its most elementary, so to say embryonic, state where the variable runs only over a finite number of values and everything, including the Parseval equation, the unicity theorem, the process of convolution, etc., is quite elementary since no limit process whatsoever is involved.

11. I am now at the end of my lecture. I am quite aware that I have not given you any deeper insight into the different proofs in question since, for instance, all the more difficult parts of the proofs have had to be omitted. The task I have set myself is a more modest one, only to try to give you some general ideas about the essentially different ways in which the theory in question may be established.

University of Copenhagen,
Copenhagen, Denmark.

QUELQUES THÉORÈMES D'UNICITÉ[1]

S. Mandelbrojt

Il y a quelques années nous avons démontré une inégalité portant sur les coefficients d'une série "asymptotiquement" liée à une fonction holomorphe dans une bande. L'application de cette inégalité aux problèmes du prolongement analytique, à la théorie générale de la quasi-analyticité et aux problèmes gravitant autour de celui de Watson nous a permis, à l'époque, de trouver des résultats d'un caractère essentiellement plus général que les résultats classiques. Quelques-uns de ces résultats ont été exposés devant l'American Mathematical Society [1].

Or cette même inégalité nous a permis depuis d'aborder des questions relevant d'autres branches d'Analyse. Elles concernent en particulier l'approximation polynomiale sur l'axe entier, différents problèmes des moments, la détermination des noyaux itérés d'une équation intégrale du type de Carleman [2].

Les problèmes en question, ainsi d'ailleurs que le problème général de la quasi-analyticité dont il a été question plus haut et auquel nous reviendrons plus tard, portent tous un caractère d'unicité.

Commençons d'abord par énumerer quelques problèmes et résultats classiques.

1°. Le théorème de S. Bernstein: Soit $\{\beta_n\}$ une suite positive croissante telle que $\sum \beta_n^{-1} = \infty$, $\sum \beta_n^{-2} < \infty$, et posons

$$F(x) = \prod \left(1 + \frac{x^2}{\beta_n^2}\right).$$

Quelle que soit la fonction $f(x)$, continue sur toute la droite, avec $\lim_{|x|=\infty} f(x) \,/\, F(x) = 0$, à tout $\epsilon > 0$ correspond un polynome $P(x)$ tel que

$$| f(x) - P(x) | < \epsilon F(x) \qquad (-\infty < x < \infty).$$

2°. Soit $\{m_n\}$ une suite telle que le problème de Stieltjes—trouver une fonction croissante $V(t)$ $(t \geqq 0)$ telle que $\int_0^\infty t^n dV(t) = m_n$ $(n \geqq 0)$—admette une solution. La condition $\sum m_n^{-1/2n} = \infty$ est suffisante pour que cette solution soit unique ("substantiellement unique"). Ce théorème est dû à Carleman.

3°. Un théorème semblable pour le problème des moments de Hamburger.

4°. Si $\{M_n\}$ est une suite positive avec $\lim M_n^{1/n} = \infty$ (le cas où $\liminf M_n^{1/n} < \infty$ est trivial), une condition nécessaire et suffisante pour que de $| f^{(n)}(x) | \leqq M_n$ $(n \geqq 0)$, $f^{(n)}(0) = 0$ $(n \geqq 0)$ résulte que $f(x) \equiv 0$ est que $\sum M_n^c / M_{n+1}^c = \infty$. Ici $\{\log M_n^c\}$ est la régularisée convexe de la suite $\{\log M_n\}$. Ce théorème, sous une forme un peu différente, est dû à Carleman. C'est le théorème classique

[1] Cette communication était mentionnée sur le programme imprimé sous le titre *Théorèmes d'unicité de la théorie des fonctions*.

de la quasi-analyticité. Une prémière condition suffisante, générale, a été donnée par Denjoy.

Nous généraliserons ces problèmes et résultats. Mais nous désirons tout d'abord envisager sous un nouvel aspect les questions traitées dans 1° et 4°.

Si $f \in L$ sur $(-\infty, \infty)$ (on écrira $\| f \| = \int_{-\infty}^{\infty} | f(x) | \, dx$, on suppose donc $\| f \| < \infty$) on désignera par $\pi(f)$ le sous-ensemble de L sur $(-\infty, \infty)$ dont les éléments φ jouissent de la propriété suivante: à tout $\epsilon > 0$ correspondent un entier N et des constantes $a_1, a_2, \cdots, a_N, \xi_1, \xi_2, \cdots, \xi_N$ tels que

$$\| \varphi(x) - \sum_1^N a_n f(x + \xi_n) \| < \epsilon.$$

Soit maintenant f une fonction indéfiniment dérivable sur $(-\infty, \infty)$, telle que $f^{(n)} \in L$ $(n \geqq 0)$. On désignera par $\omega(f)$ le sous-ensemble de L (sur $(-\infty, \infty)$) dont les éléments φ jouissent de la propriété suivante: à tout $\epsilon > 0$ correspondent un entier N et des constantes $b_1, b_2, \cdots, b_N$ tels que

$$\| \varphi(x) - \sum_1^N b_n f^{(n)}(x) \| < \epsilon.$$

Si $\{M_n\}$ est une suite positive, on désignera par $L\{M_n\}$ l'ensemble de toutes les fonctions indéfiniment dérivables sur $(-\infty, \infty)$ pour lesquelles $\| f^{(n)} \| \leqq M_n$ $(n \geqq 0)$.

On voit facilement que si $f^{(n)} \in L$ $(n \geqq 0)$ on a l'inclusion $\omega(f) \subset \pi(f)$. On a donc aussi $\omega(f') \subset \pi(f')$. Mais, en général, la relation $\pi(f') \subset \omega(f')$ n'a pas lieu. On peut néanmoins démontrer le théorème suivant:

5°. *Une condition nécessaire et suffisante pour que l'on ait pour chaque fonction f de $L\{M_n\}$ la relation*

$$\omega(f') \equiv \pi(f')$$

est que la classe $L\{M_n\}$ soit quasi-analytique, c'est-à-dire, que l'une des deux conditions suivantes soit satisfaite: a) $\liminf M_n^{1/n} < \infty$, b) $\lim M_n^{1/n} = \infty$, *avec*

$$\sum \frac{M_n^c}{M_{n+1}^c} = \infty.$$

Ainsi, donc, si φ est, dans la topologie de L, un point-limite des combinaisons linéaires des translatées de f', φ est aussi un point-limite des combinaisons linéaires des dérivées $f^{(n)}$ $(n \geqq 2)$.

Désignons par ϕ la transformée de Fourier de φ et par G la transformée de Fourier de f. La fonction $iuG(u)$ est alors la transformée de Fourier de f'. Il résulte du Taubérien général de Wiener, très légèrment généralisé, que si G ne s'annule pas et si $\phi(0) = 0$, φ est un point-limite des translatées de f'. Si donc $\lim M_n^{1/n} = \infty$ avec $\sum M_n^c / M_{n+1}^c = \infty$, et si $f \in L\{M_n\}$, il résulte de 5° qu'à tout $\epsilon > 0$ correspondent un entier N et des constantes $c_1, c_2, \cdots, c_N$ tels que

$$\text{(1)} \qquad \operatorname*{Max}_u \left| \phi(u) - \sum_2^N i^n c_n u^n G(u) \right| \leqq \left\| \varphi - \sum_2^N c_n f^{(n)} \right\| < \epsilon.$$

Cette inégalité fournit, on le voit immédiatement, un théorème d'un caractère plus général que le théorème 1°, car, d'une part, en posant $F(u) = 1/G(u)$ (la fonction f de 1° devient ici $\phi(u)F(u)$) on voit qu'on demande beaucoup moins de la régularité de F et, d'autre part, on obtient ici une inégalité (entre le second et le troisième membre de (1)) qui est plus précise que l'inégalité correspondante (entre le premier et le troisième membre de (1)) dans 1°.

Dans ce qui suit nous généralisons les problèmes donnés dans 1°, 2°, 3°, 4°, 5°.

Soit $\{\lambda_n\}$ une suite positive croissante. La manière dont les λ_n sont distribués sera caractérisée par les fonctions suivantes:

$$N(\lambda) = \sum_{\lambda_n<\lambda} 1, \qquad D(\lambda) = N(\lambda)/\lambda, \qquad D^*(\lambda) = \text{borne}\sup_{x \geqq \lambda} D(x),$$

$$\nu(D) = \text{borne}\sup_{\lambda>0} \int_0^\lambda (D(x) - D)\, dx.$$

On posera $D^* = \limsup D(\lambda) = \lim D^*(\lambda)$. Il sera aussi utile d'introduire les fonctions et quantités moyennes, à savoir:

$$\bar{D}(\lambda) = \lambda^{-1} \int_0^\lambda D(x)dx, \qquad \bar{D}^*(\lambda) = \text{borne}\sup_{x \geqq \lambda} \bar{D}(x),$$

$$\bar{D}^* = \limsup \bar{D}(\lambda) = \lim \bar{D}^*(\lambda).$$

Nous supposerons que $D < \infty$. Soit $p(\sigma)$ une fonction croissante tendant vers $+\infty$, et soit a une constante, $a > \bar{D}^*$. S'il existe une fonction continue non-croissante $h(u)$ avec $\lim h(u) \geqq \bar{D}^*$, et une fonction non-décroissante $C(u)$ telles que (pour σ grand)

$$2\nu[h(\sigma)] - p(\sigma) < -C(\sigma), \qquad \int^\infty C(\sigma) \exp\left[-\frac{1}{2}\int^\sigma du/(a - h(u))\right] d\sigma = \infty,$$

on dira que la condition d'unicité $\mathrm{U}(\lambda_n, p(\sigma), a)$ est satisfaite.

La condition d'unicité $\mathrm{U}(\lambda_n, p(\sigma), a)$ est satisfaite si, par exemple, il existe une constante positive γ telle que:

$$\int^\infty p(\sigma) \exp\left[-\frac{1}{2}\int^\sigma du \Big/ (a - D^*(\gamma p(u))\right] d\sigma = \infty,$$

$$a > D^*.$$

La condition $\mathrm{U}(\lambda_n, p(\sigma), a)$ est encore satisfaite si dans les relations précédentes D^* est remplacé par $\bar{D}^*$ (c'est-à-dire, si l'expression $D^*(\gamma p(u))$ est remplacée par l'expression $\bar{D}^*(\gamma p(u))$ et si $a > D^*$ est remplacé par $a > \bar{D}^*$).

On peut démontrer les théorèmes suivante [2].

I. *Soit $\{\nu_n\}$ une suite d'entiers positifs croissants, et désignons par $\{\lambda_n\}$ la suite d'entiers positifs croissants complémentaires à la suite $\{\nu_n\}$ par rapport à la suite de tous les entiers positifs. Soit $F(x)$ une fonction positive, continue, croissante, paire, $p(\sigma) = \log F(e^\sigma)$ étant une fonction convexe de σ. Supposons que la condition*

$U(\lambda_n, p(\sigma), 1/2)$ *soit satisfaite. Soit $f(x)$ une fonction continue sur la droite entière, telle que*

$$\lim_{|x|=\infty} \frac{f(x)}{F(x)} = 0.$$

Quel que soit $\epsilon > 0$, il existe un polynome de la forme

$$P(x) = a_0 + a_1 x^{\lambda_1} + \cdots + a_n x^{\lambda_n}$$

tel que

$$| f(x) - P(x) | < \epsilon F(x) \qquad (-\infty < x < \infty).$$

Primitivement les hypothèses de notre théorème contenaient aussi une condition de dérivabilité de F. Cette hypothèse supplémentaire a pu être supprimée grâce aux remarques de MM. Horvath et Agmon.

Il résulte immédiatement de ce théorème que si $F(x)$ est une fonction continue, positive, paire, $\log F(x)$ étant une fonction convexe de $\log x$ ($x > 0$), et si

$$(\alpha) \qquad \int^{\infty} \frac{\log F(x)}{x^2}\, dx = \infty,$$

alors, quelle que soit $f(x)$, continue sur la droite entière, avec $\lim_{|x|=\infty} f(x)/F(x) = 0$, à tout $\epsilon > 0$ correspond un polynome $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ tel que

$$| f(x) - P(x) | < \epsilon F(x).$$

On voit, en effet, que la condition $U(n, \log F(e^{\sigma}), 1/2)$ est alors satisfaite.

Or la fonction $F(x)$ figurant dans 1° satisfait certainement aux conditions qu'on vient de préciser ($\log F(x)$ est une fonction convexe de $\log x$, $F(x)$ étant le module-maximum d'une fonction entière, la condition (α) résulte de la divergence de $\sum \beta_n^{-1}$). Ainsi donc le théorème I est plus général que le théorème 1°, même lorsqu'on n'envisage que le cas particulier $\lambda_n = n$ ($n \geqq 1$). Remarquons d'ailleurs que, dans ses travaux, S. Bernstein a bien pressenti que la forme particulière qu'il a donnée à sa fonction $F(x)$ ($F(x) = \prod(1 + x^2\beta_n^{-2})$ était superflue.

II. *Soit $\{\nu_n\}$ une suite d'entiers positifs croissants, et soit $\{\lambda_n\}$ la suite complémentaire à la suite $\{\nu_n\}$ par rapport à la suite de tous les entiers positifs. Soit $\{m_n\}$ une suite réelle. Supposons qu'il existe une fonction non-décroissante $V = V(t)$ ($t \geqq 0$), $V(0) = 0$, telle que*

$$\int_0^{\infty} dV = m_0, \qquad \int_0^{\infty} t^{\nu_n}\, dV = m_n \qquad (n \geqq 1).$$

Si, en posant

$$p(\sigma) = \text{borne sup}_{q \geqq 1} (\nu_q \sigma - \log m_q)$$

la condition $U(\lambda_n, p(\sigma), 1)$ est satisfaite, cette fonction V est unique.

III. *Les* ν_n , λ_n , m_n *étant définis comme dans le premier alinéa de* II, *supposons qu'il existe une fonction non-décroissante* $V = V(t)$ $(-\infty < t < \infty)$, $V(-\infty) = \lim_{t=-\infty} V(t) = 0$, *telle que*

$$\int_{-\infty}^{\infty} dV = m_0, \qquad \int_{-\infty}^{\infty} t^{\nu_n}\, dV = m_n \qquad\qquad (n \geqq 1).$$

Posons $M_n = m_n$ *si* ν_n *est pair, et* $M_n = \infty$ *si* ν_n *est impair, et posons*

$$p(\sigma) = \operatorname*{borne\,sup}_{q \geqq 1} (\nu_q \sigma - \log M_q).$$

Si la condition $\mathrm{U}(\lambda_n , p(\sigma), 1/2)$ *est satisfaite, cette fonction* V *est unique.*

En posant dans chacun des théorèmes II, III, $\nu_n = n$ $(n \geqq 1)$ on obtient les théorèmes connus de Carleman sur l'unicité des problèmes de Stieltjes et de Hamburger. Notons que des problèmes semblables au problème II ont dejà été traités par Boas et W. H. Fuchs. Leurs résultats, différents du nôtre (même lorsque les ν_n sont entiers), portent sur des ν_n positifs, non nécessairement entiers. Le résultat de Fuchs (pour plus de bibliographie voir [2]) contient également le théorème de Carleman.

IV. *Soit* $f(x)$ *une fonction indéfiniment dérivable sur la demi-droite* $x \geqq 0$, *et supposons que* $| f^{(n)}(x) | \leqq M_n$ $(n \geqq 0, M_n < \infty)$. *Définissons les suites* $\{\nu_n\}$ *et* $\{\lambda_n\}$ *comme dans* II. *Si* $f(0) = f^{(\nu_n)}(0) = 0$ $(n \geqq 1)$, *et si, en posant*

$$p(\sigma) = \operatorname*{borne\,sup}_{n \geqq 1} (n\sigma - \log M_n),$$

la condition $\mathrm{U}(\lambda_n , p(\sigma), 1/2)$ *est satisfaite, la fonction* $f(x)$ *est identiquement nulle*·

Ce théorème contient comme cas particulier la partie du théorème 4° qui fournit une condition suffisante de la quasi-analyticité.

Voici enfin un théorème généralisant le résultat 5° (du moins la partie de ce théorème qui fournit une condition suffisante pour que $\pi(f') \subset \omega(f')$).

V. *Les suites* $\{\nu_n\}$ *et* $\{\lambda_n\}$ *étant définies comme dans* II, *supposons que* $f(x)$ *appartienne sur la droite entière à* $L\{M_n\}$, *et soit* $p(\sigma)$ *la fonction définie dans* IV. *Si* $\varphi \in \pi(f')$ *et si la condition* $\mathrm{U}(\lambda_n , p(\sigma), 1/2)$ *est satisfaite, à tout* $\epsilon > 0$ *correspondent un entier* N *et des constantes* b_1 , b_2 , $\cdots$, b_N *tels que*

$$\left\| \varphi - \sum_{1}^{N} b_n f^{(\nu_n)} \right\| < \epsilon.$$

Tous les résultats cités peuvent être obtenus à partir du théorème suivant [4]:

VI. *Soit* $F(s)$ *une fonction holomorphe et bornée dans un domaine* Δ *défini par* $\sigma > \sigma_0$, $| t | < \pi g(\sigma)$, *où* $g(\sigma)$ *est une fonction continue, croissante, bornée* $(\sigma > \sigma_0)$. *Soit dans* Δ:

$$| F(s) | \leqq M.$$

Soit $\{\lambda_n\}$ *une suite positive croissante, avec* $\bar{D}^* < \lim g(\sigma)$. *Soit* $\{d_n\}$ *une suite telle que, pour n donné,*

$$\underset{m \geqq n}{\text{borne inf}} \ \underset{\sigma \geqq x, s \in \Delta}{\text{borne sup}} \left| F(s) - \sum_1^m d_k e^{-\lambda_k s} \right| \leqq e^{-p(x)},$$

où $p(x)$ *est une fonction croissante tendant vers* $+\infty$. *Supposons qu'il existe une fonction continue non-croissante* $h(\sigma)$ *avec* $\lim h(\sigma) \geqq \bar{D}^*$ *et une fonction non-décroissante* $C(\sigma)$ *telles que*

$$2\nu[h(\sigma)] - p_n(\sigma) < -C(\sigma),$$

$$\int^{\infty} C(\sigma) \exp\left[-\frac{1}{2}\int^{\sigma} du/(g(u) - h(u))\right] d\sigma = \infty.$$

Dans ces conditions il existe une constante $A > 0$ *et une constante réele u, toutes les deux ne dépendant que de* $\{\lambda_n\}$ *et* Δ, *telles que*

$$| d_n | \leqq A \Lambda_n M e^{\lambda_n u}$$

où

$$\Lambda_n = \lambda_n \prod_{m \neq n} \frac{\lambda_m^2}{| \lambda_n^2 - \lambda_m^2 |}.$$

Ce théorème n'est d'ailleurs qu'un cas particulier d'un théorème beaucoup plus général démontré dans un Mémoire publié en 1946 dans les Ann. École Norm. [4].

Dans son étude sur les équations intégrales de la forme

$$\varphi(x) - \lambda \int_a^b K(x, y)\varphi(y)\, dy = f(x),$$

où $[a, b]$ est un intervalle fini, Carleman considère les noyaux reéls, symétriques $K(x, y)$ ayant les propriétés suivantes:

1. L'expression

$$K^2(x) = \int_a^b K^2(x, y)\, dy$$

a un sens et est finie, et

$$\lim_{x' = x} \int_a^b [K(x, y) - K(x', y)]^2\, dy = 0$$

pour toutes les valeurs de x exceptées celles d'un ensemble dénombrable $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ au plus, cette suite admettant un nombre fini de points limites.

2. Il existe au plus un nombre fini de valeurs ξ_i qu'on désignera par $\eta_1, \eta_2, \cdots, \eta_m$ telles que, quel que soit $\delta > 0$, en désignant par I_δ le complément des intervalles $[\eta_i - \delta, \eta_i + \delta]$ $(i = 1, \cdots, m)$ par rapport à $[a, b]$, l'expression

$$\int_{I_\delta} K^2(x)\, dx$$

existe et est finie.

3. Le noyau $|K(x, y)|$ admet des noyaux itérés de tout ordre $n \geqq 2$, $\bar{K}^{(n)}(x, y)$, léfinis pour $x \neq \xi_i$, $y = \xi_i$, et tels que pour $x \neq \xi_i$

$$\int_a^b [\bar{K}^{(n)}(x, y)]^2 \, dy < \infty.$$

Jn noyau ayant les propriétés 1, 2, 3 sera appelé noyau C. Il est évident ɩu'un noyau C admet des noyaux itérés de tout ordre $n \geqq 2$, que nous désignerons ›ar $K^{(n)}(x, y)$ (ces noyaux sont définis pour $x \neq \xi_i$, $y \neq \xi_i$). Un point (x, y) du arré Δ: $a \leqq x \leqq b$, $a \leqq y \leqq b$ est dit régulier s'il n'appartient à aucune droite $: = \xi_i$, $y = \xi_i$ ni à aucune droite limite de telles droites.

En nous basant encore sur le théorème VI et sur une inégalité de Carleman ɩous démontrons le théorème suivant:

VII. *Soit $K(x, y)$ un noyau C défini dans le carré Δ: $a \leqq x \leqq b$, $a \leqq y \leqq b$. Soit (x_0, y_0) un point régulier de Δ. Soit $\{\nu_n\}$ $(\nu_1 = 2)$ une suite d'entiers positifs roissants contenant tous les entiers positifs pairs, et soit $\{\lambda_n\}$ la suite complémentaire ; la suite $\{\nu_n\}$ par rapport à tous les entiers $n \geqq 2$. Supposons que*

$$K^{(\nu_n)}(x_0, y_0) = 0 \qquad (n \geqq 1).$$

Si, en posant

$$p(\sigma) = \operatorname*{borne\,sup}_{n \geqq 1} [2n\sigma - \tfrac{1}{2} \log (K^{2(n+1)}(x_0, x_0) K^{2(n+1)}(y_0, y_0))],$$

a condition $\mathrm{U}(\lambda_n, p(\sigma), 1/2)$ *est satisfaite, on a*

$$K^{(p)}(x_0, y_0) = 0 \qquad (p \geqq 2).$$

BIBLIOGRAPHIE

. S. MANDELBROJT, *Analytic continuation and infinitely differentiable functions*, Bull. Amer. Math. Soc. t. 54 (1948) pp. 239–248.
. ———, *Théorèmes d'unicité*, Ann. École Norm. (3) t. 45 (1948) pp. 101–138.
. ———, *Un théorème de fermeture*, C. R. Acad. Sci. Paris. t. 231 (1950) pp. 16–18.
. ———, *Sur une inégalité fondamentale*, Ann. École Norm. (3) t. 43 (1946) pp. 352–378.

COLLÈGE DE FRANCE,
PARIS, FRANCE.
RICE INSTITUTE,
HOUSTON, TEXAS, U. S. A.

ADDITIVE ALGEBRAIC NUMBER THEORY[1]

HANS RADEMACHER

Additive number theory in the rational field uses, since Euler (*De partitione numerorum* 1748) the tool of power series, based on the fact that

$$x^m \cdot x^n = x^{m+n}.$$

The power series

$$\sum a_n x^n$$

can first be understood in a purely *formal* way, as Euler himself did, an interpretation which led him to numerous results in additive theory and to such a fundamental identity as

$$\prod_{m=1}^{\infty} (1 - x^m) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda x^{\lambda(3\lambda+1)/2}.$$

The new method which Hardy, Littlewood, and Ramanujan introduced from 1917 on was based on the *function-theoretical* interpretation of the power series, which permitted in particular the application of Cauchy's integral theorem. The abundant results in Waring's problem, Goldbach's problem, and the theory of partitions, which ensued, are well-known.

The analytic additive theory in algebraic number fields could not be developed before we had the equivalent of power series for the algebraic fields. This was found by Erich Hecke, who in turn developed ideas of Hilbert and Blumenthal.

Let us for the sake of simplicity confine our attention to totally real fields, i.e., such which together with their conjugate fields are real. An algebraic number ν is called totally positive if it together with all its conjugates is positive; we write $\nu \succ 0$. Then

$$(1) \quad f(t) = f(t^{(1)}, t^{(2)}, \cdots, t^{(n)}) = \sum_{\nu \succ 0} a(\nu) \exp\left(-(\nu^{(1)}t^{(1)} + \cdots + \nu^{(n)}t^{(n)})\right)$$

is a "power series" in an algebraic field of degree n, where $\nu^{(1)}, \cdots, \nu^{(n)}$ are the conjugates of the integer ν and $t^{(1)}, t^{(2)}, \cdots, t^{(n)}$ are independent complex variables. Convergence depends on the coefficients $a(\nu)$, of course. For $a(\nu) = 1$ we obtain the "geometric series", for which convergence is easily established in the domain $\mathfrak{R}(t^{(j)}) > 0$, $j = 1, 2, \cdots, n$, which we, in an obvious extension of the meaning of the symbol for total positivity, may also indicate by $\mathfrak{R}(t) \succ 0$.

For the power series in one variable

$$f(t) = \sum_{n=0}^{\infty} a_n e^{-nt}, \qquad \mathfrak{R}(t) > c,$$

[1] This address was listed on the printed program under the title *Remarks on the theory of partitions*.

he Cauchy formula can be written as

$$a_n = \int_0^1 f(z + 2\pi i\varphi)e^{n(z+2\pi i\varphi)}\, d\varphi, \qquad \Re(z) > c.$$

The analogue for a power series for algebraic fields (1) demands the introduction of a basis $(\rho_1, \rho_2, \cdots, \rho_n)$ of the ideal $\mathfrak{d}^{-1}$, where $\mathfrak{d}$ is the ramification ideal, with $N(\mathfrak{d}) = |\, d \,|$, d the discriminant of the algebraic field. In the space $(y^{(1)}, y^{(2)}, \cdots, y^{(n)})$ (real coordinates) the points of $\mathfrak{d}^{-1}$ form a point lattice, whose parallelepiped ve may call E (it corresponds to the unit interval in the rational case). Then ve get

$$\text{2)} \quad a(\nu) = N(\mathfrak{d})^{1/2} \int \underset{E}{\cdots} \int f(z + 2\pi iy) \exp\left(\sum_q \nu^{(q)}(z^{(q)} + 2\pi i y^{(q)})\right) dy^{(1)} \cdots dy^{(n)},$$

ince $\mathfrak{d}^{-1}$ has the property that for any number δ in $\mathfrak{d}^{-1}$ the trace $S(\delta)$ is a rational nteger.

The extension of the Hardy-Littlewood method to algebraic number fields equires one more tool: the analogue of the Farey dissection of order N. This neans here a suitable dissection of E, each Farey piece belonging to a fraction γ uch that $\gamma\mathfrak{d}$ has a denominator $\mathfrak{a}$, with $N(\mathfrak{a}) \leqq N$. Siegel carried out the details y means of an approximation theorem of Minkowski.

The first example of a power series of type (1) was the ϑ-series

$$\vartheta(t) = \sum_\nu \exp\left(-(\nu^{(1)2}t^{(1)} + \cdots + \nu^{(n)2}t^{(n)})\right), \qquad \Re(t) \succ 0,$$

nvestigated by Hecke. Here ν runs over all integers of the field; in special ases ν may be restricted to numbers of a given ideal. Hecke used these ϑ-series or two important purposes, namely, to establish the existence of the Dedekind -functions in the whole plane, and secondly, for the discussion of Gaussian ums in algebraic fields, which in turn lead to the reciprocity laws of quadratic esidues.

Siegel realized that these ϑ-functions could be used for the analytic theory of the representation of totally positive numbers as sums of squares. If $A_r(\nu)$ s the number of representations of ν as a sum of squares of integers, Siegel btained for the real-quadratic field (and similarly in other algebraic number ields) the result, for $r \geqq 5$

$$\text{3)} \qquad A_r(\nu) = \frac{\pi^r}{\Gamma^2(r/2)\, d^{(r-1)/2}} S_r(\nu) N(\nu)^{r/2-1} + o(N(\nu)^{r/2-1}).$$

Here S is the "singular series" which reflects the arithmetical properties of ν. There exist two positive constants C_r' and C_r'' independent of ν so that $0 < C_r' < S < C_r''$ for each $r \geqq 5$, unless $S = 0$, which takes place for those ν which cannot e represented as a sum of any number of squares. (That such numbers exist s seen in the example $R(2^{1/2})$, where a square has always the form $(a + b\, 2^{1/2})^2 = a^2 + 2b^2 + 2ab\, 2^{1/2}$ which shows that the integer $t + u\, 2^{1/2}$ cannot be a sum of quares if u is odd.)

The form of the result (3) shows complete analogy to the results in rational cases: the formula for $A_r(\nu)$ shows, besides coefficients depending solely on the field, two factors, one determining the order of magnitude depending on $N(\nu)$, the other, $S_r(\nu)$, which is a product over the prime ideals of the field, involving the arithmetical properties of ν.

In the case of the sum of squares, the power series, which are here ϑ-functions, are so well known that we obtain excellent approximations to them by elementary functions in each Farey piece of E. A distinction corresponding to that between major and minor arcs is therefore not necessary.

The same favorable situation prevailed in the Hardy-Littlewood treatment of Goldbach's problem. There the goodness of approximation was established by fiat, namely by the assumption of the generalized Riemann hypothesis concerning all $L(s, \chi)$-functions. For algebraic fields the situation was similar, no distinction between minor and major Farey pieces was necessary if a Riemann hypothesis was introduced covering the Dedekind ζ-function and all the Hecke $\zeta(s, \lambda\chi)$-functions of the field. In the case of a real quadratic field I obtained for the number $B_r(\nu)$ of representations of a totally positive integer ν as the sum of r totally positive prime numbers (i.e., numbers whose principal ideal is a prime ideal) the result

$$B_r(\nu) = C_r S_r(\nu) \frac{N(\nu)^{r-1}}{(\log N(\nu))^r} + o\left(\frac{N(\nu)^{r-1}}{(\log N(\nu))^r}\right), \qquad r \geqq 3,$$

where $S_r(\nu)$ is the singular series

$$S_r(\nu) = \prod_{\mathfrak{p} \mid \nu, N(\mathfrak{p}) > 2} \left(1 + (-1)^r \frac{N(\mathfrak{p})}{(N(\mathfrak{p}) - 1)^r - (-1)^r}\right),$$

and where C_r is a constant depending on the field and on the number r. The analogy to the Hardy-Littlewood formula in the rational case is complete.

Unfortunately this result shares with that of Hardy and Littlewood also the shortcoming that it is hypothetical, being based on a generalized Riemann hypothesis. For the rational field this objection has been met by Vinogradoff, Linnik, and Tchudakoff. Whereas the former had to distinguish between the major arcs, on which the Hardy-Littlewood method could be carried through unconditionally, and the minor arcs, on which he used his famous estimates of exponential sums, the latter two could remodel the Hardy-Littlewood method without using that distinction. They utilized the fact that if the $L(s, \chi)$-functions should have zeros close to the line $\sigma = 1$, these zeros are relatively few and could be accounted for in the estimations. The corresponding work has not yet been carried out in the algebraic case. It seems that we would need analogues for Page's theorems concerning the paucity of undesirable zeros of the L-functions and also some sort of estimation for the $\zeta(s, \lambda)$-functions as the "approximate functional equations" provide for $\zeta(s)$ and the $L(s, \chi)$-functions in the rational case. These seem to me worthy objects of function-theoretical research.

Recently Siegel has been able to use the analogues of the major and minor arcs together, in the treatment of Waring's problem. The problem is to represent a totally positive ν as a sum of kth powers of totally positive integral numbers

$$\nu = \lambda_1^k + \lambda_2^k + \cdots + \lambda_r^k .$$

The result is again as to be expected: the number of representations is

$$A_r(\nu) = C_r S_r(\nu) N(\nu)^{r/k-1} + o(N(\nu)^{r/k-1})$$

for $r > (2^{k-1} + n)k$, where n is the degree of the field K, C_r depends on K, k, r only, and $S_r(\nu)$ is the singular series, which again appears as an infinite product over prime ideals of K. The singular series vanishes if and only if ν does not belong to the ring J_k which consists of the integers obtainable as sums of any number of kth powers.

There remains one important problem of additive number theory to be investigated for algebraic number fields: the problem of partitions. Whereas so far we were guided by analogies to the rational cases, the situation here is completely different and I can make today only small contributions to its solution.

In the rational field we have for $p(n)$, the number of partitions of n, Euler's generating function

$$1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-1}.$$

The same reasoning leads here for $P(\nu)$, the number of partitions of the totally positive integer ν into totally positive integers, to the identity

$$1 + \sum_{\nu \succ 0} P(\nu)e^{-S(\nu t)} = \prod_{\mu \succ 0} (1 - e^{-S(\mu t)})^{-1},$$

with

$$S(\nu t) = \nu^{(1)}t^{(1)} + \cdots + \nu^{(n)}t^{(n)},$$

where $t^{(1)}, \cdots, t^{(n)}$ are independent complex variables of positive real part. Let us consider, for simplicity's sake, only the real quadratic field, in which case we have

$$f(t, t') = 1 + \sum_{\nu \succ 0} P(\nu)e^{-\nu t - \nu' t'} = \prod_{\mu \succ 0} (1 - e^{-\mu t - \mu' t'})^{-1}. \tag{4}$$

That this is the right generating function and that we are dealing with the proper generalization of the ordinary $p(n)$ is shown also by the fact that a well-known device gives a statement analogous to an elementary theorem of Euler:

$$1 + \sum_{\nu \succ 0} Q(\nu)e^{-\nu t - \nu' t'} = \prod_{\mu \succ 0} (1 + e^{-\mu t - \mu' t'})$$

$$= \frac{\prod (1 - e^{-2\mu t - 2\mu' t'})}{\prod (1 - e^{-\mu t - \mu' t'})} = \prod_{\mu \succ 0, 2 \nmid \mu} (1 - e^{-\mu t - \mu' t'})^{-1},$$

in words: the number $Q(\nu)$ of partitions of ν into different parts is equal to the number of partitions with repetition into parts which have no divisor 2.

Let us now, before discussing $P(\nu)$ further, remember the formula for $p(n)$ in the rational field. We need only the asymptotic formula

$$p(n) \sim \frac{1}{4\cdot 3^{1/2}n} \exp\,(\pi 2n/3))^{1/2}).$$

It is of interest to note that here the arithmetical properties of n are not used in the main term. They are put in evidence only in the subsequent terms of lower order of a convergent series which we do not need here.

We might now expect from our previous experience with analogies a hypothetical formula like

$$P(\nu) \sim \frac{C}{N(\nu)} \exp\,(c(N(\nu)^{1/2}),$$

but such a formula does certainly not exist. As a matter of fact, I can show

$$P(\nu) = O(\exp N(\nu)^{2/5})$$

and something better than that.

The discrepancy becomes clearer by the observation that

$$\prod_{m=1}^{\infty} (1 - e^{-mt})$$

is essentially a modular function, whereas

$$\prod_{\mu > 0} (1 - e^{-\mu t - \mu' t'})$$

is *not* the analogue of a modular function for the real quadratic field. Such analogues exist, but they are of a different structure. We can quickly see the difference by taking the logarithms:
With

$$f(t) = \prod_{m=1}^{\infty} (1 - e^{-mt})^{-1}$$

we have

$$\log f(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n} e^{-mnt},$$

and through Mellin's formula:

$$\log f(t) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{\Gamma(s)}{n(mnt)^s}\, ds$$

$$= \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} t^s \Gamma(s) \zeta(s) \zeta(1 + s)\, ds, \qquad \sigma_0 > 1,$$

which in virtue of the functional equation for $\zeta(s)$ yields

$$\log f(t) = \frac{1}{4\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} (2\pi)^s t^{-s} \frac{\zeta(1-s)\zeta(1+s)}{\cos(\pi s/2)} \, ds. \tag{5}$$

Here we have symmetry s and $-s$, which sends $t/2\pi$ into $2\pi/t$. This shows that $f(t)$ is essentially a modular function.

On the other hand, for the function $f(t, t')$ in (4) we obtain

$$\log f(t, t') = \sum_{\mu > 0} \sum_{n > 0} \frac{1}{n} e^{-n(\mu t + \mu' t')}.$$

Here the Mellin formula is not directly applicable, but a formula due to Hecke furnishes

$$\begin{aligned} &\log f(t, t') \\ &= \frac{1}{2\pi i \log \epsilon_1} \sum_{n=-\infty}^{\infty} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{\Gamma(s + inc_1)\Gamma(s - inc_1)}{t^{s + inc_1} t'^{s - inc_1}} Z(s, \lambda^n)\zeta(1 + 2s) \, ds \end{aligned} \tag{6}$$

where ϵ_1 is the totally positive fundamental unit,

$$c_1 = \frac{\pi}{\log \epsilon_1},$$

$$Z(s, \lambda^n) = \sum_{(\mu)_1, \mu > 0} \frac{\lambda^n(\mu)}{N(\mu)^s}$$

with

$$\lambda(\mu) = \left| \frac{\mu'}{\mu} \right|^{ic_1}$$

and $(\mu)_1$ indicates that of a set of numbers μ which differ by powers of ϵ_1 as factors only one is taken. In order to leave aside complications which do not belong to the core of the matter let us now assume that

1) the fundamental unit ϵ is *not* totally positive, so that $\epsilon_1 = \epsilon^2$,

2) the class-number of the field is 1 (these two conditions are fulfilled, e.g., in the field $R(2^{1/2})$).

Then we obtain after Hecke and Siegel a functional equation

$$\begin{aligned} &Z(s, \lambda^n)\Gamma(s + inc_1)\Gamma(s - inc_1) \\ &\quad = \frac{d^{1/2}}{4} \frac{(2\pi/d^{1/2})^{2s}}{\cos(\pi/2)(s + n + inc_1)\cos(\pi/2)(s - n - inc_1)} Z(1 - s, \lambda^{-n}), \end{aligned}$$

through which (6) goes over into

$$\begin{aligned} \log f(t, t_1) = \frac{d^{1/2}}{8\pi i \log \epsilon_1} \sum_{n=-\infty}^{\infty} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{(2\pi/d^{1/2})^{2s}}{t^{s + inc1} t'^{s - inc1}} \\ \cdot \frac{Z(1 - s, \lambda^{-n})\zeta(1 + 2s)}{\cos(\pi/2)(s + n + inc_1)\cos(\pi/2)(s - n - inc_1)} \, ds. \end{aligned} \tag{7}$$

Although this formula corresponds to (4), we find here no longer symmetry in s and $-s$.

Let us now return to $P(\nu)$. Applying (2) to (4) we obtain

$$P(\nu) = d^{1/2} \iint_E f(z + 2\pi iy, z' + 2\pi iy')e^{\nu(z+2\pi iy, z'+2\pi iy')} \, dy \, dy'. \tag{8}$$

From (7) we can insert here $f(t, t')$. An estimate gives me then so far only

$$P(\nu) = O(\exp N(\nu)^{5/14+\epsilon}),$$

whereas, on the other hand, it is evident that for rational integers

$$P(n) \geqq p(n) \sim \frac{C}{n} \exp (\pi(2n/3)^{1/2})$$

and therefore, since $N(n) = n^2$

$$P(\nu) = \Omega(N(\nu)^{-1/2} \exp (\pi(2/3)^{1/2}N(\nu)^{1/4})).$$

I have not been able to evaluate the integral (8) for $P(\nu)$ asymptotically. A new feature, however, appears clearly in my calculations: $P(\nu)$ cannot be approximated by a function of $N(\nu)$ alone in combination with arithmetical properties of ν, but such a function must involve also the "angular character" $\lambda(\nu)$ of ν.

The $\lambda(\nu)$ enter into the discussion through $Z(s, \lambda^n)$. This they do also in the previously mentioned problems of additive algebraic number theory, but there their influence is restricted to the (additive) error terms, whereas they appear here in a multiplicative way. We can therefore not even expect a formula

$$P(\nu) \sim CN(\nu)^{\alpha} \exp (N(\nu)^{\beta})$$

with suitable α,β, but have to assume that $\lambda(\nu)$ will also appear in the presumptive asymptotic formula for the number of partitions in an algebraic number field.

University of Pennsylvania,
Philadelphia, Pa., U. S. A.

ON VISUALIZATION OF DOMAINS IN THE THEORY OF FUNCTIONS OF TWO COMPLEX VARIABLES

STEFAN BERGMAN

1. Introduction. The difficulty of acquainting the student with domains of four (and more) dimensional space, and of conveying to him the ability to operate with these domains, is a serious handicap in teaching the theory of functions of two (and several) complex variables. The models of four-dimensional domains which arise in this theory, as indicated in [1; 8],[1] are a valuable aid in overcoming this handicap. In particular, such models permit the student to obtain a more intuitive grasp of the geometrical situation, and, as is the case in dealing with the theory of one complex variable, this will later motivate his research.

In constructing models, we interpret one of the four coordinates, x_1, y_1, x_2, y_2, say y_2, as *time*, so that a four-dimensional domain is represented as a moving picture, each still representing the intersection of the four-dimensional body with $y_2 = \text{const.}$[2] In most cases, the moving picture is replaced by a finite number of stills, and three-dimensional bodies are pictured by projecting them on the plane.[3]

In order to use these models successfully, we need a certain training in interpreting the shape and features of our stills as geometrical properties of four-dimensional domains. Thus, it is our ultimate aim that after certain training, the student will be able to imagine the form of various models and how they vary, *without actually building them*, so that he can make conclusions about some geometrical properties without actually carrying out experiments. This ability can be considered as an artificial "intuition" for four-dimensional space, and such an intuition is indeed valuable for teaching and research. See [8].

For such training, a large number of models must be made available: there exist many different domains of interest, and we need to see how their shapes change if the same domain undergoes some geometrical transformation, e.g., rotation, stretching in some direction, etc. The immediate procedure which consists of first computing the shape of the model, then building it, and finally a drawing made by a draftsman, is a long and costly process. Also, a large enough collection of such models would require a tremendous amount of space for exhibition and study. Thus, we have two immediate tasks to perform:

(1) To develop procedures which will permit us, with comparatively little work, to prepare models corresponding to the different domains which are of

[1] The numbers in brackets refer to the bibliography at the end of this paper.

[2] We speak here of one type of models only. It goes without saying that for different purposes, it is useful to have other kinds of models which are not discussed in the present paper.

[3] A very successful presentation of the intersection of a four-dimensional body with three-dimensional space has been given by A. C. Schaeffer and D. C. Spencer [13].

interest, as well as models of the same domain undergoing various geometrical transformations.

(2) To develop a drawing technique which will make it possible for us to present three-dimensional bodies in *relief*, so that one can grasp from this plane picture the actual form of the three-dimensional body.

Task (2) has a more general character, and can be solved by modern techniques in photography and drawing, so that it need not be discussed here. Task (1) can be met satisfactorily by the use of modern computational devices, as we shall show.

The shape of a still can be obtained by computing the coordinates of a large number of points of its boundary. Even if we have to compute coordinates of a very large number of points, which means an enormous amount of numerical computation, this can be done conveniently with, say, punch-card machines or large scale digital machines.

For instance, if we have a model, and it undergoes a geometrical operation, this can be formulated, often, as a linear transformation of points of the domain. Computationally, this means that some transformation has to be applied to a great number of coordinate vectors, i.e., a given matrix has to be multiplied by a number of vectors. Using punch-card machines, one can carry out such a multiplication for a thousand or more vectors in about one-half hour, so that coordinates of the resulting domain may be obtained with reasonable speed. After these coordinates are known, we have to get their projection on the plane. Again, such an axionometric projection means a linear transformation, and the two-dimensional coordinates of the projected points can easily be determined by the use of punch-card machines. In the following, we shall describe this technique more in detail, considering examples which arise in the theory of two complex variables.

Remark. In principle we can visualize in a similar way domains in six dimensions arising in the theory of functions of three complex variables. In this case, instead of a one-parameter sequence of two-dimensional projections of three-dimensional bodies, we have a three-parameter family of such domains. The models become complicated, but it would still be worthwhile to investigate to what extent they may be useful in acquainting the student with the domains of the space of functions of three complex variables.

The author wishes to express his thanks to Dr. George Springer for various valuable suggestions in connection with the preparation of this paper.

2. Rotation of a domain. As was indicted in [1; 2; 3; 5; 6; 9, p. 146ff.; 10] the domains bounded by finitely many analytic hypersurfaces (see [9 p. 144ff.]) play a very important role in the theory of functions of two complex variables, and an understanding of the structure and geometrical properties of its boundaries is of great interest. In [9, p. 146ff.], there has been described a domain of such a type, which is bounded by four analytic hypersurfaces h_k^3, whose parametric representation follows:

$$\text{(2.1a)} \quad z_1 = Z_1\,, \qquad z_2 = 1 + i\lambda_1\,, \qquad -\infty < \lambda_1 < \infty\,,$$

$$Z_k = X_k + iY_k\,,$$

$$\text{(2.1b)} \quad z_1 = Z_2\,, \qquad z_2 = -1 + i\lambda_2\,, \qquad -\infty < \lambda_2 < \infty\,,$$

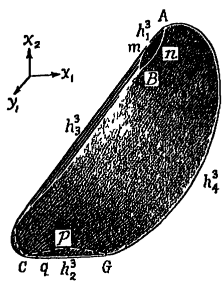


FIGURE 1.1

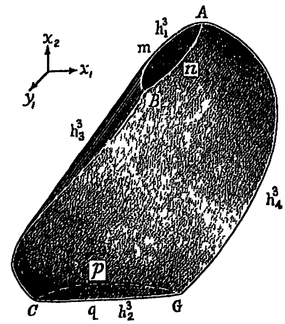


FIGURE 1.2

$$\text{(2.1c)} \quad z_1 = Z_3 + e^{i\lambda_3}, \qquad z_2 = Z_3\,, \qquad 0 \leqq \lambda_3 < 2\pi,$$

$$\text{(2.1d)} \quad z_1 = -Z_4 + e^{i\lambda_4} + \cos\lambda_4\,, \qquad z_2 = Z_4\,, \qquad 0 \leqq \lambda_4 < 2\pi.$$

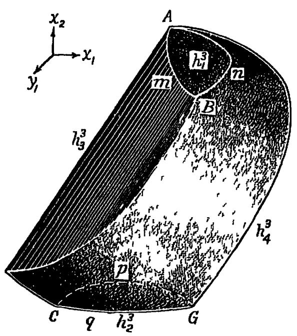


FIGURE 1.3

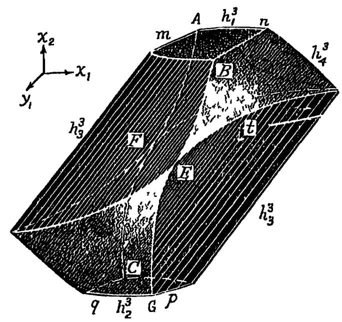


FIGURE 1.4

As was indicated in [9, Chap. 11, §6], on the boundary of the domain bounded by finitely many analytic hypersurfaces, there is a distinguished boundary surface (locus of points common to the intersection of each two analytic hypersurfaces), and the distinguished boundary line (locus of points common to the intersection of at least three analytic hypersurfaces). In the above-mentioned model, the distinguished boundary surface and the distinguished boundary line appear as edges $[BmA]$, $[BnA]$, $[CqG]$, etc., and as points A, B, C, and G, respectively. (See Figs. 1.1 through 1.4.) In [9, p. 146 ff.], the reader will find a more detailed explanation[4] of the interpretation of different geometrical objects, say segments h_k^3 of analytic hypersurfaces belonging to the boundary, segments D_{km}^2 of the distinguished boundary surface, etc., as features of the model. The above model was obtained

[4] Models of other domains may be found in [9, pp. 133, 134] and [1].

as a result of carrying out computations[5] which were done by punch-card machines, but since they involve too many technical details, it is preferable to discuss the technique of determining the shape of the stills of the domain which we obtain by rotating the model, shown in Figs. 1.1 through 1.4. As a result of computation, we have, say, 1000 cards with coordinates $x_1^{(\gamma)}$, $y_1^{(\gamma)}$, $x_2^{(\gamma)}$, $y_2^{(\gamma)}$, $\gamma = 1, 2, \cdots, 1000$, of the points of the boundary of our domain, which can be sorted so that the machine can deliver to us separately points belonging to the distinguished boundary line, the distinguished boundary surface, to each segment of the boundary, etc.

In the following will be discussed construction of the stills of the domains resulting from domains pictured in Figs. 1.1 through 1.4 by the transformation

$$\text{(2.2a)} \quad \zeta_1 = (1 + i)(z_1 + z_2)/2, \qquad \text{(2.2b)} \quad \zeta_2 = -(1 + i)(z_1 - z_2)/2,$$

which represents an (analytical) rotation.

In order to obtain the new coordinates $\zeta_1^{(\gamma)} = \xi_1^{(\gamma)} + i\eta_1^{(\gamma)}$, $\zeta_2^{(\gamma)} = \xi_2^{(\gamma)} + i\eta_2^{(\gamma)}$, $\gamma = 1, 2, \cdots, 1000$, we have, clearly, to substitute for z_1, z_2, the values $(z_1^{(\gamma)}, z_2^{(\gamma)})$ determined previously—a task which is done by punch-card machines. To get these points which will appear in some particular still, we must select those vectors whose $\eta_2^{(\gamma)}$ coordinates have some specific value. Practically, this is done in such a way that we sort out those vectors whose η_2 coordinates lie in the neighborhood of some specific value, say between 0.98 and 1.03. This sorting can be done in only a few moments. In order to obtain a two-dimensional picture, it now remains to determine the coordinates of the axionometric projection of our points $(\xi_1^{(\gamma)}, \eta_1^{(\gamma)}, \xi_2^{(\gamma)})$, which means that we have to compute the values

$$\text{(2.3)} \qquad \Xi = \xi_1 + \frac{1}{2}\eta_1 \cos 30^0, \qquad \mathrm{H} = \xi_2 + \frac{1}{2}\eta_1 \sin 30°.$$

The stills of the domain for $\eta_2 = -\frac{3}{4}, = -\frac{1}{2}, = -\frac{1}{4}, = -\frac{1}{10}$, which we obtain in this way are pictured in Figs. 2.1–2.4, p. 367. Again, the distinguished boundary line appears as endpoint of the edges, the distinguished boundary surface as the edges themselves, etc.

REMARK. We note that in our particular case it is quite easy to compute the equations of the h_k^3 in the new coordinates. The hypersurfaces $h_3^3 = \mathrm{E}[z_1 = e^{i\lambda_3} + z_2]$ and $h_4^3 = \mathrm{E}[z_1 = e^{i\lambda_4} + \cos\lambda_4 - z_2]$ become $h_3^3 = \mathrm{E}[\,|\,\zeta_2\,| = 2^{-1/2}]$ and $h_4^3 = \mathrm{E}[5\xi_1^2 + 5\eta_1^2 - 6\xi_1\eta_1 = 4]$, respectively. The hypersurfaces $h_1^3 = \mathrm{E}[x_2 = 1]$ and $h_2^3 = \mathrm{E}[x_2 = -1]$ become $h_1^3 = \mathrm{E}[\xi_1 + \eta_1 + \xi_2 + \eta_2 = 2]$ and $h_2^3 = \mathrm{E}[\xi_1 + \eta_1 + \xi_2 + \eta_2 = -2]$, respectively.

3. The images H_k^3 of segments h_k^3 of analytic hypersurfaces in the X_k, Y_k, λ_k -space. In the present section, we shall determine the images H_k^3 of the

[5] In order to compare the work needed and results obtained by different procedures the three-dimensional models of the domains considered in the present paper have also been actually built. Mr. Joseph Hilsenrath, who built these models, has developed an interesting technique for preparing three-dimensional domains. For details, see [11]. The author wishes to thank Mr. Hilsenrath for his work.

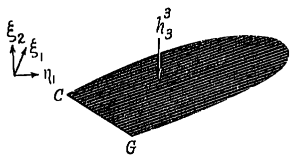


FIGURE 2.1

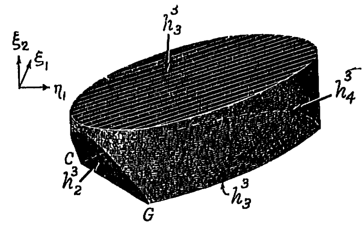


FIGURE 2.2

segments h_k^3 in the X_k, Y_k, λ_k -space, for $k = 3$ and 4. In order to determine the shapes of the boundary, say of H_3^3, we have to determine the intersections $D_{3m}^2 = H_3^3 \cap H_m^3$, $m = 1, 2, 4$. For $m = 1$, we obtain $Z_3 \equiv X_3 + iY_3 = 1 + i\lambda_1$, $0 \leqq \lambda_3 < 2\pi$, i.e.,

$$D_{31}^2 = \mathrm{E}[X_3 = 1, Y_3 = \lambda_1 \,(-\infty < \lambda_1 < \infty), 0 \leqq \lambda_3 < 2\pi] \tag{3.1}$$

i.e., a plain strip, $X_3 = 1$, cut off between $\lambda_3 = 0$ and $\lambda_3 = 2\pi$.[6] For $m = 2$, we obtain the plain strip

$$D_{32}^2 = \mathrm{E}[X_3 = -1, Y_3 = \lambda_2 \,(-\infty < \lambda_2 < \infty), 0 \leqq \lambda_3 < 2\pi]. \tag{3.2}$$

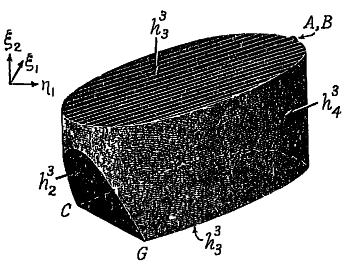


FIGURE 2.3

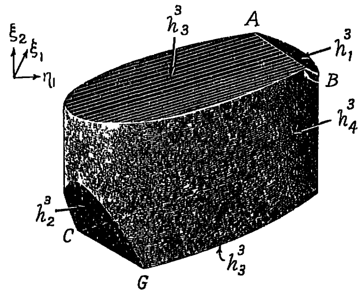


FIGURE 2.4

For $m = 4$, we have to determine the locus of the points, given by

$$X_3 = \frac{1}{2}(-\cos \lambda_3 + 2 \cos \lambda_4), \tag{3.3a}$$

$$Y_3 = \frac{1}{2}(-\sin \lambda_3 + \sin \lambda_4), \qquad 0 \leqq \lambda_4 < 2\pi, \tag{3.3b}$$

$$0 \leqq \lambda_3 < 2\pi. \tag{3.3c}$$

For each horizontal plane $\lambda_3 = $ const., $0 \leqq \lambda_3 < 2\pi$, the cross-section of H_3^3 is an ellipse with semi-major axis 1 and semi-minor axis $\frac{1}{2}$, with the center at the points $(-\frac{1}{2}\cos \lambda_3, -\frac{1}{2}\sin \lambda_3, \lambda_3)$, i.e., at a point on a circle of radius $\frac{1}{2}$ about the λ_3 axis making an angle $\lambda_3 + \pi$ with the positive X_3 axis. Thus we obtain for D_{43}^2 the mantle surface of a twisted elliptical cylinder, which is cut off, see Fig. 3, p. 368, by planes $X_3 = 1$ and $X_3 = -1$.

[6] Of course, only a subdomain indicated in fig. 3 of this strip belongs to the boundary of H_3^3. In figs. 3 and 4, the segments D_{31}^2, D_{41}^2 and D_{32}^2, D_{42}^2 are denoted by D_1^2, D_2^2 respectively. The axes λ_3, X_3, Y_3 and λ_4, X_4, Y_4 are denoted by λ, X, Y.

REMARK. It should be stressed that in our presentation we have chosen for λ_3 the straight (vertical) direction. In order to present the true (topological) situation we have to identify the upper ($\lambda_3 = 2\pi$) and the lower ($\lambda_3 = 0$) in-

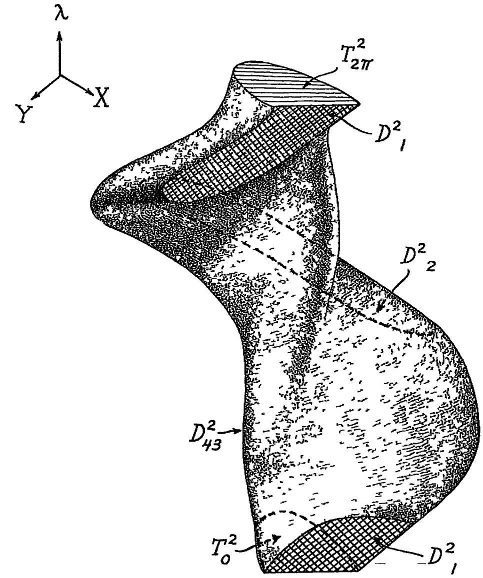


FIGURE 3

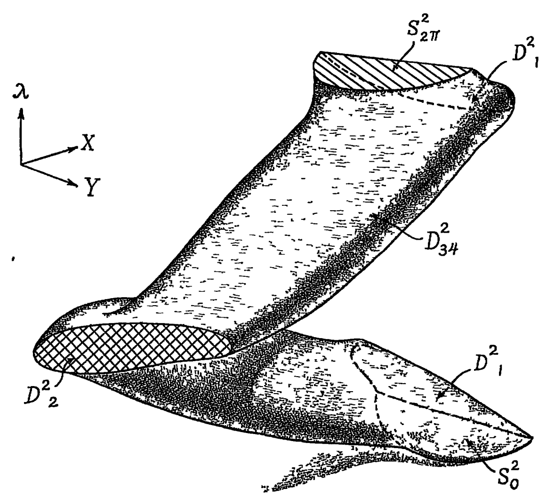


FIGURE 4

tersections, denoted in Fig. 3 by T^2_0 and $T^2_{2\pi}$ with each other. (It would be preferable to choose for λ_3 the periphery of the unit circle, so that the image of H^3_3 would appear as a twisted torus. The body becomes, however, so involved, that it would be very difficult to depict from the two-dimensional projection the geometrical situation.)

Analogous discussion of the image H_4^3 of h_4^3 in the X_4, Y_4, λ_4-space yields

$$D_{41}^2 = \mathrm{E}\,[X_4 = 1, \quad Y_4 = \lambda_1 \quad (-\infty < \lambda_1 < \infty), \quad 0 \leqq \lambda_4 < 2\pi] \tag{3.4}$$

$$D_{42}^2 = \mathrm{E}[X_4 = -1, \quad Y_4 = \lambda_2 \quad (-\infty < \lambda_2 < \infty), \quad 0 \leqq \lambda_4 < 2\pi] \tag{3.5}$$

$$\begin{aligned} D_{34}^2 = \mathrm{E}\Big[& X_4 = \frac{1}{2}(-\cos\lambda_3 + 2\cos\lambda_4), \\ & Y_4 = \frac{1}{2}(-\sin\lambda_3 + \sin\lambda_4) \quad (0 \leqq \lambda_3 < 2\pi), \quad 0 \leqq \lambda_4 < 2\pi\Big]. \end{aligned} \tag{3.6}$$

Thus, for H_4^3 (image of h_4^3) we again obtain a twisted circular cylinder (with the end surfaces S_0^2 and $S_{2\pi}^2$ to be identified with each other) which is cut off by the plane $X_4 = 1$ and $X_4 = -1$. The intersection of this plane with the plane $\lambda_4 = \text{const.}$ is a circle of radius $\frac{1}{2}$ with the center at the point $(\cos\lambda_4, \frac{1}{2}\sin\lambda_4, \lambda_4)$. (That is, the center lies on an ellipse with major axes 1 and $\frac{1}{2}$). See Fig. 4.

Analogous pictures of H_1^3 and H_2^3 (images of h_1^3 and h_2^3) are much simpler and we shall not discuss them here.

In the case where the equations of our hypersurfaces are more complicated, the corresponding discussions are more involved: in these cases, it is preferable to determine the shape of the domains by use of computational devices.

4. Domains of regularity of solutions of partial linear differential equations. If one continues to complex values of the arguments, a solution of a linear differential equation

$$\varphi_{x_1x_1} + \varphi_{x_2x_2} + a_1(x_1, x_2)\varphi_{x_1} + a_2(x_1, x_2)\varphi_{x_2} + a_3(x_1, x_2)\varphi = 0, \qquad \varphi_{x_1} = \partial\varphi/\partial x_1, \ldots \tag{4.1}$$

where $a_k(z_1, z_2)$, $z_k = x_k + iy_k$ are entire functions of two complex variables, then it is known that they can be singular only on characteristic planes (see [4, p. 1178]):

(4.2a) $\quad z_1 + iz_2 = \text{const.}, \qquad$ or $\qquad$ (4.2b) $\quad z_1 - iz_2 = \text{const.}$

Therefore, if a solution $\varphi(x_1, x_2)$ is regular in some domain, say B^2, of the (real) x_1, x_2-space, it must be regular when extended to complex values in the hull $H^4(B^2)$ which will be formed when we let pass through every point, say (x_1^0, x_2^0), of the boundary b^1 of B^2, two characteristic planes

$$x_1 - y_2 = x_1^0, \qquad y_1 + x_2 = x_2^0, \tag{4.3}$$

and

$$x_1 + y_2 = x_1^0, \qquad y_1 - x_2 = x_2^0. \tag{4.4}$$

Let us remark that the intersection of the domain $H^4(B^2)$ with the plane $z_1 - iz_2 = 0$ is a plane domain whose z_1-coordinates are $z^*/2$ and z_2-coordinates are $-iz^*/2$, where $z^* = (x_1^* + ix_2^*)/2$ and (x_1^*, x_2^*) varies over the domain B^2.

If our domain B^2 is a circle, the still for $y_2 = 0$ will have the form indicated in Fig. 5, see p. 371. Every characteristic plane will appear as a straight line. The still for $y_2 = a$, $a > 0$, will be obtained by shifting the lines (4.3) in the positive x_1-direction, and the lines (4.4) in the negative x_1-direction by an amount a. Figs. 6.1–6.4 represent the stills of the domains corresponding to a rectangle from which an elliptic hole has been cut out.[6]

If we apply the transformation

$$\zeta_1 = (z_1 + iz_2)/2, \qquad \zeta_2 = (z_1 - iz_2)/2, \qquad \zeta_k = \xi_k + i\eta_k \tag{4.5}$$

(a rotation), the new domain will appear as a product,[7] $R^4 = R_1^2 \times R_2^2$, of the

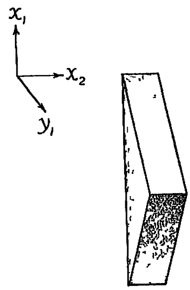


FIGURE 6.1

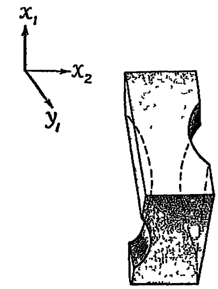


FIGURE 6.2

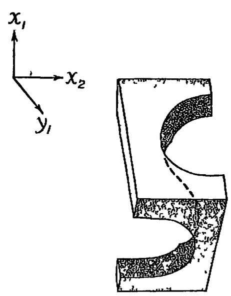


FIGURE 6.3

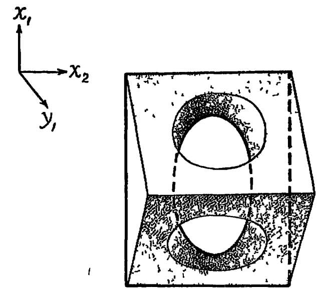


FIGURE 6.4

domains R_1^2 and R_2^2 of exactly the same shape, see Fig. 7, p. 371, B_k^2 lying in the ζ_k-plane, $k = 1, 2$.

5. Concluding remarks. Although the models seem to be a valuable educational aid, great care should be taken when acquainting the student with the geometric objects of the theory of functions, in order to avoid his subconsciously identifying the models with the objects themselves. Mathematics deals with abstract notions, and (geometric) theorems are correctly and completely expressive of the full content only if the geometric notions have been axiomatized,

[6] In the diagrams 6.1–6.4 only the parts lying in $y_1 \geqq 0$ of the domains are drawn. By reflecting the body on the x_1, x_2-plane, the remaining parts can be obtained. The axes x_1 and x_2 in Figs. 6.1–6.4 should be interchanged.

[7] R^4 will consist of all points ζ_1, ζ_2, where ζ_1 varies over R_1^2, and ζ_2 over R_2^2.

i.e., described in a completely abstract manner. In order to define and operate with these objects, it is necessary to build up a symbolism. For different purposes (say when we study topological structures or investigate differential geometrical properties, etc.) we need different symbolisms appropriate to these tasks.[8] When formulating geometric results, or checking them, as experience shows, in most cases it is preferable even to eliminate the inborn geometrical intuition and try to operate with symbols only.[9] It seems, however, that for geometrically minded students, when acquainting them with the geometry of the space of two complex variables, it is preferable to use, simultaneously, both approaches: operating with symbols and developing "artificial intuition." To be able to do this requires a considerable flexibility of mind. (Different types of mathematical minds, of course, will require different mental processes to get acquainted with the space of several variables.) Consequently, this situation suggests that we choose the symbolism in such a manner as to facilitate as much as possible the transition from models to symbols.

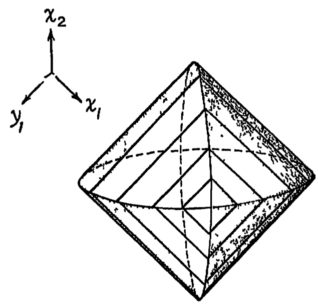


FIGURE 5

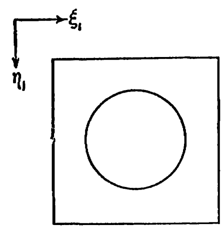


FIGURE 7

The present article is devoted to the problem of becoming acquainted with, and operating in many-dimensional domains. The acquisition of this ability may play an important part and be useful in other fields as well. Let us mention one example: When carrying out experimental work, so as to determine a physical law which can be expressed by the relation $y = y(x)$, y being a function of one real variable, with the experiments yielding the values of $y(x)$ only for a discrete sequence of points, we often draw a diagram depicting $y = y(x)$. This facilitates our grasping qualitative properties, i.e., discovering the physical law governing the phenomenon which we are investigating. In an analogous situation, when y

[8] A symbolism developed for the study of the topological structure of the domain bounded by finitely many analytical hypersurfaces can be found in [2; 3; 5; 6; 10]. For the symbolism used in differential geometric investigations, see Mémorial des Sciences Mathématiques vols. 106, 108.

[9] This, of course, does not deny that in the process of learning or carrying out research the geometrical intuition can be of greatest value. The shortness of this article prevents describing in greater detail what we mean by "geometrical intuition", or discussing different kinds of possible intuitions. See also [12], particularly Part I. Naturally, an intuition for operating with symbols can and should be developed in a student. The two approaches do not exclude each other, by any means, and in some cases they even overlap each other; but in general they mutually supplement each other.

depends upon three (or more) real parameters, the use of models and procedures similar to those described earlier in this paper may be employed successfully to aid us in drawing conclusions.[10]

A further example when a similar training may be useful would be for fliers in high altitude. Owing to the great speed and the altitude all objects appearing in the vision seem to be strangely flattened and hardly recognizable. Previous training in "seeing", under these circumstances, the objects all around may be valuable.

Let us finally remark that the approach discussed here can be interpreted also from a somewhat different point of view. In order to make geometrical conclusions, we often have to carry out some geometrical transformation of the body in question, e.g., we turn it, stretch it, etc. A real experiment may mean considerable work, and conducting and carrying this out in the imagination only may require much mental effort. The procedure described in the present paper can mean an economy in reaching the desired results and even serve to increase our capacity for drawing conclusions. Thus the use of modern computational devices may mean a great advantage in teaching and research.[11] (See [8, §4].) The shortness of this paper prevents a discussion of the extent to which these methods might increase our ability to operate with the domains of four-dimensional space.

To obtain the answers to various questions connected with the teaching, and to find the best way of presenting the geometry of the space of two variables under different circumstances, psychological experiments and studies would have to be conducted. The author hopes that this article may serve to increase the interest of educators and psychologists in these problems.

Bibliography

1. S. Bergman, *Ueber die Veranschaulichung der Kreiskörper und Bereiche mit ausgezeichneter Randfläche*, Jber. Deutschen Math. Verein. vol. 42 (1933) pp. 238–252.

2. ———, *Ueber eine in gewissen Bereichen mit Maximumfläche gültige Integraldarstellung der Funktionen zweier komplexer Variablen*, Math. Zeit. vol. 39 (1935) pp. 76–94, 605–608.

3. ———, *Ueber eine Integraldarstellung von Funktionen zweier komplexer Veränderlichen*, Rec. Math. (Mat. Sbornik) N. S. vol. 1 (1936) pp. 851–862.

4. ———, *Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen*, Rec. Math. (Mat. Sbornik) N. S. vol. 2 (1937) pp. 1169–1198.

5. ———, *Ueber uneigentliche Flächenintegrale in der Theorie der analytischen Funktionen von zwei komplexen Veränderlichen*, Revista de Ciencias, no. 438 pp. 675–682, no. 439 pp. 131–140, no. 441 pp. 377–394, 1941–1942.

6. ———, *On the surface integrals of functions of two complex variables*, Amer. J. Math. vol. 63 (1941) pp. 295–318.

7. ———, *The vizualization of domains of the theory of functions of two complex variables*, Journal of Mathematics and Physics vol. 20 (1941) pp. 107–117.

[10] The described models may be especially useful when explaining to a *non*-mathematician some qualitative or quantitative dependences of a function upon several variables.

[11] The importance of computational devices in similar connections is also discussed by Wiener, see [14].

8. ———, *Models in the theory of several complex variables*, Amer. Math. Monthly vol. 53 (1946) pp. 495–501.

9. ———, *The kernel function and conformal mapping*, Mathematical Surveys no. 5, New York, American Mathematical Society, 1950.

10. ———, *Geometric and potential-theoretical methods in the theory of functions of several complex variables*, Proceedings of the International Congress of Mathematicians, vol. 2, pp. 165–173.

11. J. Hilsenrath, *On techniques for modeling four-dimensional regions connected with the theory of functions of two complex variables* (to appear).

12. H. Poincaré, *La valeur de la science*, E. Flamarion, Editeur, Paris.

13. A. Schaeffer, D. C. Spencer, *Coefficient regions for schlicht functions*, Amer. Math. Soc. Colloquium Publications, vol. 35, New York, 1950.

14. N. Wiener, *Cybernetics*, New York, John Wiley and Son, Inc., 1948.

Harvard University,
Cambridge, Mass., U. S. A.

FUNCTIONS OF REAL VARIABLES

FUNCTIONS REPRESENTABLE AS DIFFERENCES OF SUBHARMONIC FUNCTIONS

MAYNARD G. ARSOVE

A function w will be termed *δ-subharmonic* on an open set Ω provided w has domain D $(\subset\Omega)$ and there exist subharmonic functions u and v on Ω such that $w = u - v$ holds in the extended sense on D and both u and v are $-\infty$ on $\Omega - D$. The Riesz decomposition theorem extends to such functions.

We show that if m is any generalized mass distribution on Ω, then there exists a δ-subharmonic function w having m as its generalized mass distribution. It follows that a function δ-subharmonic in the neighborhood of each point of Ω is δ-subharmonic on Ω. Among all representations (u, v) of w as the difference of subharmonic functions there exists a *canonical representation* (u^*, v^*) with u^* and v^* having minimal subharmonicity in the following sense: to each representation (u, v) of w there corresponds a subharmonic function s such that $u = u^* + s$ and $v = v^* + s$ hold on the domains of u and v, respectively. The functions u^* and v^* are unique to within a common additive harmonic function.

Sufficient conditions for δ-subharmonicity are established in terms of Blaschke (Privaloff) operators, and decompositions preserving specific properties are considered. E.g., there exists a continuous δ-subharmonic function w for which no representation (u, v) is available with u and v continuous.

A characteristic function similar to those of Nevanlinna and Privaloff is defined as follows. For (u, v) a canonical representation of w let λ be the upper envelope of u and v. We define the *characteristic function* $T_r(w, z)$ as the circumferential mean of λ on a circle $(\subset\Omega)$ with center z and radius r. Immediate applications are to the study of isolated singularities of δ-subharmonic functions and to the problem of decomposition as the difference of negative subharmonic functions. However, the characteristic function finds its most important application in the theory of entire δ-subharmonic functions. *Order*, *exponent of convergence*, *genus* are defined along classical lines, with integrals replacing infinite products. Weierstrass and Hadamard representation theorems are derived and the representation theorem of Heins for functions of order zero obtained as a consequence.

Functions of *potential type* are defined as complete entire δ-subharmonic functions of order zero having mass distributions of finite total variation. This class of functions includes potentials but has more extensive closure properties. A necessary and sufficient condition that a complete entire δ-subharmonic function w be of potential type is that $\lim_{r\to\infty}[T_r(w, 0)/\log r]$ exist finitely. If further w is subharmonic, then this limit is the total variation of the mass distribution for w.

BROWN UNIVERSITY,
PROVIDENCE, R. I., U. S. A.

METRIC SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

THØGER BANG

To a given sequence of positive numbers $\{M_n\}$ corresponds a Hadamard-class of infinitely differentiable functions, defined by the inequalities $|f^{(n)}(x)| < k^n M_n$, where k is a constant and x belongs to a given interval. A function has in each point x an "element", viz. the sequence $f(x), f'(x), \cdots$.

It is possible to organize the set of elements of functions from a given Hadamard-class as a metric space, by introducing a distance depending on the sequence $\{M_n\}$. If we in the same point x consider elements of two functions from the class, then it can be seen that their distance is a continuous function of x, and its derivative is limited by certain inequalities.

In this way it is possible to give a simple proof of the Denjoy-Carleman theorem on quasi-analytic functions. If the definition of the distance is slightly refined, we can obtain theorems on the densities of the zeros of successive derivatives of an infinitely differentiable function. Those theorems contain as special cases well-known results on the densities of zeros of derivatives of special classes of analytic functions on an interval (thus proved in a "real value" way).

UNIVERSITY OF COPENHAGEN,
COPENHAGEN, DENMARK.

ON SINGULAR INTEGRALS IN THE THEORY OF THE POTENTIAL

A. P. CALDERÓN AND A. ZYGMUND

Let $\mu(E)$ be a positive mass distribution in a bounded region D of the plane, and let $K(s, t)$ be one of the kernels

$$st(s^2 + t^2)^{-2}, \qquad s(s^2 + t^2)^{-3/2}.$$

The integrals

$$(*) \qquad \tilde{f}(x, y) = \int_D K(s - x, y - t)\, d\mu$$

occur in the theory of the potential. The former in connection with the problem of the existence of the derivatives of order two of the logarithmic potential

$$(**) \qquad u(x, y) = \int_D \log(1/r)\, d\mu, \quad r^2 = (x - s)^2 + (y - t)^2,$$

the latter in connection with the existence of the tangential derivative at $z = 0$ of the Newtonian potential

$$U(x, y, z) = \int_D (r^2 + z^2)^{-1}\, d\mu.$$

It can be shown that in all these cases the integral (*) exists almost everywhere in (x, y) as an improper integral (we eliminate from the integral in (*)

an ϵ-neighborhood of the point (x, y), and then make ϵ tend to 0). Moreover if μ is the indefinite integral of a function $f \in L^p$, $1 < p < \infty$, then also $\tilde{f} \in L^p$ If $f \log^+ f \in L$, then $\tilde{f} \in L$. These facts are included in general theorems con cerning a wide class of singular integrals in any number of dimensions, and in particular the conjugate integrals of the theory of Fourier series. In the case when μ is the indefinite integral of a function f such that $f \log^+ f$ is integrable the function u in $(^*_*)$ has derivatives of order 2 (and even a second differential almost everywhere.

UNIVERSITY OF CHICAGO,
CHICAGO, ILL., U. S. A.

FONCTIONS CROISSANTES D'ENSEMBLES ET CAPACITÉS

GUSTAVE CHOQUET

Définition. Dans un espace topologique E, on appelle ensemble K_σ toute réunion dénombrable de compacts K; ensemble $K_{\sigma\delta}$ toute intersection dénom brable de K_σ, etc.

Capacités. Soit E un espace topologique dans lequel toute intersection d'un ouvert et d'un compact soit un K_σ. Supposons qu'à tout compact $K \subset E$ on ait associé un nombre réel fini vérifiant les conditions:

1) *Croissance*: Si $K_1 \subset K_2$, on a $f(K_1) \leqq f(K_2)$;

2) *Convexité forte*: $f(K_1 \cup K_2) + f(K_1 \cap K_2) \leqq f(K_1) + f(K_2)$;

3) *Continuité à droite*: Pour tout K et tout $\epsilon > 0$, il existe un voisinage $\mathcal{V}$ de K tel que si $K \subset K' \subset \mathcal{V}$, on ait $f(K') \leqq f(K) + \epsilon$;

On dit alors que f définit une *capacité.*

Capacités interieures et extérieures. 1) Pour tout $A \subset E$, on pose $f_*(A) = \sup_{K \subset A} f(K)$;

2) Pour tout $A \subset E$, on pose $f^*(A) = \inf_{A \subset \omega} f_*(\omega)$, où ω désigne un ouvert Si l'on a $f^*(A) = f_*(A)$, on dit que A est *capacitable.*

LEMMES. 1) *La capacité extérieure vérifie l'inégalité de convexité forte.*

2) *La limite des capacités extérieures d'une suite croissante d'ensembles égale la capacité extérieure de l'ensemble limite.*

3) *Toute réunion dénombrable d'ensembles capacitables est capacitable.*

THÉORÈME. *Tout ensemble $K_{\sigma\delta\sigma}$ est capacitable.*

Application aux capacités ordinaires. Si E est un espace euclidien ou un domaine d'un tel espace et si $f(K)$ désigne la capacité ordinaire relative à la fonction de Green de E, on montre que les trois conditions précédentes sont satisfaites. D'ou

COROLLAIRE. *Pour les capacités ordinaires, tout $F_{\sigma\delta\sigma}$ (donc tout $G_{\delta\sigma}$) est capacitable*

THE SORBONNE,
PARIS, FRANCE.

THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF DIFFERENCE EQUATIONS

S. Chowla

In conjunction with other workers (Herstein, Moore, Scott) I have recently nvestigated the asymptotic behaviour of solutions of difference equations. The 'ollowing is a sample of results proved: if $U_n = U_{n-1} + (n - 1)U_{n-2}$, $U_1 = 1$, $U_2 = 2$, then

$$U_n \sim \alpha \left(\frac{n}{e}\right)^{n/2} e^{n^{1/2}}.$$

The point of the investigation is the determination of the constant α, which s fixed by $\alpha = 2^{-1/2}e^{-1/4}$. A difference equation introduced by Kerawala in the problem of the 3-deep latin rectangle (solved by Riordan, Erdös, and Kaplansky) is also discussed.

University of Kansas,
Lawrence, Kan., U. S. A.

A THEOREM ON CONDITIONAL EXTREMES WITH AN APPLICATION TO TOTAL DIFFERENTIALS

L. M. Court

This is a kind of reciprocity theorem where if one function is maximized subect to certain conditions, a related function is minimized subject to certain related conditions. Formally it resembles Mayer's law of reciprocity for isoperimetric problems in the calculus of variations.

Suppose that $\phi(q_1, \cdots, q_n)$ and ${}_\alpha G(q_1, \cdots, q_n; p_1, \cdots, p_n)$ $(\alpha = 1, \cdots, n; m < n)$ and their first and second partial derivatives are defined and continuous for $Q \equiv (q_1, \cdots, q_n) \in \mathfrak{A}$ and $P \equiv (p_1, \cdots, p_n) \in \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are domains in their respective spaces. Suppose furthermore that there is a one-to-one correspondence between the points of $\mathfrak{A}$ and $\mathfrak{B}$; that is, to each $Q \in \mathfrak{A}$ there is a $P \equiv (p_1, \cdots, p_n) \equiv (p_1^Q, \cdots, p_n^Q) \in \mathfrak{B}$ such that $\phi(q_1, \cdots, q_n)$ is rendered stationary at Q subject to the m constraints ${}_\alpha G(q_1, \cdots, q_n; p_1^Q, \cdots, p_n^Q) = 0$, the p's being regarded as parameters or inactive variables in these constraints. Let $q_i = q_i(p_1, \cdots, p_n)$ $(i = 1, \cdots, n)$ be the relations that render ϕ conditionally stationary at Q—they express the correspondence between the points of $\mathfrak{A}$ and $\mathfrak{B}$. Then these relations $q_i = q_i(p_1, \cdots, p_n)$—or what amounts to the same thing, their inverses $p_i = p_i(q_1, \cdots, q_n)$—will also render the related function $\psi(p_1, \cdots, p_n) \equiv \phi[q_1(p_1, \cdots, p_n), \cdots, q_n(p_1, \cdots, p_n)]$ stationary at P subject to the m constraints ${}_\alpha G(q_1^P, \cdots, q_n^P; p_1, \cdots, p_n) = 0$, the q's now being regarded as inactive variables. Moreover, if the second order conditions in Lagrange multiplier form for the proper or improper maximization (minimization) of $\phi(q_1, \cdots, q_n)$ subject to ${}_\alpha G(q_1, \cdots, q_n; p_1^Q, \cdots, p_n^Q) = 0$ $(\alpha = 1, \cdots, m)$, the p's inactive, are satisfied at Q, then the second order conditions in the same form for the proper or improper minimization (maximization) of $\psi(p_1, \cdots, p_n)$ subject to the constraints ${}_\alpha G(q_1^P, \cdots, q_n^P; p_1, \cdots, p_n) = 0$, the q's inactive, will be satisfied at P.

As an addendum, let us suppose that the total differential $\sum_{i=1}^n C_i(q_1, \cdots, q_n)dq_i$ is integrable so that $\gamma(q_1, \cdots, q_n) \sum_{i=1}^n C_i dq_i = \sum_{i=1}^n \phi_i dq_i = d\phi$. Suppose now we can find m functions ${}_\alpha G(q_1, \cdots, q_n; p_1, \cdots, p_n)$ such that the $m + n$ equations ${}_\alpha G = 0$, $C_i - \sum_{\alpha=1}^m {}_\alpha\mu \partial_\alpha G/\partial q_i = 0$ can be solved for the $m + n$ "unknowns" $q_1, \cdots, q_n, {}_1\mu, \cdots, {}_m\mu$ in terms of the p's. Then, because of the relationship between ϕ, ψ, and the ${}_\alpha G$ alluded to in the earlier theorem, the total differential $\sum_{k=1}^n D_k(p_1, \cdots, p_n)dp_k$, where $D_k \equiv \sum_{\alpha=1}^m {}_\alpha\mu\partial_\alpha G/\partial p_k$ and the q's in the $\partial_\alpha G/\partial p_k$ have been replaced by their values in terms of the p's, will be integrable.

Rutgers University,
New Brunswick, N. J., U. S. A.

ON APPROXIMATING TO FUNCTIONS OF SEVERAL VARIABLES BY FUNCTIONS OF FEWER VARIABLES

S. P. Diliberto and E. G. Straus

Let C_R denote the set of all real-valued continuous functions $z = z(x, y)$ defined on the unit rectangle R: $0 \leqq x, y \leqq 1$. Let C_S denote the subset of C_R composed of all functions of the form $w = w(x, y) = f(x) + g(y)$. For $z \in C_R$ define $\| z \| = \mathrm{Max}_{(x,y)\in R} | z(x, y) |$; and for $z \in C_R$ define $\mu[z]$ (the distance from z to C_S in the space C_R) by $\mu[z] = \inf_{w \in C_S} \| z - w \|$.

The following two problems (proposed by RAND Corp.) are solved: Problem 1. For $z \in C_R$ and $\delta \geqq 0$, $\epsilon > 0$ give a method for determining whether

$$| \mu[z] - \delta | < \epsilon.$$

Problem 2. For $z \in C_R$ and $\epsilon > 0$ give a method for determining $f(x) + g(y) \in C_S$ such that $\| z(x, y) - f(x) - g(y) \| - \mu[z] < \epsilon$. In addition, the results are extended to z which are functions of n-variables—and discontinuous.

We shall define the two principal devices introduced and state how they are used to solve the problems. First, let $\mathfrak{L}$ denote the class of all closed polygonal paths in R (not necessarily simple)—with sides alternately perpendicular and parallel to the x-axis. For each $l \in \mathfrak{L}$, with vertices $\{p_i\}$ $(i = 1, \cdots, n)$—numbered in consecutive order on l, define for each $z \in C_R$ the functionals

$$\Pi_l[z] = (1/n) \left| \sum_{i=1}^n (-1)^i z(p_i) \right| ; \qquad \Pi[z] = \sup_{l \in \mathfrak{L}} \Pi_l[z].$$

One establishes that for $l \in \mathfrak{L}$, $\Pi_l[z] \leqq \mu[z]$ and hence $\Pi[z] \leqq \mu[3]$.

Second, we shall say $z \in C_R$ is "level" if for each y, $\mathrm{Max}_x z(x, y) = -\mathrm{Min}_x z(x, y)$ and for each x, $\mathrm{Max}_y z(x, y) = -\mathrm{Min}_y z(x, y)$. From properties of $\Pi[z]$ it follows that if z is level, then $\Pi[z] = \mu[z] = \| z \|$ and this motivated our definition of the "leveling sequence for z": Given $z' = z'(x, y) \in C_R$ define inductively sequences $\{z^i\}$ and $\{w^i\}$ by (a) $z^{i+1}(x, y) = z^i(x, y) - w^i(x, y)$, (b) i odd: w^i is a function of y alone given by $w^i(y) = (1/2)\{\mathrm{Max}_x z^i(x, y) + \mathrm{Min}_x z^i(x, y)\}$, (c) i even: w^i is a function of x alone given by

$$w^i(x) = (1/2)\{\mathrm{Max}_y z^i(x, y) + \mathrm{Min}_y z^i(x, y)\}.$$

A first result concerning leveling is that $\| z^i \| \geqq \| z^{i+1} \| \geqq \mu[z]$; and hence $M[z] = \lim_{i\to\infty} \| z^i \| \geqq \mu[z]$. Our principal result is the

THEOREM. $\Pi[z] = \mu[z] = M[z]$.

The left-hand half of the above equations solves Problem 1 and the right-hand half solves Problem 2. Other results included are the equicontinuity of the leveling sequence and estimates on the rate at which $\| z^i \| \to \mu[z]$.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF., U. S. A.
UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIF., U. S. A.

STIELTJESSCHE INTEGRALE

KERIM ERIM

In dieser Mitteilung wollen wir die zuerst (1937) von H. Copeland für den eindimensionalen Fall aufgestellte und später (1939–1941) von uns auf die zwei- und dreidimensionalen Fälle übertragene neue Definition des Stieltjesschen Integrals auf den k-dimensionalen Fall systematisch erweitern.

Die Bildung einer besonderen Folge: $f(x_i)$ sei eine im Intervall $I(\alpha_i \leqq x_i \leqq \beta_i$, $i = 1, \cdots, k)$ stetige Funktion von x_i $(i = 1, \cdots, k)$. Man kann unter gewissen Bedingungen sehr leicht auf den Fall eingehen, wo $f(x_i)$ nicht mehr stetig ist. Wir werden hier den Fall, wo f eine monoton wachsende Funktion ist, behandeln. Ausserdem setzen wir voraus, dass

$$f(\beta_1, \cdots, \beta_k) = 1 \text{ und } f(x_1, \cdots, x_{i-1}, \alpha_i, x_{i+1}, \cdots, x_k) = 0 \qquad (i = 1, \cdots, k)$$

sind (diese Voraussetzungen sind unwesentlich). Ferner setzen wir

$$n_1^{(k)} = 1, \qquad n_\nu^{(k)} - n_{\nu-1}^{(k)} = \nu^k \qquad (\nu = 1, 2, \cdots).$$

Zur gegebenen Funktion f bilden wir nun eine Stellenfolge P_m. Wir betrachten die ν^k Stellen der Folge P_m, numeriert von $n_{\nu-1} + 1$ bis n_ν (in beliebiger Anordnung), mit den Koordinaten

$$x_1 = x_1(\nu, \lambda_1), \qquad x_2 = x_2(\nu, \lambda_1, \lambda_2), \cdots, \qquad x_k = x_k(\nu, \lambda_1, \cdots, \lambda_k)$$

$(\lambda_1, \lambda_2, \cdots, \lambda_k = 1, \cdots, k)$; wobei die $x_q(\nu, \lambda_1, \cdots, \lambda_q)$ der Reihe nach bestimmt sind als die unteren Grenzen der Zahlen; die den folgenden Bezeihungen genüngen: Zunächst sind die $x_1(\nu, \lambda_1)$ bestimmt durch

$$f[x_1(\nu, \lambda_1), \beta_2, \cdots, \beta_k] = (2\lambda_1 - 1)/2\nu$$

und allgemein die $x_q(\nu, \lambda_1, \cdots, \lambda_q)$ durch $(k \geqq q > 1)$,

(1) $\Delta_f I_{q-1} = 2\lambda_q - 1/2\nu^q$, I_{q-1} bedeutet das Intervall $x_i'(\nu, \lambda_1, \cdots, \lambda_i - 1) \leqq x_i \leqq x_i'(\nu, \lambda_1, \cdots, \lambda_i)$ $(i = 1, \cdots, q - 1)$,

(2) mit $\Delta_f I_{q-2} = \lambda_{q-1}/\nu^{q-1}$, die Bedeutung von I_{q-2} ist ähnlich der von I_{q-1}, wobei $\Delta_f I_{q-1}$ die Variation von f im Intervall I_{q-1} derart bedeutet, dass man in der Funktion f, die in (1) auftritt, ausser den Variabeln $x_1, \cdots, x_{q-1}$ die andern durch $x_q = x_q(\nu, \lambda_1, \cdots, \lambda_q)$ und $x_i = \beta_i$ für $i \geqq q + 1$, und in der Funktion f, die in (2) auftritt, ausser den $x_1, \cdots, x_{q-2}$ die andern durch $x_{q-1} = x_{q-1}'(\nu, \lambda_1, \cdots, \lambda_{q-1})$ und $x_i = \beta_i$ für $i \geqq q$ ersetzen muss. Ferner ist $\Delta_f I_0 = f$.

Die Folge P_m hat die Grenzhäufigkeit $f(\zeta_i)$. D.h.wenn h_m die Anzahl derjenigen Elementen unter den ersten m Elementen der Folge P_m ist, die in das Teilintervall $I'(\alpha_i \leqq x_i \leqq \zeta_i ; i = 1, \cdots, k)$ fallen, so ist

$$\lim_{m\to\infty} h_m/m = f(\zeta_i).$$

Die neue Definition: Es sei g eine andere beschränkte Funktion. So ist

$$\lim_{m\to\infty}[g(P_1) + \cdots + g(P_m)]/m$$

die neue Definition des Stieltjes-Integrals von g in Bezug auf f über das Intervall I.

Diese Definition lässt sich auf den Fall übertragen, wo f eine Funktion von beschränkter Schwankung ist. Es lässt sich ebenso zeigen, dass die neue Definition, wenn die klassische Definition einen Sinn hat, mit ihr übereinstimmt. Aber es gibt den Fall, in dem zwar die klassische Definition versagt, die neue Definition aber einen Sinn hat.

ISTANBUL, TURKEY.

ON TRANSFINITE RADIUS

MICHAEL FEKETE

Call $g(r)$ a generator function if (a) $g(r)$ is continuous for $0 < r < \infty$; (b) $g(r)$ is strictly decreasing with r^{-1}; (c) $\lim_{r\to 0} g(r) = \infty$. Call the locus L of a variable point P of a Euclidean space E_q of $q \geqq 1$ dimensions a "lemniscatoid with $g(r)$" if the "mean value with $g(r)$" of its distances r_ν, $1 \leqq \nu \leqq n$, to a finite number $n \geqq 1$ of fixed (not necessarily distinct) points P_ν, $1 \leqq \nu \leqq n$, of E_q (the "foci" of L) is $\leqq$ a positive constant r_0 (the "radius of L"), i.e., (*) $\sum_{\nu=1}^{n} g(r_\nu) \geqq ng(r_0)$. C being a limited-closed pointset of E_q, put $r^* =$ minimum of r_0 when the "confocal" lemniscatoids (*) contain C. Call the minimum

$\rho_n = \rho_n(C, g)$ of $r^* = r^*(C; g; P_\nu)$, when P_ν, $1 \leqq \nu \leqq n$, is a set of $n \geqq 1$ variable points of E_q, the "radius of order n of C with $g(r)$". Since $(n + m)g(\rho_{n+m}) \geqq ng(\rho_n) + mg(\rho_m)$, there exists $\lim_{n\to\infty} g(\rho_n)$ and thus also $\lim_{n\to\infty}\rho_n = \rho = \rho(C, g)$, the "transfinite radius of C with $g(r)$". For arbitrary $n \geqq 1$ and $g(r)$, $\rho_n(C, g) \leqq \delta_{n+1}(C, g)$, the "diameter of order $(n + 1)$ of C with $g(r)$" = maximum of the mean value with $g(r)$ of the $C_{n+1,2}$ mutual distances of Q_μ, Q_ν when $n \geqq 1$ and Q_i, $1 \leqq i \leqq n$, is a subset of C; thus $\lim_{n\to\infty}\delta_n = \delta = \delta(C, g)$, the "transfinite diameter of C with $g(r)$" is $\geqq \rho(C, g)$. (Cf. the author's paper *On generalized transfinite diameter*, Bull. Amer. Math. Soc. Abstract 53-7-288.) In case $q = 2$, $g(r) = \log 1/r$ and $q = 3$, $g(r) = 1/r$ holds $\rho = \delta$. Like $\delta(C, g)$, $\rho(C, g)$ also possesses the properties of monotonicity and continuity, considered l.c. and obeys the "law of subadditivity": If $C = C' + C''$ and $\rho(C', g) = \rho'$, $\rho(C'', g) = \rho''$; σ = maximum of the distance of Q', Q'' when $Q' \in C'$, $Q'' \in C''$, then $(g(\rho) - g(\sigma))^{-1} \leqq (g(\rho') - g(\sigma))^{-1} + (g(\rho'') - g(\sigma))^{-1}$. This leads, in case $q = 3$, $g(r) = 1/r$, to a new improved form of Kellogg's rule of conductor capacities.

HEBREW UNIVERSITY,
JERUSALEM, ISRAEL.

SUB-BIHARMONIC FUNCTIONS

R. N. HASKELL

If $U(M)$ is of class C^4 in a domain D, then $U(M)$ is sub-biharmonic in D if $\nabla^4 U(M) \leqq 0$ at all points M of D where $\nabla^4 U = \nabla^2(\nabla^2 U)$ and $\nabla^2 U = U_{xx} + U_{yy}$. Thus $\nabla^2 U$ is superharmonic in D and therefore the areal mean $A_r(\nabla^2 U; x_0, y_0) = (1/\pi r^2)\int_0^r\int_0^{2\pi}\nabla^2 U(x_0 + \rho\cos\varphi, y + \rho\sin\varphi)\rho d\rho d\varphi$ is a decreasing function of r for fixed (x_0, y_0). Now if r and λr are the radii of two circles in D with centers at (x_0, y_0), where λ is a fixed constant between 0 and 1, we easily derive by Green's theorem that

$$U(x_0, y_0) \leqq \frac{1}{1 - \lambda^2} L_{\lambda r}(U; x_0, y_0) - \frac{\lambda^2}{1 - \lambda^2} L_r(U; x_0, y_0) = \mathfrak{L}_r(U; x_0, y_0)$$

where $L_r(U; x_0, y_0) = (1/2\pi)\int_0^{2\pi} U(x_0 + r\cos\theta, y_0 + r\sin\theta)\, d\theta$. From this inequality we easily derive that

$$U(x_0, y_0) \leqq \frac{1}{1 - \lambda^2} A_{\lambda r}(U; x_0, y_0) - \frac{\lambda^2}{1 - \lambda^2} A_r(U; x_0, y_0) = \mathfrak{A}_r(U; x_0, y_0)$$

where $A_r(U; x_0, y_0) = (2/r^2)\int_0^r L_\rho(U; x_0, y_0)\,\rho d\rho$. It is easily seen that the necessary and sufficient conditions that a function of class C^4 be sub-biharmonic are that the above inequalities hold for all points (x_0, y_0) of D and for all concentric circles of radii r and λr with (x_0, y_0) as center which lie in D. If $U(M)$ is biharmonic, only the equality signs hold.

Now suppose r_1 and r_2 are the radii of any two concentric circles in D with $r_1 < r_2$. By Green's theorem

$$\mathfrak{L}_{r_2}(U; x_0, y_0) - \mathfrak{L}_{r_1}(U; x_0, y_0)$$
$$= \frac{\lambda^2}{2(1-\lambda^2)} \int_{r_1}^{r_2} [A_{\lambda r}(\nabla^2 U; x_0, y_0) - A_r(\nabla^2 U; x_0, y_0)] r\, dr$$

and since $\nabla^2 U$ is superharmonic, $\mathfrak{L}_{r_1}(U; x_0, y_0) \leqq \mathfrak{L}_{r_2}(U; x_0, y_0)$ and since $\mathfrak{A}_r(U; x_0, y_0) = (2/r^2)\int_0^r \mathfrak{L}_\rho(U; x_0, y_0)\, \rho d\rho$, we also have $\mathfrak{A}_{r_1}(U; x_0, y_0) \leqq \mathfrak{A}_{r_2}(U; x_0, y_0)$. Thus for each point (x_0, y_0) of D, $\mathfrak{L}_r(U; x_0, y_0)$ and $\mathfrak{A}_r(U; x_0, y_0)$ are increasing functions of r.

We now restrict somewhat Nicolesco's (M. Nicolesco, *Sur un nouveau théorème de moyenne pour les fonctions polyharmoniques*, Bulletin Mathématique de la Société Roumaine des Sciences vol. 38 II, p. 116) definition of generalized sub-biharmonic functions and define them as those functions $U(M)$ which are bounded, summable superficially and on the circumferences of all circles in D, and such that at each point (x_0, y_0) of D, $U(x_0, y_0) = \lim_{r=0} A_r(U; x_0, y_0)$, and which for all points (x_0, y_0) and pairs of concentric circles with (x_0, y_0) as center, $U(x_0, y_0) \leqq \mathfrak{L}_r(U; x_0, y_0)$. From this we have immediately $U(x_0, y_0) \leqq \mathfrak{A}_r(U; x_0, y_0)$. Now let $U_r^{(1)}(x_0, y_0) = A_r(U; x_0, y_0)$. Then $U_r^{(1)}(x, y)$ is sub-biharmonic and continuous for r fixed in a domain D' entirely in D. Similarly we may define $U_r^{(2)} = A_r(U_r^{(1)}; x_0, y_0)$, $U_r^{(3)}$, $U_r^{(4)}$, and $U_r^{(5)}$. Then $U_r^{(5)}$ is of class C^4 in $D^{(5)}$ and $\mathfrak{L}_{r_1}(U_r^{(5)}; x_0, y_0) \leqq \mathfrak{L}_{r_2}(U_r^{(5)}; x_0, y_0)$ if $r_1 < r_2$. Now because of the properties of $U(M)$ we may let $r = 0$ and derive that $\mathfrak{L}_{r_1}(U; x_0, y_0) \leqq \mathfrak{L}_{r_2}(U; x_0, y_0)$. Therefore $\mathfrak{A}_r(U; x_0, y_0)$ is also an increasing function of r and $\mathfrak{A}_r(U; x, y)$ is a continuous sub-biharmonic function for r fixed. Moreover we have that $U(x_0, y_0) \leqq \mathfrak{A}_r(U; x_0, y_0) \leqq \mathfrak{A}_{rr}(U; x_0, y_0) \leqq \cdots \leqq \mathfrak{A}_{rrrrr}(U; x_0, y_0) = \mathfrak{A}_r^{(5)}(U; x_0, y_0)$. The function $U_r^{(5)}(x, y)$ is sub-biharmonic of class C^4 and by means of a proof similar to that of Evans (Trans. Amer. Math. Soc. vol. 37 (1935) p. 237) for subharmonic functions we show that $U(M) = \int_{D'} \log(1/MP)\rho(P)d\sigma_P + H_1(M)$ where D' is entirely in D, $\rho(M)$ is subharmonic and $H_1(M)$ is harmonic in D'. By the theorem of Riesz, $\rho(M) = -\int_{D'} \log(1/MP)d\mu(e_P) + H_2(M)$; M in D', where $H_2(M)$ is harmonic and $\mu(e)$ is a positive mass distribution on sets e in D measurable Borel. We thus derive that a necessary and sufficient condition that $U(M)$ be sub-biharmonic in D is that

$$U(M) = -\int_{D'} (MP)^2 \log(1/MP)\, d\mu(e_P) + B(M); \qquad M \text{ in } D',$$

where $B(M)$ is biharmonic. As a corollary we see that $U(M)$ is of class C'.

UNIVERSITY OF TEXAS,
AUSTIN, TEX., U. S. A.

DIFFERENTIABLE INEQUALITIES, CONVEXITY, AND MIXED VOLUMES

Edgar R. Lorch

It is proposed here to develop a new approach to the theory of convexity (or concavity). The principal applications of our methods will be made to the theory of inequalities and to the theory of volumes and mixed volumes of convex bodies. In the latter connection, some functions and relations are obtained which seem fundamental. It is believed that the solution of certain principal problems in this domain will be facilitated by the exploitation of these methods. Their strength is derived from the fact that they are based on the highly developed subjects of positive definite quadratic forms, homogeneous differentiable functions, and homeomorphisms.

First a definition of convexity is introduced. Let $\mathfrak{S}$ be a surface, not necessarily closed, in n-space with an equation $\phi(x) \equiv \phi(x_1, \cdots, x_n) = 1$ where $\phi(x)$ is positively homogeneous of degree 1. It is assumed that the derivatives of ϕ, denoted by ϕ_i, ϕ_{ij}, etc., exist in sufficient number. Let $r > 1$, and consider the function $G(x) = r^{-1}\phi^r(x)$. Let $G(x)$ be defined over an open convex ray space Ω. Consider the transformation $T: x_i^* = G_i(x)$ which maps $x \in \Omega$ into a vector x^* belonging to a set Ω^* of the space adjoint to the x space. Consider also the quadratic form Q: $\sum G_{ij}(x)z_iz_j$. We define: $\mathfrak{S}$ is convex if Q is positive definite for every $x \in \Omega$. In that case, T^{-1} exists and there is a function $F(x^*)$ positively homogeneous of order r', $1/r + 1/r' = 1$, such that T^{-1} is given by: $x_i = F_i(x^*)$, $x^* \in \Omega^*$. The surface $\mathfrak{S}^*$: $\psi^{r'}(x^*) = r'F(x^*) = 1$ is the adjoint (polar) of $\mathfrak{S}$: $rG(x) = 1$.

It is shown how to obtain all bilinear inequalities of the form $\sum x_i^* y_i \leqq \psi(x^*)\phi(y)$, $y \in \Omega$, $x^* \in \Omega^*$. Also, it is proved that the surface bounding a body convex in the classic sense may be approximated by surfaces convex in our sense.

If $\mathfrak{S}_0$ is a closed surface bounding a convex body $\mathfrak{K}_0$ defined by $rG(x) \leqq 1$, and if V^* denotes the volume bounded by $\mathfrak{S}^*$, then $V^* = \lim_{r\to 1}[n(r-1)]^{-1} \cdot \int_\omega | G_{ij}(x) | \, d\omega$ where the integration is over the unit sphere. Similarly if $rH(x) = 1$ defines a surface $\mathfrak{S}_1$ bounding $\mathfrak{K}_1$, and if V_ν^* represents the νth mixed volume of the body $\lambda\mathfrak{K}_0^* + \mu\mathfrak{K}_1^*$, where the addition of the indicated bodies is performed in the classical sense, then

$$C_{n,\nu}V_\nu^* = \lim_{r\to 1}[n(r-1)]^{-1} \int_\omega | G_{ij}(x) | \, T(\| H_{ij}(x) \| \cdot \| G_{ij}(x) \|^{-1})^{(\nu)} \, d\omega.$$

The symbols are defined as follows: T means "trace." If $\| a_{ij} \|$ is a nonsingular matrix, then $\| a_{ij} \|^{(\nu)}$ is its νth compound matrix. It is the matrix induced by $\| a_{ij} \|$ on the Grassmannian coordinates of order $C_{n,\nu}$; thus $\| a_{ij} \|^{(\nu)}$ is the matrix of the ν rowed minors of $\| a_{ij} \|$.

Columbia University,
New York, N. Y., U. S. A.

A UNIFORM METHOD FOR INTEGRATING (a) $\int dx/Q_1(Q_2)^{1/2}$ AND (b) $\int dx/(Q_1Q_2)^{1/2}$, WHERE Q_1 AND Q_2 ARE DISTINCT QUADRATIC FUNCTIONS OF x

Harris F. MacNeish

The substitution $y^2 = A(x - r_1)^2/(A(x - r_1)^2 + B(x - r_2)^2)$, where r_1 and r_2 are the roots of the Wronskian determinant of the two quadratics reduces (a) to forms integrable by standard formulas, and (b) to an elliptic integral of the first type.

The following theorems are important in the development of the method used.

Theorem I. *Any two quadratics which do not have a common root are each linear functions of* $(x - r_1)^2$ *and* $(x - r_2)^2$, *where* r_1 *and* r_2 *are the roots of the Wronskian determinant of the two quadratics.*

Theorem II. *If the discriminant of the Wronskian determinant of two quadratics vanishes, the two quadratics have a common root and conversely.*

University of Miami,
Coral Gables, Fla., U. S. A.

A MINIMAL PROBLEM FOR HARMONIC FUNCTIONS IN SPACE

E. P. Miles, Jr.

Let V be the interior of the sphere S with radius a and center at the origin. For $U \in C'$ in V, let $A[U] = \iiint_V | \nabla U(P) |^2 dV_P$ where $| \nabla U(P) |$ is the magnitude of grad U at P. The author considers the minimizing of $A[U]$ in the class Γ determined as follows. Let $h_1, h_2, \cdots, h_k$ be a set of arbitrary real numbers associated with k fixed numbers $0 = \theta_1 < \theta_2 < \cdots < \theta_k = \pi$. Let $\lambda_1(\theta)$ $[\lambda_2(\theta)]$ be the step function which equals the smaller [larger] of the two numbers h_j, h_{j+1} on the interval $\theta_j \leqq \theta < \theta_{j+1}$, $j = 1, 2, \cdots, k - 1$. Let $\lambda_1(\pi) = \lambda_1(\theta_{k-1})$, $\lambda_2(\pi) = \lambda_2(\theta_{k-1})$. Assume max $\lambda_1(\theta) >$ min $\lambda_2(\theta)$, then $U(P) = U(r, \theta, \phi) \in \Gamma$ if U is harmonic in V and $\lambda_1(\theta) \leqq \liminf_{r\to a^-} U(r, \theta, \phi) \leqq \limsup_{r \to a^-} U(r, \theta, \phi) \leqq \lambda_2(\theta)$. The author uses the results of his forthcoming paper (*Certain properties of functions harmonic within a sphere* (see Bull. Amer. Math. Soc. Abstract 54-3-139)) and techniques employed by J. J. Gergen and F. G. Dressel in their papers (*A minimal problem for harmonic functions*, Duke Math. J., vol. 14 (1947); and *Mapping by p-regular functions* submitted to Duke Math. J.) to establish the existence of a unique function U' which minimizes $A[U]$ in Γ, and to study the properties of U'. In particular U' is shown to be independent of ϕ and to minimize $A[U]$ in a larger class containing Γ. The function $u(x, y) = u(r, \theta) = U'(r, \theta, 0)$ is p-harmonic with $p = y$ in the sense of Gergen and Dressel in the semicircular section of V where $\phi = 0$ and $y \geqq 0$. By complex variable and integral equation methods the function u (and hence U') is shown to have the product of its normal and tangential derivative approach zero at all boundary points except those for which $\theta = \theta_j$, $j = 0, 1, \cdots, k$.

ALABAMA POLYTECHNIC INSTITUTE,
AUBURN, ALA., U. S. A.

NOTE ON UNIFORM CONTINUITY

ANTONIO A. MONTEIRO AND M. MATOS PEIXOTO

If E is a metric space, it is shown that the following conditions are equivalent: (1) Every continuous real-valued function defined on E is also uniformly continuous; (2) The distance between any two disjoint closed sets of E is positive; and (3) There exists a Lebesgue number for every open covering of E. A previous characterization of metric spaces for which condition (1) holds was given by R. Doss (Proceedings of the Mathematical and Physical Society of Egypt vol. 3 (1947) pp. 1–6) and requires that every sequence of points $\{x_n\}$, for which there exists another disjoint sequence $\{y_n\}$ such that the distance $d(x_n, y_n) \to 0$, contains a convergent subsequence $\{x_{n_i}\}$.

UNIVERSITY OF CHICAGO,
CHICAGO, ILL., U. S. A.

ON THE STRONG DIFFERENTIATION OF THE INDEFINITE INTEGRAL

ATHANASIOS PAPOULIS

Given a summable function $f(x, y)$ defined everywhere in the plane, we form the Lebesgue integral $W(I) = \iint_I f(x, y)\, dx\, dy$ with I any interval on the plane. It is well known that the *ordinary* derivative $W'(x, y)$ of the set function $W(I)$ exists and equals $f(x, y)$ p.p. (almost everywhere). This is not true in general for the *strong* derivative $W_s(x, y)$ of $W(I)$ unless further assumptions are made on $f(x, y)$. It was established by B. Jessen, J. Marcinkiewicz, and A. Zygmund that if $f \log |f|$ is summable, then the indefinite integral $W(I)$ is almost everywhere differentiable in the strong sense.

In this paper an example of an integrable function $f(x, y)$ is given, whose indefinite integral $W(I)$ has no strong derivative at any point of a set S of positive measure; in fact, $\overline{W}_s(x, y) = +\infty$ for every point (x, y) of S (see also H. Busemann and W. Feller, Fund. Math. (1934)). This example is based on a construction due to H. Bohr with which he established the necessity of the regularity of the intervals for the validity of Vitali's covering theorem.

With $F(I) = \iint_I |f(x, y)|\, dx\, dy$ it is next shown that if $F(I)$ is strongly differentiable p.p., then so is $W(I)$. The proof is due to Professor B. Jessen and was kindly communicated to the author by Professor A. Zygmund.

The main part of the paper shows that the converse is not true; i.e., there exist integrable functions $f(x, y)$ for which $W(I)$ is strongly differentiable p.p., but the strong derivative $F_s'(x, y)$ of $F(I)$ exists nowhere on a set S of positive measure; in fact, $\bar{F}_s(x, y) = +\infty$ for every point (x, y) of S. An example of such a function is given based as before on the construction by H. Bohr.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA., U. S. A.

A CRITERION FOR TOTAL POSITIVITY OF MATRICES

ANNE M. WHITNEY

A matrix $A = \| a_{ij} \|$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ is called totally positive whenever the elements of A together with those of all its adjugates are non-negative; if all these elements are actually positive, A will be called strictly totally positive.

A criterion for total positivity is obtained by means of a reduction process. In these considerations the following approximation theorem is useful: Given an $\epsilon > 0$ and a totally positive matrix A, there exists a strictly totally positive matrix B such that $b_{ij} \geqq a_{ij}$ and $b_{ij} - a_{ij} < \epsilon$; i.e., any totally positive matrix can be approximated arbitrarily well, elementwise, by a strictly totally positive matrix.

The first column of a totally positive matrix either consists entirely of zeros or it consists of positive elements at the top, followed by zero elements, if any. Suppose the pth column to be the first nontrivial one and the qth row to contain the last positive element in this column. Starting with the qth row, moving upward, and ending with the 2nd row, we obtain a zero in the pth column by subtracting $a_{ip}/a_{i-1,p} \times$ the $(i - 1)$st row from the ith row. Let C be the matrix obtained from this reduction of A by deleting the first p columns. We may then state: A will be totally positive if and only if C is totally positive.

This reduction theorem allows us to obtain a canonical decomposition of every $n \times n$ strictly totally positive matrix into a product of at most $2n + 1$ matrices each of which is either a diagonal matrix or the inverse of a Jacobi of a special kind. It also affords simpler proofs for some theorems of Gantmakher and Krein (*Sur les matrices complètement non-negatives et oscillatoires*, Compositio Math. vol. 4 (1936) pp. 445–476).

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA., U. S. A.

THE CHANGE OF RESISTANCE UNDER CIRCULAR SYMMETRIZATION

VIDAR WOLONTIS

Let E_1 and E_2 be two disjoint closed bounded plane sets such that the complement R of their union is connected and bounded by a finite number of analytic curves. There exists a continuous function $u(z)$ which is equal to 0 on E_1 and 1 on E_2 and is harmonic in R. If $D(u)$ is the Dirichlet integral of u over R, its reciprocal $1/D(u)$ is the *resistance* $\lambda(E_1, E_2)$ *between* E_1 *and* E_2. For the case of less restricted boundary of R a broader definition of λ is given; the final conclusion of the paper holds for any closed sets E_1, E_2.

Let L be a straight line through the origin, dividing the plane into the half planes H and $\bar{H}$, and denote by $\bar{E}_1$ and $\bar{E}_2$ the sets obtained by reflecting E_1 and E_2, respectively, in L. It is proved that the resistance between the sets $E_1' = (E_1 \cap \bar{E}_1) \cup (E_1 \cap \bar{H}) \cup (\bar{E}_1 \cap \bar{H})$ and $E_2' = (E_2 \cap \bar{E}_2) \cup (E_2 \cap H) \cup (\bar{E}_2 \cap H)$ is not less than $\lambda(E_1, E_2)$.

Now let E_1^* be the set whose intersection with every circle $|z| = r$ is an arc whose midpoint is on the positive real axis and whose length is equal to the measure of the intersection of $|z| = r$ with E_1, and E_2^* the corresponding set for E_2 and the negative real axis. By a sequence of applications of the above result the following theorem is proved: $\lambda(E_1^*, E_2^*) \geqq \lambda(E_1, E_2)$.

The proof of the inequality $\lambda(E_1', E_2') \geqq \lambda(E_1, E_2)$ is based on the following representation of $\lambda(E_1, E_2)$ in terms of a generalized logarithmic potential:

$$\lambda(E_1, E_2) = \frac{1}{2\pi} \max_{\mu_i} \min_{z_i \in E_i} \int_{E_1} \int_{E_2} \omega(z_1, z_2, \xi_1, \xi_2) \, d\mu_1(\xi_1) \, d\mu_2(\xi_2),$$

where the maximum is taken with respect to distributions μ_i of unit mass on E_i $(i = 1, 2)$ and $\omega(z_1, z_2, \xi_1, \xi_2)$ is a function harmonic exterior to $F = (E_1 \cap \bar{E}_1) \cup (E_2 \cap \bar{E}_2)$, with logarithmic poles at ξ_1 and ξ_2, and with equal values at symmetric boundary points of F and opposite values of the normal derivative. The existence of this function is inferred by defining an abstract Riemann surface by identification of symmetric boundary points of F; on this Riemann surface ω is an Abelian integral of the third kind.

UNIVERSITY OF KANSAS,
LAWRENCE, KAN., U. S. A.

SURFACE AREA AND HOMOTOPY

J. W. T. YOUNGS

If X is a 2-cell and $f: X \to E^3$ is a mapping (usually given in analysis as a triple of real-valued continuous functions of two variables) from X into Euclidean 3-space, then f determines a surface $[f]$. (See Youngs, *Curves and surfaces*, Amer.

Math. Monthly vol. 51 (1944) pp. 1–11.) Attention is devoted to the case ir which the Lebesgue area of the surface $[f]$ — written $L([f])$ — is zero. Wel known examples show that one can have $L([f]) = 0$ and yet $|f(X)|$, the 3-dimen sional Lebesgue measure of the image of X under f, can be positive. Indeed, th(classical example is one in which $f(X)$ is a cube in E^3. To the layman this is para doxical in view of the justifiable impression that for a surface to have "zer(area" the "set occupied by the surface" (i.e., $f(X)$) should be "slender," fo: example, consist of nothing more than a finite number of arcs. The purpose o: this abstract is to announce a result which shows that the layman is correct, s(to speak, modulo an ϵ-homotopy. Specifically, if $L([f]) = 0$ and $\epsilon > 0$, then ther(is a homotopy h^ϵ; $(X \times I) \to E^3$ such that: 1. $h^\epsilon(x, 0) = f(x)$, $x \in X$; 2. Fo: fixed x, the diameter of the set $h^\epsilon(x \times I)$ is less than ϵ; 3. If $g^\epsilon(x) = h^\epsilon(x, 1)$ $x \in X$, then $g^\epsilon(X)$ lies in a 1-dimensional polytope. The converse of this resul' is also true.

Indiana University,
Bloomington, Ind., U. S. A.
The RAND Corporation,
Santa Monica, Calif., U. S. A.

FUNCTIONS OF COMPLEX VARIABLES

ON A CLASS OF MEROMORPHIC FUNCTIONS

LENNART CARLESON

For the theory of meromorphic functions in the unit circle, which has mainly been created by R. Nevanlinna, the growth of the characteristic function $T(r)$ is of fundamental importance. The limiting class of functions for this theory has thus been the class T_0 for which $T(r)$ is bounded. By imposing on the growth of the spherical area $A(r)$ of the image of a circle $|z| < r < 1$ a condition of integrability, the author [L. Carleson, *On a class of meromorphic functions and its associated exceptional sets*, Uppsala, 1950] has divided T_0 into subsets T_α, characterized by the property that $(1 - r)^{-\alpha}A(r)$ is integrable, $0 < \alpha < 1$. It turns out that there are analogues of most of the properties of T_0 in T_α, although proofs and results are more complicated in this case.

Concerning the existence of boundary values, the following theorem can be proved: a function in T_α has radial limits outside a set on the unit circle of vanishing $(1 - \alpha)$-capacity. This theorem corresponds to Fatou's theorem for bounded functions, which implies the corresponding result for T_0, since a function in T_0 is the quotient of two bounded functions. For the classes T_α it is not possible to reduce the problem to that for bounded functions in the class, for although a function in T_α is the quotient of two bounded functions belonging to every class T_β with $\beta < \alpha$, these do not necessarily belong to T_α itself.

The situation concerning the distribution of values is here also considerably more complicated. In the classical theory one meets exceptional values which are taken comparatively rarely by the function. Here the reverse is possible: the function can take certain values exceptionally frequently; the frequency is measured by convergence or divergence of the integral

$$\int_0^1 \frac{n(r, a)}{(1 - r)^\gamma}\, dr \tag{1}$$

where $n(r, a)$ denotes the number of a-values in $|z| \leqq r$. The dimension for a function in T_α of the set S_γ, $0 < \gamma \leqq \alpha$, where the integral (1) diverges, cannot exceed $2\gamma/\alpha$.

If a function $w(z)$ maps $|z| < 1$ conformally onto the universal covering surface of a schlicht domain and $w(z)$ belongs to T_0, then the boundary of the domain has positive logarithmic capacity. If the function belongs to T_α, the primary consequence is not a metric "increase" in the boundary. It is shown that instead the isolated subsets of vanishing capacity of the boundary are in a certain sense regularized. If $w(z)T_\alpha$, $\alpha > 1/2$, the boundary cannot contain any such subsets, e.g. isolated points, at all. A further investigation to discover the decisive properties of the boundary would no doubt be of great interest; from the above it is clear that one will have to take into account properties of a non-metric nature.

UNIVERSITY OF UPPSALA,
UPPSALA, SWEDEN.

BOUNDARY THEOREMS FOR FUNCTIONS MEROMORPHIC IN THE UNIT CIRCLE

E. F. COLLINGWOOD AND M. L. CARTWRIGHT

Let $f(z)$ be meromorphic in $|z| < 1$. The *cluster set* $C(f)$ is the set of a such that $\lim_{n\to\infty} f(z_n) = a$ for some sequence $\{z_n\}$ where $|z_n| < 1$, $\lim_{n\to\infty} |z_n| = 1$. $C(f)$ is closed and connected. The *range of values* $R(f)$ is the set of values a for which $f(z_n) = a$ in a similar sequence $\{z_n\}$. The *asymptotic set* $\Gamma(f)$ is the set of asymptotic values a such that $\lim_{t\to\infty} f(z(t)) = a$ on a continuous path $z = z(t)$, $0 < t < \infty$, where $\lim_{t\to\infty} |z(t)| = 1$. The "end" of the path is either a point or a closed arc of $|z| = 1$. Closures of the above sets are denoted by $\bar{C}(f)$, etc., complements by $\mathcal{C}C(f)$, etc., and frontiers by $\mathcal{F}C(f)$, etc. We prove:

THEOREM I. *If $f(z)$ is meromorphic in $|z| < 1$, then* (i) *in general* $\mathcal{F}R(f) \cup \mathcal{F}C(f) = \overline{\mathcal{C}R}(f) \cap C(f) \subseteq \bar{\Gamma}(f)$; *while* (ii) *if $\Gamma(f)$ is of linear measure zero, then $\mathcal{C}C(f)$ is void and $\mathcal{C}R(f) \subseteq \Gamma(f)$.*

It follows from (i) that in general $\mathcal{C}R(f) \cap C(f) \subseteq \bar{\Gamma}(f)$; and this result is best possible in the sense that $\bar{\Gamma}(f)$ cannot be replaced by $\Gamma(f)$, as we show by an example. Theorem I may be regarded as the analogue in the large, i.e., with respect to the total boundary $|z| = 1$, of Iversen's theorem stating that for a function meromorphic in the finite plane $|z| < \infty$ an excluded value is an asymptotic value. By (ii), if $\Gamma(f)$ is finite or enumerable, then so is $\mathcal{C}R(f)$. In particular, if $f(z)$ has no asymptotic value, then it has no excluded value $a \in \mathcal{C}R(f)$ as was first proved by Noshiro [Journal of the Faculty of Science. Hokkaido Imperial University. Ser. I vol. 7 (1938) pp. 149–159, Theorem 4].

Denote by F the set of (Fatou) points on $|z| = 1$ at which $f(z)$ has a unique asymptotic value to which it tends uniformly in any angle at the point and contained in $|z| < 1$; and by P the set of (Picard) points on $|z| = 1$ in every neighborhood of which, in $|z| < 1$, $f(z)$ takes all values with at most two exceptions. F' is the derivative of F. Then we have:

THEOREM II. *If $f(z)$ is meromorphic in $|z| < 1$, then every point of $|z| = 1$ belongs either to P or to F'.*

This theorem belongs to a type of results proved by Plessner [J. Reine Angew. Math. vol. 158 (1927)] and Littlewood [see M. L. Cartwright, Math. Ann. vol. 112 (1936) p. 181].

LILBURN TOWER,
ALNWICK, ENGLAND.
CAMBRIDGE UNIVERSITY,
CAMBRIDGE, ENGLAND.

ON RITT'S REPRESENTATION OF ANALYTIC FUNCTIONS AS INFINITE PRODUCTS

B. EPSTEIN AND J. LEHNER

In 1930 (Math. Zeit. vol. 32 (1930) pp. 1–3) J. F. Ritt proved the following theorem: Let $f(x)$ be regular and zero-free for $|x| < 1$, $f(0) = 1$; let $\sup|c_n| = 1$, where $-f'(x)/f(x) = c_1 + c_2x + \cdots$. Then $f(x)$ can be expanded in a unique manner in a product, $f(x) = \prod_1^\infty (1 - a_nx^n)$, valid at least for $|x| < \rho$, where ρ is a certain universal constant. Ritt proved that $\rho > 1/6$. Obviously $\rho \leqq 1$; the authors show that $\rho = 1$. They employ Ritt's expression for the $\{a_n\}$ in terms of the $\{c_n\}$, from which different estimates on $|a_n|$, depending on the arithmetic character of n, are obtained. In all cases, $|a_n| \leqq 1$; when $n = 2^k$, the upper bound may be realized, but if $n \neq 2^k$, $|a_n| \leqq 2/3$. The approach to the problem is thus purely arithmetic; the authors have not found any way to introduce function-theoretic methods.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA., U. S. A.

SCHLICHT GAP SERIES WHOSE CONVERGENCE ON THE UNIT CIRCLE IS UNIFORM BUT NOT ABSOLUTE

PAUL ERDÖS, FRITZ HERZOG, AND GEORGE PIRANIAN

There exist Taylor series (1) $\sum a_k z^{n_k}$ which converge uniformly, but not absolutely, on the unit circle C. On the other hand, if the function represented by a series (1) is bounded and schlicht in the unit disc, then $\sum n_k |a_k|^2 < \infty$; therefore the condition of schlichtness and uniform, nonabsolute convergence on C imposes restrictions on the gaps which may occur in (1)—for example, it prohibits the convergence of the series $\sum n_k^{-1}$. As a contribution toward the determination of the extent of these restrictions, the authors exhibit a series (1) which is schlicht in the unit disc, converges uniformly but not absolutely on C, and has the property $n_{k+1} - n_k \to \infty$. The series maps the unit disc into a disc whose periphery bears many short, highly branched needles; the needles and their branches point away from the center of the disc.

UNIVERSITY OF ABERDEEN,
ABERDEEN, SCOTLAND.
MICHIGAN STATE COLLEGE,
EAST LANSING, MICH, U. S. A.
UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

FUNDAMENTAL REGIONS FOR DISCONTINUOUS GROUPS OF LINEAR TRANSFORMATIONS

LESTER R. FORD

This paper presents the basic properties of properly discontinuous groups of linear transformations, $w = (az + b)/(cz + d)$, $ad - bc = 1$, using methods of the greatest simplicity. Assuming that one point, which we may take to be $z = \infty$, has no congruent points in its neighborhood, it follows that the isometric circles $|cz + d| = 1$ exist (excluding naturally the identical transformation) and their centers $-d/c$ are bounded.

The tool that is new and that leads to essential simplifications in the treatment is the deformation $r = |dw/dz| = 1/|cz + d|^2$. This is the factor by which lengths at z are multiplied when the transformation is applied. The isometric circle is the locus $r = 1$. The fundamental region R consists of all those points p of the plane such that some neighborhood of p is exterior to all isometric circles of the group. In R we have $r < 1$ for all transformations of the group except the identity.

The covering of the plane by R and its congruent regions is treated as follows. An ordinary point z_0 not in R lies within or on a finite number of isometric circles. Arrange the deformations at z_0 in order, $r_1 \geqq r_2 \geqq r_3 \geqq \cdots \geqq r_m \geqq 1$, the corresponding transformations being $T_1, T_2, \cdots, T_m$. If $r_1 > r_2$, then T_1^{-1} carries R into a region covering z_0. If $r_1 = r_2 > r_3$, then T_1^{-1} and T_2^{-1} carry R into regions abutting along a curve passing through z_0. This curve, which is the locus of $r_1 = r_2$, is a circle or a straight line. If $r_1 = r_2 = r_3 = \cdots = r_s > r_{s+1}$, then $T_1^{-1}, \cdots, T_s^{-1}$ carry R into s regions with common vertex at z_0 and filling up the area about z_0.

If the boundary of R consists in whole or in part of ordinary points, which is the interesting case, we prove at once by taking z_0 on the boundary that the sides of R are congruent in pairs and that its vertices fall into congruent cycles.

ILLINOIS INSTITUTE OF TECHNOLOGY,
CHICAGO, ILL., U. S. A.

AN EXTENSION OF THE RIEMANN MAPPING THEOREM ASSOCIATED WITH MINIMAL SURFACES

A. GELBART

Weierstrass showed that a parametric form of the solution for the minimal surface equation $(1 + \varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1 + \varphi_x^2)\varphi_{yy} = 0$ is: 1) $x = \operatorname{Re} f_1(\zeta)$, 2) $y = \operatorname{Re} f_2(\zeta)$, 3) $\varphi = \operatorname{Re} f_3(\zeta)$, where $f_i(\zeta)$ are any analytic functions satisfying 4) $\sum_1^3 f_i'^2(\zeta) = 0$. The author shows by adding 1) and 2) and making use of 4) that

$$x + iy = z = \int g(\zeta)\, d\zeta - \overline{\int p^2(\zeta)/g(\zeta)\, d\zeta} = T(g),$$

where $g(\zeta) = [f_1'(\zeta) + if_2'(\zeta)]/2$ and $p(\zeta) = f_3'(\zeta)/2$ [NACA T.N. No. 1170, March, 1947] and establishes the theorem: Given a simple closed analytic curve Γ in the z-plane; then fixing $p(\zeta)$ ($p(\zeta)$ regular in $|\zeta| \geqq 1$), there always exists a $g(\zeta)$, regular in $|\zeta| \geqq 1$, such that $T(g)$ maps $|\zeta| \geqq 1$ simply onto the closed domain exterior to Γ, infinity going into infinity. This is done by showing that the nonlinear functional equation in g: $\log |\phi[T\{g(\zeta)\}]| \equiv 0$, $|\zeta| = 1$, has a solution, where ϕ is the simple analytic function mapping the closed exterior of Γ onto $|\zeta| \geqq 1$. This theorem clearly has an association with the Plateau problem. When $p \equiv 0$, the theorem reduces to the Riemann mapping theorem.

Syracuse University
Syracuse, N. Y., U. S. A.

AN ALTERNATIVE APPROACH TO THE THEORY OF ELLIPTIC FUNCTIONS

Mario O. González

Let $w = \mathrm{Tan}(z, \lambda)$ (generalized tangent function) be the solution of the differential equation $(dw/dz)^2 = 1 + 2\lambda w^2 + w^4$ (z, w, λ complex) with the conditions $w = 0$, $w' = +1$ for $z = 0$. This is an elliptic function of order two having at $mK + niK'$ a zero when $m + n$ is even, a pole when $m + n$ is odd, $2K$ and $2iK'$ being a primitive pair of periods. This function possesses the simple addition theorem $\mathrm{Tan}(u + v) = (\mathrm{Tan}\, u\, \mathrm{Tan}'v + \mathrm{Tan}'u\, \mathrm{Tan}\, v)/(1 - \mathrm{Tan}^2u\, \mathrm{Tan}^2v)$ and the series expansion about the origin $\mathrm{Tan}\, z = \sum_{n=0}^{\infty} c_0^n z^{2n+1}/(2n + 1)!$, $c_r^{n+1} = 2(2r + 1)[(r + 1)c_{r+1}^n + (2r + 1)\lambda c_r^n + rc_{r-1}^n]$, $c_s^n = 0$ if $s < 0$ or $s > n$, and $c_0^0 = 1$. On the other hand $z = \mathrm{Tan}^{-1}w = \sum_{n=0}^{\infty} (-1)^n P_n(\lambda) w^{2n+1}/(2n + 1)$, where $P_n(\lambda)$ denotes the Legendre polynomial of order n and $|w| \leqq r$, $|\lambda| \leqq h$, $2hr^2 + r^4 < 1 - \epsilon$. Other interesting properties are the following: $\mathrm{Tan}(iz, \lambda) = i\,\mathrm{Tan}(z, -\lambda)$, $\mathrm{Tan}(z + (2m + 1)K + 2niK') = -1/\mathrm{Tan}\, z$, $\mathrm{Tan}(z + 2mK + (2n + 1)iK') = 1/\mathrm{Tan}\, z$, $\mathrm{Tan}(z + (2m + 1)K + (2n + 1)iK') = -\mathrm{Tan}\, z$, $\mathrm{Tan}(K/2) = 1$, $\mathrm{Tan}(iK'/2) = i$, $\mathrm{Tan}((2m + 1)K/2 + (2n + 1)iK'/2) = (-1)^m k + (-1)^n ik'$, where m and n are any integers including zero and k and k' are such that $k'^2 - k^2 = \lambda$, $k'^2 + k^2 = 1$. The Jacobian and Weiertrassian elliptic functions can be defined in terms of the tangent function as follows: $\mathrm{sn}\, z = 2\,\mathrm{Tan}(z/2)/(1 + \mathrm{Tan}^2(z/2))$, $\mathrm{cn}\, z = (1 - \mathrm{Tan}^2(z/2))/(1 + \mathrm{Tan}^2(z/2))$, $\mathrm{dn}\, z = 2\,\mathrm{Tan}'(z/2)/(1 + \mathrm{Tan}^2(z/2))$, $\mathrm{tn}\, z = 2\,\mathrm{Tan}(z/2)/(1 - \mathrm{Tan}^2(z/2))$, $\wp(z) = e_1 + \gamma^2/\mathrm{Tan}^2\gamma z$, $\gamma^4 = (e_1 - e_2)(e_1 - e_3)$, and their properties easily derived from those of $\mathrm{Tan}\, z$.

Havana University,
Havana, Cuba.

A CLASS OF MULTIVALENT FUNCTIONS

A. W. Goodman and M. S. Robertson

Let $T(p)$ denote the class of functions $f(z) = \sum_{n=1}^{\infty} b_n z^n$ regular in $|z| < 1$ and having the following properties: (a) all the coefficients b_n are real, (b) there is a $\rho > 0$ such that for each fixed r in the interval $\rho < r < 1$, the imaginary part of $f(re^{i\theta})$ changes sign $2p$ times as θ varies from 0 to $2\pi + 0$. Functions of this kind are called typically-real of order p and $T(1)$ is just the class of typically-real functions discussed by W. Rogosinski (Math. Zeit. vol. 35 (1932) pp. 93–121). The sharp upper bound for $|b_n|$ in terms of $|b_1|, |b_2|, \cdots, |b_p|$, is determined for all integers n and p such that $n > p \geqq 1$. This bound is the same as one conjectured earlier for the class of p-valent functions (Trans. Amer. Math. Soc. vol. 63 (1948) pp. 175–192). Since $T(p)$ contains all the starlike p-valent functions with real coefficients, this result substantiates to some degree the earlier conjecture.

University of Kentucky,
Lexington, Ky., U. S. A.
Rutgers University,
New Brunswick, N. J., U. S. A.

ON THE GROWTH OF MINIMAL POSITIVE HARMONIC FUNCTIONS IN A PLANE REGION

Bo Kjellberg

Consider an unbounded plane region D; the case of infinite connectivity being most interesting. There may exist positive harmonic functions, tending to zero in the vicinity of every finite boundary point of D.

Suppose first that there is a finite number $n \geqq 2$ of linearly independent such functions u_ν, $\nu = 1, \cdots, n$; this number n may be called "the harmonic multiplicity" of the boundary point $z = \infty$. We measure the growth of u_ν as usual by the order ρ_ν, being defined as lim sup of $\log u_\nu(z)/\log |z|$, as $z \to \infty$ in D.

Then it is true that

$$\sum_{\nu=1}^{n} \frac{1}{\rho_\nu} \leqq 2.$$

Essential in this connection is the concept of a *minimal positive harmonic function,* which can dominate no other positive harmonic functions than its own submultiples (R. S. Martin, 1941). The theorem is established by the aid of a proof of the Denjoy-Carleman-Ahlfors theorem about the number of asymptotic paths of an integral function of given order.

Leaving the finite case we can say: the set of minimal positive harmonic functions is enumerable and the above inequality still holds if n is changed to ∞.

UNIVERSITY OF UPPSALA,
UPPSALA, SWEDEN.

ON THE BOUNDARY BEHAVIOR OF ANALYTIC FUNCTIONS

OLLI LEHTO

THEOREM 1. *Let $F(z)$ be single-valued and analytic in the unit circle, and let*

$$\int_0^{2\pi} |F'(re^{i\varphi})|\, d\varphi \qquad (z = re^{i\varphi})$$

be uniformly bounded for $r < 1$. Then $\lim F(z)$ *for $z \to e^{i\varphi}$ along any path in $|z| < 1$ exists for each φ and is a continuous function of φ.*

It can first be proved that the above limit is of bounded variation. Then the main idea of the proof is to show that a discontinuity of the real part of the limit implies the unboundedness of its imaginary part.

When is the function $F(z)$ also analytic on $|z| = 1$? We shall give a criterion useful e.g. in application of orthogonal functions to function-theoretic problems.

THEOREM 2. *The function $F(z)$ is analytic on $|z| = 1$ if and only if*

$$\int_{|z|=1-0} \frac{\bar{F}'(z)}{z - \zeta}\, dz \qquad (|\zeta| > 1)$$

is analytic still on $|\zeta| = 1$.

The essential point of the proof is the following: In the above integral we can replace $\bar{F}'(z)$ by the analytic function $g(z)$ defined by $g(z) = \bar{F}'(\bar{z}^{-1})$, and carry out the limitation process from outside $|z| < 1$.

In view of the applications the following observation is of great importance: The Theorems 1 and 2 can easily be generalized, without the use of mapping theorems, for every schlicht domain G which is of finite connectivity and whose boundary components are analytic curves.

To illustrate how to apply Theorem 2 let us consider the following problem: Which of all functions f single-valued and analytic in G and with $f'(t) = 1$ $(t \in G)$ possesses the smallest Dirichlet integral over G? The derivatives of all analytic functions with a finite Dirichlet integral form a Hilbert space R if the scalar product is defined by

$$(g', h') = \iint_G g'\bar{h}'\, dx\, dy \qquad (z = x + iy).$$

Thus the derivative of the solution f_0 of the above problem is orthogonal with respect to the linear manifold comprising all R-functions η' which vanish in t.

Hence, transforming the expression of the scalar product to a curve integral extended along the boundary of G,

$$\int \bar{f}_0(z)\eta'(z)\,dz = 0.$$

On the basis of Theorem 2 in its generalized form, it follows easily from this relation that $f_0(z)$ is analytic still on the boundary of G.

This and various similar results can effectively be utilized in proving certain mapping theorems.

HELSINKI, FINLAND.

RIEMANN SURFACES AND ASYMPTOTIC VALUES ASSOCIATED WITH CERTAIN REAL ENTIRE FUNCTIONS

G. R. MACLANE

Let S be any Riemann surface obtainable, as a covering of the w-plane, by the following construction: 1) commencing with the Riemann surface of $z = (\arccos w)^2$, the branch points are displaced in an arbitrary fashion over the real axis; 2) this surface is cut along a finite number of disjoint rays extending to infinity, these cuts being placed in the surface symmetrically with respect to the natural symmetry of the surface across the real axis. Along each such cut a pair of logarithmic ends is added; 3) the process 2) is repeated a finite number of times, the new cuts lying in the logarithmic ends already added. Let C_k be the class of those surfaces in which exactly k pairs of logarithmic ends are added to the basic surface.

THEOREM. *All the Riemann surfaces in the classes* C_k $(k = 0, 1, \cdots)$ *are parabolic, and, with the proper normalization, correspond* (1–1) *with the functions* $w = f(z)$ *where*

$$f'(z) = e^{p(z)} \prod_{r=1}^{\infty} E(z/b_r, q),$$

$$0 < b_r \uparrow \infty, \qquad \sum b_r^{-q} = \infty, \qquad \sum b_r^{-q-1} < \infty,$$

$$p(z) = c_1 z + \cdots + c_\sigma z^\sigma, \qquad c_k \textit{ real}, \qquad c_\sigma \neq 0.$$

If $\gamma = \max(\sigma, q)$, *this entire function yields a surface of class* C_γ, *unless* $\sigma > q$, $c_\sigma < 0$, *in which case the surface is of class* $C_{\gamma-1}$.

The asymptotic tracts of $f(z)$ fill the plane with the possible exception of a neighborhood of the positive real axis. Two adjacent tracts have one boundary curve in common, running from $z = 0$ to $z = \infty$, and any one tract is bounded by two such curves. If one of these tracts is mapped onto $\Re\zeta > 0$, and $f(z) = F(\zeta)$, then $F(\zeta) \to a$ uniformly in $|\arg \zeta| \leqq \pi/2 - \epsilon$, where a is the appropriate asymptotic value.

If $\sigma > q$, the bounding curves are straight lines; if $\sigma \leqq q$, each tract contains a fixed sector depending only on q, but the bounding curves need not be rays. Examples show that the oscillation of arg z along each such curve is restricted only by the fact that these curves must remain outside of the fixed sectors. The actual specification of these sectors is somewhat complicated.

RICE INSTITUTE,
HOUSTON, TEX., U. S. A.

FONCTIONS MÉROMORPHES ET DÉRIVÉES

H. MILLOUX

On connait le rôle important joué par l'inégalité fondamentale de R. Nevanlinna dans l'étude de la distribution des valeurs d'une fonction méromorphe. J'ai montré que cette inégalité s'étend à l'étude comparative des valeurs prises par la fonction et par sa dérivée (ou plus généralement, par l'une quelconque de ses dérivées); par exemple, sous la forme suivante:

$$(1) \qquad T(r, f) < N\left(r, \frac{1}{f - a}\right) + N\left(r, \frac{1}{f - b}\right) + N\left(r, \frac{1}{f' - c}\right) + S(r)$$

où a et b sont des constantes distinctes quelconques (l'une d'elles pouvant être infinie) et où c est une constante finie quelconque distincte de zéro. On peut utiliser l'inégalité (1) dans le problème qui consiste à rechercher s'il existe des cercles de remplissage communs à la fonction méromorphe et à sa dérivée. Ce problème se présente d'une façon très simple et est déjà résolu lorsque la fonction étudiée est une fonction entière d'ordre infini. Lorsqu'il s'agit d'une fonction entière d'ordre fini (non nul), M. Biernacki a démontré qu'il existe au moins une direction de Julia commune à la fonction et à ses dérivées successives. Aux méthodes utilisées par M. Biernacki, on peut en substituer d'autres, basées sur l'inégalité (1); celles-ci sont plus simples, plus systématiques, et conduisent à des résultats plus précis. Elles présentent en outre l'avantage de permettre d'aborder les mêmes problèmes pour les fonctions méromorphes. En particulier, pour les fonctions entières d'ordre fini non nul, il existe au moins une direction de Borel commune à une telle fonction et à ses dérivées successives.

UNIVERSITY OF BORDEAUX,
BORDEAUX, FRANCE.

SOME EXTREMAL PROBLEMS INVOLVING SINGLE-VALUED ANALYTIC FUNCTIONS

ZEEV NEHARI

In this paper a number of extremal problems are solved, of which the following is typical: Let D be a finite schlicht domain in the complex z-plane which is bounded by n closed analytic curves $\Gamma_1, \cdots, \Gamma_n$ ($\Gamma_1 + \cdots + \Gamma_n = \Gamma$) of combined length L, and let A be an open subset of Γ of measure α ($\alpha < L$); let further C denote the class of regular and single-valued analytic functions in D which satisfy $|\operatorname{Re}\{f(z)\}| < 1$, $z \in D$, while $\limsup |\operatorname{Re}\{f(z)\}| \leqq \lambda < 1$ if z tends angularly to a point of A; to find a function $f_0(z) \in C$ with the extremal property $|f_0'(\zeta)| = \operatorname{Max} |f'(\zeta)|$, $f(z) \in C$, $\zeta \in D$. The solution of this and a number of other problems belonging to the same order of ideas is effected by suitable adaptation of a method developed by the author in a paper entitled *Extremal problems in the theory of bounded analytic functions* (Amer. J. Math. vol. 73 (1951) pp. 78–106).

WASHINGTON UNIVERSITY,
ST. LOUIS, MO., U. S. A.

ON OPEN RIEMANN SURFACES

LEO SARIO

For analyzing the multitude of open Riemann surfaces, I proposed, in the summer of 1946 (in a paper cited by Nevanlinna in the Copenhagen Congress), the study of surfaces on the basis of the existence of certain characteristic functions. I gave a criterion for a surface without such functions of a certain class. In the following we shall consider in schematic form the results attained so far in this direction (Ahlfors, Beurling, Lehto, Myrberg, Nevanlinna, Pfluger, Royden, Sario, Virtanen). Let us denote: H = harmonic single-valued nonconstant, A = analytic single-valued nonconstant, S = schlicht analytic single-valued nonconstant, B = bounded, D = with a finite Dirichlet integral. Further, F = an open Riemann surface of genus p and boundary Γ, G = a noncompact part of F (schlichtartig if $p < \infty$) bounded by Γ and a simple analytic curve γ.

We compare surface classes of nonexistence (N) qualities with classes of removability (R), uniqueness (U), and metric (M) qualities. In the scheme each class is included in those below it. The identical classes are joined together. The classes (R) and (M) concern, according to their nature, the surfaces with $p < \infty$, Γ being realized in a schlicht mapping of G. It is so far unknown whether the inequalities $N_{HB} \subset N_{HD}$ and $M_1 \subset N_{AB}$ are strict and whether $N_{HD} \subset N_{AB}$ holds for $p = \infty$.

M_n	Γ is a denumerable pointset
M_l	logarithmic measure of Γ vanishes
N_G	no Green's functions on F
N_ω	ω-measure of Γ vanishes ($\omega = HB$ in G, $\omega = 0$ on γ, $\int_\gamma (\partial\omega/\partial n)\, ds \neq 0$)
M_C	logarithmic capacity of Γ vanishes
M_F	Fekete's transfinite diameter of Γ vanishes
N_{HB}	no HB functions on F
N_β	β-measure of Γ vanishes ($\beta = HB$ in G, $\beta = 0$ on γ, $\int_\gamma (\partial\beta/\partial n)\, ds = 0$)
R_{HB}	Γ is a removable singularity for each HB in G
N_{HD}	no HD functions on F
N_δ	δ-measure of Γ vanishes ($\delta = HD$ in G, $\delta = 0$ on γ, $\int_\gamma (\partial\delta/\partial n)\, ds = 0$)
R_{HD}	Γ is removable for each HD in G
U_D (real)	Abelian integrals D are uniquely determined by their real periods
M_ϵ	ϵ-dimensional (ϵ small > 0) measure of Γ vanishes
M_1	linear measure of Γ vanishes
N_{AB}	no AB functions on F
R_{AB}	Γ is removable for each AB in G
$M_{1+\epsilon}$(abs)	$(1 + \epsilon)$-dimensional measure of Γ vanishes in all its realizations
$M_{2-\epsilon}$(abs)	$(2 - \epsilon)$-dimensional measure of Γ vanishes in all its realizations
N_{AD}	no AD functions on F
R_{AD}	Γ is removable for each AD in G
R_{SD}	Γ is removable for each SD in G
R_{SB}	Γ is removable for each SB in G
U_D (tot)	Abelian integrals D are uniquely determined by their total periods
U_m	mapping of F on a part of a closed surface is essentially unique ($p < \infty$)
M_S	Schiffer's span of F vanishes ($p = 0$)
M_2 (abs)	area of Γ vanishes in all its realizations
N_{SD}	no SD functions on F ($p = 0$)
N_{SB}	no SB functions on F ($p = 0$)
M_d (abs)	Γ is totally disconnected in all its realizations
M_d (rel)	Γ is totally disconnected in at least one of its realizations

For $p < \infty$ we have moreover $N_G = N_{HB} = N_{HD}$.

HELSINKI, FINLAND.

ÜBER NULLGEBILDE ANALYTISCHER FUNKTIONEN ZWEIER VERÄNDERLICHER, DIE IN SINGULÄREN PUNKTEN MÜNDEN

HERMANN SCHMIDT

Bei der Untersuchung der durch eine analytische Gleichung $f(x, y)$ implizit erklärten Funktionen im kleinen geht man gewöhnlich von einer bekannten Lösung (x_0, y_0) aus, in deren Vollumgebung sich $f(x, y)$ regulär verhält; der Weierstrasssche Vorbereitungssatz zusammen mit dem Satz von den Puiseuxschen Entwicklungen gibt dann Auskunft über die Lösungen in der Nachbarschaft. Wir fragen im Gegensatz dazu nach der Existenz und analytischen, insbesondere asymptotischen Darstellung von Nullgebilden in der Umgebung einer *singulären* Stelle, die nach $(0, 0) = O$ gelegt sei. Ist z.B. $f(x, y)$ durch eine *Laurentsche Reihe* $\sum_{-\infty}^{+\infty} a_{\nu\lambda}x^{\nu}y^{\lambda}$ gegeben, deren Konvergenzgebiet O zum Randpunkt hat, so kann man, wie Verf. früher gezeigt hat (Math. Zeit Bd. 43 (1938) S. 533–552), dann auf den klassischen Fall zurückkommen, wenn die Gitterpunkte (ν, λ) für die $a_{\nu\lambda} \neq 0$ ausfällt, sämtlich in einer Halbebene gelegen sind. Das Verfahren liefert z.B. die Entwicklung *hypergeometrischer Funktionen* mit mehreren Parametern nach solchen mit geringerer Parameterzahl, wie etwa Entwicklungen Kummerscher und Legendrescher Funktionen nach Besselfunktionen, und damit Reihendarstellungen der entsprechenden Nullgebilde nach Potenzen eines Parameters. Darin sind insbesondere asymptotische und zugleich konvergente Entwicklungen für *Nullstellen Legendrescher, Hermitescher, und Laguerrescher Polynome* hohen Grades enthalten, wie sie später auf andere Weise Tricomi wiedergefunden hat (vgl. z.B. Annali di Matematica, Bologna IV.S. Bd. 26 (1947) S. 141–175, 283–300).

Eine andere Untersuchungsrichtung beschäftigt sich mit Reihen der Form $f(x, y) = \sum_0^\infty x^{\nu}f_{\nu}(y)$. Eine solche hat einen sogenannten vollkommenen Hartogsschen Körper $|x| < \rho(y)$ zum Gebiet gleichmässiger Konvergenz. Enthält dieser ein Gebiet $|x| < r$ (r fest), $y \in T$, wo T einen Winkelraum mit dem Scheitel $y = 0$ bedeutet, besitzen ferner für $y \to 0$ in T einige der $f_{\nu}^{(\lambda)}(y)$ Randwerte $A_{\nu\lambda}$ (mit $A_{00} = 0$), während die übrigen $f_{\nu}^{(\lambda)}(y)$ nicht zu stark anwachsen, so lassen sich Lösungen nachweisen, die für $x \to 0$ (in geeigneten Winkelräumen) nach 0 streben und (endliche oder unendliche) asymptotische Puiseux-Entwicklungen gestatten. In einer gemeinsam mit meinem inzwischen verstorbenen Schüler G. Lockot durchgeführten Untersuchung (von der ein Teil inzwischen erschienen ist: Math. Ann. Bd. 122 (1951) S. 411–423) wurde schliesslich auch der Fall behandelt, dass $\rho(y)$ mit y gegen Null geht, wofern dies nur nicht allzu schnell erfolgt. Anstelle eines Winkelraums T kann auch das Zwischengebiet zweier sich nicht zu eng berührender Kurven treten. Eine zulässige Bedingung für den Winkelraum T ist im einfachsten Falle die folgende:

Es sei für ein festes k und positive t, $y \in T$, $H_{\nu}(t) = \lim \sup_{t \leq |y| < r_0} |f_{\nu+k}(y)|$; dann soll es eine (hinreichend grosse) Zahl ϑ derart geben, dass für $t \to 0$ $\vartheta^{\nu}H_{\nu}(t) = o(t^{-\nu/n_1})$, gleichmässig hinsichtlich ν. Die natürliche Zahl n_1 ist dabei der Anfangsexponent eines Polynoms $P(x)$ höchstens $k$ten Grades, das bis auf gewisse Fehlerglieder die gegebene Gleichung formal erfüllt, und der anschauliche Sinn der Bedingung ist im wesentlichen, dass die durch $y = P(x)$ erklärte charakteristische Fläche des R_4 das Konvergenzgebiet der gegebenen Reihe in einer gewissen Umgebung von $x = 0$ nicht verlässt.

TECHNISCHE HOCHSCHULE,
BRAUNSCHWEIG, GERMANY.

THE COEFFICIENT PROBLEM FOR SCHLICHT MAPPINGS OF THE EXTERIOR OF THE UNIT CIRCLE

GEORGE SPRINGER

Let $E(q)$ represent the domain consisting of the whole w-plane with the exception of q ($q \geqq 2$) symmetrically oriented radial slits, each of length $2^{2/q}$, emanating from the origin with one ray lying along the positive real axis. The function mapping $E(q)$ conformally onto $|z| > 1$ will be denoted by $\phi_q(w) = w + c_1(q)w^{-1} + c_2(q)w^{-2} + \cdots$, where $jc_j(q) = -C_{2j/q,k}$ if $qk = j + 1$, $k =$ integer, and $c_j(q) = 0$ otherwise. Let S represent the class of all functions $\phi(w) = w + b_1w^{-1} + b_2w^{-2} + \cdots$ which are schlicht in a simply connected domain D containing $w = \infty$ and mapping D conformally onto $|z| > 1$. Also let $S[a_1, a_2, \cdots, a_{n-1}]$ represent the subclass of S for which $b_j = a_j$, $j = 1, \cdots, n - 1$. Then for all ϕ in S, $|b_3| \leqq -c_3(2) = 1$ with equality only in the case of the functions $e^{ia}\phi_2(e^{-ia}w)$. For all ϕ in $S[c_1(q), c_2(q), \cdots, c_{n-1}(q)]$ where q divides $n + 1$, $\mathrm{Re}(-b_n) \leqq -c_n(q)$, with equality only for the function $\phi_2(w)$. These results are obtained by using variational methods similar to those employed by Schaeffer, Schiffer, and Spencer.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASS., U. S. A.

ÜBER DEN VERZERRUNGSSATZ VON KOEBE

NAZIM TERZIOĞLU

Es werden neue Beweise für den folgenden Satz von Koebe angegeben:

SATZ. *Wenn die Potenzreihe* $\tau(t) = t + a_2t^2 + \cdots$ *im Einheitskreis* $|t| < 1$ *konvergiert und ihn auf ein schlichtes Gebiet* D_τ *abbildet, welches den unendlichfernen Punkt nicht enthaelt, so ist die kürzeste Entfernung* d *des Randes von* D_τ *vom Nullpunkt grösser als* $1/4$.

Durch die Funktion

$$s(t) = \frac{d^{1/2} - [d - e^{-i\delta}\tau(t)]^{1/2}}{d^{1/2} + [d - e^{-i\delta}\tau(t)]^{1/2}} = \frac{e^{-i\delta}}{4d}\, t + \cdots$$

wird E_t ($|t| \leqq 1$) auf ein schlichtes Gebiet D_s abgebildet, welches die beiden Eigenschaften besitzt: a) D_s hat $+1$ als Randpunkt und enthaelt die reelle Strecke $0 \leqq s < 1$ in seinem Inneren; b) Wenn s irgendeinn innerer Punkt von D_s ist, so ist $1/s$ bestimmt ein äusserer Punkt.

Wir schlitzen D_s laengs $0 \leqq s < 1$ und E_t laengs derjenigen Kurve C, welche jener Strecke entspricht. Diese Schlitzgebiete gehen wieder durch $w = \log s$, $z = \log t$ ($w = u + iv$, $z = x + iy$) in zwei Halbstreifen über. Die harmonische Funktion $\log | s(t)/t | = u(x, y) - x$ gestattet eine Entwicklung der Form:

$$u(x, y) - x = \lambda + \sum_{n=1}^{\infty} (\alpha_n \cos ny + \beta_n \sin ny)e^{nx} \qquad \left(\lambda = \log \frac{1}{4d}\right).$$

Es gilt bekanntlich

$$\int_{x=\text{konst.}} (u(x, y) - x)\, dy = 2\pi\lambda,$$

$$\int_{x=\text{konst.}} (u(x, y) - x)^2\, dy = 2\pi\lambda^2 + \pi\sum (\alpha_n^2 + \beta_n^2)e^{2nx}.$$

Durch $m^2(x) = \int_{x=\text{konst.}} u^2\, dy$ wird eine positive Hilfsfunktion $m(x)$ eingeführt. Es gilt

$$m(x)m'(x) = \int_{x=\text{konst.}} u_x u\, dy.$$

Der Flaecheninhalt $S(x)$ des Streifenteiles, der rechts des Abbildes der Strecke $x =$ konstant liegt, wird ausgedrückt durch $S(x) = m(0)m'(0) - m(x)m'(x)$. Aus der obengenannten Eigenschaft b) folgt leicht, dass $m'(0) \leqq 0$ ist. Das kann aber nur dann der Fall sein, wenn $\lambda = \log (1/4\, d) \leqq 0$ ist, woraus die Behauptung folgt. Man kann aber auch so schliessen: Es lassen sich folgende Ungleichungen sehr leicht zeigen: $-2\pi(x + \lambda) \geqq S(x) \geqq -2\pi x$, woraus wiederum die Behauptung folgt.

University of Istanbul,
Istanbul, Turkey.

ON THE INCOMPLETE GAMMA FUNCTION

Francesco G. Tricomi

Sponsored by the ONR

In the preparation of a monograph on the confluent hypergeometric functions for the "Bateman Manuscript Project" I have had occasion to point out many probably new properties of the incomplete gamma function. For this investigation I put in the foreground the single-valued, entire function

$$\gamma^*(\alpha, x) = [\Gamma(\alpha + 1)]^{-1}e^{-x}{}_1F_1(1; \alpha + 1; x)$$

to which the incomplete gamma function $\gamma(\alpha, x)$ and the related ones $\gamma_1(\alpha, x)$ and $\Gamma(\alpha, x)$ are connected by means of the formulae

$$\gamma(\alpha, x) = \Gamma(\alpha) - \Gamma(\alpha, x) = \Gamma(\alpha)x^{\alpha}\gamma^*(\alpha, x), \qquad \gamma_1(\alpha, x) = \Gamma(\alpha)x^{\alpha}\gamma^*(\alpha, -x).$$

The more important among these new properties concern the *asymptotic behavior* of γ(or γ_1 or Γ) as $|x|$ and $|\alpha|$ grow *simultaneously* at infinity, where the behavior of the quotient $z = \alpha^{1/2}:(x - \alpha)$ is decisive. If $|z| \to 0$ and $|\arg z| < 3\pi/4$, we obtain the asymptotic expansion (in the sense of Poincaré):

$$\Gamma(1 + \alpha, x) \sim e^{-x}x^{1+\alpha}\sum_{n=0}^{\infty} F^{(n)}(0)(x - \alpha)^{-n-1}, \qquad F(t) = e^{-\alpha t}(1+t)^{\alpha}.$$

On the contrary if $|z| \to \infty$ or $z = O(1)$, one must distinguish two subcases according to the sign of $\operatorname{Re}\alpha$, but the results are just as exhaustive as in the previous case, although less simple. Consequently, we limit ourselves to indicate here only the first terms of the corresponding asymptotic expansions, supposing α and x real:

$$\left.\begin{aligned} &\gamma(1 + \alpha, \alpha + (2\alpha)^{1/2}y) \\ &\qquad = \Gamma(1 + \alpha)\left[\frac{1}{2} + \frac{1}{\pi^{1/2}}\operatorname{Erf}(y) + O(\alpha^{-1/2})\right] \\ &\Gamma(\alpha)\gamma_1(1 - \alpha, \alpha + (2\alpha)^{1/2}y) \\ &\qquad = -\pi\operatorname{cotg}(\alpha\pi) + 2\pi^{1/2}\operatorname{Erfi}(y) + O(\alpha^{-1/2}) \end{aligned}\right\} \ \alpha > 0, \quad y = O(1)$$

where

$$\operatorname{Erf}(y) = \gamma(1/2, y^2)/2 = \int_0^y e^{-t^2}\,dt, \qquad \operatorname{Erfi}(y) = \gamma_1(1/2, y^2)/2 = \int_0^y e^{t^2}\,dt.$$

Further new results concern the as yet less considered *real zeros* of the incomplete gamma function when α and x are also real. Considering properly the function $\gamma^*(\alpha, x)$ we find that there are:

no real zeros, if $\alpha \geqq 0$ or $\alpha = -1, -2, -3, \cdots$
one negative zero x'_1, if $1 - 2n < \alpha < 2 - 2n$ $(n = 1, 2, \cdots)$
one negative zero x' and one positive zero x'', if $-2n < \alpha < 1 - 2n$.

Moreover the previous asymptotic formulae allow a fairly good estimation of these zeros. Precisely (giving here the first terms only of the corresponding asymptotic formulae) we have

$$x' = \alpha + O(|\alpha|^{-1/2}), \qquad x'' = -\tau\alpha + O(\log|\alpha|)$$

where $\tau = 0.278463\cdots$ is the unique positive root of the transcendental equation $1 + x + \log x = 0$.

Finally there are some rapidly convergent new series expansions, for example, the following:

$$\operatorname{Erf}(x^{1/2}) = \left(\frac{\pi}{2}\right)^{1/2} \sum_{n=1}^{\infty} (-1)^{[n/2]} I_{n-1/2}(x),$$

$$\operatorname{Erfi}(x^{1/2}) = \left(\frac{\pi}{2}\right)^{1/2} \sum_{n=0}^{\infty} (-1)^{[n/2]} I_{n+1/2}(x),$$

where the (modified) Bessel functions $I_{n\pm 1/2}$ are in reality elementary functions.

California Institute of Technology,
Pasadena, Calif., U. S. A.

HANKEL DETERMINANTS OF SECTIONS OF A TAYLOR'S SERIES

Joseph L. Ullman

Let $f(z)$ be a Taylor's series with unit radius of convergence, let $f_n(z)$ denote the sections of $f(z)$, and let $f_n^p(z)$ denote the Hankel determinant of order p whose elements are the sections of $f(z)$, with first element $f_n(z)$. Let Δ denote the derived set of the zeros of $f_n(z)$ and let Δ_p denote the derived set of zeros of $f_n^p(z)$. A theorem of Jentzsch states that for any $f(z)$, every point of the unit circle belongs to Δ. If the function represented by $f(z)$ has only poles on the circle of convergence, say of total multiplicity k, the following generalization is possible. Let the circle of radius ρ be the greatest circle containing no singularites of $f(z)$ aside from the poles on the circle of convergence. Then, if Δ_{k+1} does not contain the point $z = 0$ as a limit of points different from zero, it contains every point on the circumference of the circle of radius ρ. The proof depends on the formula due to Hadamard for the circle of meromorphy of an analytic function applied to the generating function of the sections of $f(1/z)$. The same method yields results on the zeros of Hankel determinants whose elements are Faber polynomials.

University of Michigan,
Ann Arbor, Mich., U. S. A.

FONCTIONS MÉROMORPHES D'ORDRE NUL

Georges Valiron

Les propriétés de distribution des valeurs des fonctions méromorphes d'ordre nul que j'ai données récemment (C.R. Acad. Sci. Paris vol. 230 (1950) pp. 40–41) s'étendent aisément dans le sens envisagé autrefois par Borel et précisé par ses successeurs. Soit $f(z)$ une fonction méromorphe, non rationnelle, dont la fonction caractéristique $T(r, f)$ vérifie la condition de régularité de l'énoncé II de la note citée, ce qui a toujours lieu si $T(r, f) = O[(\log r)^2]$. Considérons l'ensemble des fonctions méromorphes g (qui peuvent être rationelles ou constantes finies ou non) pour lesquelles $T(r, g) = o[T(r, f)]$. Pour tout couple g_1, g_2 de fonctions distinctes de cet ensemble, la fonction $N(r, g_1, g_2)$ est asymptotiquement égale à $T(r, f)$, $N(r, g_1, g_2)$ désignant pour chaque r la plus grande des moyennes de Jensen $N(r)$ relatives respectivement aux zéros de $f - g_1$ et $f - g_2$. En particulier, si le nombre des zéros de l'une des fonctions $f - g$ est déficient au sens de Nevanlinna, la distribution des zéros de toutes les autres fonctions $f - g$ est normale; la fonction $N(r)$ correspondante est asymptotiquement égale à $T(r, f)$. On voit d'autre part que, si la fonction $f(z)$ jouissant de la propriété imposée ci-dessus admet une valeur déficiente, cette valeur est valeur asymptotique pour $f(z)$. Ce fait, indépendant de ce qui précède, établirait de suite le théorème de Pham du début de ma note citée lorsque $T(r, f) = O[(\log r)^2]$ puisqu'on sait alors que $f(z)$ ne peut avoir qu'une seule valeur asymptotique (C.R. Acad. Sci. Paris vol. 200 (1935) pp. 713–715). En outre, toute fonction $T(r)$ donnée a priori, croissante, convexe par rapport à $\log r$, dont le quotient par $\log r$ tend vers l'infini et qui jouit de la propriété de régularité, est asymptotiquement égale à la fonction caractéristique d'une famille de fonctions méromorphes aisées à construire, mais telles que l'ensemble $f - g$ considéré ci-dessus renferme une fonction dont les zéros sont déficients. Ces quelques faits très simples conduisent à poser un certain nombre de questions. Naturellement, dans le cas des fonctions les plus générales d'ordre nul, le théorème III de ma note s'étend dans le sens de Borel. Parmi les fonctions $f - g$ avec $T(r, g) = o[T(r, f)]$ une seule peut avoir des zéros déficients.

THE SORBONNE,
PARIS, FRANCE.

ON ROUCHÉ'S THEOREM AND THE INTEGRAL-SQUARE MEASURE OF APPROXIMATION

J. L. WALSH

The classical theorem of Rouché can be qualitatively expressed as follows: *Let the functions $f(z)$ and $F(z)$ be analytic in a closed region R whose boundary is B, with $f(z) \neq 0$ on B. Then if the metric* $\max [\, |f(z) - F(z)|, z \text{ on } B]$ *is sufficiently small, the functions $f(z)$ and $F(z)$ have the same number of zeros in R.* This formulation suggests an analogue, where the metric is now defined as $[f(z), F(z)] \equiv \int_B |f(z) - F(z)|^2 \, |dz|$. The analogue is studied in the relatively simple case that R is $|z| < 1$ and $f(z) \equiv z^\mu$. It is not possible to deduce the precise number of zeros of $F(z)$ in R, but we prove: *Suppose $\epsilon < \mu^\mu/(1 + \mu)^{1+\mu}$, $\mu > 0$, and denote by r_1 and r_2 $(0 < r_1 < r_2 < 1)$ zeros of the equation $r^{2\mu}(1 - r^2) = \epsilon$. If $F(z)$ is of class H_2 and if we have $[z^\mu, F(z)] < 2\pi\epsilon$, then $F(z)$ has precisely μ*

zeros in the region $|z| < r_1$ *and no zeros in the annulus* $r_1 \leqq |z| \leqq r_2$. *In the case* $\mu = 0$, *the inequality* $[1, F(z)] < 2\pi\epsilon = 2\pi(1 - r^2) < 2\pi$ *implies that* $F(z)$ *has no zeros in* $|z| \leqq r$. Here H_2 denotes the class of functions $F(z)$ analytic in R for which $\int |F(re^{i\varphi})|^2 d\varphi$ is bounded, $0 < r < 1$. The conclusion can be sharpened if $F(z)$ is a polynomial of given degree.

The methods used are straightforward and involve the Cauchy algebraic inequality and Rouché's Theorem itself. An interesting open problem is the following: Determine the largest number ϵ_μ, $\mu > 0$, such that the inequality $[z^\mu, F(z)] < \epsilon_\mu$ for a function $F(z)$ of class H_2 implies that $F(z)$ has at least one zero in R.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., U. S. A.

ON THE EFFECTIVE DETERMINATION OF THE MAPPING FUNCTION IN CONFORMAL MAPPING

S. E. WARSCHAWSKI

Suppose that C is a closed rectifiable Jordan curve with continuously turning tangent represented by the equation $z = z(s)$, where the arc length s serves as parameter, $0 \leqq s \leqq L$. Suppose that $w = f(z)$ maps the interior R of C conformally onto the circle $|w| < 1$ so that $z = z_1$ in R and $z = z_2$ on C correspond to $w = 0$ and $w = 1$, respectively. The function $\theta(s) = \arg f(z(s))$ satisfies the linear integral equation

$$(*) \qquad \theta(s) = \frac{1}{\pi}\int_0^L \theta(t)K(s, t)\,dt - 2\beta(s) \equiv B\{\theta(s)\}.$$

Here $\beta(s) = \arg (z_2 - z(s))/(z_1 - z(s))$, the kernel

$$K(s, t) = \frac{\sin(\alpha - \varphi)}{|z(s) - z(t)|}$$

where $\alpha = \alpha(t)$ is the angle which the tangent line at $z(t)$ forms with the positive real axis and $\varphi = \arg (z(t) - z(s))$. The integral in (*) exists as a suitable principal value. [See S. Gerschgorin, Rec. Math. (Mat. Sbornik) N.S. vol. 40 (1933) pp. 48–58; G. F. Carrier, Quarterly of Applied Mathematics vol. 5 (1947) pp. 101–104].

The integral equation (*) may be used for the purpose of determining the function $\theta(s)$ which gives the correspondence of the boundaries in the conformal transformation $w = f(z)$. It is shown in the paper that the integral equation may be solved by *iteration* for the following class of ("nearly convex") regions: Let C_0 be a simple closed *convex* curve with continuously turning tangent; with C_0 there is associated a constant λ, $0 < \lambda < 1$ (Neumann's constant). Let C be a simple closed curve which has continuous tangents and satisfies the additional properties: (i) C has the same length L as C_0 ; (ii) if $K(s, t)$ and $K_0(s, t)$ are the kernels of C and C_0, respectively, then $(1/\pi)\int_0^L | K(s, t) - K_0(s, t) | \, dt \leqq \epsilon < \lambda$ for $0 \leqq s \leqq L$. If $\theta_0(s)$ is an arbitrary continuous function, $\theta_n(s) = B\{\theta_{n-1}(s)\}$, then it is proved that $| \theta_{n+1}(s) - \theta_n(s) | \leqq V \cdot (1 + \epsilon) \cdot (1 - \lambda + \epsilon)^n$ for $0 \leqq s \leqq L$, where the constant $V \leqq \Omega_0 + 2\pi$, Ω_0 being the oscillation of $\theta_0(s)$ in $(0, L)$; if $\theta_0(s)$ is nondecreasing and $\Omega_0 = 2\pi$, then $V = 2\pi$. $\theta(s) = \theta_0(s) + \sum_{n=0}^{\infty} (\theta_{n+1} - \theta_n)$ is a solution of (*). This method is particularly effective in the case where C_0 is a *circle*; then $K(s, t) = 1/2$, $\lambda = 1$. Comparison is made here with the Theodorsen-Garrick method and examples are calculated.

UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINN., U. S. A.

ON THE EXISTENCE OF SUMMATION FUNCTIONS FOR A CLASS OF DIRICHLET SERIES

SHMUEL AGMON

Let $f(s)$ be analytic in the half-plane $\mathcal{R}(s) \geqq \sigma_0$ in part of which it is represented by the Dirichlet series $f(s) = \sum a_n e^{-\lambda_n s}$. A function $\omega(x)$ continuous in the interval [0, 1] is said to be a summation function of $f(s)$ in the half-plane $\mathcal{R}(s) \geqq \sigma_0$ if in each bounded set of the half-plane we have uniformly:

$$\lim_{x \to \infty} \sum_{\lambda_n \leqq x} a_n \omega\left(\frac{\lambda_n}{x}\right) e^{-\lambda_n s} = f(s). \tag{1}$$

The well known method of summation by the typical means due to M. Riesz which is applicable whenever $f(s)$ is of a finite order in the half-plane (that is to say we have there uniformly: $f(\sigma + it) = O(|t|^k)$) is of the type (1) with

$$\omega(x) = (1 - x)^{k'} \qquad (k' > k).$$

The object of this paper is to point out that summation functions satisfying (1) exist for a much wider class of functions whose order may be infinite. We prove:

THEOREM. *Let $f(s)$ be defined as above. Suppose that for $\mathcal{R}(s) \geqq \sigma_0$ we have uniformly : $f(\sigma + it) = O(e^{\beta(|t|)})$ where $\beta(u)$ is an increasing function of u such that: $\int^\infty \beta(u)u^{-2}\,du < \infty$. Then there exists a summation function $\omega(x)$ depending only on $\beta(u)$ such that* (1) *is satisfied.*

Another type of theorem is obtained (also generalizing a result of M. Riesz) when we suppose that $f(s)$ is holomorphic only in the open half-plane $\mathcal{R}(s) > \sigma_0$ and continuous in the strip $\mathcal{R}(s) \geqq \sigma_0$, $|t - t_0| < \delta$. If, furthermore, we have: $|f(\sigma + it)| < A \exp[\varphi(1/\sigma - \sigma_0) + \beta(|t|)]$ $(\sigma > \sigma_0)$ where $\varphi(u)$ and $\beta(u)$ are increasing functions of u such that $\int^\infty (\varphi(u) + \beta(u))u^{-2}\,du < \infty$, then a summation function $\omega(x)$ depending only on $\varphi(u)$ and $\beta(u)$ exists such that (1) is satisfied uniformly in any segment $\sigma = \sigma_0$, $|t - t_0| \leqq \delta' < \delta$. ($\omega(x)$ is also independent of t_0.)

RICE INSTITUTE,
HOUSTON, TEX., U. S. A.

GENERATING FUNCTIONS FOR TOTALLY POSITIVE SEQUENCES. PRELIMINARY REPORT

M. Aissen, I. J. Schoenberg, and A. Whitney

A sequence of reals, $\{a_n\}$, $(n = \cdots, -1, 0, 1, 2, \cdots)$ is called totally positive (t.p.) if the four-way infinite matrix, $A = \| a_{i-k} \|$, is totally positive; i.e. all the minors of A are non-negative.

I. J. Schoenberg (Courant Anniversary Volume, 1948) made the following

Conjecture. If $a_{-k} = 0$ $(k > 0)$, $a_0 \neq 0$, and $\{a_n\}$ is t.p., then $\sum_{n=0}^{\infty} a_n x^n$ is the Taylor series of a function of the form

$$f(x) = Ce^{\omega x} \frac{\prod_1^\infty (1 + \alpha_n x)}{\prod_1^\infty (1 - \beta_m x)} \tag{1}$$

where C, ω, α_n, β_m are non-negative constants $(C \neq 0)$ and $\sum^\infty \alpha_n$, $\sum^\infty \beta_m$ converge. That such functions generate t.p. sequences can be proved without difficulty.

We prove the following weaker form of the conjecture.

Theorem. *If $a_{-k} = 0$ $(k > 0)$, $a_0 \neq 0$, and $\{a_n\}$ is t.p. $\|$ then $\sum_0^\infty a_n x^n$ is the Taylor series of a function of the form*

$$f(x) = e^{g(x)} \frac{\prod_1^\infty (1 + \alpha_n x)}{\prod_1^\infty (1 - \beta_m x)}$$

where α_n, β_m satisfy the same conditions as in the conjecture, and $g(x)$ is an entire function. Furthermore $e^{g(x)}$ generates a t.p. sequence.

University of Pennsylvania,
Philadelphia, Pa., U. S. A.

ON THE SUMMATION OF MULTIPLE FOURIER SERIES

K. Chandrasekharan

Let $f(x) = f(x_1, \cdots, x_k)$ be a function of Lebesgue class L_1 and of period 2π in each variable. Let $f_0(x, t)$ be the spherical mean of f, of order zero, at x, and let $S^\delta(x, R)$ be the Riesz mean of order $\delta \geqq 0$ of the multiple Fourier series of f when summed spherically (cf. S. Bochner, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207). Then it is a fundamental result of Bochner that (i) $f_0(x, t) = o(1)$ as $t \to 0$ implies $S^\delta(R) = o(1)$ as $R \to \infty$ for $\delta > (k - 1)/2$, and (ii) that for a fixed x, no local hypothesis on f will yield the conclusion for $\delta \leqq (k - 1)/2$ the value $(k - 1)/2$ being the critical exponent. Now, spherical means of order

$p \geqq 0$ have been defined (cf. K. Chandrasekharan, Proc. London Math. Soc. vol. 50(1948) pp. 210–229), and it seems plausible, in analogy with (ii), that for $p > 0$, no hypothesis of the form $f_p(t) = o(1)$ as $t \to 0$ will yield the conclusion $S^\delta(R) = o(1)$ as $R \to \infty$ for $\delta \leqq p + (k-1)/2$. We show that this is not the case by proving a number of results with varying local hypotheses on $f_p(t)$ leading to conclusions about $S^\delta(R)$ for $h + (k-1)/2 < \delta \leqq p + (k-1)/2$ where h is the greatest integer less than p. We also consider the converse problem where we postulate the behaviour of $S^\delta(R)$ as $R \to \infty$ and deduce that of $f_p(t)$ as $t \to 0$.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH,
BOMBAY, INDIA.

ON THE PARTIAL SUMS OF A TAYLOR SERIES

V. F. COWLING

The geometric technique of value region considerations as introduced by Leighton and Thron (Duke Math. J. vol. 9 (1942) pp. 763–772) for continued fractions is adapted to Taylor series to prove the following typical result: Let $s_n(z) = a_0 + a_1 z + \cdots + a_n z^n$. If $| a_n/a_{n-1} | \leqq 1$, $n = 1, 2, \cdots$, and if z lies within the elliptical region E_k: $(x - (k-2)/2k)^2 + (k^2/4(k-1))y^2 = 1/4$ $(z = x + iy)$ for $k \geqq 2$ then $| s_n(z) - a_0 k/2 | \leqq | a_0 | k$, $n = 0, 1, 2, \cdots$.

UNIVERSITY OF KENTUCKY,
LEXINGTON, KY., U. S. A.

SUR LES THÉORÈMES TAUBÉRIENS POUR LES SÉRIES DE DIRICHLET

HUBERT DELANGE

Soit la série de Dirichlet $\sum_1^{+\infty} a_n e^{-\lambda_n s}$, où $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ et $\lim \lambda_n = +\infty$. Supposons cette série convergente pour $\mathcal{R}[s] > 0$ et désignons par $f(s)$ la fonction qu'elle représente.

Le résultat suivant est bien connu:

ρ étant un nombre positif ou nul, si, quand s tend vers zéro par valeurs réelles positives, $f(s) = As^{-\rho} + o[s^{-\rho}]$, on peut en conclure que, pour n infini, $a_1 + a_2 + \cdots + a_n = [A/\Gamma(\rho + 1)]\, \lambda_n^\rho + o[\lambda_n^\rho]$, pourvu que l'une des deux conditions suivantes soit satisfaite:

(1) $| a_n | \leqq M(\lambda_n - \lambda_{n-1})\lambda_n^{\rho-1}$ pour $n \geqq 2$,
(2) a_n réel, $a_n \geqq -M(\lambda_n - \lambda_{n-1})\lambda_n^{\rho-1}$ pour $n \geqq 2$, et $\liminf_{n\to+\infty} \lambda_n^{-\rho} a_n \geqq 0$.

Notre but est d'indiquer comment, si l'on renforce l'hypothèse sur $f(s)$ en faisant intervenir son comportement pour les valeurs complexes de s, soit simplement au voisinage du point $s = 0$, soit au voisinage de tous les points de la droite $\mathcal{R}[s] = 0$, on peut remplacer les conditions (1) et (2) par de plus larges.

Dans nos énoncés, il est entendu que seules interviennent les valeurs de s satisfaisant à $\mathcal{R}[s] > 0$, et que $s^{-\rho} = r^{-\rho}e^{-\rho i\theta}$ si $s = re^{i\theta}$ avec $|\theta| < \pi/2$.

1. Si l'on suppose que, quand s tend vers zéro par valeurs réelles ou complexes, $f(s) = As^{-\rho} + O[r^{1-\rho}\varphi(r)]$, où $\varphi(r)$ est une fonction positive non-croissante pour $r > 0$ et telle que l'intégrale $\int_0 \varphi(r) \log (1/r)\, dr$ dans le cas $\rho > 0$, ou $\int_0 \varphi(r)\, dr$ dans le cas $\rho = 0$, soit convergente, on peut remplacer (1) et (2) par

(1′) $a_n = o[(\lambda_n - \lambda_{n-1})\lambda_n^\rho] \qquad (n \to +\infty)$,
(2′) a_n réel, $\text{Min}\,(a_n, 0) = o[(\lambda_n - \lambda_{n-1})\lambda_n^\rho]$, et $\liminf_{n\to+\infty} \lambda_n^{-\rho} a_n \geqq 0$.

2. À l'hypothèse indiquée ci-dessus, ajoutons l'une des suivantes suivant que $\rho = 0$ ou $\rho > 0$:

a. Pour le cas $\rho = 0$: $f(s)$ reste bornée au voisinage de chaque point de la droite $\mathcal{R}[s] = 0$,

b. Pour le cas $\rho > 0$: Pour chaque $y \neq 0$, quand s tend vers zéro, on a $f(iy + s) = O[r^{1-\rho}\psi(r)]$, où $\psi(r)$ est une fonction positive non-croissante pour $r > 0$ et telle que l'intégrale $\int_0 \psi(r)\, dr$ soit convergente (cette fonction pouvant dépendre de y).

Alors (1) et (2) peuvent être remplacées par:

(1″) $|a_n| \leqq M(\lambda_n - \lambda_{n-1})\lambda_n^\rho$ pour $n \geqq 2$,
(2″) a_n réel, $a_n \geqq -M(\lambda_n - \lambda_{n-1})\lambda_n^\rho$ pour $n \geqq 2$, et $\liminf_{n\to+\infty} \lambda_n^{-\rho} a_n \geqq 0$.

En fait, ces résultats sont établis comme conséquences d'énoncés plus généraux relatifs à l'intégrale de Laplace.

N.B. Depuis que cette communication a été présentée au Congrès, l'auteur a reconnu que les hypothèses sur $f(s)$ dans (2) peuvent être élargies. Cf. C. R. Acad. Sci. Paris t. 232, pp. 589–591.

UNIVERSITY OF CLERMONT-FERRAND,
CLERMONT-FERRAND, FRANCE.

LES PERMUTATIONS CLIVÉES

ARNAUD DENJOY

I désignant l'ensemble des entiers positifs, (N) son ordination dans le sens de la croissance, toute ordination différente étant appelée une *permutation* de (N), la somme (imbriquée) d'une suite de permutations P_k $(k = 1, 2, \cdots)$ est la permutation P des entiers n décrivant I:

$$n = f_1(k \mid j) = \frac{(k + j)(k + j - 1)}{2} + j \qquad (j \geqq 0,\ k + j \geqq 2)$$

définie par

$$f_1(k \mid 0) < f_1(k \mid j) < f_1(k+1 \mid 0) \text{ si } j \geqq 1,$$
$$f_1(k \mid j) < f_1(k \mid j') \text{ si } j < j' \text{ selon } P_k .$$

Posons

$$f_p(S \mid j) \equiv f_p(k_1, \cdots, k_p \mid j) = f_{p-1}(k_1, \cdots, k_{p-1} \mid j_{p-1}) \text{ si } j_{p-1} = f_1(k_p \mid j).$$

Une permutation P est dite *clivée* si elle est soit la permutation-unité (N non permutée) soit une somme de permutations clivées (c'est une définition antirécurrente).

G étant une famille de suites finies d'entiers positifs $S = (k_1, \cdots, k_p)$, une suite S de G est dite *ouverte* ou *close* dans G selon qu'elle commence ou non une autre suite de G; G est dite *progressive* si toute suite commençant une suite de G est dans G; douée du *caractère* (*A*) si, pour toute suite ouverte S de G, toutes les suites (S, κ) $(\kappa \geqq 1)$ sont dans G.

A toute permutation clivée P correspond une famille G progressive dont les suites S sont ainsi caractérisées que les nombres $n = f_p(S \mid j)$ où j décrit I, forment une section ordinale de P, immédiatement précédée par $f_p(S \mid 0)$ et suivie par $f_p(S_{p-1}, k_p + 1 \mid 0)$. La suite S est close ou bien ouverte dans G selon que la permutation des j semblable à l'ordination des $n = f_p(S \mid j)$ par P est ou non identique à (N).

Réciproquement, soit G une famille de suites $S = (k_1, \cdots, k_p)$ et $\Phi(G)$ l'ensemble des formes ϕ, savoir $f_p(S \mid 0)$ si $k_p \geqq 2$ et $f_p(S \mid j)(j \geqq 1)$ si S est close. 1° Si G est progressive et vérifie la condition (A): L'ensemble Ψ des valeurs n des formes ϕ est identique à I; 2° $\Phi(G)$ étant ordonné alphabétiquement selon la croissance des arguments des ϕ et Ψ semblablement à $\Phi(G)$, la permutation obtenue est clivée. Pour que la permutation clivée P soit bien ordonnée, il faut et il suffit que G vérifie la condition (B): Toute suite indéfinie $\kappa_1, \kappa_2, \cdots$ d'entiers positifs commence par une suite close de G.

University of Paris,
Paris, France.

A CLASS OF NONHARMONIC FOURIER SERIES

R. J. Duffin and A. C. Schaeffer

A sequence $\{\lambda_n\}$, $n = 0, \pm 1, \pm 2, \cdots$ of real or complex numbers is said to have uniform density 1 if there exist constants L and δ such that $|\lambda_n - n| \leqq L$ and $|\lambda_n - \lambda_m| \geqq \delta > 0$ for $n \neq m$. It was shown by the authors (Amer. J. Math. vol. 67 pp. 141–154) that if an entire function $f(z)$ of exponential type γ, $\gamma < \pi$, is uniformly bounded at the points $\{\lambda_n\}$, then it is uniformly bounded on the real axis. This result was applied to problems concerning the coefficients of power series. The central result of this paper is that if $\sum |f(\lambda_n)|^2 < \infty$, then

$f(x) \in L_2(-\infty, \infty)$. An essentially equivalent statement of this result is that if $g(x) \in L_2(-\gamma, \gamma)$, then there are positive constants A and B which depend exclusively on γ, L, and δ such that

$$A \int_{-\gamma}^{\gamma} |g(x)|^2 \, dx \leqq \sum_{-\infty}^{\infty} \left| \int_{-\gamma}^{\gamma} g(x) \exp(i\lambda_n x) \, dx \right|^2 \leqq B \int_{-\gamma}^{\gamma} |g(x)|^2 \, dx.$$

This is referred to as the *frame* condition. If $\lambda_n = n$, then $A = B = 2\pi$ is Parseval's theorem. The proof that a constant B exists is quite direct. The proof of the existence of the constant A is made to depend on a closure theorem for functions analytic in a circle. The existence of A implies that the set $\{\exp(i\lambda_n x)\}$ is closed on the interval $(-\gamma, \gamma)$; however, closure on this interval does not imply the existence of A or of B.

Abstract considerations in Hilbert space show that the frame condition gives nonharmonic Fourier series of the form $\sum c_n \exp(i\lambda_n x)$ quite similar properties to ordinary Fourier series. However, the situation is more complicated because the set $\{\exp(i\lambda_n x)\}$ is highly dependent on an interval of length less than 2π. Most of the previous studies of nonharmonic Fourier series have been for the independent case. In view of this fact, an L_2 theory of these series is developed which considers properties of conjugate frames, expansion coefficients, mean convergence, and pointwise convergence. Some of these results overlap those of Paley and Wiener, Levinson, and Boas.

Carnegie Institute of Technology,
Pittsburgh, Pa., U. S. A.
University of Wisconsin,
Madison, Wis., U. S. A.

THE GENERAL FORM OF HYPERGEOMETRIC SERIES OF TWO VARIABLES

A. Erdélyi

Sponsored by the ONR

A formal double power series $\sum A_{mn} x^m y^n$ is called a hypergeometric series if $A_{m+1,n}/A_{mn} = f(m,n)$ and $A_{m,n+1}/A_{mn} = g(m,n)$ are rational functions of m and n. The most general form of such series has been subject to controversy and as far as the author knows the explicit result is given here for the first time.

Appell, Birkeland, Horn, Kampé de Fériet, Mellin, and others have investigated series in which A_{mn} is a gamma product, i.e. of the form

$$\gamma_{mn} = \prod_i \{\Gamma(c_i + u_i m + v_i n)/\Gamma(c_i)\} \tag{1}$$

where the c_i are arbitrary (possibly complex) constants and the u_i and v_i are (positive, negative, or zero) integers. The question has been asked whether this is the most general form of the coefficients of a hypergeometric series.

Clearly,

$$f(m,n)g(m+1,n) = f(m,n+1)g(m,n) \tag{2}$$

for all non-negative integers m, n and hence identically in m and n, and conversely, it is easily seen that every rational solution of (2) generates a hypergeometric series. Birkeland (C. R. Acad. Sci. Paris vol. 185 (1927) p. 923) stated that every rational solution of (2) can be decomposed into linear factors, and this leads essentially to (1) as the most general form. However, O. Ore (C. R. Acad. Sci. Paris vol. 189 (1929) p. 1238) noted that Birkeland's result is not entirely general and gave (Journal de Mathématiques (9) vol. 9 (1930) p. 311) a thorough analysis of the rational solutions of (2).

Ore's results should enable one to construct the most general hypergeometric series, but it seems that this construction has not been carried out. In the present paper Ore's theorems on the rational solutions of (2) are analyzed and their bearing on the structure of A_{mn} is investigated. It turns out that the only factors disregarded by Birkeland are rational functions of m and n. More precisely, the conclusion is reached that the coefficients of the most general hypergeometric series of two variables are of the form $A_{mn} = R(m,n)\gamma_{mn}a^m b^n$ where R is a fixed rational function of its two variables, a and b are constants, and γ_{mn} is a gamma product. This is equivalent to saying that the most general hypergeometric series of two variables results from the application of a rational differential operator $R(x(\partial/\partial x), y(\partial/\partial y))$ to a hypergeometric series of the Horn-Birkeland type. This answers a question left open by Horn (Math. Ann. vol. 105 (1931) p. 381) and justifies the general practice of restricting attention to series of the Appell-Horn-Birkeland type.

California Institute of Technology,
Pasadena, Calif., U. S. A.

ON THE ASYMPTOTIC DISTRIBUTION OF CERTAIN SUMS

N. J. Fine

Let $((t)) = t - [t] - 1/2$, the square brackets denoting the greatest integer function. It has been shown by Kac [J. London Math. Soc. vol. 13 (1938) pp. 131–134] that the distribution of the sums $N^{-1/2}\sum_{n<N}((2^n t))$ is asymptotically normal with mean 0 and variance 1/4. The author, in §8 of his dissertation [Trans. Amer. Math. Soc. vol. 65 (1949) pp. 372–414], considered the sums $\sum_{n<N}((2^n t - 1/2))$ and proved that they are bounded uniformly in N and t. The question was therefore raised as to the behavior of $N^{-1/2}\sum_{n<N}((2^n t - \beta))$, $0 \leqq \beta < 1$. It is proved here that the distribution of these sums is asymptotically normal with mean 0 and variance $\sigma^2 = \sum_{n\geqq 1}2^{-n}((\beta - \beta_n))^2$, where β_n is the fractional part of $2^n\beta$. Thus $\beta = 1/2$ is the only case with zero variance. Exlicit results are obtained for the joint distribution of pairs of sums of the above ype.

Similar results are obtained for the relative frequency with which the fractional part of $2^n t$, $0 \leqq n < N$, falls in a given interval. Here, again, the distribution is asymptotically normal. If the interval is $(0, \beta)$, then the mean is β nd the variance is τ^2/N, where $\tau^2 = \beta - \beta^2 + 2\sum_{n \geqq 1} 2^{-n}\{\min(\beta, \beta_n) - \beta\beta_n\}$.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA., U. S. A.

A CLOSURE CRITERION FOR ORTHOGONAL FUNCTIONS

ROSS E. GRAVES

In this paper the author gives a simple necessary and sufficient condition for sequence of orthogonal functions to be closed in L_2. In theory, the question f closure is reduced to the evaluation of certain integrals and the summation f an infinite series whose terms depend only upon the index n. The simplest orm of the final result reads

THEOREM Ia. *Let* $\{\varphi_n\}$ *be a set of orthonormal functions on a finite interval* $a, b)$ *and let* c *be any number such that* $a \leqq c \leqq b$. *Then*

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_c^x \varphi_n(t)\, dt \right|^2 dx \leqq \tfrac{1}{2}[(a-c)^2 + (b-c)^2],$$

here equality holds if and only if $\{\varphi_n\}$ *is closed in* L_2 *on* (a, b).

Actually, the above theorem is a special case of a more general result.

THEOREM I. *Let* $p(t)$ *be a function whose zeros and discontinuities have Jordan ontent zero such that for* $x \in (a, b)$, $p(t) \in L_2$ *on* $\min(c, x) < t < \max(c, x)$, *here* $a \leqq c \leqq b$ *(*a *and* b *may be infinite;* c *is fixed). Let* $w(x)$ *be a measurable unction almost everywhere finite and positive such that* $w(x) \int_c^x |p(t)|^2\, dt \in L_1$ *on* $a, b)$. *Then for any family of functions* $\{\varphi_n\}$ *orthogonal and normal on* (a, b),

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_c^x p(t)\varphi_n(t)\, dt \right|^2 w(x)\, dx \leqq \int_a^b \left| \int_c^x |p(t)|^2\, dt \right| w(x)\, dx,$$

here equality holds if and only if $\{\varphi_n\}$ *is closed in* L_2 *on* (a, b).

The insertion of the functions $p(t)$ and $w(x)$ serves two purposes. First, it nables us to extend the results of Theorem Ia to the case where the interval $a, b)$ is infinite; and, second, proper choice of these functions greatly facilitates ne calculation of the integrals involved and the summation of the resulting eries.

Applications of Theorem Ia are made to the trigonometric and Legendre unctions, while the closure of the Hermite and Laguerre functions is established y means of the more general Theorem I.

UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINN., U. S. A.

SUR UNE NOTION DE CONTINUITÉ RÉGULIÈRE AVEC APPLICATION AUX SÉRIES DE FOURIER

J. KARAMATA

Soit $\lambda(t)$ une fonction logarithmico-exponentielle définie pour $t > 0$ et telle que

$$\lambda(t) \to 0 \text{ lorsque } t \to 0.$$

Nous dirons que $f(x)$ est régulière d'ordre $\lambda(t)$ au voisinage du point x, si en posant

$$\varphi(t) = f(x + t) + f(x - t) - 2f(x),$$

on a

$$\varphi(t') = O\{\lambda(t)\} \text{ pour } t \leqq t' \leqq t + t\cdot\lambda(t),\ t \to 0.$$

Plus $\lambda(t)$ tend rapidement vers zéro, plus la fonction $f(x)$ est régulière.

Lorsque la fonction satisfait à la condition de Lipschitz

$$|\varphi(t)| \leqq M\cdot\lambda(t),$$

elle est certainement régulière d'ordre $\lambda(t)$, mais elle peut l'être d'un ordre plus élevé.

THÉORÈME. *Lorsque la fonction $f(x)$, continue au point x, y est régulière d'ordre $\lambda(t)$, sa série de Fourier convergera en ce point si*

$$\int_{+0} \frac{\lambda(t)}{t}\, dt \quad \textit{converge.}$$

UNIVERSITY OF GENEVA,
GENEVA, SWITZERLAND.

ON IRREGULAR POINTS OF NORMAL CONVERGENCE AND M-CONVERGENCE FOR SERIES OF ANALYTIC FUNCTIONS

BENJAMIN LEPSON

A series of complex-valued functions is said to be *M-convergent* on a set S if the sum of the least upper bounds on S of the terms is finite, while, following Laurent Schwartz, the series is said to be *normally convergent* on S if the series of absolute values is uniformly convergent on S. Given a series of functions each analytic in a domain D of the complex plane which converges absolutely at every point of D, we define the sets of irregular points in D of normal convergence and of M-convergence in the expected manner, and denote these sets by N and M respectively. Let U be the set of irregular points in D in the sense of Montel for the sequence of partial sums of the given series. It was shown previously by the author (Bull. Amer. Math. Soc. Abstract 56-3-247) that $N = M$ and that M is nowhere dense. It is clear that M contains U. The following example shows that M may actually be larger than U.

Let D be the interior of the square bounded by the axes and the lines $x = 1$ and $y = 1$. Let $P_n(z)$ be a polynomial such that $|P_n(z)| < 1/n^2$ in that portion of D with $y \leqq 1/2 + 1/n$ and $|P_n(z) - 1| < 1/n^2$ in the portion of D with $y \geqq 1/2 + 2/n$. Let $Q_n(z) = P_{n+1}(z) - P_n(z)$. Let $\{N_n\}$ be a sequence of positive integers such that $|Q_n(z)| < N_n$ for all z in D, and put $R_n(z) = Q_n(z)/(nN_n)$. Let $S_n(z)$ be the nth term of the series whose first N_1 terms are each $R_1(z)$, whose next $2N_2$ terms are each $R_2(z)$, etc. Then the series $S_1(z) - S_1(z) + S_2(z) - S_2(z) + \cdots$ is the desired example, since U is empty while M is that portion of the line $y = 1/2$ contained in D.

If the domain D is simply connected, the set M can be characterized geometrically following the method of Hartogs and Rosenthal. It is found that the necessary and sufficient conditions that a set be the set of irregular points in D of M-convergence of an absolutely convergent series of analytic functions and that it be the set of irregular points in D in the sense of Montel for a convergent sequence of analytic functions are the same.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

DIRECT THEOREMS ON METHODS OF SUMMABILITY

G. G. LORENTZ

By a direct theorem we understand an assertion that under some simple conditions which restrict the structure of s_n, this sequence is summable by a given method $A = (a_{mn})$, that is $\lim_{m\to\infty}\sum_n a_{mn}s_n$ exists. Thus, regularity of A is a direct theorem. But besides convergent sequences, all useful methods possess wide classes of summable divergent sequences, which can be easily described. Thus if (and only if) the variation of the mth row of (a_{mn}) tends to 0, all "almost convergent" sequences s_n described by

$$F: \qquad \lim_{p\to\infty} \frac{1}{p}(s_{n+1} + \cdots + s_{n+p}) = s \qquad \text{uniformly in } n$$

are A summable. We further call an increasing function $\Omega(n) \to \infty$ a summability function (s.f.) of the first or second kind, according to which of the following conditions implies A summability of s_n : $s_n = 0$ except for a subsequence s_{n_ν}, whose counting function $\omega(n)$ (i.e. the number of $n_\nu \leqq n$) is $\leqq \Omega(n)$; or $s_1 + \cdots + s_n = O(\Omega(n))$. The following conditions characterize method A which have $\Omega(n)$ as a s.f. of the first or second kind: $\lim_{m\to\infty} A(m, \Omega) = 0$, $A(m, \Omega)$ being the least upper bound of $\sum_\nu |a_{mn_\nu}|$ for all sequences n_ν with $\omega(n) \leqq \Omega(n)$, or $\lim_{m\to\infty}\sum_n \Omega(n)\,|a_{mn} - a_{m,n+1}| = 0$. And in order that A have *some* s.f. of the first or the second kind it is necessary and sufficient that $a_{mn} \to 0$ uniformly for $m \to \infty$ or that $\sum_n | a_{mn} - a_{m,n+1} | \to 0$. Since F has no s.f., it follows that any method $A \supset F$ is strictly stronger than F even for bounded sequences. Replacing summability by absolute A summability in the above definitions, absolute s.f. are obtained. A method A has absolute s.f. of the first kind if and only if the variation of the nth column converges to 0. The statement that a condition on s_n is not a Tauberian condition for a given method A is a direct theorem. Therefore, theorems and methods described above are a good means of showing that a certain Tauberian condition is the best possible of its kind. We have: (a) If $\Omega(n)$ is a s.f. for A, then $s_n - s_{n-1} = o(\Omega(n)^{-1})$ is not a Tauberian condition for A; (b) If $n_1 < n_2 < \cdots$, $c_n \geqq 0$ and

$$\lim_{m\to\infty} \left\{ \operatorname{Max}_{\nu} \sum_{n_\nu+1}^{n_{\nu+1}} | a_{mn} | \right\} = 0, \qquad \sum_{n_\nu+1}^{n_{\nu+1}} c_n \geqq \delta > 0,$$

then $s_n - s_{n-1} = O(c_n)$ is not a Tauberian condition. To apply (a) and as a problem interesting in itself we find all s.f. of the Abel, Riesz, Euler, Borel, Riemann, Hausdorff methods. The determination of the absolute s.f. is more difficult.

University of Toronto,
Toronto, Ont., Canada.

THE DETERMINATION OF CONVERGING FACTORS FOR THE ASYMPTOTIC EXPANSIONS FOR THE WEBER PARABOLIC CYLINDER FUNCTIONS

J. C. P. Miller

When determining the numerical value of a function from an expansion in series, the method commonly used is to evaluate the series in the form

$$S = S_n + R_n$$

in which S_n is a partial sum, and R_n the corresponding remainder term. For many series, n may be readily chosen so that R_n is negligible and S may be taken as approximately equal to S_n. When, however, convergence is slow, or, as in the case of asymptotic expansions, the series is divergent, it may be necessary to evaluate R_n closely, rather than to reduce its value below a prescribed, negligible upper limit. J. R. Airey introduced the idea of a *converging factor*; i.e., he expressed R_n in the form $\pm\ C_n u_n$, where $\pm u_n$ is the nth term in the series, and concentrated on the expansion of C_n in the form of a series suitable for computation. The methods by which he obtains expansions for C_n are empirical in nature, but numerical tests have justified his results.

The purpose of this note is to present expressions for the converging factors appropriate to the asymptotic expansions for the parabolic cylinder functions. These are solutions of Weber's differential equation

$$\text{(A)} \qquad \frac{d^2y}{dx^2} = (\tfrac{1}{4}x^2 + a)y.$$

There are two independent expansions, both satisfying the equation (A), and each such that individual terms of the expansions satisfy a two term recurrence relation. It follows that in each case C_n satisfies a second order differential equation and a first order difference equation. If u_n is the term of least modulus in either series it is found that $h(= \frac{1}{2}x^2 \pm a + 1 - n)$ is small; the sign depends on which of the two series is considered. From either, or both, of the equations, differential or difference, it satisfies, C_n can be determined in the form of a series of the form

$$\alpha_0(h) + \frac{\alpha_1(h)}{x^2} + \frac{\alpha_2(h)}{x^4} + \cdots$$

in which n does not occur explicitly. Results have been determined up to and including $\alpha_4(h)$ in each case, apart from the term independent of h in α_4 in the case of that asymptotic expansion which ultimately has all its terms of similar sign. The evaluation of this constant would require the preliminary evaluation $d\alpha_5/dh$.

National Bureau of Standards,
Washington, D. C., U. S. A.

MODULAR TRANSFORMATION OF CERTAIN SERIES

Saul Rosen

The Hardy-Littlewood method as modified by Rademacher can be used to obtain an infinite series expansion for modular functions. These series themselves will then exhibit an absolute or relative invariance under modular transformations. The problem of applying the transformations:

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$$

directly to the series has previously been considered in the case of the absolute modular invariant $J(\tau)$ by Rademacher. This is now extended to a consideration of functions belonging to modular subgroups, in which the invariance is not absolute, where certain roots of unity appear in the transformation equations. The function

$$f(x) = \prod_{n=1}^{\infty} (1 + x^n) = e^{-\pi i\tau/12} \frac{\zeta(2\tau)}{\zeta(\tau)}, \qquad x = e^{2\pi i\tau}$$

is considered in detail. This function belongs to $\Gamma_0(2)$, the modular subgroup for which $c \equiv 0 (\mathrm{mod}\, 2)$. The Hardy-Littlewood method leads to the expansion $f(x) = \sum_{n=0}^{\infty} Q_n x^n$ where

$$Q_n = \frac{1}{2^{1/2}} \sum_{k=1, k \mathrm{odd}}^{\infty} \sum_{h \bmod k, (h,k)=1} \Omega_{h,k} e^{-2\pi i h n/k} \sum_{\mu=1}^{\infty} \frac{2\pi}{\mu!(\mu-1)!} \left(\frac{\pi}{24k^2}\right)^{\mu} \cdot \left[2\pi\left(n + \frac{1}{24}\right)\right]^{\mu-1}.$$

The $\Omega_{h,k}$ are rather complicated roots of unity related to those which appear in the transformation theory for the η function.

In this paper the series above is shown to belong to $\Gamma_0(2)$ by direct application of the transformations which are the generators of $\Gamma_0(2)$: $\tau' = \tau + 1$ and $\tau' = \tau/(2\tau + 1)$. In the latter case it was found convenient to consider $\tau' = \tau/(2\tau + 1)$ as the resultant of the transformations $\tau' = -1/\tau$, $\tau' = \tau - 2$, and $\tau' = -1/\tau$. $\tau' = -1/\tau$ is not in $\Gamma_0(2)$. It is thus necessary to introduce series corresponding to the cosets of $\Gamma_0(2)$, which is of index 3 in the full modular group. These series are the expansions, by the Hardy-Littlewood method of the associated functions $f_1(x)$ and $f_2(x)$, related to $f(x)$ by modular transformation. A reciprocity law for the roots of unity $\Omega_{h,k}$ is derived and used in the transformation of the series.

DREXEL INSTITUTE OF TECHNOLOGY,
PHILADELPHIA, PA., U. S. A.

UNIQUENESS THEORY FOR HERMITE SERIES

WALTER RUDIN

We consider series of the form $S = \sum_0^{\infty} a_n \phi_n(x)$, where $\phi_n(x)$ is the normalized Hermite function $c_n^{-1} e^{x^2/2} D^n(e^{-x^2})$, $c_n = (-1)^n (2^n n! \pi^{1/2})^{1/2}$.

DEFINITIONS. (A) Let $F(t)$ be defined for $|x - t| < \delta$. For $0 < h < \delta$, let $y(t) = y(t; F, h)$ be the solution of $y'' - (t^2 + 1)y = 0$, such that $y(x + h) = F(x + h)$, $y(x - h) = F(x - h)$. Put $\Delta_h F(x) = y(x; F, h) - F(x)$, and $\Lambda F(x) = \lim_{h \to 0} 2\Delta_h F(x)/h^2$. $\Lambda^* F(x)$, $\Lambda_* F(x)$ are defined likewise with lim sup, lim inf in place of lim. If $F''(x)$ exists, then $\Lambda F(x) = F''(x) - (x^2 + 1)F(x)$. In particular, $\Lambda \phi_n(x) = -(2n + 2)\phi_n(x)$. (B) We write $f \in H_p$ if $f \in L$ on every finite interval, and if $\int_{-\infty}^{\infty} |x^p f(x)| e^{-x^2/2} dx < \infty$. If $f \in H_p$ for every $p \geqq 0$, we write $f \in H$. If $f \in H$, and $a_n = \int_{-\infty}^{\infty} f(x)\phi_n(x)\, dx$, we write $S = S(f)$. (C) If $f \in H_0$, put $\Omega f(x) = -\int_{-\infty}^{\infty} f(t) K(x, t)\, dt$, where $K(x, t)$ is the Green's function of the system $y'' - (x^2 + 1)y = 0$, $y(\pm\infty) = 0$. (D) We write $S \in K$ if

$$-\sum_0^{\infty} a_n \phi_n(x)/(2n + 2) = S(F),$$

and F is continuous. (E) $f^*(x)$, $f_*(x)$ denote the upper and lower Poisson sums of S.

RESULTS. I. Let $p \geqq 0$ be given. Suppose (1) $F(x)$ is continuous and $F \in H_p$; (2) $\Lambda^*F(x) > -\infty$ and $\Lambda_*F(x) < +\infty$, except possibly on countable sets E_1 and E_2; (3) $\limsup_{h\to 0}\Delta_h F(x)/h \geqq 0$ on E_1, $\liminf_{h\to 0}\Delta_h F(x)/h \leqq 0$ on E_2; (4) there exists $y \in H_p$ such that $y(x) \leqq \Lambda^*F(x)$. Then $\Lambda F \in H_p$ and $F(x) = \Omega\Lambda F(x)$ for all x.

II. Suppose $S \in K$. If F satisfies (2), (3), and if there exists $y \in H$ such that $y(x) \leqq \Lambda^*F(x)$, then $S = S(\Lambda F)$.

III. Suppose $S \in K$. If (i) $f_*(x) > -\infty$ and $f^*(x) < +\infty$, except possibly on countable sets E_1 and E_2; (ii) F satisfies (3); (iii) there exists $y \in H$ such that $y(x) \leqq f_*(x)$, then $f^*(x) = f_*(x)$ p.p. and $S = S(f^*) = S(f_*)$.

IV. Suppose $a_n = o(n^{1/4})$. If $-\infty < y(x) \leqq f_*(x) \leqq f^*(x) < +\infty$, where $y \in H$, then $S = S(f^*) = S(f_*)$.

V. If S converges to a finite function $f \in H$, then $S = S(f)$.

DUKE UNIVERSITY,
DURHAM, N. C., U. S. A.

GENERALIZED FOURIER INTEGRALS

WARD C. SANGREN

It is well known that the Sturm-Liouville expansion of an integrable function $f(x)$ behaves as regards convergence in the same way as an ordinary Fourier series. It is shown that there exists a parallel situation in the case of certain generalized Fourier integrals and the ordinary Fourier integrals. The generalized Fourier integrals which are dealt with are special cases of the more general types of expansions obtained by E. C. Titchmarsh (*Eigenfunction expansions associated with second-order differential equations*, Chapter III).

In a previous paper presented to the American Mathematical Society, expansions were obtained which are associated with second-order differential equations with step-function coefficients. These integral expansions are slight extensions of ordinary Fourier integrals. Upon considering coefficients which are discontinuous at interfaces and satisfy certain restrictions, it is possible to obtain integral expansions which extend these expansions in a fashion similar to the way the generalized Fourier integrals of the previous paragraph extend ordinary Fourier integrals. In particular, the equi-convergence properties continue to hold.

MIAMI UNIVERSITY,
OXFORD, OHIO, U. S. A.

LES FONDEMENTS D'UNE THÉORIE GÉNÉRALE DE SÉRIES DIVERGENTES

RICARDO SAN JUAN LLOSÁ

L'indétermination d'une fonction par son développement asymptotique et l'illegitimité de la dérivation qui, remarquées par Poincaré dans son mémoire original, furent, peut-être, la cause de ne pas remplacer sa convergence asymptotique par l'ordinaire de Cauchy [G. H. Hardy, *Divergent series*, Oxford, 1949, p. 28], on peut les reparer moyennant *l'approximation asymptotique optime* (a.a.o.), c'est-à-dire, dont les bornes m_n du développement asymptotique $f(z) \sim \sum_{n=0}^{\infty} a_n z^n$ dans un domaine renfermant l'origine soient ordonnément plus petites, $m_n < m_n'$, que les homologues m_n' d'une de toute autre fonction holomorphe quelconque dans le même domaine, qui remplissent la condition d'unicité de Carlemann-Ostrowski. L'existence de cette fonction est la condition nécessaire et suffisante pour la coincidence de toutes les fonctions holomorphes dans le domaine où elles vérifient cette condition, coincidence que certainement n'a pas lieu toujours. [R. San Juan Llosá, Acta Mathematica t. 75, pp. 247–254.] Approximations optimes sont la prolongation analytique ordinaire, la somme de Stieltjes, celles qui remplissent la condition de Watson-Nevanlinna, etc.

La dérivée et primitive de l'a.a.o. en $| z^{1/\alpha} - \rho e^{i\phi_0/\alpha} | < \rho$, $\alpha > 0$, $\rho_0 - \delta < \rho < \rho_0 + \delta$ c'est l'a.a.o. de la série dérivée et primitive respectivement, si les bornes m_n sont croissantes: $m_n < m_{n+1}$. Elle a aussi d'autes propriétés formelles.

En généralisant la méthode de décomposition appliquée premièrement [R. San Juan Llosá, Ibid.] a $e^{-t^{1/4}}$, il en résulte que toute fonction $\psi(t)$ avec $\int_0^\infty t^n \, d\psi(t) < \infty$ peut se décomposer en deux $\alpha(t) + \bar{\alpha}(t) = \psi(t)$ avec $\mu_n = \int_0^\infty t^n \, d\psi(t) < \infty$, telles que $\sum_{n=1}^{\infty} 1/(\mu_{qn+r} \cdot p_n)^{1/n} = \infty$, $\sum_{n=1}^{\infty} 1/(\bar{\mu}_{qn+r} \cdot p_n)^{1/n} = \infty$, $p_n > 0$. étant une suite positive quelconque telle que $\sum_{n=1}^{\infty} 1/p_n^{1/n} = \infty$. Comme corollaire immédiat il en résulte une ample généralisation d'un théorème de M. Mandelbrojt [S. Mandelbrojt, Acta Mathematica t. 72, p. 16]. L'application aux coefficients de séries de fractions simples avec des dérivées nulles à l'origine conduit, par la transformation de Laplace, à des séries avec des approximations optimes différentes en deux domaines $R' < R$. L'existence de a.a.o. différentes dans domaines R et R' emprétants ($R \cap R' \neq 0$) ou $R' < R$ nous met devant un dilèmme: ou bien on s'abstient d'approximations optimes aussi naturelles que celles de Stieltjes, ou on étendre la notion de somme d'une série potentielle $\sum_{n=0}^{\infty} a_n z^n$ dans un domaine, étant admis que celle-ci n'est pas un concept inherent à la série exclusivement, mais à celle-ci plus le domaine.

Ceci n'empêche pas d'adopter comme prolongation radaile $f(z)$ sur $(0, 1)$ l'a.a.o. dans un domaine $| z^{1/\alpha} - 1 | < 1$, $\alpha > 0$, si celle-ci coincide avec une approximation holomorphe quelconque qui réalise la condition d'unicité dans un autre d'une ampleur plus petite $\alpha' < \alpha$. On définisse $\sum_{n=0}^{\infty} a_n = \lim_{z \to 1^-} f(z)$.

Quelques propriétés formelles subsistent ainsi et d'autres avec des restrictions, suffisantes pour développer une théorie de séries divergentes, dans laquelle une autre quelconque soit contenue, à cause de la nécessité des conditions d'unicité adoptées.

UNIVERSITY OF MADRID,
MADRID, SPAIN.

ON A CLASS OF INTEGRAL-VALUED DIRICHLET SERIES

ERNST G. STRAUS

It is shown that if the Dirichlet series $f(s) = \sum_{k=1}^{\infty} a_k^s$ is integral-valued for $s = 1, 2, \cdots$, then either $a_k = 0$ for almost all k, or $\sum_{k=1}^{\infty} | a_k |^{1-\epsilon}$ diverges for every $\epsilon > 0$. Similar theorems are obtained if $f(s)$ is known to be rational or algebraic for $s = 1, 2, \cdots$. A theorem of the same type is obtained for $f(s) = \sum_{k=1}^{\infty} a_k^s - \sum_{k=1}^{\infty} b_k^s$, and hence (since we did not assume the a_k or b_k to be distinct) for every $f(s) = \sum_{k=1}^{\infty} n_k a_k^s$, where n_k is integral.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIF., U. S. A.

TAUBERIAN THEOREMS FOR SUMMABILITY (R_1)

OTTO SZÁSZ

Denote by s_n the nth partial sum of a series $\sum_1^{\infty} a_n$; if the series

$$(2/\pi) \sum n^{-1} s_n \sin nh = R(h)$$

converges for small positive h, and if $R(h) \to s$ as $h \to 0$, then $\sum a_n$ is called summable (R_1). This summation method has particular significance for Fourier series; it was recently investigated in a paper by Hardy and Rogosinski (Proc. Cambridge Philos. Soc. (1949)). The method is not regular; thus additional conditions are needed in order that convergence imply summability (R_1). We establish the condition (*) $\sum_n^{2n} (| a_\nu | - a_\nu) = O(1)$. We also prove that condition (*) and Abel summability imply R_1 summability. For Fourier series we prove summability (R_1) at each point of continuity. Finally we compare (R_1) with Cesàro summability.

NATIONAL BUREAU OF STANDARDS,
LOS ANGELES, CALIF., U. S. A.

REPRESENTATION OF AN ANALYTIC FUNCTION BY GENERAL LAGUERRE SERIES. PRELIMINARY REPORT.

OTTO SZÁSZ AND NELSON YEARDLEY

The authors prove the following theorem for the general Laguerre expansion $f(z) \sim \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(z)$ of the analytic function $f(z)$ in the complex xy-plane, where $a_n^{(\alpha)}$ are the Fourier-Laguerre coefficients and $\{L_n^{(\alpha)}(x)\}$ is the set of general Laguerre polynomials of order α and degree n, orthogonal over the interval $(0, \infty)$ with respect to the weight function $e^{-x}x^{\alpha}$. In order that $f(z)$ possess a general Laguerre expansion of order α ($\alpha > -1$) which converges to it for every point z inside the parabola $p(d_\alpha)$: $y^2 = 4d_\alpha^2(x + d_\alpha^2)$ (where $d_\alpha = -\limsup (2n^{1/2})^{-1} \log | a_n^{\alpha} |$) it is necessary and sufficient that $f(z)$ be analytic in $p(d_\alpha)$ and that to every b_α, $0 \leqq b_\alpha < d_\alpha$, there correspond a positive number $B(\alpha, b)$ such that $| f(z) | \leqq B(\alpha, b) \exp \{x/2 - | x |^{1/2}[b_\alpha^2 - (r - x)/2]^{1/2}\}$ (where $r = (x^2 + y^2)^{1/2}$) for every point z in and on the parabola $p(b_\alpha)$. The above theorem is equivalent to a generalization of a theorem by Pollard (see Ann. of Math. vol. 48 (1947) pp. 358–365) and the methods used by the authors are essentially generalizations of those employed by Pollard for the case $\alpha = 0$ in his paper referred to above.

UNIVERSITY OF CINCINNATI,
CINCINNATI, OHIO, U. S. A.
NATIONAL BUREAU OF STANDARDS,
LOS ANGELES, CALIF., U. S. A.
PURDUE UNIVERSITY,
LAFAYETTE, IND., U. S. A.

SUMMABILITY MATRICES COINCIDENT WITH REGULAR MATRICES, BANACH SPACE METHODS

ALBERT WILANSKY

Let A be a conservative summability matrix (i.e. one summing all convergent sequences). With a natural assumption (normality) we ask if there is a regular matrix summing exactly the sequences of A. If so, call A admissible. Methods of Mazur (Studia Mathematica vol. 2) and the author (Trans. Amer. Math. Soc. vol. 67) bring known Banach space techniques to bear on summability. (1) Admissibility of A is shown equivalent to continuity (where defined) of f, where $f(x) = \lim x_n$, by use of the Hahn-Banach extension theorem. This reduces the problem to examination of a metric. Hence (2) A is admissible if there is a regular matrix summing all the sequences of A. The author (loc. cit.) has shown that if A is admissible, then (3) $\lim_n \sum_k a_{nk} \neq \sum_k \lim_n a_{nk}$. Assume this in the following. Let B be the matrix inverse to A. Let $a_k = \lim_n a_{nk}$. (4) If $\sum | a_k x_k |$ converges whenever x is summable A, then (2) implies that A is admissible. (5) If $\sum\sum | a_n b_{nk} |$ converges, then A is admissible. A straightforward argument shows that denial of (1) denies (3). (6) If $\sum_{k=0}^{m(p)} a_{n(p),k}x_k$ is bounded if x is summable A for each choice of sequences $m(p)$, $n(p)$, then A is admissible. An obvious application of the principle of condensation of singularities (Banach's treatise, p. 180) gives a boundedness condition on the biorthogonal sequence $\{L_n\}\{d_n\}$ (where $L_n(x) = x_n$, d_n = sequence whose nth term is 1, others zero) which implies that $\{d_n\}$ is a basis for the space. This contradicts (3) if (1) is denied.

LEHIGH UNIVERSITY,
BETHLEHEM, PA., U. S. A.

DIFFERENTIAL AND INTEGRAL EQUATIONS

THE UNIQUENESS FOR THE EQUATION $p = f(x, y, z, q)$ WITH THE CAUCHY DATA

Emilio Baiada

The Cauchy problem for the equation (i) $p = f(x, y, z, q)$ consists in finding a solution $z(x, y)$ of (i) in a suitable domain, such that $z(\bar{x}, y) = \omega(y)$, where $\omega(y)$ is a given function and $z(x, y)$ has continuous derivatives of the first order. A number of authors, in particular, A. Haar and A. Rosenblatt have given theorems of unicity for this problem (for instance when f satisfies a Lipschitz condition with respect to z and q).

The uniqueness exists however even if we do not assume that $z(x, y)$ has continuous derivatives everywhere, thus proving that this Cauchy problem is not well stated.

University of Cincinnati,
Cincinnati, Ohio, U. S. A.

FUNCTIONAL INVARIANTS OF INTEGRO-DIFFERENTIAL EQUATIONS

I. A. Barnett

Consider the linear homogeneous integro-differential equation of the second order

$$(1) \qquad y_{tt}(x; t) + \int_0^1 L(x, u; t)y_t(u; t)\,du + \int_0^1 M(x, u; t)y(u; t)\,du = 0,$$

where L and M are two given kernels and y is the unknown function. If y is subjected to an arbitrary Fredholm transformation

$$(2) \qquad y(x) = z(x) + \int_0^1 K(x, u)z(u)\,du,$$

where the kernel K has a nonvanishing Fredholm determinant, we may show that equation (1) is transformed into an equation of similar form in z.

By a suitable determination of K it is possible to transform equation (1) into one in which the first derivative y_t is missing, that is, into an equation of the form

$$z_{tt}(x; t) = \int_0^1 N(x, u; t)z(u; t)\,du.$$

To this end we find a kernel K satisfying the linear integro-differential equation

$$K_t(x, u; t) = -\frac{1}{2}\left[L(x, u; t) + \int_0^1 L(x, s; t)K(s, u; t)\, ds\right],$$

which is known to possess an infinitude of solutions.

Using the Lie theory of continuous groups in function space, we give a method for finding functionals of L, M, and their derivatives L_t, M_t, L_{tt}, M_{tt}, etc., which remain invariant under all the transformations of the form (2). If we define the kernel

$$G(x, u; t) \equiv 2L_t(x, u; t) - 4M(x, u; t) + \int_0^1 L(x, s; t)L(s, u; t)\, ds,$$

we may show that all the traces $\int_0^1 G_t(x, x; t)\, dx$ are invariant under (2). Hence, for sufficiently small λ's the Fredholm determinant $F(\lambda; G)$ is a functional invariant involving L, M, and L_t.

To obtain a functional invariant involving higher derivatives, we consider the kernel

$$H(x, u; t) \equiv 2G_t(x, u; t) + \int_0^1 L(x, s; t)G(s, u; t)\, ds - \int_0^1 G(x, s; t)L(s, u; t)\, ds.$$

We may again show that the Fredholm determinant $F(\lambda; H)$ is a functional invariant for sufficiently small λ's. This process may be continued and at each step we obtain a new functional invariant involving the derivatives of L and M to as high an order as desired.

University of Cincinnati,
Cincinnati, Ohio, U. S. A.

SUBFUNCTIONS AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

E. F. Beckenbach and Lloyd K. Jackson

The notion of subharmonic function is generalized in the direction of elliptic partial differential equations, and the methods exploited by Perron, Radó, Riesz, and others in solving the Dirichlet problem for the Laplace equation are extended to more general elliptic equations. A similar program from a different point of view has been carried out by G. Tautz, *Zur Theorie der elliptischen Differentialgleichungen*, Math. Ann. vol. 117(1940) pp. 694–726; vol. 118(1941) pp. 733–770.

University of California,
Los Angeles, Calif., U. S. A.
National Bureau of Standards,
Los Angeles, Calif., U. S. A.

SUR LES ÉQUATIONS DIFFÉRENTIELLES LINÉAIRES À COEFFICIENTS VARIABLES

Charles Blanc

On sait que l'on peut obtenir, par des opérations élémentaires, une intégrale particulière d'une équation différentielle linéaire à coefficients constants dont le second membre est une exponentielle e^{st}; cette intégrale particulière est même parfois la seule intéressante, par exemple dans certains problèmes où $s = i\omega$ est imaginaire pure et où les intégrales de l'équation homogène tendent toutes vers zéro lorsque la variable indépendante t augmente indéfiniment.

Dans le cas où l'équation est à coefficients variables, il n'est pas possible de procéder aussi simplement; il n'est même pas possible de déterminer sous forme finie une intégrale de l'équation, sauf dans quelques cas très particuliers. Partant de la remarque que si ω augmente, l'effet de la variation des coefficients tend à diminuer, il est naturel de chercher à exprimer une intégrale par un développement asymptotique en $1/s$, valable au moins pour $\mathcal{R}_s \geqq 0$. Pour y parvenir, on considère tout d'abord l'adjointe $\sum c_k(t)z^{(k)}(t) = 0$ de l'équation proposée et on forme la fonction

$$J(s, t) = 1/\sum (-1)^k c_k(t)s^k;$$

si les coefficients sont constants, J est constant en t et l'intégrale particulière mentionnée est $u(t) = J \cdot e^{st}$; si les coefficients sont variables, il existe une intégrale $u(t) = Y(s, t)e^{st}$, où la fonction $Y(s, t)$ (*l'admittance instantanée*) peut s'évaluer par la relation

$$Y(s, t)/J(s, t) = 1 - \frac{\partial}{\partial s}\left[J \frac{\partial}{\partial t} (1/J)\right] + o(1/s^2);$$

cette intégrale s'obtient donc, à partir des coefficients de l'équation, par des opérations rationnelles et des dérivations, avec une imprécision qui diminue comme $1/\omega^2$ si ω augmente indéfiniment. Si les coefficients sont constants, on a exactement $Y = J$; on retrouve ce cas limite si l'on fait tendre ω vers l'infini.

Ces considérations trouvent une application particulièrement féconde dans l'intégration approchée d'équations à coefficients périodiques, dont le second membre est une fonction sinusoïdale d'une fréquence notablement plus élevée que celle des coefficients.

University of Lausanne,
Lausanne, Switzerland.

SUR L'EVOLUTION DU PROBLÈME DE DIRICHLET

Marcel Brelot

Pour élargir et affiner le problème de Dirichlet dans l'espace euclidien, on est amené à changer la topologie de la frontière qui se trouvera définie par complétion à partir d'une structure uniforme choisie sur le domaine en compatibilité avec la topologie euclidienne. Ainsi en prenant dans un domaine borné Ω, la distance de Mazurkiewicz (borne inférieure du diamètre des arcs joignant deux points sur Ω) ce qui donne une structure uniforme plus fine, on fait disparaître les points inaccessibles dont l'ensemble était justement de mesure harmonique nulle (selon de la Vallée Poussin) c'est-à-dire inutile pour le problème de Dirichlet; j'ai pu avec ce langage (Annales de l'Université de Grenoble t. 29 (1946)) approfondir le problème "ramifie" correspondant étudié par Perkins et de la Vallée Poussin. Un pas de plus consistait à traiter le problème avec la métrique géodésique, encore plus fine. Les propriétés des lignes de Green ou trajectoires orthogonales des surfaces de Green $G = C^{te}$ étudiées en collaboration avec Choquet dans ce but (C.R. Acad. Sci. Paris t. 228 pp. 1556, 1790) montrent que les points inaccessibles par chemin de longueur finie forment un ensemble de mesure harmonique encore nulle, ce qui permet de réaliser l'extension du problème de Dirichlet avec la représentation intégrale, la même généralité et les mêmes détails que dans le cas ramifié. Or, les lignes de Green s'étudient de même dans des espaces plus généraux qu'un domaine euclidien comme les surfaces de Riemann, ou les espaces métriques connexes localement euclidiens, pourvus d'une fonction de Green; et on peut espérer étendre à ces cas généraux la théorie précédente du problème de Dirichlet. On peut aussi songer à des conditions-frontière relatives à des limites suivant les lignes de Green et cela, sans même savoir traiter un problème de Dirichlet correspondant, permet déjà d'obtenir des inégalités de fonctions améliorant beaucoup des résultats connus sur la valeur ou la convergence de fonctions sousharmoniques ou holomorphes moyennant des conditions-frontière bien plus réduites et d'ailleurs valables dans les espaces généraux considérés plus haut.

University of Grenoble,
Grenoble, France.

SOME TOPOLOGICAL PROBLEMS CONNECTED WITH FORCED OSCILLATIONS

M. L. Cartwright and J. E. Littlewood

The study of certain nonlinear differential equations of the second order representing forced oscillations leads to the study of a (1, 1), continuous, orientation-preserving transformation, T, of the plane into itself which leaves a certain set I invariant. In typical cases C(I), the complement of I, is simply connected if the point at infinity is included in it, and then there is a (1, 1) conformal mapping Z of C(I) on $|z| > 1$ so that the point at infinity corresponds to itself. It can be shown that $\tau = ZTZ^{-1}$ is a (1, 1) continuous mapping of $|z| \geqq 1$ on itself. A point π on $|z| = 1$ corresponds to a prime end $\mathcal{P}$ of C(I), and the set P of points of $\mathcal{P}$ is either a single point or a continuum.

We have discussed various problems connected with the transformation of prime ends. For instance does a fixed prime end necessarily contain a fixed point? It can be shown that if $\tau(\pi) = \pi$, then P contains a fixed point. This is a new fixed point theorem, for the continuum P is not necessarily locally connected nor decomposable, but the proof, which is long, uses the fact that T is defined outside the set P, and involves a discussion of stability of prime ends. Stability is used in a sense which corresponds to the stability of the solutions of the differential equation.

Another problem is whether, if no prime end is fixed, $\mathrm{F}(I)$, the frontier of I, can contain more than one fixed point. Rotation about one accessible point is obviously possible, but if there is more than one fixed point on $\mathrm{F}(I)$, and yet no fixed prime end, then the fixed points are all inaccessible points of the same prime end. Results of this type suggest strongly that for the set I corresponding to van der Pol's equation

$$\ddot{x} - k(1 - x^2)\dot{x} + x = b\lambda k \cos \lambda t,$$

$\mathrm{F}(I)$ contains an indecomposable continuum when k is large and b takes certain values between 0 and 2/3.

Girton College, Cambridge University,
Cambridge, England.

LA SOLUTION DU PROBLÈME DES TROIS CORPS PAR SUNDMAN, ET SES CONSÉQUENCES

Jean Chazy

Pendant longtemps les astronomes avaient cherché à représenter la solution du problème des trois corps par des séries généralisant les séries de Fourier, et comportant des combinaisons linéaires à coefficients entiers de quatre fonctions linéaires du temps; puis Poincaré avait démontré que de tels développements ne peuvent être uniformément convergents, et notamment entraîner des conséquences à l'infini au point de vue de la stabilité des solutions.

L'astronome finlandais Sundman a obtenu en 1912 la solution générale du problème des trois corps, en exprimant les neuf coordonnées cartésiennes et le temps, variant de $-\infty$ à $+\infty$, en fonctions holomorphes d'une variable: par une extension de la représentation classique du mouvement elliptique. Le expressions de Sundman sont valables, et donnent une intégration quantitative parfaite, au sens de la théorie des fonctions, du problème des trois corps, sous une condition très simple. Elles restent valables si deux corps au cours du mouvement deviennent voisins, et même s'ils se choquent, et dans ce cas elles définissent un prolongement analytique du mouvement au delà de l'instant du choc.

On a essayé d'appliquer les développements de Sundman pour représenter les mouvements des planètes et de la Lune, mais le résultat a été plus décevant.

la convergence a été plus lente que tout ce qu'on aurait pu imaginer: mieux vaut en pratique employer les séries divergentes antérieures que les séries convergentes de Sundman.

Une fois calculées les éphémérides, il reste à résoudre des questions qualitatives et des questions asymptotiques, notamment à savoir si tel mouvement du problème des trois corps est stable, si les distances des planètes au Soleil et de la Lune à la Terre deviendront beaucoup plus grandes ou beaucoup plus petites que leurs valeurs actuelles, ou conserveront indéfiniment ou pendant très longtemps des ordres de grandeur comparables. La solution de Sundman ne donne pas de réponse directe à de telles questions, parce que, étant générale, elle contient, mélange en quelque sorte toutes les allures possibles du mouvement.

Mais un progrès fondamental comme la découverte de cette solution donne une face nouvelle au problème des trois corps, et nécessairement a des conséquences indirectes et profondes. Dès maintenant l'étude des trajectoires comportant des chocs ou des approches successives de deux corps a permis de mieux concevoir et répartir l'ensemble des trajectoirs du problème des trois corps, — de même que la considération des points singuliers est capitale dans l'étude d'une fonction analytique.

THE SORBONNE,
PARIS, FRANCE.

AN EXTREMUM PRINCIPLE FOR SOLUTIONS OF A CLASS OF ELLIPTIC SYSTEMS OF DIFFERENTIAL EQUATIONS WITH CONTINUOUS COEFFICIENTS

AVRON DOUGLIS

THEOREM. *Let R be a bounded domain in the xy-plane in which the functions $u(x, y)$, $v(x, y)$ are defined and of class C' and satisfy an elliptic system of partial differential equations of the form*

$$\frac{\partial u}{\partial x} + a(x, y)\frac{\partial u}{\partial y} - b(x, y)\frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial v}{\partial x} + a(x, y)\frac{\partial v}{\partial y} + b(x, y)\frac{\partial u}{\partial y} = 0,$$

where $a(x, y)$, $b(x, y)$ are continuous, and $b(x, y) > 0$, in R. Suppose, further, that $u(x, y)$ is continuous in the closure of R and that $u \leqq 0$ on the boundary of R. Then $u \leqq 0$ in R.

The proof is based upon the fact that the Jacobian of the transformation $u = u(x, y)$, $v = v(x, y)$ is *positive definite* in the sense that it is non-negative in R and can vanish in R only where the derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ all vanish. Recognizing that any point (u, v) for which $u > 0$ lies in the outer component of the complement of the image of the boundary of R, we make application of the following:

LEMMA. *Let $u(x, y)$, $v(x, y)$ be of class C' in the closure of a bounded domain R, and suppose the Jacobian of the transformation $u = u(x, y)$, $v = (x, y)$ is non-negative in R. Let U be a connected open set contained in the image of R whose closure is disjoint from the image of the boundary of R. Then in U the degree of the mapping of R into the plane is necessarily positive.*

CALIFORNIA INSTITUTE OF TECHNOLOGY,
PASADENA, CALIF., U. S. A.

SOME SINGULAR INTEGRAL EQUATIONS OF THE CAUCHY TYPE

JOANNE ELLIOTT

The paper deals with the integral equation

$$g(t) = \text{P.V.}\, \frac{1}{2\pi} \int_A \frac{f(s)\zeta'(s)}{\zeta(s) - \zeta(t)}\, ds. \tag{1}$$

Here $\zeta = \zeta(s)$ is a curve in the complex plane referred to its arc lengths; the domain of integration is the entire curve, half of it, or a finite number of intervals on it; the integral is taken in the sense of a Cauchy principal value.

The function $g(s)$ is given only on the set A, and the problem consists in expressing $f(s)$ in terms of these values of g only. It is assumed that $g(t) \in L_p$ and $L_{p-\epsilon}$ for some $p > 1$ and some ϵ between 0 and 1.

If the integration extends over the entire real axis, the problem reduces to the inversion of the Hilbert transform. This case has been completely solved by Titchmarsh for $p = 2$ and by M. Riesz for $p > 1$ (even without the restriction $g \in L_{p-\epsilon}$).

In connection with the theory of conjugate series in L_p, M. Riesz has also considered the case of a circle. The case of a finite interval is well known in airfoil theory and has been treated under different conditions by H. Söhngen. Similarly, the case of two disjoint real intervals occurs in connection with the theory of biplanes.

The general problem is solved using Riesz's results for Hilbert transforms and the complex variable methods introduced by Carleman in treating certain singular integral equations. We assume that the curve has a bounded curvature satisfying a Lipschitz condition and the difference ratio $|\zeta(s) - \zeta(t)|/|s - t|$ is bounded away from zero.

Formally, this equation is a special case of an integral equation with a Cauchy kernel. This type of equation has been considered by W. Trjitzinsky, N. I. Mushelishvili, Vecoua, F. Gahoff, and others.

CORNELL UNIVERSITY,
ITHACA, N. Y., U. S. A.

METHODS FOR SOLVING LINEAR FUNCTIONAL EQUATIONS, DEVELOPED BY THE ITALIAN INSTITUTE FOR THE APPLICATIONS OF CALCULUS

Gaetano Fichera

The aim of the contributed paper is to give a short review of the matters considered in a longer paper. It deals with some methods for quantitative and existential analysis of linear functional equations which have been investigated by the Italian Institute for the Applications of Calculus during a quarter of a century.

In the first chapter, procedures and results are exposed referring to bounds for a solution of a differential equation; the second chapter is devoted to methods based on minimal principles, viz. the method of least squares, the so-called variational method, and some procedures for the numerical calculation of eigenvalues.

The third chapter deals with methods of integration of partial differential equations which, by means of some convenient transformation of the solution, convert the problem into the integration of a differential equation for a function with a lower number of variables than the original unknown function. In particular the Laplace and Fourier transformations are considered, together with many results to which they have led.

The methods of integration considered in the next chapter apply functional transformations of the solutions, with a parameter varying in a discrete set of values.

These methods make it possible to establish, for example, many qualitative properties of solutions of equations of mathematical physics.

The fifth chapter deals with some modern developments of integration with the aim of converting the problem to an integral equation of the first kind; these methods are based on a suitable integral form of the solution, or on a functional interpretation of the reciprocity theorem of the solutions of the linear differential equations. The researches of functional analysis connected to this method are also exposed.

Many existence theorems for the boundary value problems of mathematical physics follow from these developments, as, for example, the determination of the strain of an isotropic and homogeneous elastic body, partially clamped and partially supported at its edge and under a given system of stresses.

The same method can also be applied to the linear integral-differential equations of mechanics and to the boundary problems for Cartan's differential forms.

University of Rome,
Rome, Italy.

ÉQUATIONS DE MONGE-AMPÈRE, DU TYPE ELLIPTIQUE, ET PROBLÈMES RÉGULIERS DU CALCUL DES VARIATIONS

PAUL P. GILLIS

Soit une équation aux dérivées partielles du second ordre, à deux variables indépendantes, du type

$$rt - s^2 + ar + 2bs + ct + d = 0, \tag{1}$$

où r, s, t, représentent les dérivées du second ordre de la fonction inconnue $z(x, y)$, et où a, b, c, d sont des fonctions de (x, y, p, q), p et q étant les dérivées du premier ordre de $z(x, y)$. Lorsque l'équation est du type elliptique, M. F. Rellich (cf. Math. Ann. t. 107 (1933) p. 505) a montré que, sous des conditions très générales, il ne pouvait exister plus de deux solutions distinctes, prenant sur la frontière d'un domaine D du plan (x, y) les mêmes valeurs (problème de Dirichlet). Le problème peut admettre deux solutions distinctes, même lorsque a, b, c, d ne dépendent que de (x, y) ou se réduisent à des constantes.

Nous avons étudié des problèmes réguliers du calcul des variations dont l'équation d'Euler-Lagrange est du type (1). Pour ces problèmes il existe tout au plus deux solutions, un maximum et un minimum, prenant sur la frontière du domaine D les mêmes valeurs. Nous retrouvons ainsi, pour l'équation d'Euler-Lagrange du problème variationnel, la proposition établie par M. F. Rellich pour l'équation (1). Il est intéressant, croyons-nous, de faire appel au problème variationnel pour démontrer, par une méthode directe, certains théorèmes d'existence pour des équations du type (1), dérivant du calcul des variations.

Nous pouvons considérer également des équations de Monge-Ampère, du type elliptique, à quatre ou $2n$ variables indépendantes. Soit, par exemple, l'équation

$$\begin{gathered}(z_{11} + z_{22})(z_{33} + z_{44}) - (z_{13} + z_{24})^2 - (z_{14} - z_{23})^2 = d, \\ z_{ij} = \frac{\partial^2 z}{\partial x_i \partial x_j} \qquad (i, j = 1, 2, 3, 4),\end{gathered} \tag{2}$$

où $d(x_1, x_2, x_3, x_4)$ est une fonction continue et positive dans un domaine D de l'espace (x_1, x_2, x_3, x_4). Nous avons montré qu'une telle équation ne peut admettre plus de deux solutions distinctes, prenant sur la frontière de D les mêmes valeurs. La proposition subsiste pour des équations beaucoup plus générales, à quatre ou $2n$ variables indépendantes. M. Th. Lepage a étudié la nature algébrique de telles équations (cf. Académie Royale de Belgique Classe des Sciences t. 35 (1949) p. 694).

Nous pouvons encore étudier le problème de Dirichlet pour des intégrales du calcul des variations dont l'équation d'Euler-Lagrange est du type mentionné ci-dessus. Lorsque le problème est régulier, il existe tout au plus deux solutions, un maximum et un minimum; dans des cas particuliers, on montre que les deux solutions existent. La considération du problème variationnel permet aussi d'aborder l'étude de certains théorèmes d'existence pour des équations du type (2).

UNIVERSITY OF BRUSSELS,
BRUSSELS, BELGIUM.

"EXPLOSIVE" SOLUTIONS OF FOKKER-PLANCK'S EQUATION

EINAR HILLE

This note deals with partial differential equations of parabolic type having certain singular solutions implying non-uniqueness of corresponding boundary value problems and the existence of solutions "exploding" after a preassigned time.

We consider the equations of Kolmogoroff connected with temporally homogeneous stochastic processes of one degree of freedom, involving the arbitrary functions $a(x)$ and $b(x)$ where the variance $b(x) > 0$ for all x. If $a(x) \equiv 0$ for the sake of simplicity, the progressive (Fokker-Planck) equation becomes $[b(x)\ T]_{xx} = T_t$ with $T(x, t)$ to be defined for all $t > 0$ and tending in the mean of order one to a preassigned function $g(x) \in L(-\infty, \infty)$ when $t \to +0$. We have shown elsewhere that the solution is not unique if $x/b(x) \in L(-\infty, 0)$ or $L(0, \infty)$, though the adjoint problem for the retrospective equation in $C[-\infty, \infty]$ has a unique solution.

If $x/b(x) \in L(-\infty, \infty)$, the function $P(x, \lambda)$, satisfying the equation

$$[b(x)y]_{xx} = \lambda y$$

with $P(0, \lambda) \equiv 0$, $P'(0, \lambda) \equiv 1$, is in $L(-\infty, \infty)$. Here $P(x, \lambda)$ is an entire function of λ of order 1/2 for fixed x; the L-norm of $P(\cdot\,; \lambda)$ has a similar property, at least if $b(x)$ is suitably restricted. Under the same assumptions $\| P(\cdot\,; \lambda) \| \to 0$ when $\lambda \to -\infty$. Let $\mu(\lambda)$ be any entire function of order <1, positive when $\lambda > 0$, and let $S(x, t)$ be the Laplace transform of $P(x, \lambda)\mu(\lambda)$. This is an entire function of $1/t$ and $\| S(\cdot\,; t) \| \leqq \exp (C/ | t |)$. Further $S(x, t) \to \infty \operatorname{sgn} x$ and $\| S(\cdot\,; t) \| \to \infty$ when $t \to +0$.

Now $S(x, t_0 - t)$ is a solution of Fokker-Planck's equation for any t_0. Taking $t_0 > 0$ and $g(x) = S(x, t_0)$, we see that among the corresponding solutions of the boundary problem, there is also $S(x, t_0 - t)$ which "explodes" after t_0 seconds. The initial values producing such "explosive" solutions may be dense in $L(-\infty, \infty)$ in view of the arbitrariness of $\mu(\lambda)$. Related phenomena have been noted by J. L. Doob for the equations of Kolmogoroff connected with Markoff chains.

YALE UNIVERSITY,
NEW HAVEN, CONN., U. S. A.

ON THE INITIAL VALUE PROBLEM FOR THE NAVIER-STOKES EQUATIONS

EBERHARD HOPF

This paper is a contribution to the theory of generalized solutions of partial differential systems. The author considers parabolic space-time systems that possess an "energy-integral" and gives a simple construction of the solution of the usual mixed initial value and boundary value problem. A simple proof is given for the existence at all times $t > 0$ of the flow of an incompressible fluid within given walls and with given initial velocities.

UNIVERSITY OF INDIANA,
BLOOMINGTON, IND., U. S. A.

ÉQUATIONS SEMI-CANONIQUES

MAURICE JANET

Le passage des équations d'Euler-Lagrange aux équations "canoniques" par la transformation de Legendre est bien connu. J'attire l'attention sur l'utilité d'autres transformations (de telles transformations ont été considérées autrefois par Routh [see Levi-Civita, Meccanica Razionale II_2 p. 447]), qui permettent de passer simplement du principe d'Hamilton au principe de Maupertuis, et d'obtenir par un procédé régulier une série de résultats analogues; elles mettent aussi en évidence dans sa généralité le fait analytique utilisé essentiellement par la "mécanique ondulatoire" dans ses débuts.

1. Donnons-nous une fonction L de $2(n + 1)$ variables x_α, z_α, homogène de degré 1 par rapport à ces $n + 1$ dernières. Les extrémales relatives à $\int L du$ où $z_\alpha = dx_\alpha/du$ (ou lignes admettant $\int L_{z_\alpha} \delta x_\alpha$ comme invariant intégral relatif) sont données, si l'on pose $L + M - pz_{n+1} = 0$, $L_{z_{n+1}} - p = 0$ (définissant M en fonction des x, de p, des z où l'indice est $\leq n$; transformation d'Ampère) par

$$\frac{dx_k}{z_k} = \frac{dx_{n+1}}{M_p} = \frac{dM_{z_h}}{M_{x_h}} = \frac{dp}{-M_{x_{n+1}}} \quad (k, h = 1, 2, \cdots, n)$$

(M homogène, degré 1, par rapport aux z_h, $L_{z_\alpha}\delta x_\alpha = -M_{z_h}\delta x_h + p\delta x_{n+1}$). Si L ne dépend pas de x_{n+1}, M n'en dépend pas non plus, p est intégrale première: les extrémales relatives à une valeur $\bar{p}$ de p ont pour supports celles qui correspondent à la fonction $\bar{M}$ des $2n$ variables x_h, z_h. On trouve M *en exprimant* que $L + M - pz_{n+1} = 0$ *a une racine double* en z_{n+1}. Exemples: 1.)

$$L = A_0 z_{n+1} + A_1 + A_2/z_{n+1}$$

où A_0 est indépendant des z, A_1, A_2 formes du 1^e et du 2^e degré en z_h (cas de la mécanique classique). $(M + A_1)^2 - 4(A_0 - \bar{p})A_2 = 0$. 2.) $L = f(\rho, x)z_{n+1}$ où ρ

$= A/z_{n+1}$, A fonction homogène degré 1 par rapport aux z_h; M est donnée par l'élimination de ρ entre $-M/A = (f - p)/\rho = f_\rho$; M s'écrit $\phi(x_1 \cdots x_n p)A$.

2. Au lieu du changement de variables précédent, on peut en faire un portant sur deux des z, par exemple

$$L + M - pz_{n+1} - qz_n = 0, \qquad L_{z_{n+1}} - p = 0, \qquad L_{z_n} - q = 0$$

(M fonction des $x_{h'}$, p, q, $z_{k'}$; h', $k' = 1, 2, \cdots, n - 1$). Alors, $dx_{k'}/z_{k'} = dx_n/M_q = dx_{n+1}/M_p = dM_{z_{h'}}/M_{x_{h'}} = dq/-M_{x_n} = dp/-M_{x_{n+1}}$. Si L ne dépend ni de x_n ni de x_{n+1}, q, p sont intégrales premières; et les extrémales relatives à $\bar{q}$, $\bar{p}$ ont pour supports celles relatives à M fonction de $2(n - 1)$ variables. Généralisations évidentes: On arrive finalement aux équations canoniques en faisant le changement de variables sur n des z.

3. Partons inversement d'une fonction M des x_k, z_k, homogène degré 1 par rapport aux z, et d'un paramètre p. Si p est *constant* et x_{n+1} une nouvelle variable, on peut choisir X pour que $\omega = -M_{z_h}\delta x_h + p\delta x_{n+1}$ soit invariant intégral *absolu* pour $dx_k/z_k = dM_{z_h}/M_{x_h} = dx_{n+1}/X$. On trouve $X = \partial M/\partial p$. Si p est *variable*, on demandera seulement que ω soit invariant intégral *relatif* pour $dx_k/z_k = dM_{z_h}/M_{x_h} = dx_{n+1}/Y = dp/P$. On trouve $Y = \partial M/\partial p$, $P = 0$. Si par exemple $M = \phi(x_1 \cdots x_n p)(a_{kh}z_k z_h)^{1/2}$; soit sur chaque extrémale une variable σ définie par $d\sigma = (a_{kh}dx_k dx_h)^{1/2}$; les valeurs V, U trouvées pour $d\sigma/dx_{n+1}$ dans les deux questions précédentes sont liées par $1/U = \partial(p/V)/\partial p$, formule générale d'analyse que les créateurs de la "mécanique ondulatoire" ont rapprochée curieusement, avec succès, de l'expression de la "vitesse de groupe" due à Lord Rayleigh.

UNIVERSITY OF PARIS,
PARIS, FRANCE.

ON THE FUNDAMENTAL SOLUTION OF LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS

FRITZ JOHN

Let $L(u)$ be an elliptic differential operator with analytic coefficients of order m, acting on a function $u = u(x_1, \cdots, x_n)$. The characteristic form $Q(\xi)$ of L is definite. A solution u of $L(u) = 0$ is said to have a "pole of order s" at a point y, if u is of class C^m for $x \neq y$ in a neighbourhood of y and if s is the smallest integer such that for every derivative $u^{(m)}$ of u of order m

$$u^{(m)} = O(|x - y|^{-s}) \qquad (s < \infty).$$

A "fundamental" solution is a solution with a pole of order n. Such a fundamental solution can be constructed from solutions of ordinary Cauchy problems as follows.

Let $x \cdot \xi = p$ denote a hyperplane with unit normal ξ. Let $v = v(x, \xi, p)$ be the solution of $L(v) = 0$, for which, on $x \cdot \xi = p$, v and its first $m - 2$ normal derivatives vanish, the $(m - 1)$st normal derivative having the value $1/Q(\xi)$. Let $f(\lambda)$ be the function defined by

$$f(\lambda) = \begin{cases} -\tfrac{1}{2}(2\pi i)^{1-n}\lambda, \text{ if } n \text{ is odd} \\ 2(2\pi i)^{-n} \log \lambda, \text{ if } n \text{ is even.} \end{cases}$$

Then, if Δ is the ordinary Laplace operator with respect to y,

$$K(x, y) = \Delta^{[(n+1)/2]} \int_{(y-x)\cdot\xi>0, |\xi|=1} d\omega_\xi \int_0^{(y-x)\cdot\xi} v(x, \xi, y\cdot\xi - \lambda) f(\lambda)\, d\lambda$$

is a fundamental solution with pole y. K is analytic in x, y for $x \neq y$ and x sufficiently near to y. If the highest coefficients of L are constant and the others are entire functions, K is analytic for all real $x \neq y$. If all coefficients of L are constant, K can be obtained by quadratures.

If u is a solution with a pole of order s at y, then u can be written as a linear combination of a solution regular at y and of derivatives of $K(x, y)$ with respect to y of the orders $0, 1, \cdots, s - n$. A pole of order $s < n$ is a removable singularity. Every fundamental solution with pole y is of the form $cK(x, y) + w(x)$, where c is a constant, and w is regular analytic at y.

New York University,
New York, N. Y., U. S. A.

MOMENT THEORY AND ORTHOGONAL POLYNOMIALS

Samuel Karlin

The n-dimensional reduced moment space D_n is defined to be the set of all (μ_i), $i = 1, \cdots, n$, which are the first n moments of a distribution on the interval $(0, 1)$, i.e., $\mu_i = \int_0^1 t^i dF$. An approach to the theory of orthogonal polynomials by means of the geometry of supporting planes to the moment space is furnished. Let $\mu = (\mu_1 \cdots \mu_{2n-1})$ represent an interior point to D_{2n-1} and μ^*_{2n} and μ_{*2n} the two extreme values of any extension of μ, where an extension preserves the first $2n - 1$ moments and represents a point of D_{2n}. The space of supporting planes to the cone generated by D_{2n} by dropping the normalization of $\mu_0 = 1$ can be described as the space of all polynomials of degree $2n$ non-negative over the interval $(0, 1)$. The unique supporting planes $x(1 - x)Q^2_{n-1}(x)$ and P^2_n at μ_{*2n} and μ^*_{2n} are such that P_n constitute an orthogonal system with respect to ρ and Q_n with respect to $x(1 - x)\rho$ where ρ is the distribution giving rise to the moments μ. The pairs (P_n, Q_{n-1}) are called associative and generalize the classical system of associative orthogonal polynomials arising from second order differential equations. They satisfy recursion formulas which on combination yield the differential equation for the classical polynomials. The Jacobi distributions $x^\alpha(1 - x)^\beta/B(\alpha + 1, \beta + 1)$ fill out the full interior of the two-dimensional moment space and the corresponding family of distributions related to the fourth moment space are given by $x^{\alpha}(1-x)^{\beta}e^{-\gamma/x}e^{-\delta/(1-x)}$, $\gamma > 0, \delta > 0$. The $2n$-dimensional moment space has an n-dimensional subset of symmetry which generalizes the ultraspherical distributions. Corresponding statements can be made for odd-dimensional moment spaces. Much of the standard theory of orthogonal polynomials can be interpreted in this way.

California Institute of Technology,
Pasadena, Calif., U. S. A.

SUCCESSIVE UPPER AND LOWER APPROXIMATIONS

Joseph P. LaSalle

The motivation behind the problem to be considered is its application to the study of solutions of differential equations. In order to avoid some details which only conceal essential features, we shall describe the problem in more general terms. Let E be the space of continuous functions f defined on the closed unit interval $I = [t; 0 \leqq t \leqq 1]$. The algebraic operations in E are as usual and convergence in E is uniform convergence of the functions over I. We define $f \geqq g$ if $f_i(t) \geqq g_i(t)$ for all $t \in I$ and $i = 1, \cdots, n$. If T is a nondecreasing, totally continuous (maps bounded sets into compact sets) mapping of E into itself and if $l, u \in E$ $(l \leqq u)$ satisfy $l \leqq T(l)$ and $u \geqq T(u)$, then it is a simple result that $T^n(l)$ $(T^n(u))$ is a nondecreasing (nonincreasing) sequence converging to a fixed point of T. When T has a unique fixed-point, $\lim_{n\to\infty}T^n(l) = \lim_{n\to\infty}T^n(u)$ and one obtains successive approximations which bound the fixed-point from above and below with the bounds moving monotonically toward the fixed-point. The major restriction on this result is that T be nondecreasing. Dropping this restriction on T, a result of the above type can still be given. If T is totally continuous and (a) there is a nondecreasing, totally continuous mapping T^* of $E \times E$ into itself (in $E \times E$, $(f_1, g_1) \leqq (f_2, g_2)$ means $f_2 \leqq f_1 \leqq g_1 \leqq g_2$) satisfying $T^*(f, f) = (T(f), T(f))$, (b) (l_0, u_0) satisfies $(l_0, u_0) \leqq T^*(l_0, u_0)$, and (c) T^* has at most one fixed-point; then $T^{*n}(l_0, u_0)$ converges to (y, y) as $n \to \infty$ and y is a fixed-point of T. In the particular case of a system of differential equations satisfying Lipschitz's condition, it can be shown that T^* and (l_0, u_0) satisfying (a)—(c) can always be constructed. Thus it is always possible, for such a system, to apply this method to obtain successive upper and lower approximations of solutions which monotonically approach the solution.

University of Notre Dame,
Notre Dame, Ind., U. S. A.

ON THE GREEN'S FUNCTIONS OF ORDINARY DIFFERENTIAL SYSTEMS

KENNETH S. MILLER AND MENAHEM M. SCHIFFER

We consider a linear differential system $Lu = 0$, $U_\alpha(u) = C_\alpha$, $\alpha = 1, 2, \cdots, n$, where L is an nth order linear differential operator with coefficients continuous in some closed finite interval $[a, b]$ and $U_\alpha(u)$ are any set of two point boundary conditions. (1) Applying Green's identity with respect to a Green's function and a solution $\phi(x)$ of $Lu = 0$ we obtain a representation of $\phi(x)$ in terms of the Green's function and its various derivatives with respect to the second argument evaluated at the end points of the interval. In particular, we explicitly exhibit a fundamental system for $Lu = 0$ in terms of a Green's function for $Lu = 0$ and any set of incompatible boundary conditions. (2) If the end points a and b, the operator L, and the boundary conditions U_α are subjected to slight variations δa, δb, δL, δU_α respectively, we obtain finite comparison and infinitesimal variational formulas for the Green's function of the perturbed system $L + \delta L$, $U_\alpha + \delta U_\alpha$ in $[a + \delta a, b + \delta b]$. A typical result: When a and b are varied, L is of even order $n = 2r$ and the boundary conditions are (*) $U_\alpha(u) = u^{(\alpha-1)}(a)$, $U_{r+\alpha}(u) = u^{(\alpha-1)}(b)$, $\alpha = 1, 2, \cdots, r$, then

$$\delta G(x, \xi) = (-1)^r \left[\frac{\partial^r}{\partial x^r} G(b, \xi) \frac{\partial^r}{\partial \xi^r} G(x, b) p_0(b) \delta b - \frac{\partial^r}{\partial x^r} G(a, \xi) \frac{\partial^r}{\partial \xi^r} G(x, a) p_0(a) \delta a \right].$$

Here $p_0(x)$ is the coefficient of $u^{(n)}$ in the expression for Lu. This formula is analogous to Hadamard's variational formula for the Green's function of the Laplacian in n-space. While in the Hadamard formula one has to deal with functional-derivative equations, we obtain, in the case of one variable, a differential equation for the Green's function as a function of the end points of the interval. (3) Systems of nonlinear differential equations can be deduced for the nonvanishing derivatives of the Green's function where both arguments have been evaluated at the end points of the interval. (4) Certain combinations of a Green's function and its derivatives can be constructed which vary monotonically with the interval; and inequalities for these expressions can be easily derived. A simple typical example in this direction is the inequality: $G(x, \xi) \leqq \frac{1}{2}[G(x, x) + G(\xi, \xi)]$ where $G(x, \xi)$ is a Green's function for a self-adjoint operator of order $n = 4r$ and with boundary conditions of the type (*). Positive definite quadratic forms are established whose coefficients are derivatives of the Green's function. (5) Consider the finite comparison formula for the case in which the operator L is varied to L', $(\delta L = L' - L)$ and G and G' are the corresponding Green's functions. If L_ζ means that L operates on functions of ζ only, then $G'(x, \xi) = G(x, \xi) - \int_a^b G'(x, \zeta) \delta L_\zeta G(\zeta, \xi)\, d\zeta$. This may be regarded as an integral equation for $G'(x, \xi)$ where the kernel depends on the known Green's function $G(\zeta, \xi)$. Special classes of differential equations and types of variations are studied for which this integral equation can be solved by Neumann series. Green's function for a relatively complicated operator can thus

be obtained in terms of the Green's function of a simple differential equation. (6) Variational formulas for the eigenvalues and eigenfunctions of nth order Sturm-Liouville systems are derived and studied.

New York University,
New York, N. Y., U. S. A.
Princeton University,
Princeton, N. J., U. S. A.

A SYSTEM OF FUNCTIONAL EQUATIONS

Edmund Pinney

A system of functional equations of the type

$$\frac{d^m y}{dx^m} = F\left[x, y, \frac{dy}{dx}, \cdots, \frac{d^{m-1}y}{dx^{m-1}}; f(x), f'(x), \cdots, f^{(p)}(x); f(y), f'(y), \cdots, f^{(n-1)}(y)\right],$$

$$f^{(n)}(y) = G\left[x, y, \frac{dy}{dx}, \cdots, \frac{d^{m-1}y}{dx^{m-1}}; f(x), f'(x), \cdots, f^{(p)}(x); f(y), f'(y), \cdots, f^{(n-1)}(y)\right],$$

where the function f and the relation between x and y are unknown, may be transformed to difference-differential equations by means of the parametric representation

$$x = \phi(s), \qquad y = \phi(s + 1), \qquad f(x) = \psi(s), \qquad f(y) = \psi(s + 1).$$

This is proved by reducing the above system to a system of first order differential equations in which the functions of y are dependent variables and x and $f(x)$ are expressed as arbitrary functions $\phi(s)$ and $\psi(s)$, respectively, for $0 < s < 1$. This second system may be solved, and the dependent variables corresponding to y and $f(y)$ may be designated by $\phi(s + 1)$ and $\psi(s + 1)$, effectively extending the definition of $\phi(s)$ and $\psi(s)$ to the interval $1 < s < 2$. The process may be repeated, extending the definition of $\phi(s)$ and $\psi(s)$ to $2 < s < 3$, etc.

A typical example of the application of this method is to the equation $f(f(x)) = x$. This is equivalent to the system $y = f(x)$, $f(y) = x$. The solution is found to be $x = p(s)$, $f(x) = p(s + 1)$, where $p(s)$ is any continuous periodic function of period 2.

University of California,
Berkeley, Calif., U. S. A.

BOUNDARY VALUE PROBLEMS FOR A PARTIAL DIFFERENTIAL EQUATION OF MIXED TYPE

MURRAY H. PROTTER

Consider the equation $K(y)\ U_{xx} - U_{yy} = 0$ where $K(y)$ is a non-negative, monotone continuous function for $y \geqq 0$ and $K(0) = 0$. Let (a, b) be an interval along the x-axis and let $y = g_1(x)$, $y = g_2(x)$ be the equations of those characteristics going through the points $(a, 0)$, $(b, 0)$, respectively, and which intersect on the line $2x = a + b$. We denote by $D(a, b)$ the domain bounded by the curves $g_1(x)$, $g_2(x)$ and the interval (a, b) of the x-axis. The equation is hyperbolic in $D(a, b)$ except along the x-axis which is a parabolic line.

The Cauchy problem for this equation with the data given along the parabolic line has been solved by Bers (N.A.C.A. T.N.). The boundary value problem, in which the solution u is given along one characteristic and the parabolic line, has been solved. Bounds are obtained for the solution in terms of the boundary values and the first derivative of the boundary values. The solution to the problem in which u is given along one characteristic and the normal derivative is given along the parabolic line has been obtained. Finally, the solution to the two-characteristic problem is extended to include the parabolic line.

The method of proof consists of approximating $K(y)$ by a step function $K^*(y)$ with a finite number of steps and then solving the equation

$$K^*(y)U_{xx} - U_{yy} = 0.$$

In this case the domain $D(a, b)$ may be broken into strips in each of which the solution is given by an expression of the form $f(x + \lambda_i y) + g(x - \lambda_i y)$ where $K^*(y) = \lambda_i^2$ in the ith strip. Conditions are imposed on these functions by the requirement that u, u_x, u_y are to be continuous in $D(a, b)$. From these conditions bounds can be obtained for U and U_y along the parabolic line; and then a result of Bers for the corresponding Cauchy problem yields bounds for the whole domain. If we consider a sequence of step functions $K_n(y) \to K(y)$, we can show that the corresponding solutions converge to a function $u(x, y)$ which satisfies the original equation.

SYRACUSE UNIVERSITY,
SYRACUSE, N. Y., U. S. A.

THE CAUCHY-KOWALEWSKI EXISTENCE THEOREM

PAUL C. ROSENBLOOM

The proofs of the classical existence theorems by the use of the majorant method yield estimates for the domain of existence of solutions which depend strongly upon the dimensions, both of the space of independent variables and of the space of the unknown functions. A new proof is given based on a modification of the majorant method, which applies equally well to Hilbert space and to Banach spaces. The estimates for the domain of existence of solutions depend on the degree of smoothness of the unit sphere. An essential step in the proof is a generalization to Banach spaces of Bernstein's theorem on the derivatives of polynomials bounded in the unit circle.

Let B_1 and B_2 be complex Banach spaces and let B_3 be the space of linear transformations of B_1 into B_2. The differential of an analytic function u on B_1 and B_2 can be regarded as an analytic function on B_1 to B_3. Let f be an analytic function on the unit sphere of $C \times B_1 \times B_2 \times B_3$ to B_2 where C is the space of complex numbers. Let $|f| \leqq M$ there, and let f vanish at the origin. This is an inessential normalization of the problem.

We consider the equation

$$\frac{\partial u}{\partial t} = f(t, x, u, du),$$

and we seek a function u on $C \times B_1$ to B_2, analytic in some neighborhood of the origin, and such that $u(0, x) \equiv 0$.

We find that such a solution exists, and is uniquely determined.

The method of proof is to consider the expansion of f as a series of homogeneous polynomials in t, u, and du, and the expansion of u as a power series in t. A recursion formula is obtained for the coefficients of the latter series. The desired estimates are now easy to obtain from the above mentioned extension of Bernstein's theorem.

SYRACUSE UNIVERSITY,
SYRACUSE, N. Y., U. S. A.

AN ELEMENTARY TREATMENT OF UNIQUENESS FOR THE HAMBURGER MOMENT PROBLEM

HERMAN RUBIN

The author obtains many of the customary conditions for the solution of the Hamburger moment problem with moments m_i, $i = 0, 1, 2, \cdots$, to be unique by purely elementary methods. By means of the author's treatment of the Daniell-Stone integral (Bull. Amer. Math. Soc. Abstract 55-11-539), the uniqueness problem is reduced to the determination of the integral of $|P|$ for an arbitrary polynomial P. Using Newton's method for the square root, this is reduced to the determination of the integral of $(x - \lambda)^{-1}$, for nonreal λ. This yields, in the standard manner, the sufficient condition that $\rho(\lambda) = 0$ for all (one) nonreal (complex) λ. If $\rho(\lambda) \neq 0$ for a nonreal λ, it is shown by introducing the moment problems $m_i(\tau) = m_i + \tau\lambda^i + \bar{\tau}\bar{\lambda}^i$ that the original moment problem has two distinct solutions. The condition $\rho(0)\rho^{(2)}(0)$, which is easily seen to be sufficient, is also shown to be necessary by means of the inequality

$$1/\rho(\lambda) \leqq 1/m_0 + |\lambda|^2/\rho^{(2)}(\lambda) + 1/\rho(0)(1 + m_0^{1/2}|\lambda|/(\rho^{(2)}(\lambda))^{1/2})^2,$$

which is obtained by introducing as a basis for the space of polynomials $\{1, x\omega_n^{(2)}(x)\}$, where $\omega_n^{(2)}$ is the nth orthonormal polynomial for the problem $m_n^{(2)} = m_{n+2}$.

STANFORD UNIVERSITY,
STANFORD, CALIF., U. S. A.

SU UNA CLASSE DI EQUAZIONI DI LIÉNARD AVENTI UNA SOLA SOLUZIONE PERIODICA

GIOVANNI SANSONE

Se nell'equazione

$$\frac{d^2i}{dt^2} + \omega f(i)\frac{di}{dt} + \omega^2 i = 0$$

è $\omega > 0$; $f(i)$ è continua in $(-\infty, \infty)$; $f(i) < 0$ per $\delta_{-1} < i < \delta_1$, $\delta_{-1} < 0$, $\delta_1 > 0$; $f(i) > 0$ per $i < \delta_{-1}$, $i > \delta_1$; $|f(i)| \leqq L < 2$; $f(i)$ non crescente da $-\infty$ a 0, non decrescente da 0 a ∞; allora essa possiede una sola soluzione periodica.

L'equazione caratteristica

$$\frac{du}{di} = -f(i) - \frac{i}{u}$$

ammette un solo ciclo stabile, e il punto (0, 0) del piano (i, u) ed il punto all'infinito dello stesso piano sono fuochi instabili.

UNIVERSITY OF FLORENCE,
FLORENCE, ITALY.

SOME NUCLEI OF CONTOUR INTEGRALS WHICH SATISFY LINEAR DIFFERENTIAL EQUATIONS

S. B. SARANTOPOULOS

1. It is known that a linear differential equation

$$L_\kappa(y) \equiv \sum_{q=0}^{q=\mu} P_q(\kappa)\frac{d^q y}{d\kappa^q} = 0, \tag{1}$$

$$[P_q(\kappa) = \alpha_{q,\nu}\kappa^\nu + \alpha_{q,\nu-1}\kappa^{\nu-1} + \cdots + \alpha_{q,0},\ \alpha_{\mu,\mu} \neq 0]$$

is satisfied by a contour integral

$$y = \int_C K(\kappa, z)v(z)\,dz \tag{2}$$

where the nucleus $K(\kappa, z)$ is $e^{z\kappa}$, $v(z)$ a solution of the equation

$$\bar{H}_z(v) \equiv \sum_{r=0}^{r=\nu} (-1)^r \frac{d^r(Q_r v)}{dz^r} = 0, \tag{3}$$

$$[Q_r(z) = \alpha_{\mu,r} z^\mu + \alpha_{\mu-1,r} z^{\mu-1} + \cdots + \alpha_{0,r}]$$

and C a contour properly chosen.

Besides the nucleus e^{κ}_z (Laplace transformation), others have been used, for instance $(z - \kappa)^\alpha$ (Euler transformation), $z\kappa$ (Mellin transformation), etc.

2. In my contributed paper I apply the nucleus $e^{z\kappa}$ and others, and I obtain different general conclusions and some partial ones which can be used in order to solve completely some linear differential equations of special forms. Some of my results are the following:

a) When $K(\kappa, z) = e^{z\kappa}$, we see that the equations (1), (3), and their adjoint equations $\bar{L}_\kappa(v) = 0$, $H_z(v) = 0$ are connected with each other, so that if we know a solution of one of them, we are in a position to solve them all completely.

b) The solution of the adjoint of (1), wherein appear only the derivatives $y^{(\mu_s)}(s = 0, 1, \cdots, h;\ \mu_0 = \mu > \mu_1 > \cdots > \mu_{h-1} > \mu_h = 0)$ with P_{μ_s} having as coeffiicents $\alpha_{\nu_s}, \alpha_{\nu_s-1}, \cdots, \alpha_{\mu_s}(\mu_s - 1 = \nu_s,\ s = 0, 1, \cdots, h)$, if the condition $\sum_{s=0}^{s=h} \Delta_\rho^{\mu_s} \alpha_{\mu_s} = 0$ $(\Delta_\rho^\lambda = \rho(\rho - 1) \cdots (\rho - \lambda - 1)$ is satisfied, is given by (2) where $K(\kappa, z) = e^{z\kappa}$ and $v(z)$ is a polynomial of degree ρ.

c) When $K(\kappa, z) = (c_0 + c_1 \kappa + \cdots + c_\nu \kappa^\nu)e^{z\kappa}$, the function $Y = c_0 v - c_1 v' + \cdots + (-1)^\nu c_\nu v^{(\nu)}$ must in general be a solution of the equation $\bar{H}(Y) = 0$ and C properly chosen. If $Y = 0$, namely if v is a solution of

$$c_0 v - c_1 v' + \cdots + (-1)^\nu c_\nu v^{(\nu)} = 0,$$

the given equation (1) reduces to a special form and has as a general solution

$$y = \sigma(\kappa) \sum_{i=0}^{i=\mu} \frac{C_i}{\kappa + \gamma_i} e^{\alpha_i \kappa} \qquad \text{(if } \nu = \mu),$$

where γ_i $(i = 1, 2, \cdots, \mu)$ are the μ roots of the polynomial $\sigma(\kappa) = c_0 + c_1\kappa + \cdots + c_\nu \kappa^\nu$, α_i suitable numbers and C_i arbitrary constants.

d) Finally for $K(\kappa, z) = [\sigma_1(\kappa)v_1(z) + \cdots + \sigma_h(\kappa)v_h(z)]e^{z\kappa}$ with $v(z) = 1$, where $\sigma_n(\kappa) = c_0^n + c_1^n \kappa + \cdots + c_{\nu_n}^n \kappa^n$ $(n = 1, 2, \cdots, h)$ are given polynomials and $v_n(z)$ functions to be determined, we find that the function

$$Y = \sum_{i=1}^{i=h} [c_0^i v_i - c_1^i v_i' + \cdots + (-1)^{\nu_i} c_{\nu_i}^i v_i^{(\nu_i)}]$$

must be a solution of $\bar{H}(Y) = 0$ and the contour C properly chosen. There are different cases to be distinguished.

UNIVERSITY OF ATHENS,
ATHENS, GREECE.

ÉQUATION DIFFÉRENTIELLE DES SYSTÈMES ISOTHERMES

FERNAND SIMONART

Étant donné, sous forme finie ou par son équation différentielle, un système de courbes à un paramètre, on peut exprimer, d'une manière invariante au moyen des opérateurs différentiels, que ce système se réduit à une famille isotherme. Le résultat s'étend à l'équation différentielle quadratique d'un réseau isotherme et trouve son application dans l'étude des surfaces ou congruences harmoniques et dans les problèmes relevant de la géométrie textile.

UNIVERSITY OF LOUVAIN,
LOUVAIN, BELGIUM.

ON THE SOLUTION OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS

J. J. SLADE AND H. PORITSKY

Linear differential equations with trigonometric polynomial coefficients are considered. As is well known, basic solutions exist of the form $e^{\lambda x}P(x)$, where P is a periodic function and λ a proper constant. Expanding the periodic factor into a Fourier series and substituting into the differential equations leads to proper linear difference equations for the coefficients. By equating to zero the determinants Δ_N of order $2N$, obtained from the infinite matrix of coefficients and solving for λ, then letting N become infinite, one obtains in the limit characteristic exponents λ. The present paper is concerned with the speeding up of the convergence of this process.

The determinant $\Delta_n(\lambda)$ described above does not in general converge to a finite limit; however, two numbers a, c may be found such that $N^c a^N \Delta_N(\lambda)$ does converge to a function $\Delta(\lambda)$. It is know that $\Delta(\lambda)$ can be constructed as a rational trigonometric function of λ and the calculation is reduced to determination of its coefficients.

Difference equations for $\Delta_n(\lambda)$ are obtained by utilizing the Laplace expansion, thus obtaining certain difference equations not unlike the original difference equations for the coefficients of $P(x)$.

By approximating these difference equations certain infinite products are introduced similar to the infinite products expansions of trigonometric functions. Replacing these products by their limits suggests the introduction of certain convergence factors P_n, which though they approach unity and do not change the limits, speed up the convergence and render the calculation more practical.

RUTGERS UNIVERSITY,
NEW BRUNSWICK, N. J., U. S. A.
GENERAL ELECTRIC COMPANY,
SCHENECTADY, N. Y., U. S. A.

SUR LES THÉORÈMES DE KLEIN DANS LES ÉQUATIONS DIFFÉRENTIELLES LINÉAIRES

IVAN VIDAV

Dans l'ordre d'idées de Klein (F. Klein, *Bemerkungen zur Theorie der linearen Differentialgleichungen zweiter Ordnung*, Math. Ann. t. 64), j'ai traité dans mon Mémoire (I. Vidav, *Kleinovi teoremi v teoriji linearnih diferencialnih enačb* (*Kleinsche Theoreme in der Theorie der linearen Differentialgleichungen*) Ljubljana, 1941) le problème suivant: l'équation différentielle de Fuchs du second ordre à 5 points singuliers, dont tous sont situés sur l'axe réel, contient deux paramètres accessoires. Ces deux paramètres doivent être déterminés de manière que le rapport de deux solutions de l'équation différentielle se reproduise par la circulation de la variable indépendante autour de deux points singuliers consécutifs. Les exposants de tous les points singuliers soient entre 0 et 1. La condition pour la solution du problème est que les exposants des deux points singuliers autour desquelles le rapport se reproduit, soient égaux. J'ai trouvé que le problème admet toujours une infinité de solutions: Si l'intervalle entre les deux points singuliers est k-fois oscillatoire (à savoir que l'intégrale de l'équation différentielle, s'annulant en point initial, a encore k zéros sur cet intervalle), le problème admet précisément $2k + 1$ solutions.

Dans cette conférence, je traite un problème analogue: Les trois paramètres accessoires de l'équation différentielle à six points singuliers doivent être déterminés de manière que le rapport de deux intégrales se reproduise par la circulation autour de trois points singuliers consécutifs. J'ai réussi à résoudre le problème dans le cas seulement, où dans un des triples des points singuliers la somme des exposants $= 1$. Désignons par a, b, c ($a < b < c$) ce triple. Alors deux cas peuvent se présenter: Au premier cas il y a $2(i + k) + 1$ solutions, dans le deuxième nous en avons $2(i + k) - 1$, si l'intervalle (a, b) est précisément i-fois et (b, c) k-fois oscillatoire.

Ce n'est que le théorème fondamental, où on a $i = k = 0$, qui est important pour la théorie de fonctions automorphes. Comme au deuxième cas c'est toujours $i > 0$ et $k > 0$, il n'y a qu'un seul théorème fondamental. Ce théorème conduit—les exposants étant les inverses de nombres entiers—aux fonctions automorphes uniformes dont le domaine d'existence est limité par une infinité de cercles.

UNIVERSITY OF LJUBLJANA,
LJUBLJANA, YUGOSLAVIA.

FUNCTIONAL ANALYSIS

ON CONVEX SETS IN LINEAR SPACES

William D. Berg and Otton M. Nikodým

Let V denote an arbitrary infinite-dimensional real linear space, and let E denote a convex set in V. By an interior point of E we mean a point $x \in E$ having the property that each line through x contains an open segment in E containing x. By a convex body we mean a convex set possessing at least one interior point. By a linearly bounded set we mean a set having the property that each line meets it in a subset of a bounded segment. (The intersection may be empty.) We shall indicate some paradoxical properties of convex bodies.

Let B denote a Banach space possessing an everywhere dense linear subvariety V' isomorphic to V. If W is a linear subvariety of B and S is the unit sphere in B with center 0, then the counterimage in V (through the isomorphism) of the set $(W + S) \cap V'$ is a convex body. $(W + S \equiv \{w + s \mid w \in W, s \in S\}.)$ We have the lemma: A necessary and sufficient condition that the counterimage of $(W + S) \cap V'$ be linearly bounded in V is that $\overline{W} \cap V' = (0)$.

We use this result in the special case when our Banach space is a real Hilbert space H (which need not be separable) to construct linearly bounded convex bodies A, B in V with the property that the smallest convex set $c(A \cup B)$ containing their union is not linearly bounded.

Take an $x_1 \notin V'$ and let W_1 be any linear variety in H containing x_1 for which $\overline{W}_1 \cap V' = (0)$. From the lemma the counterimage A of $(W_1 + S) \cap V'$ is a convex body which is linearly bounded in V. By a reflection of x_1 with respect to a line in V' through 0 we find an $x_2 \notin V'$. If W_2 is any linear variety containing x_2 for which $\overline{W}_2 \cap V' = (0)$, the counterimage B of $(W_2 + S) \cap V'$ will be a linearly bounded convex body. We can find a sequence in A whose reflection will be a sequence in B and such that the sequence of midpoints, which will lie in $c(A \cup B)$ and on the counterimage of the line of reflection in V', is unbounded. Thus $c(A \cup B)$ is linearly unbounded.

Another paradoxical result: If V is a real infinite-dimensional linear space and E is a convex body, there exists a family of disjoint ("parallel") hyperplanes filling V such that every hyperplane of the family meets E in a nonempty set.

(These results represent part of the work of the authors under a cooperative contract between the Atomic Energy Commission and Kenyon College.)

Kenyon College,
Gambier, Ohio.

APPROXIMATELY MULTIPLICATIVE OPERATORS

D. G. Bourgin

Let $C(S)$ denote the ring of continuous functions over the compact space S. Let U map $C(S_1)$ into $C(S_2)$. Designate by $d(xy)$ the norm of the deviation of the map of xy from the product of the maps of x and of y. U is multiplicative or approximately multiplicative according as $d(xy)$ vanishes identically or not. Suppose $d(xy)$ is dominated by the product of the minima of $| x |$ and $| y |$. This is condition I. Condition II requires that $d(xy)$ vanish if x or y is not regular. Condition III demands $d(xy)$ be dominated by a constant plus the minimum of $| xy |$. We indicate some characteristic results:

Except if S_1 and S_2 contain a single point, I implies U is multiplicative.

Suppose the two spaces are compacta with at least two points, U is continuous, satisfies II, is "onto," and at least one nonregular element is not mapped into the zero of $C(S_2)$. It then follows that U is actually multiplicative. The argument involves a representation of a continuous positive multiplicative functional on (a) the multiplicative semi-group of the non-negative elements of $C(S)$ to the multiplicative semi-group of the non-negative reals. Such functionals are included in the continuous multiplicative functionals on (b) the multiplicative group of the positive elements of $C(S_1)$ to the multiplicative group of the positive reals and hence have the representation exp $\int_{S_1} \log X(S) d\mu$. Continuity considerations for the case (a) require that the Borel measure be concentrated on an at most denumerable closed subset of S_1. The case S_i compact is also treated and a somewhat similar result obtained. With other restrictions on S_i it has been possible to assert the result even when U is no longer required to be continuous.

If III is satisfied and the map is 1-1 and some pair of nonregular elements maps into a pair with disjoint zero sets, then S_1 and S_2 are homeomorphic.

University of Illinois,
Urbana, Ill., U. S. A.

ADDITIVE FUNCTIONALS ON A SPACE OF CONTINUOUS FUNCTIONS

R. H. Cameron and Ross E. Graves

This paper deals with functionals defined (and finite) on the space C of real functions which are continuous on $I: 0 \leqq t \leqq 1$ and vanish at $t = 0$. A functional $F(x)$ is called additive if $F(x) + F(y) = F(x + y)$ for all x and y in C, homogeneous if $F(\lambda x) = \lambda F(x)$ for all real λ, and linear if it is both additive and homogeneous. The paper gives certain uniqueness theorems for additive functionals and characterization and representation theorems for essentially additive functionals (i.e., functionals which are equal almost everywhere to additive functionals in the sense of Wiener's measure on C), as well as certain results of other types.

Among the uniqueness theorems are the following. *If two additive functionals are equal almost everywhere on C, they are equal everywhere on the set of absolutely continuous functions of C having derivatives of class L_2 on I.* Again, *two measurable additive functionals on C are equal either almost everywhere or almost nowhere.* The latter result is an immediate consequence of the following "zero or one" theorem. *Every measurable linear manifold of functions of C over the field of rational numbers is either of measure zero or of measure unity.*

The representation theorem is the following. *A necessary and sufficient condition that a functional $F(x)$ be essentially additive and of class $L_2(C)$ is that it be expressible in the form $F(x) =$ (PWZ) $\int_0^1 f(t)\, dx(t)$, where the right member is a Paley Wiener Zygmund Riemann Stieltjes integral and $f(t) \in L_2(I)$, and the equation holds almost everywhere on C. Moreover for almost all t in I, the function $f(t)$ is given by*

$$f(t) = 2 \frac{d}{dt} \int_C^W F(x)x(t)\, d_W x.$$

The characterization theorem states that *a necessary and sufficient condition that a function of class $L_2(C)$ be essentially additive is that it have only first degree terms in any one of its orthogonal developments in Fourier-Hermite functionals.* This theorem implies the following theorem connecting additivity and linearity. *If a functional of class $L_2(C)$ is essentially additive on C, it is essentially linear on C.* It is shown by a counterexample that this theorem becomes false if the word "essentially" is omitted from both the hypothesis and the conclusion.

University of Minnesota,
Minneapolis, Minn., U. S. A.

ON MAPPINGS OF A UNIFORM SPACE ONTO ITSELF

Albert Edrei

Let E be a totally bounded, separated, uniform space and let $\mathcal{F}(E, E)$ be the set of all mappings of E into itself. Introducing in $\mathcal{F}(E, E)$ the structure of uniform convergence [N. Bourbaki, *Topologie Générale*, chap. X, p. 5], families of mappings can be considered as sets of points of the functional space $\mathcal{F}_u(E, E)$. Let $H = \{h\}$ be a family such that $h(E) = E$, for every h. Consider H', the derived set of H; H'_c, the set of all uniformly continuous members of H'; H^*_c, the set of all 1-1 mappings of H'_c. If the elements of H are 1-1, let H^{-1} be the set of the inverse mappings of H and form $(H^{-1})'$, $(H^{-1})'_c$, $(H^{-1})^*_c$ [respective analogues of H', H'_c, H^*_c]. The product of two mappings, $fg = h$, is the mapping $f(g(x)) = h(x)$; $e(x)$ [$\equiv x$] is the identical-mapping.

We say that H is:

(i) *iterative* if in each infinite subset $\{g\}$ of H there exist two distinct mappings g_1 and g_2 such that the equation $ug_1 = g_2$ admits of a solution u, in H.

(ii) *strongly iterative* if to every h [ϵ H] there corresponds in every infinite subset $\{g\}$ of H a mapping g such that the equation $uh = g$ admits of a solution u, in H.

Let $\mathfrak{A}$ be the filter of vicinities [we translate "entourage" by vicinity] which defines the structure of E; we denote vicinities by U and V.

We say that h is *V-continuous* if there exists U such that $h(U(x)) \subset V(h(x))$ for all $x \in E$.

We say that

(iii) H is *almost smooth* if to every V there corresponds a non-negative integer $n(V)$ such that all but $n(V)$ members of H are V-continuous;

(iv) H is *equally almost smooth* if to every V there corresponds U and a non-negative integer $n(V)$ such that all but $n(V)$ members of H satisfy

$$h(U(x)) \subset V(h(x)); \tag{1}$$

(v) H has *scattered smoothness* if to every V there corresponds U, and an infinity of distinct members of H, for which (1) is true.

These definitions yield

1. *If H is iterative, then $e \in H'$ if and only if H has scattered smoothness.*

2. *If H is an iterative semi-group of scattered smoothness, then $(H \cup e) \subset H'$; H_c' is then a semi-group with a unit element.*

3. *If H is strongly iterative and has scattered smoothness, then every mapping of H is* 1-1. *Moreover, if H is also a semi-group, then $(H^{-1} \cup e \cup H) \subset H'$.*

4. *Let E be compact and H be a strongly iterative semi-group:*
   (a) *if H has scattered smoothness, then $H_c^* = (H^{-1})_c^*$ is a group;*
   (b) *if H is commutative, has scattered smoothness, and is almost smooth, then $H' = (H^{-1})'$ is a commutative group;*
   (c) *if H is equally almost smooth, then $H' = (H^{-1})'$ is a group and its elements are uniformly equicontinuous.*

UNIVERSITY OF SASKATCHEWAN,
SASKATOON, SASK., CANADA.

THE CONTINUATION METHOD FOR FUNCTIONAL EQUATIONS

F. A. FICKEN

Let $x \to T(x)$ be a transformation of a Banach space into itself. We wish to solve $T(x) = 0$. Suppose that we can find a family of transformations $T(s, x)$ $(0 \leqq s \leqq 1)$ such that $T(1, x) \equiv T(x)$ and $T(0, x) \equiv T^*(x)$, where the equation $T^*(x) = 0$ has a unique solution $x(0) = x^*$ inside the sphere S^N: $\| x \| \leqq N$.

Let $T(s, x + u) - T(s, x) = L(s, x, u) + \| u \| R(s, x, u)$ where the differential L

is linear in u. Assume that, *uniformly in s and x*: $T(s, x)$ is continuous in s, $L(s, x, u)$ is continuous in s and x and has a bounded inverse, and $R(s, x, u) \to 0$ as $u \to 0$. Suppose now that $T(s_0, x_0) = 0$ with $\| x_0 \| < N$. Under these local conditions we can determine a constant k and a number $\sigma(x_0)$, depending only on min $(k, N - \| x_0 \|)$, such that a continuous locally unique solution $x(s)$ can be found (by an iterative method) for $| s - s_0 | \leqq \sigma(x_0)$. Starting with x^* we obtain a solution first for $0 \leqq s \leqq \sigma(x^*)$, then for $\sigma(x^*) \leqq s \leqq \sigma(x^*) + \sigma(x(\sigma(x^*)))$, and so on. This is called the continuation method. If a local extension y elds a solution that remains interior to S^N, then a further local extension is possible, but the length σ of the step toward $s = 1$ may decrease.

Let Σ denote the set of those x in S^N such that $T(s, x) = 0$ for some s. Assume that there is a $\rho > 0$ such that x in Σ implies $\| x \| \leqq N - \rho$ (strong a priori bound). Under this global condition the solution stays inside S^N. Moreover, there is then a positive lower bound (determined by min $(k, N - \rho)$) for the step-length σ. Hence a continuous solution $x(s)$ that is everywhere locally unique can be constructed for $0 \leqq s \leqq 1$ in a *finite* number of steps. Uniqueness in S^N can easily be established.

Thus the foregoing conditions suffice to guarantee the existence, continuity, and uniqueness in S^N of the solution $x(s)$ of $T(s, x) = 0$ for $0 \leqq s \leqq 1$. Under these conditions, moreover, $x(s)$ can be generated by the continuation method from $x^* = x(0)$ in a finite number of local extensions.

Under the further global assumption that Σ is compact, it suffices to assume that x in Σ implies $\| x \| < N$ (weak a priori bound) and to impose far weaker local requirements. That Σ is compact is implied either by the familiar condition that $x - T(s, x)$ be completely continuous or by a local requirement of a new type.

New York University,
New York, N. Y., U. S. A.
University of Tennessee,
Knoxville, Tenn., U. S. A.

THE ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES AND EIGENFUNCTIONS OF A GENERAL VIBRATION PROBLEM

Lars Gårding

Let $p(\xi) = p(\xi_1, \cdots, \xi_n)$ be a real homogeneous polynomial of positive degree m' such that $p(\xi) > 0$ when ξ is real and not zero. Because $p(-\xi) = (-1)^{m'}p(\xi)$, it then follows that $m' = 2m$ is even. Elsewhere (C. R. Acad. Sci. Paris vol. 230 (1950) pp. 1030–1032) I have outlined the solution of Dirichlet's problem for the differential equation

$$pf(x) = (-1)^m p(\partial/\partial x)f(x) = 0$$

in a bounded open region S in real n-space. The object of the present note is the corresponding eigenvalue problem, in particular the asymptotic distribution of the eigenfunctions and eigenvalues. Let H be the set of all real functions $f = f(x)$ of class C_∞ which vanish outside compact subsets of S. If f and g are in H and $F(\xi) = \int e^{ix_1\xi_1+\cdots+ix_n\xi_n} f(x)\, dx$ and $G(\xi)$ are the Fourier transforms of f and g, the analogue of Dirichlet's integral is

$$(f, g) = (2\pi)^{-n} \int F(\xi)\overline{G(\xi)}p(\xi)\, d\xi.$$

Closing H in the norm $(f, f)^{1/2}$ we obtain a Hilbert space $\mathfrak{H}$. It turns out that the quadratic form $\int f(x)^2 dx$ $(f \in H)$ has a unique bounded positive and completely continuous extension to $\mathfrak{H}$. The corresponding operator T, which may be called Green's transformation, has eigenelements φ_ν $(\nu = 1, 2, \cdots)$ which constitute a basis of $\mathfrak{H}$ and are mutually orthogonal. They can be described as analytic functions $\varphi_\nu(x)$ whose derivatives of order $\leqq m$ are square integrable in S. Corresponding to every φ_ν there is an eigenvalue $1/\lambda_\nu > 0$ such that $T\varphi_\nu = \varphi_\nu/\lambda_\nu$ and one finds that $p\varphi_\nu(x) = \lambda_\nu\varphi_\nu(x)$ $(x \in S)$. We label the eigenelements in such a fashion that $\lambda_\nu \leqq \lambda_{\nu+1}$ for all ν and normalize them so that $\int_S \varphi_\nu(x)^2 dx = 1$. Then one has the asymptotic formulas

$$\lim_{N\to\infty} N^{-1} \sum_1^N \varphi_\nu(x)^2 = \bar{S}^{-1}, \qquad \lim_{N\to\infty} N^{-1} \sum_1^N \varphi_\nu(x)\varphi_\nu(y) = 0 \qquad (x, y \in S, x \neq y)$$

$$N(t) = \sum_{\lambda_\nu \leqq t} 1 = (2\pi)^{-n}\omega_p \bar{S} t^{n/2m}(1 + o(1)),$$

where $o(1) \to 0$ as $t \to \infty$, $\bar{S} = \int_S dx$ is the volume of S and $\omega_p = \int_{p(\xi)<1} d\xi$ that of the region $p(\xi) < 1$. In the cases $p(\xi) = \xi_1^2 + \cdots + \xi_n^2$ and $p(\xi) = (\xi_1^2 + \xi_2^2)^2$ these formulas are well known. They are due to H. Weyl, T. Carleman, R. Courant, and A. Pleijel. Our proof uses the method of Carleman.

LUND UNIVERSITY,
LUND, SWEDEN.

A NOTE ON SEPARABLE NORMED LINEAR SPACES

D. B. GOODNER

A normed linear space X has property P_s, $s \geqq 1$, if and only if for every normed linear space $Y \supset X$ there exists a projection T, $\|T\| \leqq s$, of Y onto X. Let H be a normal space such that the space $C(H)$ of real-valued bounded continuous functions on H is separable. Then $C(H)$ has property P_1 if every open F_σ-set has a closure which is open. A separable normed linear space X which has property P_1 and which has an extreme point on its unit sphere is finite-dimensional. This latter result, in conjunction with a known theorem on separable spaces, shows that an extremally disconnected compact metric space consists of a finite number of points. The space of continuous real-valued functions on an extremally disconnected compact Hausdorff space H is separable if and only if H consists of a finite number of points.

FLORIDA STATE UNIVERSITY,
TALLAHASSEE, FLA., U. S. A.

ON THE REDUCTION OF A COMPLETELY CONTINUOUS LINEAR TRANSFORMATION IN HILBERT SPACE

HANS LUDWIG HAMBURGER

Let C be a completely continuous linear transformation in the Hilbert space $\mathfrak{H}$ such that the eigenvalues $\lambda_\nu \neq 0$ of C are all simple.

Let $\mathfrak{S}$ be the subspace spanned by the eigenelements φ_ν of C belonging to λ_ν, and $\mathfrak{S}^*$ the subspace spanned by the eigenelements ψ_ν of C^* belonging to $\bar{\lambda}_\nu$. We further write $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{S}^*$, $\mathfrak{N}^* = \mathfrak{H} \ominus \mathfrak{S}$. We then state

THEOREM 1. *We have* $C(\mathfrak{S}) \subseteq \mathfrak{S}$, $C(\mathfrak{N}) \subseteq \mathfrak{N}$; *moreover, C is quasi-nilpotent in* $\mathfrak{N}$.

One notices that $\mathfrak{N} \oplus \mathfrak{S} = \mathfrak{H}$ if and only if $\mathfrak{N}^* \cap \mathfrak{S}^* = 0$. The necessary and sufficient conditions for $\mathfrak{N} \cap \mathfrak{S} = 0$ are formulated in

THEOREM 2. $\mathfrak{N} \cap \mathfrak{S} = 0$ *if and only if the biorthogonal complement* $\{\psi_\nu'\}$ *in* $\mathfrak{S}$ *of the system* $\{\varphi_\nu\}$ *spans* $\mathfrak{S}$.

Here the biorthogonal complement in $\mathfrak{S}$ of $\{\varphi_\nu\}$ denotes the uniquely determined system of elements ψ_ν' such that (i) $\psi_\nu' \in \mathfrak{S}$, (ii) $(\varphi_\nu, \psi_\mu') = \delta_{\nu\mu}$. If P denotes the orthogonal projector of $\mathfrak{H}$ on $\mathfrak{S}$, one readily finds $\psi_\nu' = P\psi_\nu$.

Besides the case $\mathfrak{N}^* \cap \mathfrak{S}^* = 0$, the case $C^*(\mathfrak{N}^* \cap \mathfrak{S}^*) = 0$ requires a special interest, since $C^*(\mathfrak{N}^* \cap \mathfrak{S}^*) = 0$ implies that $\mathfrak{R} \subseteq \mathfrak{N} \oplus \mathfrak{S}$, where $\mathfrak{R}$ denotes the range of C. The two following theorems deal with this case:

THEOREM 3. If $\sum_{\nu=1}^{\infty} |\lambda_\nu| \, \|\varphi_\nu\| \, \|\psi_\nu\| < \infty$, *then* $Cx = \sum_{\nu=1}^{\infty} \lambda_\nu(x, \psi_\nu)\varphi_\nu$ *for every* $x \in \mathfrak{S}$; *besides we have* $C(\mathfrak{N} \cap \mathfrak{S}) = 0$ *and* $C^*(\mathfrak{N}^* \cap \mathfrak{S}^*) = 0$.

THEOREM 4. *If* $C(\mathfrak{N} \cap \mathfrak{S}) \neq 0$, *then a completely continuous linear transformation B with simple eigenvalues can be determined in $\mathfrak{H}$ such that* (i) $BC = CB$, (ii) $B(\mathfrak{N}) = B^*(\mathfrak{N}^*) = 0$.

We finally give an explicit example of a completely continuous linear transformation C with simple eigenvalues, where $C(\mathfrak{N} \cap \mathfrak{S}) \neq 0$, and where the subspaces $\mathfrak{N} \cap \mathfrak{S}$ and $\mathfrak{N}^* \cap \mathfrak{S}^*$ are both of infinitely many dimensions.

UNIVERSITY OF ANKARA,
ANKARA, TURKEY.

SPECTRAL SYNTHESIS OF BOUNDED FUNCTIONS

HENRY HELSON

It is known that a bounded measurable function defined on a locally compact abelian group is a constant multiple of a character if its spectral set consists of a single point. This result is extended to the following theorem: a bounded measurable function is the weak limit of linear combinations of characters chosen from its spectral set assuming the spectral set to be denumerable and discrete. The proof consists in considering the function as an operator on L^1 and L^2 by multiplication, and then passing to the dual group by taking Fourier transforms. The dual operator so induced can be approximated by linear combinations of translations, and the analysis of such approximations leads first to the known result on characters and then to the extension.

UNIVERSITY OF UPPSALA,
UPPSALA, SWEDEN.

ON CONVOLUTION ALGEBRAS

EDWIN HEWITT AND H. S. ZUCKERMAN

Let G be any group, let $\mathfrak{F}$ be a linear space of complex-valued functions on G, and let $\mathfrak{L}$ be a linear space of linear functionals on $\mathfrak{F}$. Let $L_y(f(xy))$ denote $L(F(y))$, where $F(y) = f(xy)$ considered as a function of y, whenever $F(y) \in \mathfrak{F}$. If, for every $f \in F$, $f(xy)$, considered as a function of y, is in $\mathfrak{F}$, and if for every $L \in \mathfrak{L}$, $L_y(f(xy)) \in \mathfrak{F}$, then elements of $\mathfrak{L}$ admit an operation of convolution, $L*M(f) = L_x(M_y(f(xy)))$, making $\mathfrak{L}$ into an algebra over the complex numbers K. The following special cases are examined. (1) G an arbitrary locally compact group, $\mathfrak{F} = \mathfrak{C}_\infty(G) =$ all continuous complex-valued functions on G which are arbitrarily small outside of compact sets; $\mathfrak{L} =$ all bounded linear functionals on $\mathfrak{C}_\infty(G)$, denoted by the symbol $\tilde{\mathfrak{C}}_\infty(G)$. This algebra has nil radical. The space of maximal ideals in $\tilde{\mathfrak{C}}_\infty(R)$ (R the additive group of real numbers) contains 2^c elements, and various facts concerning the topology of this space are obtained. Identification of all maximal ideals remains an open problem. (2) G as in (1); $\mathfrak{F} = \mathfrak{D}_\infty(G) =$ all functions on G of the form $\alpha + f$, where $\alpha \in K$ and $f \in \mathfrak{C}_\infty(G)$, $\mathfrak{L} = \tilde{\mathfrak{D}}_\infty(G) =$ all bounded linear functionals on $\mathfrak{D}_\infty(G)$. The algebra $\tilde{\mathfrak{D}}_\infty(G)$ has radical consisting of all functions βL_∞, where $L_\infty(\alpha + f) = \alpha$ ($f \in \mathfrak{C}_\infty(G)$). (3) $G = R$, $\mathfrak{F} = \mathfrak{A} =$ all functions $f(x) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{it_n x}$ such that $|||f||| = \sum_{n=-\infty}^{+\infty} |\alpha_n| < \infty$; $\mathfrak{L} = \tilde{\mathfrak{A}} =$ all linear functionals on $\mathfrak{A}$ bounded in the norm $|||f|||$. $\tilde{\mathfrak{A}}$ is realized as the algebra of all continuous complex-valued functions on a certain compact Hausdorff space, and thus the structure of all closed ideals in $\tilde{\mathfrak{A}}$ is known. (4) G as in (1), $\mathfrak{F} = \mathfrak{C}(G) =$ all complex continuous functions on G; $\mathfrak{L} = \tilde{\mathfrak{C}}(G) =$ all linear functionals on $\mathfrak{C}(G)$ of the form $\int f(x)\, d\gamma(x)$, where γ is a Borel measure on G confined to a compact set. Again, $\tilde{\mathfrak{C}}(G)$ has nil radical. Other examples of the same general type are also studied.

UNIVERSITY OF WASHINGTON,
SEATTLE, WASH., U. S. A.

PROJECTIONS AND BASES IN BANACH SPACES

Robert C. James

A sequence of elements $\{x_n\}$ is a basis for a Banach space B if and only if for each x of B there is a unique series $x = \sum_1^\infty a_i x_i$, where $\lim_{n\to\infty} \| x - \sum_1^n a_i x_i \| = 0$. A fundamental sequence $\{x_n\}$ is known to be a basis if and only if for some number N and each n the projection defined by $P_n(\sum_1^\infty a_i x_i) = \sum_1^n a_i x_i$ satisfies $\| P_n \| \leqq N$. Consequently, a separable Banach space B and each of its subspaces have a basis if for some number N and for each finite-dimensional subspace M of B there is a projection P of B onto M with $\| P \| \leqq N$. It is shown that if B has a basis and there is a projection onto each subspace of B, then, for some number N, there is a projection of norm less than N on each finite-dimensional subspace. This can be extended to subspaces M having a basis if M is either reflexive or if the basis for M is unconditionally convergent and no subspace of M is isomorphic with (c_0).

If there is a projection of norm less than N on each finite-dimensional subspace of B, then for any fundamental sequence $\{y_n\}$ of B there is a basis $\{x_n\}$ of B such that $x_n \in y_1 \dotplus \cdots \dotplus y_n$ for each n. The following shows that this is not possible in general: Let L be a linear space with a countable Hamel's basis, and let $\{y_n^p\}$, $p = 1, 2, \cdots$, be a countable set of Hamel's bases for L. Then L can be normed and completed so as to be a Banach space B for which there exist no numbers N and p and basis $\{x_n\}$ for B with $x_n \in y_1^p \dotplus \cdots \dotplus y_{n+N}^p$ for each n. Furthermore, B can be $l^{(p)}$ $(1 < p \neq 2)$ or a subspace of any Banach space not having a projection onto each subspace.

University of California,
Berkeley, Calif., U. S. A.

BROWNIAN MOTIONS AND DUALITY OF LOCALLY COMPACT ABELIAN GROUPS

Shizuo Kakutani

Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space (i.e. a measure space with $\mu(\Omega) = 1$), and $(S, \mathfrak{M}, m)$ a measure space with $m(S) \leqq \infty$. Let $\mathfrak{M}_0 = \{M \mid M \in \mathfrak{M}, m(M) < \infty\}$. Let $L^2(\Omega)$, $L^2(S)$ be the corresponding complex L^2-spaces. A complex-valued function $x(M, \omega)$ defined for $M \in \mathfrak{M}_0$, $\omega \in \Omega$, is a generalized Brownian motion on $(S, \mathfrak{M}, m)$ if the following conditions (i), (ii) are satisfied: (i) for any $M \in \mathfrak{M}_0$, $x(M, \omega)$ is a Gauss function with variance $\sigma = m(M)$. (A Gauss function is a $\mathfrak{B}$-measurable function of ω which has a complex Gauss distribution with mean 0. $x(\omega) \equiv 0$ is considered as a Gauss function with $\sigma = 0$.) (ii) if $M_k \in \mathfrak{M}_0$, $k = 1, \cdots, n$, are disjoint, then $x(M_k, \omega)$, $k = 1, \cdots, n$, are independent and $x(\bigcup_{k=1}^n M_k, \omega) = \sum_{k=1}^n x(M_k, \omega)$ almost everywhere on Ω. It is easy to see that every generalized Brownian motion $x(M, \omega)$ determines a linear isometric mapping $f(s) \to x_f(\omega)$ of $L^2(S)$ onto a Gauss subspace of $L^2(\Omega)$ (i.e. a closed linear subspace of $L^2(\Omega)$ consisting only of Gauss functions) such that (iii) $f_M(s) \to x(M, \omega)$, where $f_M(s)$ is the characteristic function of a set $M \in \mathfrak{M}_0$. Conversely, every linear isometric mapping $f(s) \to x_f(\omega)$ of $L^2(S)$ onto a Gauss subspace of $L^2(\Omega)$ determines a generalized Brownian motion $x(M, \omega)$ by (iii). Thus a generalized Brownian motion on $(S, \mathfrak{M}, m)$ may be considered as a linear isometric embedding of $L^2(S)$ into $L^2(\Omega)$ as a Gauss subspace.

In this paper, the case is discussed when $S = G$ is a locally compact abelian group and $(G, \mathfrak{M}, m)$ is a Haar measure space on G. Let G^* be the character group of G, and let $(G^*, \mathfrak{M}^*, m^*)$ be a Haar measure space on G^* with a suitable normalization. Since, because of the generalized Plancherel theorem, the Fourier transform gives a linear isometric correspondence between $L^2(G)$ and $L^2(G^*)$, there exists a natural duality between generalized Brownian motions on $(G, \mathfrak{M}, m)$ and $(G^*, \mathfrak{M}^*, m^*)$. If we consider the case when G is the additive group of real numbers mod 1 with the usual compact topology, and when G^* is the character group of G (i.e. the additive group of integers) with discrete topology, then the representation of a generalized Brownian motion on $(G, \mathfrak{M}, m)$ obtained by using this duality is nothing but the representation of a Brownian motion obtained by Paley-Wiener (*Fourier transforms in the complex domain*, chap. IX).

YALE UNIVERSITY,
NEW HAVEN, CONN., U. S. A.

SUR L'ANALYSE HARMONIQUE DES FONCTIONS À CARRÉ MOYEN FINI

J. KAMPÉ DE FÉRIET

1. Le problème de l'analyse harmonique d'une fonction ayant un carré moyen fini a été résolu par Norbert Wiener, qui a donné la définition du spectre d'énergie ou periodogramme d'une telle fonction. Cette définition dépend essentiellement des propriétés de la fonction à l'infini; or, dans les applications (en particulier dans la théorie de la turbulence), on ne connait jamais la fonction que pour un intervalle fini; il est donc intéressant de définir le spectre d'énergie pour une fonction tronquée, nulle au dehors d'un intervalle fini et de partir de ce spectre pour construire le spectre total de la fonction.

2. Soit $u(t)$ une fonction complexe de la variable réelle t $(-\infty < t < +\infty)$ à carré sommable dans tout intervalle fini; considérons la fonction tronquée $u_T(t) = u(t)$, $|t| < T$; $u_T(t) = 0$, $|t| > T$. Nous définirons pour u_T un coefficient de corrélation fonction continue de h, définie positive, en posant:

$$\rho_T(h) = \frac{1}{2T}\int_{-T+|h|/2}^{T-|h|/2} u\left(t+\frac{h}{2}\right)u^*\left(t-\frac{h}{2}\right)dt \qquad |h| \leqq 2T$$

$$= 0 \qquad |h| \geqq 2T.$$

Le coefficient de Fourier de u_T :

$$\alpha_T(\omega) = \frac{1}{2T}\int_{\infty}^{+\infty} e^{-i\omega t}u_T(t)\,dt = \frac{1}{2T}\int_{-T}^{T} e^{-i\omega t}u(t)\,dt$$

appartient à $L^2(-\infty, +\infty)$ d'après le théorème de Plancherel; en appliquant la formule de Parseval on a:

$$\rho_T(h) = \int_{-\infty}^{+\infty} e^{i\lambda h}\,dS_T(\lambda), \qquad S_T(\lambda) = \frac{T}{\pi}\int_{-\infty}^{\lambda} |\,\alpha_T(\omega)\,|^2\,d\omega.$$

C'est la fonction $S_T(\lambda)$ qui définit le spectre d'énergie de u_T.

3. Si l'on fait maintenant l'hypothèse que:

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} u(t+h)u^*(t)\,dt = \rho(h)$$

existe pour tout h, on voit immédiatement que: $\rho(h) = \lim_{T\to+\infty} \rho_T(h)$.
Si l'on suppose en outre, que $\rho(h)$ est continue pour $h = 0$, un théorème de Paul Lévy permet d'affirmer que $S_T(\lambda)$ tend vers une fonction non décroissante bornée $S(\lambda)$ telle que:

$$\rho(h) = \int_{-\infty}^{+\infty} e^{i\lambda h}\,dS(\lambda).$$

On retrouve ainsi, par une voie très naturelle la fonction spectrale de $u(t)$ sous une forme:

$$S(\lambda) = \lim_{T\to+\infty}\frac{T}{\pi}\int_{-\infty}^{\lambda} |\,\alpha_T(\omega)\,|^2\,d\omega$$

qui parait particulièrement intéressante pour les applications.

University of Lille,
Lille, France.

ON COMMUTATIVE SELF-ADJOINT OPERATOR ALGEBRAS

J. M. G. Fell and J. L. Kelley

Let A be a weakly closed, self-adjoint, commutative algebra of operators on a Hilbert space H and suppose $1 \in A$. Let X be the space of maximal ideals of A with the usual topology, and let T be the isomorphism carrying the continuous complex-valued functions on X into A.

If B is the algebra of all complex Borel functions on X modulo the functions vanishing outside a set of the first category, then T can be extended to an isomorphism of B to an algebra of not necessarily bounded normal operators (if a little care is used defining sum and composition of operators).

A multiplicity function m on X to the cardinals is defined, which is continuous in the order topology for the cardinals. The space X, together with m, determine A and H to a unitary equivalence. Briefly: because of its special structure there is a measure μ on X such that the algebra A' of μ-essentially bounded functions on X is isomorphic to A. If Y is the union for $x \in X$ of $\{x\} \times m_x$, and H' is the space of functions on Y which are square integrable with respect to the (relativized) product of μ by counting measure, then A' operates on H' in a natural way and forms a model for A on H.

The above and other results on the structure of A, H, and X are intended as extension and simplification of various results of M. H. Stone and recent results of Rohlin and Plesner, and von Neumann.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF., U. S. A.

A NOTE ON BANACH ALGEBRAS

J. L. KELLEY AND R. L. VAUGHT

1. For A, a complex Banach $*$-algebra with unit satisfying $\|x\| = \|x^*\|$, let the positive cone P be the closure of the set of all finite sums of elements of the form xx^*. Let $N(x) = \text{dist}^{1/2}(-xx^*, P) = \inf \| xx^* + y \|^{1/2}$ for $y \in P$. Then N is the norm defined by Gelfand and Naimark as $\sup f^{1/2}(xx^*)$ for functionals f of positive type and the "$*$-radical" is the set of x with $N(x) = 0$. If A is symmetric ($e + xx*$ always has an inverse), then P is also the closure of the open cone, defined by Raikov, of self-adjoint elements with positive spectra.

2. Let B be a commutative real Banach algebra with unit, P the closure of the set of all finite sums of squares, and Σ the set of functionals of positive type of norm at most 1. The extreme points of Σ are precisely the homomorphisms of B into the reals, and $\text{dist}(-x, P) = \sup f(x)$ for real homomorphisms f. Thus the "real radical" I of B is the set of x such that x and $-x \in P$, and under the norm $\text{dist}^{1/2}(-x^2, P)$, B/I is a (possibly incomplete) algebra of continuous functions. There are obvious corollaries for commutative $*$-algebras.

3. Application: Let G be a locally compact Abelian group and R the Banach $*$-algebra of all bounded Radon measures m on G which are the sum of a discrete measure and a measure absolutely continuous with respect to Haar measure, under convolution and with $m^*(A) = \overline{m(A^{-1})}$ for $A \subset G$. If F_m is the Fourier transform of an $m \in R$, then $\sup | F_m(\alpha) |$ for α a continuous character of $G = \inf \| mm^* + nn^* \|^{1/2}$ for $n \in R$.

These results extend and sharpen known results of Gelfand, Naimark, Raikov, and others.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF., U. S. A.

EINE EINFACHE KLASSE LOKALKONVEXER LINEARER RÄUME

GOTTFRIED KÖTHE

In neuester Zeit haben neben den Banachschen Räumen allgemeinere lineare topologische Räume Interesse gefunden. Eine allgemeine Theorie der lokalkonvexen Räume wurde von J. Dieudonné (Ann. École Norm. (3) Bd. 59 (1942) S. 107–139) und G. W. Mackey (Trans. Amer. Math. Soc. Bd. 57 (1945) S. 155–207; Bd. 60 (1946) S. 519–537) aufgestellt. Ein grosser Teil ihrer Resultate ergibt sich durch Verallgemeinerung der Beweise der Sätze, die O. Toeplitz und ich (J. Reine Angew. Math. Bd. 171 (1934) S. 519–537) für die vollkommenen Räume bewiesen haben. Ist $\mathfrak{L}$ ein linearer Raum, dessen Elemente Vektoren $x = (x_1, x_2, \cdots)$ mit komplexen Koordinaten sind, so besteht der duale Raum $\mathfrak{L}^*$ aus allen $u = (u_1, u_2, \cdots)$ mit $\sum |u_i||x_i| < \infty$ für alle $x \in \mathfrak{L}$. $\mathfrak{L}$ ist vollkommen, wenn $\mathfrak{L} = \mathfrak{L}^{**}$. Ein vollkommener Raum $\mathfrak{L}$ und sein dualer $\mathfrak{L}^*$ bilden ein Linearsystem im Sinn von Mackey. Die Räume (l_p) und (m) sind vollkommen, aber auch der Raum der ganzen transzendenten Funktionen ist ein Beispiel. In jedem vollkommenen Raum sind die Topologien T_s (schwache), T_b (starke), und T_k erklärt, letztere durch die Umgebungen der Null $\sup_{u \in K} |ux| < \epsilon$, K eine schwach kompakte Menge in $\mathfrak{L}^*$. Jede schwache Cauchyfolge eines vollkommenen $\mathfrak{L}$ hat einen Limes in $\mathfrak{L}$, $\mathfrak{L}$ ist vollständig bezüglich T_k (jeder Cauchyfilter hat einen Limes in $\mathfrak{L}$). $\mathfrak{L}$ ist bezüglich T_k separabel, ferner dann und nur dann stark separabel, wenn in $\mathfrak{L}^*$ jede beschränkte schwach abgeschlossene Menge schwach kompakt ist. In den vollkommenen Räumen sind die schwach kompakten Mengen identisch mit den schwach folgenkompakten. Die linearen stetigen Abbildungen eines vollkommenen Raumes in sich bilden stets einen maximalen Ring im endlicher Matrizen (eine einfache Neubegründung dieser Theorie erscheint in den Math. Nachrichten Berlin). Das Interesse dieser Theorie liegt darin, dass sie einfache Beispiele nicht metrischer lokalkonvexer Räume liefert. Besonders wichtig sind hier die konvergenzfreien Räume (Math. Ann. Bd. 111 (1935) S. 229–258), in denen eine Reihe von grundlegenden Sätzen der Banachschen Theorie nicht mehr gilt. Schliesslich ist die Theorie der vollkommenen Räume die natürliche Grundlage für das Studium der Auflösung von linearen Gleichungen mit unendlich vielen Unbekannten (Math. Zeit. Bd. 51 (1948) S. 317–345, dort ist weitere Literatur zitiert).

UNIVERSITY OF MAINZ,
MAINZ, GERMANY.

SUR LES ENSEMBLES PARTIELLEMENT ORDONNÉS

GEORGE KUREPA

Soit $(S; \leq)$ un ensemble ordonné ($\equiv$ partiellement ordonné); désignons: par $O(S)$ la classe de tous les ensembles totalement ordonnés maximaux $\subsetneqq S$

et par $\check{O}(S)$ la classe de tous les ensembles $A \subseteq S$ sans points distincts comparables et tels que $A \cap M \supset v$ (vide) pour tout $M \in O(S)$.

Soit f une transformation de S (donc $f(x)$ est un ensemble vide ou non vide pour tout $x \in S$).

L'opération (A) *généralisée ou l'opération généralisée suslinienne* relativement à S et f consistera en formation de l'ensemble

$$(1) \qquad \bigcup_M \bigcap_{x \in M} f(x) \qquad (M \in O(S));$$

c'est que si S est l'ensemble des complexes finis de nombres naturels ordonnés de telle manière que, quels que soient les complexes A, B, la relation $A \leqq B$ veut dire que A est une section (portion) commençante de B et si f est telle que pour tout $x \in S$, $f(x)$ soit un ensemble fermé (ouvert) d'un espace, les ensembles (1) coincident avec les ensembles analytiques de Suslin-Luzin relativement à l'espace (C. R. Acad. Sci. Paris t. 164 (1917) pp. 88–91).

En choisissant d'autres $(S; \leqq)$ convenables on obtient de différentes cathégories d'ensembles (1). (Cf. aussi Sierpinski, Bulletin International de l'Academie Polonaise des Sciences, Cracovie (1918) pp. 161–167.)

L'opération (Γ) *généralisée ou l'opération généralisée de M. P. Alexandroff* relativement à S et f consistera à former l'ensemble

$$\bigcup_A \bigcap_{t \in A} \mathrm{C}f(t) \qquad (A \in \check{O}(S)),$$

le complément $\mathrm{C}f(t)$ étant pris relativement à un ensemble quelconque E englobant les $f(x)$, $x \in S$.

M. Alexandroff (Rec. Math. (Mat. Sbornik) N.S. t. 31 (1923) pp. 310–318; aussi Fund. Math. t. 6 (1924) pp. 160–165) a prouvé que dans le cas de S de tout à l'heure l'ensemble (2) est un complémentaire analytique.

Dans l'étude du problème de Suslin on est amené à considérer des ensembles ordonnés S vérifant la condition (C) que voici:

L'ensemble des prédécesseurs d'un point quelconque de S est totalement ordonné, c'est-à-dire ne contient aucun couple de points incomparables (cf. Kurepa, Thèse, Paris, 1935, ou Publ. Math. Belgrade t. 4 (1935) pp. 1–138 en particulier pp. 69, 127; aussi Revista de Ciencias (Lima) (1941) pp. 483–500, en particulier p. 487).

Théorème. *Quels que soient l'ensemble* $(S; \leqq)$ *et la transformation* f *de celui-ci, on aura* (2) $\subseteq$C(1) *c'est-à-dire l'ensemble* (2) *est une partie du complément de l'ensemble* (1)*; et dualement. L'égalité* (2) $=$ C(1) *et son duale auront lieu dans les trois cas suivants*: I. *S est totalement ordonné*; II. *S est totalement anti-ordonné* (*c'est-à-dire sans points comparables*); III. *S est bien ordonné et vérifie la condition* (C) *précédente.*

Dans les cas I *et* II *les égalités respectives se réduisent aux formules de De Morgan. Dans le cas* III, *la supposition que S vérifie* (C) *est essentielle.*

University of Zagreb,
Zagreb, Yugoslavia.

AXIOMATIC PERRON INVERSION

Roy B. Leipnik

We apply the following to Fubini theorems for Perron integration, sets of uniqueness of expansions, and existence of solutions of functional equations.

Let A_* and A^* be operators, upon the set S' of functions on X' to a countably complete (hence Archimedean) lattice-ordered Abelian group Y', to the set S of functions on X to a partially-ordered Abelian group Y. Denote by $A \ominus B$ the set of $f - g$ for $f \in A \subset S'$ and $g \in B \subset S'$. Let z' and z be the unit functions in S' and S. We call A_* and A^* a Newton pair whenever they satisfy: Axiom 1. $A_*(f + g) \leqq A_*(f) + A^*(g)$; and Axiom 2. There are nonvoid subsets U and L of S' such that $U \ominus L \subset E_f[z \leqq A_*(f)$ implies $z' \leqq f]$.

Using U and L for the Perron "ober" and "unter" process we define, for each $\phi \in S$, a $Q_*(\phi) \in S'$ and a $Q^*(\phi) \in S'$ and prove that $Q_*(\phi) \leqq Q^*(\phi)$.

If $A_*(f) = A^*(f)$, we call f gaugeable and define $A(f) = A_*(f)$. Note by Axiom 1 that A is convex on gaugeable functions. If $A(f) = A(g)$, we say g is equi-gaugeable with f. We prove that there is at most one $g \in U \cap L$ equi-gaugeable with f. If one such g exists, we call f centered and write $g = M(f)$.

If $Q_*(\phi) = Q^*(\phi)$, we say ϕ is invertible, and define $Q(\phi) = Q_*(\phi)$. We prove that if f is centered, then $A(f)$ is invertible, $Q(A(f)) = M(f)$, and $A(Q(A(f))) = A(f)$. We also prove that if ϕ is invertible, then $A_*(Q(\phi)) = \phi \leqq A^*(Q(\phi))$ except on a $\sigma\delta$ subset of X, which is void if A_* is bounded.

If A^* satisfies also: Axiom 3. $A^*(f + g) \leqq A^*(f) + A^*(g)$, we say A_* and A^* form a Perron pair. Axiom 1 and Axiom 3 coincide on gaugeable functions.

We then prove that if ϕ is invertible, then $Q(\phi)$ is centered, $A(Q(\phi)) = \phi$ except on a $\sigma\delta$ set which is void if A_* and A^* are bounded, and $Q(A(Q(\phi))) = Q(\phi)$.

"Vector" operators A_* and A^* satisfy the axioms if all their components do. If Y is the reals, A_* (A^*) are defined by lim inf (sup) method; then Axioms 1 and 3 hold always.

Institute for Advanced Study,
Princeton, N. J., U. S. A.

ON A CLASS OF BOUNDED MATRICES

Wilhelm Magnus

The matrices are those investigated by I. Schur (J. Reine Angew. Math. vol. 140 (1911) pp. 1–28) namely $A(\theta)$ and $H(\theta)$ with the general elements $(-n + m + \theta)^{-1}$ and $(n + m + \theta)^{-1}$ respectively, where $n, m = 0, 1, 2, \cdots$ denote the rows and columns and where θ is a real parameter. The exact boundaries for the spectrum of $H(\theta)$ for $\theta > 0$ are given, generalisations of Hilbert's inequality are proved and it is shown that every point of $(0, \pi)$ belongs to the spectrum of $H(1)$ and that the spectrum is purely continuous. It is shown that the formal inverse of $A(\theta)$ which was given by E. H. Linfoot and W. M. Shepherd, Quart. J. Math. Oxford Ser. vol. 10 (1939) pp. 84–98 is bounded if and only if $-1/2 < \theta < 1/2$. For a combination C of the identity and $H^2(1 + \theta)$ the following result is proved: C has a bounded inverse B and a formal unbounded inverse U where B and U define the same operator in a certain linear subspace of Hilbert space.

For the proofs use has been made of the fact that $H(\theta)$ can be expressed as an integral operator, and also of the Laplace transformation and of certain identities from the theory of the gamma function and the hypergeometric function.

California Institute of Technology,
Pasadena, Calif., U. S. A.

DIFFERENTIABILITY PROPERTIES OF THE SOLUTIONS OF VARIATIONAL PROBLEMS FOR MULTIPLE INTEGRALS

Charles B. Morrey, Jr.

Let $z_0(x)$ $(x = (x^1, \cdots, x^\nu), z = (z^1, \cdots, z^N)$ be a vector function of class C' on the closure $\bar{G}$ of a domain G and suppose that $f(x, z, p)$ $(p = p^i_\alpha, \alpha = 1, \cdots, \nu, i = 1, \cdots, N)$ is of class C'' in its arguments and satisfies the *condition of regularity*

$$f_{p^i_\alpha p^j_\beta} \lambda_\alpha \lambda_\beta \xi^i \xi^j > 0$$

for all (x, z, p) in a domain D containing all the points $(x, z_0(x), p_0(x))$ for $x \in \bar{G}$ and all λ and ξ for which $\lambda_\alpha \lambda_\alpha > 0$ and $\xi^i \xi^i > 0$ (a repeated index is summed, the Greek indices running from 1 to $\nu \geqq 2$ and the Roman from 1 to $N \geqq 1$). Let $I(z)$ denote the multiple integral of f over G. Suppose that the first variation $I(z_0, \zeta)$ of $I(z)$ is zero for all functions ζ of class C' on $\bar{G}$ which vanish on the boundary G^*.

Under the assumptions above, the following further differentiability properties of z_0 are proved:

(i) Each $z^i_{0x^\alpha}$ is of class $\mathfrak{P}_2$ on each domain Δ with $\bar{\Delta} \subset G$ and each $z^i_{0x^\alpha}$ satisfies a uniform Hölder condition with constant $L(\mu)$ and exponent μ for each μ, $0 < \mu < 1$, on each such Δ; moreover the Euler differential equations are satisfied almost everywhere.

(ii) If f is of class $C^{(n)}$, $n \geqq 2$, and its nth derivatives all satisfy uniform Hölder conditions with exponent μ, $0 < \mu < 1$, on D, then z is of class $C^{(n)}$ on G and its nth derivatives satisfy uniform Hölder conditions with exponent μ on each domain Δ with $\bar{\Delta} \subset G$.

It is first proved that z_0 satisfies Haar's equations on G. Using a device of Liechtenstein, it is found that difference quotients of z_0 satisfy linear systems of Haar's equations. The study of these is begun with the special case arising when f is of the form

$$\cdot f(x, z, p) = a_{ij}^{\alpha\beta} p_\alpha^i p_\beta^j, \qquad a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha} = a_{ij}^{\beta\alpha} = a_{ji}^{\alpha\beta}$$

in which the $a_{ij}^{\alpha\beta}$ are constants satisfying the regularity condition

$$m\lambda_\alpha\lambda_\alpha\xi^i\xi^i \leqq a_{ij}^{\alpha\beta}\lambda_\alpha\lambda_\beta\xi^i\xi^j \leqq M\lambda_\alpha\lambda_\alpha\xi^i\xi^i, \qquad 0 < m \leqq M.$$

In this case it is shown that

$$mD_2(z, G) \leqq I(z, G) \leqq MD_2(z, G), \qquad D_2(z, G) = \int_G z_{x^\alpha}^i z_{x^\alpha}^i \, dx,$$

for all vectors z of class $\mathfrak{P}_2$ on G *which vanish on* G^*. Using this striking fact, the Dirichlet problem and related nonhomogeneous problems are solved uniquely. The more general linear equations are treated by operator theory. Boundedness properties are established on interior regions which allow the passage to the limit in the difference quotients for z_0. Repetition of the process yields the higher differentiability properties of z_0 corresponding to those for f.

University of California,
Berkeley, Calif., U. S. A.

ON THE CONTINUITY OF POSITIVE LINEAR TRANSFORMATIONS

Leopoldo Nachbin

A topological ordered vector space is a locally convex real topological vector space which is also an ordered vector space such that (1) the cone of all positive elements is closed and (2) given any neighborhood V of 0, there is another neighborhood W of 0 such that $0 \leqq x \leqq y \in W$ imply $x \in V$. Theorem 1: Let $\mathfrak{E}$ be a topological ordered vector space such that (1) $\mathfrak{E}$ has a countable base of neighborhoods at 0 and is complete in the Cauchy sense, and (2) every element of $\mathfrak{E}$ can be expressed as the difference of two positive elements. Then, for any topological ordered vector space $\mathfrak{X}$, every positive linear transformation from $\mathfrak{E}$ into $\mathfrak{X}$ is continuous. Let A be a completely regular space: if A is complete under the weakest uniform structure with respect to which all real-valued continuous functions on A are uniformly continuous, we say that A is saturated. A regular Hausdorff space in which every open covering contains a countable subcovering is normal saturated. Let $\mathfrak{C} = \mathfrak{C}(A)$ be the topological ordered vector space of all real-valued continuous functions on A. Theorem 2: Given A, then for every topological ordered vector space $\mathfrak{X}$, all positive linear transformations from $\mathfrak{C}$ into $\mathfrak{X}$ are continuous if and only if A is saturated. Corollary: If A is saturated, $\mathfrak{X}$ is normed, and $\Phi:\mathfrak{C} \rightarrow \mathfrak{X}$ is positive linear, there is a compact set $K \subset A$ such that $\Phi(f) = 0$ whenever $f \in \mathfrak{C}$ and $f(x) = 0$ for every $x \in K$; if A is not saturated, this may be false even for functionals Φ. If $\mathfrak{E}$ is an ordered vector space, among all topologies making $\mathfrak{E}$ into a topological ordered vector space, there is a strongest one: call it the natural topology of $\mathfrak{C}$. Theorems 1 and 2 can then be restated as follows. If $\mathfrak{C}$ is as in Theorem 1, then the natural topology of $\mathfrak{C}$ as an ordered vector space is the topology already given on $\mathfrak{C}$. The natural topology of $\mathfrak{C}$ as an ordered vector space is the compact open topology if and only if A is saturated.

UNIVERSITY OF CHICAGO,
CHICAGO, ILL., U. S. A.

SPACES OF INTEGRAL FUNCTIONS

A. C. OFFORD

It was shown by Fréchet (*Espaces abstraits*, p. 87) that the integral functions form a metric space in which the distance between any two functions $f(z)$ and $g(z)$ can be defined by

$$(f, g) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(f, g)_n}{1 + (f, g)_n}$$

where $(f, g)_n = \sup_{|z| \leqq n} |f(z) - g(z)|$.

I have shown that for the functions of this space, with the exception of those of a set of the first category, there will be certain relations between the distribution of their a-values and their order of magnitude for large values of $|z|$. It is not possible to state precisely in a few words what these relations are, but they can be described roughly in the following way.

A function $f(z)$ is said to behave typically in the annulus $n \leqq |z| \leqq n + 1$ if, broadly speaking, it has all the properties described in Theorems 3 and 4 of the paper by Littlewood and Offord (Ann. of Math. vol. 49 (1948) pp. 885–952 (888)) within this annulus. The functions $N(R)$ and $m(R)$ of Littlewood and Offord will, of course, now depend on the particular function $f(z)$, and it is necessary to make a simple extension to functions of infinite order of the properties described by Littlewood and Offord for functions of finite order. Now let $\nu(f)$ denote the number of annuli for which $n \leqq \nu$ and in which a given function $f(z)$ fails to have this typical behaviour. Let $\phi(\nu)$ be any monotone increasing function which tends to infinity with ν. Then the set $E(\phi)$ of functions $f(z)$ for which $\nu(f)/\phi(\nu)$ tends to infinity with ν is a set of the first category.

BIRKBECK COLLEGE, UNIVERSITY OF LONDON,
LONDON, ENGLAND.

WIENER INTEGRALS OF nTH VARIATIONS

MARGARET OWCHAR

Let C be the space of real continuous functions $x(t)$ defined on $0 \leqq t \leqq 1$ such that $x(0) = 0$. Let the nth variation of a functional $F(x)$ with respect to $y_1(t), \cdots, y_n(t) \in C$ be defined as

$$\delta^{(n)}F(x \mid y_1, \cdots, y_n) = \frac{d}{dh}\,\delta^{(n-1)}F(x + hy_n \mid y_1, \cdots, y_{n-1})\Big|_{h=0}.$$

Under appropriate conditions on $F(x)$, on the n variations, $\delta^{(i)}F(x \mid y_1, \cdots, y_i)$, $i = 1, 2, \cdots, n$, and on $y_1(t), \cdots, y_n(t)$, with respect to which these variations are taken, we prove that the Wiener integral of the nth variation is equal to a sum of Wiener integrals of products of $F(x)$ and integrals of the type $\int_0^1 y_i'(t)\, dx(t)$, with constants involving products of integrals of $y_j'(t)y_k'(t)$ with respect to t. This result yields formulas for the integration of certain functionals over C, and also gives an expression for the Fourier-Hermite coefficient, in the expansion of a functional in $L_2(C)$, in terms of $\int_c^w \delta^{(n)}F\,(x \mid y_1, \cdots, y_n)\, d_w x$. It is also proved that under conditions which imply the existence of the first n Volterra derivatives, $F^{(i)}(x \mid t_1, \cdots, t_i)$, $i = 1, 2, \cdots, n$,

$$\int_c^w F^{(n)}(x \mid t_1, \cdots, t_n)\, d_w x = (-2)^n \frac{\partial^{2n}}{\partial t_n^2 \cdots \partial t_1^2} \int_c^w F(x) \prod_{i=1}^{n} [x(t_i)]\, d_w x$$

for almost all $\{t_i\}$ on $[0, 1]$.

UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINN., U. S. A.

A GENERAL SPECTRAL THEORY

R. S. PHILLIPS

Let R be the ring of all bounded linear transformations on a separable Hilbert space into itself. The traditional approach to the spectral problem begins with a non-negative ring homomorphism T_1 on some subring M_1 (with unit) of the ring M of real bounded functions on an abstract set Σ to R. It is then required to obtain a realization of $T_1(f)$ on M_1 as a Riemann-Stieltjes or Lebesgue-Stieltjes integral $T_1(f) = \int f(s)\, dE$, where $E(\sigma)$ is a finitely or completely additive set function on a family of subsets of Σ with values in an abelian set of projection operators such that $E(\sigma_1)\, E(\sigma_2) = 0$ if $\sigma_1 \cap \sigma_2 = 0$. A solution to this problem is obtained by first showing that any real abelian subset S of R is contained in a complete lattice abelian subring $H(S)$ of R; here $H(S)$ is the real part of $(S')'$ where A' consists of all elements of R which commute with each element of A. It is thus possible to extend T_1 to be a non-negative and linear transformation T on M to $H(T_1(M_1))$. It is shown that the class of sets σ, whose characteristic functions are mapped by T into projection operators, is sufficiently large to define the Riemann-Stieltjes integral $\int f(s)\, dE$ on a subring of M containing M_1. This leads to a "Riemann" extension of the non-negative ring homomorphism T_1. If Σ is a locally bicompact Hausdorff space and if we are concerned only with the subring M_0 of continuous functions in M_1 vanishing outside of compact subsets of Σ, then a completely additive measure function $E(\sigma)$ can be found such that the Lebesgue-Stieltjes integral of the measurable functions of M gives a "Lebesgue" extension of T_0 (retraction of T_1 to M_0). This theorem may be applied to obtain the usual spectral theorems for bounded real operators, unitary operators, abelian sets of bounded normal operators, and one-parameter semigroups of bounded real or unitary operators.

UNIVERSITY OF SOUTHERN CALIFORNIA,
LOS ANGELES, CALIF., U. S. A.

ON THE CHARACTERIZATION OF THE ROWS AND COLUMNS OF THE REPRESENTATIONS OF THE SEMI-SIMPLE LIE GROUPS

GIULIO RACAH

1. If $\sum_{q=1}^{r} e^q X_q$ is the general infinitesimal element of a semi-simple Lie group, the problem is considered of characterizing its irreducible representations by the eigenvalues of a set of operators $f(X_q)$, which are commutative with all the operators of the group. A particular operator of this kind was given by Casimir [H. Casimir, Proceedings of the Royal Society of Amsterdam vol. 34 (1931) p. 844].

A correspondence was found between these operators and the invariants of the adjoint group, and an integrity basis was constructed, both for these operators and for the invariants of the adjoint group. If the rank of the group is l, this integrity basis consists of l independent operators, the eigenvalues of which are sufficient for characterizing the different irreducible representations of the group [G. Racah, Rendiconti Lincei vol. 8 (1950) p. 108].

2. The problem is also considered of characterizing the different rows and columns of an irreducible representation by the eigenvalues of a commutative set of operators $k(X_q)$. It is known that the weights of Cartan are not sufficient for this purpose, and it is shown that the eigenvalues of at least $(r - 3l)/2$ additional operators are needed. In some cases such a number of additional commutative operators was actually found.

HEBREW UNIVERSITY,
JERUSALEM, ISRAEL.

ALGEBRAIC FUNCTIONS IN AN ABELIAN NORMED RING

R. K. RITT

Let R be an Abelian normed ring, with zero radical; let $K = R(e, a)$, the subring of R generated by the identity, e, and an arbitrary element, a, of R. Let $\sigma(a)$ be the spectrum of a relative to K. We consider the equation (1) $u^n = b$. (a) If for $b = a$, and for some integer $n > 1$, there exists a u in K which satisfies (1), $\sigma(a)$ does not separate a neighborhood of the origin from a point exterior to the circle $|z| = ||a||$. (b) If for every z in $\sigma(a)$, and $b = a - ze$, (1) has a solution in K, $\sigma(a)$ is nowhere dense. (c) If for every b in K, (1) has solutions, $\sigma(a)$ does not separate the plane and is nowhere dense.

K is said to be conservative if for every sequence of polynomials, $p_n(z)$, which converges uniformly on $\sigma(a)$, the sequence $p_n(a)$ converges in the ring topology. (d) If K is conservative and if $\sigma(a)$ is homeomorphic to a denumerable number of Jordan arcs, K is identical to the ring of continuous functions on the unit interval, and then contains a solution of (1) for all n and all b in K. (e) If K is conservative and $\sigma(a)$ does not separate the plane and is nowhere dense, K is the ring of continuous functions on $\sigma(a)$, but (1) may not have solutions. As a typical application, the existence of "roots" of normal transformations in a Hilbert space can be shown to depend on their spectra.

By using an implicit function theorem the elements of R for which (1) has solutions can be shown to form an open set, B. It is then possible to think of (1) as defining an "algebraic function" which is analytic (in the sense of Lorch) on B. The "Riemann surface" of such a function may have a nondenumerable number of "sheets".

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

LEAST SQUARE ERROR AND VARIANCE

ARTHUR SARD

Let $m(x)$ be a nondecreasing function defined for all real x. Let $\mathcal{H}$ be the space of those complex functions $f(x)$ of the real variable x which are measurable relative to $m(x)$ and such that the Lebesgue-Stieltjes integral $\int_{-\infty}^{\infty} |f(x)|^2 \, dm(x)$ is finite, two functions being equivalent if equal almost everywhere $m(x)$. For $f, g \in \mathcal{H}$, put $(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dm(x)$ and $||f|| = (f, f)^{1/2}$.

A *functional* ϕ is *admissible* if, for $f, g \in \mathcal{H}$, ϕf is a complex number, $\phi(f + g) = \phi f + \phi g$, and $\phi f \to 0$ as $||f|| \to 0$. An *operator* U is *admissible* if, for $f, g \in \mathcal{H}$, $Uf \in \mathcal{H}$, $U(f + g) = Uf + Ug$, and $||Uf|| \to 0$ as $||f|| \to 0$. If U is an admissible operator, U^* is the admissible operator such that $(Uf, g) = (f, U^*g)$, $f, g \in \mathcal{H}$. An operator U is *non-negative* if U is admissible, $U = U^*$, and $(Uf, f) \geqq 0$, $f \in \mathcal{H}$; U is *positive* if, in addition, $(Uf, f) = 0$ only when $||f|| = 0$.

In each of the three problems described below an f in $\mathcal{H}$ is considered and an approximation g of f is to be determined. The error δf in f is assumed to be a stochastic process such that: 1) δf is a function of ω, x, $\omega \in \Omega$, measurable relative to $p(\omega)m(x)$, where Ω is a probability space with mass function $p(\omega)$. 2) $E(\| \delta f \|^2) < \infty$, where $Ez = \int_\Omega z\, dp(\omega)$. 3) $\| \overline{E\delta f} \| = 0$. Let $Vf = \int_{-\infty}^{\infty} v(x, y)f(y)\, dm(y)$, $f \in \mathcal{H}$, where $v(x, y) = E[\delta f(x)\overline{\delta f(y)}]$. Then the operator V is non-negative. Assume further: 4) V is positive.

Let Π be the space spanned by n fixed elements h_i of $\mathcal{H}$. (Here and elsewhere $i, j = 1, \cdots, n$.) Let W be a fixed non-negative operator such that the $n \times n$ matrix $((Wh_i, h_j))$ is nonsingular.

THE LEAST SQUARE PROBLEM. To determine n constants α_i so as to minimize $(W(f - g), f - g)$, where $g = \sum_i \alpha_i h_i$.

THE CURVE FITTING PROBLEM. To determine n admissible functionals ϕ_i, independent of f, such that $E(g + \delta g) = f$ whenever $f \in \Pi$ and such that the total variance t_α is minimal, where $\alpha_i = \phi_i f$, $\delta\alpha_i = \phi_i \delta f$, $g = \sum_i \alpha_i h_i$, $\delta g = \sum_i \delta\alpha_i h_i$, $t_\alpha = E\sum_i | \delta\alpha_i |^2$.

THE APPROXIMATION PROBLEM. To determine an admissible operator C, independent of f, such that $E(g + \delta g) = f$ whenever $f \in \Pi$ and such that the total variance t_g is minimal, where $g = Cf$, $\delta g = C\delta f$, $t_g = E(\| \delta g \|^2)$.

The least square problem has a unique solution. The curve fitting and approximation problems are equivalent: If either has a solution, the other does also; both solutions are unique and lead to the same g for any f in $\mathcal{H}$.

THEOREM. *All three problems are equivalent if and only if* $VWh_i \in \Pi$, $i = 1, \cdots, n$.

Heretofore only the following, as applied to the conventional case, has been known: All three problems are equivalent if $W = V^{-1}$ [A. C. Aitken 1945; Gauss].

QUEENS COLLEGE,
FLUSHING, N. Y., U. S. A.

ON THE NUMBER OF VARIATIONS OF SIGNS IN A SEQUENCE OF LINEAR FORMS

I. J. SCHOENBERG

Let (1) $y_i = a_{i1}x_1 + \cdots + a_{in}x_n$ $(i = 1, \cdots, m)$ be a given real linear transformation. Let r be the rank of its matrix $A = \| a_{ik} \|$ and let $v(y)$ denote the number of variations of signs in the sequence $\{y_i\}$. Let $s = \sup v(y)$ for all choices of $\{x_k\}$. Clearly $s \geqq r - 1$, for certain r among the y_i are independent and can be given alternating signs. The main result is *Theorem* 1. *We have that* $\sup v(y) = r - 1$ *if and only if A enjoys the following property: A does not have two minors of order r coming from the same r columns of A and having opposite signs*. As an application of this theorem a new proof is derived for a theorem of Th. Motzkin giving necessary and sufficient conditions in order that the transformation (1) should have the property that the inequality $v(y) \leqq v(x)$ always holds. Moreover, Theorem 1 admits a geometric interpretation concerning polygonal lines in n-dimensional space which opens the possibility of a new analytic approach to the theory of curves of order n in the sense of Juel, Haupt, Scherk, and others.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA., U. S. A.

THE SECOND ADJOINT OF A C^* ALGEBRA

S. SHERMAN

Let $\mathfrak{A}$ be a C^* algebra (in the sense of Segal) with unit, i.e., a uniformly closed, self-adjoint, complex algebra of bounded linear operators on a complex Hilbert space. If we consider $\mathfrak{A}$ as a Banach space, then $\mathfrak{A}^{**}$ its second adjoint is isomorphic to the Banach space of a weakly closed (in the sense of von Neumann) algebra of operators with unit, i.e., of a ring of operators (in the sense of von Neumann) on some complex Hilbert space (possibly nonseparable) and so of a C^* algebra. In the case that $\mathfrak{A}$ is commutative, then $\mathfrak{A}$ is isomorphic (algebraically and metrically) to $C(X)$ the algebra of all continuous complex-valued functions (with the usual metric) on compact, Hausdorff space X and the theorem of the above sentence is consistent with a result of Kakutani which states that $C^{**}(X)$ is isomorphic, as a Banach space, to $C(Y)$, where Y is a totally disconnected, compact, Hausdorff space. The fact that Y is *totally disconnected* corresponds crudely to the fact that $\mathfrak{A}^{**}$ is isomorphic to the Banach space of a *weakly closed* algebra of operators. In the case that $\mathfrak{A}$ is the C^* algebra of all completely continuous operators on a complex Hilbert space, then by the work of Schatten, *The cross space of linear transformations*, Ann. of Math. vol. 47 (1946) pp. 73–84; von Neumann and Schatten, *The cross space of linear transformations* II, Ann. of Math. vol. 47, (1946) pp. 608–630; and Dixmier, *Les functionelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert*, Ann. of Math. vol. 51 (1950) pp. 387–408; we know that $\mathfrak{A}^{**}$ is isomorphic as a Banach space to $\mathfrak{B}$ the weakly closed algebra of all bounded linear operators on our complex Hilbert space, again a situation consistent with the theorem. We may regard our theorem as a noncommutative generalization of Kakutani's theorem.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

ON VARIATIONAL ANALYSIS IN THE LARGE

MAX SHIFFMAN

A theory of critical points of a function on a manifold, and of extremals of single integral variational problems has been developed by M. Morse in his theory of the calculus of variations in the large. The basic analytical steps are however not applicable to variational problems involving multi-dimensional varieties. In a single integral problem the Euler-Lagrange equation is an ordinary differential equation, while for a multi-dimensional problem it is a partial differential equation. The use and simplicity of the theory of ordinary differential equations is therefore no longer possible. Also, there is a basic geometric method used, namely, a given curve is decomposed by a finite number of points into small portions, and then these points varied, leading directly to a reduction to a finite number of variables. This is no longer possible for multi-dimensional varieties.

In a note in Proc. Nat. Acad. Sci. U. S. A. (1948), the author sketched a theory for single integral problems, which was, however, in principle applicable to a variational problem of any dimension. This theory consists in considering in addition a finite number of certain linear integral expressions. There is a unique solution to the variational problem subject to the side conditions that these linear integral expressions have specified values. By then varying these given values, the reduction of the problem to one depending on a finite number of real variables is obtained, and the topological methods shown to be applicable. Of great importance in this theory is the establishment of the positivity of the Weierstrass E-function for an integrand essentially formed from the original integrand by taking its first variation. We carry through the above theory for double integral variational problems.

Mention should also be made of the special theory for the problem of unstable minimal surfaces bounded by a given closed curve in space, which was developed by the author, by Morse and Tompkins, and by Courant.

STANFORD UNIVERSITY,
STANFORD, CALIF., U. S. A.

THE SPECTRUM AND THE OPERATIONAL CALCULUS FOR A FAMILY OF OPERATORS

M. H. STONE

Let X be an ordered algebra with elements $x, y, z, \cdots$ which has an ordinally and algebraically isomorphic representation as an algebra of bounded real functions on a suitable domain. The theory of compactification shows that there is no loss of generality in requiring the domain to be a compact space and the representative functions to be continuous, as we shall do here. X may be given or interpreted as an algebra of operators. If $\{x_\alpha\}$ is a family of elements of X and $\{f_\alpha\}$ the corresponding family of representative functions, we define the *spectrum* of that family to be the range of the mapping which carries the general point p of the domain of the representative functions into the point $\{f_\alpha(p)\}$ in the topological product of replicas of the real number-system, one for each index α. The spectrum is topologically a compact subspace of this product-space. *This definition is independent of the particular representation considered.* The spectrum of an element x, as usually defined, is identical with the spectrum of the family consisting of that element alone, as here defined. If φ is any continuous real function defined on the spectrum of $\{x_\alpha\}$, we define $\varphi(\{x_\alpha\})$ to be that element in X(if any exist) which is represented by the continuous function $\varphi(\{f_\alpha\})$. *This definition is independent of the particular representation considered.* It suffices for the elaboration of an operational calculus of the usual type. Extensions of this calculus to the case of bounded, and even nonbounded, functions of Baire can be made in certain cases—in particular to the case where X is an algebra of commuting self-adjoint operators. (Cf. M. H. Stone, Proc. Nat. Acad. Sci. U. S. A. vol. 26 (1940) pp. 280–283; Canadian Journal of Mathematics vol. 1 (1949) pp. 176–186.)

University of Chicago,
Chicago, Ill., U. S. A.

NOTE ON A FUNCTIONAL EQUATION

H. P. Thielman

The functional equation

$$\phi[F(x, y)] = \phi(x) + \phi(y) \tag{1}$$

is considered. When $F(x, y) = x + y$ or xy we have the classical cases of Cauchy's functional equations. The case when $F(x, y)$ is a general linear function, and also the case when it is $x + y + nxy$ have been considered in recent mathematical literature [J. Aczél, Comment. Math. Helv. vol. 21 (1948) pp. 247–252; H. P. Thielman, Amer. Math. Monthly vol. 56 (1949) pp. 452–457]. In this paper the case when $F(x, y)$ is a polynomial is settled by the following theorem: *If $F(x, y)$ is a real polynomial of degree greater than unity for which the functional equation* (1) *has a real continuous monotone increasing solution, then $F(x, y)$ must be of the form*

$$F(x, y) = \frac{(ax + b)(ay + b) - b}{a}, \qquad a > 0.$$

In this case every real, continuous solution (defined for all $x > -(b/a)$) of the functional equation (1) *is of the form* $\phi(x) = k \log (ax + b)$ *where* k *is an arbitrary real constant.*

Applications of this result are made to commutative functions.

IOWA STATE COLLEGE,
AMES, IOWA, U. S. A.

A SET OF UNITARY REPRESENTATIONS OF THE GROUP OF CONTACT TRANSFORMATIONS

L. VAN HOVE

We consider the group Γ of one-to-one, infinitely differentiable point transformations of the space $(s, p_1, \cdots, p_n, q_1, \cdots, q_n)$, leaving the differential form $ds - \sum_1^n p_j\, dq_j$ invariant. These transformations being measure-preserving, a unitary representation of Γ is provided by

$$U_\gamma \varphi(s, p, q) = \varphi(s', p', q'),$$

where γ is an element of Γ and φ is square-integrable. The point (s', p', q') is the transformed of (s, p, q) by γ^{-1}. By Fourier transformation in s, the Hilbert space $\mathfrak{H}$ of the φ can be reduced into a direct integral $\mathfrak{H} = \int_{-\infty}^{\infty} \mathfrak{H}^{(\alpha)}(d\alpha)^{1/2}$ (J. von Neumann, Ann. of Math. vol. 50 (1949) p. 401) of Hilbert spaces $\mathfrak{H}^{(\alpha)}$, the elements of which are square-integrable functions of p, q. Each $\mathfrak{H}^{(\alpha)}$ is invariant for the U_γ, which are thus direct integrals $\int U_\gamma^{(\alpha)}$. For each value of α, the $U_\gamma^{(\alpha)}$ give a unitary representation $R^{(\alpha)}$ of Γ.

The infinitely differentiable functions $f(p, q)$ are in one-to-one correspondence with the infinitesimal transformations of Γ, and hence with the self-adjoint operators $H^{(\alpha)}[f]$ representing the latter in $R^{(\alpha)}$. This correspondence satisfies

$$H^{(\alpha)}[f_1]H^{(\alpha)}[f_2] - H^{(\alpha)}[f_2]H^{(\alpha)}[f_1] = -iH^{(\alpha)}[(f_1, f_2)],$$

where (f_1, f_2) denotes the Poisson bracket. Furthermore $H^{(\alpha)}[1] = \alpha$.

Assume $\alpha \neq 0$. Let $\mathfrak{A}$ be the weakly closed algebra of bounded operators in $\mathfrak{H}^{(\alpha)}$ generated by the bounded functions of the $2n$ operators $H^{(\alpha)}[p_j]$, $H^{(\alpha)}[q_j]$, $(j = 1, \cdots, n)$. $\mathfrak{A}$ is isomorphic to the algebra of operators in a Hilbert space $\mathfrak{K}$ of square-integrable functions of n variables, $H^{(\alpha)}[q_j]$ corresponding to multiplication by the jth variable, and $(1/i)H^{(\alpha)}[p_j]$ to partial differentiation. The commutator $\mathfrak{A}'$ of $\mathfrak{A}$ in $\mathfrak{H}^{(\alpha)}$ has the same structure, and each operator in $\mathfrak{H}^{(\alpha)}$ is a function of elements in $\mathfrak{A}$ and $\mathfrak{A}'$.

In particular, the operators $H^{(\alpha)}[f(q)]$ (f arbitrary function of q) are expressible by means of $H^{(\alpha)}[q_j]$, Q_j', $(j = 1, \cdots, n)$, where the Q' are n commuting elements of $\mathfrak{A}'$. Similarly, the $H^{(\alpha)}[f(p)]$ are functions of $H^{(\alpha)}[p_j]$, P_j', where the P' are again commuting elements of $\mathfrak{A}'$. The isomorphism of $\mathfrak{A}'$ with the algebra of operators in $\mathfrak{L}$ can be made such that Q'_j corresponds to multiplication by the jth variable, and $(1/i)P'_j$ to partial differentiation.

Let A be an operator in $\mathfrak{H}^{(\alpha)}$ commuting with all operators of the representation $R^{(\alpha)}$. It belongs to $\mathfrak{A}'$ and commutes with Q'_j, P'_j $(j = 1, \cdots, n)$; hence it is a multiple of identity. The irreducibility of $R^{(\alpha)}(\alpha \neq 0)$ follows. The representation $R^{(0)}$, which has an entirely different structure, is also irreducible.

UNIVERSITY OF BRUSSELS,
BRUSSELS, BELGIUM.

ÜBER DIE EIGENWERTAUFGABEN MIT REELLEN DISKRETEN EIGENWERTEN

HELMUT WIELANDT

Bei den Eigenwertaufgaben der Analysis befindet man sich oft in der folgenden Lage: Es ist leicht zu zeigen, dass es keinen komplexen Eigenwert und keinen Häufungspunkt von Eigenwerten gibt, dass also alle möglicherweise vorhandenen Eigenwerte reell und diskret sind; es ist dagegen schwer zu zeigen, dass es überhaupt Eigenwerte gibt. Der Wunsch, in dieser Richtung Sätze von möglichst umfassendem Geltungsbereich aufzustellen, hat u. a. die Entwickelung der Theorie der vollstetigen quadratischen Formen von unendlich vielen Veränderlichen und der vollstetigen hermiteschen Operatoren veranlasst. Jedoch passen alle diese allgemeinen Theorieen bisher nicht beispielsweise für die selbstadjungierten gewöhnlichen Differentialgleichungen der Gestalt $My = \lambda Ny$. Die Schwierigkeit liegt in dem Nachweis, dass die Voraussetzung der Vollstetigkeit erfüllt ist. Es lohnt daher, nach Existenz-und Entwickelungssätzen zu suchen, die frei von topologischen Voraussetzungen sind und z.B.im Fall der erwähnten Differentialgleichungen ohne weiteres angewandt werden können.

Ein solcher Satz kann auf elementar funktionentheoretischem Wege in Anlehnung an den Beweis des Spektralsatzes von Doob und Koopman bewiesen werden. Er lautet: $\mathfrak{R}$ sei ein komplexer linearer Raum mit Skalarprodukt (f, g), $(f, f) > 0$ für $f \neq 0$. L sei eine Abbildung von ganz $\mathfrak{R}$ in sich mit $L(\alpha f + \beta g) = \alpha Lf + \beta Lg$ und $(Lf, g) = (f, Lg)$. Für jede komplexe Zahl λ sei die Maximalzahl $v(\lambda)$ linear unabhängiger Lösungen von $f = \lambda Lf$ endlich, und es gebe $v(\lambda)$ lineare Bedingungen für h, deren Erfülltsein für die Lösbarkeit von $g - \lambda Lg = h$ hinreicht. Die "Eigenwerte" λ mit $v(\lambda) > 0$ seien diskret (sie sind von selbst reell). Unter diesen Voraussetzungen gibt es so viele Eigenlösungen, dass jedes in der Form $f = Lg$ darstellbare Element von $\mathfrak{R}$ in eine Reihe nach ihnen entwickelt werden kann (wobei die Konvergenz im Sinne der Metrik $\|f\|^2 = (f, f)$ zu verstehen ist), und die Eigenwerte besitzen die üblichen Extremaleigenschaften. Diese Behauptungen gelten insbesondere für Integralgleichungen, die in einem sehr weiten Sinn symmetrisierbar sind. Z.B.darf man für $\mathfrak{R}$ den Raum $L^2(0, 1)$ und für L einen Integraloperator mit beliebigem quadratisch integrablem Kern, wählen, welcher nur bezüglich irgend eines in $\mathfrak{R}$ erklärten Skalarproduktes symmetrisch sein muss. (Ausführliche Darstellung in Math. Nachrichten Berlin Bd. 4 (1951) pp. 308–314.)

UNIVERSITY OF MAINZ,
MAINZ, GERMANY.

PERTURBATION OF ANALYTIC OPERATORS

FRANTIŠEK WOLF

$A(\lambda)$ will denote a bounded operator in Banach space which depends analytically on the complex parameter λ, $A(\lambda) = \sum_0^\infty A_k\lambda^k$. A_0 will be supposed to have an isolated point μ_0 in its spectrum $\sigma(A_0)$. Then, for small enough λ the following holds. There exists a bounded idempotent analytic operator $E(\lambda)$ with range $\mathfrak{M}(\lambda)$ such that (i) $\mathfrak{M}(0)$ is the eigenspace of A_0 at μ_0, (ii) $A(\lambda)E(\lambda) = E(\lambda)A(\lambda)$, (iii) the spectrum of $A(\lambda)$ in $\mathfrak{M}(\lambda)$, $\sigma(A(\lambda); \mathfrak{M}(\lambda))$ is identical with $\sigma(A(\lambda))$ in the neighborhood of μ_0. If A_0 in $\mathfrak{M}(0)$ is a scalar operator, and if $\sigma(E(0)A_1E(0); \mathfrak{M}(0))$ has n disconnected components σ_k, then $E(\lambda) = \sum_1^n E_k(\lambda)$. $E_k(\lambda)$ are again bounded analytic idempotents with range $\mathfrak{M}_k(\lambda)$, commutative with $A(\lambda)$ such that $\sigma(A(\lambda); \mathfrak{M}_k(\lambda))$ is a portion of $\sigma(A(\lambda))$ near μ_0 which behaves asymptotically like the set $\{\mu \mid (\mu - \mu_0)/\lambda \in \sigma_k\}$. If σ_k is a single point and $E(0)A_1E(0)$ in $\mathfrak{M}_k(0)$ is a scalar, then this procedure of splitting the analytic eigenspace $\mathfrak{M}_k(\lambda)$ can be continued further and $\sigma(E_k(0)A_2E_k(0))$ will be involved. In all cases we have $\lim_{\lambda\to 0} \dim \mathfrak{M}_k(\lambda) = \dim \mathfrak{M}_k(0)$. If $\dim \mathfrak{M}_k(0) = 1$, then $\sigma(A(\lambda); \mathfrak{M}_k(\lambda))$ consists of a single point which depends analytically on λ. These results properly combined can serve to prove Rellich's result on the perturbation of self-adjoint operators in Hilbert space. They also suggested that the same result may hold even for normal operators. This has been proved recently by S. L. Jamison (Ph.D. Thesis, 1950, University of California in Berkeley). The case when A_0 in $\mathfrak{M}(0)$ is not a scalar has also been investigated. Without loss of generality we may suppose $\mu_0 = 0$. Then A_0 is quasi-nilpotent. Suppose it is nilpotent and $A_0^{\mathfrak{M}} = \emptyset$. Then $A^{\mathfrak{M}}(\lambda) = \sum_0^\infty B_k\lambda^k$ satisfies the conditions of the theorem and will generate idempotents commutative with $A(\lambda)$ itself. The possibility of splitting $E(\lambda)$ will exist if $\sigma(E(0)B_1E(0))$ has disconnected components. It has been proved that this spectrum is n-fold. It is still an open question whether all analytic idempotents generated by $A(\lambda)$ can be obtained in this way.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF., U. S. A.

GENERALIZED PARAMETRIC SURFACES

L. C. YOUNG

Our object, as in "generalized curves", is to ensure that solutions exist for variational problems hitherto beyond our reach. New definitions of equivalence and convergence of surface-representations $x(u, v)$ replace Fréchet's.

We use x, J for vectors in m and $m(m - 1)/2$ dimensions, (u, v) for a point of $R \equiv [0 \leqq u \leqq 1, 0 \leqq v \leqq 1]$. The $x(u, v)$ are to be sufficiently elementary, e.g. Dirichlet representations, so that the Jacobian $J(u, v)$ exists almost everywhere. We write F for the space of continuous functions $f = f(x, J)$ such that $f(x, kJ) = kf(x, J)$ for $k \geqq 0$. With $x(u, v)$ we associate in F the linear functional

$$L(f) = \iint_R f\{x(u, v), J(u, v)\}\, du\, dv. \tag{1}$$

Equivalence of two $x(u, v)$ shall mean that they define the same linear functional $L(f)$. Convergence of a sequence $\{x_n(u, v)\}$ shall mean convergence for each $f \in F$ of the associated linear functionals $L_n(f)$. Since equivalent $x(u, v)$ are regarded as defining a same surface, we speak of an elementary surface $L(f)$ when $L(f)$ has the form (1); similarly with convergence of sequences of elementary surfaces $L_n(f)$. The limit $L(f)$ of such a sequence is termed generalized parametric surface: for a linear $L(f)$ to be this, it is necessary and sufficient that $L(f) \geqq 0$ when $f \geqq 0$.

Given a set E of elementary surfaces, let the set of limits of its convergent subsequences be $\bar{E}$. Classical variational problems concern, for a fixed $f_0 \in F$, the infimum μ of $L(f_0)$ for surfaces $L(f) \in E$. We term generalized problem that of the infimum $\bar{\mu}$ of $L(f_0)$ for $L(f) \in \bar{E}$. Then $\mu = \bar{\mu}$, and if the (relevant) members of E have areas and (suitably selected) representations uniformly bounded, then $\bar{\mu}$ is attained. This is the existence theorem.

When the members of E have a fixed boundary curve C, those of $\bar{E}$ are said to have the boundary C and more information is available. Thus a generalized parametric surface $L(f)$ with simple rectifiable boundary C has a cyclic decomposition $L(f) = L^*(f) + \sum L_n(f)$. Here $L^*(f)$ is "singular"—a limiting sum of innumerable minute closed polyhedra; the $L_n(f)$ are closed except $L_1(f)$ which has the boundary C, and each is incapable of further decomposition without altering the boundary; we term $L_1(f)$ hemi-cycle, the other $L_n(f)$ cycles. Both $L^*(f)$ and the $L_n(f)$ can be characterized by expressions such as occur in "generalized curves"; thus a cycle or hemi-cycle has a "track" $x(u, v)$ possibly mildly discontinous. In many problems, however, minimizing $L(f)$ have continuous tracks; when Weierstrass's condition holds, they provide solutions in the classical sense.

UNIVERSITY OF WISCONSIN,
MADISON, WIS., U. S. A.

MEASURE THEORY

ON HAUSDORFF MEASURES

ARYEH DVORETZKY

Let $y = h(x)$ be defined for $0 < x < \infty$ and assume values in $0 \leqq y \leqq +\infty$. Let S be any linear set of points and ρ an arbitrary positive number. Cover S by a countable number of open intervals $I_1, I_2, \cdots$ of lengths $x_1, x_2, \cdots$ each of which is less than ρ, and denote by $m_\rho(S; h)$ the lower bound of $h(x_1) + h(x_2) + \cdots$ for all such coverings of S. Then $m(S; h) = \lim_{\rho \downarrow 0} m_\rho(S; h)$ is called the (linear, exterior) *Hausdorff measure of S with respect to $h(x)$*. (F. Hausdorff [Math. Ann. vol. 79 (1918) pp. 157–179] originally considered the most important case of continuous monotone $h(x)$ with $\lim_{x\downarrow 0} h(x) = 0$. For later studies see e.g. G. Bouligand [*Les définitions modernes de la dimension*, Actualités scientifiques et industrielles, no. 274, Paris, 1935] and A. Dvoretzky [Proc. Cambridge Philos. Soc. vol. 44 (1948) pp. 13–16].)

If $m(S; h) < \infty$ implies $m(S; g) < \infty$ we write $g \prec h$. If either $g \prec h$ or $h \prec g$ we say that $g(x)$ and $h(x)$ are *comparable*; if both relations hold we write $g \sim h$ (read: *equivalent*). If $m(S; g) = m(S; h)$ for all S we write $g \approx h$ (read: *strictly equivalent*). Given any $h(x)$ it can be shown that *there exists a continuous monotone* (nondecreasing) *$g(x)$ strictly equivalent to $h(x)$*.

Whatever $h(x)$ we put $h^*(x) = x \inf_{0<t\leqq x} h(t)/t$. *Then $h^* \sim h$ or, more precisely, for all S we have $m(S; h^*) \leqq m(S; h) \leqq 2m(S; h^*)$.*

Using this result it can be shown that $g \prec h$ *if and only if*

$$\limsup_{x\downarrow 0} \frac{g^*(x)}{h^*(x)} < \infty.$$

(Hence $g \prec h$ could have been defined also by the requirement that $m(S; g) > 0$ imply $m(S; h) > 0$.) Thus, *$g(x)$ and $h(x)$ are equivalent if and only if $h^*(x)/g^*(x)$ is bounded and bounded away from zero as $x \downarrow 0$*; moreover, the bounds of this expression may be employed to obtain explicit bounds for $m(S; h)/m(S; g)$.

By constructive methods similar to those used in proving the above results it can be shown that *if $\lim_{x\downarrow 0} h(x) = 0$ and $\lim_{x\downarrow 0} h(x)/x = \infty$, then there exist a monotone, continuous and convex $g(x)$, and perfect nowhere dense sets S_1, S_2, S_3 such that $0 = h(S_1) < h(S_2) < h(S_3) = \infty$ and $\infty = g(S_1) > g(S_2) > g(S_3) = 0$.* Thus the only functions $h(x)$ comparable to all others are either 1) those for which $\liminf_{x\downarrow 0} h(x) = \alpha > 0$, when $m(S; h)$ is α times the number of points in S; or 2) those for which $\liminf_{x\downarrow 0} h(x)/x = \beta < \infty$, when $m(S; h)$ is β times the exterior Lebesgue measure of S. Similar results may be obtained for sequences of functions $h(x)$.

Hausdorff measures of nonlinear sets can be treated in a similar manner.

HEBREW UNIVERSITY,
JERUSALEM, ISRAEL.

LUSIN'S THEOREM FOR ONE TO ONE MEASURABLE TRANSFORMATIONS

CASPER GOFFMAN

A one to one transformation $T_n: (f(x), f^{-1}(y))$ of the closed unit n cube I onto the closed unit n cube I^{-1} is said to be measurable if $f(x)$ and $f^{-1}(y)$ are both measurable functions; it is said to be of Baire class $\leqq \alpha$ if $f(x)$ and $f^{-1}(y)$ are both of Baire class $\leqq \alpha$. It is shown, for every $n \geqq 1$, that if T_n is measurable then, for every $\epsilon > 0$, there are closed sets $E \subset I$ and $E^{-1} \subset I^{-1}$ of measure greater than $1 - \epsilon$ such that T_n is a homeomorphism between E and E^{-1}; and there is a $T'_n: (g(x), g^{-1}(y))$ of Baire class $\leqq 2$ such that $f(x) = g(x)$ and $f^{-1}(y) = g^{-1}(y)$ almost everywhere. If $n \geqq 2$ then, for every $\epsilon > 0$, there is a homeomorphism $T'_n: (g(x), g^{-1}(y))$ between I and I^{-1} such that $f(x) = g(x)$ and $f^{-1}(y) = g^{-1}(y)$ on sets of measure greater than $1 - \epsilon$. This result does not hold for $n = 1$. Attention is called to a result the author obtained several years ago concerning arbitrary one to one transformations (Duke Math. J. vol. 10 (1943) pp. 1–4) and to a number of unsolved problems.

UNIVERSITY OF OKLAHOMA,
NORMAN, OKLA., U. S. A.

EXTENSION OF MEASURE

OTTON M. NIKODÝM

Known measure extension devices for sets are studied on abstract Boolean lattices. Let (A) be a finitely genuine Boolean sublattice of a Boolean σ-lattice (A'),that is, for elements a, b, c of (A) the equalities

$$a + b = c, \qquad a +' b =' c; \qquad a \cdot b = c, \qquad a \cdot' b =' c$$

are equivalent respectively, and $0 =' 0'$, $1 =' 1'$. The primed symbols refer to (A'), the unprimed to (A). The above four conditions are independent. A given finitely additive measure $\mu(a) \geqq 0$ on (A) generates the "Lebesguean exterior measure" on (A') defined by

$$\mu'_e(a') = \inf \sum_{n=1}^{\infty} \mu(x_n)$$

for all coverings $\{x_n\}$ of a', that is, $a' \subseteq' \sum'^{\infty}_{n=1} x_n$. The "interior measure" is defined by $\mu'_i(a') = \mu'_e(1') - \mu'_e(co'a')$. Let (L') be the class of all $a' \in (A')$ for which $\mu'_e(a') = \mu'_i(a')$. $\mu'_e(a')$ is greater than or equal to any convex Carathéodory's measure on (A') for which $\mu'_e(a) \leqq \mu(a)$ on (A). Let (C') be the class of all μ'_e-"Carathéodory-measurable" elements of (A'). Let us say that a', b' do not differ by more than $\epsilon > 0$ if the symmetric difference $a' \dot{+} b'$ has a covering

$\{x_n\}$ with $\sum_{n=1}^{\infty} \mu(x_n) \leqq \epsilon$. Define (S') as the class of all $a' \in (A')$ such that for every $\epsilon > 0$ there exist $a \in (A)$ not differing from a' by more than ϵ. This device was studied by the author in Académie Royale de Belgique, 1938.

It can be proved that (1) the classes (L'), (C''), (S') are identical and contain (A), (2) they too are σ-lattices, and (3) μ_e' is, on them, a denumerably additive measure. It may be less than μ on some $a \in (A)$.

If we define on (A') a "fundamental" sequence $\{a_n'\}$ as a sequence such that a_p' does not differ from a_q' by more than $\epsilon_{p,q} > 0$ with $\epsilon_{p,q} \to 0$, and if we use the corresponding Cantor-Ch. Meray completing process, we obtain the same Boolean lattice (B') as if we had used MacNeille's well known measure extension device (Proc. Nat. Acad. Sci. U. S. A. (1938)) upon the convex measure μ_e'.

(These remarks represent part of the work of the author under a cooperative contract between the Atomic Energy Commission and Kenyon College.)

Kenyon College,
Gambier, Ohio, U. S. A.

Section III

GEOMETRY AND TOPOLOGY

SECTION III. GEOMETRY AND TOPOLOGY

INTEGRAL GEOMETRY IN GENERAL SPACES

L. A. SANTALÓ

Let E be a space of points in which a locally compact group of transformations G operates transitively. Let dx be the left invariant element of volume in G. Let H_0 and K_0 be two sets of points in E and denote by xH_0 the transformed set of H_0 by x ($x \in G$). Let us first assume that the identity is the only one transformation of G which leaves H_0 invariant. If $F(K_0 \cap xH_0)$ is a function of the intersection $K_0 \cap xH_0$, the main purpose of the so-called Integral Geometry (in the sense of Blaschke) is the evaluation of integrals of the type

$$I = \int_G F(K_0 \cap xH_0)\, dx \tag{1}$$

and to deduce from the result some geometrical consequences for the sets K_0 and H_0.

Let us now suppose that there is a proper closed subgroup g of G which leaves H_0 invariant. The elements $H = xH_0$ will then be in one to one correspondence with the points of the homogeneous space G/g. If there exists in G/g an invariant measure and dH denotes the corresponding element of volume, the Integral Geometry also deals with integrals of the type

$$I = \int_{G/g} F(K_0 \cap H)\, dH \tag{2}$$

from which it tries to deduce geometrical consequences for K_0.

In what follows we shall give some examples and applications of the method.

1. Immediate examples. Let us assume G compact and therefore of finite measure which we may suppose equal 1. In order to define a measure $m(K_0)$ of a set of points K_0, invariant with respect to G, we choose a fixed point P_0 in E and set

$$m(K_0) = \int_G \varphi(x)\, dx \tag{3}$$

where $\varphi(x) = 1$ if $xP_0 \in K_0$ and $\varphi(x) = 0$ otherwise.

If the measures $m(K_0)$, $m(H_0)$, and $m(K_0 \cap xH_0)$ exist, it is then known and easy to prove that

$$\int_G m(K_0 \cap xH_0)\, dx = m(K_0)m(H_0), \tag{4}$$

and since $\int_G dx = 1$, the mean value of $m(K_0 \cap xH_0)$ will be $m(K_0)m(H_0)$. Therefore we have: *Given in E two sets K_0, H_0, there exists a transformation x of G such that $m(K_0 \cap xH_0)$ is equal to or greater than $m(K_0)m(H_0)$.*

If K_0 consists of N points P_i ($i = 1, 2, \cdots, N$) and call $\nu(K_0 \cap xH_0)$ the number of points P_i which belong to xH_0, we want to evaluate $\int_G \nu(K_0 \cap xH_0)\, dx$. We set $\varphi_i(x) = 1$ if $xP_i \in H_0$ and $\varphi_i(x) = 0$ otherwise. According to (3) and the invariance of dx we have

$$m(H_0) = \int_G \varphi_i(x)\, dx = \int_G \varphi_i(x^{-1})\, dx$$

where $\varphi_i(x^{-1}) = 1$ if $P_i \in xH_0$ and $\varphi_i(x^{-1}) = 0$ otherwise. Consequently we have

$$\int_G \nu(K_0 \cap xH_0)\, dx = \sum_1^N \int_G \varphi_i(x^{-1})\, dx = Nm(H_0). \tag{5}$$

Thus the mean value of ν is equal to $Nm(H_0)$ and we have: *Given N points P_i in E and a set H_0 of measure $m(H_0)$, there exists a transformation x of G such that xH_0 contains at least $Nm(H_0)$ of the given points; it contains certainly a number greater than $Nm(H_0)$ if H_0 is closed.*

2. An application to convex bodies. Let E be now the euclidean 3-space and G the group of the unimodular affine transformations which leave invariant a fixed point O. Let H be the planes of E. The subgroup g will consist of all affinities of G which leave invariant a fixed plane H_0. Each plane H can be determined by its distance p to O and the element of area $d\omega_2$ on the unit 2-sphere corresponding to the point which gives the direction normal to H. The invariant element of volume in G/g is then given by

$$dH = p^{-4}\, dp\, d\omega_2 . \tag{6}$$

Let K_0 be a convex body which contains O in its interior, and let $p(\omega_2)$ be the support function of K_0 with respect to O. If we set $F(K_0 \cap H) = 0$ if $K_0 \cap H \neq 0$ and $F(K_0 \cap H) = 1$ if $K_0 \cap H = 0$, (2) reduces to

$$I(O) = \int_{K_0 \cap H = 0} dH = \frac{1}{3} \int p^{-3}\, d\omega_2 \tag{7}$$

where the last integral is extended over the whole 2-sphere. If O is an affine invariant point of K_0 (for instance, its center of gravity), (7) gives an affine invariant for convex bodies (with respect to unimodular affinities). The minimum of I with respect to O is also an affine invariant which we shall represent by I_m.

By comparing I_m with the volume V and the affine area F_a of K_0 the following theorem can be shown: *Between the unimodular affine invariants I_m, F_a, and V of a convex body the inequalities*

$$I_m V \leqq (4\pi/3)^2, \qquad I_m F_a^2 \leqq (2^6/3)\pi^3 \tag{8}$$

hold, where the equalities hold only if K is an ellipsoid.

For the analogous relations for the plane see [3]. I do not know if in (8) I_m can be replaced by the invariant $I(O)$ corresponding to the center of gravity of K_0.

3. The group of motions in a space of constant curvature. The best known case is that in which $E \equiv S_n$ is an n-dimensional space of constant curvature k and G is the group of motions in it. In this case the invariant element of volume dx in G is well known. If (P_0, e_0^i) $(i = 1, 2, 3, \cdots, n)$ denotes a fixed n-frame (i.e., a point P_0 and n unit mutually orthogonal vectors with the origin at P_0), any motion x can be determined by the n-frame $(P = xP_0, e^i = xe_0^i)$. Let dP be the element of volume in S_n at P and let $d\omega_{n-1}$ be the element of area on the unit euclidean $(n-1)$-sphere corresponding to the direction of e^1; let $d\omega_{n-2}$ be the element of area on the unit euclidean $(n-2)$-sphere orthogonal to e^1 corresponding to the direction of e^2 and so forth. Then dx can be written

$$dx = [dP\, d\omega_{n-1}\, d\omega_{n-2} \cdots d\omega_1]. \tag{9}$$

Let C_p be a p-dimensional variety $(p \leqq n)$ of finite p-dimensional area A_p and C_q a q-dimensional variety $(q \leqq n)$ of q-dimensional area A_q. Assuming $p + q \geqq n$, let $A_{p+q-n}(C_p \cap xC_q)$ be the $(p + q - n)$-dimensional area of $C_p \cap xC_q$. Then the formula

$$\int_G A_{p+q-n}(C_p \cap xC_q)\, dx = \frac{\omega_{p+q-n}}{\omega_p\, \omega_q}\, \omega_1 \omega_2 \cdots \omega_n A_p A_q \tag{10}$$

holds, where ω_i denotes the area of the euclidean unit i-sphere, that is,

$$\omega_i = \frac{2\pi^{(i+1)/2}}{\Gamma((i+1)/2)}. \tag{11}$$

If $p + q = n$, $A_{p+q-n}(C_p \cap xC_q)$ denotes the number of points of the intersection $C_p \cap xC_q$. Notice that (10) is independent of the curvature k of S_n.

If, instead of C_q, we consider a q-dimensional linear subspace L_q^0 of S_n, and denote by $A_{p+q-n}(C_p \cap L_q)$ the area of the intersection of C_p with $L_q = xL_q^0$, we obtain

$$\int_{G/g} A_{p+q-n}(C_p \cap L_q)\, dL_q = \frac{\omega_{p+q-n}\, \omega_{n-q}}{\omega_p} A_p \tag{12}$$

where g is the group of motions which leaves invariant L_q^0 and dL_q is the invariant element of volume in the homogeneous space G/g normalized in such a way that the measure of the L_q which cut the unit $(n - q)$-sphere in S_n be its area (depending upon the curvature k of the space).

We shall give two applications:

a) Let S_n be the euclidean n-sphere; G is then compact and according to (9) its total volume will be $\int_G dx = \omega_1\omega_2 \cdots \omega_n$. In this case the mean value of $A_{p+q-n}(C_p \cap xC_q)$ will be $\omega_{p+q-n}\omega_p^{-1}\omega_q^{-1}A_pA_q$. Consequently we have: *Given on the euclidean n-sphere two varieties C_p, C_q of dimensions p, q $(p + q \geqq n)$ and finite areas A_p, A_q, there exists a motion x such that the area of the intersection $C_p \cap xC_q$ is equal to or greater than $\omega_{p+q-n}\omega_p^{-1}\omega_q^{-1}A_pA_q$.*

b) We shall now give an application to the elementary non-euclidean geometry.

Let T be a tetrahedron in the 3-dimensional space of constant curvature k and let L_2 be the planes of this space. One can show that

$$\int_{T\cap L_2\neq 0} dL_2 = \frac{1}{\pi}\left(\sum_1^6 (\pi - \alpha_i)l_i + 2kV\right) \tag{13}$$

where l_i are the lengths of the edges of T and α_i the corresponding dihedral angles; V is the volume of T.

On the other hand (12) applied to the edges of T gives

$$\int_{G/g} N(T \cap L_2)\, dL_2 = 2 \sum_1^6 l_i \tag{14}$$

where $N(T \cap L_2)$ denotes the number of edges which are intersected by L_2 and therefore is either $N = 3$ or $N = 4$. From (13) and (14) we can evaluate the measures of the sets of planes L_2 corresponding to $N = 3$ and $N = 4$. These measures being non-negative we get the inequalities (for the euclidean case see [5])

$$2\sum_1^6 \alpha_i l_i - 4kV \leqq \pi \sum_1^6 l_i \leqq 3\sum_1^6 \alpha_i l_i - 6kV$$

which for $k = 1$, $k = -1$ gives the following inequalities for the volume V of a tetrahedron in non-euclidean geometry

$$\frac{1}{4}\sum^6 (2\alpha_i - \pi)l_i \leqq V \leqq \frac{1}{6}\sum_1^6 (3\alpha_i - \pi)l_i \quad \text{for the elliptic space,}$$

$$\frac{1}{6}\sum_1^6 (\pi - 3\alpha_i)l_i \leqq V \leqq \frac{1}{4}\sum_1^6 (\pi - 2\alpha_i)l_i \quad \text{for the hyperbolic space.}$$

These inequalities may have some interest because, as is known, V cannot be expressed in terms of elementary functions of l_i and α_i.

4. A definition of p-dimensional measure of a set of points in euclidean n-space. Let E be now the euclidean n-dimensional space E_n. The methods of Integral Geometry can be used in order to give a definition of area for p-dimensional surfaces (see Maak [3], Federer [2], and for a comparative analysis Nöbeling [4]). The idea of the method is as follows. The formulas (10) and (12) hold for varieties which have a well-defined p- and q-dimensional area in the classical sense. For more general varieties the same formulas (10), (12) can be taken as a definition for A_p (taking for C_q a variety with A_q well-defined), *provided the integrals on the left-hand sides exist.* The problem is therefore to find the conditions of regularity which C_p must satisfy in order that the integrals (10) or (12) exist.

We want to give an example.

Let C be a set of points in E_n and let dP_i be the element of volume in E_n at the point P_i. Let N_s be the number of common points of C with s unit $(n - 1)$-

spheres whose centers are the points $P_1, P_2, P_3, \cdots, P_s$. Let us consider the following integrals (in the sense of Lebesgue)

$$I_s = \int N_s \, dP_1 \, dP_2 \, dP_3 \cdots dP_s \qquad (s = 1, 2, 3, \cdots) \tag{15}$$

extended with respect to each P_i to the whole E_n.

The q-dimensional measure of C can be defined by the formula

$$m_q(C) = \frac{\omega_q}{2\omega_n^q} I_q. \tag{16}$$

Notice that if I_r is the first integral in the sequence $I_1, I_2, I_3, \cdots$ which has a finite value (may be zero), then $m_q(C) = \infty$ for $q < r$ and $m_q(C) = 0$ for $q > r$. The number r can be taken as the definition for the *dimension* of C.

If C is a q-dimensional variety with tangent q-plane at every point, (16) gives the ordinary q-dimensional area of C (as may be deduced from (10)). The definition (16) may be applied whenever the integrals (15) exist. Following a method used by Nöbeling [4] in similar cases, it is not difficult to prove that the integrals I_s exist if C is an analytic set (or Suslin set).

5. Application to Hermitian spaces. Let $E \equiv P_n$ be now the n-dimensional complex projective space with the homogeneous coordinates $\xi_0, \xi_1, \cdots, \xi_n$ and let G be the group of linear transformations which leaves invariant the Hermitian form $(\xi\bar{\xi}) = \sum \xi_i \bar{\xi}_i$.

If we normalize the coordinates ξ_i such that $(\xi\bar{\xi}) = 1$, every variety C_p of complex dimension p posseses an invariant integral of degree $2p$, namely $\Omega^p = (\sum [d\xi_i d\bar{\xi}_i])^p$ (see Cartan [1]). Let us put

$$J_p(C_p) = \frac{p!}{(2\pi i)^p} \int_{C_p} \Omega^p. \tag{17}$$

It is well known that if C_p is an algebraic variety of dimension p, $J_p(C_p)$ coincides with its order.

If C_p is an analytic variety ("synectic" according to Study, i.e., defined by complex analytic relations) the methods of Integral Geometry give a simple interpretation of the invariant J_p. Let L_{n-p}^0 be a linear subspace of dimension $n - p$ and put $L_{n-p} = xL_{n-p}^0$. If g is the subgroup of G which leaves L_{n-p}^0 invariant, and dL_{n-p} means the invariant element of volume in the homogeneous space G/g normalized in such a way that the total volume of G/g is equal 1, the formula

$$\int_{G/g} N(C_p \cap L_{n-p}) \, dL_{n-p} = J_p(C_p) \tag{18}$$

holds, where $N(C_p \cap L_{n-p})$ denotes the number of points of intersection of C_p with L_{n-p}.

A more general formula, assuming $p + q \geqq n$, is the following

$$\int_{G/g} J_{p+q-n}(C_p \cap L_q)\, dL_q = J_p(C_p) \tag{19}$$

which coincides with (18) for $q = n - p$.

If dx is the element of volume in G normalized in such a way that the total volume of G is equal 1, given two analytic varieties C_p, C_q we also have

$$\int_G J_{p+q-n}(C_p \cap xC_q)\, dx = J_p(C_p) J_q(C_q) \tag{20}$$

which may be considered as the generalization to analytic varieties of the theorem of Bezout.

For $p + q = n$, the foregoing formula (20) is a particular case of a much more general result of de Rham [6].

6. Integral geometry in Riemannian spaces. The Integral Geometry in an n-dimensional Riemannian space R_n presents a different aspect. Here we do not have, in general, a group of transformations G. However, if we take as geometrical elements the geodesic curves Γ of R_n, it is possible to consider integrals analogous to (2), though conceptually different, and to deduce from them geometrical consequences.

Let $ds^2 = g_{ij}\, du^i\, du^j$ be the metric in R_n and let us set $\varphi = (g_{ij}\dot{u}^i\dot{u}^j)^{1/2}$ and $p_i = \partial\varphi/\partial\dot{u}^i$. The exterior differential form $d\Gamma = (\sum[dp_i\, du^i])^{n-1}$ of degree $2(n-1)$ is invariant under displacements of the elements u^i, p_i on the respective geodesic. Therefore we can define the "measure" of a set of geodesic curves as the integral of $d\Gamma$ extended over the set.

Let us consider a bounded region D_0 in R_n and let different arcs of geodesic contained in D_0 be taken as different geodesic lines. Let us consider a geodesic Γ which intersects an $(n - 1)$-dimensional variety C_{n-1} contained in D_0 at the point P. Let $d\sigma$ denote the element of $(n - 1)$-dimensional area on C_{n-1} at P. If $d\omega_{n-1}$ denotes the element of area on the unit euclidean $(n - 1)$-sphere corresponding to the direction of the tangent to Γ at P and θ denotes the angle between Γ and the normal to C_{n-1} at P, the differential form $d\Gamma$ may be written in the form $d\Gamma = |\cos\theta|\,[d\omega_{n-1} d\sigma]$. If C_{n-1} has a finite $(n - 1)$-dimensional area A_{n-1} and $N(C_{n-1} \cap \Gamma)$ denotes the number of intersection points of C_{n-1} and Γ, from the last form for $d\Gamma$ it follows that

$$\int_{D_0} N(C_{n-1} \cap \Gamma)\, d\Gamma = \frac{\omega_{n-2}}{n-1} A_{n-1} \tag{21}$$

where the integral is extended over all geodesics of D_0.

If dt denotes the element of arc on Γ and dP is the element of volume in R_n at P, clearly we have $[d\Gamma\, dt] = [dP\, d\omega_{n-1}]$. From this relation if we consider all arc elements (Γ, t) with the origin within a given region D (contained in D_0)

of finite volume $V(D)$ and call $L(D \cap \Gamma)$ the length of the arc of Γ which lies within D, we obtain

$$\int_{D_0} L(D \cap \Gamma)\, d\Gamma = \frac{1}{2}\omega_{n-1} V(D). \tag{22}$$

Some consequences of the formulas (21) and (22) for the case $n = 2$ have been given in [8]. They have particular interest for the Riemannian spaces of finite volume whose geodesic lines are all closed curves of finite length (for $n \geqq 3$ it seems, however, not to be known if such spaces, other than spheres, exist). For instance, one can easily show: *If the geodesic lines of a Riemannian space R_n of finite volume V are all closed curves of constant length L and there exists in R_n an $(n-1)$-dimensional variety of area A_{n-1} which intersects all the geodesic curves, the inequality*

$$LA_{n-1} \geqq \frac{(n-1)\omega_{n-1}}{2\omega_{n-2}} V$$

holds (*equality for the elliptic space*).

REFERENCES

1. E. CARTAN, *Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces*, Annales de la Société Polonaise de Mathematiques vol. 8 (1929) pp. 181–225.
2. H. FEDERER, *Coincidence functions and their integrals*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 441–466.
3. W. MAAK, *Oberflächenintegral und Stokes Formel im gewöhnlichen Raume*, Math. Ann. vol. 116 (1939) pp. 574–597.
4. G. NÖBELING, *Ueber den Flächeninhalt dehnungsbeschränkter Flächen*, Math. Zeit. vol. 48 (1943) pp. 748–771.
5. G. PÓLYA and G. SZEGÖ, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, vol. 2, p. 166.
6. G. DE RHAM, *Sur un procède de formation d'invariants intégraux*, Jber. Deutschen Math. Verein. vol. 49 (1939) pp. 156–161.
7. L. A. SANTALÓ, *Un invariante afin para las curvas planas*, Mathematicae Notae vol. 8 (1949) pp. 103–111.
8. ———, *Integral geometry on surfaces*, Duke Math. J. vol. 16 (1949) pp. 361–375.

UNIVERSITY OF ROSARIO,
ROSARIO, ARGENTINA.

ARITHMETICAL PROPERTIES OF ALGEBRAIC VARIETIES

Beniamino Segre

While classical algebraic geometry deals essentially with algebraic varieties in the complex field, in the new trend of ideas pursued by van der Waerden, Zariski, and André Weil the algebraic varieties are considered in an arbitrary commutative field, possibly not algebraically closed and of arbitrary characteristic. In this connection, questions of an arithmetical character arise, which are often very difficult to solve. In fact the new field of research does not only include a good deal of the old diophantine analysis, but also leads to questions of a completely new type, as, for instance, those related with the subvarieties of a given algebraic variety in a given field. Hence the study of such problems has been, up to now, confined to comparatively simple ones, and I shall now deal briefly with a few of them.

Given an arbitrary field γ, of characteristic p say, let

$$F(x, y, z) = 0 \tag{1}$$

be a cubic nonhomogeneous equation with coefficients in γ. The problem of finding conditions for the existence of solutions of such an equation in γ has not been solved, even in the case when γ is the rational field. However, it has been proved that, when the cubic surface F represented by (1) is not a cone, the existence of any special solution of (1) in γ implies the existence of a rational parametric solution with coefficients in γ, and so—when γ is infinite—of an infinity of solutions in γ. As a very particular consequence of this result, we obtain the well-known possibility of expressing any given rational number, r say, as the sum of the cubes of three rational numbers; for the cubic surface $x^3 + y^3 + z^3 = r$ is nonsingular and has three rational points at infinity.

The result just given shows, moreover, that the conditions for the solvability of (1) in γ coincide with those for the rational representability of F upon a plane in γ. The conditions for the birational representability of F upon a plane in γ depend as follows on the study of the straight lines lying on F.

First of all, it is easily seen that, if F is nonsingular and γ has the characteristic $p \neq 2$, then—in a convenient algebraic extension of γ—F contains exactly 27 lines, forming a well-known configuration. The number and the configuration of those among these lines which belong to γ may present 10 different cases, each of which actually occurs if γ is the rational field, while only 4 or 1 of them take place in the real or complex field respectively.

In order that F be representable birationally upon a plane in γ, it is necessary that F contain at least one line, or a couple or triplet or sextuplet of lines skew in pairs, belonging to γ. This is also the necessary and sufficient condition for the existence on F of a nontrivial curve belonging to γ, i.e., of a curve which is not the complete intersection of F with another surface belonging to γ.

The results just indicated remain not all true in fields of characteristic $p = 2$. A similar exception arises also in the following question. Consider, over a field γ which we suppose at first of characteristic $p \neq 2$, a quadratic form f of nonzero discriminant, in any number n of variables, and let A be any automorphism of f in γ, i.e., a linear substitution on the variables, with coefficients in γ, transforming f into itself. It is then always possible to decompose A into a product of at most n automorphisms, each of which is involutory in character, and so gives a harmonic homology transforming the quadric f into itself. This result, established long ago in the complex field by Voss and C. Segre, has been proved recently by Dieudonné, by means of an elegant geometric argument which extends the one previously given in the real field by E. Cartan. When $p = 2$, Dieudonné has shown the possibility—with a single exception—of decomposing any automorphism of a nonsingular quadric in n variables into a product of involutory automorphisms. The number of factors in this product can be supposed $\leqq 2n$ if γ has more than two elements, and $\leqq 5n/2$ otherwise; but it can probably be reduced, and it would be of some interest to find the best possible result. The exception I have alluded to concerns the case when γ contains only two elements, and f is a nonsingular quadric, ruled in γ and belonging to a three-dimensional space ($n = 4$).

I shall deal in some detail with this case, in order to show how geometric ideas can play a useful role in questions of this type. At present the ambient space of f contains in all 15 points, 9 of which lie on f, while the remaining 6 points ("external" with respect to f) are distributed in two triplets of collinear points, the lines l, l' which contain these triplets being skew and nonsecant with respect to f in γ. The quadric f contains 6 lines belonging to γ, distributed in two systems of three lines two by two skew; the three lines of one system meet the three lines of the other system at the 9 points of f.

f has in all 72 automorphisms, $6 \cdot 6 = 36$ of which transform into itself each system of generators, permuting arbitrarily the three generators of each system; the remaining automorphisms interchange the two systems of generators, and their number is likewise 36. The 72 automorphisms of f can be distributed in another manner in two sets, each including 36 transformations, according as to whether they transform each of the two triplets of collinear external points into itself or interchange them. There are in all 6 involutory automorphisms of f, i.e., 6 homologies transforming f into itself, whose centres are the 6 external points; each of them transforms both l and l' into itself, and they generate by multiplication the first of the two sets of 36 automorphisms just considered. Hence none of the automorphisms of the second set can be obtained as a product of involutory ones.

The study of the linear spaces lying on a quadric in an arbitrary field is very significant in the arithmetical theory of quadratic forms. By means of it, one can obtain geometrically several results about the isomorphisms and the representations of quadratic forms, as I have shown in the Liège Colloquium on algebraic geometry. I shall not enter into that again now, but shall confine myself to a few properties of the linear spaces lying on a quadric, which may be useful in a number of questions of arithmetical or algebraic character.

It is well-known that, given in S_r a nonsingular quadric f over an arbitrary field γ, there are k-dimensional linear spaces S_k lying on f and belonging to γ or to a convenient extension of γ, if, and only if, k satisfies the conditions

$$0 \leqq k \leqq \frac{r-1}{2}. \tag{2}$$

The totality, H say, of these S_k is algebraic, i.e., on the Grassmann variety G of the S_k's of S_r, H is represented by an algebraic variety, H^* say, both H and H^* having the dimension

$$d = \frac{(k+1)(2r-3k-2)}{2}.$$

It can be shown that each of the projective characters $H_{a_0,a_1,\cdots,a_k}$ of H is zero, except for

$$H_{1,3,\cdots,2k+1} = 2^{k+1},$$

where $a_0, a_1, \cdots, a_k$ are any $k+1$ integers such that

$$0 \leqq a_0 < a_1 < \cdots < a_k \leqq r$$

$$a_0 + a_1 + \cdots + a_k = (k+1)^2,$$

and $H_{a_0,a_1,\cdots,a_k}$ denotes the number of S_k's of H contained in a general Schubert fundamental form $[a_0, a_1, \cdots, a_k]$. As a consequence of this result, it can be proved that H^* has the order

$$\nu = \frac{2^{(k+1)(k+2)/2}1!2!\cdots k!\, d!}{(r-1)!(r-3)!\cdots(r-2k-1)!}.$$

It is well-known that, according as to whether the second equality sign does not or does hold in (2), the variety H^* (and so also H) is absolutely irreducible or splits up into two irreducible homographic components, H^* or each of these components being birationally equivalent to a d-dimensional linear space. In the second case, the two components of H^* are the intersections of G with two linear spaces, which are skew to each other, of dimension

$$\binom{2k+1}{k} - 1$$

and belonging to a quadratic extension γ_1 of γ. If $p \neq 2$, then

$$\gamma_1 = \gamma([(-1)^{k+1}D]^{1/2}),$$

where D is the discriminant of f; this gives the geometric significance of the condition

$$(-1)^{k+1}D = \square,$$

which intervenes in several arithmetical questions.

Some important problems of arithmetical character are suggested by the study of the birationality or unirationality of an algebraic variety, and some of them have already been investigated by M. Noether, Castelnuovo, and Enriques in the case of surfaces. (I remark, incidentally, that I prefer to call birational a variety which is rational in the usual sense, reserving the word rational to designate a variety having the ground field γ as its field of definition.) Thus, for instance, it can be shown that Castelnuovo's theorem, on the birationality of every plane involution in the complex field, cannot be extended to the real or rational field.

A further example has been obtained recently by Morin and Predonzan, who have shown that, given any k positive integers $\nu_1, \nu_2, \cdots, \nu_k$, the variety V represented by a system of algebraic equations,

$$f_1(x_1, x_2, \cdots, x_n) = 0, f_2(x_1, x_2, \cdots, x_n) = 0, \cdots,$$
$$f_k(x_1, x_2, \cdots, x_n) = 0,$$

of respective degrees $\nu_1, \nu_2, \cdots, \nu_k$ and coefficients generic in a field γ, is unirational in an extension γ^* of γ, provided that n be sufficiently large compared to $\nu_1, \nu_2, \cdots, \nu_k$. From a result by R. Brauer and myself, γ^* can then be obtained from γ by adjoining a finite number of roots of elements of γ, whose indices are not greater than the greatest of the ν's.

Another interesting class of algebraic varieties has been considered, from different points of view, first by Severi and then by François Châtelet and myself. Châtelet has studied these varieties by means of a certain Brauer algebra, and I have called them Severi-Brauer varieties and extended some of their properties to a larger class of varieties. Here I shall confine myself to a brief reference to the Severi-Brauer varieties, namely to the algebraic varieties V_d defined over a field γ, which—in a convenient extension of γ—can be put into a (1, 1) algebraic correspondence without exception with a linear space S_d. The important question of referring V_d and S_d birationally in a given extension γ^* of γ is equivalent to the simpler one of determining on V_d a simple point belonging to γ^*. This result can be proved by extending an argument which, for $d = 1$, is due to Noether, thus replacing V_d by the projective model representing its anticanonical system.

I hope that, despite the sketchy way in which, for lack of time, I have been compelled to deal with my subject, I may have conveyed its many-sidedness and interest.[1]

University of Bologna,
Bologna, Italy.

[1] For a more comprehensive treatment, cf. B. Segre, *Arithmetical questions on algebraic varieties*, London, The Athlone Press, 1951.

GEOMETRY

QUASICONVEX POLYHEDRA

C. Arf

In an oriented euclidean plane let E denote a finite set of points, no three of which are on a straight line. A polygon with vertices in E is called quasiconvex in E provided no points of E are in the interiors of the smallest positive angles formed by consecutive sides of the polygon. In a previous paper (*Un théorème de géométrie élémentaire*, Revue de la Faculté des Sciences de l'Université d'Istanbul, Ser. A. vol. 12, pp. 153–160) the author established a translation property of such polygons and in addition proved that any rotation of the plane that superposes E onto itself also superposes each closed polygon quasiconvex in E upon itself. In the present paper the notion of quasiconvex polygon is extended to define a polyhedron quasiconvex in a finite subset of the (oriented) euclidean three-space. Both properties of quasiconvex polygons mentioned above have their analogues valid for quasiconvex polyhedra. Thus, for example, each closed polyhedron quasiconvex in a finite subset F of euclidean three-space is transformed into itself by any motion of the space that carries F into itself.

Istanbul, Turkey.

CONVEX BODIES ASSOCIATED WITH A CONVEX BODY

Preston C. Hammer

Let C be a closed planar convex body. Let b be a boundary point of C and let r be a positive real number. We define $C_b(r)$ as the convex body obtained by a similitude of C with center b and ratio r. For $r \leqq 1$ we define $C(r) = \prod_b C_b(r)$ and for $r \geqq 1$, $C(r) = \sum_b C_b(r)$ where b varies over all boundary points of C. Let x be an interior point of C and let $r(x)$ be the maximum ratio into which x divides any chord of C passing through it. We take the ratio as the larger part divided by the whole chord. Then it is known that $r(x)$ reaches a minimum value r^* at a unique point x^*. Some results are:

1. $C(r)$ is a closed convex body, $C(r) = 0$ for $r < r^*$, $C(r^*) = x^*$, $C(r_1) \supset C(r_2)$ for $r_1 > r_2$, and the boundaries of $C(r)$ simply cover the entire plane.

2. For $r < 1$, the boundary of $C(r)$ consists precisely of those points x of C such that $r(x) = r$.

3. For $r > 1$, $C(r)$ is the minimal convex body such that the maximum ratio into which boundary points of C divide chords of $C(r)$ is $r/(2r - 1)$. Any diameter of C determines a diameter of $C(r)$ by expansion about its midpoint with a ratio $2r - 1$. If $r_2 > r_1 > 1$, then the boundary of $C(r_1)$ consists precisely of those points dividing chords of $C(r_2)$ in a maximum ratio $(r_1 + r_2 - 1)/(2r_2 - 1)$ and $C(r_2)$ is minimal with respect to this property.

4. The sets $C(r)$ are affine invariants.

Let $B = C(r)$ for $r < 1$. We say C is *completely reducible* if $B(r/(2r - 1)) = C$ for every r between r^* and 1. We say C is *reducible to a ratio* r_i if $B(r_i/(2r_i - 1)) = C$, but $B(r/(2r - 1)) \neq C$ for $r < r_i$. If $r_i = 1$, we say that C is *irreducible*. Then the following results are obtained:

5. C is completely reducible if and only if its boundary is a central curve ($r^* = 1/2$) and then the bodies $C(r)$ are obtained from C by similitudes about the central point x^*.

6. If $r^* > 1/2$, then there exists a unique ratio $r_i > r^*$ to which C is reducible. The body $C(r_i)$ is irreducible.

N.B. The results appear to extend to higher dimensions although some proofs as we have conceived them may become unwieldy.

(This document is based on work performed at Los Alamos Scientific Laboratory of The University of California under U. S. Government contract W-7405-Eng-36.)

Los Alamos Scientific Laboratory,
Los Alamos, N. M., U. S. A.

THE CLASSIFICATION OF CONGRUENT POLYHEDRA THAT CAN BE STACKED TOGETHER SO AS TO FILL SPACE COMPLETELY

Arjeh Kroch

Two congruent solids can always be joined together so as to be brought into complete coincidence in space: 1) by translation of one solid to another in a certain direction; 2) by rotation around an axis; 3) by a screw motion if the axis of rotation is parallel to the direction of translation.

Congruent polyhedra that fill space completely can be arranged: 1) in parallel layers, generally by translation; and 2) in concentric girdles, generally by rotation.

I. Congruent polyhedra arranged in parallel layers limited by parallel planes, such as:

a) Right prisms of which the bases are congruent and convex polygons that may cover a plane completely:

1) in rows by fulfilling the condition: $2n\pi/\sum\alpha_i = 2n\pi/(n - 2)\pi \geqq 3$ where $n = 3, 4, 5, 6$. If $n > 2n/(n - 2)$, one pair of the polygon sides must be parallel;

2) in concentric rings by covering the areas of equal angles with a common vertex, where $2\pi/\alpha$ is an integer.

b) Oblique prisms of which the bases are parallelograms or hexagons with three pairs of parallel sides.

II. Congruent polyhedra arranged in parallel layers not limited by parallel planes, such as:

Parallelohedra that could be erected by starting from any convex quadrihedral angle σ_{abcd}. The parallelohedra are sorted:

a) Parallelohedra built by 12 parallelograms: 2[*ab*], 2[*bc*], 2[*cd*], 2[*da*], 2[*ac*], 2[*bd*].

b) Parallelohedra built by 8 parallelograms: 2[*ab*], $\cdots$, 2[*da*] and 4 hexagons: 2[*alc*], 2[*bld*] —*l* is the straight line common to the two diagonal planes (*ac*) and (*bd*) of the solid angle σ_{abcd}.

c) Parallelohedra built by 8 hexagons and 6 parallelograms. (Any convex quadrihedral angle could be cut by a plane in a parallelogram [*mn*].) The faces are: 2[*mn*], 2[*amb*], 2[*bnc*], 2[*cmd*], 2[*dna*], 2[*ac*], 2[*bd*].

d) Parallelohedra built by 10 parallelograms and 12 hexagons: 2[*mn*], 4[*ml*], 4[*nl*], 2[*amb*], 2[*bnc*], 2[*cmd*], 2[*dna*], 2[*alc*], 2[*bld*]. This solid is not convex.

III. Congruent polyhedra, arranged in concentric girdles, by filling the spaces of congruent solid angles with a common vertex, that occupy the whole space surrounding this vertex. The necessary condition for congruent solid angles to fill space are: $4\pi/(\sum \beta_i - (m - 2)\pi) = n$ and $\sum r_i\beta_i = 2m\pi$. The β_i are dihedral angles of the solid angle n, and the r_i are positive integers $r_i > 2$.

From such solid angles only trihedral angles *abc* ($m = 3$) can be filled completely by congruent solids as the parallelopipeds 2[*ab*], 2[*bc*], 2[*ca*].

IV. Congruent polyhedra arranged in concentric girdles by filling the spaces of congruent regular or vertically regular polyhedra, that themselves fill space. The cubes are the only regular polyhedra and the rombo-dodecahedra the only vertically regular polyhedra filling space, and these two kinds could be cut up into congruent pyramids that can be stacked together so as to fill space completely from any single point. Instead of pyramids, bipyramids arranged in concentric girdles can be used.

INSTITUTE OF TECHNOLOGY,
HAIFA, ISRAEL.

A METHOD OF VISUALIZING FOUR-DIMENSIONAL ROTATIONS

JOHN KRONSBEIN

Elliptic 3-space with rectangular coordinates, radially projected upon a sphere R_4 in 4-space which is then stereographically projected upon inversion 3-space, establishes a correspondence between elliptic kinematics and quaternary rotations, and also between elliptic kinematics and that inversion subgroup preserving the property of "diametral points."

Great circles on R_4 map upon circles through diametrically opposite points of a fixed sphere R_i in inversion space. That the great circles are either linked or meet in two diametrically opposite points becomes intuitive.

Elliptic planes map into spheres through great circles of R_i, and the original coordinate system upon a triple set of circles terminating in ends on R_i, of Cartesian coordinate axes, suggesting new elliptic parallels. R_i assumes the role of an inaccessible plane, and kinematics may be confined to inside R_i by identifying diametral points. Rotations of R_4 induce linear fractional transformations of elliptic coordinates with coefficients components of corresponding quarternary orthogonal matrices. Quadrics in elliptic space have only two distinct shapes.

If the transverse generating circles of a torus intersect R_i orthogonally, it becomes an elliptic hyperboloid of revolution of one sheet; the reguli become "right" and "left" elliptic straight lines—oblique torus generator circles. All rotations of R_4 are now visible as elliptic movements, in particular the 6 "simple" rotations of Cartan's "biplan."

The torus is a Clifford surface whose reguli are right and left Clifford parallels. Visual examination of any elliptic line moving into another exhibits decomposition of quaternary rotation into *two*, about skew axes—a "*Study* line cross": (but for a transformation) a ternary euclidean rotation of R_i generating a torus by transverse generator circles; and a "pseudorotation" about an elliptic "infinitely distant" (inaccessible) line, a great circle on R_i in a plane orthogonal to axis of the first, retaining invariant individual transverse generator circles of torus.

Suitably orienting elliptic lines, each ternary rotation is composed of a *negative* "left translation" along a left Clifford parallel, followed by an equal *positive* "right translation" along a right parallel. Similarly, a pseudorotation is composed of *positive* left followed by *positive* right translation. These operations in R_i permit correlation of elliptic movements and quaternary rotations to quaternions $| A | | B | X' = \bar{A}XB$, where $| A |$ is the modulus of A. Euclidean rotations are represented by well known $\bar{A}XA$, pseudorotations belong to AXA. These are commutative. A limiting process generates ordinary euclidean movements as derived by E. Study with Clifford dual numbers. Slides illustrate the kinematics.

EVANSVILLE COLLEGE,
EVANSVILLE, IND., U. S. A.

GEOMETRICAL PROPERTIES OF TWO-DIMENSIONAL WAVE MOTION

L. A. MACCOLL

By a sinusoidal wave is meant a solution of the wave equation

$$V_{xx} + V_{yy} = c^{-2}V_{tt}$$

of the form

$$V = e^{\alpha(x,y)} \sin [\omega t + \beta(x, y)].$$

The curves α = constant are called curves of constant amplitude, or α-curves; and the curves β = constant are called curves of constant phase, or β-curves. In this paper there is determined a set of properties which is necessary and sufficient in order that two given families of curves shall be, respectively, the family of α-curves and the family of β-curves for some sinusoidal wave. When β is a constant, we have the case of a standing wave. Analogous results are obtained for this case, and also for the case in which α is a constant. A few other properties of the families of α-curves and β-curves are obtained incidentally.

BELL TELEPHONE LABORATORIES,
MURRAY HILL, N. J., U. S. A.

COVERING THEOREMS FOR SYSTEMS OF SIMILAR SETS OF POINTS

R. RADO

Corresponding to any point set S in a euclidean space R^n define numbers $F_r(S)$, $f_r(S)$ as follows. Let Ω be any system of a finite number of sets similar and homothetic to S, and $\|\Omega\|$ the volume (measure) of the union of the members of Ω. Call a subsystem Ω' of Ω discrete if no two members of Ω' have a point in common. The rank $\rho(\Omega^*)$ of any subsystem Ω^* is the least number r such that the members of Ω^* can be divided into r discrete systems. Put $\sigma_r(\Omega)$ = upper bound $\|\Omega^*\| \cdot \|\Omega\|^{-1}$ where Ω^* ranges through all subsystems of Ω with $\rho(\Omega^*) \leqq r$, and $F_r(S)$ = lower bound $\sigma_r(\Omega)$ for all Ω. If the members of Ω are required to be congruent instead of similar to S, the number corresponding to $F_r(S)$ is $f_r(S)$. The following estimates of F_r and f_r can be proved (to appear in the Proc. London Math. Soc.) assuming S to be bounded and convex.

(i) If T is the union of all sets congruent and homothetic to S and containing the origin, and $|A|$ denotes the volume of any set A, $\Delta(A)$ the critical determinant of A as defined in the geometry of numbers, then

$$\frac{|S|}{2^n\Delta(T)} \leqq f_1(S) \leqq \frac{|S|}{|T|}.$$

There is equality in both places if T is a body representing the boundary case of Minkowski's theorem on lattice points in convex bodies, e.g., if $n = 2$ and S is a triangle or a centrally symmetric hexagon.

(ii) $0 < f_1(S) < f_2(S) < \cdots$ if $n > 1$, $f_r(S) \to 1$ as $r \to \infty$.

(iii) $(n+2)^{-n} < F_1(S) \leqq f_1(S) \leqq 2^{-n}$.

(iv) If S is a cylinder with base B, and if $f_r(B)$ refers to B embedded in an R^{n-1}, then $f_r(B)/2 \leqq f_r(S) \leqq (r/2)f_1(B)$, $f_1(S) = f_1(B)/2$.

(v) If S is a cube, then $f_r(S) = r \cdot 2^{-n}$ $(r = 1, 2)$,

$$(1 - 2r^{-1/n})^n < f_r(S) \leqq [r - (r-1)^{1-n}(r-2)^n](n+1)^{-1} \qquad (r \geqq 2^n),$$

$$1 - 2nr^{-1/n} + O(r^{-2/n}) \leqq f_r(S) \leqq 1 - \frac{n(n-1)}{2(n+1)} r^{-1} + O(r^{-2}) \qquad (r \to \infty).$$

For any arbitrary sets S and T of finite measure $|S|$, $|T|$ respectively we have

(vi) $|S|^{-r} f_r(S) \leqq |T|^{-r} f_r(T)$ whenever $S \subset T$, and $r = 1, 2, \cdots$.

Some of the estimates above can be sharpened, and other estimates can be deduced for special classes of sets S. In the whole of this work the author is indebted to A. S. Besicovitch for some of the results and some methods of proof.

King's College, University of London,
London, England.

DIFFERENTIAL GEOMETRY

THE EDGE OF REGRESSION OF PSEUDOSPHERICAL SURFACES

HOWARD W. ALEXANDER

Given a curve Γ in space for which: (a) the position vector $x(s)$ is an analytic function of s, the arc length measured from a particular point P, and (b) $\tau^2 \neq 1$ at every point, where $1/\tau$ is the torsion of the curve, then there exists an analytic function $\theta(u, v)$ and a corresponding pair of pseudospherical surfaces Σ_1 and Σ_2 with the following properties: (1) Σ_1 and Σ_2 meet in the curve Γ to form a cuspidal edge, or an edge of regression. The curve Γ is the envelope of both sets of asymptotic lines, which cross from Σ_1 to Σ_2 as they touch Γ. Along Γ, the mean curvature of Σ_1 and Σ_2 is infinite, and the geodesic curvature of Γ is equal to its space curvature; (2) the function θ is the angle between the asymptotic lines, which is zero along Γ, and u and v are the arc lengths along the asymptotic lines. Thus $\theta(u, v)$ is a solution of $\partial^2\theta/\partial u\partial v = \sin\theta$. The function $\theta(u, v)$ is analytic in the neighborhood of P. The conditions of analyticity may be replaced by suitable differentiability conditions. The effect of these results is to establish an essentially one-to-one correspondence between the totality of pseudospherical surfaces and the totality of twisted curves in space.

ADRIAN COLLEGE,
ADRIAN, MICH., U. S. A.

POLYGENIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

JOHN DECICCO

Polygenic functions are single-valued continuous complex functions of the form $w = F(z) = F(z^1, \cdots, z^n) = \phi(x^1, \cdots, x^n; y^1, \cdots, y^n) + i\psi(x^1, \cdots, x^n; y^1, \cdots, y^n)$, with continuous partial derivatives in a certain region of the $2n$-dimensional space R_{2n} of the n complex variables $z^\alpha = x^\alpha + iy^\alpha$, for $\alpha = 1, \cdots, n$, such that the Cauchy-Riemann equations are not necessarily obeyed. The Taylor formula with remainder of a polygenic function is discussed together with applications to analytic polygenic functions. The representations of a polygenic function and its mean and phase derivatives as limits of integrals are obtained. The derivative dw/dz relative to a conformal surface S_2 and a complex variable z is a congruence of clocks possessing the three properties developed by Kasner. The clocks, corresponding to all possible S_2 through a point P of R_{2n}, form a linear family depending essentially on $2(n - 1)$ real constants. If dw/dz is unique relative to n conformal surfaces through P, not all of which belong to the same pseudo-conformal manifold of $2(n - 1)$-dimensions, then w is monogenic at P. The multiharmonic functions relative to an S_2 are also studied. Kasner's pseudo-angle completely characterizes the pseudo-conformal group within the polygenic group in R_{2n}. If two real functions ϕ and ψ are conjugate-multiharmonic, then any curve in ϕ = const. is pseudo-orthogonal to the manifold ψ = const. Many other applications of the pseudo-angle are obtained.

DePaul University,
Chicago, Ill., U. S. A.

A CORRESPONDENCE PRINCIPLE IN CONFORMAL GEOMETRY

Aaron Fialkow

In earlier papers (especially *Conformal differential geometry of a subspace*, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 309–433) an analytic apparatus was developed and used to study the properties of a subspace V_n $(1 \leqq n \leqq m - 1)$ imbedded in any enveloping space V_m which remained unchanged under conformal transformations of V_m. The present paper establishes a correspondence principle between the classical differential geometry of a subspace V_n in euclidean space R_m and the conformal geometry of a subspace $\bar{V}_n$ in a conformally euclidian space $\bar{R}_m$. By means of this principle, conformal analogues may be found corresponding to any classical result "in the small." It is shown that a certain conformally euclidean metric $dS_{(P)}$, defined throughout $\bar{R}_m$, may be associated with any point P of $\bar{V}_n$. This metric is conformally invariant and reduces to the conformal metric of $\bar{V}_n$ in a neighborhood of P on $\bar{V}_n$. The metric $dS_{(P)}$ is defined by means of a certain point P', conformally related to P, and if $\bar{R}_m$ is subjected to a conformal map onto a euclidean space R_m in which $\bar{V}_n \leftrightarrow V_n$ and the image of P' is the point at infinity, then $dS_{(P)}$ is simply the euclidean metric of this R_m. Any result concerning the classical differential geometry of this V_n in its enveloping R_m corresponds to an analogous result in the conformal geometry of the subspace. In this way, the paper defines and derives pictorial interpretations for conformal analogues of the classical geometric objects and theorems connected with the differential geometry of a neighborhood of a point of a subspace. In terms of these results, simple interpretations of the conformal measure tensors of the subspace are given. The variation of the metric $dS_{(P)}$ as P varies on $\bar{V}_n$ is shown to depend upon the deviation tensor and deviation vector which have no analogues in classical differential geometry. It is also shown that the metrics $dS_{(P)}$ of the points P of $\bar{V}_n$ induce a conformally invariant Riemannian metric upon the enveloping space in the neighborhood of $\bar{V}_n$, and that there is a correspondence between the Riemannian geometry of the subspace in its enveloping Riemann space and the conformal geometry of $\bar{V}_n$ in $\bar{R}_m$.

Polytechnic Institute of Brooklyn,
Brooklyn, N. Y., U. S. A.

ON THE UNIQUENESS OF GEODESICS

PHILIP HARTMAN

Let $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ be a positive-definite element of arc length. If the functions $E(u, v)$, $F(u, v)$, $G(u, v)$ are of class C^1 on some (u, v)-domain, one can consider the differential equations for geodesics, involving the Christoffel symbols Γ^i_{jk} as coefficients. This note answers the question raised by Professor Wintner as to whether or not geodesics are uniquely determined (locally) by initial conditions (even though the coefficient functions Γ^i_{jk} of the differential equation are not subject to a "uniqueness" restriction of the Lipschitz type).

In general, the answer is in the negative; however, (*) if ds^2 is the element of arc length on a (2-dimensional) surface of class C^2 (in a 3-dimensional Euclidean space), then a geodesic is uniquely determined by initial conditions. The proof of this assertion (*) depends on the circumstance that the theorem of Gauss-Bonnet and/or the Theorema Egregium (in a suitable form) holds on surfaces of class C^2 [H. Weyl, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich vol. 61 (1916) pp. 40–72; cf. also E. R. van Kampen, Amer. J. Math. vol. 60 (1938) pp. 129–138] and the fact that the total curvature integral has a bounded integrand, namely, the Gaussian curvature of the surface.

An extension of the arguments used in the proof of (*) shows that it is possible to introduce local geodesic polar (or parallel) coordinates on a surface of class C^2.

THE JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MD., U. S. A.

SPINOR SPACE AND LINE GEOMETRY

VÁCLAV HLAVATÝ

L_3 denotes the ideal space of a four-dimensional centered metric vector space, $\bar{Q}$ its absolute, $\bar{x}$ ($\bar{v}$) a generic point on $\bar{Q}$ (not on $\bar{Q}$), v^ν a vector on $\bar{Q}$ defined at $\bar{x}$, anc S_3 the projective three-dimensional spinor space on which L_3 is mapped by Cartan's method:

1) $\bar{x}$ is mapped on a straight line Ξ_0 in S_3. The locus of Ξ_0 (as $\bar{x}$ moves on $\bar{Q}$) is a linear congruence Γ^2, intersection of two projectively orthogonal linear complexes Γ^3 and $*\Gamma^3$.

2) A lineal element $(\bar{x}, v^\nu)$ is mapped on a couple of spinors $(v^1)^{1/2}\psi_1 + \epsilon(v^2)^{1/2}\psi_2$, where $\epsilon = \pm 1$, ψ_i ($i = 1, 2$) are focal points of Ξ_0 in Γ^2, $v^i = e^i_\lambda v^\lambda$ and e^i_λ are the null vectors of the conformal metric tensor of $\bar{Q}$. As v^ν changes, these couples constitute an involution on Ξ_0 with the double points ψ_1, ψ_2. $\bar{Q}$ as a point set is mapped on (the lines of) the congruence Γ^2, while the set of lineal elements on $\bar{Q}$ is mapped on (spinor points of) S_3.

3) The lineal elements $(\bar{x}, v^\nu)$ common to $\bar{Q}$ and its tangential cone with the

vertex $\bar{v}$ are mapped on a couple of distinct lines Ω_+, Ω_-, whose locus is Γ^3. They are conjugate polars with respect to $^*\Gamma^3$ and axis of the biaxial involution defined by the Cartan's matrix of the point $\bar{v}$. As $\bar{v}$ moves along a line in L_3 the locus of Ω_+, Ω_- is a regulus (Lie's "sphere"-line transformation).

4) The plane coordinates ξ_a of the polar plane to a generic spinor ξ^a with respect to Γ^3 are the "covariant components of ξ^a" in the theory of quanta.

5) Any projective transformation L_b^a in L_3 reproducing $\bar{Q}$ is mapped on two always distinct transformations whose coefficients are uniquely determined by L_b^a. The biaxial involution of 2) is a map of a projective transformation of lineal elements on $\bar{Q}$, induced on $\bar{Q}$ by a projective collineation with the center $\bar{v}$.

6) Dirac's equations for a wave spinor are the equations for double points of a symbolical biaxial involution (cf. 3)) on one of its symbolical axes Ω_+, Ω_-. Hence it is to be expected that the equations for the remaining double points will also have a physical interpretation.

INDIANA UNIVERSITY,
BLOOMINGTON, IND., U. S. A.

A GENERAL THEORY OF CONJUGATE NETS IN PROJECTIVE HYPERSPACE

CHUAN-CHIH HSIUNG

The purpose of this paper is to establish a general theory of the projective differential geometry of conjugate nets in a linear space S_n of $n(\geqq 4)$ dimensions. By purely geometric determinations a completely integrable system of linear homogeneous partial differential equations in canonical form is introduced, defining a conjugate net in S_n except for a projective transformation. Relative to an invariant local pyramid of reference the conditions of immovability for a point and a hyperplane in S_n are deduced. Some results of the author for conjugate nets in three- and four-dimensional spaces are extended to obtain the following theorems. (1) In a linear space S_n $(n \geqq 3)$ let N_x be a conjugate net and π be a fixed hyperplane, then the points M, $\bar{M}$ of intersection of the fixed hyperplane π and the two tangents at a point x of the net N_x describe two conjugate nets N_M, $N_{\bar{M}}$ in the hyperplane π respectively, and one of the two nets N_M, $N_{\bar{M}}$ is a Laplace transformed net of the other. (2) In a linear space S_n $(n \geqq 4)$ let N_x be a conjugate net and S_{n-2} be a fixed linear subspace, then the point of intersection of the fixed subspace S_{n-2} and the tangent plane at a point x of the surface sustaining the net N_x describes a conjugate net in the subspace S_{n-2}. (3) Conjugate nets with equal and nonzero Laplace-Darboux invariants in a linear space S_n $(n \geqq 4)$ are characterized by the property that at each point x of any one of them there exists a proper hyperquadric (and therefore $\infty^{n(n+3)/2-11}$ such hyperquadrics) having second order contact at the

Laplace transformed points x_{-1}, x_1 of the point x with both Laplace transformed surfaces S_{-1}, S_1 of the net N_x, respectively.

NORTHWESTERN UNIVERSITY,
EVANSTON, ILL., U. S. A.

ANGULAR MEASURE AND THE GAUSS-BONNET FORMULA

STANLEY B. JACKSON

Let C be a closed curve of class C'' lying in a simply connected region S of a surface, and having only a finite number of double points, all simple crossings. C cuts from S a finite number of simply connected regions R_i. From the Gauss-Bonnet formulas for the R_i the following relation is obtained:

$$\int_C ds/\rho + \sum_i x_i \iint_{R_i} K\,dS = 2\prod \left(\sum_i x_i - \sum_j y_j\right) \tag{1}$$

where $1/\rho$ is the geodesic curvature of C, and x_i and y_j are integral multiplicities assigned to region R_i and double point P_j respectively. The quantity $\Gamma = 2\pi(\sum_i x_i - \sum_j y_j)$ may be called the angular measure of C on S since, for a plane curve, Γ is precisely the plane angular measure.

A continuous deformation of a closed parametrized curve of class C' into a second such curve is said to be of class C' if every intermediate curve is of class C'. It is known that a necessary and sufficient condition that two closed plane curves of class C' can be deformed into each other by a deformation of class C' is that they have the same plane angular measure. It is shown that the class C' deformation classes in a simply connected region S of a surface are likewise completely characterized by the angular measure Γ on S.

If C is on a topological sphere, (1) is not uniquely determined, but it is shown that Γ is determined to within an integral multiple of 4π. The class C' deformation classes on the sphere are still characterized by the angular measure, for two spherical curves are shown to belong to the same deformation class if and only if their angular measures Γ_i are congruent mod 4π.

A formula analogous to (1) for curves on a torus can be obtained by first making cuts C_1 and C_2 to render it simply connected. The angular measure Γ is then uniquely determined, and two curves on the torus are shown to belong to the same class C' deformation class if and only if (a) they belong to the same homotopy class and (b) they have the same angular measure Γ.

UNIVERSITY OF MARYLAND,
COLLEGE PARK MD., U. S. A.

THE CONVERSE OF THE THEOREM OF MEHMKE-SEGRE

EDWARD KASNER

The theorem of Mehmke-Segre (originally stated by H. S. Smith) states that if two curves touch, the ratio γ_2/γ_1 of their curvatures is a projective invariant. The author proves that the only point transformations, and the only contact transformations, which leave this ratio invariant form the projective group.

Two curves in contact form a horn angle. In terms of horn angles other important groups may be characterized. Thus for the conformal group the conformal measure

$$M_{12} = \frac{(\gamma_2 - \gamma_1)^2}{\gamma_2' - \gamma_1'}$$

is invariant where the prime indicates differentiation with respect to arc length. Conversely this is characteristic of conformal transformations among all contact transformations.

For the equilong group the analogous measure is

$$M_{12}^* = \frac{(r_2 - r_1)^2}{r_2' - r_1'}$$

where r means radius of curvature and prime means differentiation with respect to inclination θ.

We show that $\gamma_2 - \gamma_1$ and $r_2 - r_1$ are relative invariants which characteiize the conformal and equilong groups.

COLUMBIA UNIVERSITY,
NEW YORK, N. Y., U. S. A.

CURVES IN MINKOWSKI SPACE

C. C. MACDUFFEE

A matrix Λ is called Lorentzian if $\Lambda^T J \Lambda = J$ for some symmetric orthogonal matrix J. If $\Lambda(s)$ is a Lorentzian matrix whose elements are differentiable functions of the real variable s, then $\Lambda(s)$ satisfies a differential equation $\Lambda'(s) = P(s)\Lambda(s)$ where $P(s)J$ is skew. Conversely every such matrix $P(s)$ having continuous elements determines uniquely a Lorentzian matrix $\Lambda(s)$ which reduces to I for $s = 0$. Orthogonal matrices are special instances for $J = I$. In particular a continuous curve in Minkowski four-space has a moving tetrahedral determined by a Lorentzian matrix $\Lambda(s)$ where s is the parameter in the equations of the curve, the differential equation $\Lambda'(s) = P(s)\Lambda(s)$ corresponding to the Frenet-Serret formulas. An osculating curve at $s = s_0$ is obtained by replacing $P(s)$ by $P(s_0)$. These osculating curves are determined for all continuous curves in the Minkowski space of special relativity. The generic case is a curve reducible to the form $t = a \sinh \mu s$, $x = a \cosh \mu s$, $y = b \sin \nu s$, $z = b \cos \nu s$ where $a^2\mu^2 - b^2\nu^2 = 1$, but some of the nongeneric cases are of special interest.

UNIVERSITY OF WISCONSIN,
MADISON, WIS., U. S. A.

CURVATURE OF CLOSED HYPERSURFACES

S. B. MYERS

It is known that no closed surface whose Gaussian curvature is everywhere nonpositive exists in euclidean 3-space E_3; also, a minimal surface in E_3 has nonpositive curvature everywhere, so that no closed minimal surface exists in E_3. The object of this paper is to generalize these facts Among the results obtained are the following. (1) If M_{n-1} is a closed hypersurface in euclidean space E_n or in hyperbolic space H_n, there exists a point in M_{n-1} at which all sectional (Riemannian) curvatures of M_{n-1} are positive; if M_{n-1} is a closed hypersurface in the n-sphere S_n of radius R, lying entirely in an open hemisphere of S_n, there exists a point in M_{n-1} at which all sectional curvatures of M_{n-1} are not less than $1/R^2$. (2) If M_{n-1} is a minimal hypersurface in a Riemannian manifold M_n, at every point of M_{n-1} the Ricci curvature of M_{n-1} in each direction does not exceed the Ricci curvature in the same direction of the tangent geodesic hypersurface to M_{n-1}. (3) Hence there does not exist a closed minimal hypersurface in E_n or in H_n, while in S_n no open hemisphere contains a closed minimal hypersurface.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

UEBER STRAHLENSYSTEME DEREN ABWICKELBAREN FLÄCHEN EINE FLÄCHE UNTER GEODÄTISCHEN LINIEN UND IHREN GEODÄTISCHEN PARALLELEN SCHNEIDEN

NILOS SAKELLARIOU

Es seien

$$\xi = x(u_1, u_2) + \lambda \cdot \zeta(u_1, u_2) \tag{1}$$

die vektorielle Gleichung eines Strahlensystems, 2ρ der Abstand der zwei Brennpunkte, x der Bestimmungsvektor des Mittelpunktes, $\zeta(u_1, u_2)$ der Einheitsvektor der lauferden geradlinie von (1), u_i ($i = 1, 2$) die Parameter der abwickelbaren Flächen von (1) und $\zeta_{u_1}^2 = e$, $\zeta_{u_1} \cdot \zeta_{u_2} = f$, $\zeta_{u_2}^2 = g$ die Koeffizienten der Fundamentalform $e\, du_1^2 + 2f\, du_1 du_2 + g\, du_2^2$. Es ist gesucht diejenige Fläche Φ zu bestimmen, welche von der abwickelbaren Flächen des (1) sich unter geodätischen Linien (u_2 = constant) und ihrer (u_1 = constant) orthogonaltrajektorien (u_1 = Bogenlänge der geodätischen Linie) schneidet. Setzt man $\lambda = \lambda(u_1, u_2)$ in (1), so hat man die vektorielle Gleichung von Φ in Bezug auf die Linien in denen sie sich mit den abwickelbaren Flächen von (1) schneidet.

Setzt man $f = 0$ and $\lambda_{u_1} + \rho_{u_1} + 2b\rho = 0$ ($b = g_{u_1} : 2g$), so hat man $\lambda = \rho \pm e^{-1/2}$, und eine hinreichende Bedingung $\rho_{u_1} + b\rho \mp e_{u_1}/4e^{3/2} = 0$, in Verbindung mit

$$(2) \qquad \rho_{u_1u_2} + a\rho_{u_1} + b\rho_{u_2} + (a_{u_1} - b_{u_2})\rho = 0 \qquad (a = e_{u_2} : 2e).$$

Setzt man $f = 0$ und $\lambda_{u_2} - \rho_{u_2} - 2a\rho = 0$, so hat man die hinreichende Bedingung $[\rho_{u_1} + 2b\rho + \int (\rho_{u_2} + 2a\rho)_{u_1}du_2 + \varphi(u_1)]^2 + [\int (\rho_{u_2} + 2a\rho)du_2 + \varphi(u_1) - \rho]e = 1$, in Verbindung mit (2); $\varphi(u_1)$ ist eine willkürliche Funktion von u_1. Die Differentialgleichung (2) reduziert sich entweder in die $\rho_{u_2} + a\rho = e^{-\int b du_1 + \psi_2(u_2)}$, wenn $b_{u_2} - ab = 0$ ist, oder in die $\rho_{u_1} + b\rho = e^{-\int a du_2 + \psi_1(u_1)}$, wenn $a_{u_1} - ab = 0$ ist; $\psi_1(u_1)$, $\psi_2(u_2)$ sind willkürliche Funktionen von u_1, u_2 entsprechend.

Im Falle $e = g$ hat man für $\zeta(u_1, u_2)$ die Differentialgleichung

$$2ee_{u_1u_2} - 3e_{u_1} \cdot e_{u_2} = 0 \text{ oder } \zeta_{u_1} \cdot \zeta_{u_2^2u_2} - 2\zeta_{u_2^2} \cdot \zeta_{u_1u_2} = 0.$$

University of Athens,
Athens, Greece.

DIRETTRICI CONGIUNTE E BICONGIUNTE DI UNA RIGATA

Alessandro Terracini

Vi sono delle proprietà geometriche che richiedono uno studio "in profondità." Sono le proprietà su alcune delle quali ebbi l'occasione di richiamare l'attenzione anche in questi ultimi anni, suscettibili di sussistere in senso approssimato, in vari gradi successivi di approssimazione. La presente comunicazione appartiene a tale ordine di idee: essa concerne le direttrici di una rigata nell'ambito della geometria della retta.

Per due direttrici C_y, C_z di una rigata non sviluppabile R di S_3 i fasci delle tangenti alla rigata in y, z e nei due punti della generatrice successiva alla yz situati rispettivamente su quelle due direttrici stanno sempre in un complesso lineare K (unico, escludendo direttrici entrambe asintotiche).

Riferite le generatrici della R a un parametro t, la proprietà enunciata vuol significare che il primo membro dell'equazione—scritta nel modo consueto—che traduce l'appartenenza dei fasci considerati ad un complesso lineare, per i valori t e $t + h$ del parametro è infinitesimo d'ordine $\leqq 5$, anzichè soltanto $\leqq 3$ come generalmente avverrebbe scegliendo altrimenti quei fasci (quell'infinitesimo non può mai essere d'ordine pari). Perciò chiamando ordine d'approssimazione σ l'eccesso di quell'ordine d'infinitesimo rispetto a 3, l'esistenza di K sussiste sempre nell'ordine $\sigma \geqq 2$.

Chiamo *congiunte*, o *bicongiunte*, quelle direttrici quando $\sigma \geqq 4$, o rispettivamente $\sigma \geqq 6$. Lo studio delle direttrici congiunte è agevole: in generale, data arbitrariamente sulla R la C_y, esistono ∞^1 direttrici C_z (dipendenti da un'equazione di Riccati) congiunte, e anzi congiunte ad un sistema ∞^1 di direttrici determinato dalla C_y (sistema che si può caratterizzare geometricamente in modo semplice). È eccezionale il caso in cui C_y è un ramo della linea flecnodale: se C_y non è rettilinea, esiste una sola direttrice congiunta, coincidente col secondo ramo in quanto distinto dal primo; se è rettilinea le direttrici congiunte sono indeterminate.

Più notevole e più riposta è la determinazione delle direttrici bicongiunte: su una rigata R, nelle ipotesi più generali, ne esistono ∞^3 coppie, determinate da un sistema di equazioni differenziali che si può scrivere effettivamente. Lo studio del caso successivo $\sigma \geqq 8$ sembra condurre a calcoli molto laboriosi.

University of Turin,
Turin, Italy.

NONLINEAR DISPLACEMENTS IN AFFINE-CONNECTED SPACE

J. L. Vanderslice

An affine connection assigned to a general analytic manifold A_n establishes through the associated affine normal coordinate systems an invariant point correspondence between the tangent space of differentials at any point P and a finite neighborhood of P in A_n. This correspondence is not preserved under the affine point displacement defined by the connection unless the latter is flat. It is here shown that the correspondence is preserved by a unique nonlinear integrable displacement $y^i_{,j} = -\delta^i_j + G^i_j(x, y)$. The y^i are affine normal coordinates associated with the point x, and G^i_j is a tensor expressed as a power series in y with certain generalized normal tensors as coefficients. Stated another way, the equation shows how the normal coordinates vary as the origin of these coordinates varies. Discarding the terms of degree greater than r furnishes a nonlinear displacement of rth order neighborhoods, in general nonintegrable. The first case of interest is $r = 2$ since $r = 1$ gives just the familiar linear point displacements between tangent spaces. These second degree displacements lead easily to differential equations for affine geodesic deviation. It is found also that they are properly projective when and only when the space is projectively flat with skew-symmetric Ricci tensor. In this case they are nonintegrable. However, when and only when the space is projectively flat there exists a modification of the normal coordinates unique to the third order and leaving them geodesic in terms of which the second degree displacements are both projective and integrable.

University of Maryland,
College Park, Md., U. S. A.

SUR CERTAINS RÉSEAUX TRACÉS SUR UNE SURFACE ET LEUR RÔLE EN GÉOMÉTRIE DIFFÉRENTIELLE

Paul Vincensini

M étant un point quelconque d'une surface S et P un point associé à M suivant une loi continue arbitraire, j'ai récemment montré l'intérêt que présente, pour l'étude de la correspondance ponctuelle $[M \rightarrow P]$, le réseau (R) de S dont les courbes correspondent avec orthogonalité des éléments linéaires aux courbes homologues sur la surface Σ lieu de P.

Dans cette étude, les deux cas où P est dans le plan tangent en M à S ou en dehors se distinguent nettement. Dans le premier cas (R) est invariant par déformation arbitraire de S, ce qui n'a pas lieu dans le deuxième. Mais, dans les deux cas, il y a lieu de noter une circonstance d'un caractère assez surprenant.

Par l'application d'un procédé uniforme, consistant à varier la loi d'association $[M \rightarrow P]$ ou à imposer à (R) telle propriété spéciale, on retrouve successivement et sous une forme renouvelée les questions de géométrie les plus diverses: Transformation de Laplace des réseaux orthogonaux, surfaces de Guichard, surfaces spirales de M. Lévy, congruences de Ribaucour, congruences W, transformations des surfaces avec conservation des aires, déformation des quadriques et des surfaces réglées, congruences de sphères de courbure $+ 1$ de A. Demoulin, solutions permanentes de l'équation de Cayley dans la théorie des systèmes triples orthogonaux, propriétés nouvelles des lignes de courbure, des asymptotiques réelles ou virtuelles, des développées moyenne ou harmonique d'une surface.

Toutes ces questions, et d'autres encore dont l'ensemble occupe une place importante dans le développement de la géométrie différentielle classique, viennent ainsi se grouper de la façon la plus directe autour de la notion de réseau (R), lui conférant un intérêt sur lequel il m'a semblé opportun d'attirer l'attention.

Pour la mise en oeuvre du procédé de groupement signalé on pourra se reporter au Mémoire: Questions liées au caractère invariant de certains réseaux (Ann. École Norm (3) vol. 64), et à un autre travail qui paraîtra prochainement au Bull. Soc. Math. France, où la notion de réseau (R) est étendue à la géométrie hyperspatiale.

University of Marseille,
Marseille, France.

AFFINE AND PROJECTIVE GEOMETRIES OF SYSTEMS OF HYPERSURFACES

KENTARO YANO

In an n-dimensional space X_n referred to a coordinate system (x^i) $(i, j, k, \cdots = 1, 2, \cdots, n)$, let us suppose that there is given a system of hypersurfaces such that there exists one and only one hypersurface passing through n given points in general position and belonging to the system. Such a system of hypersurfaces is represented by an equation of the form

$$f(x^1, x^2, \cdots, x^n; a^1, a^2, \cdots, a^n) = 0,$$

where the a's are parameters fixing the hypersurface.

Thus, the system of hypersurfaces is represented by (i) a system of coordinates (x^i) for the enveloping space X_n, (ii) a function $f(x; a)$ for the system of hypersurfaces, and (iii) n essential parameters (a^i) to determine various hypersurfaces. But the system of hypersurfaces must be an invariant geometric configuration under (1) a transformation of coordinates $\bar{x}^i = \bar{x}^i(x)$ in X_n, (2) a transformation of the function f: $\bar{f}(x; a) = \alpha f(x; a)$, and (3) a transformation of parameters $\bar{a}^i = \bar{a}^i(a)$.

The geometry in which the above factor α is assumed to be a constant will be called the affine geometry of a system of hypersurfaces while the geometry in which α is assumed to be a function of x and a is called the projective geometry of a system of hypersurfaces.

In the affine case, the system of hypersurfaces is represented by a system of partial differential equations

$$\frac{\partial^2 f}{\partial x^j \partial x^k} = \Gamma_{jk}(x, u), \qquad \left(u_i = \frac{\partial f}{\partial x^i}\right).$$

The so-called projective change of Γ_{jk} will be obtained in the form

$$\bar{\Gamma}_{jk} = \Gamma_{jk} + H_j u_k + H_k u_j.$$

The main purpose of the present paper is to determine a projectively connected space of hyperplane elements whose system of totally geodesic hypersurfaces just coincides with the given system of hypersurfaces.

UNIVERSITY OF TOKYO,
TOKYO, JAPAN.

ALGEBRAIC GEOMETRY

SOPRA ALCUNE SUPERFICIE ALGEBRICHE UNIFORMIZZABILI

Aldo Andreotti

Le superficie algebriche di cui si conosce una rappresentazione parametrica con funzioni analitiche uniformi di due variabili sono in numero molto ristretto (superficie razionali, riferibili a rigate, ellittiche, iperellittiche, superficie delle coppie di punti di una curva algebrica, superficie prodotto di due curve algebriche, $\cdots$).

Picard [in una nota dei C. R. Acad. Sci. Paris del 4. VII. 1904] studiando alcuni sistemi di trascendenti nuove, generalizzazione, sotto certo aspetto, delle classiche funzioni abeliane, affacciò per primo il sospetto che tali funzioni potessero essere utilizzate per la "uniformizzazione" di tipi di superficie più generali di quelle fin'allora parametrizzabili con funzioni note.

Si tratta qui di quei sistemi di funzioni meromorfe, studiati nel caso di una sola variabile, per la prima volta da Poincarè quindi, per più variabili, da Picard, i quali *ammettono un teorema di moltiplicazione*; con ciò s'intende che esistono certe costanti a, b, $\cdots$, l, tante quante sono le variabili u, v, $\cdots$, w, non tutte uguali all'unità, tali che i valori delle funzioni del sistema nel "punto" $(au, bv, \cdots , lw)$ si esprimono razionalmente per mezzo dei valori delle funzioni medesime nel punto $(u, v, \cdots , w)$.

Ebbene Picard, nella nota citata, enuncia, dando un cenno della dimostrazione, un teorema, secondo il quale una superficie algebrica possedente una trasformazione razionale in sè con determinate proprietà differenziali nell'intorno di un punto unito, è parametrizzabile colle funzioni suindicate. Egli osserva che alla famiglia delle superficie suddette appartengono le superficie razionali e quelle iperellittiche ed aggiunge: "il serait, je crois, intéressant de rechercher s'il y a d'autres surfaces que les precedentes . . . rentrant dans la classe sur la quelle certaines équations fonctionnelles appellent ainsi l'attention."

In un lavoro di prossima pubblicazione (negli atti dell' Accademia delle Scienze detto dei XL) si dà risposta al problema precedente. Anzitutto si dimostra compiutamente il teorema di Picard in discorso; si prova quindi la possibilità di uniformizzare colle nominate funzioni la superficie generica di numerose famiglie di superficie algebriche (quali ad es. le superficie del 4° ordine contenenti una coppia di rette, incidenti o non; le superficie di genere $P_g = 0$, bigenere $P_2 = 1$, trigenere $P_3 = 0$ (superficie di Enriques) che non sono nè razionali nè iperellittiche nè appartengono ad alcuno dei tipi sopra indicati.

Al termine del lavoro si prova che le superficie algebriche irrazionali uniformizzabili colle funzioni di Picard-Poincaré hanno necessariamente il genere lineare $p^{(1)} \leqq 1$ sicchè, se si escludono le superficie razionali, ellittiche ed iperellittiche restano solo superficie con un fascio, di genere uguale all'irregolarità, di curve ellittiche o superficie regolari con tutti i generi uguali ad 1.

Gli esempi addotti mostrano che entrambi questi casi sono possibili.

UNIVERSITY OF ROME,
ROME, ITALY.

ALGEBRAIC THEORY OF INTERSECTIONS FOR CYCLES OF AN ALGEBRAIC VARIETY

I. BARSOTTI

Let S be an n-dimensional projective space over an arbitrary field k, and let z_1, z_2 be two cycles of S, of dimensions n_1, n_2 such that $n_1 + n_2 \geqq n$. If C is any maximal algebraic system of cycles of S containing z_2, let F be the image-variety of C, and call D the algebraic correspondence between F and S such that when P ranges over the points of F, the corresponding cycle $D\{P\}$ of S ranges over the whole C. If D is considered in the direction $S \rightarrow F$, then it induces a well defined correspondence D' between z_1 and F; if $\dot{P}$ is the point of F such that $z_2 = D\{\dot{P}\}$, then by definition $D'\{\dot{P}\}$ is the intersection-product of z_1 and z_2. It is proved that this definition is symmetrical in z_1, z_2, and does not depend on the choice of C; moreover, it has a meaning also when some component of the intersection has a dimension greater than $n_1 + n_2 - n$. The generalization to the case in which S is any irreducible variety does not present serious difficulty. This way of reaching a definition of multiplicity as an application of the theory of algebraic correspondences is nearer to the original geometrical method, and is made possible by the previous development, by the author, of a theory of correspondences independent of the intersection theory (Ann. of Math. vol. 52 (1950) pp. 427–464). Although the present one is a theory "in the large," one of the main results is reached by "local" methods, thus providing the link with local theories.

UNIVERSITY OF PITTSBURGH,
PITTSBURGH, PA., U. S. A.

THE SYMBOLICAL METHOD IN ALGEBRAIC GEOMETRY

E. M. BRUINS

Several problems of algebraic geometry can be solved in a very simple way using the fundamental theorems of the symbolic method. If, for example, we ask for the degree of the parabolic curve of $f = (a'x)^n = 0$ in S_3 we have, if x is on the curve, from the condition that every line in the tangent plane

$(a'x)^{n-1}(a'y) = 0$, $(u'y) = 0$ intersects $(a'x)^{n-2}(a'y)^2 = 0$ in two coincident points, immediately that

$$0 \equiv (a'b'c'u')(a'b'd'u')(a'x)^{n-2}(b'x)^{n-2}(c'x)^{n-1}(d'x)^{n-1}$$
$$\equiv (a'b'c'd')^2(a'x)^{n-2}(b'x)^{n-2}(c'x)^{n-2}(d'x)^{n-2}\cdot(u'x)^2 \qquad (\text{mod } f)$$

and the degree is $4n(n-2)$.

Again, the resultant of two binary forms a_x^n and α_x^m is of degree n in α and m in a and therefore the weight is nm. The principle of extension gives immediately: The resultant of the N-ary forms $(a_i'x)^{n_i}$, $i = 1, 2, \cdots, N$, is of degree π/n_i in a_i' and of weight $\pi = n_1n_2\cdots n_N$. Thus if we have in S_{kh} a $V_{(k-1)h}$ determined by $(a_i'x)^{n_i} = 0$, $i = 1, 2, \cdots, h$, the resultant of the system $(a_i'x)^{n-p}(a_i'y)^p = 0$, $p = 2, 3, \cdots, k$, and an intersecting $S_{(k-1)h-1}$ is of degree in x $\sum_{i=1}^{h}\sum_{p=2}^{k}((k!)^h/p)\,(n_i - p)$ and of weight $(k!)^h$, i.e., of degree $(k!)^h$ in the coordinates of the $S_{(k-1)h-1}$ in S_{kh}. Taking this $S_{(k-1)h-1} \equiv (v_1'v_2'\cdots v_h'u')$, $(v_i'y) = (a_i'x)^{n_i-1}(a_i'y)$, we find, with $\Sigma = n_1 + n_2 + \cdots + n_h$, the degree of the hyperspaces intersecting the points with $(k+1)$-fold tangents of $V_{(k-1)h}$ to be $(k!)^h[(1 + 1/2 + \cdots + 1/k)\Sigma - (hk + 1)]$ as $(k!)^h$ factors $(u'x)$ can be split off reducing modulo f_i; for example,

$h = 1, k = 2$, V_1 in S_2; $3(n-2)$; $3n(n-2)$ points of inflexion.

$h = 1, k = 3$, V_2 in S_3; $(11n - 24)$; $n = 3$, 27 lines on a cubic surface.

$h = 1, k = 4$, V_3 in S_4; $(50n - 120)$; $n = 4$, ruled surface of degree 320.

$h = 2, k = 2$, V_2 in S_4; $(6(n + m) - 20)$;

$n = 2, m = 2$, 16 lines on the intersection of two quadrics.

University of Amsterdam,
Amsterdam, Netherlands.

REGULAR SURFACES OF GENUS 2

Patrick Du Val

If $[k]$ indicates k-dimensional space, in $[n+2]$ let Γ_{n+1}^2 be a quadric cone with $[n-1]$ vertex Ω_{n-1}, and G_3^{2n} a variety whose general hyperplane section is a surface with all genera 1; their intersection is a surface F^{4n}. If F^{4n} has a rational n-ic double curve K^n lying in a generating $[n]$ of Γ_{n+1}^2, it is the bicanonical model of a regular surface of genera $p_g = p_a = 2$, $p^{(1)} = n + 1$. The canonical pencil is traced by the generating $[n]$'s of Γ_{n+1}^2; its base points are the intersections of K^n, Ω_{n-1}, and are pinch points of the surface.

As a partial converse, the bicanonical model of every regular surface of these genera (with irreducible canonical pencil) lies on a Γ_{n+1}^2 whose generating $[n]$'s trace the canonical pencil.

The cases $n = 1, 2$ are known. Surfaces F^{4n} are constructed for which the general curve of the canonical pencil is hyperelliptic, for all values of n, and some for which it is of more general type for $n = 3, 4, 5, 6$. These are shown to include all surfaces of the genera in question (with irreducible canonical pencil) for which $n \leqq 4$, or whose canonical pencil is hyperelliptic. There are two distinct families for $n = 3$, and five for $n = 4$.

UNIVERSITY OF GEORGIA,
ATHENS, GA., U. S. A.

SINGULARITÉS DES POINTS DE DIRAMATIONS ISOLÉS DES SURFACES MULTIPLES

LUCIEN GODEAUX

Soit F une surface algébrique contenant une involution cyclique I d'ordre premier $p = 2\nu + 1$, n'ayant qu'un nombre fini de points unis. On construit sur F un système linéaire $|C|$ contenant p systèmes linéaires partiels $|C_0|$, $|C_1|, \cdots, |C_{p-1}|$ appartenant à I, dont le premier est dépourvu de points-base. On construit ensuite une surface Φ, image de I, telle qu'à ses sections planes Γ_0 correspondent les C_0.

Si A est un point uni de I, cette involution détermine dans le domaine du premier ordre de A soit l'identité, soit une involution d'ordre p. Plaçons-nous dans le second cas; il existe deux directions unise a_1, a_2 issues de A. Soient C_0' les courbes C_0 passant par A; elles y ont comme tangentes a_1, a_2; C_0'' les C_0' touchant en A une direction distincte de a_1, a_2, et ainsi de suite. On obtient une suite $|C_0'|$, $|C_0''|, \cdots, |C_0^{(\nu)}|$ de systèmes de courbes dont les multiplicités en A vont en croissant. Au système $|C_0^{(\nu)}|$, fait suite un système $|C_0^{(\nu+1)}|$ dont les courbes ont en A la multiplicité p et des tangentes variables.

Au point A sont attachés deux entiers inférieurs à p, α, β, tels que $\alpha\beta - 1$ soit multiple de p.

Les courbes $C_0^{(i)}$ ont la multiplicité $\lambda_i + \mu_i$ en A, λ_i tangentes confondues avec a_1, μ_i avec a_2. On a $\lambda_i + \alpha\mu_i \equiv 0$, $\mu_i + \beta\lambda_i \equiv 0 \pmod p$.

Supposons que l'on ait $\lambda_1 + \alpha\mu_1 = hp$, $\mu_1 + \beta\lambda_1 = kp$ et soit A' le point de diramation homologue de A. Le cône tangent à Φ en A' se scinde en deux cônes rationnels si $h = k = 1$, en trois cônes rationnels si $h > 1$ et $k = 1$, ou $h = 1$, $k > 1$, en quatre cônes rationnels si $h > 1$, $k > 1$.

L'étude des courbes qui correspondent sur Φ aux C_0'', C_0''', $\cdots$ permet de montrer que les cônes se rencontrent suivant une droite ou ne se rencontrent pas, et de déterminer les points doubles éventuels infiniment voisins de A'.

Bibliographie: Ann. École Norm. (1948); Annali di Matematica (1949); Bulletin de l'Académie des Sciences, des Lettres et des Beaux-arts de Beligique (1949).

UNIVERSITY OF LIÈGE,
LIÈGE, BELGIUM.

SYSTEMS OF SINGULAR PRIMALS IN S_r

Temple R. Hollcroft

In this paper, distinct simple multiple points (those involving no multiple planes) of algebraic primals f in S_r and systems of primals with multiple points are studied. There are $C_{r+s-2,s-1}$ types of each s-fold point. For example, there are r types of double points. A double point of f at P of type k, $1 \leqq k \leqq r$, has a quadric tangent cone of species k whose vertex is an S_{k-1} tangent to f at P. P is a node, binode, unode, respectively for $k = 1, r - 1, r$. The system of primals with the equation $\sum_{i=1}^{r-k+1} \lambda_i \varphi_i^2 + \sum_{j=r-k+2}^{r} \lambda_j \psi_j^3 = 0$, wherein φ_i and ψ_j are nonsingular primals of orders 3α and 2α respectively, has $2^{k-1}3^{r-k+1}\alpha^r$ double points of type k, $1 \leqq k \leqq r$, at the intersections of the φ_i and ψ_j.

An anomalous system of primals is one whose effective dimension Δ is greater than its virtual dimension Δ_0. Its anomaly $A = \Delta - \Delta_0$. In 1929 B. Segre obtained anomalous (irregolare) systems of plane curves with dependent cuspidal invariants. Systems of primals with multiple points of any order are anomalous for certain limiting values of the order and dimension. In particular, the system $F \equiv \sum_1^r \lambda_i f_i^2 = 0$, wherein f_i are nonsingular primals of order α in S_r, has α^r nodes at the basis points of the linear system $\sum f_i = 0$. For F, $\Delta_0 = C_{2\alpha+r,r} - \alpha^r - 1$, $\Delta = rC_{\alpha+r,r} - r(r-1)/2 - 1$, and $A = rC_{\alpha+r,r} - C_{2\alpha+r,r} + \alpha^r - r(r-1)/2$. A special case is the system of surfaces $\lambda_1 f_1^2 + \lambda_2 f_2^2 + \lambda_3 f_3^2 = 0$ with α^3 distinct nodes for which $A = (\alpha - 1)(\alpha - 2)(\alpha - 3)/6$. This means that of the α^3 nodal invariants of this system, only $\alpha^3 - A$ are independent. The author has shown (J. Reine Angew. Math. vol. 159 (1928) p. 263) that some of the invariants associated with a multiple point of a surface whose only constituents are nodes may be dependent, but invariants of distinct nodes of surfaces have not heretofore been proved dependent. All nodal invariants of plane curves are independent. In S_r the system F is anomalous for the following limits: $r = 3, \alpha \geqq 4$; $r = 4, \alpha \geqq 3$; $r \geqq 5, \alpha \geqq 2$. When $r = 2$, $\Delta = \Delta_0 = \alpha(\alpha + 3)$ for all values of α, that is, F consists of two curves of order α.

Wells College,
Aurora, N. Y., U. S. A.

IMPERFECT POINT ON INVARIANT SPACE CURVES

W. R. Hutcherson

A quartic surface (Bull. Amer. Math. Soc. vol. 55 (1949) p. 1073) containing an I_{11} and invariant under $x_1':x_2':x_3':x_4' = x_1 : \epsilon x_2 : \epsilon^2 x_3 : \epsilon^3 x_4$ was cut by a general tenth degree surface, also invariant. The resulting invariant curves possessed groups of eleven points in cyclic involution. These were mapped upon a surface determined by 18 equations in a space of 20 dimensions. A simple imperfect point $P_3(0010)$ on the quartic became either a 4, 6, 7, 9, or 10 tuple point on the curves in the tangent plane at P_3. Cutting with invariant ninth degree surfaces required only an image surface of 10 equations in S_{12}. The P_3 point became either a 6, 7, 8, or 9 tuple point on the tangent plane. All branches went through P_3 in one of the two invariant directions.

UNIVERSITY OF FLORIDA,
GAINESVILLE, FLA., U. S. A.

DUALITY AND NEIGHBOR POINTS

THEODORE S. MOTZKIN

It is proved that, under certain conditions, the characteristic series of a plane curve branch B uniquely determines that of the dual branch B' if the (algebraically closed) coordinate field is of characteristic $p = 0$, and shown that this is not the case for $p \neq 0$, even if duality relates B and B' birationally.

HEBREW UNIVERSITY,
JERUSALEM, ISRAEL.

ALGEBRAIC APPROXIMATIONS OF MANIFOLDS

JOHN NASH

Consider a differentiable imbedding of a compact differentiable manifold in euclidean k-space. Seifert has shown [*Algebraische Approximation von Mannigfaltigkeiten*, Math. Zeit. vol. 41 (1936) pp. 1–17] that in a fairly large class of cases such an imbedding can be approximated by a smooth portion of an algebraic variety. Our main result is to prove this in general.

Since the terminology of classical algebraic geometry is inadequate for our purposes, we must make a few definitions.

A *real algebraic n-manifold*, (M, R), involves an analytic Riemannian n-manifold, M, and a ring, R, of single-valued, real, regular analytic functions on M satisfying the conditions: (1) at any point of M, R provides n functions with independent gradient vectors in M; (2) any $n + 1$ functions of R satisfy a nontrivial polynomial relation with rational coefficients, i.e., R has transcendence degree n over the rationals; (3) R is maximal relative to the properties required above.

A *representation* of (M, R) is an analytic topological imbedding of M in euclidean k-space, E^k, with imbedding functions in R. The point set produced will be a smooth portion of an n-dimensional algebraic variety in E^k. Any compact (M, R) has a representation as a topologically isolated portion of the corresponding irreducible algebraic variety.

If (M, R) and (M', R'), both compact, are equivalent as differentiable manifolds, then R and R' are isomorphic. We call (M, R) and (M', R') *equivalent* if $R \cong R'$ and any isomorphism of the rings will induce an analytic [algebraic] homeomorphism of the manifolds.

Any compact manifold differentiably imbedded in E^k may be approximated by a representation of a differentiably equivalent algebraic manifold.

PRINCETON UNIVERSITY,
PRINCETON, N. J., U. S. A.

THE INTERSECTION OF ALGEBRAIC VARIETIES

DANIEL PEDOE

André Weil, in *Foundations of algebraic geometry*, p. 146, proves: *Let A^r, B^s be two subvarieties of a variety U^n, and let C be a component of $A \cap B$. Then, if C is simple on U, its dimension is at least $r + s - n$.* The proof, on the surface, is very simple. A proof of equal apparent simplicity can be deduced from the methods given by Severi and van der Waerden.

A space S^d in the ambient space S^N is *independent* of a point P if S^d is generic in S^N over the field obtained by adjoining the coordinates of P to the ground field k. Let ξ be a generic point of A^r, and let S^{N-n-1} be an $(N - n - 1)$-space of S^N independent of ξ. Let η be a generic point of S^{N-n-1}, and let $\zeta = \lambda\xi + \mu\eta$, where λ, μ are new indeterminates. The variety W in S^N whose generic point is ζ is described as *the cone joining S^{N-n-1} to A^r*. It is of dimension $r + N - n$. We prove Proposition I. *$W \cap U^n$ is purely r-dimensional, and if A^r is not contained in the locus of singular points of U^n, W and U^n intersect simply along A^r.* It follows that we can write $W \cap U^n = A^r + \sum_i A_i^r$, where the summation is over irreducible algebraic varieties of dimension r. We then prove Proposition II. *If P is a point of A^r which is simple for U^n and is independent of S^{N-n-1}, P is not on any A_i^r.* We now have $W \cap U^n = A^r + \sum_i A_i^r$, and therefore $W \cap U^n \cap A^s = A^r \cap A^s + \sum_i A_i^r \cap A^s$, or, since $A^s \subseteq U^n$, $W \cap A^s = A^r \cap A^s + \sum_i A_i^r \cap A^s$. No component of $W \cap A^s$ is of dimension less than $(r + N - n) + s - N = r + s - n$ (as in Weil, this theorem is proved earlier). It follows that any irreducible component C of $A \cap B$ which is of dimension less than $r + s - n$ is necessarily embedded in a component of some $A_i^r \cap A^s$. This cannot happen if there is a point on C which is simple for U^n. For if P is such a point, we may assume it is independent of S^{N-n-1}. By Proposition II, P cannot be in any A_i^r. Hence it cannot lie on any $A_i^r \cap A^s$. That is, C is not embedded, and the theorem is proved.

UNIVERSITY OF LONDON,
LONDON, ENGLAND.

A SERIES OF NONINVOLUTORIAL CREMONA TRANSFORMATIONS IN $[n]$

EDWIN J. PURCELL

Consider the h systems of equations

$$\bar{x}_{\delta+1}\bar{x}_{11}^{(d)} + \cdots + \bar{x}_{\eta+1}\bar{x}_{1,\zeta+1}^{(d)} = 0,$$
$$\cdots\cdots\cdots$$
$$\bar{x}_{\delta+1}\bar{x}_{\zeta 1}^{(d)} + \cdots + \bar{x}_{\eta+1}\bar{x}_{\zeta,\zeta+1}^{(d)} = 0,$$

where $d = 1, 2, \cdots, h$. For $d = 1$, δ denotes zero; for $d \neq 1$, δ denotes $\beta_1 + \cdots + \beta_{d-1}$, where the β_m are positive integers and $\sum\beta_d = n$. The η and ζ represent $\beta_1 + \cdots + \beta_d$ and β_d, respectively. The $\bar{x}_j$ are parameters and $\bar{x}_{ij}^{(d)} \equiv \sum\bar{a}_{ijk}^{(d)}x_k$ (the summation to extend from $k = 1$ to $k = n + 1$) wherein, for $d = 1$ the $\bar{a}_{ijk}^{(1)}$ are constants, but for $d > 1$ the $\bar{a}_{ijk}^{(d)}$ are homogeneous of order σ_{d1} in $\bar{x}_1, \cdots, \bar{x}_{\xi+1}$, homogeneous of order σ_{d2} in $\bar{x}_{\xi+1}, \cdots, \bar{x}_{\theta+1}, \cdots$, and also homogeneous of order $\sigma_{d,d-1}$ in $\bar{x}_{\varphi+1}, \cdots, \bar{x}_{\psi+1}$; here ξ, θ, φ, and ψ mean, respectively, β_1, $\beta_1 + \beta_2$, $\beta_1 + \cdots + \beta_{d-2}$, and $\beta_1 + \cdots + \beta_{d-1}$, $(i = 1, 2, \cdots, \beta_d; j = 1, 2, \cdots, \beta_d + 1; k = 1, 2, \cdots, n + 1)$.

The homogeneous coordinates, $\bar{x}_1:\bar{x}_2:\cdots:\bar{x}_{n+1}$, of a generic point $\bar{P}$ of a flat projective space, $[\bar{n}]$, determine in the above h systems of equations a unique point $P:(x_1, \cdots, x_{n+1})$ of another flat projective space, $[n]$.

This establishes a type $\{\beta_1, \beta_2, \cdots, \beta_h\}_n$ noninvolutorial Cremona transformation between the two n-dimensional spaces, $[\bar{n}]$ and $[n]$. There are 2^{n-1} general types of these Cremona transformations in $[n]$, each of which may be specialized in many ways.

The fundamental elements in both spaces and their images are discussed. The details of the type $\{n - 1, 1\}_n$ are given.

The simplest of these transformations, type $\{n\}_n$, is due to L. Godeaux who investigated it from a different viewpoint. [L. Godeaux, *Sur une correspondance Crémonienne entre deux espaces à n dimensions*, Rendiconti del Reale Istituto Lombardo (2) vol. 43 (1910) pp. 116–119.]

UNIVERSITY OF ARIZONA,
TUCSON, ARIZ., U. S. A.

QUADRICS ASSOCIATED WITH THE CLIFFORD MATRICES

THOMAS G. ROOM

The matrices to be considered are linear combinations of the usual Clifford matrices. The matrix $C_{(n)}$ is defined inductively by:

$$C_{(1)} = \begin{pmatrix} u_0 & u_1 \\ -u_2 & -u_0 \end{pmatrix}, \qquad C_{(n)} = \begin{pmatrix} C_{(n-1)} & u_{2n-1}I_m \\ -u_{2n}I_m & -C_{(n-1)} \end{pmatrix}$$

where $m = 2^{n-1}$, and the $2n + 1$ quantities u_r are arbitrary.

The matrix $C_{(n)}$ determines a collineation in $[2m - 1]$. As the parameters (u_r) vary, the conjugate in this collineation of a given point describes in general a $[2n]$. If the point lies in the space determined by its conjugates, then this space is of dimension n only. For ease of exposition n is taken to be 4. The figure then lies in [15], and if a point lies in the space, k, of its conjugates, k is a [4].

Take the coordinates in the [15] to be $x_{\alpha_1\alpha_2\alpha_3\alpha_4}$ $(\alpha_r = 0, 1)$. The locus, K, of the [4]'s, k, is the meet of eight quadric [7]-cones, $q^{(3)}_{\beta_1\beta_2\beta_3\beta_4}$, and a nonsingular quadric $q^{(4)}$, the equations of which are derived from a basic quadratic form $q^{(3)}_{,,,} = x_{000}x_{111} - x_{001}x_{110} + x_{010}x_{101} - x_{100}x_{011}$, in the following ways: $q^{(3)}_{\beta_1\beta_2\beta_3\beta_4}$, where one of the subscripts β_r is 0 or 1, and the remaining three are commas, is obtained from $q^{(3)}_{,,,}$ by multiplying by $(-1)^{\beta_r}$ and inserting the digit β_r among the subscripts of each $x_{\alpha_1\alpha_2\alpha_3}$, before α_1 if $r = 1$, after α_3 if $r = 4$, and otherwise between α_{r-1} and α_r. For example, $q^{(3)}_{,,1,} = -x_{0010}x_{1111} + x_{0011}x_{1110} - x_{0110}x_{1011} + x_{1010}x_{0111}$. $q^{(4)} = Q_0 - Q_1$, where Q_β is obtained from $q^{(3)}_{,,,}$ by replacing each $x_{\alpha_1\alpha_2\alpha_3}$ by $x_{\alpha_1\alpha_2\alpha_3\gamma}$, where $\gamma \equiv \alpha_1 + \alpha_2 + \alpha_3 + \beta \pmod 2$.

The quadratic forms $q^{(3)}$, Q_0, Q_1 are connected by the sixteen relations: $(-1)^{\beta+\gamma+\delta}(x_{\alpha'\beta\gamma\delta}q^{(3)}_{\alpha,,,} - x_{\alpha\beta'\gamma\delta}q^{(3)}_{,\beta,,}) + (-1)^{\delta}(x_{\alpha\beta\gamma'\delta}q^{(3)}_{,,\gamma,} - x_{\alpha\beta\gamma\delta'}q^{(3)}_{,,,\delta}) \equiv -x_{\alpha\beta\gamma\delta}Q_\lambda$ where $\alpha + \alpha' = \beta + \beta' = \gamma + \gamma' = \delta + \delta' = 1$, and $\lambda \equiv \alpha + \beta + \gamma + \delta + 1 \pmod 2$.

The locus K is of dimension 10. There are on it ∞^{10} [4]'s, any two of which meet either in a line or not at all. The section of K by a tangent—[10] at a point P of itself is a cone vertex P projecting the V_6^5 which is the Grassmannian in [9] of lines of [4].

In the general figure in $[2m - 1]$ there is a set of quadric $[2m - 9]$-cones, the equations of which are obtained by inserting $n - 3$ additional digits, 0, or 1, among the subscripts of the $x_{\alpha_1\alpha_2\alpha_3}$ in the quadratic form $q^{(3)}_{,,,}$.

University of Sydney,
Sydney, Australia.

LINEAR LINE INVOLUTIONS WITHOUT A COMPLEX OF INVARIANT LINES

C. R. Wylie, Jr.

The object of this paper is the identification of the possible linear line involutions which do not possess a complex of invariant lines. Through the use of the familiar mapping relation between the lines of S_3 and the points of a nonsingular V_4^2 in S_5 the following complete classification is obtained.

If a general plane field (bundle) of lines in S_3 is transformed into a bundle (plane field) of lines, the lines which meet their images form a quadratic complex consisting of all the lines tangent to a nonsingular quadric surface. In this case the involution is uniquely determined as the line transformation associated with the polarity defined by the quadric surface.

If a general plane field (bundle) of lines in S_3 is transformed into another plane field (bundle) of lines, either a) the lines which meet their images form two singular linear complexes which intersect in a nonsingular bilinear congruence, or b) every line meets its image. In a) the involution is uniquely determined as the line transformation associated with the skew-homology defined by the axes of the congruence. In b) the invariant lines form a bundle and a plane field with no lines in common, and the involution is the line transformation associated with the harmonic homology whose center is the center of the bundle of invariant lines and whose plane is the plane of the field of invariant lines.

University of Utah,
Salt Lake City, Utah, U. S. A.

ALGEBRAIC TOPOLOGY

IRREDUCIBLE RINGS IN MINIMAL FIVE-COLOR MAPS

ARTHUR BERNHART

A long-term program of attack for the four-color problem is submitted. The key notion is the study of the colorability of the two hemispheres formed by an equatorial ring of n regions.

The number of ways of coloring one side of an n-ring corresponding to each possible coloring scheme on the bounding ring constitutes a set of unknowns. These unknowns must satisfy a system of inequalities due to the geometrical constraints of a minimal map. They must also satisfy an interesting system of linear homogeneous equations which arise from the connective structure of the Kempe chains. Both types of requirement may be conveniently set forth by arranging the unknowns in magic squares.

In order to settle the geometrical question whether a given configuration is possible on a minimal map, one solves the relevant algebraic requirements simultaneously. If no solution is possible the configuration is *reducible*. But more interesting are the configurations for which both hemispheres possess consistent complementary solutions, and which are therefore *irreducible* relative to the specified requirements. For, these irreducible configurations become the building units for constructing a minimal map. By annexing pentagons and hexagons, for instance, irreducible rings of increasing complexity may be discovered.

The completion of this program would lead to known irreducible structures for both hemispheres, providing a counter example to the four-color conjecture.

UNIVERSITY OF OKLAHOMA,
NORMAN, OKLA., U. S. A.

A GENERALIZATION OF THE HUREWICZ ISOMORPHISM THEOREM

A. L. BLAKERS

Let X be an arcwise connected topological space and let F_x denote the fundamental group at $x \in X$. Let F_0 be an invariant subgroup of F_x and let G_x denote the group ring of F_x/F_0. Steenrod [Ann. of Math. vol. 44 (1943) p. 615] has shown that the groups G_x determine, in three ways, local systems of groups in X. Let $\mathcal{G}$ denote the local system in which the automorphisms of G_x determined by elements of F_x are right translations. This local system is completely determined (up to isomorphism) by the group G_x and these automorphisms.

Let A be a subspace of X and assume that the pair (X, A) is $(n-1)$-connected, $n \geqq 2$. If $n > 2$, this means that the natural homomorphism $i: \pi_1(A, x) \rightarrow$

$\pi_1(X, x)$ is an isomorphism onto, and hence we may consider the group $F_x = \pi_1(X, \mathring{x})$ as operating on $\pi_n(X, A, x)$. If $n = 2$, then i is onto, and because of the way that elements of $\pi_1(A, x)$ in the image of the boundary homomorphism $\partial: \pi_2(X, A, x) \to \pi_1(A, x)$ operate on $\pi_2(X, A, x)$ [see A.L. Blakers, Ann. of Math. vol. 49 (1948) p. 439] if $\beta \in \pi_1(X, x)$, then operators from $i^{-1}(\beta)$ on $\pi_2(X, A, x)$ will differ by elements of the commutator subgroup. Now let $\beta \in F_0$, let $\gamma \in \pi_n(X, A, x)$, and let Ω_0 denote the least subgroup of $\pi_n(X, A, x)$ which contains all elements of the form $(\beta(\gamma) - \gamma)$, $n > 2$, and all elements $((i^{-1}\beta)\gamma - \gamma)$ and all commutators when $n = 2$. Then it is proved, by a method which follows closely that used by Eilenberg [Ann. of Math. vol. 45 (1944) p. 442] to prove the Hurewicz isomorphism theorem in the absolute case, with integer coefficients, that $H_n(X, A, \mathfrak{G}) \approx \pi_n(X, A, x)/\Omega_0$.

This result includes as special cases two previous generalizations of the Hurewicz theorem to the relative case. For the case $F_0 = F_x$ see A. L. Blakers loc. cit., p. 461, and for the case where (X, A) is n-simple, see S.T. Hu, Duke Math. J. vol. 14 (1947) p. 1022. That the result should hold when $F_0 = 1$ was first suggested to me by Dr. W. S. Massey, who also pointed out that this result implies a simple proof of a theorem of J.H.C. Whitehead [Ann. of Math. vol. 42 (1941) Theorem 3], since it shows that his group $\mathfrak{M}$ is isomorphic to the relative homotopy group $\pi_n(X^*, X)$, and the theorem follows from the exactness of the homotopy sequence of the pair (X^*, X).

LEHIGH UNIVERSITY,
BETHLEHEM, PA., U. S. A.

ON THE MAPPINGS OF A 4-COMPLEX INTO CERTAIN SIMPLY CONNECTED SPACES

R. E. CHAMBERLIN

Let R be a simply connected Hausdorff space such that $\pi_2(R)$ has a finite number of generators and $\pi_3(R) = 0$. Let K be a finite 4-dimensional complex. The problem of classifying the maps of K into R is studied, using methods similar to those of Whitney (Ann. of Math. vol. 50 (1949) pp. 270–284). Let $\pi_2(R) = \sum_{\mu=1}^{s} \pi_{2\mu}$ where the $\pi_{2\mu}$ are cyclic. Let α_μ be a generator of $\pi_{2\mu}$. Suppose f and g are two mappings of K into R, both mapping the 3-dimensional section of K into a point. Then f and g are homotopic if and only if there exists a 3-π_4-cochain Y and a 1-π_2-cochain X such that $\Delta_{f,g} = \delta Y + \sum_{\mu=1}^{s} \theta^4_{n_\mu} \cdot (X_\mu \cup \theta^2_{n_\mu} X_\mu)\eta(\alpha_\mu, \xi)$, where $\Delta_{f,g}$ is the difference cocycle and θ^r_n the operator as in Whitney, loc. cit. The X_μ are the components of a decomposition of X corresponding to the decomposition of $\pi_2(R)$ and $\eta(\alpha_\mu, \xi)$ is an element of $\pi_4(R)$ depending on α_μ and a special mapping ξ of a 4-ball into R.

If R is a space with $\pi_1(R) = \pi_3(R) = \cdots = \pi_{n-1}(R) = 0$ and $\pi_2(R)$ is isomorphic to the group of integers, the mapping classes of an n-complex K into

R which are homotopic to the constant mapping on the $(n-1)$-section of K are in $1-1$ correspondence with the elements of $H^n(K, \pi_n(R))$. Finally an application to the classification of sphere bundles is given.

UNIVERSITY OF UTAH,
SALT LAKE CITY, UTAH, U. S. A.

RÄUME MIT MITTELBILDUNGEN

BENO ECKMANN

Unter einem n-Mittel in einem zusammenhängenden topologischen Raume R, wo n eine ganze Zahl $\geqq 2$ ist, sei eine stetige Funktion f verstanden, welche jedem System von n Punkten $x_1, \cdots, x_n \in R$ einen Punkt $y = f(x_1, \cdots, x_n) \in R$ zuordnet, derart dass $f(x_1, \cdots, x_n)$ symmetrisch in $x_1, \cdots, x_n$ ist und $f(x, \cdots, x) = x$ für alle $x \in R$. Diese Begriffsbildung findet sich bei Aumann [Math. Ann. vol. 119 (1944) pp. 210–215]; die elementaren Mittelwerte reeller Zahlen und die sogenannten analytischen Mittelwerte in Gebieten der komplexen Zahlenebene sind derartige Mittel. Mit Hilfe der Homotopiegruppen $\pi_r(R)$, $r \geqq 1$, lassen sich notwendige Bedingungen für die Existenz eines n-Mittels in R angeben: (1) Gibt es in R für eine Zahl $n \geqq 2$ ein n-Mittel, dann ist $\pi_1(R)$ abelsch, und alle $\pi_r(R)$, $r \geqq 1$, sind durch n teilbar, d.h. zu jedem $\alpha \in \pi_r(R)$ existiert ein $\beta \in \pi_r(R)$, so dass $\alpha = n \cdot \beta$ ist (die Gruppen π_r seien additiv geschrieben). Ist R ein endliches Polyeder, so folgt hieraus und aus bekannten Sätzen der Homotopietheorie: (2) Gibt es in R für *jedes* $n \geqq 2$ ein n-Mittel, so ist R in sich zusammenziehbar. Umgekehrt sieht man auf Grund bekannter Erweiterungssätze leicht ein, dass es in einem in sich zusammenziehbaren Raume R für jedes n ein n-Mittel gibt. Weitere notwendige Bedingungen für die Existenz eines n-Mittels in einem Polyeder R für ein bestimmtes $n \geqq 2$ betreffen die Homologie-Eigenschaften von R. Durch Betrachtung der Cohomologiegruppen von R, derjenigen des n-fachen Cartesischen Produktes von R mit sich und der durch f induzierten Homomorphismen ergibt sich nämlich: (3) Gibt es in R ein n-Mittel, so ist in den Dimensionen $r \geqq 1$ die Ordnung jeder ganzzahligen Cohomologieklasse von R zu n teilerfremd; es sind somit die Bettischen Zahlen b_r $(r \geqq 1)$ alle $= 0$, und die Torsionszahlen von R sind zu n teilerfremd. Diese Sätze zusammen mit einigen elementaren Eigenschaften der n-Mittel gestatten verschiedene naheliegende Anwendungen auf spezielle Polyeder, geschlossene und offene Mannigfaltigkeiten usw. So sieht man insbesondere in sehr vielen Fällen (zu denen auch alle von Aumann a.a.O. behandelten gehören) leicht ein, dass es im betrachteten Raume R für *kein* $n \geqq 2$ ein n-Mittel gibt.

SWISS FEDERAL SCHOOL OF TECHNOLOGY,
ZÜRICH, SWITZERLAND.

SUR LES CONSÉQUENCES, POUR LE PROBLÈME DES QUATRE COULEURS, D'UN THÉORÈME DE M. WHITNEY

ALFRED ERRERA

Appliquant une proposition que M. H. Whitney a publiée dans *A theorem on graphs* [Ann. of Math. (1931)] et tenant compte d'un théorème bien connu de P.G. Tait [Proceedings of the Edinburgh Mathematical Society (1880)], nous avons montré ailleurs [Congrès national des sciences, Bruxelles, Juin, 1950] que le problème des quatre couleurs revient à celui-ci:

Traçons, dans le plan, deux arbres, $\mathfrak{A}$ et $\mathfrak{A}'$, à *noeuds trièdres*, et ayant n *impasses* [A. Sainte-Laguë, *Les réseaux*, Mémorial des Sciences Mathématiques (1926)], considérées dans un ordre cyclique; faisons correspondre celles-ci bi-univoquement, ce qui peut se faire de n manières. Est-il possible, quels que soient les arbres et la correspondance des impasses, de colorier $\mathfrak{A}$ et $\mathfrak{A}'$ en trois couleurs, A, B, C (chaque noeud portant une arête de chaque couleur), de façon que les impasses correspondantes aient même couleur?

Un tel arbre possède 2^{n-3} coloriages *distincts* et il s'agit d'*ajuster* les coloriages de $\mathfrak{A}$ et $\mathfrak{A}'$.

Localement, cet ajustement est aisé, car, en général, deux arêtes correspondantes présentent, chacune, un choix de deux couleurs, donc au moins une couleur commune. Mais aux *noeuds terminaux* (qui portent deux impasses), il en va autrement; dans certains cas, il y a une difficulté à vaincre; dans d'autres, une facilité pouvant augmenter le nombre des solutions.

Globalement, l'ajustement est très difficile, car les choix de couleurs dans $\mathfrak{A}$ et $\mathfrak{A}'$ ne se présentent pas dans le même ordre, sauf pour le cas trivial où $\mathfrak{A}$ et $\mathfrak{A}'$ ont seulement deux noeuds terminaux et où les autres noeuds portent leurs impasses d'un même côté; dans ce cas, on peut donner tous les ajustements possibles.

On peut d'ailleurs se demander si toute carte "du type de Whitney" possède une décomposition en deux arbres ayant seulement deux noeuds terminaux.

UNIVERSITY OF BRUSSELS,
BRUSSELS, BELGIUM.

ON THE STRUCTURE OF THE GROUP OF HOMEOMORPHISMS OF AN ARC

N. J. FINE AND G. E. SCHWEIGERT

In a previous report, abstracted in connection with the April 1950 meeting of the AMS at Washington, D.C., we initiated an investigation of this group G. Here we present a strictly algebraic characterization of G and some previously unannounced results.

In order that a group H be isomorphic to G, one must be able to associate with H a set A which eventually attains the properties of an arc, and such that the elements of H become homeomorphisms of A because of the algebraic structure of H. Our characterization postulates that H contains subgroups K_x indexed by A, which has the power of the continuum. The intersection of all the K_x's is the identity, and an inner automorphism generated by $f \in H$ carries each K_x into some K_y, with $y = x$ only if $f \in K_x$. (The prototype of K_x in G is the group which leaves x fixed.) These assumptions enable us to identify the elements of H as permutations of A, and the following ones permit us to introduce the required order in A with preservation of betweenness. The set T complementary to the union of the K_x consists of two semi-groups which are inverses and which are exchanged by inner automorphisms induced by the involutions. The set of inner automorphisms induced by elements of T is transitive on the subgroups K_x. H is generated by the involutions. (More is known for G: every element of G can be represented as a product of at most four involutions.) Further assumptions now ensure separability in the order-topology of A and absence of "gaps", and a maximality condition completes the characterization.

Some of the properties of G not mentioned in our previous abstract are the following: G has only two nontrivial normal subgroups, namely F, the elements which leave the endpoints of the arc fixed, and Q, the elements which coincide with the identity on intervals containing the endpoints. F has two further normal subgroups whose intersection is Q, and Q itself has no nontrivial normal subgroups.

We have also proved that every automorphism of G is inner.

University of Pennsylvania,
Philadelphia, Pa., U. S. A.

ON THE EMBEDDING OF TOPOLOGICAL SEMIGROUPS AND INTEGRAL DOMAINS

B. Gelbaum, G. Kalisch, and J. M. H. Olmsted

S (I) denotes a commutative topological Hausdorff semigroup with cancellation law and identity (integral domain). A study is made of general conditions and criteria for the embeddability of S (I) in its quotient structure. The equivalence relation R in $S \times S$ ($I \times I^*$) is defined by: $(a, b)R(c, d)$ if and only if $ad = bc$. R is called open if and only if the saturation of every open set is open. $Q(S)$ ($Q(I)$) denotes the quotient structure $(S \times S)/R$ ($(I \times I^*)/R$). In terms of these notions the following results are obtained: 1. If S (I) is also a group (field), then the topology of S (I) can be weakened so that a separated topological group (field) is obtained. Consequently if S is compact, then S in its given topology is a topological group. 2. If I is a polynomial domain over a nondiscrete coefficient field (whose topology is induced by that of I), if $Q(I)$ is topologized so that it is a topological field in which I is embedded topologically and if R is open, then I is dense in $Q(I)$. 3. If G is a semigroup with a two-sided cancellation law (no commutativity or existence of identity assumed) and if G is compact or countably compact in a topology relative to which multiplication is continuous in both variables, then G, in its given topology, is a topological group. (This result has also been obtained by J. E. L. Peck.) 4. If S has an invariant metric, then $Q(S)$ can be provided with an invariant metric, and if R is open, then S is weakly embeddable in $Q(S)$ thus metrized. 5. If S is complete separable metric and enjoys the following two properties: (a) if U is open, then aU is open; (b) if $a_n b_n$ and b_n converge to e (the identity), then a_n converges to e; then every continuous isomorphism of S onto S', another semigroup which is complete separable metric and which enjoys property (a), is open, i.e., bicontinuous. This theorem is applied to the embedding of certain locally compact semigroups in groups. Counterexamples are provided to emphasize the necessity of certain hypotheses employed in reaching some of the above conclusions.

University of Minnesota,
Minneapolis, Minn., U. S. A.

ON EQUIVARIANT MAPS OF SPACES WITH OPERATORS

Alex Heller

If a group G operates on spaces X and Y, it operates also on $X \times Y$ by $g(x, y) = (gx, gy)$. Denote the space of trajectories of G in X by X/G. If $f: X \to Y$ is G-equivariant, i.e., if $fg = gf$ for all $g \in G$, there is a unique f/G: $X/G \to Y/G$ such that $\nu f = (f/G)\mu$ where μ and ν are the natural maps of X and Y on X/G and Y/G. If λ is the left projection of $X \times Y$ on X and $f': X \to X \times Y$ is defined by $f'x = (x, fx)$, then f'/G is a (continuous) right inverse of λ/G. If X is normal T_1 and operates on it without fixed points, then $f \to f'/G$ is a one-one correspondence of the equivariant maps of X into Y with the right inverses of λ/G.

Applications include the proof by homological methods of the nonexistence of equivariant maps in situations implying Kakutani's frame theorem for S^2 and S^3 and evaluation of the degrees of equivariant maps of spheres with rotations, implying de Rham's necessary condition for the homeomorphism of rotations.

If G is a topological group operating on X as the group of a principal bundle and on Y continuously, the same conclusion holds. This is shown to subsume the result for finite G. It follows that the set of cross-sections of a fibre bundle is in one-one correspondence with the set of equivariant maps of the associated principal bundle into the fibre.

This result allows the demonstration of the equivalence of two definitions of the notion of tensor function on a differentiable manifold.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

THE EQUIVALENCE OF FIBRE BUNDLES

SZE-TSEN HU

Let $\Phi = \{F, G; X, B; \psi, \phi_U\}$ be a fibre bundle [Trans. Amer. Math. Soc. vol. 67 (1949) p. 286], where the base space B is a finite polyhedron and the reference group G is arcwise connected. Let S be a subset of B. A system of local maps $\{f_U\}$, defined on S with respect to Φ, consists of, for each coordinate neighborhood U, a map $f_U : S \cap U \to G$ satisfying the condition: If $b \in U \cap V$, then $f_V(b) = (\phi_{V,b}^{-1}\phi_{U,b})\cdot f_U(b)\cdot(\phi_{V,b}^{-1}\phi_{U,b})^{-1}$.

Triangulate the base space B in such a way that every closed simplex is contained in some coordinate neighborhood U. By means of the usual obstruction method, for each $n > 1$, the notion of the systems of local maps defined on the $(n-1)$-dimensional skeleton B^{n-1} of B leads to a new cohomology invariant $W^n(\Phi)$, which is a subgroup of the cohomology group $H^n(B, \pi_{n-1}(G))$, called the n-dimensional characteristic subgroup of Φ.

Let $\Phi^* = \{F, G; X^*, B; \psi^*, \phi_{U^*}^*\}$ be another fibre bundle with the same director space F, the same reference group G, and the same base space B as the fibre bundle Φ. Assume that B be triangulated in such a way that every closed simplex is contained in $U \cap U^*$ for some U and some U^*. Φ and Φ^* are said to be equivalent if there exists an admissible homeomorphism h of X onto X^* such that (i) $h(\psi^{-1}(b)) = \psi^{*-1}(b)$ for each $b \in B$, and (ii) $b \in U \cap U^*$ implies that $\phi_{U^*,b}^{*-1}h\phi_{U,b}$ is in G and depends continuously on $b \in U \cap U^*$. Φ and Φ^* are said to be n-equivalent if the partial bundles $\Phi \mid B^n$ and $\Phi^* \mid B^n$ are equivalent.

The equivalence problem of fibre bundles is to find a neccssary and sufficient condition in terms of some convenient invariants that the given fibre bundles Φ and Φ^* be equivalent. The arcwise connectedness of G implies that Φ and Φ^* are 1-equivalent. Theoretically, the equivalence problem could be considered as solved if, under the assumption that Φ and Φ^* be $(n - 1)$-equivalent, one can find a necessary and sufficient condition of the n-equivalence.

Suppose that Φ and Φ^* be $(n - 1)$-equivalent. Then $W^n(\Phi) = W^n(\Phi^*)$. Every admissible homeomorphism h of $\psi^{-1}(B^{n-1})$ onto $\psi^{*-1}(B^{n-1})$ determines an element $z^n(h)$ of $H^n(B, \pi_{n-1}(G))$, called an n-dimensional deviation element of (Φ, Φ^*). The totality of the n-dimensional deviation elements form a coset $Z^n(\Phi, \Phi^*)$ of $W^n(\Phi)$ in $H^n(B, \pi_{n-1}(G))$, called the n-dimensional deviation coset of (Φ, Φ^*). The main theorem states as follows: Φ and Φ^* are n-equivalent if and only if $W^n(\Phi) = Z^n(\Phi, \Phi^*) = W^n(\Phi^*)$.

TULANE UNIVERSITY OF LOUISIANA,
NEW ORLEANS, LA., U. S. A.

L'EMPLOI, EN TOPOLOGIE ALGÉBRIQUE, DU FORMALISME DU CALCUL DIFFÉRENTIEL EXTÉRIEUR

Jean Leray

Par *différentielle* d'un anneau A nous désignons une application linéaire δ de A en lui-même et un automorphisme α de A tels que:

$$\delta^2 = 0, \qquad \alpha\delta + \delta\alpha = 0, \qquad \delta(a \cdot a') = \delta a \cdot a' + \alpha a \cdot \delta a' \qquad \text{si } a \text{ et } a' \in A.$$

Les $a \in A$ tels que $\delta a = 0$ constituent un sous-anneau; son quotient par δA est nommé anneau d'homologie de A.

La notion d'anneau différentiel fournit une définition générale et commode de *l'anneau de cohomologie* HX d'un espace localement compact.

Elle permet d'attacher à une *application continue* ξ d'un espace X dans un espace Y un invariant topologique H_r (r un entier $\geqq 1$) de la nature suivante: H_1 est l'anneau de cohomologie de Y, relativement à un anneau variable, qui est $H\xi^{-1}(y)$ au point y de Y; H_r a une différentielle; son anneau d'homologie est H_{r+1} ; pour $r \to +\infty$, $H_r \to \sum_p H^{(p)}/H^{(p+1)}$, où $H^{(p)}$ est un sous-groupe additif de HX ayant les propriétés suivantes:

$$H^{(p)} \subset H^{(p+1)}; \qquad H^{(p)} \cdot H^{(q)} \subset H^{(p+q)}; \qquad H^{(p)} \to 0 \text{ pour } p \to -\infty \ (p \text{ un entier}).$$

A *un groupe de Lie* compact et connexe G on peut associer des différentielles opérant sur les anneaux de cohomologie des espaces sur lesquels G opère; ces différentielles sont anticommutatives; leurs produits constituent une algèbre extérieure, qui est isomorphe au module d'homologie de G muni de la multiplication de Pontrjagin.

Les notions qui précèdent permettent d'étudier les anneaux de cohomologie des *espaces homogènes*.

Bibliographie: Journal de Mathématiques vol. 29; C. R. Acad. Sci. Paris vols. 228, 229.

Collège de France,
Paris, France.

HOMOTOPY GROUPS OF THE SPACE OF CURVES WITH APPLICATION TO SPHERES

Everett Pitcher

Suppose X is a compact Riemannian manifold of class C^3 (more general spaces are admissible), Y is the space of curves (Fréchet equivalence classes) of finite length joining two points of X, and J is length on X as a function of Y. Then $\pi_r(X) \approx \pi_{r-1}(Y)$. Suppose (the general case according to Morse when X is analytic) the two end points conjugate on no extremal. Suppose b_i are the critical levels of J in increasing order and $b_i < c_i < b_{i+1}$ with c_i sufficiently near b_{i+1}. Set $Z_i = \{y \mid y \in Y \text{ and } Jy \leqq c_i\}$. Then $Y = \bigcup_i Z_i$ and (Z_i, Z_{i-1}) is in the homotopy class with $(Z_{i-1} \cup (\bigcup_j E_{i,j}), Z_{i-1})$, where there is an attached cell $E_{i,j}$ for each critical point $\gamma_{i,j}$ of J at level b_i, whose dimension is the type of $\gamma_{i,j}$. Exact homotopy sequences on pairs and triples of sets Z_i, Y and on triads for attached cells with the methods of Blakers and Massey are useful.

Specialize to $X = S^n$ with $n \geqq 2$, $Y = Y^n$ with end points at distance $\pi/2$. Following Morse, there is exactly one critical point at level $b_p = (p + 1/2)\pi$, of type $p(n-1)$. Then $\pi_r(Z_p^n) \approx \pi_r(Y^n)$ where $[p(n-1)-1] \leqq r \leqq (p+1)\cdot(n-1)-2$.

In the sequence on (Y^n, Z_1^n), the identity map on Z_1^n induces a homomorphism equivalent to the Freudenthal suspension, yielding the isomorphisms for small r and partly describing suspension generally.

In the sequence on (Z_p^n, Z_{p-1}^n), with $n \geqq 3$, the groups $\pi_r(Z_p^n, Z_{p-1}^n)$ are trivial for $r < p(n-1)$; there is an isomorphism ν of $\pi_{r-1}(S^{p(n-1)-1})$ into $\pi_r(Z_p^n, Z_{p-1}^n)$ for $2 \leqq r \leqq 2p(n-1)-3$ which is onto for $3 \leqq r \leqq (p+1)\cdot(n-1)-2$. A nontrivial element of $\pi_{p(n-1)}(Z_p^n, Z_{p-1}^n)$ and its boundary are constructed. For $p = 2$, the boundary is $\pm[\iota_{n-1}, \iota_{n-1}]$, where ι_{n-1} generates $\pi_{n-1}(Z_1^n) \approx \pi_{n-1}(S^{n-1})$.

It is readily shown by these methods that $\pi_5(S^3) \approx I_2$ (a crucial step in the argument furnished by W. S. Massey, who supplied a lemma of J. H. C. Whitehead).

LEHIGH UNIVERSITY,
BETHLEHEM, PA., U. S. A.
INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

APPLICATIONS OF A CALCULUS OF FINITE DIFFERENCES TO THE FOUR-COLOR PROBLEM

C. N. REYNOLDS

The operator carrying $f(x)$ into $x\Delta^2 f(x-1)$ together with its inverse generates a calculus comparable to ordinary differential and integral calculus. This calculus is applied in the present paper to a consideration of the problem of coloring any map on a simply connected closed surface in at most four colors in such a manner that no two regions having common boundary are assigned the same color.

UNIVERSITY OF WEST VIRGINIA,
MORGANTOWN, W. VA., U. S. A.

REDUCED POWERS OF A COCYCLE

N. E. Steenrod

A generalization is given of the author's "squaring" operations (cup-i products) (Ann. of Math. vol. 48 (1947) pp. 290–319). Let $H^q(X, A; G)$ denote the q-dimensional relative cohomology group of X mod A with coefficients in G. Let J = the integers, J_p = the integers mod p. If $p \geqq 3$ is a prime, and $i = 0, 1, 2, \cdots$, we construct homomorphisms

$$D^p_{2i}: \quad H^q(X, A; J) \to H^{pq-2i}(X, A; J_p)$$

$$D^p_{2i+1}: H^q(X, A; J) \to H^{pq-2i-1}(X, A; J).$$

The pth power of $u \in H^q(X, A; J)$ (based on cup products) reduced mod p coincides with $D^p_0 u$. When $p = 2$, the same construction gives the squaring operations. $D^p_j u$ is called the j-fold reduction of the pth power. It has order p even when j is odd. The operations D^p_j commute with homomorphisms induced by continuous maps; they are therefore topologically invariant. With respect to the coboundary operation δ: $H^q(A) \to H^{q+1}(X, A)$, we have

$$\delta D^p_j u = m D^p_{j+p-1} \delta u, \qquad m = ((p-1)/2)! \bmod p.$$

If p $\equiv 3$ mod 4, then $m \equiv \pm 1$ mod p. In any case, m has an inverse $(\pm m)$ mod p. Starting with a space with a nonzero D^p_j, and forming the join with a pair of points, we obtain a new space with a nonzero D^p_{j+p-1}. Since there are examples of nonzero D^p_0 (e.g. in complex projective spaces), there are examples of nonzero $D^p_{k(p-1)}$ for $k = 1, 2, \cdots$. If f maps the $(n+1)$-skeleton of a complex K into an n-sphere S^n, and u generates $H^n(S^n; J)$, the vanishing of all $D^p_j f^* u$ is a necessary condition for f to be extendable to all K. Thus the $D^p_j f^* u$ are candidates for obstructions of higher order.

Princeton University,
Princeton, N. J., U. S. A.

A GENERALIZATION OF A THEOREM OF PONTRJAGIN

R. L. Wilder

A generalization of the Alexander Duality Theorem was given by Pontrjagin (*Zum Alexanderschen Dualitätssatz*, Nachr. Ges. Wiss. Göttingen (1928) pp. 446–456) in the following form: Let S be an arbitrary orientable closed n-dimensional manifold, M a complex imbedded in S, $q^r(M)$ the dimension of the vector space of r-cycles (mod 2) of M that bound in S reduced modulo the subspace of cycles that bound in M, and $q^{n-r-1}(S - M)$ the number defined analogously for $S - M$. Then

$$q^r(M) = q^{n-r-1}(S - M).$$

Moreover, corresponding interlinked bases of r-cycles of M and $(n - r - 1)$-cycles of $S - M$ can be set up, starting with an arbitrary finite partial basis for either M or $S - M$. It is now well known that the extension to the case where M is an arbitrary closed subset of S offers no difficulty. For a generalized manifold S, the corresponding theorem takes on added significance, inasmuch as the small neighborhoods of points are not generally euclidean, and the necessary investigation of their homology properties is in terms of bounding cycles of S.

In the present paper this theorem is extended to the case where S is an orientable generalized closed manifold, using a field as coefficient group, and applications are made to local linking properties of the manifold. In particular, the extension of the local linking Theorem IX 4.2 of the author's book *Topology of manifolds* (Amer. Math. Soc. Colloquium Publications, vol. 32, New York, 1949) to the complete duality $p^r(M) = p^{n-r-1}(S: V - M; U - M)$, with corresponding interlinked homology classes, is an easy corollary.

In addition, the following formula (also due to Pontrjagin for the case of a classical manifold; loc. cit.) is similarly generalized:

$$p^{n-r-1}(S - M) = q^r(M) + p^{n-r-1}(S) - [p^{r+1}(M) - q^{r+1}(M)],$$

the p's in this formula being the usual Betti numbers over the field used as coefficient group, and the q's as defined above.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

FUNCTION SPACES AND POINT SET TOPOLOGY

CONTINUOUS COLLECTIONS OF CONTINUA

R. D. Anderson

A λ-collection is a continuous collection of compact continua filling up a locally compact metric continuum such that the elements of the collection are homeomorphic to each other and are mutually exclusive. A λ'-collection is a continuous collection of compact continua filling up a locally compact metric continuum such that the elements of the collection are homeomorphic to each other and there is a set of two points of the continuum which is the common part of each pair of elements of the collection.

The author studies λ-collections and λ'-collections of arcs or pseudo-arcs. There exist compact metric spaces H and K and λ-collections G_H and G_K of arcs filling up H and K respectively with G_H and G_K arcs with respect to their elements as points such that H is locally connected but contains no 2-cell and K contains no arc not a subset of an element of G_K.

If S and S' are compact metric spaces, necessary and sufficient conditions that S and S' be a 2-cell and a 2-sphere respectively are that there exist two λ-collections G_S and H_S of arcs filling up S and two λ'-collections $G_{S'}$ and $H_{S'}$ of arcs all with the same end points filling up S' such that no nondegenerate continuum is common to any pair of elements of $G_S + H_S$ or $G_{S'} + H_{S'}$ and G_S and H_S are arcs and $G_{S'}$ and $H_{S'}$ are simple closed curves with respect to their elements as points.

If M is a dendron and S is a 2-cell, there exists a λ-collection G of pseudo-arcs filling up S such that G is homeomorphic to M. By a modification of an argument previously used by the author it is shown that if S is a 2-cell, there exists a λ-collection G of pseudo-arcs filling up S such that G is homeomorphic to S. Elementary characterizations of such collections are given.

University of Pennsylvania,
Philadelphia, Pa., U. S. A.

OPERATIONS INDUCED IN CONJUGATE SPACES

Richard Arens

Let A, B, C, and K be normed linear spaces, and let m be a bounded bilinear operation from $A \times B$ to C. Let A^*, B^*, $\cdots$ be the normed linear spaces of bounded linear operations of A, B, $\cdots$, respectively, into K. Then there is a natural operation from $C^* \times A$ to B^* defined thus: $n(f, a)(b) = f(m(a, b))$. Applying this process three times yields a bounded bilinear operation p from $A^{**} \times B^{**}$ to C^{**}. It is an extension of qm, where q is the natural mapping of

C in C^{**}. When $A = B = C$ and m is the multiplication of a normed linear algebra, A^{**} becomes a normed linear algebra with the multiplication p, and p is associative if and only if m is. Returning to the general case, the question of the uniqueness of this extension p of m raises the following question. Transpose m, perform the extension as above, and transpose the resulting bilinear form,—do you then obtain p? A necessary condition is that K be the field of scalars. Assuming this condition, the general case can be reduced to the case $C = K$, the case of functionals. This case is solved for $A = C = (c_0)$ the space of null-sequences. When m is symmetric, the preceding question is whether p is also symmetric. This is settled affirmatively for a special class of m's: the multiplications in the normed linear algebras $C(X, K)$ where X is any compact Hausdorff space. The subject may be presented in axiomatic form for a suitable category of sets (A) and mappings (m), and functions of more than 2 variables can also be treated.

University of California,
Los Angeles, Calif., U. S. A.

HIGHER DIMENSIONAL HEREDITARILY INDECOMPOSABLE CONTINUA

R. H. Bing

It is shown that there are hereditarily indecomposable continua of all dimensions. This is done as follows. If a and b are two points of the Hilbert cube Q, it is shown that there is a sequence $D_1, D_2, \cdots$ of connected open subsets of Q such that (1) D_i contains $\bar{D}_{i+1}$, (2) D_i separates a from b in Q, and (3) each arc in D_i is $1/i$-crooked, where an arc is called ϵ-crooked provided each subarc pq of it contains points r and s such that r is between p and s, $\rho(p, s) < \epsilon$, and $\rho(r, q) < \epsilon$. The intersections of the $\overline{D_i}$'s is a hereditarily indecomposable continuum that separates Q. Since each compact continuum can be topologically imbedded in Q, it follows that if M is a compact continuum and A and B are mutually exclusive closed subsets of M, there is a closed point set K in M separating A from B such that each component of K is hereditarily indecomposable. Hence, each $(n + 1)$-dimensional compact continuum contains an n-dimensional hereditarily indecomposable subcontinuum.

University of Wisconsin,
Madison, Wis., U. S. A.

FIXED POINT THEOREMS FOR PSEUDO-ARCS AND CERTAIN OTHER METRIC CONTINUA

O. H. HAMILTON

This paper gives proofs for fixed point theorems for certain compact metric continua and for certain subcontinua of the Euclidean plane. A continuum M is said to have the fixed point property if every continuous transformation of M into itself leaves a point of M invariant.

Moise has defined a chain as a collection of mutually exclusive open sets (called links) $y_1, y_2, \cdots, y_k$ such that y_i and y_k have a boundary point in common if and only if i and j are identical or consecutive integers. If Y designates a chain, $C(Y)$ will designate the closure of the set of points each of which is in a link of Y, and Δy will designate the maximum diameter of a link of Y. Let $Y_1, Y_2, \cdots$ be a sequence of chains such that: (1) $C(Y_{i+1})$ is a subset of $C(Y_i)$ for each i, and (2) Limit $\Delta Y_i = 0$. It is shown that if M is a continuum which is the intersection of the elements of the sequence $[C(Y_i)]$, then M has the fixed point property. A corollary is that a pseudo-arc has the fixed point property.

A cyclic chain is a collection of open sets satisfying the definition of a chain except that the links y_1 and y_k have a boundary point in common. It is shown that if J is a continuum which is the intersection of a monotonic descending sequence of closures of cyclic chains $[C(Y_i)]$ with limit $\Delta Y_i = 0$, and M is a continuum which does not separate the plane which has J for its boundary, then M has the fixed point property. This proposition answers in part the question: Does every compact continuum which does not separate the plane and which consists of a simply connected domain and its boundary have the fixed point property?

OKLAHOMA AGRICULTURAL AND MECHANICAL COLLEGE,
STILLWATER, OKLA., U. S. A.

A REALISATION OF THE GEOMETRY OF THE HILBERT SPACE IN THE PLANE

RUDOLF INZINGER

The considerations are based on the set L^2 of periodic functions

$$h(\varphi) = h(\varphi + 2\pi)$$

which, in the sense of Lebesgue, are quadratically integrable, representing, as is well known, vectors of an abstract Hilbert space H. Every function $h(\varphi)$ from L^2 will be considered as support function of a supportable domain of a plane Π: then h signifies the signed distance of a tangent spear of the domain boundary from the origin 0 of the coordinates and φ its angle with a fixed direction, which is determined modulo 2π. Then the set M of supportable domains of the plane

Π is a geometrical realisation of the function space L^2, resp. of the abstract Hilbert space H. The metric, proceeding from L^2, resp. H as well as all concept formations, deducible from it, then may be geometrically interpreted on the domains of the set M.

The linear transformations of the convolution type in M, which also in particular include the differentiation "D" and the integration "I" of the support functions, are of particular interest. To these functional operations correspond in M the formations of the evolute resp. evolvente domains, which turn out to be special cases of the evolutoid resp. evolventoid formations and which, on their part, are the correspondents of the linear differential operators "$aD + b$" resp. of their inversions. Appropriate transitions to the limits make it possible to deduce from these constructions geometric interpretations for the operations e^{aD}, e^{aI}, e^{aD^2}, e^{aI^2} and many others, which do not require differentiability properties of the support functions. The theory of differentiability and integrability of any real order finds its geometric interpretation in a generalization of the theory of the evolute resp. evolvente domains for not necessarily integer orders. As for some other functional operations, there results an especially simple geometric interpretation for the Hilbert transformation.

There are also a number of other functional transformations, leading from L^2, which may be easily and very clearly illustrated in a geometrical way; this is especially true for the Laplace transformations, the Stieltjes transformation and their inversions.

TECHNISCHE HOCHSCHULE,
VIENNA, AUSTRIA.

CHARACTERIZATIONS OF CERTAIN SPACES OF CONTINUOUS FUNCTIONS

MEYER JERISON

In one of their characterizations of the Banach space of continuous, real-valued functions on a compact Hausdorff space, Arens and Kelley (Trans. Amer. Math. Soc. vol. 62 (1947) pp. 499–508) show that such Banach spaces satisfy the following condition: (A) If there is no element common to all members of a collection Γ of maximal convex subsets of the surface of the unit sphere, then there exist directed sets $\{C'_\alpha\}$ and $\{C''_\alpha\}$ in Γ such that

$$\inf_{c \in C''_\alpha} \| b - c \| + \inf_{c \in C'_\alpha} \| b - c \| \to 2$$

for all b in the space for which $\| b \| \leqq 1$.

A Banach space, B, satisfies condition (A) if and only if there exist a compact Hausdorff space X and an involutory homeomorphism σ of X onto itself such that B is equivalent to the space of continuous functions on X satisfying the relation $f[\sigma(x)] = -f(x)$ for all x in X. This result leads to a simplification of the theorem of Arens and Kelley; namely, a Banach space is equivalent to the space of all continuous functions on some compact Hausdorff space if and only if it satisfies condition (A) and its unit sphere has an extreme point. The space of continuous functions that vanish at infinity on a locally compact Hausdorff space is characterized by condition (A) together with the condition that all the extreme points of the unit sphere of the conjugate space lie in two sets that are diametrically opposite each other and are separated in the weak* topology.

This paper was published in full in Trans. Amer. Math. Soc. vol. 70 (1951) pp. 103–113.

University of Illinois,
Urbana, Ill., U. S. A.

A PROOF THAT HILBERT SPACE IS HOMEOMORPHIC WITH ITS SOLID SPHERE

V. L. Klee, Jr.

Suppose that C is a closed convex body in Hilbert space H, and $C \neq H$. The result stated in the title is established, and is applied to show that C must be homeomorphic with H. By use of the author's earlier result that H is homeomorphic with the surface of its sphere, it is demonstrated that the boundary of C must be homeomorphic with either H itself or the cartesian product (for some $n \geqq 0$) of H with an n-sphere. There is described a periodic homeomorphism of H whose set of fixed points has a nonempty interior and infinitely many components, and a pointwise periodic homeomorphism which is not periodic.

University of Virginia,
Charlottesville, Va., U. S. A.

TOPOLOGICAL SPACES HAVING THE SAME REGULAR OPEN SETS

Michael J. Norris

Given a fixed point set, two distinct topologies for this point set may have the same regular open sets. It is the aim of this paper to study the maximal families of topologies for a fixed point set which have the same regular open sets. For convenience, no separation properties are demanded for the topologies.

The regular open sets in any topological space may be used as a neighborhood basis for a topology on the point set. This topology is semi-regular, and its regular open sets are those of the original one. In some cases separation properties are preserved. Thus, in a family such as is considered, there is a weakest topology which is the only semi-regular topology in the family.

Let R be a subset of Q. If inclusion is required to hold only modulo R', the notion of homeomorphism for two topologies on Q by means of the identity transformation becomes what is called R-equivalence. The corresponding partition classes are the topologies between a weakest and a strongest member of the class. If R is dense for one topology of a partition class, all members of the class have the same regular open sets. Using the strongest members of the partition classes described above, it can be shown that every topology is weaker than some maximal topology in the family of topologies having the same regular open sets as the given one. The remainder of the paper is concerned with these maximal topologies and related matters.

College of St. Thomas,
St. Paul, Minn., U. S. A.

INVARIANCE AND PERIODICITY PROPERTIES OF NON-ALTERNATING IN THE LARGE TRANSFORMATIONS

Russell Remage, Jr.

Let S be a compact connected T_1-space. A D-chain is defined to be a nonvacuous intersection of proper nodal sets. $D(X)$ is the intersection of all nodal sets containing a nonvacuous subset X of S. A prime D-chain is a D-chain which contains properly no nondegenerate D-chain. Let T be a nonalternating in the large transformation of S onto itself. If A is a nodal set which is neither S nor a single point, such that both A and the complement of A meet their respective images, then either $A \cap \overline{S - A} = z$ is fixed or there is a pair H and K of invariant sets each contained in a prime D-chain such that $H \subset A$, $K \subset \overline{S - A}$. If such a cut-point z is not fixed, it cannot be nonwandering.

A D-chain C is said to be nonvariant provided there is an invariant set I such that $C = D(I)$; a nondegenerate prime D-chain C is weakly nonvariant provided C is contained in $D(CT)$. Each reduces to invariance when T is a homeomorphism. Using these definitions, it is shown that the following five conditions are pairwise equivalent: 1) each point which separates a pair of invariant sets is fixed. 2) If P and Q are weakly nonvariant prime D-chains, then every prime D-chain contained in $D(P \cup Q)$ is weakly nonvariant. 3) The union of all weakly nonvariant prime D-chains is a D-chain. 4) For every pair of distinct prime D-chains P and Q such that PT contains Q, the D-chain $D(P \cup Q)$ contains either exactly one fixed cut-point or exactly one nondegenerate nonvariant D-chain. 5) If C and C_1 are nonvariant D-chains and C_1 is properly contained in C, and if z is a cut-point which separates $C_1 - z$ from some point of C, then z is fixed.

With regard to periodicity, if $ST = S$ is pointwise nonwandering and n.a.l., then each cut-point is periodic, and moreover T is elementwise periodic on all prime D-chains which are not end-elements.

University of Delaware,
Newark, Del., U. S. A.

METRIC SPACE AND ITS GROUP OF ISOMETRIES

Hsien-chung Wang

Let E be a metric space in the sense of Fréchet. The normal subgroups of the group of all isometries of E are discussed. As consequences, we are able to determine a class of spaces which are of particular interest in metric geometry. Following Busemann, let us call a metric space with metric ρ *a* (*)*-space* if given any four points x, y, x', y' with $\rho(x, y) = \rho(x', y')$, there exists an isometry of the space carrying x, y to x', y' respectively. Evidently, most of the classical metric geometries have this property. Concerning these spaces of dimension greater than 3, very few properties are known even under further assumptions on the metric. In this paper, we eventually determine all the compact connected (*)-spaces. We first show that such a space must be finite-dimensional and locally connected. Then using theory of Lie groups, the following is proved:

Theorem I. *With one possible exceptional case, all compact, connected* (*)*-spaces fall into four classes*: (1) *spheres*, (2) *real projective spaces*, (3) *quaternion projective spaces, and* (4) *complex projective spaces.*

Let ρ and ρ' be two metrics in a same space E. They are called congruent if there is a continuous automorphism f of E such that $\rho'(x, y) = \rho(fx, fy)$. They are called *similar* if there is a real function g such that $\rho'(x, y) = g(\rho(x, y))$.

Theorem II. *According as E belongs to the class* (1), (2), (3), *or* (4), *the* (*)*-metric in E is, respectively, similar to* (*up to a congruence*) *the spherical metric, elliptic metric, a certain symmetric Riemannian metric, or the metric used by J. Weyl to study analytical curves.*

Some uniqueness theorems are proved. We shall only mention the following one.

Theorem III. *Let E be a sphere* (*real projective space*) *of even dimension with a convex metric. If there exists a transitive group of isometries, then E is congruent to the spherical space* (*elliptic space*).

Louisiana State University,
Baton Rouge, La., U. S. A.

INCOMPRESSIBILITY AND PERIODICITY

Charles W. Williams

Let T be a multiplicative topological abelian group with identity element e. We assume that T is generated by some compact neighborhood of e. Let T act as a transformation group on the topological space X in the sense that to each pair (x, t) where x is in X and t is in T, there is assigned a point of X denoted by xt such that $xe = x$, $(xt)s = x(ts)$ where s is in T, and such that the function xt defines a continuous transformation of $X \times T$ into X. A semigroup $S \subset T$ is said to be replete provided that S contains some translate of each compact subset of T. The point x in X is said to be periodic under T provided that there exists a relatively dense subgroup $G \subset T$, such that $xG = x$.

We say that the space X is incompressible under T provided that for any subset $M \subset X$, and any replete semigroup $S \subset T$ for which $MS \subset M$, it is true that $M - MS$ is a set of the first category. It is then demonstrated that if T is separable and X is incompressible under T, the set of all points of X which are periodic under T is residual in X.

WASHINGTON AND LEE UNIVERSITY,
LEXINGTON, VA., U. S. A.

Section IV

PROBABILITY AND STATISTICS, ACTUARIAL SCIENCE, ECONOMICS

SECTION IV. PROBABILITY AND STATISTICS, ACTUARIAL SCIENCE, ECONOMICS

MATHEMATICS OF FACTORIAL DESIGNS

RAJ CHANDRA BOSE

1. The basic principles of design and analysis of field trials were developed at Rothamstead by R. A. Fisher between the years 1921 and 1925 and were precisely set forth in his important work *Statistical methods for research workers* first published in 1925. Both the methods themselves and their field of application has since then been vastly extended in various directions, and the Design of Experiments has today become one of the major fields of application of statistical technique. Some of the later methodological advances have been the introduction of incomplete block designs and factorial designs together with the principles of confounding and fractional replication. The recent use of multifactorial designs may also be mentioned. In order to get efficient and easily analysable designs it is found that certain combinatorial conditions must be satisfied in every case. Thus combinatorial mathematics, which until recently has been regarded as an unimportant sideline, is seen to be of major practical importance for the theory of designs. I shall here give you some of the important combinatorial theorems in connection with the theory of confounding and fractional replications and shall discuss the problem of orthogonal arrays (useful in the construction of multifactorial designs and other situations.

2. In a factorial experiment each treatment is complex consisting of a number of factors each at a particular level. Thus if there are m factors $F_1, F_2, \cdots, F_m$, the ith factor F_i being tested at s_i levels, then the number of treatments is $v = s_1 s_2 \cdots s_m$. If each block accommodates all the v treatments, one treatment being assigned randomly to each plot in the block, we have a "complex" experiment. When, however, the number of treatments becomes large, there arises the need for a reduction of the block size in order to ensure an adequate elimination of the major source of heterogeneity (e.g., fertility differences in the case of agricultural experiments). The technique most frequently used for achieving this is confounding. Each complete replication is now allocated to v/k blocks of k plots each, where k divides v, in such a way that only unimportant treatment contrasts are confounded, i.e., become nonestimable.

There arises in this way a combinatorial problem, namely, the problem of allocating the treatments to the various blocks in such a way that only certain desired contrasts are confounded. It must, however, be recognized that an unrestricted solution of this problem may not be possible at all. This was demonstrated in the case of the 2^m experiment (i.e., an experiment with m factors each at two levels) by Barnard (1936), who showed that the confounding of any two contrasts implies the automatic confounding of another contrast which is called their generalized interaction. I shall give here the generalization of Barnard's result for the case of the symmetrical factorial design, i.e., a design in which each of the m factors is at s levels, where $s = p^n$, p being a prime. This sets a limit on what types of confounding are or are not possible. This theory was developed in a paper by Kishen and the author in (1940) and was treated by the author in further detail in (1947).

Since $s = p^n$, there exists a Galois field $GF(p^n)$ with s elements $\alpha_0, \alpha_1, \cdots, \alpha_{s-1}$. We can then set up a (1, 1) correspondence between the levels and the elements of $GF(p^n)$ in any arbitrary manner. To any treatment therefore corresponds an m-plet $(x_1, x_2, \cdots, x_m)$ of Galois field elements, which can be represented by a point in finite Euclidean space $EG(m, p^n)$. Thus the s^m points of this space represent the treatments. Given any flat space Σ of dimensions $d \leqq m - 1$ in $EG(m, p^n)$, the s^d treatments represented by the points on Σ may be said to be the treatments corresponding to Σ and may be denoted by (Σ). Consider any linear homogeneous form $U = a_1x_1 + a_2x_2 + \cdots + a_mx_m$, where the a's belong to $GF(p^n)$. Then $U = c$ represents a flat space of $m - 1$ dimensions if c is a constant. If c is supposed to vary over all the elements of the field, then $U =$ const. represents a pencil of parallel $(m - 1)$-flats. To each flat of this pencil correspond s^{m-1} treatments, and the contrasts between these s sets of treatments carry $s - 1$ degrees of freedom which may be said to belong to the pencil $U = c$. If among the a's only $a_{i_1}, a_{i_2}, \cdots, a_{i_t}$ are nonzero $(t \leqq m)$, it can be shown that the degrees of freedom belong to the t-factor interaction of F_{i_1}, $F_{i_2}, \cdots, F_{i_t}$, and since the number of pencils for which $a_{i_1}, a_{i_2}, \cdots, a_{i_t}$ are nonzero where the other a's are zero is exactly equal to $(s - 1)^{t-1}$, all the $(s - 1)^t$ degrees of freedom belonging to the t-factor interaction of $F_{i_1}, F_{i_2}, \cdots, F_{i_t}$ are accounted for.

The generalization of Barnard's result can then be given as follows:

If $U_1, U_2, \cdots, U_g$ are distinct linear forms and if degrees of freedom corresponding to the pencils

$$U_1 = \text{const.}, \qquad U_2 = \text{const.}, \cdots, U_g = \text{const.}$$

are confounded, then the degrees of freedom corresponding to all the pencils

$$\lambda_1 U_1 + \lambda_2 U_2 + \cdots + \lambda_g U_g = \text{const.}$$

are confounded. The number of these pencils is

$$N_g = \frac{s^g - 1}{s - 1},$$

and thus the number of degrees of freedom confounded equals $s^g - 1$. The sets of treatments belonging to the blocks correspond to the $(m - g)$-flats,

$$U_1 = c_1, \qquad U_2 = c_2, \cdots, U_g = c_g,$$

each particular set of c's giving rise to one block.

3. We may now extend this result to cover the case of fractional replication. Instead of taking all the s^g blocks, we take only s^r blocks $(r < g)$, given by

$$U_1 = 0, \cdots, U_r = 0, \qquad U_{r+1} = c_{r+1}, \cdots, U_g = c_g .$$

We then have a $1/s^r$ fraction of a complete replication. The degrees of freedom confounded remain as before. Let $L =$ const. be a pencil, the degrees of freedom corresponding to which are unconfounded. The set of pencils

$$L + \mu_1 U_1 + \mu_2 U_2 + \cdots + \mu_r U_r = \text{const.},$$

where the μ's are arbitrary elements of the field, may be called the complete alias set of the pencil $L =$ const. To any contrast between the sets of treatments corresponding to the flats of $L =$ const., there will be analogous contrasts between the sets of treatments corresponding to any alias pencil. These may be called the aliases of the given contrasts. Thus each contrast belongs to an alias set with s^r members. The result of using a fractional replication is that the contrasts are no longer separately estimable, but the sum of the contrasts belonging to any complete alias set is estimable. If all the contrasts of any alias set with the exception of one contrast are of sufficiently high order to be negligible, then this one contrast is estimable.

4. Let us consider a matrix $A = (a_{ij})$ with m rows and N columns, where each element a_{ij} represents one of the s integers $0, 1, 2, \cdots, s - 1$. Consider all the t-rowed submatrices which can be formed from A, $t \leqq m$. Each column of any t-rowed submatrix can be regarded as an ordered t-plet, so that each such submatrix contains N ordered t-plets. Since each component of any t-plet can be chosen in s ways, there are s^t distinct t-plets. The matrix A is called an orthogonal array (N, m, s, t) of size N, m constraints, s levels, and strength t if each of the C_t^m partial t-rowed matrices that may be formed from the array contains the s^t possible ordered t-plets each repeated λ times. Clearly $N = \lambda s^t$. The number λ may be called the index of the array. Orthogonal arrays are useful in many different situations, particularly for the construction of "multifactorial designs" in problems of physical or industrial research where it is required to ascertain the effect of quantitative or qualitative alterations in the various components of the complete assembly on some measured characteristic. To carry out a complete factorial experiment would require s^m assemblies where s is the number of values at which each component can appear and m is the number of factors or components. Plackett and Burman (1946) showed that all main effects may be determined with maximum accuracy if we measure the characteristic for the $N = \lambda s^2$ assemblies corresponding to the columns of an orthogonal array of strength two (two factor and higher order interactions being supposed to be absent). Rao (1947) showed that if the design is an array of strength t, then we can measure interactions up to the order $t/2$ or $(t - 1)/2$ according as t is even or odd. The estimates of main effects and interactions are unaffected by interactions among less than t factors, but the estimate of error is enhanced by their presence.

5. Two mathematical problems arise in connection with orthogonal arrays, the first of which may be stated as follows: Given N, s, and t (N being divisible by s^t), what is the maximum number of constraints m which can be accommodated in the array? This maximum number may be denoted by $f(N, s, t)$.

No complete answer to this problem is known, even for the simple case $t = 2$, $\lambda = 1$, when the array is abstractly identical with a set of mutually orthogonal Latin squares of m constraints (the rows and columns counting for two of the constraints). When $s = p^n$ where p is a prime we know that $f(p^{2n}, p^n, 2) = p^n + 1$, so that there are $p^n - 1$ mutually orthogonal Latin squares. In general we can write $s = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where the p's are prime and $p_1^{n_1} < p_2^{n_2} < \cdots < p_n^{n_k}$. It is then conjectured that $f(s^2, s, 2) = 1 + p_1^{n_1}$. This includes as a special case Euler's conjecture that when $s = 4n + 2$, we cannot get two mutually orthogonal Latin squares of side s. Euler's conjecture has been verified only for the case $s = 6$.

Mann (1942) has proved that $f(s^2, s, 2) \geqq 1 + p_1^{n_1}$. Mann's result can be generalized as follows:

If N_i is divisible by s_i^t for $i = 1, 2, \cdots, u$, then

$$f(N_1 N_2 \cdots N_u, s_1 s_2 \cdots s_u, t) \geqq \min(m_1, m_2, \cdots, m_u)$$

where $m_i = f(N_i, s_i, t)$.

Rao (1947) established the following interesting inequalites.

$$\text{(i)}\quad N - 1 \geqq (s-1)C_1^m + (s-1)^2 C_2^m + \cdots + (s-1)^u C_u^m \quad \text{if } t = 2u,$$

$$\text{(ii)}\quad N - 1 \geqq (s-1)C_1^m + (s-1)^2 C_2^m + \cdots + (s-1)^u C_u^m + (s-1)^{u+1} C_u^{m-1} \quad \text{if } t = 2u + 1.$$

Rao's results may be extended in various directions. Mr. Kenneth A. Bush in a recent thesis establishes for arrays of index unity the theorem $f(s^t, s, t) \leqq s + t - 1$.

The following special results for orthogonal arrays of strength 2 and 3 are of interest:

THEOREM. *If* $\lambda - 1 = a(s - 1) + b$, $0 < b < s - 1$, *and* n *is the largest positive integer (including* 0*) consistent with*

$$s(b - 2n) \geqq (b - n)(b - n + 1),$$

then for the case $t = 2$

$$m \leqq I\left(\frac{\lambda s^2 - 1}{s - 1}\right) - n - 1$$

and for the case $t = 3$

$$m \leqq I\left(\frac{\lambda s^2 + s - 2}{s - 1}\right) - n - 1.$$

Thus if $\lambda - 1$ is not divisible by $s - 1$, we can always improve the Rao bound by unity, and possibly by more up to an amount $I(b/2)$.

6. The second problem which arises in connection with orthogonal arrays is the actual problem of construction when the parameters of the array are given. I give below some theorems relating to this topic, without going into the details of their application.

THEOREM. *Let M be a module consisting of s elements. Suppose it is possible to find a scheme of $r + 1$ rows*

$$\begin{array}{cccc} a_{01}, & a_{02}, & \cdots, & a_{0n} \\ a_{11}, & a_{12}, & \cdots, & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r1}, & a_{r2}, & \cdots, & a_{rn} \end{array}$$

such that (1) *each row contains λs elements belonging to M*, (2) *among the differences of corresponding elements of any two rows each element of M occurs exactly λ times, then we can use the scheme to construct an orthogonal array* $(\lambda s^2, r + 2, s, 2)$.

This theorem may be used to construct the arrays $(18, 7, 3, 2)$ and $(32, 9, 4, 2)$. In general it can be shown that if $s = p^n$ and $\lambda = p^u$ where p is a prime, then we can construct an orthogonal array of strength 2 and size λs^2 in which the number of constraints is

$$p^{(r+1)n+d} + p^{rn+d} + \cdots + p^{n+d} + 1$$

where $u = rn + d$, $0 \leqq d < n$, $r \geqq 0$.

The following theorem also applies to the case $s = p^n$, $\lambda = p^u$ when p is a prime.

THEOREM. *If we can find a matrix (c_{ij}) of m rows and r columns, where c_{ij} are elements of $GF(p^n)$, such that every submatrix of t rows is of rank r, then we can construct the orthogonal array* (s^r, m, s, t).

The above theorem can be used for constructing the following arrays:

(1) The array $(s^3, s + 2, s, 3)$ when $s = 2^n$.

(2) The array $(s^3, s + 1, s, 3)$ when $s = p^n$ where p is an odd prime.

(3) The array $(s^4, s^2 + 1, s, 3)$ when $s = p^n$ and p is prime.

(4) The array $(s^4, s^{r-1}, s, 3)$ when $s = 2$.

REFERENCES

1. R. A. FISHER, *Statistical methods for research workers*, London, Oliver and Boyd, Ltd. 1925.

2. M. M. BARNARD, *An enumeration of the confounded arrangements in the* $2 \times 2 \times 2 \times \cdots$ *factorial designs*, Supplement to the Journal of the Royal Statistical Society vol. 3 (1936) pp. 195–202.

3. R. C. Bose and K. Kishen, *On the problem of confounding in the general symmetrical factorial design*, Sankhyā vol. 5 (1940) pp. 21–36.
4. R. C. Bose, *Mathematical theory of the symmetrical factorial design*, Sankhyā vol. 8 (1947) pp. 107–166.
5. H. B. Mann, *The construction of orthogonal Latin squares*, Ann. Math. Statist. vol. 13 (1942) pp. 428–435.
6. R. L. Plackett and J. P. Burman, *The design of optimum multifactorial experiments*, Biometrika vol. 49 pp. 303–325.
7. C. R. Rao, *Factorial experiments derivable from combinatorial arrangements of arrays*, Supplement to the Journal of the Royal Statistical Society vol. 9 pp. 128–139.

University of North Carolina,
Chapel Hill, N. C., U. S. A.

PROCESSUS À LA FOIS STATIONNAIRES ET MARKOVIENS POUR LES SYSTÈMES AYANT UNE INFINITÉ DÉNOMBRABLE D'ÉTATS POSSIBLES

PAUL LÉVY

1. Soit un système, ayant une infinité dénombrable d'états possibles A_h ($h = 1, 2, \cdots$). Soit $H(t)$ la fonction égale à chaque instant t à l'indice de l'état du système à cet instant. Si le processus dont dépend le système est markovien et stationnaire, la méthode classique consiste à le définir par les fonctions $P_{h,k}(t)$, qui indiquent la probabilité du passage en un temps $t > 0$ de l'état A_h à l'état A_k.

Nous nous proposons de montrer l'intérêt que peut présenter une autre méthode, qui consiste à séparer dans la mesure du possible l'étude de la succession des états et celle de la rapidité de son évolution.

2. Si le système est initialement dans un état A_h, il est à peu près évident que le temps T au bout duquel il le quitte est une variable aléatoire de la forme X/λ_h, λ_h étant un coefficient $\geqq 0$, et X une variable aléatoire positive, de fonction de répartition $1 - e^{-x}$.

Nous supposerons essentiellement $\lambda_h < \infty$. Disons seulement que, si λ_h est infini pour tous les états d'un cycle final et qu'un de ces états arrive à être réalisé, $H(t)$ est à partir de ce moment une fonction dont les valeurs aux différents instants sont indépendantes les unes des autres.

Tous les λ_h étant donc supposés finis, on peut définir chaque réalisation possible de $H(t)$ en se donnant la succession E des intervalles e dans chacun desquels elle est constante, et en associant à chacun d'eux la valeur de $H(t)$ qui lui correspond. Cette *succession* est un ensemble ordonné, mais pas toujours bien ordonné. Son complément E' sur l'axe des t (ou sur le demi-axe si on étudie le processus à partir d'un instant initial $t_0 > -\infty$) est un ensemble fermé, dénombrable ou non, mais toujours de mesure presque sûrement nulle. Si en effet il faut prévoir des discontinuités de $H(t)$, qui sont des sauts ou des points d'accumulation de sauts, pour tout t donné ou choisi au hasard, le système est presque sûrement dans un des états considérés comme possibles, entre lesquels par hypothèse on répartit la probabilité. C'est ce qui s'exprime dans la théorie classique par la condition

$$\sum_k P_{h,k}(t) = 1, \tag{1}$$

et ici par le fait que E' soit presque sûrement de mesure nulle.

3. Notre méthode consiste à étudier d'abord la succession des valeurs de $H(t)$, et ensuite la rapidité de l'évolution. Si, dans une succession donnée S, chaque état A_h est réalisé r_h fois, le temps nécessaire pour la parcourir est

$$T = \sum_1^\infty \sum_1^{r_h} \frac{X_{h,\nu}}{\lambda_h}, \tag{2}$$

les $X_{h,\nu}$ étant des variables aléatoires du type X, toutes indépendantes les unes des autres (on remarque que la sommation par rapport à ν donne une variable aléatoire du type III de Pearson). Elle est presque sûrement finie ou infinie en même temps que sa valeur probable

$$\mu = \sum_1^\infty \frac{r_h}{\lambda_h} E\{x\} = \sum_1^\infty \frac{r_h}{\lambda_h}. \tag{3}$$

On remarque en particulier que, pour qu'une succession infinie puisse être parcourue en un temps fini, il est nécessaire qu'aucun r_h ne soit infini, et qu'aux r_h positifs correspondent des λ_h augmentant indéfiniment avec h.

4. Soit $P_{h,k}$ la probabilité que le système, s'il est initialement dans l'état A_h, prenne par la suite au moins une fois l'état A_k.

Théorème. *Si $P_{h,k} > 0$, on a $P_{h,k}(t) > 0$ pour tout $t > 0$.*

Comme évidemment $P_{h,k} = 0$ entraîne $P_{h,k}(t) = 0$, on voit que $P_{h,k}(t)$ est, ou bien toujours $= 0$, ou bien toujours > 0 (t variant de zéro à l'infini).

Résumé de la démonstration. Si le système peut passer en un temps fini de l'état initial à un autre état A_k, le temps T au bout duquel il y arrive pour la première fois est donné par une formule de la forme

$$T = \frac{X}{\lambda_h} + \sum_1^\infty \sum_1^{Q_l} \frac{X_{l,\nu}}{\lambda_l}, \tag{4}$$

les Q_l pouvant être aléatoires, puisqu'il peut y avoir différentes successions d'états conduisant de A_h à A_k. Si $P_{h,k} > 0$, T est, dans des cas de probabilité positive, inférieur à un nombre positif arbitrairement petit ϵ. C'est d'ailleurs la somme de deux termes positifs indépendants, le premier ayant une densité de probabilité positive de 0 à ∞ ; il en est alors de même de la somme.

Soit alors $t > 0$ et $t_0 \in (0, t)$. Il y a une probabilité positive que $t_0 < T < t$, et que le système, après l'instant T, reste dans l'état A_k un temps $\geqq t - t_0$, donc que $H(t) = k$, c.q.f.d.

5. La succession des états. Les différents cas possibles. Il y a lieu de distinguer plusieurs cas bien différents, suivant que $H(t)$ a, ou non, d'autres points singuliers que des sauts.

Le cas le plus simple est le *cas fini*, où $H(t)$ n'a pas d'autres points singuliers que des sauts, qui sont alors en nombre fini dans tout intervalle fini; c'est une fonction-escalier. Les états successifs constituent une chaîne de Markoff, bien définie par la donnée des probabilités de passages $p_{h,k}$ ($p_{h,k}$ est la probabilité que l'état qui suit A_h soit A_k). Ces probabilités sont non-négatives, et telles que

$$\alpha_h = \sum_k p_{h,k} = 1 \qquad (h = 1, 2, \cdots). \tag{5}$$

La loi de l'évolution du système est bien définie par les coefficients $p_{h,k}$ et λ_h, liés aux fonctions $p_{h,k}(t)$ par les relations

$$(6) \qquad \lambda_h = -P'_{h,h}(0), \qquad \lambda_h p_{h,k} = P'_{h,k}(0) \qquad (k \neq h),$$

d'ailleurs valables dans tous les cas. Mais ce n'est que dans le cas fini que ces coefficients suffisent à définir la loi de l'évolution.

La condition nécessaire et suffisante pour qu'on soit dans le cas fini est que tous les α_h soient égaux à 1, et que la série

$$(7) \qquad \sum_1^\infty \frac{1}{\Lambda_n}$$

soit divergente [nous désignons par H_n ($n = 1, 2, \cdots$) les valeurs successives de $H(t)$ à partir d'une valeur initiale donnée h_0, et par Λ_n les valeurs correspondantes de λ_h]. On remarque que cette seconde condition est aléatoire; on peut ne pas savoir à l'avance si on sera dans le cas fini (la même remarque s'applique aux autres cas que nous distinguerons).

Si au contraire, la condition (5) restant réalisée, la série (7) est convergente, les instants T_n des changements d'état tendent, presque sûrement, pour n infini, vers une limite T_ω. Il peut y avoir d'autres valeurs (aléatoires) de t, qui soient des points d'accumulation de sauts, de sorte que pour ces valeurs $H(t - 0)$ n'existe pas. Si $H(t + 0)$ existe en tout point, c'est-à-dire qu'après chacun de ces instants $H(t)$ ait une valeur entière déterminée et ne change pas pendant un temps fini, les discontinuités forment un ensemble dénombrable et bien ordonné, auquel on peut appliquer la numération transfinie. C'est le *cas transfini*.

Il faut remarquer pue, pour qu'il soit nécessaire d'introduire des nombres transfinis élevés (tels que ω^ω), il faut (puisque chaque état ne peut être réalisé qu'un nombre fini de fois en un temps fini) qu'apparaissent indéfiniment et transfiniment de nouveaux états, correspondant à de grandes valeurs de λ_h, de sorte que, au moins en moyenne, *le film s'accélèrera.* Remarquons d'ailleurs que le nombre transfini η qui borne supérieurement les nombres ξ qu'il faut utiliser peut être aléatoire. Il est naturellement de la seconde classe, et n'a pour chaque processus qu'une infinité dénombrable de valeurs possibles ayant chacune une probabilité positive. Si une loi de probabilité donnée pour η vérifie cette condition, on peut définir un processus qui la réalise.

Nous appellerons *troisième cas* l'ensemble des cas dans lesquels il peut (avec une probabilité positive) exister des valeurs (aléatoires) de t pour lesquelles $H(t + 0)$ n'existe pas. Des subdivisions de ce cas sont utiles, mais ne peuvent pas être indiquées dans les limites du présent exposé.

Avant d'indiquer comment, dans le cas transfini et dans le troisième cas, on peut compléter la définition du processus, il peut être utile d'indiquer quelques exemples.

6. Exemples. 1°. Supposons que les valeurs possibles de $H(t)$ soient tous les entiers h, de $-\infty$ à $+\infty$, et que tous ces entiers se succèdent sûrement dans l'ordre naturel, les T_n étant seuls aléatoires. Supposons $\sum_{-\infty}^{+\infty} 1/\lambda_h < \infty$. Si, après l'instant T_ω, on repart d'un entier choisi suivant une loi donnée, on est dans le cas transfini (alors $\eta = \omega^2$). Mais si on repart de $-\infty$, on est dans le troisième cas. Dans un cas comme dans l'autre, d'ailleurs, on aura une infinité de phases successives ayant toutes une même durée probable finie.

2°. Modifions l'exemple précédent en supposant que l'état A_h ait deux formes différentes A'_h et A''_h, et qu'à chaque changement de h, l'indice supérieur ait une probabilité γ_h de changer.

Supposons d'abord $\sum_{-\infty}^{+\infty} \gamma_h$ fini. Alors, pour chaque phase, l'indice supérieur ne change qu'un nombre fini de fois; il a une valeur initiale et une valeur finale bien déterminées. Nous pouvons supposer qu'il ne change pas de la fin d'une phase au début de la suivante. Alors on peut considérer qu'au moment du changement de phases, il y a deux *états fictifs* possibles; ils sont éphémères et ne sauraient subsister un temps fini; mais chacun d'eux implique un certain souvenir du passé immédiat et sa transmission à l'avenir immédiat.

Si la série $\sum_1^\infty \gamma_h$ est divergente, les circonstances sont bien différentes. Supposons pour fixer les idées tous les γ_h égaux à 1/2. Les valeurs successives de l'indice supérieur sont alors indépendantes, et aucun souvenir des indices anciens ne peut réapparaître. Il n'y a alors qu'un seul état fictif.

3°. Établissons maintenant une correspondance biunivoque entre les indices h et les nombres rationnels $r = p/q$; $H(t)$ devient une fonction $R(t)$ à valeurs rationnelles. Nous pouvons supposer que les états se succèdent dans l'ordre des r croissants; si par exemple $\lambda_h = 1/q^3$, tous les états correspondant aux r d'un intervalle semi-ouvert $(r_0, r_0 + 1]$ se succèdent en un temps presque sûrement fini, de valeur probable $\zeta(2)/\zeta(3)$. La fonction $R(t)$ est alors continue et prend successivement toutes les valeurs réelles, rationnelles, ou irrationnelles, ces dernières correspondant à un ensemble de valeurs de t de mesure nulle, et n'ayant aucune chance d'être réalisées pour un t donné.

Physiquement, un tel processus est sans doute irréalisable. Mais nous voyons qu'une théorie mathématique, pour être complète, doit prévoir l'existence de processus comprenant une infinité non-dénombrable d'états fictifs, susceptibles d'être tous réalisés successivement. Ce sont des états de transition, mais tous distincts, chacun transmettant du passé à l'avenir un héritage différent.

7. Définition du processus; le cas transfini. Supposons $H(t)$ déterminé jusqu'à un instant τ, et que $H(\tau - 0)$ n'existe pas, de sorte que ce point est un point d'accumulation d'intervalles e. Pour définir la suite du processus, il s'agit d'abord de déterminer la probabilité q_k que $H(\tau + 0)$ existe et ait la valeur k. Si, en plus de (5), on a toujours $\sum q_k = 1$, $H(t + 0)$ existe toujours et on est dans le cas transfini.

Remarquons d'abord que, si le processus est défini par la donnée des fonctions $P_{h,k}(t)$, les q_k s'en déduisent par la formule

$$q_k = \lim_{\epsilon \downarrow 0} \lim_{t \uparrow \tau} P_{H(t),k}(\epsilon). \tag{8}$$

Pour l'appliquer, il n'est d'ailleurs pas nécessaire de connaître $H(t)$ dans un intervalle (t_0, τ); il suffit de connaître la succession des valeurs de $H(t)$, ou même une suite partielle extraite de cette succession, mais qui aille *jusqu'au bout.*

Cette formule est générale. Considérons spécialement le cas transfini et proposons-nous d'abord de définir $H(T_\omega + 0)$ indépendamment de la formule (8). À chaque état A_h correspond une probabilité $q'_{h,k}$ bien déterminée que, le système partant de cet état, on ait $T_\omega < \infty$ et $H(T_\omega + 0) = k$, et q_k peut être défini comme limite de $q'_{h,k}$, h variant comme dans la formule (8). Mais la donnée des $q'_{h,k}$ est surabondante; il suffit de connaître ces probabilités avec une erreur qui tende vers zéro quand T_ω (dont la loi dépend de h) tend en probabilité vers zéro. En outre la limite q_k de $q'_{H_n,k}$ peut n'être pas définie pour chaque suite $\{H_n\}$; il suffit que ce soit une fonction mesurable (la mesure étant ici la probabilité de la réalisation des différents suites H_n théoriquement possibles, c'est-à-dire telles que tous les $p_{H_n,H_{n+1}}$ soient > 0, et que $\sum 1/\Lambda_n < \infty$).

La loi dont dépend $H_\omega = H(T_\omega + 0)$ étant ainsi définie, il n'y a plus de difficulté à former successivement tous les $H_\xi = H(T_\xi + 0)$, tant que le nombre transfini ξ n'atteint pas ω^2. Mais, si T_{ω^2} est fini, il faut introduire de nouveaux coefficients $q''_{h,k}$ pour définir H_{ω^2}. Les remarques faites à propos de H_ω s'appliquent à nouveau. Il en sera de même pour chaque nombre transfini non accessible par l'addition (c'est-à-dire, appartenant à la suite transfinie $\omega, \omega^2, \cdots ; \omega^\omega, \omega^{\omega+1}, \cdots$); pour chacun de ces nombres ξ, si T_ξ est fini, il faut introduire de nouveaux coefficients $q^{(\xi)}_{h,k}$.

Ainsi, nous n'échappons pas aux difficultés du transfini. Cela est d'ailleurs dans la nature des choses. On peut s'en assurer en observant que

$$P_{h,k}(t) = \sum \Pr\{T_\xi \leqq t < T_{\xi+1}, H_\xi = k/H_0 = h\},$$

la sommation étant étendue à tous les nombres ξ, finis ou transfinis, pour lesquels T_ξ peut être fini. C'est un développement asymptotique, chaque terme étant, pour t assez petit, négligeable devant n'importe lequel des termes précédents.

8. Esquisse d'une théorie générale. Groupons les fonctions $H(t)$ en familles K' telles que: $H_1(t)$ défini dans (t_0, t_1) et $H_2(u)$ défini dans (u_1, u_2) appartiennent à une même famille si et seulement si on peut établir une correspondance biunivoque et monotone entre $t \in (t_1, t_2)$ et $u \in (u_1, u_2)$ telle que $H_1(t) = H_2(u)$. En d'autres termes, chaque famille K' est caractérisée par la donnée des relations d'ordre entre les différents intervalles e où $H(t)$ est constant, et celle de ses valeurs pour ces différents intervalles; les longueurs des intervalles n'interviennent pas.

Si chaque entier h ne correspond qu'à un nombre fini r_h d'intervalles e, nous dirons que K' est une famille K. L'ensemble des K a la puissance du continu.

Considérons, dans cet ensemble, le sous-ensemble C des K tels que: a. Un intervalle où $H(t) = k$ ne peut suivre immédiatement un intervalle où $H(t) = h$ que si $p_{h,k} > 0$. b. La somme (3) est finie. Seuls les $K \in C$ correspondent, pour un processus pour lequel on connaît les λ_h et les $p_{h,k}$, à une succession de valeurs pouvant être réalisées en un temps fini; C a au plus la puissance du continu.

Groupons maintenant les $K \in C$ en classes ϕ et en classes ψ telles que: K_1 et K_2 appartiennent à un même ϕ (ou ψ) s'ils sont identiques à partir d'un (jusqu'à un) certain moment; la partie commune à K_1 et K_2, éléments d'un même ϕ (ou ψ) ne peut se réduire à un intervalle que s'il y a un dernier (ou premier) intervalle;

s'il n'en est pas ainsi, elle comprend une infinité d'intervalles, mais il n'y a aucun intervalle commun à tous les K d'une classes ϕ (ou ψ).

Le processus étant markovien, il est clair que: à l'instant τ ou une succession E d'intervalles e se termine, le souvenir que le système garde du passé ne peut pas contenir autre chose que la classe ϕ définie par la succession E associée aux valeurs constantes de $H(t)$ dans chaque e. Le choix que le hasard fera à cet instant ne peut de même pas définir autre chose que la classe ψ à laquelle doit appartenir la fonction $H(t)$ dans l'intervalle commençant à l'instant τ. Donc, pour définir le processus, *il s'agit de se donner en fonction de la classe ϕ finissante la loi de probabilité dont dépend la classe ψ commençante.*

Comme l'ensemble des ψ a au plus la puissance du continu, il n'y a pas de difficulté à parler d'une loi définie dans cet ensemble. Il faut remarquer d'autre part que cette loi n'a pas besoin d'être donnée pour chaque ϕ. Il suffit qu'il soit presque sûr qu'on ne se trouvera à aucun moment (t variant de t_0 à l'infini) dans un cas où cette loi ne soit pas définie. Dans le cas transfini, nous avons obtenu un résultat plus précis; cette loi, ou plutôt la fonction de répartition qui la définit, doit être, pour chaque nombre transfini non accessible, une fonction mesurable d'une variable dont la donnée équivaut à celle de la classe ϕ. Mais dans le cas général, un énoncé analogue serait dépourvu de sens.

Enfin l'exemple 2 du *n*° 6 nous montre que la définition d'une classe ϕ peut comprendre des éléments dont le souvenir ne saurait être conservé. Il faut alors grouper les classes ϕ en classes plus étendues ϕ', éléments d'un espace $\mathfrak{S}$, telles que: t tendant vers τ, la probabilité, estimée à l'instant t, que la succession d'intervalles qui doit se terminer à l'instant τ corresponde à une classe ϕ appartenant à un domaine ouvert R situé dans $\mathfrak{S}$, tend presque sûrement vers 1 ou vers zéro suivant que finalement ϕ appartient à R ou non. On fera un groupement analogue des ψ en classes ψ', et il s'agit finalement, pour définir un processus markovien et stationnaire, de se donner pour ψ' une loi de probabilité presque sûrement bien définie en fonction de ϕ', sauf dans des cas qui n'ont aucune chance d'être jamais réalisés.

9. Conclusion. La définition des processus markoviens et stationnaires reposant sur la distinction entre les propriétés intrinsèques de $H(t)$ et la rapidité de l'évolution définie par ces propriétés n'est simple et commode que dans le cas fini, et peut-être dans le cas transfini si on se limite à un nombre transfini donné, tel que ω^2 ou ω^3. Cette distinction n'en est pas moins utile à d'autres points de vue. Elle permet de démontrer le théorème du n°4. Nous en montrerons ultérieurement une autre application, spéciale au cas transfini, relative à une extension de la notion de cycle final et de théorèmes connus de W. Doeblin et A. Kolmogoroff.

ÉCOLE POLYTECHNIQUE,
PARIS, FRANCE.

ON SOME ASPECTS OF STATISTICAL INFERENCE

S. N. Roy

Within the last fifty years or so in a broad sense, and especially within the last thirty years in a more restricted sense, enormous progress has been achieved in the theory of statistical inference which is essentially statistical philosophy. Beginning with the principles of graduation developed by Karl Pearson, passing on to Fisher's elegant theory of estimation for parametric problems, moving further into the Neyman-Pearson theory of testing of hypothesis for both parametric and nonparametric situations and the Neyman theory of estimation for parametric problems and culminating in the Wald theory of inference which is a generalization in all directions including all the previous structures as special cases, the theory of statistical inference has indeed made almost revolutionary progress. Not much of this progress—particularly at the latter stages of it—has been reflected in the various applications of statistical methods. From a certain point of view statistical activity could be conveniently arranged into three sectors: there is at one end the sector of statistical inference, there is at the other end the vast sector of applications of statistical methods to various fields of knowledge and human activity, and in between the two there is a middle sector which is concerned with the construction of concrete statistical tools based on the principles of inference and the adaptation of them to practical and numerical uses. It is the business and responsibility of workers in the middle sector to form a sort of connecting link between the two other sectors at the two ends, bringing the fruits of statistical philosophy to the applied statistician and conveying to the statistical philosopher the requirements of the applied statistician and the limitations under which he has necessarily to operate. There are very few, perhaps hardly any, who belong to all the three sectors and comparatively few who belong to any two of these sectors. Most belong to one or the other of the three sectors.

For a variety of reasons, which need not be gone into at the moment, it has so happened that within the last fifteen years or so the function of the middle sector has not been as effectively discharged as one would have wished, with the result that univariate analysis (or what is otherwise known as analysis of variance and covariance) to some extent, and multivariate analysis to a much larger extent, have not had the full benefit of the deeper and later development of the Neyman-Pearson theory of inference (which is concerned with the testing of composite hypothesis), to say nothing of the still later and more general theory due to Wald and his followers. That is why, so far as the philosophical basis is concerned, multivariate analysis, and to some extent the more involved parts of univariate analysis, are today in a far less satisfactory position than the more elementary portions of univariate analysis. In the construction of concrete statistical tools or tests of hypothesis usually involved in multivariate and univariate analysis, extensive use has been made of a principle put forward by Neyman and Pearson at a particular stage of development of their theory (before the more significant part of their theory was fully developed by them and later workers)—which is known as the "*likelihood ratio principle*," the test based thereon being known as the "*likelihood ratio test*" of a hypothesis concerning parameters in a probability distribution when its form could be supposed to be known. This "*likelihood ratio principle*" is supposed to have played the same significant part in testing of hypothesis as the "*maximum likelihood statistic*" has no doubt played in estimation. This, however, does not appear to be borne out by such studies as the speaker himself has made or is aware of as having been made by others. It should be kept in mind that neither the "*maximum likelihood statistic*" in problems of estimation nor the "*likelihood ratio principle*" in testing of hypothesis could be rationally considered to be good or bad in their own rights. They are good or bad, optimum or otherwise, in so far as they conform to or depart from criteria which we could legitimately lay down as optimum from considerations more closely tied up with probability and physical needs. For situations of estimation and for large sample problems *consistency* and *efficiency* are such criteria, in small samples, however, efficiency being replaced by "*relative loss of information*." Under rather restrictive conditions a further optimum criterion would be available (no matter what the sample size might be) which is known as sufficiency. It is well-known that under certain restrictions on the elementary probability law (which are not too stringent), the maximum likelihood statistic would be consistent and efficient (both in the large and small sample sense); it would also be sufficient if a sufficient statistic exists at all. It is really this that would justify the use of the maximum likelihood statistic in estimation—particularly in what might be technically called point estimation. Of course, even here situations could be constructed in which, under the more general theory due to Wald, more optimum estimates could be set up. But in a wide variety of situations, the maximum likelihood estimate would appear to hold the field. In the testing of hypothesis, on the Neyman-Pearson theory, a similar criterion would be, for a simple hypothesis (that is, one in which the corresponding probability is completely specified) the power of the test with regard to an alternative hypothesis (the corresponding probability being completely specified), or its complement the second kind of error. That test would be optimum (in relation to a particular alternative) which, at a given level of the first kind of error, happens to be the most powerful in the sense of minimizing the second kind of error (with regard to the particular alternative). In parametric problems, that is, those in which the *form* of the probability law is specified and the hypothesis concerns only the parameters, there is, under certain restrictions on the probability law, the further possibility of this most powerful test being such for all alternatives (in which case it is called a uniformly most powerful test) or for a class of alternatives (in which case it might be called uniformly most powerful test in a restricted sense). In such parametric problems, if it so happens (as it would more often than not) that uniformly most powerful tests are not available, we look around for a test which, for a particular level of the first kind of error, would be the most powerful test on an average, the average power of any test being computed by averaging the power function over all possible or relevant alternatives after first attaching to each alternative a suitable weight based on mathematical and physical considerations. Different systems of weight attached to the alternatives will thus lead to different optimum tests, that is, most powerful tests on an average. This concept of the most powerful test on an average (being based on any suitable system of weights attached to the alternatives) is an extraordinarily fruitful one both physically and mathematically—physically because starting from a multi-decision problem in the sense of Wald and taking a zero loss function between any two alternatives, unless one of them happens to be the so-called hypothesis to be tested, and making another plausible simplifying assumption, we can show that the multi-decision problem reduces to a two-decision situation which, by the Neyman-Pearson approach, would yield exactly what we call the most powerful test on an average; and mathematically because, in the case of several parameters, a system of suitable weights attached to the alternatives often secures for the average power (as also for the most powerful test on an average) the elegant and desirable property of invariance under a wide class of transformations of the parameters to new parameters. This is so far as simple hypothesis is concerned. For a composite hypothesis, that is, one in which part of the probability law is specified no matter what the other part might be, the situation is more complicated but more interesting. We shall consider the parametric situation of this composite hypothesis in which the form of the probability law is specified, the hypothesis (and its alternative or alternatives) would concern some of the parameters in the probability law no matter what the values of the other parameters might be. The specified parameters might be called non-free parameters and the unspecified parameters the free parameters. In such a situation, it would be desirable to be able to construct a class of tests in the specification of any of which the free parameters would not enter (but the non-free parameters ordinarily would) and for any of which the first kind of error would stay constant no matter what the free parameter might be. Such tests, when available, are known as similar region tests. The constancy of the first kind of error might be replaced by the weaker condition that the first kind of error would stay $\leqq \alpha$ (<1) in which case any such test might be called a valid test. We shall, however, mostly consider similar region tests. If such a class is available at all, it would be desirable, if possible, to be able to obtain from amongst these the most powerful test of any hypothesis about the non-free parameters against any specific alternative about the same parameters no matter what values of the free parameters might be associated with the non-free parameters in the hypothesis and in the alternatives. In other words, if θ_1 stands for the set of k non-free parameters and θ_2 for the set of s free parameters, the first drive is to construct a class of tests for the hypothesis $\theta_1 = \theta_1^0$, such that the specification and the first kind of error for any one of them might depend on θ_1^0, but not on θ_2. If this were successful, then the second drive would be to find amongst these a test which would be the most powerful test of the simple hypothesis ($\theta_1 = \theta_1^0$, $\theta_2 = \theta_2^0$) against a specific (θ_1 and θ_2) and yet whose specification and first kind of error would be independent of θ_2^0 and θ_2. The specification of such a test would of course depend upon θ_1^0 and θ_1, but if it so happens that it is independent of θ_1 or is the same for a whole class of θ_1, we have available a uniformly most powerful test (in an unrestricted or restricted sense) for a composite hypothesis. Let us now take the possibilities one by one. Under certain restrictions on the probability law a mechanism is available by which it is possible to construct an infinite number of similar region tests for $H(\theta_1 = \theta_1^0)$—in which case such tests might be called Neyman-mechanism tests. This may or may not exhaust the whole possible class of similar regions. Under certain further restrictions on the probability law there would be available from amongst such tests one which is the most powerful test of the simple hypothesis ($\theta_1 = \theta_1^0$ and $\theta_2 = \theta_2^0$) against (θ_1 and θ_2) but whose specification or first kind of error would be independent of θ_2^0 and θ_2. Under certain further restrictions this test would happen to be the uniformly most powerful test in the sense already indicated. Where this last does not happen and the most powerful test depends for its specification on θ_1, we are interested in constructing a most powerful test on an average, the averaging being over θ_1, again leaving the set θ_2 free. The weight function on θ_1 has to be suitably chosen with an eye to various requirements. This is for testing of hypothesis. The Neyman theory of estimation by confidence intervals rests upon the Neyman-Pearson theory of testing of hypothesis, and the optimum test in the latter case, under certain weak restrictions, would lead to an optimum estimate in the former case.

The foregoing is all mostly concerned with the Neyman-Pearson theory and its later developments. In this talk the speaker will not discuss the Wald theory of statistical inference and its relation to the likelihood ratio method. Such a a discussion could be appropriately held later. In the present talk, therefore, the likelihood ratio principle will be considered to be good or bad, optimum or otherwise, according as it possesses or does not possess optimum properties in terms of the Neyman-Pearson theory and its later (direct) developments. That could be the only justification for its use unless we sought such justification in terms of the Wald theory, for which the ground does not appear to be yet ready. Now, when examined from the point of view of the Neyman-Pearson theory, it is found that, while the likelihood ratio test (under fairly general conditions) possesses a number of properties, none of them happens to be a strong or optimum property. The only strong or optimum general property (irrespective of whether the sample is large or small) known to the speaker is that if in any situation there is a uniformly most powerful test (in an unrestricted or restricted sense), then the likelihood ratio test would coincide with it. But since uniformly most powerful tests are rare, it would be seldom that this fact could be brought up to uphold the use of this test. Let us go back to the other general properties to which we have just referred. Assume for simplicity, but without any essential loss of generality, that $\phi(X; \theta_1)$ is a probability density for a set of stochastic variates $X(\equiv x_1, \cdots, x_n)$ which is continuous in X and also continuous and differentiable in

$$\theta_1(\equiv \theta_{11}, \cdots, \theta_{1k}),$$

both X and θ_1 ranging over the real field. Assume now that we are going to test a simple hypothesis θ_1^0 against a simple alternative θ_1. It is well-known that the most powerful test of θ_1^0 against θ_1 would be based on a critical region ω_0 defined such that inside ω_0:

$$\phi(X; \theta_1) \geqq \lambda\phi(X; \theta_1^0), \tag{1.0}$$

where λ is so chosen that

$$\int_{\omega_0} \phi(X; \theta_1^0)\, dX = \alpha_0 \tag{1.1}$$

(a preassigned first kind of error). This ω_0 will have the properties that

$$\int_{\omega_0} \phi(X; \theta_1)\, dX \geqq \alpha_0, \tag{1.2}$$

and for all ω_j's satisfying (1.1)

$$\int_{\omega_0} \phi(X; \theta_1)\, dX \geqq \int_{\omega_j} \phi(X; \theta_1)\, dX. \tag{1.3}$$

Denoting by $\hat{\theta}_1$ (when $\hat{\theta}_1$ is really a set of statistics and not parameters) the maximum likelihood estimates of θ_1, it is well-known that the likelihood ratio test for θ_1^0 would be based on a critical region $\hat{\omega}$ such that inside $\hat{\omega}$:

$$\phi(X; \hat{\theta}_1) \geqq \mu(\theta_1^0; \alpha_0)\phi(X; \theta_1^0), \tag{1.4}$$

where μ is so chosen that

$$\int_{\hat{\omega}} \phi(X; \theta_1^0)\, dX = \alpha_0. \tag{1.5}$$

Now two general properties of the likelihood ratio test would immediately emerge out of all this.

(i) By the well-known Neyman-Pearson lemma

$$\int_{\hat{\omega}} \phi(X; \hat{\theta}_1)\, dX \geqq \int_{\omega_j} \phi(X; \hat{\theta}_1)\, dX \tag{1.6}$$

for all ω_j's satisfying (1.5). But the left-hand side of (1.6) is an upper bound of the integral of $\phi(X; \theta_1)$ over $\hat{\omega}$, and the right-hand side is an upper bound of the integral of $\phi(X; \theta_1)$ over any ω_j (satisfying (1.5))—the upper bounds having references to variation over θ_1. If these were the least upper bounds (which unfortunately they are not), then the likelihood ratio test would have had a remarkable property. As it is, (1.6) does not in general get us far. Under certain conditions, however, in large samples these would tend to be the least upper bounds, in which case the likelihood ratio test would have the well-known property of being an asymptotically most powerful test.

(ii) Consider the hypothesis θ_1^0 against θ_1 at a level of the first kind of error which is α (different from α_0). The most powerful test of θ_1^0 against θ_1 would be based on ω such that inside ω:

$$\phi(X; \theta_1) \geqq \lambda(\theta_1, \theta_1^0; \alpha)\phi(X; \theta_1^0), \tag{1.7}$$

where λ must be a function of θ_1, θ_1^0, and α, being so chosen as to satisfy

$$\int_\omega \phi(X; \theta_1^0)\, dX = \alpha. \tag{1.8}$$

Now take the boundary of ω (given by the equality sign under (1.7)) and consider the envelope of such boundaries by varying over θ_1 and α subject to the constraint that

$$\lambda(\theta_1, \theta_1^0; \alpha) = \mu(\theta_1^0; \alpha_0), \tag{1.81}$$

$\mu(\theta_1^0; \alpha_0)$ being taken over from the right-hand side of (1.4). Such an envelope would be obtained by eliminating θ_1 between

$$\frac{\partial \phi}{\partial \theta_{1j}} (j = 1, 2, \cdots, k) = 0 \quad \text{and} \quad \phi(X; \theta_1) = \mu(\theta_1^0; \alpha_0)\, \phi(X; \theta_1^0), \tag{1.82}$$

leading to

$$\phi(X; \hat{\theta}_1) = \mu(\theta_1^0; \alpha_0)\phi(X; \theta_1^0), \tag{1.83}$$

which is the boundary of the critical region for the likelihood ratio test. If however, the envelope had been obtained by keeping α constant but varying only over θ_1 this would have been a significant result. As it is, it does not get us far.

Other *general* properties are also known, including properties of invariance under transformation of parameters θ_1 to a new set. But none of these *general* properties gets us very far. We have so far discussed *likelihood ratio test* in relation to a simple hypothesis. Taking a composite hypothesis $\theta_1 = \theta_1^0$ for a probability law $\phi(X; \theta_1; \theta_2)$ where θ_2 stands for the set of free parameters $(\theta_{21}, \cdots, \theta_{2s})$, the likelihood ratio test would be based on a region $\hat{\omega}$ such that inside $\hat{\omega}$:

$$\phi(X; \hat{\theta}_1, \hat{\theta}_2) \geqq \lambda\phi(X; \theta_1^0, \hat{\theta}_2^0), \tag{1.9}$$

where λ is chosen so as to satisfy

$$\int_{\hat{\omega}} \phi(X; \theta_1^0, \theta_2)\, dX = \alpha_0, \tag{1.91}$$

and $\hat{\theta}_1$ and $\hat{\theta}_2$ are the unrestricted maximum likelihood estimates of θ_1 and θ_2, and $\hat{\theta}_2^0$ are the set of maximum likelihood estimates of the set θ_2 assuming $\theta_1 = \theta_1^0$. There is no guarantee that (1.9) subject to (1.91) would yield a similar region (specified independently of θ_2). If it does, then properties somewhat analogous to those discussed under (i) and (ii) would be available—which again are not strong or optimum properties. The restrictions on ϕ under which (1.9) would lead to a similar region test have been investigated but need not be discussed here. They are more stringent than those under which Neyman-mechanism tests are available for a composite hypothesis.

For purposes of illustration consider three simple and familiar problems from univariate and three corresponding problems from multivariate analysis. Under univariate analysis consider for two univariate normal populations with means m_1 and m_2 and standard deviations σ_1 and σ_2, (a) the composite hypothesis $\sigma_1 = \sigma_2$, and (b) the composite hypothesis $m_1 = m_2$ assuming $\sigma_1 = \sigma_2$, and (c) lastly for k univariate normal populations with means and standard deviations m_r, σ_r ($r = 1, 2, \cdots, k$), the hypothesis $m_1 = m_2 = \cdots = m_k$ assuming $\sigma_1 = \sigma_2 = \cdots = \sigma_k$ (which is the basis of analysis of variance). It is known that for each of (a), (b), and (c) Neyman-mechanism tests exist. For (a), among these tests, there is a uniformly most powerful test for the whole class of composite alternatives $\sigma_1 > \sigma_2$, and another for the whole class $\sigma_1 < \sigma_2$, both being based on the familiar F-test. The likelihood ratio method yields the same test, as it should, in this situation. For (b), among these tests, there is a uniformly most powerful one for the class of alternatives $m_1 > m_2$ and another for the class $m_1 < m_2$, both being based on the familiar t-test and being identical with that given by the likelihood ratio method (as should happen in this situation). For (c) the situation, however, is different. There is here no uniformly most powerful test (general or restricted). Among the Neyman-mechanism tests there is, however, a most powerful test of the composite hypothesis against any specific alternative for the non-free parameters (no matter what the free parameters might be). This is a test which could be based on the t-table and might thus be called a t-test. Different most powerful tests on an average could be constructed by attaching different weight functions to the space of the non-free parameters. On one such weight function, namely:

$$f\left(\sum_{r=1}^{k} n_r(m_r - \bar{m})\right)$$

(where n_r is the size of the sample from the rth population and

$$\bar{m} = \sum_{r=1}^{k} n_r m_r \Big/ \sum_{r=1}^{k} n_r$$

and where f is subject to certain plausible restrictions to make it a weight function) we have a most powerful test on an average which happens to be the F-test (known to be derivable from the likelihood ratio method for this composite hypothesis). The likelihood ratio criterion here could be justified to the same extent that the weight function f could be upheld, and the f has, of course, some grounds for justification—both mathematical and perhaps also physical. (a), and (b), therefore, will be said to possess each a test with very strong properties while (c) would possess from two different points of view two different tests with strong (but not very strong) properties. This is so far as univariate analysis goes.

Let us consider now the corresponding problems in multivariate analysis. Take two p-variate normal populations with means m_{1i}, m_{2i} and dispersion matrices (α_{1ij}) and (α_{2ij}) $(i, j = 1, 2, \cdots, p)$ and in the light of random samples of sizes N_1 and N_2 from them consider the hypothesis (a.1) $(\alpha_{1ij}) = (\alpha_{2ij})$, and (b.1) $m_{1i} = m_{2i}$ $(i = 1, 2, \cdots, p)$ assuming $(\alpha_{1ij}) = (\alpha_{2ij}) = (\alpha_{ij})$ (suppose). It is known that for each of the problems (a.1) and (b.1) there are (an infinite number of) Neyman-mechanism tests. Amongst these there does not exist either for (a.1) or (b.1) a most powerful test against a specific alternative (for the non-free parameters), let alone a uniformly most powerful test, that is, there is no test with strong properties, let alone any having very strong properties. But there would be available for both (a.1) and (b.1) optimum tests with less strong optimum properties than those for (a), (b), or even (c) discussed earlier. Such tests would be generated in the following manner. Let X_1 and X_2 be $p \cdot n_1$ $(n_1 = N_1 - 1)$ and $p \cdot n_2$ $(n_2 = N_2 - 1)$ matrices of reduced observations from the two populations such that the sample dispersion matrices (a_{1ij}) and (a_{2ij}) would be given by $n_1(a_{1ij}) = X_1X_1'$ and $n_2(a_{2ij}) = X_2X_2'$ and let

$$(n_1 + n_2)(c_{ij}) = n_1(a_{1ij}) + n_2(a_{2ij}).$$

Also let

$$(b_{ij}) = (\bar{x}_{1i} - \bar{x}_{2i})(\bar{x}_{1j} - \bar{x}_{2j})$$

where $\bar{x}_{1i}$ and $\bar{x}_{2i}$ $(i = 1, 2, \cdots, p)$ are the first and second sample means for the ith variate. Assume further that $(n_1, n_2) \geqq p$. Now for the hypothesis (a.1) and for different types of specific alternatives (concerning the non-free parameters) different nonsimilar region tests (in the sense of having to be specified by the free parameters as well) exist which are otherwise most powerful tests. The intersection of such regions, however, would lead to two similar regions and two tests based thereon such that one is to be used for one type of alternatives and the other for another type. These two tests have each, for its own purposes, a number of moderately strong properties (which is the most we could get in this situation). These tests are based respectively on the largest and smallest of the p *generally* nonvanishing roots of the determinantal equation in T:

$$T: | X_1X_1' - TX_2X_2' | = 0, \tag{2.0}$$

that is, on T_1 and T_p when $\infty > T_1 \geqq T_2 \geqq \cdots \geqq T_p \geqq 0$. The joint sampling distribution of $(T_1, \cdots, T_p)$ involves as parameters the roots $(\tau_1, \tau_2, \cdots, \tau_p)$ of the determinantal equation in τ:

$$| \alpha_{1ij} - \tau\alpha_{2ij} | = 0 \tag{2.1}$$

(assuming both (α_{1ij}) and (α_{2ij}) to be nonsingular). The distribution of each of the T_i's will however involve all the τ_j's. For any hypothesis concerning τ_j's the T_i's might be supposed to provide Neyman-mechanism tests out of which T_1 and T_p might be supposed to provide optimum tests for two different purposes. In the space of $(T_1, \cdots, T_p)$ we could also construct different most powerful tests on an average (by averaging over $\tau_1, \cdots, \tau_p$ with different weight functions). Suitable weight functions over the parametric space $(\tau_1, \cdots, \tau_p)$ would lead to optimum tests in the space of $(T_1, \cdots, T_p)$, one such being that based on the product $\prod_{j=1}^{p} T_j$. This is the one that we obtain by applying the likelihood ratio principle on (a.1). It does not possess any more optimum properties than those possessed by any of $(T_1, T_2, \cdots, T_p)$ or any of their functions including the other symmetric functions besides $\prod_{j=1}^{p} T_j$. The only justification we could put up for this would therefore be based on what one could say about the weight function on $(\tau_1, \cdots, \tau_p)$ which leads to this test as the most powerful test on an average. But T_1 and T_p, besides possessing the properties already indicated, would also be the most powerful tests on an average under weight functions on $(\tau_1, \cdots, \tau_p)$ which are more appropriate than that leading to $\prod_{j=1}^{p} T_j$, and hence T_1 and T_p would each seem to be more appropriate than $\prod_{j=1}^{p} T_j$, though this latter is also good enough for certain purposes. The situation with regard to (b.1) would be nearly the same except that here we have an optimum test with a weaker property being based on the one *generally* nonzero root of the determinantal equation in T:

$$| b_{ij} - Tc_{ij} | = 0,$$

where root T involves as a parameter in its sampling distribution the one possible nonzero root τ of the determinantal equation in τ:

$$| (m_{1i} - m_{2i})(m_{1j} - m_{2j}) - \tau\alpha_{ij} | = 0.$$

The multivariate analogue of the univariate hypothesis (c) would be as follows: suppose there are k p-variate normal populations with means m_{ri} ($r = 1, 2, \cdots, k$ and $i = 1, 2, \cdots, p$) and a common dispersion matrix (α_{ij}). The hypothesis to be tested is (c.1) the $m_{1i} = m_{2i} = \cdots = m_{ki}$ ($i = 1, 2, \cdots, p$). We have Neyman-mechanism tests for (c.1) but no most powerful test amongst this for any specific alternative, let alone any uniformly most powerful test. In the sense indicated under (a.1) there are two weak optimum tests for (c.1) which are based on the generalizations of the t-test for (c), and two other weak optimum tests which are generalizations of the F-test for (c). I shall here consider these latter. They are based on the largest and smallest roots of the pth degree equation in T:

$$| X_1X_1' - TX_2X_2' | = 0,$$

where the sample sizes from the different populations are N_r ($r = 1, 2, \cdots, k$), where $n_r = N_r - 1$, X_1 is a $p \times (k - 1)$ matrix of reduced sample means for the different variates such that

$$\sum_{r=1}^{k} N_r(\bar{x}_{ri} - \bar{x}_i)(\bar{x}_{rj} - \bar{x}_j) = X_1X_1'$$

(where $\bar{x}_{ri}$ is the mean of the ith variate for the rth sample and where $\bar{x}_i = \sum_{r=1}^{k} N_r\bar{x}_{ri}/\sum_{r=1}^{k} N_r$) and X_2 is a $p \times n$ matrix of reduced observations (where $n = \sum_{r=1}^{k} N_r - k$), such that the pooled dispersion matrix from the different samples would be given by $\sum_{r=1}^{k} n_r(a_{rij}) = X_2X_2'$. If we assume that $n > p$, the number of generally nonvanishing roots of the determinantal equation would be min $(k - 1, p)$, depending upon the rank of the matrix X_1X_1'. The joint distribution of the roots would involve as parameters the q possible nonvanishing roots $\tau_1, \cdots, \tau_q$ $(q \leqq p)$ of the pth degree determinantal equation in τ:

$$| \beta_{ij} - \tau\alpha_{ij} | = 0,$$

where

$$\beta_{ij} = \sum_{r=1}^{k} N_r(m_{ri} - \bar{m}_i)(m_{rj} - \bar{m}_j)/k - 1 \left(i, j = 1, 2, \cdots, p; \bar{m}_i = \sum_{r=1}^{k} N_r m_{ri} \Big/ \sum_{r=1}^{k} N_r\right),$$

and where (α_{ij}) is supposed to be of rank p and (β_{ij}) of rank $q \leqq \min(k - 1, p)$. Tests based on T_1 or T_p, besides having weak optimum properties already indicated, would also be most powerful tests on an average, under physically meaningful weight functions on $(\tau_1, \cdots, \tau_q)$. The sum $\sum_{j=1}^{p} T_j$ also comes out as the most powerful test on an average under another meaningful weight function on $(\tau_1, \cdots, \tau_q)$, but no meaningful weight function on $(\tau_1, \cdots, \tau_q)$ could be set up which would lead to the product $\prod_{j=1}^{p} T_j$ as the most powerful test on an average. It is, however, this product of the roots that comes out as the likelihood ratio test.

These, and other investigations which the author has not had the space or time to discuss here, would show that more attention to the Neyman theory of testing of composite hypothesis and its later developments would yield more meaningful tests than have so far been generally available on the basis of the likelihood ratio principle.

Bibliography

The papers of Neyman-Pearson, Neyman, Scheffé, Lehman, and Stein are easily available and well-known to readers in this country. In addition to these papers, use has also been made in this talk of four other papers in Sankhyā (Indian Journal of Statistics) to which references are given below:

1. *Notes on testing of composite hypothesis* (I): Sankhyā vol. 8(3) (1947).
2. *Notes on testing of composite hypotheses* (II): Sankhyā vol. 9 (1) (1948).
3. *Univariate and multivariate analysis as problems in testing of composite hypotheses* (I): Sankhyā vol. 10(1 and 2) (1950).
4. *Univariate and multivariate analysis as problems in testing of composite hypotheses* (II): Sankhyā (in the press).

University of North Carolina,
Chapel Hill, N. C., U. S. A.

PROBABILITY

A HISTORY OF PROBABILITY IN THE UNITED STATES OF AMERICA BEFORE THE TWENTIETH CENTURY

William Dowell Baten

This article contains a history of the material concerning probability which was written by people who lived in the United States of America. It begins with an article presented in 1791 and extends through 1900. During this period many papers on probability were presented to the various scientific organizations. The subject matter includes life insurance theory, mortality tables, derivations of the normal curve of error, the method of least squares, probable errors of observations, games of chance, the importance of probability problems, probability laws of functions of chance variables, derivations of various probability laws, inverse probability, properties of polynomials, mean values, gambler's ruin, geometric probability, test books, teaching concerning probability, applications, probability courses, etc.

This paper also lists the names of the people who presented this material and summarizes their contributions to the development of probability subject matter.

Michigan State College,
East Lansing, Mich., U. S. A.

ON ASYMPTOTIC EXPANSIONS OF PROBABILITY FUNCTIONS

Harald Bergström

Consider the pr. f. of a sum $X^{(1)} + X^{(2)} + \cdots + X^{(n)}$ of n independent, equally distributed random variables in the euclidean space R_k of the finite dimension k. If each variable has the pr. f. $P(E)$, the sum has the pr. f. $P^{*n}(E)$, where $P^{*n}(E)$ denotes the n-fold convolution of P with itself. In particular for 1-dimensional d.f.'s the known Edgeworth expansion has been used as an approximation of the d.f. for the sum. In the following we give a more general asymptotic expansion of P^{*n} from which the Edgeworth expansion may be determined as a special case.

If $\varphi\ (E)$ is a set function and E a set, we denote the total variation of φ in R_k by $V[\varphi]$ and put $M_E[\varphi] = \text{Max}_x \mid \varphi(E - x) \mid$, where $E - x$ denotes the set obtained by E through the translation $-x$. $g(n)$ (as well as $g_1(n, s)$) may denote a decreasing function of n with $g(n/\rho) = O[g(n)]$ for every positive constant ρ.

Let now Q be a pr. f. which shall be used for the approximation of P^{*n}. Then

$$(\mathrm{A}_1) \qquad P^{*n} = \sum_{\nu=0}^{s} \Delta_n^{(\nu)} + r_n^{(s+1)}$$

with $\Delta_n^{(\nu)} = C_{n,\nu}Q^{*n-\nu}*(P - Q)^{*\nu}$ and some remainder term $r_n^{(s+1)}$. If the condition

$$(C_1) \qquad V[\Delta_n^{(1)}] < g(n)$$

is satisfied, $\Delta_n^{(\nu)} = O[g^\nu(n)]$ for any fixed integer ν. However $\Delta_n^{(\nu)}$ may be of a smaller order of magnitude than $g^\nu(n)$. Let now (C_1) hold and let $S = \sum_{n=1}^{\infty} g^{s+1}(n)/n$ converge when s is some fixed integer $\geqq 0$, and suppose that the inequalities $M_E[\Delta_n^{(s+1)}] < \mu(E)g_1(n, s)$, $M_E(P) < \mu(E)$, $M_E(Q) < \mu(E)$ and the condition

$$(C_2) \qquad \lim_{p\to\infty} V[P^{*p}*(P - Q)] = 0$$

are satisfied. Then

$$(1) \qquad M_E[r_n^{(s+1)}] < C\mu(E)g_1(n, s),$$

where C depends on s and the functions $g(n)$ and $g_1(n, s)$ but not on n. When (C_1) and (C_2) are satisfied and S converges, then (A_1) has moreover the strictly asymptotical properties

$$(2) \qquad V[\Delta_n^{(\nu)}] = O[g^\nu(n)], \qquad V[r_n^{(s+1)}] = O[g^{s+1}(n)]$$

and (C_1) and (C_2) are necessary for the existence of these properties when S converges. The condition (C_2) is used only to that extent that $V[P^{*n}*(P - Q)]$ is smaller then some value depending on s and the functions $g(n)$ and $g_1(n, s)$ but not on n.

When P is a nonsingular d.f. with finite absolute mean values of the order λ for some λ in the interval $2 < \lambda \leqq 3$, and Q is that (nonsingular) normal d.f. which has the same mean value vector and the same moment matrix as P, then (C_1) is satisfied with

$$g(n) = n^{-(\lambda-2)/2} \log^{k/2} n$$

and $\Delta_n^{(\nu)} = o(n^{-\nu(\lambda-2)/2})$ or $O[n^{-\nu(\lambda-2)/2}]$ according as $\lambda < 3$ or $\lambda = 3$. We have

$$(3) \qquad r_n^{(s+1)} = O[n^{-1/2} + n^{-(s+1)(\lambda-2)/2}]$$

always, and

$$(4) \qquad r_n^{(s+1)} = O[n^{-(s+1)(\lambda-2)/2}]$$

when (C_2) holds. Here (C_2) may be replaced by the less restrictive condition

$$(C_3) \qquad \lim_{n\to\infty} V\{P^{*n}(x)*Q[(an)^\alpha x]*[P(x) - Q(x)]\} = 0$$

with $\alpha = (s + 1)(\lambda - 2)/2 - 1/2$ and a constant $a > 1$. In the 1-dimensional case, (C_3) is necessarily satisfied if

$$r_n^{(s+1)} = o(n^{-(s+1)(\lambda-2)/2}).$$

From (A_1) different expansions may be derived. Let P be a nonsingular pr. f. with finite absolute mean values of order λ for some λ in the interval $2 < \lambda \leqq 3$, and let Q be that normal pr. f. which has the same mean value vector and the same moment matrix as P. Further suppose that $M_E(P)/m(E)$ is bounded, when E belongs to any family, regular in the sense of Lebesgue, and $m(E) > 0$ is the measure of E. Then denoting the derivative in the sense of Lebesgue by $D[\varphi]$ for a set function φ, we have

$$D[\Delta_n^{(\nu)}] = O[n^{-(k+\nu(\lambda-2))/2}] \tag{5}$$

everywhere and, if the condition (C_2) is satisfied,

$$D[r_n^{(s+1)}] = O[n^{-(k+(s+1)(\lambda-2))/2}] \tag{6}$$

in all points, where DP^{*n} exists. If P is absolutely continuous, the condition (C_2) is necessary for the validity almost everywhere of that last relation.

CHALMERS UNIVERSITY OF TECHNOLOGY,
GOTHENBURG, SWEDEN.

RANDOM FUNCTIONS ON DIVISIA ENSEMBLE

MARIA CASTELLANI

In this paper we investigate a class of real measurable functions of a statistical Divisia ensemble. Let us consider a lattice space with a set D of elements which renew themselves at random during a certain time interval (t_0, t), because there is an inflow and outflow of elements. Let us assume that each element is associated with a real, measurable, bounded function $\delta(\theta - x)$ for $(\theta - x) \subseteq (t - t_0)$.

Let $p(\theta - x)$ be the probability function of an element which is not being eliminated from D, $d\mu_\theta$ the probability density of an element which is eliminated, $d\nu_x$ the elements which are flowing into D during the time interval $(x, x + dx)$, and λ_0 the elements in D at the time t_0. In several problems it might be more convenient to assume $p(\theta - x) \cdot \delta(\theta - x) = f(\theta - x)$. According to any given law of random flow, the expected value $E(\delta)$ is

$$E(\delta) = \lambda_0 f(t - t_0) + \lambda_0 \int_{t_0}^{t} f(\theta - t_0)\, d\mu_\theta + \int_{t_0}^{t} f(t - x)\, d\nu_x + \int_{t_0}^{t}\int_{x}^{t} f(\theta - x)\, d\mu_\theta\, d\nu_x .$$

We do not investigate this general integral equation but, instead, limit our attention to obtaining some fundamental equations of actuarial mathematics. When the δ is a linear function and we assume a Karup functional relation $p(\theta - x)\, d\mu_\theta = -\, dp(\theta - x)$, we then obtain a generalized Galbrun integral equation for sickness insurance. A generalized Cantelli equation for mathematical reserves is easily derived by considering the actual rate of elements flowing out.

The Benini-Des Essars equation for the average time spent in D by the flowing elements is derived under very restrictive conditions of continuity and linearity.

UNIVERSITY OF KANSAS CITY,
KANSAS CITY, MO., U. S. A.

AN ERGODIC THEOREM FOR STATIONARY MARKOV CHAINS WITH A COUNTABLE NUMBER OF STATES

KAI LAI CHUNG

Let X_n, $n = 0, 1, \cdots$, be the random variables of a Markov chain with a countable number of states, numbered $1, 2, \cdots$, and with stationary transition probabilities $P_{jk}^{(1)}$. As usual $P_{jk}^{(n)}$ denotes the n-step transition probability. Write $T_{jk}^{(n)} = \sum_{\nu=0}^{n} P_{jk}^{(\nu)}$. The ergodic theorem for Markov chains states that if j and k belong to the same ergodic class, then $\lim_{n\to\infty} n^{-1}T_{jk}^{(n)} = p_k$. It is known that if $p_j > 0$, then $p_k > 0$ so that the above implies $\lim_{n\to\infty} T_{jj}^{(n)}/T_{kk}^{(n)} = p_j/p_k$. If $p_j = 0$, then $p_k = 0$ and no conclusion can be drawn of the existence of the last-written limit. This question was raised by Kolmogorov and answered by Doeblin (Bull. Soc. Math. France vol. 66 (1938) pp. 210–220). More precisely he proved that if j and k are such that there exist positive integers m and m' for which $P_{jk}^{(m)} > 0$ and $P_{kj}^{(m')} > 0$, then $\lim_{n\to\infty} T_{jk}^{(n)}/T_{kk}^{(n)} = 1$, and $\lim_{n\to\infty} T_{jj}^{(n)}/T_{kk}^{(n)}$ exists and is positive and finite.

A new proof of Doeblin's theorem is given by introducing the quantities in the infinite series written below and using generating functions. This method also determines the last-written limit to be equal to λ_j/λ_k where $\lambda_j = 1 + \sum_{n=1}^{\infty} P(X_\nu \neq k \text{ for } 1 \leq \nu \leq n, X_n = j \mid X_0 = j)$ and λ_k is obtained by interchanging j and k in λ_j. With extraneous conditions on the analytic nature of $T_{kk}^{(n)}$ the theorem becomes a consequence of a Hardy-Littlewood-Karamata-Tauberian theorem. Our proof circumvents this invocation. An extension is made to Markov chains with a continuous parameter t. If e. g. the uniformity condition $\lim_{t\downarrow 0} P_{jj}(t) = 1$ uniformly in all j is assumed, Doeblin (Skandinavisk Aktuarietidskrift vol. 22 (1939) pp. 211–222) has shown that almost all sample functions are step functions, etc. Under these circumstances it is proved that $\lim_{t\to\infty} \int_0^t P_{jj}(s)\,ds/\int_0^t P_{kk}(s)\,ds$ exists and is positive and finite, in fact a determination of its value similar to the above is given. Contrary to hope this natural extension is not easily deducible from the discrete parameter case, but our method works with Laplace transforms instead of generating functions. The question of the existence of $\lim_{n\to\infty} P_{jj}^{(n)}/P_{kk}^{(n)}$ (or of $\lim_{t\to\infty} P_{jj}(t)/P_{kk}(t)$) seems to be open. Only in the special case where X_n is the sum of n independent, identically distributed random variables with mean 0, it has been proved by Erdös and the present author that this limit is one.

CORNELL UNIVERSITY,
ITHACA, N. Y., U. S. A.

ÉLÉMENTS ALÉATOIRES DE NATURE QUELCONQUE

R. FORTET

À la suite de travaux de M. Fréchet, je me suis intéressé, et j'ai incité quelques jeunes probabilistes à s'intéresser, à l'élaboration d'une théorie directe des éléments aléatoires de nature quelconque (ne se réduisant pas nécessairement à des variables aléatoires numèriques à une ou à un nombre fini de dimensions); des applications à la théorie des fonctions aléatoires ou à divers problèmes concrets justifient cette étude; il est naturel, surtout si on a en vue l'application aux fonctions aléatoires, d'aborder d'abord le cas d'un élément X prenant ses valeurs dans un espace linéaire $\mathfrak{X}$; un premier problème est alors d'étendre au cas d'un tel élément la notion d'espérance mathématique, ce qui revient à une théorie de l'intégration; dans le cas où $\mathfrak{X}$ est un espace de Banach, Mlle Mourier a montré (C.R. Acad. Sci. Paris t. 229 (1949) p. 1300) qu'une application de l'intégrale de Pettis permet de perfectionner notablement des résultats de Fréchet; tandis qu'une extension, non encore publiée, de la méthode de Daniell faite par M. Régnier permet de définir une espérance mathématique pour une large catégorie de f.a. mesurables au sens de Doob.

Un second problème est d'envisager l'addition d'éléments aléatoires X_1, $X_2, \cdots, X_n, \cdots$ à valeurs dans $\mathfrak{X}$; pour cela la notion de caractéristique semble devoir être utile; si $\mathfrak{X}$ est un espace de Banach, on peut définir la caractéristique $\varphi(x^*)$ d'un élément aléatoire X, comme fonction de la fonctionnelle linéaire x^* variable dans le dual $\mathfrak{X}^*$ de $\mathfrak{X}$, par $\varphi(x^*) = E[x^*(X)]$; des propriétés de $\varphi(x^*)$ s'établissent aisément, dont la principale est que $\varphi(x^*)$ est "définie positive"; mais une fonction définie positive $\varphi(x^*)$, même continue avec la topologie faible dans $\mathfrak{X}^*$, n'est pas forcément une caractéristique; si $\mathfrak{X}$ est séparable et réflexif, on peut donner une condition nécessaire et suffisante pour que $\varphi(x^*)$, supposée définie positive, soit la caractéristique d'un élément aléatoire X "proprement dit" (c'est-à-dire tel que Pr $[\,\|X\| < +\infty] = 1$); on peut définir un élément laplacien X par la condition que, en supposant $E(X) = \theta$ (θ étant l'élément 0 de $\mathfrak{X}$), le logarithme de la caractéristique $\varphi(x^*)$ est de la forme: $-(1/2)E[x^*(Z)]$, où Z est un élément aléatoire proprement dit quelconque à valeurs dans $\mathfrak{X}$.

UNIVERSITY OF CAEN,
CAEN, FRANCE.

A NOTE ON THE GENERAL CHEBYCHEFF INEQUALITY

SALEM H. KHAMIS

Let $\Phi(x)$, $a \leqq x \leqq b$, be a cumulative distribution function, differentiable, never-decreasing and possessing moments up to order $2n$. Let $\Phi(x)$ be defined as 0 for $x \leqq a$ and 1 for $x \geqq b$, and assume that it has at least $n + 1$ points of increase. If $\Psi(x)$, $a' \leqq x \leqq b'$, be another cumulative distribution function with similar properties and identical moments up to order $2n$, then, by the Chebycheff inequality, $|\Phi(x) - \Psi(x)| \leqq [Q_n(x)Q'_{n+1}(x) - Q'_n(x)Q_{n+1}(x)]^{-1} = \Omega_{2n}(x)$, say, where, $Q_r(x)$, $r = 0, 1, 2, \cdots$, or n, is the denominator of the rth convergent of the continued fraction associated with $\int_{-\infty}^{+\infty} d\Phi(t)/(x - t)$ or $\int_{-\infty}^{+\infty} d\Psi(t)/(x - t)$ and is a polynomial of exact degree r possessing orthogonal properties with respect to $d\Phi(x)/dx$ and $d\Psi(x)/dx$ as weight functions.

We have shown that the right-hand side of the above inequality may be improved upon, obtaining for it the value $A\Omega_{2n}(x)$, where A is given by

$$0 \leqq A = 1 + \min\left[\max\left\{-\frac{\Psi'(x)}{\Phi'(x)}\right\}, \quad \max\left\{-\frac{\Phi'(x)}{\Psi'(x)}\right\}\right] \leqq 1,$$

provided the maxima are finite.

We have shown also that when the property "never-decreasing" does not hold, the difficulty, under certain general conditions, could be overcome, thus making it possible to apply the Chebycheff inequality to approximations for distribution functions by series expansions involving negative components. Further, the general relationship between the polynomials $Q_r(x)$ and the corresponding orthogonal polynomials, as usually defined in treatise on orthogonal polynomials, is given.

The methods used in this paper are illustrated, particularly, in the case of the Pearson type III distribution function.

STATISTICAL OFFICE, UNITED NATIONS,
LAKE SUCCESS, N. Y., U. S. A.

UNLIKELY EVENTS IN GENERAL STATIONARY-TRANSITION MARKOFF CHAINS

BERNARD OSGOOD KOOPMAN

A system is considered whose possible states form a perfectly general abstract set X, the observable subsets of which form a σ-algebra Σ. It evolves stochastically through a sequence of n stages, at the kth of which it is in the state $x_k \in X$ ($k = 1, \cdots, n$). The transition probabilities $p_{n,k}(x, E) = \text{prob}(x_k \in E \mid x_{k-1} = x)$ are one-step Markoffian ($x \in X$, $E \in \Sigma$). For a certain $S \in \Sigma$, the event $[x_k \in S]$ is called "success on the kth trial". The object is to find $P(s) = \lim_{n\to\infty} P_n(s)$, where $P_n(s) =$ probability of s successes in first n trials.

It is shown that the class $\mathfrak{B}$ of all complex-valued functions $f = f(x, E)$, defined and bounded over $(x, E) \in (X, \Sigma)$, σ-additive in E (x fixed) and measurable (Σ) in x (E fixed), is a Banach algebra, provided linear combinations with complex coefficients are defined as usual, the $\mathfrak{B}$-product $fg = (fg)(x, E)$ as the abstract integral of $g(y, E)$ with respect to the measure $f(x, F_y)((y, F_y)$

being variables of integration), and norm $N(f)$ as $\sup_{x \in X}$ [absolute variation of $f(x, E)$ on X]. The unit element $u \in \mathfrak{B}$, where $u(x, E) = \chi_E(x) =$ characteristic function of set E. Accents and $\cap$ denote complements and intersections in X; and the ordinary (non $\mathfrak{B}$-) product is written $f(x, E)g(x, E)$.

The following hypotheses involve the stationarity of transitions, the improbability of success, and the Markoffian regularity of $\lim_{m \to \infty} h^m = g$ in (4):

(1) Success $\to$ success: $\chi_S(x)p_{n,k}(x, S \cap E) = a(x, E)$, $N(a) < 1$.
(2) Success $\to$ failure: $\chi_S(x)p_{n,k}(x, S' \cap E) = b(x, E)$.
(3) Failure $\to$ success: $\chi_{S'}(x)p_{n,k}(x, S \cap E) = c(x, E)/n$.
(4) Failure $\to$ failure: Writing $\chi_{S'}(x)p_{n,k}(x, S' \cap E)$ as $g_{n,k}(x, E)p_{n,k}(x, S')$,

where $g_{n,k}(x, E) = \text{prob}(x_k \in E \mid x_{k-1} = x, x_k \in S')$, we assume: (i) $g_{n,k}(x, E) = g(x, E)$; (ii) $g(x, E) \geqq \chi_{S'}(x)\nu(E)$, where $\nu(E)$ is σ-additive over Σ, $\nu(E) \geqq 0$, $\nu(S') > 0$, $\nu(S' \cap E) = \nu(E)$. By classical reasoning $g^m \to h$, where $h(x, E) = \chi_{S'}(x)\mu(E)$, and $\mu(E)$ is like $\nu(E)$, but $\mu(S') = 1$.

Then it is shown that $P(s)$ is generated by the function $\varphi(t) = \sum_{s=0}^{\infty} P(s)t^s$ obtained by first constructing the Banach-algebraic expression

$$f_t = p_0[(u - at)^{-1}b + u]\, g \exp\, [(t - 1)c(u - at)^{-1}h],$$

where $h(x, E) = \chi_S(x)\mu(E)$, and $p_0(x, E) = p_0(E)$ is the initial probability that $x_1 \in E$; and then setting $\varphi(t) = f_t(x, X)$ (which is independent of x).

An equally explicit but more complicated result is obtained in the non-stationary case.

COLUMBIA UNIVERSITY,
NEW YORK, N. Y., U. S. A.

SOME NON-NEGATIVE TRIGONOMETRIC POLYNOMIALS CONNECTED WITH A PROBLEM IN PROBABILITY

EUGENE LUKACS AND OTTO SZÁSZ

Let $0 < b_1 < b_2 < \cdots < b_n$ be n integers and let $g(\theta)$ be the Vandermonde determinant formed from $b_1^2, b_2^2, \cdots, b_n^2$ with the first row replaced by $\sin^2 b_i\theta/2$ $(i = 1, \cdots, n)$. The function $g(\theta)$ is then a cosine polynomial. In connection with a problem in probability the question arose as to when $g(\theta)$ is a non-negative trigonometric polynomial. The following results were obtained:

(A) If the b_i are the first n consecutive integers, then $g(\theta)$ is non-negative.

(B) If the b_i are the first n consecutive odd integers, then $g(\theta)$ is non-negative.

(C) If the numbers $b_1, \cdots, b_n$ are obtained from the first $(n + 1)$ consecutive integers by omitting the integer k $(1 \leqq k \leqq n)$, then the trigonometric polynomial $g(\theta)$ is non-negative if and only if $2k^2 \geqq n + 1$.

(D) If the numbers $b_1, \cdots, b_n$ are obtained from the first $(n + 1)$ consecutive odd integers by omitting the kth odd integer $2k - 1$ ($1 \leqq k \leqq n$), then the trigonometric polynomial $g(\theta)$ is non-negative if

1. $k \geqq k_1$ where $k_1 = \dfrac{1 + (1 + 2n)^{1/2}}{2}$ or if
2. $k < k_1$ but if $\int_{-x_0}^{1} (1 - t^2)^{n-1}(t^2 - x_0^2)\, dt \geqq 0$ where

$$x_0 = \left(\frac{n + 2k - 2k^2}{2n(n + 1) - 2k(k - 1)}\right)^{1/2}.$$

If neither of these conditions is satisfied, $g(\theta)$ assumes also negative values.

(E) If the numbers $b_1, \cdots, b_n$, are chosen from the first $(n + 2)$ consecutive integers by omitting two integers k and p ($1 \leqq k \leqq p - 1 \leqq n$), the non-negativeness depends on the discussion of a quadratic polynomial whose coefficients are functions of n, k, and p.

National Bureau of Standards,
Washington, D. C., U. S. A.
National Bureau of Standards,
Los Angeles, Calif., U. S. A.
University of Cincinnati,
Cincinnati, Ohio, U. S. A.

INFORMATION RETRIEVAL VIEWED AS TEMPORAL SIGNALLING

Calvin N. Mooers

The problem of directing a user to stored information, some of which may be unknown to him, is the problem of "information retrieval". Signalling theories can be applied, though this is a form of temporal signalling, which distinguishes it from the point-to-point signalling currently under study by others.

In information retrieval, the addressee or receiver rather than the sender is the active party. Other differences are that communication is temporal from one epoch to a later epoch in time, though possibly at the same point in space; communication is in all cases unidirectional; the sender cannot know the particular message that will be of later use to the receiver and must send all possible messages; the message is digitally representable; a "channel" is the physical document left in storage which contains the message; and there is no channel noise because all messages are presumed to be completely accessible to the receiver. The technical goal is finding in minimum time those messages of interest to the receiver, where the receiver has available a selective device with a finite digital scanning rate.

Classification and indexing schemes are ruled out because of gross topological difficulties, and the Batten system because of no reasonable large-scale mechanization. To avoid scanning all messages in entirety, each message is characterized by N independently operating digital descriptive terms (representing ideas) from a vocabulary V, and a selection is prescribed by a set of S terms. Conventional assignment of a digital configuration to a message requires $N \log_2 V$ binary digit places. It is a nonsingular transformation from ideas to configuration. Prescription of a selection requires $S \log_2 V$ digit places. Although only $\log_2 M$ digit places are required to differentiate (or enumerate) M messages, $S \log_2 V$ may be many times greater, indicating a digital redundancy and waste in coding. This redundancy can be removed by recoding the descriptive ideas into the digital configuration representing the message by a singular transformation (Zatocoding). The message can then be characterized with $(N/S) \log_2 M$ digit places, with a possible increase in scanning rate per message. Because Zatocoding represents each idea by a pattern ranging over the digit places for a message, and superposes these in the same coordinate frame by Boolean addition, the selection process includes the desired messages and gives a statistical exclusion of the undesired messages.

Zator Company,
Boston, Mass., U. S. A.

ON A CLASS OF TWO-DIMENSIONAL MARKOV PROCESSES

Murray Rosenblatt

Let $x(t)$ be an element of Wiener space and let $V(t, x)$ satisfy the local Hölder condition $| V(t, x) - V(t + \Delta t, x + \Delta x) | \leqq M(t, x) \{ | \Delta t |^{\alpha(t,x)} + | \Delta x |^{\alpha(t,x)} \}$, $\alpha(t, x) > 0$ everywhere except on a curve rectifiable in every finite rectangle of the (t, x) plane. Let $V(t, x)$ be bounded in every finite rectangle of the (t, x) plane. The study of the two-dimensional Markov process $(x(t), \int_0^t V(\tau, x(\tau))\, d\tau)$ is related to the study of certain differential and integral equations. Let $Q(t, x) = E\{e^{-u\int_0^t V(\tau,x(\tau))d\tau} \mid x(t) = x\} e^{-x^2/2t}/(2\pi t)^{1/2}$. If $0 \leqq V(t, x)$, $Q(t, x)$ is the solution of the differential equation $(1/2)\partial^2 Q/\partial x^2 - \partial Q/\partial t - uV(t, x)Q = 0$, subject to the conditions $Q(t, x) \to 0$, $x \to \pm\infty$, $\lim_{t\to 0} \int_{-\epsilon}^{\epsilon} Q(t, x)dx = \lim_{t\to 0} \int_{-\infty}^{\infty} Q(t, x)dx = 1$ for all $\epsilon > 0$. $Q_x(t, x)$ is continuous in (t, x) for all $(t, x) \neq (0, 0)$. Let $G(x, y, t) = \Pr\{\int_0^t V(\tau, x(\tau))d\tau \leqq y \mid x(t) = x\}$. Then $\int_y^{y+\lambda} (E(\alpha) - G(x, \alpha, t))\, d\alpha = e^{x^2/2t}(2\pi t)^{1/2} \int_0^t \int_{-\infty}^{\infty} e^{-(x-\xi)^2/2(t-\tau)-\xi^2/2\tau}/2\pi(\tau(t - \tau))^{1/2} V(\tau, \xi)\{G(\xi, y + \lambda, \tau) - G(\xi, y, \tau)\}\, d\xi\, d\tau$ where $E(y) = 1$ if $y \geqq 0$, and $=0$ if $y < 0$. This last equation is valid if one assumes that $V(t, x)$ is bounded and Borel measurable.

Cornell University,
Ithaca. N. Y., U. S. A.

DISTRIBUTION FOR THE ORDINAL NUMBER OF SIMULTANEOUS EVENTS WHICH LAST DURING A FINITE TIME

HERMANN VON SCHELLING

If we know (1) that an event of a given class may occur with the probability p per unit of time; (2) that, when it has happened, it will not occur again during the next k units of time; (3) that such an event was observed at the nth moment; we ask for the probability $w(m; n, k, p)$ that this event is the mth one since zero time.

Relative probabilities can be found immediately. Their sum $s_n(x)$ with $x = p/q^{k+1}$ satisfies the following recursion formula

$$s_{n+1}(x) - s_n(x) - xs_{n-k}(x) = 0$$

with $s_1(x) = s_2(x) = \cdots = s_{k+1}(x) = 1$. Its solution depends upon the roots of the characteristic equation

$$f(z) = z^{k+1} - qz^k - 1 + q = 0.$$

It has to be proved that (a) $f(z) = 0$ does not have multiple roots; (b) $z_1 = +1$ is the only root of $f(z) = 0$ on the unit circle; (c) the roots $z_2, z_3, \cdots, z_{k+1}$ of $f(z) = 0$ are located inside the unit circle.

Exact formulas for $w(m; n, k, p)$ and for the first and the second moment of this distribution are given as functions of the roots of $f(z) = 0$. According to (a), (b), (c) it is possible to present simple approximations which do not depend on the solution of the characteristic equation and may be evaluated easily.

The distributions $w(m; n, k, p)$ are branching from the binomial type which corresponds to the particular case $k = 0$. The final results are elementary. It is to be expected that the new distributions will become useful in many fields of science and biology since the assumption (2) is frequently adequate to observed conditions.

NAVAL MEDICAL RESEARCH LABORATORY,
NEW LONDON, CONN., U. S. A.

INFORMATION AND THE FORMAL SOLUTION OF MANY-MOVED GAMES

L. S. SHAPLEY

A game is said to be in *normalized form* if it is so formulated that each player has just one move—the selection of a *strategy*—made independently of the other players' selections. A 2-person zero-sum game with n moves in which no information passes between moves can easily be formulated in this fashion, and its solution is obtained by considering:

$$\max_{\varphi} \min_{\psi} \iint H(x_1, \cdots, x_n)\, d\varphi(x_{i_1}, \cdots, x_{i_p})\, d\psi(x_{j_1}, \cdots, x_{j_{n-p}}) \tag{1}$$

where φ is a probability distribution over the set of strategies $(x_{i_1}, \cdots, x_{i_p})$ of the maximizing player, and ψ is the same for the minimizer. We suppose that the identity of the player making the ith move, as well as the set A_i of his alternatives, does not depend on the previous course of play. We may abbreviate (1) to

$$\operatorname*{mix}_{S|I-S} H(x_1, \cdots, x_n)$$

where I is the set of integers $\{1, \cdots, n\}$ and S the set of moves assigned to the maximizer.

The introduction of *information* (i.e. knowledge, at one move, of the outcome of another) quickly makes the normalized form useless in practice, since the number of strategies increases at a prohibitive rate. Yet, in the extreme case of *perfect information* (knowledge, at each move, of all preceding moves), the game is relatively easily solved one move at a time, e.g.:

$$\max_{x_1 \in A_1} \min_{x_2 \in A_2} \min_{x_3 \in A_3} \cdots \max_{x_n \in A_n} H(x_1, x_2, x_3, \cdots, x_n). \tag{2}$$

It is of interest to ask what conditions on the informational structure are required for a game to be decomposable into a sequence of separated moves (as in (2)), or, more generally, separated subgames in normalized form. The solution in such a case would be given by:

$$\operatorname*{mix}_{T_1|U_1} \operatorname*{mix}_{T_2|U_2} \cdots \operatorname*{mix}_{T_m|U_m} H(x_1, \cdots, x_n) \tag{3}$$

where the T_k and U_k partition S and $I - S$ respectively. Using McKinsey's notion of equivalence of patterns of information, we find that a necessary and sufficient condition that a game be solvable by (3), regardless of the nature of the function H or of the sets A_i, is that it be possible to order the moves in time so that (i) the information pattern never implies knowledge of the future, (ii) each player's knowledge of his opponent's actions increases monotonically, and (iii) the accessions of information occur only between certain pairs of consecutive moves, on which occasions both players are apprised of all their opponent's actions to date. No condition is imposed on a player's awareness of his own previous actions.

THE RAND CORPORATION,
SANTA MONICA, CALIF., U. S. A.

STOCHASTIC PROCESSES BUILT FROM FLOWS

KÔSAKU YOSIDA

By virtue of the theory of semi-groups due to E. Hille and the author, we may construct stochastic processes in a separable measure space R from flows in R. A flow (= one-parameter family of equi-measure transformations in R) $F_t x$ induces a one-parameter group T_t of transition operators: $(T_t f)(x) = f(F_t x)$, $-\infty < t < \infty$. Here a linear operator T on the Banach space $L_1(R)$ to $L_1(R)$ is called a transition operator if $f(x) \geqq 0$ implies $(Tf)(x) \geqq 0, \int (Tf)(x)\,dx = \int f(x)\,dx$. T_t admits infinitesimal generator A:

$$T_t f = \exp\,(tA)f = \text{strong}\lim_{n\to\infty} \exp\,(nt((I - n^{-1}A)^{-1} - I))f, \; -\infty < t < \infty.$$

Since $(I - n^{-1}A)^{-1}$ and $(I + n^{-1}A)^{-1}$ must be transition operators for $n > 0$, $(I - n^{-1}A^2)^{-1}$ exists as a transition operator. Hence A^2 is the infinitesimal generator of a one-parameter semi-group $S_t = \exp\,(tA^2)$, $t \geqq 0$, of transition operators. Thus the Fokker-Planck equation in a Riemannian space R: $\partial f(t, x)/\partial t = A^2 f(t, x)$, $f(0, x) = f(x) \in L_1(R)$, $t \geqq 0$, is integrable stochastically if $A = p^i(x)\partial/\partial x^i$ is the infinitesimal transformation of a one-parameter Lie group of equi-measure transformations of R.

Extensions. i) If R admits several flows $\exp\,(tA_i)$, $i = 1, 2, \cdots, m$, with mutually commutative A_i, then $\exp\,(t\sum_i A_i^2)$ defines a stochastic process in R. ii) Let the group of motions in a Riemannian space R be a semi-simple Lie group with infinitesimal transformations $X_1, X_2, \cdots, X_m$ transitive in R. Then the Casimir operator $C = g^{ij}X_iX_j$ $((g^{ij}) = (g_{ij})^{-1}$, $g_{ij} = c^{\sigma}_{i\rho}c^{\rho}_{j\sigma}$, $[X_i, X_j] = c^k_{ij}X_k)$ is commutative with every X_i. In this case, at least when R is compact, $\exp\,(tC)$ defines a temporally and spatially homogeneous, continuous stochastic process in R—a Brownian motion in the homogeneous space R.

NAGOYA UNIVERSITY,
NAGOYA, JAPAN.

INTERPOLATION

NEW DEVELOPMENTS IN INTERPOLATION FORMULAE

ERICH MICHALUP

Interpolation Formulae (I.F.) have been chiefly developed along two distinct methods, of which the first was devised by Sprague, and the second by Jenkins.

Any symmetrical I.F. for equidistant intervals may be written in summation form, involving an operator and a nucleus, similar to the operand of a summation formula.

A comparison of the differences of the Jenkins' function and the nucleus gives the relationship between the constants of both, and consequently the necessary conditions which need to be satisfied by the constants of the nucleus in order to obtain I.F. with continuous derivatives, etc. In order to simplify numerical applications we have chosen as the operator for subdivision into five parts $[5]^n 10^{1-n}$, obtaining for the graduating I.F. with three continuous derivatives the operator $[5]^5 10^{-4}$ and the nucleus $[960, 877, -81, -609, -67]/75$. Applying a fifth difference I.F. we may, for theoretical reasons, base the smoothness criterion on the fifth or sixth differences of the interpolated values. We may deem the results smooth, if the size and sign of the difference series flow continuously and regularly, and if the sum of the differences, irrespective of sign, diminish from the first to the last (sixth). It seems however sufficient to investigate only the last two difference series since if both are smooth, so also must the preceding ones be.

Limiting our investigation only to the totals of differences, irrespective of sign, we may frequently arrive at an erroneous conclusion. For this reason, starting with the nucleus, we have developed various reproducing and graduating I.F. with and without continuous derivatives, minimizing the sum of the squares of the fourth to sixth differences, or the sum of the absolute values of the sixth differences. Violent breaks appearing in the fifth differences on the application of both old and new I.F., make them open to objection from a theoretical point of view and thus prove the soundness of the above established smoothness criterion. A further advantage of our method of developing I.F. is the fact that in order to obtain the fifth or sixth differences, it is not necessary to evaluate all the former difference series. It is sufficient to determine the fifth or sixth differences from the original data and calculate the last difference series by means of the nucleus. Finally we develop the conditions which need to be satisfied by the constants of the nucleus of an eight-term I.F. for subdivision into seven parts.

UNIVERSITY OF CARACAS,
CARACAS, VENEZUELA.

THE ACCURACY OF LINEAR INTERPOLATION

Hugh E. Stelson

This paper gives the derivation of formulas which express the error due to a simple linear interpolation in finding (a) an unknown function or (b) an unknown argument.

The error due to a double linear interpolation is expressed as a combination of errors for simple linear interpolations. Hence the errors for a double linear interpolation may be expressed by the formulas for simple linear interpolation.

The error due to the interpolation in finding an unknown function is defined by

$$\delta \equiv f(a) + \frac{x-a}{b-a}[f(b) - f(a)] - f(x)$$

where the function and its derivatives exist and are continuous in the interval $a < x < b$.

The following results are obtained.

I. By use of Taylor's finite expansion, it can be shown that

$$\delta \equiv \frac{(x-a)(b-x)}{2} f''(\theta) \quad \text{where} \quad a < \theta < b.$$

II. By use of Taylor's infinite expansion, it can be shown that a first approximation to the error is given by

$$\delta \doteq \frac{(x-a)(b-x)}{2} f''(x) \quad \text{where} \quad a < x < b \quad \text{and} \quad |b-a| < 1.$$

III. It can also be shown that a first approximation to the maximum error is given by

$$\delta(\max) \doteq \frac{(b-a)^2}{8} f''\left(\frac{a+b}{2}\right) \quad \text{where} \quad (a, b) \text{ is small.}$$

IV. The error in finding an unknown argument may be defined as

$$\gamma \equiv a + \frac{f(x) - f(a)}{f(b) - f(a)}(b-a) - x.$$

Similar results to I, II, and III can be obtained for the argument. For example, the exact error is expressed by

$$\gamma = a - x + \frac{(x-a)f'(a) + \dfrac{(x-a)^2}{2} f''(\theta)}{f'(a) + \dfrac{b-a}{2} f''(\theta)}, \qquad a < \theta < b.$$

V. In double linear interpolation the error is the same for either order of interpolation. If ϵ_1, ϵ_2, and ϵ_3 are simple linear interpolation errors, the error in this case is

$$\epsilon = \epsilon_3 + \epsilon_1 + \frac{x - a}{b - a}(\epsilon_2 - \epsilon_1).$$

The approximate maximum error, ϵ, can be expressed by

$$\epsilon \doteq \frac{(b - a)^2}{8} f_{xx}\left(x, \frac{c + d}{2}\right) \quad \text{at} \quad \frac{a + b}{2}$$
$$+ \frac{(d - c)^2}{16} [f_{yy}(a, y) + f_{yy}(b, y)] \quad \text{at} \quad \frac{c + d}{2}$$

where $f(a, b)$, $f(a, c)$, $f(b, d)$, and $f(b, c)$ are given fixed values and $f(x, y)$ is to be determined. (a, b) and (c, d) are small.

Michigan State College,
East Lansing, Mich., U. S. A.

STATISTICS

ON THE LOGARITHMICO-PEARSON DISTRIBUTIONS

E. CANSADO

In *Funciones caracteristicas de las distribuciones de Pearson* [E. Cansado, Revista Matematica Hispano-Americana, Madrid (1947)] I have obtained the c.f. of all the Pearson distributions. I consider now the distribution of log $|\xi|$, where ξ is a variate with a distribution belonging to the Pearson system; I give these distributions the name of logarithmico-Pearson distributions.

The c.f. of one of these variates or distributions is

$$\phi(\theta; \log|\xi|) = \mathcal{E}[e^{\theta \log|\xi|}] = \mathcal{E}[|\xi|^{\theta}] = \int_{-\infty}^{+\infty} |x|^{\theta}\, dF(x)$$

where $\theta = i\tau$, $i = (-1)^{1/2}$, τ is a real variable, and $F(x)$ a distribution function.

TABLE I

Pearson's distributions		Logarithmico-Pearson distributions	
Density functions	Ranges	Symbols	Characteristic functions
$\frac{1}{(2\pi)^{1/2}} e^{-x^2/2}$	$-\infty, +\infty$	$\log \nu$	$\frac{\Gamma((1+\theta)/2)}{\Gamma(1/2)} 2^{\theta/2}$
$\frac{1}{\Gamma(r)} e^{-x} x^{r-1}$	$0, +\infty$	$\log \gamma_1(r)$	$\frac{\Gamma(r+\theta)}{\Gamma(r)}$
$\frac{1}{\Gamma(r)} e^{-1/x} x^{-r-1}$	$0, +\infty$	$\log \gamma_2(r)$	$\frac{\Gamma(r-\theta)}{\Gamma(r)}$
$\frac{1}{B(r, s)} x^{r-1}(1-x)^{s-1}$	$0, +1$	$\log \beta_1(r, s)$	$\frac{\Gamma(r+s)\Gamma(r+\theta)}{\Gamma(r)\Gamma(r+s+\theta)}$
$\frac{1}{B(r, s)} x^{r-1}(1+x)^{-r-s}$	$0, +\infty$	$\log \beta_2(r, s)$	$\frac{\Gamma(r+\theta)\Gamma(s-\theta)}{\Gamma(r)\Gamma(s)}$
$\frac{1}{A(0, s)} (1+x^2)^{-s}$	$-\infty, +\infty$	$\log \alpha(0, s)$	$\frac{\Gamma((1+\theta)/2)\Gamma(s-(1+\theta)/2)}{\Gamma(1/2)\Gamma(s-1/2)}$

where $A(r, s) = \int_{-\infty}^{+\infty} e^{r \tan^{-1} x} (1+x^2)^{-s}\, dx$ is the alpha function which I have introduced in *Exposicion sistematica de las distribuciones de Pearson* [E. Cansado, Trabajos de Estadistica (Madrid) vol. 3 (1950) (with English summary)]. Therefore $A(0, s) = B(1/2, s-1/2)$ and $A(0, 1) = \pi$. Evidently $\alpha(0, 1)$ is the Cauchy distribution.

If ξ and η are two independent variates, we have for $\zeta = k\xi^a\eta^b$:

$$\text{(A)} \qquad \phi(\theta; \log|\zeta|) = |k|^\theta \phi(a\theta; \log|\xi|)\phi(b\theta; \log|\eta|)$$

where k, a, and b are real numbers.

From (A) and Table I we deduce that: (i) if ξ is a ν variate, $\zeta = \xi^2/2$ is a $\gamma_1(1/2)$ variate; (ii) if ξ is an $\alpha(0, s)$ variate, $\zeta = \xi^2$ is $\beta_2(1/2, s - 1/2)$ variate; and (iii) we have all the relations listed in Table II, where ξ and η are independent variates of the types specified in the first two columns.

TABLE II

ξ	η	$1/\xi$	$\xi\eta$	ξ/η
ν	ν			$\alpha(0, 1)$
$\gamma_1(r)$	$\gamma_2(s)$	$\gamma_2(r)$	$\beta_2(r, s)$	
$\gamma_1(r)$	$\gamma_1(s)$	$\gamma_2(r)$		$\beta_2(r, s)$
$\gamma_2(r)$	$\gamma_2(s)$	$\gamma_1(r)$		$\beta_2(s, r)$
$\beta_1(r, s)$	$\gamma_1(r + s)$		$\gamma_1(r)$	
$\beta_1(r, s)$	$\beta_2(r + s, s)$		$\beta_2(r, s)$	
$\beta_2(r, s)$		$\beta_2(s, r)$		

Evidently: $\chi^2/2$ is a $\gamma_1(m/2)$ variate (Pearson's χ^2 with m d. of f.); $t/(m)^{1/2}$ is an $\alpha(0, (m + 1)/2)$ variate (Student's t with m d. of f.); m_1/m_2F is a $\beta_2(m_1/2, m_2/2)$ variate (Snedecor's F with m_1 and m_2 d. of f.); and $2z + \log(m_1/m_2)$ is a $\log \beta_2(m_1/2, m_2/2)$ variate (Fisher's z with m_1 and m_2 d. of f.).

Therefore its distributions and properties may be obtained easily following the method of this paper. By means of the expansion of $\log(e^{-\gamma e}\Gamma(a)/\Gamma(a + \theta))$ we can obtain the expression of all the cumulants (Thiele's semi-invariants) of all the logarithmico-Pearson distributions, as it has been done for some of them in *The cumulants of the z and of the logarithmic and t distributions* [J. Wishart, Biometrika vol. 34 (1947) pp. 170–178] and *Cumulantes de la z de Fisher* [E. Cansado, Revista Matematica Hispano-Americana, Madrid (1947)].

Consejo Superior de Estadistica,
Madrid, Spain.

ESTIMATING PARAMETERS OF LOGARITHMIC-NORMAL DISTRIBUTIONS BY THE METHOD OF MAXIMUM LIKELIHOOD

A. C. Cohen, Jr.

Parameters a, b, and c of the logarithmic-normal distribution

$$f(z) = \frac{1}{c(z - a)(2\pi)^{1/2}} e^{-[\log^2((z-a)/b)]/2c^2}, \qquad z > a,$$

are estimated by the method of maximum likelihood. Estimating equations obtained are

$$\log b = \sum_1^n [\log (z_i - a)]/n, \tag{1}$$

$$c^2 = \sum_1^n [\log^2 (z_i - a)]/n - \left[\sum_1^n \frac{\log (z_i - a)}{n}\right]^2, \tag{2}$$

$$\left[n \sum_1^n \log (z_i - a) - \left\{n \sum_1^n \log^2 (z_i - a) - \left(\sum_1^n \log (z_i - a)\right)^2\right\}\right] \cdot \sum_1^n 1/(z_i - a) - n^2 \sum_1^n [\log (z_i - a)]/(z_i - a) = 0. \tag{3}$$

Equation (3) is solved by a simple iterative process to yield $\hat{a}$. Estimates $\hat{b}$ and $\hat{c}$ are then obtained from (1) and (2). Variances and covariances of these estimates are derived by constructing the likelihood information matrix. Kapetyn (1903) employed a method based on selected points to estimate parameters of the logarithmic-normal distribution. S. D. Wicksell (1917), E. J. Gumbel (1926), P. T. Yuan (1933), and others used the method of moments.

University of Georgia,
Athens, Ga., U. S. A.

SOME TESTS FOR COMPARING PERCENTAGE POINTS OF TWO ARBITRARY CONTINUOUS POPULATIONS

A. W. Marshall and John E. Walsh

Consider samples from two arbitrary continuous populations, the first with $100\alpha\%$ point Θ_α, the second with $100\beta\%$ point ϕ_β. The two populations are not necessarily the same or even related. Easily applied significance tests are presented for $\Theta_\alpha - \phi_\beta$ which are valid for moderate as well as large sized samples. The exact significance level of a test is not known, but its value is determined within reasonably close limits. An approximate lower bound for the efficiency of these tests is found by investigating the special case of normal populations with known ratio of variances. The tests are found to be reasonably efficient if neither α nor β is too large or too small. Since the tests are valid for moderate sized samples, they may be of practical value.

Let $x(1), \cdots, x(m)$ denote the m sample values from the first population arranged in increasing order of magnitude; $y(1), \cdots, y(n)$ denote the n sample values from the second population arranged in increasing order of magnitude. The tests compare $\Theta_\alpha - \phi_\beta$ with a given hypothetical value μ_0. The one-sided test of $\Theta_\alpha - \phi_\beta < \mu_0$ with desired significance level approximately ϵ is

$$\textit{Accept}\quad \Theta_\alpha - \phi_\beta < \mu_0 \quad \textit{if} \quad x\{\alpha m + .85K_\epsilon[\alpha(1-\alpha)m]^{1/2}\} - y\{\beta n - .85K_\epsilon[\beta(1-\beta)n]^{1/2}\} < \mu_0,$$

where K_ϵ is the standardized normal deviate exceeded with probability ϵ. Here $\alpha m + .85K_\epsilon\,[\alpha(1-\alpha)m]^{1/2}$ and $\beta n - .85K_\epsilon\,[\beta(1-\beta)n]^{1/2}$ should be integers or nearly equal to integers. If z is not an integer, $x(z) = x(\text{integer nearest } z)$; similarly for $y(z)$. An approximate lower bound for the significance level of this test is γ, where $K_\gamma = 1.25K_\epsilon$; an approximate upper bound is δ, where $K_\delta = .83K_\epsilon$. The true significance level will usually be much nearer γ than δ. The one-sided test of $\Theta_\alpha - \phi_\beta > \mu_0$ with desired significance level approximately ϵ is

$$\textit{Accept}\quad \Theta\alpha - \phi_\beta > \mu_0 \quad \textit{if} \quad x\{\alpha m - .85K_\epsilon[\alpha(1-\alpha)m]^{1/2}\} - y\{\beta n + .85K_\epsilon[\beta(1-\beta)n]^{1/2}\} > \mu_0.$$

An approximate lower bound for the significance level of this test is γ while δ is an approximate upper bound. Two-sided tests of $\Theta_\alpha - \phi_\beta \neq \mu_0$ can be obtained by combining the two one-sided tests. It is conjectured that m and n are usually large enough for the tests to be sufficiently accurate for use if min $[\alpha m, (1-\alpha)m, \beta n, (1-\beta)n] \geqq 5$ and $\epsilon \geqq .005$. The accuracy of this conjecture was checked for the special case where the populations are identical and $\alpha = \beta$.

The RAND Corporation,
Santa Monica, Calif, U. S. A.

THE DISTRIBUTION OF RANGES IN SAMPLES FROM A DISCRETE RECTANGULAR POPULATION

Paul R. Rider

The distribution of ranges in samples from a discrete rectangular population is derived. The distribution of the quotients of such ranges is worked out for special cases.

Washington University,
St. Louis, Mo., U. S. A.

ASYMPTOTICALLY SUB-MINIMAX SOLUTIONS OF STATISTICAL DECISION PROBLEMS

Herbert Robbins

Consider the following problem: $x_1, \cdots, x_n$ are independent variates, each normal with unit variance, and with means $\theta_1, \cdots, \theta_n$, where each of the unrelated parameters θ_i is ± 1. From the observed value $x = (x_1, \cdots, x_n)$ it is required to decide for every $i = 1, \cdots, n$ whether $\theta_i = 1$ or -1. For any decision rule R the risk function $L(R, \theta)$ is as the expected proportion of errors among the n decisions concerning the components of the true parameter point $\theta = (\theta_1, \cdots, \theta_n)$. The unique minimax rule is $\bar{R}$: "$\theta_i = \text{Sgn}\,(x_i)\ (i = 1, \cdots, n)$," and the risk for $\tilde{R}$ is identically $F(-1) = .1587$, where $F(x)$ is the standard normal c.d.f. Nevertheless, there are strong reasons for regarding $\tilde{R}$ as a poor rule when n is large. In fact, let $\bar{x} = (x_1 + \cdots + x_n)/n$, let $x^* = -1, \bar{x}, 1$ according as $\bar{x} \leqq -1$, between -1 and 1, or $\geqq 1$, and let R^* be the rule "$\theta_i = \text{Sgn } (2x_i - \log (1 - x^*/1 + x^*))$ $(i = 1, \cdots, n)$." It can be shown that $\lim_{n\to\infty} L(R^*, \theta) = h(p(\theta))$, where $h(p) = pF(-1 + (1/2) \log (1 - p/p)) + (1 - p)F(-1 - (1/2) \log (1 - p/p))$ $0 \leqq p \leqq 1$, and $p(\theta) = (1/2n) \sum (1 + \theta_i)$; the approach is uniform in p, and $h(p) < F(-1)$ for $p \neq .5$. In fact, $h(0) = 0$, $h(.1) = .069$, $h(.2) = .112$, $h(.3) = .139$, $h(.4) = .154$, $h(.5) = F(-1)$; $h(1 - p) = h(p)$. The exact risk function $L(R^*, \theta)$ is fairly easy to compute and shows a good approach to $h(p(\theta))$ for n of the order of 100. Of course, $L(R^*, \theta) > F(-1)$ for $p(\theta) \cong .5$, but in view of the above limiting relation we call R^* an *asymptotically sub-minimax* rule as $n \to \infty$. For large n, R^* should be preferable to $\tilde{R}$ in most applications. It is somewhat paradoxical that R^* makes the decision concerning each θ_i depend on values of x_j $(j \neq i)$ which contain no "information" about θ_i.

Asymptotically sub-minimax solutions of "product space" problems like the above often exist, and a general attack on the subject is outlined; limitations of space prevent discussion in this abstract.

UNIVERSITY OF NORTH CAROLINA,
CHAPEL HILL, N. C., U. S. A.

SOME FORMAL RELATIONS IN MULTIVARIATE ANALYSIS

GERHARD TINTNER

Let u_i $(i = 1, 2, \cdots, R)$ be a column vector with p_i components. A_{ij} and B_{ij} are two sets of matrices with p_i rows and p_j columns. Two sets of bilinear and quadratic forms are: $f_{ij} = u_i'A_{ij}u_j$ and $g_{ij} = u_i'B_{ij}u_j$. Let α_{ij} and β_{ij} be two sets of scalars. These are Lagrange multipliers in the applications. A linear combination of the bilinear and quadratic forms is: $h = \sum_{i,j}(\alpha_{ij}f_{ij} + \beta_{ij}g_{ij})$. The necessary conditions for a maximum or minimum of h are:

$$\sum_i (\alpha_{ij}A_{ij} + \beta_{ij}B_{ij})u_i = 0.$$

From this we can derive: $\sum_i(\alpha_{ij}f_{ij} + \beta_{ij}g_{ij}) = 0$.

Case A: $A_{ij} = B_{ij}$. Let $\gamma_{ij} = \alpha_{ij} + \beta_{ij}$. Then

$$\sum_i \gamma_{ij}A_{ij}u_i = 0, \qquad \sum_i \gamma_{ij}f_{ij} = 0.$$

Case A_1: $R = 2$. Case B: $p_i = p$, $A_{ij} = A$, $B_{ij} = B$. Then

$$\sum_i (\alpha_{ij}A + \beta_{ij}B)u_i = 0, \qquad \sum_i (\alpha_{ij}f_{ik} + \beta_{ij}g_{ik}) = 0.$$

Case $B_1 : R = 1$. Case $B_2 : R = 1$. Let also a be a vector. $A = a \cdot a'$, a singular matrix. From case A_1 we have: $(\gamma_{11}\gamma_{22}A_{11} - \gamma_{12}\gamma_{21}A_{12}A_{22}^{-1}A_{21})u_1 = 0$. This corresponds to case B_1.

Canonical correlation: Two linear combinations of a set of random variables X_i $(i = 1, 2, \cdots, n)$ are: $U = \sum_1^{n'} k_i X_i$ and $V = \sum_{n'+1}^{n} k_i X_i$ $(n' < n)$. Let $[a_{ij}]$ be the covariance matrix of the X_i. We want to maximize the canonical correlation between U and V while their variances are one. Then we have case A_1 with: $u_1 = \{k_1, \cdots, k_{n'}\} u_2 = \{k_{n'+1}, \cdots, k_n\}$. $A_{11} = [a_{ij}]$ $(i, j = 1, \cdots, n')$, $A_{12} = [a_{ij}]$ $(i = 1, \cdots, n', j = n' + 1, \cdots, n)$, $A_{22} = [a_{ij}]$ $(i, j = n' + 1, \cdots, n)$. $\gamma_{11} = \gamma_{22}$, $\gamma_{12} = 1$.

Principal components: The coefficients of the factor which contributes most to the variance of the X_i are $k_1, \cdots, k_n$. We have case B_1 with $u_1 = \{k_1, \cdots, k_n\}$, $A = I$, (unit matrix) $B = [a_{ij}]$.

Weighted regression: Let $X_i = M_i + y_i$, where M_i is the systematic part and y_i a random error; the covariance matrix of the y_i is $[V_{ij}]$. Let there be R relations: $k_{i0} + \sum_j k_{ij} M_j$. We apply a modification of the method of least squares. We have case B: $u_i = \{k_{i1}, \cdots, k_{in}\}$, $A = [a_{ij}]$, $B = [V_{ij}]$.

Discriminant analysis: The observations X_i are taken from two groups. Let the differences of the averages of the two groups be d_i. The discriminant function is $\sum_i k_i d_i$. The square of this function has to be maximized while its variance is constant. We have case B_2: $u_1 = \{k_1, \cdots, k_n\}$. $A = [d_i d_j]$, $B = [a_{ij}]$.

Iowa State College of Agricultural and Mechanic Arts,
Ames, Iowa, U. S. A.

A FREQUENCY SYSTEM THAT GENERALIZES THE PEARSON'S SYSTEM

Fausto I. Toranzos

We consider the system of frequency functions generated by the differential equation $y'/y = Q_{m+1}(x)/P_m(x)$ (Q and P are polynomials of degrees $m + 1$ and m respectively). The general result is a product of the Gauss normal functions and of those of the Pearson type, as well as of exponentials. The cases in which $m = 1$ and $m = 2$, are specially considered. When $m = 1$, the integral is $y = e^{\alpha(x-b)^2} x^c K$. If $\alpha < 0$, we have a bell-shaped asymmetric curve with the value zero in the points $x = 0$ and $x = \infty$, having only one maximum in this interval; this case is important in the study of the distribution of economical series of relative percentage values of prices and volumes; the curve is asymmetric but very similar to the normal curve in the neighborhood of the maximum, which is not characteristic of the Pearson's curve III. Processes are explained to calculate the constants by means of moment and of the least square systems;

k and the characteristic function are given in a function of the Whittaker's hypergeometrical confluent functions.

NATIONAL UNIVERSITY OF CUYO,
MENDOZA, ARGENTINA.

ESTIMATION IN THE ALTERNATIVE FAMILY OF DISTRIBUTIONS

JOHN W. TUKEY

Estimates t_α of the parameter θ in a one-parameter family of distributions $F(X \mid \theta)$ are often used to estimate θ in an alternate family $F^*(X \mid \theta)$. The quality of such estimates deserves study. The asymptotic situation is treated in detail for the case where t_α is asymptotically unbiased in sampling from $F^*(X \mid \theta)$ by methods which extend readily to other asymptotic cases. Asymptotic moments are defined as in Bull. Amer. Math. Soc. Abstract 54-7-281.

If a one-parameter family of *bases of estimate* $s_\alpha(X_1, X_2, \cdots, X_n)$ is given, and if $t_\alpha = f_\alpha(s_\alpha, n)$, then the asymptotic unbiasedness of t_α for $F(X \mid \theta)$ fixes the asymptotic bias and variance of t_α for $F^*(X \mid \theta)$. The choice of α to minimize the asymptotic mean square error (aMSE) of t_α in estimating any given *estimand* equal to $e(\theta)$ for $F(X \mid \theta)$ and to $e^*(\theta)$ for $F^*(X \mid \theta)$ will involve a bias in sampling from $F^*(X \mid \theta)$ of order n^{-1}. The reduction in aMSE by accepting this bias is of order n^{-2}, while the aMSE is of order n^{-1}. The choice of parametrization is unimportant.

When applied to θ = scale, $F(X \mid \theta)$ = normal distributions with known mean, $F^*(X \mid \theta)$ = contaminated distributions of fixed ratio (i.e. mixture in fixed ratio of two normal distributions with same mean), computation suggests:

(a) for pure scaling, the average of e^{-cx^2} for suitably chosen c is best;

(b) next to this, the truncated variance is preferable to other bases of estimate;

(c) the use of the variance is best restricted to analysis of variance, calculation of variance components, estimation of percentiles near 2% and 98%.

PRINCETON UNIVERSITY,
PRINCETON, N. J., U. S. A.

TWO METHODS OF RANDOMIZATION IN STATISTICS AND THE THEORY OF GAMES

ABRAHAM WALD AND JACOB WOLFOWITZ

Let $x_1, x_2, \cdots$ be respective observations on the chance variables $X_1, X_2, \cdots$. A randomized sequential decision function (s.d.f.) is a set of probability measures $\{\delta(x_1, \cdots, x_n)\}$ $(n = 1, 2, \cdots)$ on a decision space D. When the observations $(x_1, \cdots, x_n)$ have been obtained, an element of D is chosen by means of a chance mechanism with probability distribution $\delta(x_1, \cdots, x_n)$. The decision to take another observation will, in general, be an element of D. The δ's should fulfill certain measurability conditions. By a nonrandomized s.d.f. $\{\gamma(x_1, \cdots, x_n)\}$ $(n = 1, 2, \cdots)$ is meant a set of functions each of which is single-valued with $\gamma(x_1, \cdots, x_n) \in D$. (Certain measurability restrictions must be fulfilled.) Let δ be any randomized s.d.f. The authors prove a theorem a consequence of which is that, with slight restrictions on D, there exists a space A whose elements $\gamma(a)$ are nonrandomized s.d.f.'s, and a probability measure μ defined on a Borel field in A, such that δ is equivalent to proceeding according to an element $\gamma(a)$ selected from A by means of a chance mechanism with probability measure μ. By "equivalent" is meant that the probability of obtaining the observations $(x_1, \cdots, x_n)$ and then choosing an element in the measurable subset c of D is the same for both procedures, identically in c and $x_1, \cdots, x_n$. Similar applications can be made to the theory of games.

Columbia University,
New York, N. Y., U. S. A.

ON THE MEAN DURATION OF RANDOM WALKS IN n DIMENSIONS

Wolfgang R. Wasow

A particle performs a discontinuous random walk in a bounded n-dimensional domain B. Given that the particle is at P, let $F(P, Q)$ be the distribution function of its position Q after one step. Denote by $W(P)$ the mean of the number of steps in a random walk that starts at P and ends when the particle leaves B for the first time. Under certain mild conditions $W(P)$ is the unique solution of the integral equation $W(P) = \int_B W(Q)\, d_Q F(P, Q) + 1$. If the transition probability is sufficiently concentrated about P and sufficiently symmetric, it is shown by the methods of I. Petrovsky (Math. Ann. vol. 109, pp. 425–444), that $W(P)$ is approximately equal to the solution of a boundary value problem for a certain elliptic differential equation. Using this fact the inequality $W(P) \leqq \pi^{-1}\Gamma^{2/n}(n/2 + 1)s^{-2}\, V^{2/n}[1 + \epsilon(s)]$ can be proved for an important class of random walks. Here, s is the mean step length, V the volume of B, and $\lim_{s \to 0}\epsilon(s) = 0$. If B is a sphere and P its center, the equality sign holds. If B is a cube, then $W(P)$ is of lower order in n than the right member, but by a factor which is greater than $O(n^{-\delta})$, $\delta > 0$.

National Bureau of Standards,
Los Angeles, Calif., U. S. A.

LA NOZIONE DI "BENI INDIPENDENTI" IN BASE AI NUOVI CONCETTI PER LA MISURA DELLA "UTILITÀ"

Bruno de Finetti

Il confronto di "preferibilità" tra situazioni economiche non può che condurre a definire linee o varietà d'indifferenza, e quindi "indici di utilità", ma non ad attribuire a uno di essi il ruolo privilegiato di "misura dell'utilità"; a tale nozione, che sembrava perciò doversi definitivamente abbandonare, si ritorna invece, dandole un senso ineccepibile, qualora si ricorra alla considerazione della "preferibilità" nel campo ampliato delle "distribuzioni di probabilità fra situazioni economiche". Tale concetto, introdotto da J. von Neumann e O. Morgenstern, *Theory of games and economic behaviour*, è stato esposto e tradotto in forma assiomatica nel modo più sodisfacente e convincente da J. Marschak, *Rationality, uncertainty, utility*, Econometrica vol. 18 (2) Apr. 1950.

Strettamente legata alla misurabilità dell' utilità è la nozione di "indipendenza" dei beni: nella mia nota *Sui campi di ofelimità*, Rivista Ital. Sc. Economiche vol. 7 (4) Lug. 1935, avevo notato che l'usuale condizione non è invariante rispetto alla scelta dell'indice di utilità, cosicchè aveva senso soltanto una condizione più debole (di "pseudoindipendenza"), cioè la condizione per l'esistenza di un indice sodisfacente la condizione di indipendenza. Tale condizione più debole consiste nel fatto che le linee d'indifferenza sodisfino un 'equazione differenziale a variabili separabili, o, in forma finita, che ogni rettangolo $ABCD$ (lati paralleli agli assi), coi vertici A, B, C su tre linee d'indifferenza a, b, c, abbia anche il quarto vertice D sempre su una stessa linea d'indifferenza d.

Adottando il nuovo punto di vista (Neumann-Morgenstern), la nozione di indipendenza acquista un significato univoco. Alla precedente *condizione geometrica* va aggiunta una *condizione stocastica* dipendente dalla scelta dell'indice d'utilità determinata dalla considerazione delle distribuzioni di probabilità.

Tale nuova condizione può essere espressa in più forme.

Localmente, come condizione differenziale, si tratta di confrontare il grado di convessità relativa $(-f''/f')$ risultante dalle considerazioni stocastiche, con quello richiesto per l'indipendenza in base alle condizioni geometriche. Il primo si può esprimere p. es. come limite per $h \to 0$ di $h^2/2g(h)$, ove $g(h)$ sia la perdita certa indifferente con il rischio di un guadagno o perdita $\pm h$ con probabilità 1/2 e 1/2. Il secondo dipende dalla pendenza e curvatura delle linee d'indifferenza e loro traiettorie ortogonali, e vale precisamente $K_i + K_n(1 - q^2)/q$ (K_i e K_n curvature dette, q pendenza).

Globalmente, in forma finita, la condizione si esprime facilmente ed espressivamente considerando quattro situazioni (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2) $(x, y$

quantità dei due beni considerati); in un campo C si ha indipendenza se è indifferente avere il diritto con probabilità 1/2 e 1/2 alla prima o l'ultima oppure alle due intermedie fra tali quattro situazioni.

Una trattazione più dettagliata è in preparazione per il "Giornale degli economisti".

University of Trieste,
Trieste.

SECTION V

MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

SECTION V. MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS

THE REFRACTIVE INDEX OF AN IONISED GAS

SIR CHARLES G. DARWIN

1. The subject of the refractive index is not one of the exciting growing points in modern physics, but of all the considerable variety of subjects in physics with which I have been concerned in my day, though the present problem may not have been the most important, it has in fact on the whole been the one that caused me most difficulty and puzzlement. The substance of it is contained in two papers published in Proc. Roy. Soc. London. Ser. A. vol. 146 (1934) p. 17 and vol. 182 (1943) p. 152. It has a point which may appeal especially to the mathematician for it raises in a strong way one of the things loved by the pure mathematician more than the physicist. This is that there was a really difficult question about the inversion in the order of two limits. Such difficulties often occur in physical problems, but they are usually solved by seeing that one order gives an impossible answer; here either answer was credible.

The subject is how one should calculate the refractive index of a medium in which there are free electrons as well as atoms. The most typically important case is the ionosphere, where the doubts on the matter did give rise to serious questions about the actual number of electrons concerned, leaving this number doubtful to the extent of fifty per cent. The subject of metal optics also is involved, although this is so much more complicated a matter that detailed inferences are not so easy, although the broad principle holds there, too. To explain my point I must go back rather a long way. The subject of the calculation of the refractive index of a medium from the characteristics of its constituent atoms was first published by Sellmeyer, although in fact a year or two earlier Maxwell showed that he had fully mastered it by publishing the result in the most curious way ever done. He set it as an examination question—I think we may assume that he knew the answer to his own question. The result was later improved by Lorentz, and I would like briefly to consider these questions, because although they are quite classical, they are really much more difficult than many of the textbooks suggest.

2. The aim is to work out how the interaction of a set of particles—call them atoms—is to be calculated when the single ones have been fully characterised. The mutual effects depend on inverse square law combinations, which have the important character that there is an equality between the effects of near parts, middle distant parts, and very distant parts, because the numbers in these three regions go up in proportion as the single effects go down. Thus, roughly speaking, it is always necessary to sum an infinite series with very shaky convergence conditions at both ends.

Here the question is essentially the convergence of the effects of the adjacent parts on an atom. It is briefly whether the Sellmeyer or the Lorentz formula should be used for an ionised medium, and I had better begin by reminding you of how the matter stands for a dielectric. If you set down the electromagnetic equations and substitute a solution proportional to $e^{i\nu(t-\mu z/c)}$ so that μ is the refractive index, then the result is

$$\mu^2 E = E + 4\pi P.$$

Here E is the electric field strength, as it would be measured in a pipe made along its own direction, and P is the polarisation of the medium. Thus $\mu^2 - 1 = 4\pi P/E$ which is the fundamental relation. The whole question is how P is related to E. When the continuum implied by P and E is replaced by discrete atoms, what does it give as the actual force acting on an atom. Let F be the local electric force on the atom. This force F sets it in oscillation and induces a polarisation depending on the atom's nature so that adding together the polarisations of all the N atoms in a cubic centimetre

$$P = N\alpha F$$

where α is an atomic characteristic which is calculable if the nature of the atom is known. For a simple type of resonance, $\alpha = e^2/m(\nu_0^2 - \nu^2)$, where ν is the radian frequency.

The relation of P to E is thus replaced by a relation of F to E. Sellmeyer simply assumed that F was the same as E, and thus derived the formula

$$(S) \qquad \mu^2 - 1 = \frac{4\pi N e^2}{m(\nu_0^2 - \nu^2)}.$$

But Lorentz pointed out that F should not be the same as E. E is the force inside a pipe cavity after removing the atoms which should be in it, but these atoms are really there and their effects must be allowed for. He replaced them in two stages. In the first, all but a sphere tangent to the pipe are replaced, and these give an extra force $4\pi P/3$. The ones inside this sphere are then put in, and these give zero for an isotropic medium. Thus $F = E + 4\pi P/3$.
The result is

$$(L) \qquad \frac{3(\mu^2 - 1)}{\mu^2 + 2} = \frac{4\pi N e^2}{m(\nu_0^2 - \nu^2)}.$$

The problem is how these arguments are to be modified when there are free electrons, as in the ionosphere. The simplest idea is to assume that a free electron will be represented by taking $\nu_0 = 0$. In sounding the ionosphere, radio waves of steadily increasing frequency are used, and at a certain frequency, reflection occurs. This is at the frequency where $\mu = 0$. Then the S formula gives for the density of the ionosphere $N = m\nu^2/4\pi e^2$, but the L formula implies $N = (3/2)\cdot(m\nu^2/4\pi e^2)$. Thus if the wrong choice of formula is made, there will be an error of fifty per cent in the number determined, and it is very necessary to decide which one is correct.

3. When there are free electrons about, the crucial question is how their charges are neutralized. That this is important may be seen first by the fact that if they were not neutralized at all, the mutual repulsions would be so great that there would be no steady state at all, and the gas would blow up at such a rate that it is impossible to speak of a refractive index at all. If one could assume that there was a continuous positive distribution of charge, the subject becomes very simple and it emerges clearly that the S formula is applicable. But actual neutralization is done by positive ions—which it will suffice to take as protons in fixed positions to bring out the essentials of the matter. It is by no means obvious that their averaged effect is equivalent to that of a distributed continuum of charge.

Consider the simple sorts of argument one might use. The first says that an electron is not a dipole, and thus has no polarisation, and therefore the terms involving P should be omitted in calculating the force. But this is hopeless; it does not lead to the S formula, but to an even simpler one, viz., $\mu^2 = 1$. The mere existence of a refractive index implies that there is a polarisation effect to be allowed for, and the different formulae are merely minor variations on how it is to be done. This argument tells that one cannot dispense with P.

If that is so, then why not use the whole P argument, and say that the electron is an atom in which the natural frequency $\nu_0 = 0$; then the L formula should apply. But this will not do, for when ν_0 vanishes, there is nothing to annihilate the velocity of the electron, and it will wander away from the central position which it is assumed by the argument that it should continue to occupy. In other words, it is not right to say that a free electron is represented by going to the limit of ν_0.

The next point is that the definition of E has become extremely vague. It was based on cutting out a pipe cavity, but free electrons will penetrate the cavity, and then what will happen? However, I shall not waste time over these confusions, but give you, without arguing it, the outcome. The natural way of regarding polarisation is in terms of the distance between the position of the electron and the position it would have had if there had been no light acting on it. Let ξ represent this distance. The question is how this ξ is affected by the various positive charges. If the neutralizing charge is spread uniformly in space, it is not hard to show that the force due to it is $-4\pi Ne\xi/3$; for as the electron oscillates, it crosses over through this positive material, and this evokes such a force. But actually in a proton field it does not cross over any positive electricity, and there is no force of this kind. So the contradiction arises that if Av means the average effect of smearing out the positive charges into a continuum, and Lt means the limit when the displacement of the electron due to the light tends to zero in magnitude (as it is always taken to do in optics), then:

$$Lt\ Av = -\frac{4\pi}{3} Ne\xi$$

which leads to the S type of formula, but

$$Av\ Lt = 0,$$

which leads to L. This contradiction can be reconciled by noticing that in its second form it has excluded cases where the electron is very near one of the protons. This is a rare event, but the effect is enormous when it does happen, although not calculable without far greater elaboration. I shall show how it leads to a reconciliation, so that the S formula is justified.

4. This was in fact the subject of my second paper, and at first I did not see how to get at it. I therefore took the quite different line that if the business was so difficult and confusing, a wholly different attack would be best in which questions of what E, P, and F meant should not arise. To do this the whole group of atoms must be regarded as a single dynamical system, and at once there is difficulty over the optical retardations of the scattered waves in their mutual effects on the electrons. However, this can be avoided if distances are all much smaller than a wave length of the light concerned. The process was therefore to take a small sphere of radius a containing n electrons and n fixed protons and submit it to incident light defined by the *external* electric field $E_0 \sin \nu t$ and to calculate the total electric moment induced in it. This is easily related to the index by consideration of the light scattered by the sphere, and the relation is that the moment is:

$$\frac{\mu^2 - 1}{\mu^2 + 2} a^3 E_0 \sin \nu t.$$

It will be noticed that the formula has a family resemblance to the L formula, and this leads in fact to a sort of inversion of the answer, so that the apparent roles of S and L are interchanged.

I shall not describe the method in detail. There are $6n$ dynamical coordinates in a Hamiltonian, containing their mutual reactions and the external $E_0 \sin \nu t$. A set of integrals according to Gibbs' statistics can be set down for the effects. They are of course not calculable. However, owing to the inverse square attractions and repulsions, they contain a lot of singularities, corresponding exactly to the collision regions of the electrons with the protons.

The following assumptions make it possible to discuss these singularities in detail. It is assumed possible to draw around each proton a sphere of radius b, where b obeys the following conditions:

I. $b \ll a/n^{1/3}$. This signifies that there is hardly ever more than one electron at a time inside the b of any selected proton.

II. $b \ll V/\nu$ where V is the mean velocity of the electrons. This signifies that a collision time is short, and the electric field acting on the electron (apart from that of the local proton) may be taken as constant during that time.

III. $b \gg e^2/mV^2$. This signifies that at entry into b, the electron has velocity practically what it had at infinity.

These conditions, broadly speaking, define the difference between regions of collision and external regions. They are easily satisfied in the ionosphere, and also they are roughly satisfied inside a metal.

In the Gibbsian integrals the singularities can now be isolated into regions of one electron near one proton; so that the $6n$-fold integral can be separated into $6n - 6$ regular integrations and 6 involved in the singularity—three for momentum and three for the position, containing terms in $1/r$ or its differentials. When the 6-fold integrals are worked out, it is apparent that the singularities are not strong enough to give any trouble at all; they give small correction terms of relative order e^2/bmV^2 only, on top of the main terms, and by III the correction terms are negligible. Therefore the contribution of the singularity to the integral is the same as it would be if there were no proton there, or, alternatively, if there were instead a uniform positive distribution of electricity. Therefore the integrals yield the same answer as that for the uniform distribution. This uniform distribution is very easily shown to yield the S type of formula.

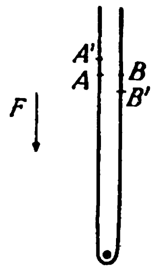
Now I am not going to claim, to mathematicians, that this is proof. There are something like $n!$ singularities in the integrals, and it has been shown only that one of them does not matter, but nothing has been shown about their cumulative effect. I do not know whether the argument would be improved by looking into this, but I think it would hardly be worthwhile, because there will in the large number n always be a few electrons going slowly, so that III fails, or electrons having two electrons in simultaneous collision, so that I fails for them. If therefore one is very critical, all one should say is that the method demolishes any possible claim for the L formula and makes the S the most natural one, but that it does not rigorously exclude some other formula.

I have not mentioned one important point the method does bring out. It is always possible that it might prove too much, and show that L is always wrong and S right. It does not do this, for if an insulated medium is taken, the forms of all the Gibbsian integrals are entirely altered and they do yield results leading to L.

5. There is little chance of gaining physical insight from this first method and that is the purpose of the second method, which directly evaluates the effect of collisions. To explain it I want to go back to the static case and give quite a different derivation of the Lorentz force on a dielectric. I take a model which is rather specialised, but it could be easily generalised much more.

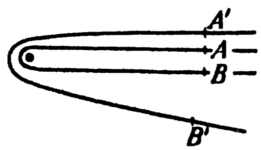
The dielectric is supposed to be composed of a vacuum containing an irregular distribution of dielectric spheres, N per cc. Each has radius d, and they are all polarised in the same direction, with moment p along the direction of the electric force E. In consequence of this polarisation there is a wildly varying force between them. Let its average be F. The force from a dipole is of order p/r^3 at r. This may range from p/d^3 to pN, since $N^{-1/3}$ is the distance between atoms. It may easily fluctuate many thousandfold, so that the evaluation of its average is a delicate business.

Note that inside any of the atoms there is a depolarising force—like the force H inside a permanent magnet, which is in the opposite direction to its B—of amount p/d^3. Thus in the outer parts the average force is F, but inside an atom it is $F - p/d^3$. Now a fraction of space $4\pi d^3N/3$ is occupied by atoms, so that the full average, inside and out, is $F - 4\pi pN/3 = F - 4\pi P/3$. If, for example, an electron is carried a long way l straight down the field, it will cut through some of the atoms and the work on it is $el(F - 4\pi P/3)$. Now another



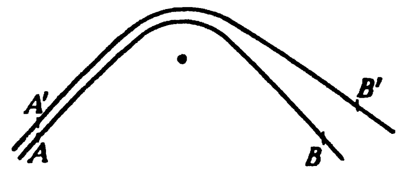
1

FIGURE 1



2

FIGURE 2



3

FIGURE 3

way of estimating this work might be to drill a long pipe down the dielectric; if it is long enough, end effects will disappear, and the work would then be eEl; so that $F = E + 4\pi P/3$, which gives the mean force F in terms of E, provided we remember that F applies outside the atoms, or, if inside an atom, then the force of that atom is to be counted separately. This result suggests the essential difference between the average force throughout space and that outside an atom; for an electron that may be outside or inside an atom the correct value to take is E, but for one locked in an atom the force is F together with the force of the atom itself.

I have always felt that this result must be identical with that of Lorentz's method of evaluating the mean force F, but it is quite surprisingly different in its argument and the connection is not obvious. It does, however, suggest that one cannot get the corresponding answer in the case of free electrons in an ionised gas without considering the effects of collisions in some detail. It was following this idea up that led to the solution.

6. There is, strictly speaking, no such thing as polarisation or the electron-proton gas, but it is obvious how to use the idea, by a comparison of the path of the actual electron, with what it would have if there were no light present. In the free parts of space the unperturbed electron would describe a straight line, the perturbed a gently sinuous curve in antiphase under the influence of the periodic electric force F. It is the difference between these two positions that defines the polarisation.

Consider what happens in a collision. I sketch a few special cases. I again adopt conditions I, II, III. A is the point of entry into and B of exit from the

b-sphere of the unperturbed electron, A', B' of the corresponding perturbed electron. AA' is initially a distance ξ along the direction of F, and the question is what the relative positions of B and B' will in consequence be. In Fig. 1 obviously ξ is reversed by the collision. In Fig. 2 not only is the polarisation reversed, but the direction of motion is also changed, so that $\dot{\xi}$ from being zero acquires a negative value. In Fig. 3 it is also changed, but this time positively. The general result is that there is a highly complicated set of changes in both ξ and $\dot{\xi}$ induced by collisions. These effects must be averaged.

It is always a surprise what troublesome mathematics are involved in hyperbolic planetary orbits, but still they can of course be done. When the means have been taken, they are multiplied by $N\pi b^2 V\Delta t$ for the number of collisions in time Δt. The results are

$$\Delta\dot{\xi} = -\xi\frac{4\pi}{3}N\frac{e^2}{m}\Delta t - \dot{\xi}\frac{4\pi}{3}N\frac{e^2}{m}\frac{b}{V}\Delta t$$

$$\Delta\xi = -\xi\frac{4\pi}{3}N\frac{e^2}{m}\frac{b}{V}\Delta t + \dot{\xi}\frac{4\pi}{3}Nb^3\Delta t.$$

Of these four terms the first is the only one that matters. The second is negligible by II compared with it. The third really expresses the same thing as the first, and the fourth is unimportant by condition I. Note that b has disappeared in the first term, which is the only one that matters.

The effect of the collisions on the average is therefore

$$\Delta\dot{\xi} = -\xi\frac{4\pi}{3}N\frac{e^2}{m}\Delta t$$

and this can be re-expressed by saying they give an average *acceleration* $-\xi(4\pi Ne^2/3m)$.

This acceleration would be given by an electric force $-\ 4\pi Ne\xi/3$. This shows that the quantity I called *Av Lt* is not zero as it seemed, but is really just the same as *Lt Av*, and thus the result is

$$\mu^2 - 1 = -\frac{4\pi Ne^2}{mV^2}$$

which is the answer to the main problem.

7. One or two points in the work should be mentioned. The calculation and averaging of the hyperbolic orbits is quite troublesome, and, in the manner I did it, gave rather difficult convergence. The integrals corresponding to the equations in $\Delta\xi$ and $\Delta\dot{\xi}$ had of course important contributions from the orbits that passed near the centre, but they also had important terms from the much more numerous orbits that only just entered the b-sphere. These second contributions consisted of a pair of terms which exactly cancelled. If they had not done so, the refraction formula would have been entirely different. There must presumably be some different integrating process which would cancel out these terms at an early stage corresponding to the fact that such orbits cannot be important, but I never could find it.

Another point I may mention is that the work of the first paper gives the rules for calculating the index of a medium partly composed of free electrons and partly of bound, but I shall not go into this here.

Finally I ought perhaps to confess that the work does not give exact definitions for the electric fields in an ionised medium. Perhaps I should have tried to do this, but I may conclude by saying frankly that all through my work I have felt that there was a danger of overstressing a subject not in the main centre of interest in physics, so that I have been content with clearing up the matter beyond all reasonable doubt without going into all its possible refinements.

CAMBRIDGE, ENGLAND.

DEVELOPMENTS AT THE CONFLUENCE OF ANALYTIC BOUNDARY CONDITIONS[1]

HANS LEWY

1. I wish to discuss two prototypes of a class of problems which are frequently encountered in the applications of analysis to elasticity and hydrodynamics as well as in the study of complex functions defined by geometric properties of the induced conformal mapping. The choice of the two prototypes is personal; I was led to the one example by an investigation of the nature of water waves near a horizontal obstacle floating on the surface of the liquid, but I noticed that the essence of the treatment of this problem applies to a larger group of problems. Roughly speaking, we are dealing with functions of two real variables which satisfy an elliptic analytic differential equation (such as Laplace's) and which, on the boundary of their domain of definition, satisfy two analytic boundary conditions C_1 and C_2, with C_1 holding on a portion of the boundary and C_2 holding on an adjacent portion. The problem is to describe the behaviour of the solution of the differential equation near the point of confluence of the two boundary conditions, according to rising orders of vanishing of functions usable in the development, in a way similar to the development of a function, regular for $z = 0$, with the aid of ascending powers of z.

More precisely the two prob'ems here to be discussed are:

Problem A: The conformal map of a portion of the upper half of a complex $z = x + iy$ plane on a domain of the upper $\zeta = u + iv$ plane, bounded (partially) by a portion of the negative real axis and a portion of a parabola Γ of form $v = 2u^2$, the two meeting at the origin under an angle π so that the tangent of the boundary remains continuous at the origin.

Problem B: The harmonic function $u(x, y)$ is continuous in a neighborhood of the origin for $y \leqq 0$ and satisfies on the negative x-axis the condition $\partial u/\partial y = 0$ and on the positive x-axis the condition $\partial u/\partial y = u$.

Problem B is a linear problem while A is nonlinear. Let $F(z)$ be the complex function which yields the desired conformal mapping of A, resp. the analytic function of the complex argument $z = x + iy$ whose real part is the function u occurring in B. An immediate simple observation tells us that $F(z)$ can in both cases be analytically extended into a neighborhood of the origin of an infinitely leaved Riemann surface with a logarithmic branch point at the origin. It can be proved that $F(z)$ remains continuous as $z \to 0$ provided this is interpreted to mean that $|\operatorname{Im} \log z|$ remains bounded as $z \to 0$.

Denote by z and z^{-1} points of the Riemann surface such that

$$\log z^{-1} = \log z + 2\pi i,$$

[1] For proofs and further results consult the author's memoir of the same title, University of California Publications in Mathematics, vol. 1 (1950) pp. 247–280.

and set, for brevity, $F(z) = F$, $F(z^+) = F^+$. For real positive z the defining property of $F(z)$ in A leads to the equation

$$F^+ - F + i(F^+ + F)^2 = 0, \tag{A1}$$

while in case B we find

$$(F^+)' - (F)' + i(F^+ + F) = 0. \tag{B1}$$

These equations must remain valid everywhere on the Riemann surface since they hold for positive z. (B1) is a difference-differential equation connecting the values of the same function (and its derivative) on successive leaves of the Riemann surface. (A1) is a difference equation of the same kind; since z does not enter explicitly, the relation between F, F^+, F^{++}, $\cdots$ is a simple functional iteration in which z plays the rôle of a uniformizing parameter.

2. To formulate briefly the answers to the problems stated above, consider power products of z and $z \log z$. Denote by (m, n) the degree of the product $z^m(z \log z)^n$; write $(m, n) = (m', n')$ if $m = m'$, $n = n'$, and write $(m, n) < (m', n')$ if $z^{m'}(z \log z)^{n'}/[z^m(z \log z)^n]$ tends to 0 as $z \to 0$. As m and n range over the non-negative integers, the degrees (m, n) are thus lined up in the order of their increase. It makes sense to talk of a function $F(z)$ as asymptotically equal to a polynomial of degree (m, n),

$$F(z) \sim \sum_{(0,0)}^{(m,n)} A_{\mu\nu} z^\mu (z \log z)^\nu,$$

if the difference between the left and right sides, divided by $z^m(z \log z)^n$, tends to zero as $z \to 0$. Obviously the last equality implies all those asymptotic relations which are obtained by striking successively from the right side the terms of highest degree. Accordingly a relation $F(z) \sim \sum_{(0,0)}^{\infty} A_{\mu\nu} z^\mu (z \log z)^\nu$ with infinite sum can be interpreted as the infinitude of relations obtained by equating the left side to any section of finite degree on the right, and no statement about convergence of the infinite sum is necessary to make this asymptotic relation meaningful.

Now with this terminology there holds the following theorem:

THEOREM. *Every solution $F(z)$ of problem* A *or* B *is asymptotically equal to an infinite double series in z and $z \log z$, whose coefficients are uniquely determined by $F(z)$.*

While I cannot reproduce here the proof of this theorem, I state at least the essential lemma from which it follows.

LEMMA. *Suppose that for positive t and real* $\log t$ *a function $F_1(t) \sim M_1(t, t \log t)$, where $M_1(t, t \log t)$ is a polynomial of degree (m, n). Let a be positive and $F_1(t)$*

be bounded and measurable in $0 < t < a$. *Set*

$$F_2(z) = \int_0^a F_1(t) \log (t - z)\, dt$$

with $0 \leqq |z| < a$, $0 \leqq \operatorname{Im} \log z < 2\pi i$, *and with* $\log (t - z)$ *that branch of the function which is real for* $t > z$. *Then there exists a polynomial* $M_2(z, z \log z)$ *of degree* $(m, n + 1)$ *such that*

$$F_2(z) \sim M_2(z, z \log z).$$

3 Returning to the double power series in z and $z \log z$ which is the asymptotic representation of $F(z)$, we shall call it $P(z, z \log z)$ and indicate a simple way of finding its coefficients. First let $F(z)$ be a solution of B. Set formally

$$P(z, z \log z) = \varphi_0(z \log z) + z\varphi_1(z \log z) + z^2\varphi_2(z \log z) + \cdots .$$

Then evidently

$$P(z^+, z^+ \log z^+) = \varphi_0(z \log z + 2\pi i z) + z\varphi_1(z \log z + 2\pi i z) + \cdots .$$

Applying Taylor's series to $\varphi_0(z \log z + 2\pi i z)$, $\varphi_1(z \log z + 2\pi i z)$, $\cdots$, we get

$$P(z^+, z^+ \log z^+) = P(z, z \log z) + 2\pi i z\varphi_0'(z \log z) + 2\pi i z^2\varphi_1'(z \log z) + \cdots ,$$

$$\frac{dP}{dz}(z, z \log z) = (\log z + 1)\varphi_0'(z \log z) + \varphi_1(z \log z)$$
$$+ z(\log z + 1)\varphi_1'(z \log z) + \cdots ,$$

$$\frac{dP}{dz}(z^+, z^+ \log z^+) = (2\pi i + \log z + 1)(\varphi_0'(z \log z) + 2\pi i z\varphi_0''(z \log z) + \cdots)$$
$$+ \varphi_1(z \log z) + \cdots$$

(B1) begins formally with the terms free of z (i.e., depending only on $z \log z$),

$$\text{(B1.0)} \qquad 2\pi i\varphi_0'(z \log z) + 2\pi i z \log z\varphi_0''(z \log z) + 2i\varphi_0(z \log z).$$

Next come the terms divisible by z with coefficients that are functions of $z \log z$; these terms involve only φ_0 and φ_1 and their derivatives, etc. Setting (B1.0) equal to zero, we see that φ_0, being a power series of $z \log z$, must be proportional to the Bessel function $J_0[((z \log z)/\pi)^{1/2}]$. Similarly φ_1 can be determined as a solution of a linear differential equation whose known members depend on φ_0, found previously. The recursive evaluations of $\varphi_0, \varphi_1, \varphi_2, \cdots$ introduce a constant of integration at each step, which agrees with the fact that there are infinitely many solutions of the proposed problems.

It is of interest that a less formal and more structural relation between $F(z)$ and $\varphi_0 = J_0[((z \log z)/\pi)^{1/2}]$ can be given. Consider a fixed value $\alpha \neq 0$ and the infinitely many solutions of the equation

$$\text{(S)} \qquad z \log z = \alpha.$$

They may be arranged according to increasing values of $|\operatorname{Im} \log z|$. As $\operatorname{Im} \log z$ tends to ∞ or to $-\infty$, the value z tends necessarily to 0. Let $z_1, z_2, \cdots$ be such a sequence of solutions of (S) with $z \to 0$ and $\operatorname{Im} \log z \to \infty$. Then

$$\lim_{n\to\infty} F(z_n) = \varphi_0(\alpha),$$

$$\lim_{n\to\infty} (F(z_n) - \varphi_0(\alpha))/z = \varphi_1(\alpha),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

In view of this relationship, I propose to name $\varphi_0, \varphi_1, \cdots$ the "shadows" of $F(z)$ at $z = 0$ of order $0, 1, \cdots$.

Analogous considerations for A lead to the recursive determination of the shadows of $F(z)$ in A. We find without difficulty

$$\varphi_0 = 0,$$

$$\varphi_1 = (c + 2\pi^{-1} z \log z)^{-1}, \qquad c \neq 0 \text{ a constant.}$$

$$\cdots\cdots\cdots\cdots\cdots$$

4. Since with each $F(z)$ is associated a unique double power series $P(z, z \log z)$, the natural question is whether or not there is a way of regaining $F(z)$ when $P(z, z \log z)$ is known. The answer is positive in case B and is covered by the following theorem.

THEOREM. *A solution $F(z)$ of* (B1) *which is regular for $z \neq 0$, $|z| \leqq \rho$, $\rho > 0$, and continuous for $z \to 0$, can be developed in a double power series in z and $z \log z$ which converges uniformly and absolutely to $F(z)$ for all z with $|z| \leqq \rho$ and bounded $|\operatorname{Im} \log z|$.*

The double power series involved is of course the same one which yields the asymptotic development.

A further development of $F(z)$ is based on the observation that if $G_0(z)$ is a solution of (B1) which is continuous for $z \to 0$, then $G_1(z) = \int_0^z G_0(t)\,dt$ is another such solution, and so are $G_2(z) = \int_0^z G_1(t)\,dt, \cdots$. Assume $G_0(0) \neq 0$. The following theorem holds:

THEOREM. *Any $F(z)$, solution of* (B1) *and continuous at the origin, can be developed in a series*

$$F(z) = \sum_{\nu=0}^{\infty} c_\nu G_\nu(z)$$

which converges uniformly and absolutely in a proper concentric subcircle of the common domain of existence of $F(z)$ and $G_0(z)$.

This theorem illustrates the perfect analogy with the developability of a regular function of z in a power series in z, where likewise the functions used are the iterated integrals of the first of them, namely the constant 1.

To prove the quoted theorems I had to devise some new analytic tools which fit the special singularity of the functions $F(z)$. It would take more than the allotted time to describe them in this short address. Unfortunately, these tools depend strongly on the linearity of the equation (B1), and I have no hope of extending them to the case A. I do not know whether $P(z, z \log z)$ converges in case A, and I am inclined to doubt it.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF., U. S. A.

STÖRUNGSTHEORIE DER SPEKTRALZERLEGUNG

Franz Rellich

Wir betrachten lineare selbstadjungierte Operatoren $A(\varepsilon)$ in einem Hilbertschem Raum, die von einem reellen Parameter ε abhängen und fragen, wie die Spektralzerlegung von $A(\varepsilon)$ analytisch von ε abhängt. Wenn $A(\varepsilon)$ regulär analytisch von ε abhängt (was das heisst muss allerdings erst gesagt werden) dann pflegt man Eigenwerte und Eigenfunktionen von $A(\varepsilon)$ als Potenzreihen in ε anzusetzen und durch eine Störungsrechnung die Koeffizienten dieser Potenzreihen der Reihe nach zu berechnen. (Lord Rayleigh, E. Schrödinger). Sicher geschieht dies bei Problemen der mathematischen Physik häufig auch dann, wenn von einer Konvergenz des Verfahrens keine Rede sein kann.

Im folgenden schildere ich, was von seiten der Mathematik seit 1936 an gesicherten Resultaten zu dieser Frage erarbeitet worden ist.

1. Störungstheorie im endlichdimensionalen Raum. Eine erste Orientierung gewinnt man beim Studium eines selbstadjungierten Operators $A(\varepsilon)$ in einem endlichdimensionalen (komplexen) Vektorraum. Nach Einführung eines Koordinatensystems ist der Operator durch eine Hermitesche Matrix $A(\varepsilon) = ((a_{ik}(\varepsilon)))$, $i, k = 1, 2, \cdots, h$ gegeben. Die $a_{ik}(\varepsilon)$ seien reguläre Potenzreihen der reellen Veränderlichen ε in der Umgebung von $\varepsilon = 0$ und es sei dort $a_{ik}(\varepsilon) = \overline{a_{ki}(\varepsilon)}$. Wie hängen Eigenwerte $\lambda(\varepsilon)$ und Eigenelemente $\phi(\varepsilon)$ der Matrix $A(\varepsilon)$ von ε ab? Die Eigenwerte $\lambda(\varepsilon)$ genügen der algebraischen Gleichung h-ten Grades

$$(-1)^h \begin{vmatrix} a_{11}(\varepsilon) - \lambda, & a_{12}(\varepsilon), & \cdots, & a_{1h}(\varepsilon) \\ a_{21}(\varepsilon), & a_{22}(\varepsilon) - \lambda, & \cdots, & a_{2h}(\varepsilon) \\ \cdots & \cdots & \cdots & \cdots \\ a_{h1}(\varepsilon), & a_{h2}(\varepsilon), & \cdots, & a_{hh}(\varepsilon) - \lambda \end{vmatrix} \equiv \lambda^h + p_1(\varepsilon)\lambda^{h-1} + \cdots + p_h(\varepsilon) = 0$$

wobei die Koeffizienten $p_1(\varepsilon), \cdots, p_h(\varepsilon)$ in der Umgebung von $\varepsilon = 0$ konvergente Potenzreihen von ε sind. Die Wurzeln einer solchen algebraischen Gleichung sind im allgemeinen nicht Potenzreihen von ε, sondern Potenzreihen von $\varepsilon^{1/h}$. Danach könnte man meinen, der in der Störungsrechnung übliche Ansatz $\lambda = \lambda^{(0)} + \lambda^{(1)}\varepsilon + \lambda^{(2)}\varepsilon^2 + \cdots$ sei falsch und sei durch $\lambda = \lambda^{(0)} + \lambda^{(1)}\varepsilon^{1/h} + \lambda^{(2)}\varepsilon^{2/h} + \cdots$ zu ersetzen. Tatsächlich aber ist unser Polynom $\lambda^h + p_1(\varepsilon)\lambda^{h-1} + \cdots + p_h(\varepsilon)$ dadurch eingeschränkt, dass es identisch ist mit der Säkulardeterminante einer Hermiteschen Matrix. Seine Wurzeln sind also reell für reelle ε. Daraus lässt sich leicht herleiten, dass alle Wurzeln (auch die, welche für $\varepsilon = 0$ mehrfach sind) nicht nur konvergente Potenzreihen von $\varepsilon^{1/h}$ sondern sogar von ε sind.

Damit ist gezeigt, dass alle Eigenwerte von $A(\varepsilon)$ reguläre Potenzreihen in der Umgebung von $\varepsilon = 0$ sind und dasselbe lässt sich für geeignet gewählte Eigenelemente (= Eigenvektoren) beweisen [9].

Für die Anwendungen wäre es erwünscht eine Störungsrechnung auch für zwei oder mehr Störungsparameter $\varepsilon_1, \varepsilon_2, \cdots$ zu besitzen. Im letzten Abschnitt wird an dem Beispiel der endlichen Hermiteschen Matrix gezeigt, welches die Schwierigkeiten einer solchen Theorie bereits im algebraischen Falle sind. Im folgenden wird daher (abgesehen vom letzten Abschnitt) ausschliesslich von *einem* Parameter ε die Rede sein.

2. Der Begriff des regulären Operators. Für einen beschränkten, im ganzen Hilbertschen Raum erklärten Operator $A(\varepsilon)$ wird Regularität in naheliegender Weise dadurch erklärt, dass man für jedes (von ε unabhängige) Element u des Hilbertschen Raumes $A(\varepsilon)u = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$ verlangt, wo die f_n von ε unabhängige Elemente sind und die Reihe im Sinne der Metrik des Hilbertschen Raumes konvergiert. Weniger zwingend ist die Definition der Regularität für nicht notwendig beschränkte Operatoren. Die in [11] gegebene Definition ist so geartet, dass ein selbstadjungierter Operator $A(\varepsilon)$ jedenfalls dann regulär ist, wenn für ein komplexes c der Operator $A(\varepsilon) + c$ eine beschränkte in der Umgebung von $\varepsilon = 0$ reguläre Reziproke besitzt; dann hat $A(\varepsilon) + c'$ in einer Umgebung von $\varepsilon = 0$ eine reguläre beschränkte Reziproke für jedes komplexe c', für welches $A(0) + c'$ eine beschränkte Reziproke hat. Es ist eine Hauptaufgabe für die Regularität eines nichtbeschränkten Operators praktisch verwendbare Kriterien anzugeben. Das geschieht in [11], [12], [13].

Um einen Begriff von der Natur dieser Kriterien zu geben, sei das folgende genannt: *Der Operator $A(\varepsilon) = A_0 + \varepsilon A_1$ in $\mathfrak{A}$ ist in einer Umgebung von $\varepsilon = 0$ regulär, wenn A_0 in $\mathfrak{A}$ selbstadjungiert und A_1 in $\mathfrak{A}$ Hermitesch ist.* Die Anwendbarkeit dieses Kriteriums kann insbesondere bei mehrdimensionalen Differentialoperatoren dadurch erschwert sein, dass der Teilraum $\mathfrak{A}$, in dem der ungestörte Operator A_0 selbstadjungiert ist, nicht genau genug bekannt ist. Sei statt dessen ein engerer Raum $\mathfrak{D} \subseteq \mathfrak{A}$ bekannt, in welchem A_0 wesentlich selbstadjungiert ist (zu jedem u aus $\mathfrak{A}$ gibt es eine Folge u_n aus $\mathfrak{D}$ mit $\| u_n - u \| \to 0$, $\| A_0 u_n - A_0 u \| \to 0$, $n \to \infty$). Es gilt das modifizierte Kriterium, das nur die Kenntnis von $\mathfrak{D}$ verlangt: *Der Operator $A(\varepsilon) = A_0 + \varepsilon A_1$ in $\mathfrak{A}$ ist in einer Umgebung von $\varepsilon = 0$ regulär, wenn A_0 in $\mathfrak{D}$ wesentlich selbstadjungiert, A_1 in $\mathfrak{D}$ Hermitesch ist und wenn $\| A_1 u \| \leqq k\{ \| u \| + \| A_0 u \| \}$ ist für alle u aus $\mathfrak{D}$ mit einer geeigneten Konstanten k.* (Diese Ungleichung verlangt nicht zu viel, denn es lässt sich beweisen: *Wenn A_0 in $\mathfrak{A}$ selbstadjungiert und A_1 in $\mathfrak{A}$ Hermitesch ist, dann gibt es immer eine Konstante k, sodass $\| A_1 u \| \leqq k\{ \| u \| + \| A_0 u \| \}$ für alle u aus $\mathfrak{A}$.*) Dieses Kriterium (in beiden Formen) ist insofern speziell als es nur reguläre Operatoren $A(\varepsilon)$ erfasst, die in einem *von ε unabhängigen Teilraum* $\mathfrak{A}$ selbstadjungiert sind.

3. Reguläre Störung isolierter Punkteigenwerte. Der selbstadjungierte, in einer Umgebung von $\varepsilon = 0$ selbstadjungierte Operator $A(\varepsilon)$ habe für $\varepsilon = 0$

einen isolierten Eigenwert h-facher Vielfachheit, er heisse λ. *Dann besteht das Spektrum des gestörten Operators $A(\varepsilon)$ in der Nähe von λ genau aus h in der Umgebung von $\varepsilon = 0$ regulären analytischen Punkteigenwerten $\lambda_1(\varepsilon), \cdots, \lambda_h(\varepsilon)$ mit regulären analytischen Eigenfunktionen.* Dieser Satz kann entweder dadurch bewiesen werden, dass man mit einem seit Poincaré geläufigen Kunstgriff ein Polynom h-ten Grades angibt, dessen Wurzeln gerade $\lambda_1(\varepsilon), \cdots, \lambda_h(\varepsilon)$ sind [9], [11], [3]. Diese Zurückführung hat den Vorteil, die funktionentheoretische Abhängigkeit der $\lambda(\varepsilon)$ von ε sofort klar zu machen. Ein anderer und vom Standpunkt der Spektraltheorie besonders einfacher Beweis wurde von Nagy [7] angegeben. Er verwendet die Darstellung der Spektralschar durch ein komplexes Integral über die Resolvente $[A(\varepsilon) - z]^{-1}$. Wenn $\mu_1 \leqq \lambda \leqq \mu_2$ ein Intervall der Spektralachse ist, dessen Endpunkte nicht zum Spektrum des ungestörten Operators $A(0)$ gehören (auch nicht Häufungspunkte des Spektrums sind), dann kann die Regularität von $E_{\mu_2}(\varepsilon) - E_{\mu_1}(\varepsilon)$ bewiesen werden, wo $E_\lambda(\varepsilon)$ die Spektralschar von $A(\varepsilon)$ ist. Insbesondere werden isolierte Punkteigenwerte (jetzt sogar von unendlicher Vielfachheit) erfasst. Denselben Beweisgedanken verwendet Kato [4] ohne [7] zu kennen.

4. Fehlerabschätzungen. Für die Koeffizienten in den Potenzreihen

$$\lambda(\varepsilon) = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \cdots,$$
$$\phi(\varepsilon) = \phi^{(0)} + \varepsilon\phi^{(1)} + \cdots$$

der Eigenwerte und Eigenfunktionen sind Rekursionsformeln aufgestellt und Fehlerabschätzungen angegeben worden ([12], verbesserte Abschätzungen [7]). Dabei darf der ungestörte Eigenwert $\lambda^{(0)}$ mehrfach sein. Die erste Näherung $\lambda^{(1)}$ wird aber ausdrücklich als *einfach* vorausgesetzt. Diese Voraussetzung besagt folgendes. Es sei $A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots$ und es seien ϕ_i, $i = 1, 2, \cdots, h$ die Eigenelemente des ungestörten Eigenwertes $\lambda^{(0)}$. Dann werden Aussagen nur über solche Eigenwerte $\lambda(\varepsilon) = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \cdots$ gewonnen, bei denen $\lambda^{(1)}$ einfacher Eigenwert der Matrix $(\phi_i, A_1\phi_k)$, $i, k = 1, 2, \cdots, h$ ist. Häufig besitzt diese Matrix aber keine einfachen Eigenwerte (z.B., nicht im Falle $A(\varepsilon)u = -u'' + \varepsilon \cos xu = \lambda u$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ für höhere Eigenwerte als den tiefsten Ebener Rotator im elektrischen Feld). Es wären Fehlerabschätzungen ohne diese einschränkende Voraussetzung erwünscht.

5. Nichtreguläre Eigenwerte bei regulärer Störung. Das Eigenwertproblem

$$A(\varepsilon)u \equiv \int_0^1 \left[\frac{1}{2}(x + y) - \frac{1}{2}|x - y| - \frac{xy}{1 + \varepsilon}\right] u(y)\, dy = \lambda u(x)$$

für den in der Umgebung von $\varepsilon = 0$ regulären Integraloperator $A(\varepsilon)$ (es ist aequivalent mit $-u'' = \lambda^{-1}u$, $0 \leqq x \leqq 1$, $u(0) = 1$, $\varepsilon u'(1) + u(1) = 0$) hat für $\varepsilon = 0$ die Eigenwerte $\lambda_n = (n^2\pi^2)^{-1}$ und die in $0 \leqq x \leqq 1$ vollständigen Eigenfunktionen $u_n(x) = 2^{1/2} \sin(n\pi x)$, $n = 1, 2, \cdots$. Für $-1 < \varepsilon < 1$ hat er die

Eigenwerte $\lambda_n(\varepsilon) = \nu_n^{-2}(\varepsilon)$, wobei $\nu_n(\varepsilon)$ die positiven Wurzeln der Gleichung $\operatorname{tg} \nu = -\varepsilon\nu$ sind. Die zugehörigen Eigenfunktionen sind $u_n(x, \varepsilon) = 2^{1/2} \sin [\nu_n(\varepsilon)x]$. Eigenwerte und Eigenfunktionen hängen regulär analytisch von ε ab und gehen für $\varepsilon = 0$ in die entsprechenden ungestörten Eigenwerte bzw. Eigenfunktionen über. *Aber die $u_n(x, \varepsilon)$ sind nicht vollständig, wenn ε negativ ist.* Für $-1 < \varepsilon < 0$ gibt es nämlich noch die Eigenfunktion $u_*(x, \varepsilon) = \sinh [\nu_*(\varepsilon)x]$, wobei $\nu_*(\varepsilon)$ die positive Wurzel der Gleichung $\tanh \nu = -\varepsilon\nu$ ist. Der zugehörige Eigenwert ist $\lambda_* = -\nu_*^{-2}(\varepsilon)$. Es strebt λ_* gegen Null, wenn ε durch negative Werte nach Null geht (aber 0 ist *nicht* Eigenwert von $A(0)$). Das Auftreten solcher nichtregulären Bestandteile des Spektrums, die prinzipiell durch die übliche Störungsrechnung nich erfasst werden können, ist unerwünscht. Es entsteht die Frage, welche zusätzlichen Bedingungen $A(\varepsilon)$ (über die Regularität in ε hinaus) erfüllen muss, damit solche nichtregulären Bestandteile *ausgeschlossen* bleiben. Solche Bedingungen sind aufgestellt worden für Operatoren, deren Spektrum im ungestörten Zustand den einzigen Häufungspunkt $\lambda = 0$ (vollstetige Operatoren) bzw. die einzigen Häufungspunkte $\lambda = \pm\infty$ (Operatoren mit diskretem Spektrum) besitzen [13]. Für Operatoren $A(\varepsilon)$ mit diskretem Spektrum reicht es z.B. aus, dass der Teilraum, in welchem $A(\varepsilon)$ selbstadjungiert ist, *von ε unabhängig sei*. (Vgl. das in §2 angeführte Kriterium). Bei den in ε regulären Eigenwertproblemen gewöhnlicher oder partieller Differentialgleichungen deren Koeffizienten keine Singularitäten aufweisen (und deren Spektrum diskret ist), wird man also vor dem Auftreten nichtregulärer Bestandteile solange sicher sein, als der Störungsparameter nicht in den Randbedingungen auftritt.

6. Das kontinuierliche Spektrum. Die Störungstheorie für das kontinuierliche Spektrum ist bisher nicht mit derselben Vollständigkeit entwickelt worden, wie die des diskreten Spektrums. Es sind jedoch in typischen Fällen wichtige Resultate von Friedrichs [1], [2] gewonnen worden. Das Augenmerk ist dabei auf die *Spektraldarstellung* des Operators und ihre analytische Abhängigkeit von ε gerichtet. Diese neue Wendung sei an dem folgenden Problem beschrieben. Der ungestörte Operator A_0 sei beschränkt und besitze ein einfaches kontinuierliches Spektrum von a bis b. Wählt man als Hilbertschen Raum den Raum der in $a \leqq \lambda \leqq b$ erklärten Funktionen $u(\lambda)$ mit $\int_a^b |u(\lambda)|^2 d\lambda < \infty$, dann ist es keine wesentliche Beschränkung der Allgemeinheit vorauszusetzen, dass die Anwendung von A_0 in diesem Funktionenraum in der Multiplikation mit der unabhängigen Variabeln besteht, also $A_0 u = \lambda u(\lambda)$. Der gestörte Operator sei $A(\varepsilon) = A_0 + \varepsilon A_1$ mit $A_1 u = \int_a^b k(\lambda, \mu) u(\mu)\, d\mu$, $\overline{k(\lambda, \mu)} = k(\mu, \lambda)$. Die Spektraldarstellung von $A(\varepsilon)$ finden, heisst die Funktionen $u(\lambda)$ so auf Funktionen $v(\lambda)$ desselben Hilbertschen Raumes abbilden, $u = U(\varepsilon)v$, dass $B(\varepsilon)v = \lambda v$ wird mit $B(\varepsilon) = U^{-1}(\varepsilon)A(\varepsilon)U(\varepsilon)$. Erwünscht wäre der Nachweis, dass die Abbildung $u = U(\varepsilon)v$ durch einen unitären, in der Umgebung von $\varepsilon = 0$ regulären Operator erfolge: $U(\varepsilon) = 1 + \varepsilon U^{(1)} + \cdots$, $U^*(\varepsilon)U(\varepsilon) = U(\varepsilon)U^*(\varepsilon) = 1$. Er gelingt unter der Voraussetzung, dass $k(\lambda, \mu)$ einer Hölderbedingung genügt und dass $k(\lambda, \mu)$ am Rande des Quadrates $a \leqq \lambda, \mu \leqq b$ verschwindet. Die Voraussetzung des

Verschwindens von $k(\lambda, \mu)$ am Rande wird illustriert durch das Beispiel $k(\lambda, \mu) \equiv 1$, für welches der Satz nicht mehr richtig ist, weil der gestörte Operator für $\varepsilon \neq 0$ kein rein kontinuierliches Spektrum hat, vielmehr ein in ε nicht regulärer Punkteigenwert auftaucht [1]. Die Überlegungen lassen sich auf unendliches Grundintervall übertragen. In [2] werden weiter typische Beispiele studiert, in denen durch die Störung Punkteigenwerte entstehen oder vergehen und es werden so mathematisch exakte Modelle für den Auger-Effekt und die Theorie der natürlichen Linienbreite von Weisskopf und Wigner gewonnen. Es wird auf die Möglichkeit hingewiesen, dass, selbst wenn die Spektraldarstellung eines Operators nicht regulär analytisch in ε ist, die Lösung der zugehörigen zeitabhängigen Schrödingergleichung regulär analytisch von ε abhängen kann.

7. Asymptotische Entwicklungen. Für die Eigenwertaufgabe

$$A(\varepsilon)u \equiv -u'' + (q(x) + \varepsilon s(x)) = \lambda u, \qquad 0 < x < \infty$$

(und geeigneten Randbedingungen) mit diskretem Spektrum sind Sätze vom Typus $\lambda(\varepsilon) = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \cdots + \varepsilon^k\lambda^{(k)} + O(\varepsilon^{k+1})$ und entsprechend für die Eigenfunktionen bewiesen worden (Titchmarsh [15]). Man kann so auch Operatoren $A(\varepsilon)$ erfassen, die nicht regulär von ε abhängen; man kann sich auch auf $\varepsilon \geqq 0$ beschränken. Sehr viele in den Anwendungen wichtigen Operatoren hängen nicht regulär von ε ab. Vielfach darf man nicht einmal asymptotische Gesetze der Form $\lambda(\varepsilon) = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \cdots + \varepsilon^k\lambda^{(k)} + O(\varepsilon^{k+1})$ erwarten wie bei dem Lord Rayleighschen Beispiel: $-u'' + \varepsilon u'''' = \lambda u$, $0 \leqq x \leqq 1$ (Saite mit kleiner Steifheit). Allgemeine Sätze fehlen (vgl. allerdings den nächsten Abschnitt) und wären sehr erwünscht. Die Voraussetzungen in Kato [5] dürften gerade bei nichtregulärer Störung schwer nachprüfbar sein.

8. Stetige Abhängigkeit vom Störungsparameter. Wenn $A(\varepsilon)$ nur stetig von ε abhängt, ist es konsequent Folgen $A^{(n)} \to A$ zu betrachten. Die Operatoren A, $A^{(n)}$, $n = 1, 2, \cdots$ seien selbstadjungiert, aber nicht notwendig in einem gemeinsamen Definitionsbereich; in einem Teilraum $\mathfrak{D}$ ihrer Definitionsbereiche sei A wesentlich selbstadjungiert (d.h., dass zu jedem u aus dem Definitionsbereich von A eine Folge u_n aus $\mathfrak{D}$ gehört mit $\| u_n - u \| \to 0$, $\| Au_n - Au \| \to 0$). Unter dieser Voraussetzung sagen wir $A^{(n)}$ *strebt gegen* A, wenn $\lim_{n\to\infty} \| A^{(n)}u - Au \| = 0$ für jedes u aus $\mathfrak{D}$. Wir sagen $A^{(n)}$ *strebt gleichmässig gegen* A, wenn es eine Nullfolge ε_n gibt, sodass

$$\| A^{(n)}u - Au \| \leqq \varepsilon_n\{ \| u \| + \| Au \| \}$$

ist für alle u aus $\mathfrak{D}$. Bezeichnet $E_\lambda^{(n)}$ die Spektralschar von $A^{(n)}$ und E_λ die von A, dann gilt: 1) *Wenn $A^{(n)}$ gegen A strebt und λ_0 nicht Punkteigenwert von A ist, dann strebt $E_{\lambda_0}^{(n)}$ gegen E_{λ_0} für $n \to \infty$.* 2) *Wenn $A^{(n)}$ gleichmässig gegen A strebt und das Spektrum von A zum Intervall $\lambda_0 - p < \lambda < \lambda_0 + p$ leer ist, dann strebt $E_0^{(n)}$ gleichmässig gegen E_{λ_0}.* Die in [10] und [8] gegebenen Beweise für die zweite Behauptung sind nicht ausreichend, wie ich einer Mitteilung von Herrn Nagy verdanke; sie sind aber ausreichend unter der zusätzlichen Annahme, dass die

Operatoren A, $A^{(n)}$ beschränkt seien. Ein vollständiger Beweis auch für nichtbeschränkte Operatoren findet sich in der Dissertation von E. Heinz (*Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. erscheint demnächst).

9. Zwei und mehr Störungsparameter. Es sei angedeutet, welche funktionentheoretischen Fragen mit einer Störungstheorie von mehr Störungsparametern $\varepsilon_1, \varepsilon_2, \cdots$ verknüpft sind. Wir beschränken uns auf Operatoren in einem endlichdimensionalen Vektorraum. Die Eigenwerte eines solchen Operators sind die Wurzeln der Säkulardeterminante einer für reelle $\varepsilon_1, \varepsilon_2, \cdots$ Hermiteschen Matrix, deren Koeffizienten in der Umgebung von $\varepsilon_1 = \varepsilon_2 = \cdots = 0$ reguläre Potenzreihen von $\varepsilon_1, \varepsilon_2, \cdots$ sind. Das Beispiel der Matrix

$$A(\varepsilon_1, \varepsilon_2) = \begin{pmatrix} 1 + 2\varepsilon_1 & \varepsilon_1 + \varepsilon_2 \\ \varepsilon_1 + \varepsilon_2 & 1 + 2\varepsilon_2 \end{pmatrix}$$

mit den beiden Eigenwerten $\lambda = 1 + \varepsilon_1 + \varepsilon_2 \pm 2^{1/2}(\varepsilon_1^2 + \varepsilon_2^2)^{1/2}$ zeigt, dass die Eigenwerte einer solchen Matrix im allgemeinen nicht mehr reguläre Potenzreihen in einer Umgebung von $\varepsilon_1 = \varepsilon_2 = 0$ sind. Die Eigenwerte sind wieder die Wurzeln der algebraischen Gleichung

$$(-1)^h \det ((a_{ik}(\varepsilon) - \lambda\delta_{ik})) \equiv \lambda^h + p_1(\varepsilon_1, \varepsilon_2)\lambda^{h-1} + \cdots + p_h(\varepsilon_1, \varepsilon_2) = 0$$

wobei die Koeffizienten $p_1, \cdots, p_h$ reguläre Potenzreihen in einer Umgebung von $\varepsilon_1 = \varepsilon_2 = 0$ sind. Die Singularitäten der Wurzeln λ in der Umgebung von $\varepsilon_1 = \varepsilon_2 = 0$ können bekanntlich mannigfacher Art sein. Es entsteht die Frage, in wie weit die Singularitäten dadurch eingeschränkt werden, dass nicht eine beliebige Gleichung $\lambda^h + p_1(\varepsilon_1, \varepsilon_2)\lambda^{h-1} + \cdots + p_h(\varepsilon_1, \varepsilon_2) = 0$ mit regulären Koeffizienten vorliegt, sondern dass die linke Seite der Gleichung identisch ist mit einer Hermiteschen Säkulardeterminante, deren Koeffizienten regulär analytisch von $\varepsilon_1, \varepsilon_2$ abhängen. Jedenfalls sind alle Wurzeln der Gleichung dadurch reell für reelle $\varepsilon_1, \varepsilon_2$. Ist das die einzige Einschränkung? Im Falle *eines* Störungsparameters ε ist es die einzige. Wenn nämlich die Wurzeln einer Gleichung $\lambda^h + p_1(\varepsilon)\lambda^{h-1} + \cdots + p_h(\varepsilon) = 0$ mit regulären Koeffizienten für reelle ε reell sind, dann sind sie reguläre Potenzreihen von ε in einer Umgebung von $\varepsilon = 0$, wie wir in §1 festgestellt haben. Also

$$\lambda^h + p_1(\varepsilon)\lambda^{h-1} + \cdots + p_h(\varepsilon) \equiv \prod_{i=1}^{h} (\lambda - \lambda_i(\varepsilon)) \equiv (-1)^h \det (((\lambda_1(\varepsilon) - \lambda)\, \delta_{ik})).$$

Die linke Seite der Gleichung ist also die Säkulardeterminante der Hermiteschen Matrix $a_{ik}(\varepsilon) = \lambda_i(\varepsilon)\, \delta_{ik}$, deren Glieder $a_{ik}(\varepsilon)$ reguläre Potenzreihen von ε in einer Umgebung von $\varepsilon = 0$ sind. Bei zwei Störungsparametern lässt sich beweisen, dass jedes Polynom zweiten Grades $\lambda^2 + p(\varepsilon_1, \varepsilon_2) + q(\varepsilon_1, \varepsilon_2)$, dessen Nullstellen reell sind für reelle $\varepsilon_1, \varepsilon_2$ und dessen Koeffizienten p, q reguläre Potenzreihen von $\varepsilon_1, \varepsilon_2$ in einer Umgebung von $\varepsilon_1 = \varepsilon_2 = 0$ sind, in der Form

$$\lambda^2 + p\lambda + q \equiv \begin{vmatrix} a - \lambda & \alpha + i\beta \\ \alpha - i\beta & b - \lambda \end{vmatrix}$$

geschrieben werden kann, wo die a, b, α, β in der Umgebung von $\varepsilon_1 = \varepsilon_2 = 0$ reguläre Potenzreihen von ε_1, ε_2 mit reellen Koeffizienten sind. Das lässt sich so einsehen. Wenn diese Darstellung möglich ist, dann ist $a + b = -p$, $ab - \alpha^2 - \beta^2 = q$, also

$$(a - b)^2 + 4\alpha^2 + 4\beta^2 = p^2 - 4q.$$

Es muss sich also $p^2 - 4q = F(\varepsilon_1, \varepsilon_2)$ als Summe von 3 Quadraten von Funktionen schreiben lassen, die in der Umgebung von $\varepsilon_1 = \varepsilon_2 = 0$ konvergente Potenzreihen mit reellen Koeffizienten sind. Weil $F(\varepsilon_1, \varepsilon_2)$ in der Umgebung von $\varepsilon_1 = \varepsilon_2 = 0$ nirgends negativ ist, kann man F tatsächlig sogar als Summe von zwei Quadraten der verlangten Eigenschaft schreiben [14], [6]. Also

$$F = [G(\varepsilon_1, \varepsilon_2)]^2 + [H(\varepsilon_1, \varepsilon_2)]^2.$$

Man kann also $\beta \equiv 0$ (d.h., die Matrix reell) wählen. Es wird $\alpha = H/2$, $a = (G - p)/2$, $b = -(G + p)/2$ eine Lösung der Aufgabe. Ob ein Polynom h-ten Grades, $h > 2$, als Säkulardeterminante einer Hermiteschen Matrix (mit regulären Gliedern) geschrieben werden kann, wenn seine Wurzeln alle reell sind, weiss ich nicht. Die vorgetragene Überlegung zeigt aber, dass dies bereits für $h = 2$ nicht immer möglich ist im Falle von s Veränderlichen $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s$, $s > 2$. Eine Funktion $F(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s)$ die reell, nicht negativ und regulär in der Umgebung von $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_s = 0$ ist lässt sich nämlich im allgemeinen nicht als Summe von 3 Quadraten von Potenzreihen schreiben, die in einer Umgebung von $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_s = 0$ regulär sind und reelle Koeffizienten haben. Die Tatsache, dass die Eigenwerte $\lambda(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s)$ einer Hermiteschen Matrix Wurzeln einer Gleichung

$$\det((a_{ik} - \lambda\delta_{ik})) = 0$$

sind kann also stärkere Einschränkungen für die Möglichkeiten von Singularitäten der Funktionen $\lambda(\varepsilon_1, \cdots, \varepsilon_s)$ bei $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_s = 0$ bedeuten als die Einschränkung, die sich allein aus der Realität der Wurzeln ergeben würde und die für $s = 1$ Singularitäten sogar ausschliesst.

Bibliographie

1. K. Friedrichs, *Über die Spektralzerlegung eines Integraloperators*, Math. Ann. Bd. 115 (1938) S. 249–272.
2. ———, *On the perturbation of continuous spectra*, Communication on Applied Mathematics Bd. 1 (1948) S. 361–406.
3. E. Hölder, *Über die Vielfachheiten gestörter Eigenwerte*, Math. Ann. Bd. 113 (1936) S. 620–629.
4. T. Kato, *On the convergence of the perturbation method* I, Progress of Theoretical Physics Bd. 4 (1949) S. 514–523.
5. ———, *On the convergence of the perturbation method* II, Progress of Theoretical Physics Bd. 5 (1950) S. 95–101, 207–212.
6. H. Kneser, *Lösung der Aufgabe* 267, Jber. Deutschen Math. Verein. Bd. 50 (1940). S 29–31.
7. B. de Sz. Nagy, *Perturbations des transformations autoadjointes dans l'espace de Hilbert*, Comment. Math. Helv. Bd. 19 (1946) S. 347–366.

8. ———, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin, 1942, insb. Kap. 9, §3.
9. F. Rellich, *Störungstheorie der Spektralzerlegung* I, Math. Ann. Bd. 113 (1936) S. 600–619.
10. ———, *Störungstheorie der Spektralzerlegung* II, Math. Ann. Bd. 113 (1936) S. 677–685.
11. ———, *Störungstheorie der Spektralzerlegung* III, Math. Ann. Bd. 116 (1939) S. 555–570.
12. ———, *Störungstheorie der Spektralzerlegung* IV, Math. Ann. Bd. 117 (1940) S. 356–382.
13. ———, *Störungstheorie der Spektralzerlegung* V, Math. Ann. Bd. 118 (1942) S. 462–484.
14. ———, Aufgabe 267, Jber. Deutschen Math. Verein. Bd. 48 (1938).
15. E. C. Titchmarsh, *Some theorems on perturbation theory*, Proc. Roy. Soc. London. Ser. A. Bd. 200 (1949) S. 34–46 und A Bd. 201 (1950) S. 473–479.

University of Göttingen,
Göttingen, Germany.

MECHANICS

THE NATURE OF SOLUTIONS OF A RAYLEIGH TYPE FORCED VIBRATION EQUATION WITH LARGE COEFFICIENT OF DAMPING

PAUL BROCK

The Rayleigh equation $\ddot{y} + \epsilon(\dot{y}^3/3 - \dot{y}) + y = A \cos \omega t$ written in the phase plane is: $dv/d\xi = -(\epsilon^2/v)[v^3/3 - v + (\xi - (A/\epsilon) \cos \omega t)]$ where $y = \epsilon\xi$ and $v = \dot{y}$. If the fundemental cubic $v^3/3 - v + \xi = 0$ is approximated by a piecewise linear function, a Rayleigh type equation $\ddot{y} + (a\epsilon/b)\dot{y} + y = A \cos \omega t - c\epsilon/b$ is the result. The approximating equation is explicitly integrated. The solution has the characteristic relaxation time $O(1/\epsilon)$ and traverse time $O(\epsilon)$. It is shown that under initial conditions at $t = t_0$; $y = \epsilon(2/3 + (A/\epsilon) \cos \omega t_0)$ and $\dot{y} = 1$, periodic solutions will exist for amplitude and frequency values related by the equation:

$$ke^{-k\pi/\epsilon\omega} = \frac{1 - (A/\epsilon)\omega B^2(\epsilon\omega C/k - S)}{AC/\epsilon + 1/k + 4/3 - (AB^2/\epsilon)(C + \epsilon\omega S/k)}$$

where $C = \cos \omega t_0$, $S = \sin \omega t_0$, $B^2(1 + \epsilon^2\omega^2/k^2) = 1$, k is a constant. It is also shown that in the (v, ξ) phase plane, all solutions after a certain short initial time will remain in a framelike region bounded on the exterior by the lines:

$$v - 1 = k\left[\xi - \left(\frac{2}{3} + \frac{A}{\epsilon}\right)\right] \qquad v + 1 = k\left[\xi + \left(\frac{2}{3} + \frac{A}{\epsilon}\right)\right]$$

$$\xi = \frac{2}{3} + \frac{A}{\epsilon} \qquad \xi = -\left(\frac{2}{3} + \frac{A}{\epsilon}\right)$$

and on the interior by the lines:

$$v - 1 = k\left[\xi - \left(\frac{2}{3} - \frac{A}{\epsilon}\right)\right] \qquad v + 1 = k\left[\xi + \left(\frac{2}{3} - \frac{A}{\epsilon}\right)\right]$$

$$\xi = \frac{2}{3} - \frac{A}{\epsilon} \qquad \xi = -\left(\frac{2}{3} - \frac{A}{\epsilon}\right).$$

Qualitative verification of the results of this paper was obtained on the Reeves Electronic Analog Computer (REAC).

(This paper is an abstract of the author's doctoral dissertation, New York University, 1950.)

REEVES INSTRUMENT CORP.,
NEW YORK, N. Y., U. S. A.

ROTORS IN SPHERICAL POLYGONS

MICHAEL GOLDBERG

A rotor in a spherical polygon is defined as a convex closed spherical curve which remains tangent to all the sides (great circles) of a fixed spherical polygon during a complete rotation of the curve. Plane rotors in plane polygons and solid rotors in polyhedra have been studied by Meissner [Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich vol. 54 (1909) pp. 309–329; vol. 63 (1918) pp. 544–551], Fujiwara [Tôhoku Science Reports vol. 8 (1919) pp. 221–246, Goldberg [Amer. Math. Monthly vol. 55 (1948) pp. 393–402], and others. On the sphere, however, only the curve of constant width (a rotor in a lune, which is a two-sided spherical polygon) has been studied by Blaschke [Verhandlungen der Sächsischen Akademie der Wissenschaften vol. 67 (1915)] and Santaló [Bull. Amer. Math. Soc. vol. 50 (1944) pp. 528–534]. In the present work, a kinematic construction is given for rotors in regular spherical polygons of any number of sides and of any arc length. Conversely, any regular spherical polygon can be a rotor in a closed noncircular spherical curve. The plane rotors are limiting cases of the spherical rotors.

BUREAU OF ORDNANCE, NAVY DEPARTMENT,
WASHINGTON, D. C., U. S. A.

SOME NEW CASES OF INTEGRATION OF DIFFERENTIAL EQUATIONS OF EXTERIOR BALLISTICS BY QUADRATURES

E. LEIMANIS

The principal problem in exterior ballistics consists in the determination of forms of the resistance function of the air for which the differential equations of exterior ballistics are integrable by quadratures. Under the assumption of constant density and temperature of the air, the problem was completely solved by Drach (Ann. École Norm. ser. 3 vol. 37 (1920) pp. 1–94). Two particular integrable cases of variable density and constant temperature were given by Legendre (*Dissertation sur la question de balistique,* proposée par l'Acad. Roy. Sci. et Belles Lettres de Prusse pou le prix de 1782) and Cavalli (Mémoires de l'Art. Française vol. 2 (1923) pp. 421–455).

Assuming the density and the temperature of the air to be variable quantities it is shown in this paper that by the application of infinitesimal transformations all the known integrable cases, and in addition, new integrable cases of a more general character can be thus derived.

UNIVERSITY OF BRITISH COLUMBIA,
VANCOUVER, B. C., CANADA.

MECHANICS: ELASTICITY AND PLASTICITY

SU ALCUNE QUESTIONI DI ELASTICITÀ EREDITARIA

Dario Graffi

Numerose ricerche, anche recenti, dimostrano che alcune proprietà dei corpi possono rappresentarsi mediante la teoria della elasticità ereditaria di Boltzmann-Volterra. In questa teoria, mentre le altre equazioni della elasticità restano invariate, la legge di Hooke, per i corpi isotropi, si scrive:

$$D\alpha(t) = \frac{1+\sigma}{E}\beta(t) - \frac{\sigma}{E}I_1(\beta(t)) + \int_0^t g(t,\tau)\beta(\tau)\,d\tau + \int_0^t h(t,\tau)I_1(\beta(\tau))\,d\tau$$

dove $D\alpha(t)$, $\beta(t)$ sono, rispettivamente, le omografie vettoriali (tensori) delle deformazioni e degli sforzi calcolate ambedue all'istante t, $I_1(\beta(t))$ l'invariante lineare delle predette omografie, E e σ i moduli di Young e Poisson, $g(t, \tau)$, $h(t, \tau)$, due funzioni di t e della variabile τ che verranno chiamate funzioni di Volterra. Supposte assegnate le forze agenti su un corpo e trascurabili i termini di inerzia, ricerco i casi in cui le tensioni del corpo sono le stesse come nel caso non ereditario (funzioni di Volterra nulle) e trovo che ciò accade quando $I_1(\beta)$ è funzione lineare delle coordinate, (condizione questa soddisfatta nei più importanti problemi di elasticità) oppure quando fra le funzioni di Volterra passa una relazione molto semplice. Anzi, in questo ultimo caso, le tensioni rimangono le stesse anche se in certe parti del contorno del corpo, dove gli spostamenti sono nulli, non si conoscono le forze. Termino con alcune osservazioni sui sistemi piani e sul caso in cui, invece delle forze, sono dati gli spostamenti alla superfice del corpo.

University of Bologna,
Bologna, Italy.

ON THE PRESSING OF A RIGID STAMP INTO AN ELASTIC-PLASTIC BODY IN PLANE STRAIN

H. J. Greenberg, D. C. Drucker, and W. Prager

A rigid stamp is pressed with increasing force P into an incompressible elastic-plastic (Prandtl-Reuss) body under conditions of plane strain. Symmetry of body and stamp with respect to the line of entry is assumed. For sufficiently small P there is associated with each value of P a unique displacement of the stamp into the body; for $P = P_{cr}$ the stamp will move into the body at a constant velocity v with no increase in P (since there are no time effects, the value of v plays no role). The problem is to determine the critical force P_{cr} which must be applied to the stamp in order to initiate this plastic flow. (Note: for plástic flow it is not necessary that the entire body be plastic—there will, in general, be elastic regions and plastic regions.) No general analytical methods for a solution of this problem based on tracing the development of the elastic-plastic stress distribution exist. In the present paper extremum principles are given which make it possible to obtain upper and lower bounds for P_{cr} directly. Assuming either perfectly rough or perfectly lubricated contact between stamp and body, the velocity distribution at plastic flow over the surface of contact is known. In addition, over the remaining surface, a combination of zero tractions and zero velocities is assumed. Under these boundary conditions it is shown that P_{cr} is greater than or equal to the force which would be computed from any assumed stress distribution satisfying equilibrium conditions, the Mises yield condition (as an inequality or equality), and whatever stress boundary conditions are imposed. At the same time it is shown that $P_{cr} \leqq (k/v) \int_v \gamma \, dv$ where k is the yield stress in simple shear, v is the total volume of the body, and γ is the maximum shear rate corresponding to any assumed velocity distribution which is continuous and satisfies the boundary conditions imposed on velocities and the condition of incompressibility. (This minimum principle is extended to include the important case of discontinuous velocity distributions; an additional term is then required.) Direct application of these principles furnishes upper and lower bounds for the load P_{cr}. The relation to the Saint-Venant Mises theory of plastic flow is discussed and examples are considered.

Carnegie Institute of Technology,
Pittsburgh, Pa., U. S. A.
Brown University,
Providence, R. I., U. S. A.

REMARKS ON THE DIRECT INTEGRATION OF THE EQUATIONS OF ELASTICITY

George H. Handelman and Albert E. Heins

It is shown that with the aid of the vector form of Green's Theorem and the use of the Green's tensor (free space), appropriate to the linear equations of elasticity, (cf. A. E. H. Love, *Mathematical theory of elasticity*, Dover Publications, 1944, p. 245), it is possible to formulate some of the boundary value problems of elasticity as a system of linear, nonhomogeneous integral equations. The method is applied to the Kirsch problem (G. Krisch, Zeit V. D. I. vol. 42 (1898) pp. 797–807) of an infinite elastic medium with a stress-free circular hole, subject to tensile stress at infinity perpendicular to the axis of the cylinder. Other types of conditions at infinity may be treated by the same method.

This class of problems is treated as strictly three-dimensional in character and it is shown how the dependence of the displacements along the generators of the cylinder can be computed. For the case treated, the Poisson integral theorem is an important tool. The method of formulation applies to any reasonably shaped body or cavity, but it is not clear yet which class of geometries result in integral equations which can be effectively solved.

CARNEGIE INSTITUTE OF TECHNOLOGY,
PITTSBURGH, PA., U. S. A.

THE METHOD OF CHARACTERISTICS APPLIED TO PROBLEMS OF STEADY MOTION IN PLANE PLASTIC STRESS

PHILIP G. HODGE, JR.

The stress and strain distribution in a thin sheet which is strained plastically in its plane, under conditions of plane stress is discussed. It is shown that three types of problems may be distinguished, as follows. (1) In the case of steady motion where the thickness is known, the stress distribution may be found by solving three equations for the three stress components, provided that the boundary conditions in stresses are sufficient to determine the problem. (2) For general steady motion problems, it is necessary to solve six equations for three stress components, two velocity components, and the thickness. If initial isotropy is assumed, these equations can be reduced to a system of five first order, quasi-linear partial differential equations in as many unknowns. The characteristics of this set of equations are found to consist of two families of double characteristics and one distinct family. Conditions which must be valid along the characteristics are obtained. (3) For the general case of nonsteady motion, it is necessary to use a step-by-step procedure in time.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIF., U. S. A.

THE PROBLEM OF ATLANTIS

W. S. JARDETZKY

The geophysical reconstructions of the earth's face in the past periods show that there was a land connecting the New and Old World in the region of the Atlantic ocean. The existence of the mid-Atlantic ridge reveals a possible form of this connection if we assume that the upper parts of the ridge and its offshoots were once above the sea level. This assumption is conform to the viewpoint of F. B. Taylor and to the writer's theory of formation of continents. The process of sinking of the mid-Atlantic ridge leads to many interesting problems of mechanics, since it may be formulated in different ways. The most simple approach is given by considering an elastic strip (sial layer between America, Europe, and Africa) supporting a load (the ridge and shelves), whose upper bound is represented by a cosine-line, and subjected to a pressure along its bottom face exerted by the underlying sima layer. The equation for the deflecting curve of an elemental strip is known from the theory of plates. Then the vertical displacement of a particle may be given in terms of the distance x from a border of the strip, the length l of the strip, the bottom pressure p, the height of the ridge h, the density ρ, and the thickness d of the strip. The case in which the ridge could sink is given by the condition $p < \rho g(h/2 + d)$. The maximum displacement at the midpoint varies from 0.6 to 2 km, if we assume that the bottom of the continents lies 5 to 15 km deeper than the bottom face of the layer stretched by continents. We can imagine that this layer having properties of a plastic body was stretched by continents moving away. Since it became thinner, the pressure of the substratum has decreased. If the substratum was also plastic, a certain bending of the sialic layer was also possible. The strip of the uppermost solid layer could sink, if the pressure of the substratum would correspond to the isostatic state or would be less. But the question whether the upper face could be once really a land is connected with the initial conditions in the region of the Atlantic Ocean.

MANHATTAN COLLEGE,
NEW YORK, N. Y., U. S. A.

THE ANALYSIS OF PLASTIC FLOW IN PLANE STRAIN WITH LARGE STRAINS

E. H. LEE

An ideally plastic material is considered, and strains of the order of elastic strains are neglected. In regions of plastic flow the equilibrium equations and the flow limit determine a pair of first order hyperbolic equations for two stress variables. The stress strain—increment relation determines another pair of equations for velocities of flow which contain stress variables, and have the same characteristics as the stress equations. Solutions of these equations must be combined with the rigid body motion of regions subjected to strains of elastic order only.

Solutions of the stress equations have in the past dominated the literature. To each there correspond infinities of velocity solutions, with velocity boundary conditions limited only to such restrictions as being monotonically increasing. A combined stress and velocity field is called a complete solution, and this, not a stress field only, is required to represent a particular physical problem.

A study of types of boundary value problems shows that boundary values specified in terms of stresses only, leading to statically determinate solutions of the stress equations, do not arise in practice. Examples are given to demonstrate this. Moreover, the free-boundary problem of locating rigid regions in the field would prevent direct solution of the stress equations even in the case of boundary values expressed entirely in terms of stresses.

Examples are given to emphasize the importance of checking the compatibility of the stress and strain-rate tensors before even a complete solution is physically acceptable.

The work of Christianovitch on the solution of the stress equations for a closed domain is discussed in terms of the above concepts. It is shown that the concept of lines of rupture, although of mathematical interest in the theory of differential equations, has no physical significance in plastic flow.

As an example of a complete solution of a problem of technological interest, the deformation of a notched bar pulled in tension is presented.

The need for studying the system of four partial differential equations with mixed boundary values in stresses and velocities lends considerably greater mathematical interest to the theory of plastic flow in plane strain, compared with the statically determinate case which has been thoroughly treated. It is hoped that this will lead to more intensive work in this field by those interested in differential equations.

Brown University,
Providence, R. I., U. S. A.

SOME REMARKS ON THE METHOD OF SOLVING TWO-DIMENSIONAL ELASTIC PROBLEMS

Sigeiti Moriguti

It is well known that the general solution of two-dimensional elastic problems can be expressed by means of two analytic functions of a complex variable $z = x + iy$, where x and y are the Cartesian coordinates. In the present note the nature of these functions is first discussed from the viewpoint of the theory of functions of complex variables. In particular, the analytic continuation of the functions beyond a free boundary is made a subject of the investigation by applying the principle of image. This is found to be very useful in particular problems, since simple problems can be solved in a closed form at once, while the solutions of more complicated problems are expressible in such a legitimate form that there is no risk of labor being wasted in unfruitful calculation.

Next, the method of perturbation is taken up for carrying out the calculation in some problems. It differs little from the method of alternate cycle of approximation, as was used by R. J. C. Howland. But it seems that the former method has the advantage as compared with the latter, namely: 1. The solution is obtained in a form of convergent power series of a parameter, so that graphical or other procedures of interpolation can be dispensed with. 2. A number of terms to be taken into account in the process of numerical calculation is automatically arranged in a regular manner.

As an example, the present method is applied to find the stress distribution in a pulled strip with an infinite row of circular holes; the results are simply interpretable. Other examples worked out by Makoto Isida are added: a strip perforated by an elliptic hole, a strip with circular arc notches on both sides, etc.

Lastly, the author has the view of extending the investigation to the case when two boundaries are very near to each other. With this aim, some results of preliminary considerations are presented: 1. An asymptotic expression of the stress concentration at the periphery of a circle near the straight boundary of semi-infinite plate. 2. Integral equations for the problem.

UNIVERSITY OF TOKYO,
TOKYO, JAPAN.

ON THE GEOMETRY OF TWO-DIMENSIONAL ELASTIC STRESS SYSTEMS

A. W. SÁENZ AND P. F. NEMÉNYI

The problem of plane elastic strain is discussed from the standpoint of the geometry of the principal stress trajectories. One of the present authors has proved in 1933 that, in the absence of body forces, any isometric orthogonal net of plane curves can be interpreted as the system of orthogonal stress trajectories of a five-parameter family of two-dimensional elastic stress systems. Such stress systems we call "isometric". Wegner has shown in 1934 that, again assuming absence of body forces, any two-dimensional elastic stress system can be obtained through superposition of two isometric stress systems, one of which can be of the simple type called "harmonic".

In the present paper all deductions are based on the more general assumption that the field of body forces is laminar and solenoidal. Starting from the representation of the Airy stress function in the known form $R\{\bar{z}G(z) + H(z)\}$ the relation

$$\tan 2\vartheta = -\frac{I\{\bar{z}G''(z) + H''(z)\}}{R\{\bar{z}G''(z) + H''(z)\}}$$

is obtained, where ϑ is the direction angle of the net, and the other notations are those usual in the theory of complex functions. From this the generalized Wegner theorem is deduced by aid of elementary properties of analytic functions.

The deductions leading to the formula for $\tan 2\vartheta$ yield as a corollary the result that Prager's representation of the stress deviator

$$\tau e^{-2i\vartheta} = \bar{z}g(z) + h(z)$$

remains valid if a body force field of the type mentioned is present, and that therefore the first invariant of the stress tensor is determined to within one additive constant by τ and ϑ.

Finally we have investigated how far the knowledge of ϑ alone determines the stress field in the non-isometric case, and we have proved that the family of two-dimensional stress systems compatible with a given system of trajectories contains from two to five independent parameters. The proof starts from the above representation of the stress deviator. Supposing that the direction field ϑ is defined by a specific system σ_{x0}, σ_{y0}, τ_{xy0}, such that for it

$$\tau_0 e^{-2i\vartheta} = \bar{z} g_0(z) + h_0(z),$$

one can construct two analytic functions of the two complex variables z, z':

$$\text{(a)} \qquad \begin{aligned} D_0(z, z') &\equiv z' g_0(z) + h_0(z) + z\bar{g}_0(z') + \bar{h}_0(z') \\ N_0(z, z') &\equiv z' g_0(z) + h_0(z) - z\bar{g}_0(z') - \bar{h}_0(z'). \end{aligned}$$

From (a) and from the fact that $D_0(z, \bar{z})N(z, \bar{z}) = N_0(z, \bar{z})D(z, \bar{z})$, D and N being the corresponding functions for any other stress system having the same set of stress trajectories, it follows on the basis of a known theorem in the theory of functions of several complex variables that

$$\text{(b)} \qquad D_0(z, z')N(z, z') = N_0(z, z')D(z, z')$$

for z and z' in suitable domains. The proof is completed by the aid of analytic continuation.

NAVAL RESEARCH LABORATORY,
WASHINGTON, D. C., U. S. A.
UNIVERSITY OF MARYLAND,
COLLEGE PARK, MD., U. S. A.

A SIMPLE CASE OF "INCOMPATIBILITY" BETWEEN LINEAR ELASTICITY AND THE THEORY OF FINITE DEFORMATIONS

ANTONIO SIGNORINI

The most general theory of elasticity has been investigated in several directions by myself and my co-workers [for a systematic exposition of my inquiries see *Trasformazioni termoelastiche finite*, Annali di Matematica vol. 22 (1943) pp. 33–143, vol. 30 (1949) pp. 1–72], but my communication will deal only with the anomalies with respect to the classical theory. Consider a free elastic body which is held strained by given external forces. For the components of the elastic displacement the classical theory presupposes tacitly but systematically the existence of series expansions whose first terms alone are to be retained. In the complete theory a set S_2 of differential equations is written for the second terms. Many years ago I pointed out that S_2 must satisfy integrability conditions which are *generally* propitious to the classical theory owing to the fact that they can be satisfied by a suitable choice of the rigid rotation R accompanying the strain, and it becomes therefore possible *to eliminate a traditional indetermination*. There are, however, cases of "incompatibility" in which the integrability conditions of S_2 cannot be satisfied by any choice of R, and the series expansions, the existence of which is assumed by the linear theory, *do not exist*, independently of any question of convergence. I am now taking up the subject again because recently I did find a case of incompatibility considerably more simple and striking than those which I had previously found. It is the case of a rectangular plate simultaneously bent by: 1.) couples applied to one pair of opposite edges and having their axes parallel to those edges; and 2.) couples applied to the other pair of edges but having their axes normal to the plate [see Rendiconti dei Lincei vol. 8 (1950) pp. 276–281].

UNIVERSITY OF ROME,
ROME, ITALY.

A NEW APPROACH TO CANTILEVER-STRUT PROBLEMS

CHARLES P. WELLS AND RICHARD A. BETH

Consider an originally straight uniform rod, built in or clamped at one end and acted upon at the other end by a force r, making an angle θ_0 with the tangent to the rod at the clamped end. The classical approach to determining large deflections of such rods depends on the manipulation of certain elliptic integrals and gives the coordinates x, y of the deflected end of the rod only implicitly in terms of r and θ_0. This method is further complicated by the manner in which the parameter of the elliptic integrals enters the equations. The authors have developed a method, believed to be new, in which the following four quantities are determined explicitly in terms of r and θ_0: the coordinates x and y of the free end of the rod, the tangent angle θ_1 at the free end, and the elastic energy V of the bent rod. This procedure is based on the energy relations $\partial V/\partial r = -r\partial y/\partial r$ for constant θ_0 and $\partial V/\partial \theta_0 = -r\partial y/\partial \theta_0 - rx$ for constant r. Making use of the dimensionless quantities $X = x/L$, $Y = y/L$, $W = V/rL$, $\rho^2 = rL^2/B$ where $B = EI$, we define $S(\rho, \alpha) = \rho(1 + Y + W)$ with $\alpha = \cos\theta_0$ and show that $S(\rho, \alpha)$ satisfies the equation $\partial S/\partial \rho = 1 + \alpha - (\beta^2/2)(\partial S/\partial \alpha)^2$, where $\beta = \sin\theta_0$. Further it is shown that $X = (\beta/\rho)(\partial S/\partial \alpha)$, $Y = 1/2(S/\rho + \partial S/\partial \rho) - 1$, $\cos\theta_1 = \partial S/\partial \rho - 1$ and $W = 1/2(S/\rho - \partial S/\partial \rho)$. Solutions of the differential equation for $S(\rho, \alpha)$ which satisfy the boundary conditions $S(0, \alpha) = 0$ and $S(\rho, -1) = 0$ are found formally in the form $S = \sum_0^\infty A_n(\alpha)\rho^{2n+1}$. The convergence of the series is not determined, but for small values of ρ the results agree well with those obtained from elliptic integrals. It is conjectured that the series converge for $|\rho| < \pi/2$, i.e., for values of $|\rho|$ less than the Euler critical load. For large values of ρ, a solution is obtained in the form $S = \sum_0^\infty P_n e^{-2n\rho}$ where $P_n = P_n(\rho, z)$, $z = (1 - \sin(\theta_0/2))/(1 + \sin(\theta_0/2))$.

MICHIGAN STATE COLLEGE,
EAST LANSING, MICH., U. S. A.
WESTERN RESERVE UNIVERSITY,
CLEVELAND, OHIO, U. S. A.

MECHANICS: HYDRODYNAMICS

ON THE STABILITY OF THE BOUNDARY LAYER OVER A BODY OF REVOLUTION

RICHARD H. BATTIN

Mangler has shown that the steady state laminar boundary layer problem for a body of revolution may be transformed to that of the two-dimensional boundary layer. Thus, to each axially symmetric body one may associate a two-dimensional body having an appropriate pressure distribution. Then these two problems will be represented mathematically by the same system of equations with the same boundary conditions. In the present paper the equations for small disturbances in the boundary layer over a body of revolution are examined and they are found to be identical with those for the two-dimensional problem, within the limitation of usual approximations. Therefore, the local stability characteristics of the boundary layer for a body of revolution can be investigated by the existing work of Lees and Lin. The comparison of the two problems must be made for the same Reynolds number with the boundary layer thickness as the appropriate reference length. A complete correspondence between the two- and three-dimensional problems is possible when the downstream steady state pressure gradient is zero. This condition is satisfied in the case of a cone and a flat plate in a supersonic stream.

In the investigation of the downstream development of a disturbance of given time frequency, the axially symmetric case is certainly different from the two-dimensional one because the boundary layer thickness varies with downstream distance in different manners. However, it is possible to compare the two problems, from this standpoint, for bodies with zero downstream pressure gradient. In carrying through this analysis two methods of representing a disturbance are employed. In one case the disturbance amplitude is considered to be time dependent while in the other the amplitude is regarded as a function of downstream distance. The second representation must be indistinguishable from the first to an observer moving downstream with a "group" velocity appropriate for the transient disturbance. An analysis of this group velocity is made for disturbances of given time frequency and of given wave length.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASS., U. S. A.

SUR L'ÉCOULEMENT D'UN FLUIDE VISQUEUX AUTOUR D'UN OBSTACLE

RATIP BERKER

I. Considérons l'écoulement permanent d'un fluide visqueux incompressible autour d'un obstacle fixe de dimensions finies, le mouvement au loin étant un courant uniforme de vitesse U_0 parallèle à Ox par exemple. Je dirai qu'un mouvement est régulier d'ordre k à l'infini si les produits $r^k(u - U_0)$, $r^k v$, $r^k w$ $(r = (x^2 + y^2 + z^2)^{1/2}$ restent bornés en valeur absolue et s'il existe des conditions analogues pour les dérivées partielles des deux premiers ordres de u, v, w. Je démontre que si l'écoulement considéré ci-dessus est régulier d'ordre k à l'infini, il faut que $k \leqq 2$, il n'existe pas d'écoulement pour lequel $k > 2$.

II. Je montre qu'en tout point d'une paroi en contact avec un fluide visqueux incompressible la composante normale de l'effort est égale à p et la composante tangentielle est égale à 2μ fois le vecteur tourbillon tourné de 90° dans le plan tangent.

III. Si on calcule les efforts exercés par le fluide sur l'obstacle dans l'écoulement considéré en I, on trouve, en utilisant entre autres les résultats énoncés en II, que la résultante de ces efforts est nulle si $k > 2$; mais dans ce cas d'après I il n'existe pas d'écoulement. Ainsi se trouve éclairci la question du paradoxe d'Alembert qu'on a cru avancer pour l'écoulement considéré en I.

TECHNICAL UNIVERSITY OF ISTANBUL,
ISTANBUL, TURKEY.

ON SOME CAVITATIONAL FLOWS IN LUBRICATION

A. CHARNES AND E. SAIBEL

The theory of lubrication as developed by Reynolds [O. Reynolds, Philos. Trans. Roy. Soc. London Ser. A. vol. 177 (1886) p. 157] may be considered a "hydraulic" approximation to viscous incompressible Stokes flow, which results mathematically in a boundary value problem of the first kind for a second order elliptic equation in the pressure averaged across the film thickness.

Vogelpohl [G. Vogelpohl, Zeitschrift für Angewandte Mathematik und Mechanik vol. 17 (1937) p. 362] and others note that the set of boundary conditions imposed by Reynolds leads to results which do not agree adequately with experiment. The difficulty appears to be that there are too many physical conditions to be satisfied than can be imposed as boundary conditions. Gumbel has suggested three possible sets of boundary conditions, solutions for all of which have appeared by Vogelpohl and others only for infinitely wide bearings, i.e., no side leakage. (We shall shortly report on finite bearings under a variety of conditions.) At present the experimental evidence [G. Vogelpohl, loc. cit.] favors the conditions which we note to be a limiting case of flow with free surfaces. Wannier [G. Wannier, Quarterly of Applied Mathematics vol. 8 (1950) p. 1], too, following Duffing and H. Reissner, solves the infinite cylindrical full bearing case in Stokes flow, and emphasizes the need for a theory which will embrace cavitation in a coherent manner.

We consider Stokes two-dimensional flow adjacent to moving rigid boundaries connected by free boundaries (arising physically from cavitation) and a simply connected fluid region. The boundary conditions are $p = 0$, vanishing viscous stress on the free surface, and on the moving rigid boundaries adherence of the fluid. Function-theoretic methods are indicated since pressure and vorticity are conjugate functions and since the problem is a mixed boundary value problem leading to the biharmonic equation for the stream function. We show by properties of analytic functions that no useful solutions are possible satisfying all these boundary conditions since these imply that the pressure-vorticity function is zero, i.e., the stream function is harmonic.

By use of Muscehlisvili's technique in a semi-inverse manner (prescribing pressure distribution, for example), a class of flows corresponding to flows about a finite slider may be obtained. We shall report on the computational results elsewhere.

CARNEGIE INSTITUTE OF TECHNOLOGY,
PITTSBURGH, PA., U. S. A.

EXACT SOLUTIONS FOR FLOW OF A PERFECT GAS IN A TWO-DIMENSIONAL LAVAL NOZZLE

T. M. CHERRY

Our concern is with the steady irrotational isentropic flow of a perfect gas in two dimensions, investigated by the hodograph method. The conditions defining a "Laval-type" flow are: (i) there is a straight streamline Ox, $\theta = 0$, about which the flow-field is symmetrical; on this axis, the velocity increases steadily with x, from subsonic to supersonic values, (ii) the velocity coordinates τ, θ are analytic functions of x, y throughout the field, so there are no shocks. The essential condition, thence deduced, for the Legendre potential Ω is that for supersonic τ it be three-valued; the "central" branch joins the other two along branch lines issuing from the axial sonic point $\tau = \tau_s$, $\theta = 0$.

To obtain such an Ω we need to guess a likely form and then verify that it is as desired. The guess is guided by an analogy between the hodograph series $\Omega = \sum A_\nu \chi_\nu(\tau) e^{i\nu\theta}$ and Kapteyn series; $\chi_\nu(\tau)$ is a hypergeometric function. $\sum_{-\infty}^{\infty} J_n(nt) \cos n\theta$ has, for $t < 1$, the sum $(1 - t\cos\phi)^{-1}$, where $\phi - t\sin\phi = \theta$, and the sum-function is, for $t > 1$, three-valued as desired for Ω. We can choose t as a function of τ so that $\chi_\nu(\tau)$ and $J_\nu(\nu t)$ are, apart from trivial factors, asymptotically equal for $\nu \sim \infty$ and all τ; and $\chi_\nu(\tau)$ is thence expressible in terms of $J_\nu(\nu t)$. Hence a hodograph series becomes a double Kapteyn series; the appropriate guess for Ω is evident; and its analytic continuation can be found via manipulations of Kapteyn series and similar integrals.

There are four formulae covering the different branches of Ω, e.g. for $\tau < \tau_s$,

$$\Omega = \sum_1^\infty \left\{\frac{(n/e)^n}{\delta^n\Gamma(n+1)} + \frac{\delta^n h_n \Gamma(n)}{(n/e)^n}\right\}\chi_n(\tau) \cos n\theta, \qquad (\delta, h_n \text{ certain constants});$$

the branches for $\tau > \tau_s$ are given by integrals, not series. Thence x, y are found by elementary processes. An approximate solution for the neighbourhood of the axial sonic point gives the forms

$$\tau - \tau_s = A\{y^2 + C(x - x_s)\}, \qquad \theta = By\{y^2 + 3C(x - x_s)\}.$$

Other Laval-type flows can be obtained from this by superposition.

A full account is to be published in Proc. Roy. Soc. London.

UNIVERSITY OF MELBOURNE,
MELBOURNE, AUSTRALIA.

A MODIFIED EQUATION OF DIFFUSION

R. V. CHURCHILL

The classical partial differential equation of diffusion implies that a disturbance of the concentration in one part of a body has an immediate effect upon the concentration at other points. In this sense the equation is not physically realistic. The defect arises from the fundamental postulate that the flux of the diffusing material in any direction is proportional to the directional derivative of the concentration in that direction. If in place of that postulate we assume that the driving force that acts on the diffusing substance is proportional to the gradient of the concentration and that this force is accompanied by a high resistance proportional to the flux, then the resulting modified equation of diffusion is of hyperbolic type and it displays a high but finite velocity of propagation of disturbances. Its solutions under simple boundary conditions are compared to the corresponding solutions of the classical equations and to experimental results. The modified equation is more difficult to solve than the classical one. The presence of the additional coefficient in the modified equation, however, gives an advantage in the comparison of theoretical and experimental results. The modification also applies to the heat equation.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

POHLHAUSEN'S METHOD FOR THREE-DIMENSIONAL BOUNDARY LAYERS

J. C. COOKE

Wild [J. M. Wild, Journal of the Aeronautical Sciences vol. 16 (1949) p. 41] has extended to three dimensions the well-known Karman-Pohlhausen method for the laminar boundary layer flow over a fixed obstacle, and used the method for an infinite yawed elliptic cylinder in a stream. He showed that the boundary layer equations in this case reduce to the usual two-dimensional ones which can be solved in many ways including Pohlhausen's, together with a third equation which he solves by an extension of Pohlhausen's method, too, assuming a quartic velocity distribution as is usual. The labor involved is fairly heavy, and so it would seem advisable to test the method before pursuing it further. In this paper this is done for the case of an infinite yawed cylinder when the velocity outside the boundary layer over the surface normal to the generators is of the form $U = cx^m$. The solution is known in this case [Proc. Cambridge Philos. Soc. vol. 46 (1950) p. 645].

A table of the skin friction, displacement thickness, and momentum thickness is given for various values of β $[=2m/(m + 1)]$, and the agreement is found to be fairly good for $\beta > 0$ (accelerated flow) but not so good for $\beta < 0$ (retarded flow); it is better, however, than the method gives in the two-dimensional case, comparisons for this also being given in the paper.

The method involves first solving the two-dimensional problem by Pohlhausen's method. This can be avoided in the problem discussed, as the two-dimensional problem has a known solution [V. M. Faulkner and S. W. Skan, A.R.C. Reports and Memoranda no. 1314 (1930); The Philosophical Magazine Ser. 7 vol. 12 (1931) p. 865; D. R. Hartee, Proc. Cambridge Philos. Soc. vol. 33 (1937) p. 223]. Using this solution and Pohlhausen's method for the third dimension involves the numerical solution of an integral equation, which has been done roughly in this paper for the case $\beta = -0.1$ only; the agreement is improved in some respects and made worse in others. It is impossible to bring all three quantities into close agreement, since they are simple functions of each other, and improving one may make another worse. In fact the Pohlhausen quartic is not good enough in these cases.

UNIVERSITY OF MALAYA,
SINGAPORE, MALAYA.

SIMPLE WAVES IN TWO-DIMENSIONAL COMPRESSIBLE FLOW

ABOLGHASSEM GHAFFARI

This paper is concerned with a general and exact solution of the two-dimensional equation of motion of a compressible fluid, which is steady, irrotational, and isentropic. The solution arrived at is the generalization of the Prandtl-Meyer "wedge solution" to include an expansion around any smooth (or abrupt) convex boundary. The solution obtained is the solution of equations for the most general type of motion in which the Cartesian components of velocity are connected by some functional relation throughout the field of flow. This type of motion has been discussed by Lees [F. Lees, *On a case of steady flow of a gas in two dimensions*, Proc. Cambridge Philos. Soc. vol. 22 (1924)] and Bateman [H. Bateman, *The irrotational motion of a compressible inviscid fluid*, Proc. Nat. Acad. Sci. U.S.A. vol. 16 (1930)], but neither investigation brings out the relation of the flow to the wedge-solution of Prandtl-Meyer [G. I. Taylor and W. MacColl, *Aerodynamic theory*, ed. by Durand, vol. 3, Div. H, chapter 4, 1935]. Taking into account the conditions of continuity, zero rotation, and Bernoulli's equation, together with the fundamental assumption, which implies that the components of fluid velocity are each a function of fluid speed only, it is found that the motion is necessarily *supersonic* everywhere. It is shown that the lines of equal speed, pressure (isobars), and density are all *straight lines*, although they are not necessarily concurrent as in the original Prandtl-Meyer theory, and further that the component of fluid velocity perpendicular to the isobar is equal to the local velocity of sound. The extended Prandtl-Meyer solution for the flow around an arbitrary fixed boundary is obtained analytically. The complete investigation may be found in the author's paper *A simplified theory of "simple waves"*, The Aeronautical Quarterly vol. 1 (1949) pp. 187–194, and *The hodograph method in gas dynamics*, Tehran University publications no. 85, 1950.

TEHRAN UNIVERSITY,
TEHRAN, ISRAEL.

ON DIFFUSION BY DISCONTINUOUS MOVEMENTS, AND ON THE TELEGRAPH EQUATION

SYDNEY GOLDSTEIN

At time $t = 0$ a large number of noninteracting particles start from an origin and move with a uniform velocity v along a straight line for an interval of time τ. To begin with, half move in each direction. Thereafter, and at the end of each successive interval of time τ, each particle starts a new partial path; it still moves with speed v, and there is a probability p that it will continue to move in the same direction as in its previous path, and a probability $q(= 1 - p)$ that the direction of its velocity will be reversed, so the directions in any two consecutive intervals are correlated with a correlation coefficient $c = p - q$. The partial correlations for nonconsecutive intervals are zero. The difference equation is found for the fraction $\gamma(n, \nu)$ of the number of particles at a distance $y = \nu v \tau$ from the origin after a time $t = n\tau$; it is shown how $\gamma(n, \nu)$ may be computed; asymptotic formulae for large n are found, both for a fixed value of p/q and for a fixed value of nq/p. The limiting density distribution (and the limiting characteristic function) are found when $n \to \infty$, $\tau \to 0$, with $n\tau = t$, $\nu v \tau = y$, and in the limiting operation $c = 1 - \tau/A$, with A constant, so that $c \to 1$, and the speed v is kept constant; the limiting form of the difference equation is the telegraph equation, with $v^2 = (LC)^{-1}$, $A = L/R$, where L, C, and R are the self-inductance, capacitance, and resistance per unit length; and the limiting density distribution is the solution of this equation for an instantaneous source. If $p + q \neq 1$, and there is, at the end of each interval of time τ, a nonzero probability, $1 - p - q$, that a particle will escape from the system, then the same limiting operation, with $\tau/(1 - p - q) = G$, G constant, leads to the telegraph equation with leakage, the leakage resistance being G/C. The solution of the telegraph equation is further considered (and in particular is given in terms of Lommel's function of two variables) when one end of a long cable is held at a constant potential for $t \geqq 0$.

INSTITUTE OF TECHNOLOGY,
HAIFA, ISRAEL.

SOME GEOMETRICAL PROPERTIES OF PLANE FLOWS

A. G. HANSEN AND M. H. MARTIN

The geometric configuration consisting of five straight lines tangent to the *stream line* (Ψ = const.), *isobar* (p = const.), *isocline* (θ = const.), *isovel* (q = const.), and *isopycnic* (ρ = const.) at a point P in the physical plane has a number of interesting properties for a wide class of fluids (which includes polytropic gases and incompressible fluids). The configuration changes radically with the Mach number M at P according as $M \leqq 1$ or $\geqq 1$. For $M > 1$ the Mach directions at P have an interesting separation property, in view of the formula $\cot \alpha \cot \beta = \cot^2 \mu$, where α, β, μ denote the angles from the direction of the stream line to the directions of the isocline, isobar, and Mach line respectively. The cross ratio of the pencil formed by the tangents, taken in the proper order, to the stream line, isovel, isobar, and isopycnic for rotational, isoenergetic flow of a polytropic gas always equals the adiabatic exponent γ. The general results of the theory are illustrated by examples, e.g. Lighthill's channel flow.

UNIVERSITY OF MARYLAND,
COLLEGE PARK, MD., U. S. A.

A MOVING BOUNDARY FILTRATION PROBLEM OR "THE CIGARETTE PROBLEM"

MURRAY S. KLAMKIN

The partial differential equation for the concentration of any component in the tobacco as a function of the length of a cigarette and position in the cigarette is derived assuming steady inhalation, constant absorption coefficient, and continuous destruction of a constant fraction of the component at the burning tip. The equation is solved, yielding the following result:

If two cigarettes, identical, except for length are smoked down under steady inhalation to the same length, then the amount of component absorbed by the smoker per unit length is the same for both cigarettes provided the burning fraction is taken to be zero. This implies that if a longer cigarette is more effective in removing a component, it is due to the burning fraction. This appears to be contrary to claims of certain cigarette manufacturers.

To approximate more closely to actual smoking conditions, the equation is then solved for a discontinuous inhalation.

POLYTECHNIC INSTITUTE OF BROOKLYN,
BROOKLYN, N. Y., U. S. A.

ON THE STABILITY OF ZONAL WINDS OVER A ROTATING SPHERICAL EARTH

C. C. LIN

In the mathematical study of the large scale motions in the atmosphere, we consider the two-dimensional motion of a thin layer of incompressible ideal fluid over a rotating spherical earth. It then follows that a fluid element maintains the "vertical component" of its absolute vorticity throughout its motion. By regarding the flow field as maintained by the mutual interaction of a collection of vortex filaments, we are able to calculate the acceleration, in a zonal atmosphere motion, of a fluid element possessing an excess or defect of vorticity over its surroundings. Such a calculation indicates that the zonal motion is stable provided that there exists no extremum of absolute vorticity in the main flow. Otherwise, sustained oscillatory atmospheric motions may be expected. This acceleration formula may also be used to account for the meridional movements of cyclonic and anti-cyclonic disturbances in the atmosphere.

The existence of oscillatory atmospheric motions had led to the investigation of exact solutions of the vorticity equation representing neutral oscillations of finite amplitudes propagating without change in shape. The only known set of such solutions is found to be inadequate for explaining the observed oscillations in the atmosphere. A general formulation of the problem is now made, and it shows that other solutions of this type are presumably all very complicated. It is, therefore, natural to develop the theory for small oscillations in order to obtain oscillations of more general types to account for observed meteorological phenomena.

The theory is developed for small oscillations about the zonal motion over a rotating earth. It is found that the mathematical problem is similar to that for the stability of two-dimensional jets. The only difference is that the vorticity gradient of the main motion is not related to the velocity distribution in the same manner. But here it is again shown that the vanishing of the gradient of the vorticity of the main flow is a necessary condition for instability. It is also shown to be sufficient for velocity distributions of the general nature actually occurring in the atmosphere. There often occur both a maximum and a minimum of vorticity. The wave velocity of the neutral oscillation is shown to be lying between the velocity of the zonal winds at those latitudes. Transfer of momentum and energy between the main flow and the disturbance is also discussed.

Massachusetts Institute of Technology,
Cambridge, Mass., U. S. A.

THE KINETIC STRUCTURE OF PLANE SHOCK WAVES

Knox Millsaps and Karl Pohlhausen

The structure of shock waves was studied for a wide range of Mach numbers after the classical method of Becker for the thermal dependence (the power laws and Sutherland's formula) of viscosity and conductivity. Following the work of Thomas, the effects due to the first order nonlinear terms formulated by Burnett using the kinetic theory of Chapman and Enskog were estimated by numerical methods by assuming the Becker solution as a first approximation. The convergence of the method was not examined.

Alabama Polytechnic Institute,
Auburn, Ala., U. S. A.
Office of Air Research,
Dayton, Ohio, U. S. A.

THE COLLAPSE OR GROWTH OF A SPHERICAL BUBBLE OR CAVITY IN A VISCOUS FLUID

H. Poritsky

The growth or collapse of cavities in a fluid is of interest in underwater explosions, in cavitation near propellers, and in the initial stages of foaming in oil. Lord Rayleigh has treated the problem of collapse of a spherical bubble in a perfect (nonviscous) fluid. In this paper the collapse or growth of a spherical bubble in a viscous incompressible fluid is studied.

The viscosity in the fluid motion equations shows up in the terms $\mu\nabla(\nabla\cdot\bar{V})/3 + \mu\nabla^2\bar{V}$; now the first term vanishes due to the incompressibility of the fluid, while the second term vanishes due to the assumed spherical symmetry of the flow and its incompressibility. Hence, apparently, *no* difference exists throughout the fluid between the equations governing the collapse of a spherical bubble in a viscous and in a perfect fluid. In fact one may derive a Bernoulli equation for the pressure (defined as the negative mean of the principal stress).

The resolution of the above paradox is brought about by noting that while μ fails to appear in the equations of motion inside the fluid, it does appear in the expression for the stresses and in the pressure. Thus μ reenters the scene in the formulation of the pressure inside the spherical cavity. One obtains the following second order differential equation for the cavity radius R

$$\frac{P - p_0}{\rho} = RR'' + \frac{3}{2}(R')^2 + \frac{4\mu}{\rho}\frac{R'}{R} \tag{1}$$

where P is the pressure inside the cavity, p_0 the pressure at infinity, and $'$ denotes d/dt.

For $R = R_0$, $R' = 0$ initially the above may be integrated into

$$\frac{P - p_0}{3}(R_0^3 - R^3) + \frac{\rho}{2}R'R^3 + 4\mu\int_0^t R(R')^2\,dt \tag{2}$$

which also follows from energy considerations. For $\mu = 0$ equation (2) agrees with Lord Rayleigh's results.

Equation (1) is put in dimensionless form and integrated numerically for several specific values of the dimensionless constants involved.

General Electric Company,
Schenectady, N. Y., U. S. A.

ELECTROMAGNETIC MEASUREMENTS OF THE FLOW VELOCITY OF A FLUID IN A PIPE OF ELLIPTICAL CROSS SECTION

H. Poritsky and H. Weil

A well-known method used to measure fluid flow in a straight insulated pipe consists in applying a uniform magnetic field, $\bar{H}$, across the pipe and normal to the pipe axis and measuring the potential difference induced across the pipe in the direction perpendicular to $\bar{H}$. For a pipe of circular cross section this potential is proportional to the average velocity, $\bar{u}$, of the fluid across the section and is independent of the velocity distribution in the pipe; thus it measures the net flow whether the flow be laminar or turbulent. (B. Thüerlemann, Helvetica

Physica Acta vol. 14, pp. 383–419.) This property enables the method to be used even when the flow state of the fluid in the pipe is not known.

It is of interest to know whether, for pipes of noncircular cross section, the induced potential is independent of the velocity distribution. The present investigation shows that in general this is not the case.

For a pipe of elliptical cross section the induced potentials are calculated for a laminar velocity profile and for a constant velocity profile. The latter is an approximation to the turbulent velocity profile except in a thin laminar layer at the boundary.

Let the ellipse be $x^2/a^2 + y^2/b^2 = 1$, let $\bar{H}$ be parallel to the x-axis, and let the potential difference V be measured at $(0, b)$, $(0, -b)$. It is shown in Gaussian units that

$$V = 8\, \bar{u}Hb(a^2 + 2b^2)/3C(a^2 + b^2) \qquad \text{(laminar flow)}$$

$$V = 2\, H\bar{u}b/C \qquad \text{(uniform flow)},$$

where C is the velocity of light.

These results are obtained by solving Maxwell's field equations subject to the condition of vanishing normal conduction current at the boundary. The problem reduces to solving the Poisson's equation $\nabla^2\varphi = (H/C)\ \partial u/\partial y$, for the potential φ, with a prescribed normal derivative $\partial\varphi/\partial n$ on the boundary.

For the laminar case, $\partial\varphi/\partial n = 0$ at the boundary and the problem is solved by superposing a particular solution dependent on y only and a proper polynomial solution of Laplace's equation.

For uniform flow, φ satisfies Laplace's equation with a prescribed $\partial\varphi/\partial n \neq 0$ on the boundary. The solution is simply proportional to y.

General Electric Company,
Schenectady, N. Y., U. S. A.

LINEARISIERTE ÜBERSCHALLSTRÖMUNG UM LANGSAM SCHWINGENDE DREHKÖRPER

Robert Sauer

Die linearisierte Überschallströmung um einen schwingenden Drehkörper wird untersucht unter der Annahme, dass die Frequenz der Schwingung klein ist im Vergleich zu der durch die Länge des Körpers dividierten Strömungsgeschwindigkeit. Wenn man dann das Geschwindigkeitspotential nach Potenzen der Frequenz entwickelt und nur die ersten Koeffizienten dieser Entwicklung berücksichtigt, erhält man eine geschlossene elementare Darstellung der Strömung. Hierdurch ergeben sich mit sehr geringem Arbeitsaufwand Näherungswerte der Dämpfungskoeffizienten (pitching force, damping moment) schwingender Drehkörper. Das analoge Verfahren für die zweidimensionale Überschallströmung (langsam schwingender Tragflügel) liefert elementare Näherungen der bekannten von Borbély und anderen angegebenen Lösungen.

TECHNICAL UNIVERSITY OF MUNICH,
MUNICH, GERMANY.

SYNTHETIC METHOD FOR NONLINEAR PROBLEMS

B. R. SETH

The solution of a large number of physical problems has now reached a stage when any further appreciable advance needs fresh mathematical methods. More than a century has been spent on linearised problems in which any number of solutions can be superposed on one another. But a natural phenomenon is seldom the result of linearised superposed effects. Any event is the result of a number of others dovetailing into one another, and hence any attempt at the exact formulation of a physical problem produces nonlinearity in our equations.

The present study of nonlinear problems is the result of what are called boundary or edge effects. The general solution of nonlinear equations is almost an impossible task. They are therefore subjected to a simplifying process so that they narrow down to the investigation of a particular aspect of an event. An over-all picture is not obtained and a number of important effects remain unknown.

All analytical solutions are limiting cases of the actual state of a phenomenon. The limiting state can be approached in an infinite number of ways. The reducing method has been very much in fashion. It has not been generally appreciated that it is possible that, instead of narrowing down a field, its extension might give exact solutions of nonlinear problems. Such a method, which may be called synthetic, will not only give known solutions as limiting cases, but will also throw light on a number of allied problems.

When applied to Navier-Stokes equations of viscous flow this method gives very interesting results. No exact solution of the equations of motion is known even for a sphere or a cylinder moving uniformly through a viscous liquid. By the introduction of an external force X in the direction of motion, an exact solution can be obtained. When X is made to approach zero we get the following results (B. R. Seth, Philosophical Magazine ser. 7 vol. 27 (1939) pp. 212–220):

(1) The irrotational solution.

(2) Motion due to Hill's spherical vortex.

(3) Stokes' solution and its relationship to Oseen's solution.

(4) An expression for the drag which depends on the relative values of the viscosity, velocity, and size of the body.

(5) Prandtl's boundary layer theory.

This method has been extended to the case of a sphere or a cylinder moving uniformly through a compressible fluid. It can be profitably used in allied fields of research.

Hindu College,
Delhi, India.

ON THE SOLUTION IN THE LARGE OF A CAUCHY PROBLEM (WITH SPECIAL REFERENCES TO THE COMPRESSIBLE FLOW AFTER A STATIONARY SHOCK)

Shien-siu Shü

The problem of the compressible flow after a given shock wave is considered. Two cases are investigated: namely, (i) the Taylor-MacColl conical flow after a conical shock wave and (ii) the two-dimensional flow after a curved shock wave.

The existence of a conical flow with a conical shock due to a solid cone is established. The classical method of Poincaré and Liapounoff for nonlinear differential equations does not apply to the present instance for the solution in the large. Certain information, such as the derivative of the solution possesses a positive zero, is necessary to establish the existence of the solution of the problem. The present method gives constructively a sequence of functions which converges monotonely and uniformly to the solution of the problem.

The question of extension of the solution into the complex domain is investigated. A practical solution of Taylor-MacColl's equation, which is valid in the whole region concerned, is obtained in a series of a rational function of the semi-apex angle. With seven terms worked out, the results for both the velocity components are in excellent agreement with the refined numerical computations by Kopal's group at M. I. T. for the Taylor-MacColl cone.

The second instance is an attempt at finding an approximate analytical solution for the two-dimensional subsonic flow after a shock. The effect of vorticity is neglected and a simple pressure-density relation (as first suggested by Tschaplygin for subsonic flows and elaborated by many others) is assumed. The solution in the large for the flow is then explicitly worked out by means of Schwarz formulas for Björling problem of a minimal surface. The solution, thus obtained, satisfies the exact shock conditions, yet it satisfies the original partial differential equations only approximately.

The general problem of finding an explicit analytic representation of the solution in the large in terms of the given shock conditions for the flow after a curved shock wave is still wide open.

Illinois Institute of Technology,
Chicago, Ill., U. S. A.

AN EXACT SOLUTION TO THE NONLINEAR DIFFERENTIAL EQUATION DESCRIBING PASSAGE OF PLANE WAVES OF SOUND THROUGH AIR

KEEVE M. SIEGEL

The equation describing the motion of sound waves in air ($\gamma = 1.4$) is

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} \bigg/ \left(1 + \frac{\partial \xi}{\partial x}\right)^{\gamma+1}. \tag{1}$$

The substitution

$$\xi = X(x)T(t) - x \tag{2}$$

allows us to separate the nonlinear partial differential equation into two nonlinear ordinary differential equations. Exact solutions of these ordinary nonlinear equations are found and substituting these solutions into (2) yields an exact solution to the nonlinear partial differential equation (1).

Applications of the solution to certain boundary value problems are discussed and comparison made with the approximations to (1) which already exist in the literature.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH., U. S. A.

THE DISPERSION, UNDER GRAVITY, OF A COLUMN OF FLUID SUPPORTED ON A RIGID HORIZONTAL PLANE

C. K. THORNHILL

An interesting hydrodynamical problem is that of the dispersion under gravity of a fluid column immersed in a surrounding fluid medium of lower density and resting on a rigid horizontal plane. Examples of such a dispersive motion are afforded by the bursting of a dam wall, the sudden shattering of a vessel containing liquid, or the spread of pancake mixture in a frying pan.

The corresponding ideal symmetric problem of incompressible potential flow, neglecting viscosity, ground friction, and turbulence, is easily shown to be of parabolic type in three independent variables (two space variables and time); for the velocity potential, which is a function of all three variables, satisfies Laplace's equation in the two space variables. The principal boundary conditions are that, at every point of the unknown boundary of the fluid column, the two fluid pressures must be equal, and the two normal fluid velocities must both be equal to the normal velocity of the boundary.

The solution may be expressed, in general, in terms of infinite series of cosines (plane symmetry) or spherical harmonics (axial symmetry), in which the coefficients are unknown functions of the time. A numerical solution for the early stages of the motion may be obtained by limiting the above series to a finite number of terms. The boundary conditions can then be satisfied only at a finite number of points, instead of everywhere, on the column boundary, but, in this way, the solution may be reduced numerically to that of a finite number of simultaneous linear equations with constant coefficients at each small interval of time. The fewer terms retained in the series and thus the fewer points at which the boundary conditions are satisfied, the shorter is the duration in time before the numerical solution ceases to satisfy reasonably the principles of conservation of mass and energy.

The numerical solutions given are for fluid columns of initially semi-circular cross-section in vacuo, for which the number of simultaneous equations to be solved is much reduced. In all cases, the column spreads out rapidly from the base, as would be expected.

An approximate solution, appropriate to the later stages of such a dispersive motion, or to columns which are initially very squat, may be obtained by neglecting vertical accelerations in the fluid motion. In this case the problem reduces to one of hyperbolic type in two independent variables (one space variable and time) and a solution is readily obtained by the numerical method of characteristics. It is shown that, with this approximation, the dispersive motions of all columns which are similar, except for a scale factor in height, are derivable from the same characteristic solution, and that differences in the relative densities of the column and the surrounding medium correspond to different values of gravity.

Numerical examples are given in which vertical accelerations have been neglected *ab initio*, and also in which initial solutions, obtained as above, have been continued by this approximate method.

The author gratefully acknowledges the considerable contributions made by Dr. W. G. Penney, F.R.S., to the ideas and methods described in this paper.

ARMAMENT RESEARCH ESTABLISHMENT,
FORT HALSTEAD, ENGLAND.

A NEW VORTICITY THEOREM

C. TRUESDELL

Our objective is to relate and generalize the vorticity theorems for plane and rotationally-symmetric flows of an inviscid fluid. We shall call a continuous barotropic flow of an inviscid fluid subject to conservative extraneous force a *C-flow*, a continuous steady flow of an inviscid fluid devoid of heat sources or heat flux and subject to no extraneous force a *G-flow*. Let $\epsilon = 0$ for plane flow, $\epsilon = 1$ for rotationally-symmetric flow. Then the previously known results are (1) in a C-flow $w/r^{\epsilon}\rho$ = const. for each particle (d'Alembert (1761), Svanberg (1841)), and (2) in a G-flow of a perfect gas ($p = R\rho\theta$) $w_C/r^{\epsilon}p$ = const.

along each stream-line, $\mathbf{w}_G$ being the curl of the Crocco reduced velocity vector (Crocco-Prim (1936, 1947)).

While the theorems (1) follow as special cases from the Helmholtz flux theorem, this latter in the general case concerns integrals or differentials, rather than finite local variables. No analogous generalization exists for the theorems (2). We desire a *finite scalar conservation law* including both (1) and (2). A natural generalization of plane and rotationally-symmetric flow is complex-laminar flow, characterized by the existence of surfaces normal to the stream-lines. Since the theorems (1) and (2) despite their formal similarity concern situations which are dynamically different, to relate them a purely kinematical generalization is required. The appropriate theorem of kinematics, containing a complete solution to our problem, is as follows:

Given a continuous complex-laminar motion whose vortex-lines are steady, let h be defined by $ds^2 = h^2dx_w^2 + \cdots$, where x_w is arc-length along the vortex-line; let v_0 be any substantially constant function; let $\mathbf{v}_C \equiv \mathbf{v}/v_0$, $\mathbf{w}_C \equiv$ curl $\mathbf{v}_C$; let A be any solution of v_0 grad $(A/v_0) \times \partial\mathbf{v}_C/\partial t = \mathbf{w}_C\,\partial A/\partial t$; let v be any solution of $\partial v/\partial t + v_0 \operatorname{div}(v\mathbf{v}_C) = 0$; then if it be possible to select from the infinite number of functions v_0 and A satisfying the foregoing requirements one pair such that

$$(*) \qquad \mathbf{w}_C \cdot \operatorname{curl}\left[A\left(\frac{1}{v_0}\frac{\partial \mathbf{v}_C}{\partial t} + \mathbf{w}_C \times \mathbf{v}_C\right)\right] = 0,$$

it follows that

$$\frac{Aw_C}{hv} = \text{const.}$$

for each particle.

Application I. In a C-flow Kelvin's circulation theorem holds. An analytical expression of this fact is curl $(\partial\mathbf{v}/\partial t + \mathbf{w} \times \mathbf{v}) = 0$. Hence $(*)$ is satisfied by the choice $A = 1$, $v_0 = 1$, and we obtain a generalization of the d'Alembert and Svanberg theorems (1): *in any complex-laminar C-flow with steady vortex-lines, $w/h\rho =$ const. for each particle.*

Application II. A Prim gas is one whose equation of state is $\rho = P(p)H(\eta)$, η being the entropy; a perfect gas is a special case. In a G-flow of a Prim gas an ultimate speed v_{ult} exists and can be taken as v_0. The continuity equation then assumes the form div $[P(p)\mathbf{v}_C] = 0$, while the condition of integrability for the dynamical equation becomes curl $[\mathbf{v}_C \times \mathbf{w}_C/(1 - v_C^2)] = 0$. Hence the choice $A = (\int P(p)^{-1}dp)^{-1}$ satisfies $(*)$, and we obtain a two-fold generalization of the Crocco-Prim theorems (2): *in any complex-laminar G-flow of a Prim gas,*

$$w_C P/h \int P\, dp = \text{const.}$$

along each stream-line.

Application of the new kinematical theorem to other special cases of interest, particularly in meteorology, should be possible by suitable choices of the parameters v_0 and A.

NAVAL RESEARCH LABORATORY,
WASHINGTON, D. C., U. S. A.
UNIVERSITY OF MARYLAND,
COLLEGE PARK, MD., U. S. A.

THE APPLICATION OF VARIATIONAL METHODS TO THE COMPRESSIBLE FLOW PROBLEMS

CHI-TEH WANG

Approximate solutions of compressible flow past bodies of arbitrary shape have been carried out by variational methods. Three methods have been used, namely, the Rayleigh-Ritz method [1, 2], the Galerkin method [3, 4], and the Biezeno-Koch method [5]. In the case of potential flows, many examples have been worked out; they are the flow past a circular cylinder [1, 2, 3, 4, 5], elliptical cylinders [6], a symmetrical profile [2], a sphere, and ellipsoids of revolution [7]. The results are found to check excellently with those computed by other methods. From these numerical examples, it is noted that the variational methods give good approximate solutions at both high and low Mach numbers and for flows past both thick and thin bodies. In the case of transonic flows with shock waves [8] and compressible viscous flows [4], work is in progress and the results will be reported in later papers.

Among the three methods tried, the Rayleigh-Ritz method is found to require the least amount of computational labor if the variational principle is known. In the case of potential flows, it has the disadvantage however that the ratio of specific heats γ must be such that $\gamma/(\gamma - 1)$ is an integer. The Galerkin method and the Biezeno-Koch method, on the other hand, do not impose any restriction on the value of γ taken, but the calculation then becomes more lengthy. In problems where the variational principles have not been formulated, these two methods may be applied advantageously because they do not require the use of such principles.

In carrying out the variational methods, no linearization of the fundamental equations is necessary and the calculations may be performed in the physical plane. It therefore gives the stipulation that a study of such solutions may throw some light on the nonlinear characteristic of the fundamental equations. Such a study has been made [9] and it is found that beyond a certain Mach number there does not exist any physically possible solution. At this limiting Mach number, it can be shown that unsymmetrical flow patterns may appear even in an otherwise symmetrical flow.

If only the velocity and pressure distributions on a given profile at various Mach numbers are to be found, one may study only the linearized variational integral. Comparisons are made between the linearized and nonlinear solutions in the examples worked out. It is seen that in all these cases the linearized solutions give close agreement to the nonlinear ones up to the limiting Mach number. By working with the linearized integral, the labor saved is enormous.

REFERENCES

1. CHI-TEH WANG, *Variational method in the theory of compressible fluid*, Journal of the Aeronautical Sciences vol. 15 (1948) pp. 675–685.
2. ———, *Two-dimensional subsonic compressible flows past arbitrary bodies by the variational method*, NACA Tech. Note 2326, March, 1951.
3. CHI-TEH WANG and R. F. BRODSKY, *Application of Galerkin's method to compressible fluid flow problems*, Journal of Applied Physics vol. 20 (1949) pp. 1255–1256.
4. ———, *Approximate solution of compressible fluid flow problems by Galerkin's method*, Journal of the Aeronautical Sciences vol. 17 (1950) pp. 660–666.
5. CHI-TEH WANG and PEI-CHI CHOU, *Application of Biezeno-Koch method to compressible fluid flow problems*, Journal of the Aeronautical Sciences vol. 17 (1950) pp. 599–600.
6. G. V. R. RAO, *Subsonic compressible flow past elliptic cylinders*, to be published.
7. CHI-TEH WANG and S. DE LOS SANTOS, *Approximate solutions of compressible flows past bodies of revolution by variational method*, to be published in Journal of Applied Mechanics as paper no. 50-A-33.
8. CHI-TEH WANG, *A variational method for transonic flows with shock waves*, Report submitted to NACA, June 1949.
9. CHI-TEH WANG and G. V. R. RAO, *A study of the nonlinear characteristic of compressible flow equations by means of variational method*, Journal of the Aeronautical Sciences vol. 17 (1950) pp. 343–348.

NEW YORK UNIVERSITY,
NEW YORK, N. Y., U. S. A.

A CONSTRUCTIVE THEORY FOR THE EQUATIONS OF FLOWS WITH FREE BOUNDARIES

EDUARDO H. ZARANTONELLO

The plane irrotational steady flow of an incompressible perfect fluid with free boundaries past a convex obstacle is essentially determined by an equation of the form: (*) $\lambda(\sigma) = \nu(\sigma)\kappa[\theta(\sigma)]e^{-\tau(\sigma)}$, $0 \leqq \sigma \leqq \pi$, where τ and θ are the boundary values of two conjugate harmonic functions in the unit circle and $\lambda = -d\theta/d\sigma$; ν is a non-negative continuous function with a p-integrable first derivative ($p > 1$), and $\kappa(\theta)$ is the curvâture of the obstacle as a function of the direction of the tangent, it is positive for convex obstacles. The function ν contains several parameters describing the geometrico topological structure of the flow. According to their definition θ and τ can be expressed in terms of λ as $\theta = T\lambda$, $\tau = K\lambda$, where T and K are totally continuous linear integral operators, the second being also self-adjoint and positive definite. Now under the condition $d\kappa/d\theta \leqq [\text{l.u.b. } \nu(\sigma)]^{-1}$, the following results are proved:

a) The iterates $S_\alpha^k\lambda$ of the operator $S_\alpha : S_\alpha\lambda = \alpha\nu\kappa(T\lambda)e^{-K\lambda} + \beta\lambda, \alpha + \beta = 1$, applied to a non-negative continuous function converge uniformly towards a solution of (*) provided that α is a sufficiently small positive number. The solution is unique.

b) The set of equations: $\lambda(\sigma_k^{(n)}) = \nu(\sigma_k^{(n)})\kappa[\theta(\sigma_k^{(n)})]e^{-\tau(\sigma_k^{(n)})}, \sigma_k^{(n)} = k\pi/(n + 1)$, $k = 1, 2, \cdots, n$, have, for every n, a unique solution of the type: $\lambda(\sigma) = \sum_1^n a_j \sin j\sigma$, which can also be obtained by an iteration process as in a).

c) The trigonometric polynomials $\lambda^{(n)}(\sigma)$, solutions of the preceding equations, tend uniformly towards the solution of (*). Numerical applications are given.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., U. S. A.

MATHEMATICAL PHYSICS

UPPER AND LOWER BOUNDS FOR QUADRATIC FUNCTIONALS

J. B. DIAZ

(This paper was listed in the program of the Congress under the title *Concerning scalar products and certain minimum and maximum principles of mathematical physics.*)

In many problems of mathematical physics, it is desired to find or to estimate the numerical value of a quadratic integral of an unknown function u, where u is a solution of a certain boundary value problem consisting of a linear partial differential equation plus linear boundary conditions. The quadratic integral in question is associated with a certain bilinear integral, which may be employed as a scalar product in a suitable linear vector space naturally associated with the boundary value problem. Consider, for example, the determination of the torsional rigidity (see the formula given by J. B. Diaz and A. Weinstein, *The torsional rigidity and variational methods*, Amer. J. Math. vol. 70 (1948) pp. 107–116) where it is required to estimate the Dirichlet integral $\int_D(u_x^2 + u_y^2)dxdy$ of the warping function u, which is a solution of a Neumann problem. Here the bilinear integral is $\int_D(\varphi_x\psi_x + \varphi_y\psi_y)dxdy$. It has been shown previously (J. B. Diaz and A. Weinstein, *Schwarz inequality and the methods of Rayleigh-Ritz and Trefftz*, Journal of Mathematics and Physics vol. 26 (1947) pp. 133–136) that upper and lower bounds for the quadratic integral of the unknown function u follow immediately from Schwarz' inequality in the linear vector space naturally associated with the boundary value problem. Justifying a remark made in the conclusion of the same paper, it is shown that monotone nonincreasing sequences of upper bounds and monotone nondecreasing sequences of lower bounds for the desired quadratic integral follow at once from Bessel's inequality in the associated linear vector space.

INSTITUTE FOR FLUID DYNAMICS AND APPLIED MATHEMATICS,
UNIVERSITY OF MARYLAND,
COLLEGE PARK, MD., U. S. A.

ON INTEGRALS OF MOTION OF THE RUNGE TYPE IN CLASSICAL AND QUANTUM MECHANICS

A. W. SÁENZ

The general objective of this paper has been to study the significance of the Runge and allied integrals of motion in both classical and quantum mechanics. The results obtained can be summarized in the following fashion.

In the domain of classical mechanics, which is the subject of Part I, the integrals of motion of the Kepler and isotropic harmonic oscillator problems have been constructed by a particularly simple and satisfying kinematical procedure. The existence of $(2n - 1)$ independent algebraic integrals in a holonomic system of n degrees of freedom, such as in the two cases cited above, has been related to the reentrant nature of all the bound orbits, thus putting in evidence the privileged position of the said cases among one-body problems with spherically symmetric potentials. A larger class of systems has also been treated, which includes the plane oscillator, the Kepler problem, and the spherical top with no forces as special cases, and a natural interpretation of its integrals of motion has been obtained by introducing the appropriate coordinates. These integrals satisfy Poisson bracket relations identical to those of the Runge vector and the angular momentum in the ordinary Kepler problem.

In Part II, detailed quantum mechanical calculations of the effect of the Runge integral on the eigenfunctions of hydrogen in polar and parabolic coordinates have been performed, and similar computations have been carried out for the analogues of the integrals of the plane oscillator and spherical top, mentioned previously. It has been shown that in all these cases any eigenfunction of a given energy level can be reached by application of these integrals to any other eigenfunction with the same eigenvalue as the first one.

In Part III, devoted to the Dirac Kepler problem, general conditions on the integrals of motion have been obtained, from which it has been possible to demonstrate that no integrals linear in the momenta, outside of those already known, are possible. It has also been proved that no integrals exist with commutation relations analogous to those of the ordinary Runge vector. These considerations have shown that the j-degeneracy can not be due to invariance of the Hamiltonian under four-dimensional rotations.

Naval Research Laboratory,
Washington, D. C., U. S. A.

MATHEMATICAL PHYSICS: OPTICS AND ELECTROMAGNETIC THEORY

THE NUMBER OF SERIES-PARALLEL NETWORKS

RONALD M. FOSTER

The number of two-terminal series-parallel networks was investigated by Macmahon in 1892, and subsequently studied in considerable detail by Riordan and Shannon (Journal of Mathematics and Physics vol. 21 (1942) pp. 83–93). In the present paper it is shown that the number C_n of essentially different series-parallel networks of n elements, without specification of any accessible terminals, can be derived from the known number S_n of essentially different two-terminal series-parallel networks of n elements by the following relation:

$$C_n = S_n - \tfrac{1}{2} \sum_{i=1}^{n-1} S_i\, S_{n-1} - \tfrac{1}{2} S_{n/2}$$

where $S_{n/2}$ is taken to be equal to zero if n is odd. The derivation of this formula depends upon a process of counting the number of ways of separating into two parts a series-parallel network, and also its dual network, at all possible pairs of terminals. From the available tabulation of the S_n numbers, it is found by means of the above relation that the C_n numbers are, for values of n from 2 to 10, equal to 1, 2, 3, 6, 12, 26, 59, 146, 368, respectively. These agree with the results obtained by Tellegen by a process of systematic enumeration through $n = 7$ (Philips Technical Review vol. 5 (1940) pp. 324–330); the values for 8, 9, and 10 have been checked by a similar process. A closely related problem is the enumeration of series-parallel networks of n elements with r of the elements individually marked; somewhat similar formulae are derived in this case. Finally, the asymptotic behavior of certain of these numbers is studied.

POLYTECHNIC INSTITUTE OF BROOKLYN,
BROOKLYN, N. Y., U. S. A.

AN OPTICAL MODEL OF PHYSICS

MAX HERZBERGER

This paper attempts to transfer the methods developed in optics to general physical problems.

1. Optics can be considered in three phases: *Geometrical optics* determines in three-space, with the help of a variation principle (Fermat), a Finsler geometry in which the light rays are the straight lines and the wave surfaces the spheres. *Diffraction optics* singles out in this geometry a point disturbance and a manifold of light rays forming a field. Along them we have a transversal disturbance

of given frequency characterized by an axial vector. Huygens' construction (Green's Theorem) gives the value of the disturbance vector at a given point. If, because of absorption, the wave surfaces are "in phase" but with different amplitudes, the diffraction effect can be calculated. (The superposition of diffraction effect from a *finite* number of disturbances gives a sufficient approximation to optics.)

Absorption and emission are concerned with the selection of frequencies due to the vibration of the emitting and absorbing elements. The Einstein effect can be explained by the eigenvalue theory of the modified wave equation.

2. Physics can be considered in three parallel phases in Euclidean space-time: A variation principle gives a *Finsler space* which describes the world lines for every physical event. It will be shown that general relativity is only a special example of this Finsler space. However, some of its assumptions are taken over into this more generalized geometry.

Diffraction theory must be based on lines of disturbance in four-space. The antisymmetric part of the field given by a line disturbance gives the electromagnetic theory and Maxwell's equations. This is the sole carrier of energy. Interference and coherence are explained on this level.

Absorption and emission are guided by the eigenvalue theory of the generalized Schrödinger equation without its probability interpretation. The photoelectric effect of Einstein, as well as his famous mass-energy equivalence hypothesis, can be embodied here.

The theory, though in Euclidean space, has all the necessary invariance. The assumption of line discontinuities is thought of as an approximation. Investigation of the neighborhood of these discontinuities will show a structure. The theory does not assume existence of photons and gravitational waves. The mathematical methods are: Finsler geometry on the first level (W. R. Hamilton); Dirichlet problem (generalized) on the second; and eigenvalue problems of the second-order partial differential equations on the third. It discourages the use of probability (outside statistical problems) and of complex and hypercomplex numbers for real problems.

Eastman Kodak Research Laboratory,
Rochester, N. Y., U. S. A.

THE REALIZATION OF THE TRANSFER FUNCTION OF THE FINITE, FOUR-TERMINAL NETWORK

Robert Kahal

The open circuit transfer function of a finite, four-terminal network is defined as the ratio of the output voltage to the input voltage. The transfer function, $\theta(p)$, is a rational fractional function of the complex frequency variable p. The following theorem is proved. Theorem: the necessary and sufficient conditions that a rational, fractional function, $\theta(p)$, shall be the transfer function of a finite, four-terminal network are:

(1) $\theta(p)$ is real when p is real.

(2) $\theta(p)$ has no poles in $\mathrm{Re}(p) > 0$.

(3) Poles of $\theta(p)$ on $\mathrm{Re}(p) = 0$ are simple. The real part of the residue at such a pole must be zero.

It is shown that $\theta(p)$ can always be realized with a symmetrical lattice structure and not more than one pair of perfectly coupled coils. This canonical form was suggested by R. M. Foster.

POLYTECHNIC INSTITUTE OF BROOKLYN,
BROOKLYN, N. Y., U. S. A.

THE CORRELATION TENSOR OF THE ELECTROMAGNETIC FIELD IN CAVITY RADIATION

CHAIM L. PEKERIS

We consider the correlation of any two of the six components of the electric vector $\mathbf{E}$ and magnetic vector $\mathbf{H}$ in cavity radiation, taken at two points in space distant r apart and separated by the interval τ in time. On representing the electromagnetic field by the antisymmetrical tensor F in four-space:

$$(1)\quad (x, y, z, ict) = (x_1, x_2, x_3, x_4);\ (E_x, E_y, E_z) = -i(F_{41}, F_{42}, F_{43}),$$

$$(2)\quad (H_x, H_y, H_z) = (F_{23}, F_{31}, F_{12}),\ R_{iklm} = \frac{\langle F_{ik}(\mathbf{x}, t)F_{lm}(\mathbf{x} + r, t + \tau)\rangle}{\langle \mid F^2 \mid \rangle},$$

we find that

$$(3)\quad R_{iklm} = \left(\delta_{kl}\frac{\partial^2}{\partial x_i \partial x_m} + \delta_{im}\frac{\partial^2}{\partial x_k \partial x_l} - \delta_{il}\frac{\partial^2}{\partial x_k \partial x_m} - \delta_{km}\frac{\partial^2}{\partial x_i \partial x_l}\right)(g/2r),$$

where

$$(4)\quad g(r) = \frac{45}{4\gamma^3}\left[\left(\coth\alpha - \frac{1}{\alpha}\right) + \left(\coth\beta - \frac{1}{\beta}\right)\right],$$

$$\alpha = \gamma(r + c\tau), \qquad \beta = \gamma(r - c\tau), \qquad \gamma = (\pi kT/hc),$$

$\langle\ \rangle$ denotes average over $\mathbf{x}$ and t, and $\langle \mid F^2 \mid \rangle$ represents the average of the absolute value of a single component of F. Maxwell's equations are satisfied by (3) provided that

$$(5)\quad \Box(g/r) = 0.$$

The particular solution (4) of (5) is determined from Planck's formula for the spectral intensity of cavity radiation.

Our correlation tensor is formally similar to the commutation tensor in quantum electrodynamics derived by Jordan and Pauli [P. Jordan and W. Pauli, Jr., Zeitschrift für Physik vol. 47 (1928) p. 163], where $g(r)$ is replaced by the delta function $[\delta(r + ct) - \delta(r - ct)]$. It follows that field components which commute have zero correlation in cavity radiation, while those which do not commute are correlated. This is consistent with the fact that commuting variables can be measured simultaneously without mutual disturbance.

WEIZMANN INSTITUTE,
REHOVOTH, ISRAEL.

A NOTE ON THE RESPONSE OF SYSTEMS WITH OR WITHOUT NONLINEAR ELEMENTS

REINHARDT M. ROSENBERG

A semi-inverse method is used for discussing the behavior of mechanical systems or electrical circuits when the governing differential equation is known. This equation is assumed to be ordinary, nonlinear (or not), and it must satisfy the well known conditions for uniqueness and existence of solutions for initial conditions through a regular point.

The idea exploited here has the objective of exploring rapidly a variety of expected solutions to problems of considerable difficulty. Suppose $x = \xi_1(t)$ satisfying the conditions $\xi_1^{(i)}(t_0) = \xi_{10}^i$, $i = 1, 2, \cdots, n$, is expected on physical grounds to correspond to the behavior of the system under consideration whose equation is of nth order. If ξ_1 is substituted into this equation in which a function of the dependent or independent variable has been left undetermined, an algebraic equation results which defines the undetermined function uniquely. A systematic variation in the ξ_j will yield a table of correspondences between the solutions and the undetermined function which is useful in disclosing the performance of the system. It is usually necessary to investigate, in addition, solutions when the undetermined function is defined by ξ_j, but $\xi_j^i(t_0) \neq \xi_{j0}^i$. In second order equations, such investigations can frequently be carried out in the large by the methods of Poincaré. When this is impossible, solutions neighboring on ξ_j can always be discussed by one of the methods of small parameters. At present some very well known unsolved problems are being attacked in this manner, and valuable clues regarding the behavior of the systems (aside from many particular solutions) have been gathered. An example of a nonlinear problem arising in the landing of airplanes is presented.

The above idea leads naturally to the inquiry what conditions an equation must satisfy so that its solution can be given in terms of elementary functions. In linear equations, such conditions take the form of certain differential conditions which are demonstrated for the particular equation $\ddot{x} + f(t)x = 0$.

UNIVERSITY OF WASHINGTON,
SEATTLE, WASH., U. S. A.

BICONICAL ANTENNAS OF ARBITRARY ANGLE

S. A. SCHELKUNOFF

This paper presents a solution of an electromagnetic boundary-value problem for two perfectly conducting equal coaxial cones of finite length with a common apex. The solution is based on a subdivision of space into two regions in each of which the general solution of Maxwell's equations may be expressed as a series of orthogonal functions satisfying some of the required boundary conditions. The coefficients of these series are then determined to obtain the particular solution which satisfies all the given boundary conditions. The input impedance of the biconical antenna is expressed in terms of a "terminal impedance" which is obtained in the form of an infinite series of functions involving Bessel and Legendre functions.

BELL TELEPHONE LABORATORIES,
MURRAY HILL, N. J., U. S. A.

FORMAL SOLUTIONS OF AN INTEGRO-DIFFERENTIAL EQUATION FOR MULTIPLY SCATTERED RADIATION

WILLIAM C. TAYLOR

Study of the scattering and absorption of light in a plane stratified medium has motivated the consideration of certain integro-differential equations (S. Chandrasekhar, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 641–711). One of the simplest of these is $\mu\partial I/\partial\tau + I = \lambda\int_{-1}^{1} G(\mu, \mu')I(\tau, \mu')d\mu'$, $-1 \leqq \mu \leqq 1$, where $I = I(\tau, \mu)$ is sought, and $G(\mu, \mu') = \sum_n (2n + 1)g_n P_n(\mu)P_n(\mu')$, where $P_n(\mu)$ is the Legendre polynomial. The g_n are given, defined by what is called the phase function, $g(\mu) = \sum_n(2n + 1)g_nP_n(\mu)$, $-1 \leqq \mu \leqq 1$, which is not here restricted to be a polynomial. A formal solution involving an arbitrary function has been obtained by applying to the equation the Stieltjes transform in the variable μ and then separating variables. Attempts to fit boundary conditions have not yet met with success.

UNIVERSITY OF CINCINNATI,
CINCINNATI, OHIO, U. S. A.
BALLISTIC RESEARCH LABORATORIES,
ABERDEEN PROVING GROUND, MD., U. S. A.

MATHEMATICAL PHYSICS: RELATIVITY, GRAVITATION, AND FIELD THEORY

COVARIANT QUANTIZATION OF NONLINEAR FIELD THEORIES

PETER G. BERGMANN

Sponsored by the ONR

In three preceding papers, the classical canonical formulation of covariant field theories has been developed in general terms and carried out for the general theory of relativity. [P. G. Bergmann, The Physical Review vol. 75 (1949) p. 680; P. G. Bergmann and J. H. M. Brunings, Reviews of Modern Physics vol. 21 (1949) p. 480; Bergmann, Penfield, Schiller, and Zatzkis, The Physical Review vol. 78 (1950) p. 329.] These papers approach the problem of quantization without solving it completely. The purpose of this paper is to present a complete and self-consistent scheme of quantization of a covariant, nonlinear field theory.

That the commutation relations between canonical variables are covariant has been shown previously [P. G. Bergmann and J. H. M. Brunings, loc. cit.]. The remaining difficulty was to find operators satisfying both the commutation relations and the algebraic constraints (true identities [L. Rosenfeld, Annales de l'Institut Henri Poincaré vol. 2 (1932) p. 25]) typical for covariant and gauge invariant theories. It is well known that no observable appearing in a commutation relation with a c-number right-hand side possesses normalizable eigenfunctions; therefore, no state vector satisfying the algebraic constraints can be normalized in a linear vector space composed of both the state vectors satisfying the constraints and those disobeying them. However, normalization is possible in the subspace of those state vectors obeying all the constraints imposed in a given theory. In a typical field theory, the number of these constraints is always transfinite.

The following procedure is self-consistent and invariant. Through a suitable canonical transformation, all the constraints are made canonical variables, preferably momentum densities. If these constraints include the Hamiltonian constraint [P. G. Bergmann and J. H. M. Brunings, loc. cit.], the remaining variables are subject solely to the commutation relations. The usual condition represented by the Schrödinger equation is included among the constraints. In the typical wave-mechanical representation, the state vector is then a normalizable functional of all the "coordinate" variables, except those conjugate to the constraints, and except for its normalizability arbitrary. Once the theory has been constructed in this representation, the most general representation can be obtained by carrying out an arbitrary canonical (unitary) transformation, involving all the variables including the constraints.

The next step in this investigation will be the application of this general formalism to specific problems.

Syracuse University,
Syracuse, N. Y., U. S. A.

GRAVITATIONAL SHIFT IN THE SOLAR SPECTRUM

A. J. Coleman

This paper studies the interaction between a Schwarzschild gravitational field and the electric field of the nucleus of a hydrogen-like atom. It is known that within the framework of general relativity an invariant form of quantum mechanics leads, in a first approximation, to the formula for the spectral shift which Einstein deduced merely by treating the atom as a symmetric clock. Carrying the quantum-mechanical discussion to a higher approximation reveals a perturbation of the Coulomb potential of the nucleus which is directionalized with respect to the gravitational field. This new term is of the correct order of magnitude and general character to explain the limb-effect, that is, the fact, known for more than forty years and hitherto unexplained, that the shift in lines of the solar spectrum is different at the edge and center of the sun's disc. It may possibly also explain the discrepancies between the spectral shifts at the center predicted by Einstein and the recent observations of M. G. Adam.

University of Toronto,
Toronto, Ont., Canada.

LA RECESSION DES NÉBULEUSES EXTRA-GALACTIQUES

P. Drumaux

L'étude mathématique de la gravitation, faite à la lumière de la relativité, montre que le monde cosmique qui nous environne, à savoir l'ensemble des nébuleuses extra-galactiques, ne correspond pas à l'idée qu'on s'en était fait. Les nébuleuses n'ont pas seulement une vitesse radiale de recession mais en outre une vitesse transversale de même ordre de grandeur. Il faut d'autre part faire une distinction entre le mouvement relatif des nébuleuses par rapport à la Voie lactée et le mouvement général d'entraînement de l'ensemble des nébuleuses, y compris la Voie lactée, sous l'effet du champ gravifique cosmique dans lequel cet ensemble est plongé. Le calcul montre que la vitesse de ce mouvement d'entraînement est de l'ordre de 100,000 km/sec.

Le calcul conduit d'autre part à la connaissance des trajectoires des nébuleuses qui sont soit des spirales gauches elliptiques, soit des courbes exponentielles apériodiques. La détermination astronomique de ces trajectoires résultera de la mesure des effets Doppler et des magnitudes apparentes pour des paires de nébuleuses situées dans la même direction et cela pour au moins six directions différentes.

Le calcul montre qu'il est alors possible de déterminer les vitesses transversales et aussi les trajectoires à condition que les mesures puissent se faire avec haute précision.

Or l'incertitude actuelle dans les magnitudes absolues est un grand obstacle à surmonter. Un premier pas serait fait en mesurant les écarts dans les magnitudes absolues d'une nébuleuse à l'autre et cela paraît possible en opérant sur deux nébuleuses situées dans la même direction et ayant approximativement même effet Doppler ainsi que même type spectral. On peut alors mesurer les écarts susdits tout en ignorant les magnitudes absolues elles-mêmes. La connaissance de ces écarts permettrait la mesure des rapports de distance de nébuleuses situées dans une même direction, ce qui conduirait à la détermination des trajectoires.

UNIVERSITY OF GHENT,
GHENT, BELGIUM.

AN EXPANSION OF A FOUR-DIMENSIONAL PLANE WAVE IN TERMS OF EIGENFUNCTIONS

KATHLEEN SARGINSON

The problem is to find the expansion of a four-dimensional plane wave $\exp i\{\mathbf{p}\cdot\mathbf{r} - Et\}$ in terms of those solutions of the scalar relativistically invariant wave equation $\Box^2\psi = k^2\psi$ in which R, α, θ, ϕ, where $r = R\cosh\alpha$, $t = R\sinh\alpha$ and r, θ, ϕ are ordinary spherical polar coordinates, are taken as independent variables. The solution of the wave equation in terms of these coordinates may be expressed in terms of associated Legendre functions and Bessel functions, the orders of the appropriate functions being determined by the condition of quadratic integrability.

The nature of the expansion of the plane wave depends on whether $(\mathbf{r}, t)$, $(\mathbf{p}, E)$ are space-like vectors or time-like vectors. When $(\mathbf{r}, t)$ and $(\mathbf{p}, E)$ are both space-like, the expansion consists of a finite sum of terms together with an integral. When either $(\mathbf{r}, t)$ or $(\mathbf{p}, E)$ is time-like, or when both $(\mathbf{r}, t)$ and $(\mathbf{p}, E)$ are time-like, the finite sum is not present in the expansion, and this consists of an integral only.

SOMERVILLE COLLEGE,
OXFORD, ENGLAND.

ON THE INTERPRETATION OF THE PARAMETERS OF THE PROPER LORENTZ GROUP

EDWARD JAY SCHREMP

The six essential parameters of the proper Lorentz group are commonly known to be equivalent to two vectors in ordinary Euclidean 3-space, one vector representing a spatial rotation, and the other representing the characteristic relative velocity of a rotation-free Lorentz transformation. (What are here provisionally pictured as *vectors* are actually specialized forms of the *quaternions* described below.) Unless these two vectors are collinear, however, their specification is not altogether unique, because of the familiar properties of noncommutativity of the transformations which they represent. The non-uniqueness which thus characterizes this physical mode of interpretation would seem to reflect a certain want of perfection.

There are, of course, alternative mathematical representations of these parameters wherein this non-uniqueness may be removed. The purpose of this note is to single out one particular geometrical interpretation of these parameters which appears to be of intrinsic physical interest.

This geometrical interpretation depends upon the well-known fact that any proper Lorentz matrix L is expressible in the form

$$L = Q\bar{Q} = \bar{Q}Q,$$

where Q is a suitable complex matrix, and $\bar{Q}$ is the corresponding complex conjugate matrix. It then turns out that one may regard every matrix Q as a representation of a complex quaternion

$$q = e_0q^0 + e_1q^1 + e_2q^2 + e_3q^3$$

whose norm is of unit magnitude. Accordingly, the three ratios q^1/q^0, q^2/q^0, q^3/q^0 constitute a set of three complex essential parameters of the proper Lorentz group. The geometrical interpretation of these parameters is then evidently equivalent to the geometrical interpretation of a general complex quaternion.

As in the case of real quaternions, the geometrical interpretation of complex quaternions involves a consideration not only of the quaternion group itself but also of the adjoint group of the quaternion group. The complex quaternion group defines, with its reciprocal group, a complex 3-dimensional group space with homogeneous coordinates (q^0, q^1, q^2, q^3); while its adjoint group defines a complex 3-dimensional vector space which is an immediate generalization of the Euclidean 3-space of ordinary experience. In this complex vector space, there is a unique geometrical interpretation of a general proper Lorentz transformation. In the terminology appropriate to this geometry, such a transformation would be called a complex rotation about a complex direction.

The terminology and the properties of the two foregoing complex geometries have been systematized in the present work, following the principle that these geometries are specializations of the general complex projective geometry of three dimensions. In the course of this study it has appeared that to every individual geometrical concept there belongs an intrinsic physical content, the physical ideas of time and motion being reflected in the complex character of the geometry.

NAVAL RESEARCH LABORATORY,
WASHINGTON, D. C., U. S. A.

EMPTY SPACE-TIMES ADMITTING THREE-PARAMETER GROUPS OF MOTION

A. H. TAUB

An empty space-time is a four-dimensional Riemannian space of signature (+ - - -) which has a vanishing Ricci tensor and no singularities in the metric tensor. The existence of empty space-times admitting three-parameter groups of motion which are not flat spaces is shown. In special coordinate systems the metric tensor of these spaces has components which are not bounded for all values of the spatial coordinates (including infinite ones). However, it is unlikely that this is due to an "essential" singularity in the metric tensor. These singularities are to be attributed to the coordinate system used since the hypersurfaces on which they occur have transitive groups operating on them. It is further shown that a space-time which admits the three-parameter group of Euclidean translations, which has a vanishing Ricci tensor, and which has the further property that the curvature tensor is not singular along the time axis is a flat space. The Einstein field equations for vanishing stress-energy tensor are integrated under the assumption that the space-time admits a three-parameter group of motions with minimum invariant varieties consisting of two-dimensional surfaces of constant curvature. There are three such cases, the Schwarzchild one and two others. It is shown that all three cases have many common properties. In particular, all three are static.

UNIVERSITY OF ILLINOIS,
URBANA, ILL., U. S. A.

INTEGRAL RELATIONSHIPS BETWEEN NUCLEAR QUANTITIES

ENOS E. WITMER

One-eleventh of the electron mass appears to be the natural unit of mass for the masses of nuclei. This unit we designate *the prout.* We [Enos E. Witmer, The Physical Review vol. 78 (1950) p. 641] have introduced the hypothesis that the masses of all those nuclei in the ground state, which are not subject to β-decay, are an integral number of prouts. Furthermore some of the nuclei subject to β-decay appear to follow this integral rule. It is a consequence of this that a large class of nuclear reaction energies should be an integral number of prouts. It is only in the case of the lighter nuclei that some of the masses and reaction energies are known with sufficient accuracy to test this hypothesis. However, the agreement in this case is as good as can be expected.

The best tables of nuclear reaction energies and nuclear masses are probably those in a recent article [Tollestrup, Fowler, and Lauritsen, The Physical Review vol. 78 (1950) p. 372]. The reaction energies in this table conform to our rule quite well, taking account of the probable errors. Furthermore almost all of these reaction energies approximate an *even* number of prouts.

Using this and other data we have made a table of nuclear masses in prouts up to Ne^{20} and a few beyond that. We find that all except two of these nuclear masses are an even number of prouts. A number of them contain many powers of 2 as factors. Thus in 24 nuclear masses there are a total of 55 powers of 2. In particular, all masses of stable nuclei consisting of $2n$ protons and $2n$ neutrons are divisible by 4 when expressed in prouts.

We take the number of prouts in the nucleus of O^{16} to be 320616. It follows from this that 1 prout is 0.00004989024831 m.u. Also 1 prout is 46.453 kev. The isotopic masses of n^1, H^1, and H^2 recomputed by these ideas are 1.0089804, 1.0081322, and 2.0147179 m.u. respectively.

A number of nuclear magnetic moments are represented by the formula

$$\mu = \frac{k}{671}\left(1 - \frac{1}{861}\right)\mu_N,$$

where k is an integer.

These results are in accord with our ideas of the importance of integers in the nuclear domain.

University of Pennsylvania,
Philadelphia, Pa., U. S. A.

NUMERICAL METHODS

ALMOST-TRIANGULAR MATRICES

Franz L. Alt

A square matrix of order n with elements a_{ij} will be called almost-triangular of degree t if $a_{ij} = 0$ whenever $j - i > t$, where t is an integer between 0 and $n - 1$. Similarly, it will be called almost-diagonal of degree t if $a_{ij} = 0$ when $|j - i| > t$. The present paper deals with numerical methods for the common matrix operations which take advantage of the special character of almost-triangular matrices. It turns out that the most efficient methods in this case are quite different from those in more general cases.

Matrices of this kind occur frequently in applications: Replacement of a system of linear ordinary, or in some cases of partial, differential equations by an approximating system of finite-difference equations, mechanical structures, other problems of mechanics, such as gear trains, and design of electrical circuits. One of the most important applications is in the field of "linear programming."

Solution of n simultaneous linear equations whose matrix A is non-singular and almost-triangular of degree t: The matrix A' pertaining to the first $n - t$ equations and the last $n - t$ unknowns is triangular. For brevity, assume that A' is not singular. Assign arbitrary values to the first t unknowns. The remaining unknowns can be determined very simply from A'. These values will, in general, not satisfy the last t equations, but will, when substituted in them, give rise to "errors" $e_1, e_2, \cdots, e_t$. A different choice of $x_1, \cdots, x_t$ will result in different errors e_i, and these are linear functions of the chosen $x_1, \cdots, x_t$. Make t different choices for the first t unknowns, forming a linearly independent set (for instance, a t-rowed unit matrix). Then, by merely solving a set of t linear equations (with matrix E formed by the "errors" e) we can determine a linear combination of the chosen x_i such that all corresponding errors vanish, so that we obtain a solution of the original system with matrix A. It can be shown that E is singular if and only if A is singular.

Finding the inverse of A can be accomplished by solving n systems of linear equations by the above method. The problem of finding the characteristic roots λ of A is attacked by choosing a number of values for λ and attempting for each of them to solve the system $(A - \lambda I)X = 0$, by the process described above. For the existence of a nontrivial solution it is necessary and sufficient that the t-rowed matrix E vanish. It can be shown that E is a polynomial of degree n in λ. In practice one evaluates E for somewhat more than $n + 1$ trial values of λ, extrapolates by differences if necessary, and finds zeros by inverse interpolation.

The methods indicated apply likewise to matrices which become triangular by striking out any t rows and columns.

National Bureau of Standards,
Washington, D. C., U. S. A.

NEW ADVANCE IN THE STRUCTURAL STUDY OF MULTIPLANE NOMOGRAMS

GEORGES R. BOULANGER

1. Present time development of nomographic computing techniques emphasizes the steadily increasing importance of multiplane nomograms (most elaborate type of nomograms now in use). Attempts to study the structure of these nomograms have been made in the past by Maurice d'Ocagne [1] and more recently by René Lambert [2]. Thosc attempts failed in the sense that, as a consequence of the lack of an efficient investigation tool, only partial results could have been obtained.

2. We have given the principle of a method for carrying on the structural study of multiplane nomograms [3]. It consists of a schematic-symbolic representation of the structures, with lattices for planes and contacts, wherein the geometric elements are represented by symbols situated at the strategic crossing points. The schemes thus obtained we named "forms" (see [4] or [5] for terminology).

As a first approach to the general morphologic study, we established in [5] the most important structural properties of the nomograms whose elements are all numbered lines and whose contacts are of the tangential type. This was based on the study of the so-called "complete tangential forms".

3. We have extended now to all types of forms the results we obtained in that particular case. We want to emphasize in this paper one aspect of this extension: the systematic generation of the forms.

4. The basic notion is that of fixed and arbitrary elements. In all types of forms, if we arrange the contacts in a systematic order, the elements automatically separate into fixed elements (always in the same positions for a given type of form) and arbitrary ones (placed arbitrarily in determinate areas). We have given the design of the fixed elements in both cases of independent and grouped planes, and this permits an immediate generation of all the complete forms (tangential, punctual, combined).

5. We consider, in the complete forms so-obtained, the possibilities of degeneracy (lines becoming points) and derivation (numbered elements going into constant ones). This gives the noncomplete forms in a quite logical way.

That procedure implies a preliminary study of the contacts. We have indicated in [6] that all the nomographic contacts may be constructed out of 18 elementary ones, which we proposed to consider as fundamental.

6. The systematic generation of the forms leads to a rational classification of the corresponding nomograms and eliminates the divergencies that arose in the past from empirical considerations.

REFERENCES

1. M. D'OCAGNE, *Traité de nomographie*, Paris, Gauthier-Villars, 1921.
2. R. LAMBERT, *Structure générale des nomogrammes et des systèmes nomographiques*, Paris, Hermann, 1937.

3. G. Boulanger, *Un mode de représentation schématique de la structure des abaques à plans mobiles superposés*, C. R. Acad. Sci. Paris vol. 224 (1947) pp. 181–183.
4. ———, *Sur la classification des formes. Terminologie*, C. R. Acad. Sci. Paris vol. 224 (1947) pp. 1693–1694.
5. ———, *Contribution à la théorie générale des abaques à plans superposés* (Thèse Doctorale Université de Paris), Bruxelles, Robert Louis, et Paris, Gauthier-Villars, 1949.
6. ———, *Sur la notion de contact nomographique*, C. R. Acad. Sci. Paris vol. 229 (1949) pp. 971–973.

University of Brussels,
Brussels, Belgium.

ASYMPTOTICALLY ERGODIC OUTPUT UNDER ERGODIC INPUT OF DELAY DIFFERENTIAL MACHINES

Frank H. Brownell

We consider here the behavior of a linear autonomous delay differential machine subjected to an ergodic input, the governing equation thus being

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \int_{0 \leq h}^{\infty} x^{(k)}(t-h)\, dF_k(h) = y(t) \tag{1}$$

where $y(t)$ is the input and $x(t)$ the output. If the machine coefficient functions $F_k(h)$ satisfy a few reasonable conditions, and if the resulting system is stable, we can show that if $y(\xi, t)$ is an ergodic process for ξ over a probability space, then $x(\xi, t) = x_1(\xi, t) + x_2(\xi, t)$ where $x_1(\xi, t)$ decays exponentially for each ξ and $x_2(\xi, t)$ is an ergodic process. Also the power spectrum of $x_2(\xi, t)$ can be obtained as one would formally expect from that of $y(\xi, t)$.

The proof uses the Laplace transform representation of the solution of (1), which can easily be justified, and from which the conclusion is almost trivial. The result is implicit for the significance of much of Wiener's *Times series*. Doob in the Berkeley Symposium, 1945–1946, proves for equation (1) without delay terms that a stationary input implies an asymptotically stationary output, but does not discuss ergodicity.

University of Washington,
Seattle, Wash., U. S. A.

REMARKS ON AN ALGEBRAIC METHOD FOR NUMERICALLY SOLVING THE FREDHOLM INTEGRAL EQUATION $y - \lambda\mathfrak{K}y = f$

Hans Bückner

Let $\mathfrak{H}$ be the set of all functions to which $\mathfrak{K}$ (Hermitian) is applicable. Let $\mathfrak{P}_n$ be a linear operator for $\mathfrak{H}$ and $\mathfrak{H}_n = \mathfrak{P}_n\mathfrak{H}$ a set of stepwise constant functions with at most n steps. $\mathfrak{H}_n$ shall contain the sum of any two elements of it. In order to find approximations for y and for eigenfunctions and eigenvalues, $\mathfrak{K}y$ may be substituted by a finite sum in the sense of numerical quadrature. This gives a system of say n ordinary linear equations for n ordinates of y.

Results: Any such system is equivalent to another Fredholm integral equation $\eta - \lambda\mathfrak{K}_n\eta = \mathfrak{P}_n f$ with η and the corresponding eigenfunctions belonging to $\mathfrak{H}_n$ for a suitable $\mathfrak{P}_n$. If $\lim_{n\to\infty}\psi(n)(\mathfrak{K}_n\mathfrak{P}_n - \mathfrak{P}_n\mathfrak{K})z = \mathfrak{M}z$; $z \subset \mathfrak{H}' \subset \mathfrak{H}$ exists for a suitable subset $\mathfrak{H}'$ and for a suitable function $\psi(n) \to +\infty$ with $n \to \infty$, a linear operator $\mathfrak{M}$ is defined for $\mathfrak{H}'$. Let y and the eigenfunctions of $\mathfrak{K}$ belong to $\mathfrak{H}'$. Then the deferred approach to the limit for the a.m. method gives the first order solutions of the following problem of disturbances

$$y - \lambda(\mathfrak{K} + \epsilon\mathfrak{M})y = f; \qquad \epsilon = \frac{1}{\psi(n)}.$$

Hence the deferred approach to the limit is a link between the algebraic method and the method of disturbances, and it seems that this statement is not restricted to integral equations.

Minden, Germany.

ON A RELAXATION METHOD FOR EIGENVALUE PROBLEMS

Stephen H. Crandall

The relaxation method given by Vazsonyi (Journal of Applied Physics vol. 15 (1944) pp. 598–606) for eigenvalue problems in the plane is considered for the general eigenvalue problem of real symmetric matrices. The convergence of the relaxation method is proved by means of a convergent iterative process constructed within the framework of the relaxation method. The method seems to be particularly well suited to exploratory computations where a rough (slide rule accuracy) location of the latent roots is desired, since the same computational apparatus is used on all modes and no orthogonalizations or inversions are necessary.

If A is a real symmetric matrix, the eigenvectors x and the corresponding eigenvalues λ satisfy $Ax = \lambda x$. If an arbitrary vector v is substituted in both sides of this equation, instead of a single scalar λ, a set of n numbers L_i is obtained. It is shown that there is always an eigenvalue included within the range defined by the extreme L_i. A standard relaxation technique is used to alter the initial v until this range is made as narrow as desired. There is a weighted average of the L_i, R, such that $R - \lambda$ is $O(\epsilon^2)$ when for the individual L_i, $L_i - \lambda$ is $O(\epsilon)$. A numerical example and some remarks regarding the actual technique of the relaxation process are given.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASS., U. S. A.

NEW MATRIX TRANSFORMATIONS FOR OBTAINING CHARACTERISTIC VECTORS

WILLIAM FELLER AND GEORGE E. FORSYTHE

Let A be a square matrix, symmetric or not, with linear elementary divisors. Suppose λ is a known characteristic value of A, with associated column vector C and row vector R; and suppose ν is an unknown characteristic value of A, with associated vectors X, Y. Let γ, ρ, β, t be arbitrary complex-valued parameters. Explicit formulas are given for a new family F of matrix transformations of the type $A' = U(A - tCR)U^{-1}$, where $U = U(\gamma, \rho, \beta)$. In terms of the characteristic value ν' and vectors X', Y' of A', simple formulas are given for the corresponding quantities $\nu = \nu'$, $X = U^{-1}X'$, $Y = Y'U$ of A. It is shown that F includes the special transformations of Hotelling ("deflation," 1933), Duncan and Collar (1934), Semendiaev (1943), and Blanch (unpublished). Two new special transformations which are both *order-reducing* and *symmetry-preserving* lead to promising numerical procedures for getting subdominant characteristic roots of a symmetric matrix whose dominant root is known. For arbitrary matrices it may sometimes be useful to have the wide choice of transformations which the family F affords. A numerical example is given.

PRINCETON UNIVERSITY,
PRINCETON, N. J., U. S. A.
NATIONAL BUREAU OF STANDARDS,
LOS ANGELES, CALIF., U. S. A.

THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

L. FOX

The successful use of relaxation methods for the numerical solution of ordinary differential equations with two point boundary conditions has led to a re-examination of the classical step-by-step methods for the solution of one point boundary condition problems.

If finite-difference formulae of differentiation are used throughout, the problem can be reduced to the solution of a set of algebraic equations, typified by $L(y) + \Delta(y) = 0$, where $L(y)$ is a function of values of the required function at adjacent pivotal points, and $\Delta(y)$ is the "difference correction," a known function of higher differences of y.

In relaxation methods the solution y is obtained iteratively from successive approximations y_r, according to the scheme $L(y_r) + \Delta(y_{r-1}) = 0$, $\Delta(y_0) = 0$. At each step the set of equations is treated, and solved by relaxation, as simultaneous algebraic equations, linear if the given differential equation is linear.

If all the boundary conditions are given at the same point the algebraic equations can be regarded and solved as recurrence relations, Δ being introduced iteratively as before. This technique is a fundamental departure from classical step-by-step methods. No estimation, extrapolation, or verification is required, and only the function, its differences, and Δ need be recorded. This process, moreover, has the merit of all iterative methods, that it is not necessary to work throughout with the number of figures finally required; more can be added as approximate solutions become more accurate. In some cases the original finite-difference equations can be turned into new forms, only slightly more complicated than the old, applicable to both recurrence and relaxation, and with a very small difference correction.

For linear and certain types of nonlinear equations this new technique is a distinct improvement over the old. Interesting problems occur in which relaxation and recurrence are used side-by-side over different parts of the range of integration. When recurrence is not accurate because of building-up error, relaxation is at its best; when relaxation is laborious because the simultaneous equations are ill-conditioned, recurrence is very satisfactory. Since the same finite-difference equations are used throughout there is no bump in the differences of approximate solutions, where recurrence and relaxation meet, so that Δ can be satisfactorily calculated over the whole range.

NATIONAL PHYSICAL LABORATORY,
TEDDINGTON, ENGLAND.

AN ALGORITHM FOR THE CONSTRUCTION OF A POLYNOMIAL REPRESENTING A GIVEN TABULAR FUNCTION

F. N. FRENKIEL AND H. POLACHEK

Sponsored by the ONR

A simple scheme (algorithm) is developed for calculating the coefficients $a_1, a_2, \cdots, a_n$ of the polynomial

$$Y = a_1 + a_2X + a_3X^2 + \cdots + a_nX^{n-1}$$

passing through n arbitrary points $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$. The form of the expressions obtained is such that any given coefficient a_i may be easily computed from the succeeding coefficients, a_{i+1}, a_{i+2}, $\cdots$, a_n. Consequently, in applying the algorithm, a_n is computed first, then a_{n-1}, a_{n-2}, $\cdots$, finally a_1. The polynomial thus obtained may be used in place of an initially prescribed mathematical or physical function for the purpose of carrying out interpolation, differentiation, or other mathematical operations.

NAVAL ORDNANCE LABORATORY,
SILVER SPRING, MD., U. S. A.

ITERATIVE METHODS FOR OBTAINING SOLUTIONS OF BOUNDARY VALUE PROBLEMS

MAGNUS R. HESTENES

Consider two bounded symmetric linear operators A and B on a Hilbert space over the reals. Suppose B is positive and completely continuous and that there is a number k such that $A\ kB$ is positive definite. Let $\mu(x) = (x, Ax)/(x, Bx)$. It is shown that there exist positive numbers b, c such that if we set $x_{n+1} = x_n - \alpha_n\xi_n$, $\mu_n = \mu(x_n)$, $\xi_n = Ax_n - \mu_n Bx_n$ where $b \leqq \alpha_n \leqq c$, then $\{\mu_n\}$ converges to a characteristic value and $\{x_n\}$ converges strongly in subsequence to a corresponding characteristic vector. The case when α_n is chosen so as to minimize $\mu(x_n - \alpha\xi_n)$ is also considered. As an alternate method set $\rho_n = |\xi_n|^2/(x_n, Bx_n)$, $\sigma_n = 2(\xi_n, Bx_n)/(x_n, Bx_n)$, $\eta_n = A\xi_n - \mu_n B\xi_n - \sigma_n\xi_n - \rho_n Bx_n$ and $x_{n+1} = x_n - \alpha_n\eta_n$. Under certain conditions $\{\mu_n\}$ will again converge to a characteristic value and $\{x_n\}$ will converge in subsequence to a characteristic vector. These results are applicable to self-adjoint systems of differential equations.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIF., U. S. A.
NATIONAL BUREAU OF STANDARDS,
LOS ANGELES, CALIF., U. S. A.

A METHOD OF GRADIENTS FOR THE CALCULATION OF THE CHARACTERISTIC ROOTS AND VECTORS OF A REAL SYMMETRIC MATRIX

M. R. HESTENES AND W. KARUSH

The least characteristic root of a real symmetric matrix A is $\lambda_1 = \min \mu(x)$, where $\mu(x) = (x, Ax)/(x, x)$, $x \neq 0$. The gradient of μ is proportional to $\xi(x) = Ax - \mu(x)x$. Thus we are led to the iteration scheme $x^{i+1} = x^i - \alpha\xi(x^i)$ (where $\alpha > 0$ may depend upon x^i) for possible convergence to a (minimum) characteristic vector belonging to λ_1. Several methods for prescribing α are given, and proofs of convergence are made. Theorems on rate of convergence are established. Analogous results hold for a maximum characteristic vector. A generalized iteration scheme is shown to yield convergence to all the characteristic vectors of the smallest invariant subspace containing the initial vector x^0. These methods have been effective in several numerical cases; they are especially adapted to automatic computing machines.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIF., U. S. A.
NATIONAL BUREAU OF STANDARDS,
LOS ANGELES, CALIF., U. S. A.
UNIVERSITY OF CHICAGO,
CHICAGO, ILL., U. S. A.

DETERMINATION OF INSTRUCTION CODES FOR AUTOMATIC COMPUTERS

ROBERT F. SHAW

The development of automatically sequenced computers has resulted in a renewed interest in the basic logic of mathematical processes. Since such computers can respond to only such instructions and consequently perform only such elementary operations as have been provided for in their construction, the careful choice of such "built-in" instructions becomes a matter of the greatest importance. A well-chosen set of instructions, or "code," will simplify programming (preparation of problems for the computer), reduce operating time, and in general will widen the scope of work which is economically practicable.

A fully automatic computer capable of performing any arithmetic or logical operation can probably be built with only three elementary instructions. One of these is arithmetic (addition, subtraction, or the equivalent); the second is logical (such as the choice of two alternate sequences of operations dependent on some criterion which in turn depends on the results of previous operations); the third is physical (the transfer of the result of an arithmetic operation to a storage position for future reference). Such a computer, however, would be found very difficult to program and would be so inefficient as to be impractical, even though theoretically workable.

In the code of any practical computer, it is certain that both addition and subtraction would be included, and also multiplication (except possibly for certain special-purpose machines). Division might also be provided, thus completing the four elementary operations of arithmetic. Other instructions which are quite useful and require little additional equipment are an unconditional transfer of control; unconditional and conditional stops, right and left shifts, an instruction to record the point of leaving the main sequence of instructions on a transfer of control and some type of extraction operation, to facilitate the separation of certain digits of a number from others. The choice of additional instructions, such as those associated with input and output of information and those concerned with moving data from point to point in the computer, will be determined partly by the type of computer and partly by the type of work to be performed.

In the majority of problems, the amount of programming devoted to actual computation is small compared to that required for the control of the processes. The most promising line of development in the choice of an instruction code therefore appears to involve the selection of complex instructions which will simplify the programming of the control operations.

ECKERT-MAUCHLY COMPUTER CORPORATION,
PHILADELPHIA, PA., U. S. A.

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY MEANS OF CONTINUOUS-VARIABLE MATHEMATICAL MACHINES

HENRY WALLMAN

In a recently published paper by the author, *On an electronic integral transform computer, with application to integral equations* in the Journal of the Franklin Institute, 1950, a comparatively simple continuous-variable (analog) mathematical machine is described. The present paper discusses a simplified version capable of solving certain partial differential equations of elliptic type with boundary conditions, such as the Dirichlet potential problem. In a recent doctoral thesis by Alan B. Macnee, *An electronic differential analyzer*, Proceedings of the Institute of Radio Engineers, 1949, another simple but flexible continuous-variable mathematical machine is described, and in the present paper it is shown how this can be used for solving partial differential equations of parabolic or hyperbolic type; the important point is the ready adaptability of the electronic differential analyzer (as contrasted with the mechanical differential analyzer) to solving the "two-point" boundary value problem for ordinary differential equations.

CHALMERS INSTITUTE OF TECHNOLOGY,
GOTHENBURG, SWEDEN.

IMPROVED ADAMS METHOD OF NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

P. W. ZETTLER-SEIDEL

The differential equation ("D.E.") $y^{(N)} = F(y^{(N-1)}, \cdots, y, x)$ is solved by iteratively integrating a suitable approximation $f(t)$ of $y^{(N)}(x)$ $(t = (x - x_0)/h$, where x_0 and $x_{-1} = x_0 - h$ are the "last" and "next last" values of x). The iterative integration of an approximation of the highest derivative both facilitates the numerical procedure and is more accurate than approximating each derivative $y^{(N)}(x), \cdots, y'(x)$ independently, as usually done. It is thus desirable to have N as large as feasible in order to increase the accuracy of the computation. The "interpolative integration" (from $t = -1$ to $t = 0$) improves the values at $x = x_0$, the "extrapolative integration" (from $t = 0$ to $t = \lambda$, i.e. from $x = x_0$ to $x = x_1 = x_0 + \lambda h$) performs the next step of the numerical integration of the D.E. Formulae are derived for both the interpolative and extrapolative integrations for any integrable $f(t)$, and they are also specialized for $f(t)$ being an arbitrary polynomial. These formulae are written such that in the actual application all confusing usage of foregoing values in any column of the computation form is avoided. The most suitable polynomials $f(t)$ are found to be: (A) the quadratic determined by three consecutive values of $y^{(N)}(t)$, (B) the quadratic determined by two consecutive values of $y^{(N)}(t)$ and its derivative at either point, and (C) the cubic determined by 2 consecutive values of $y^{(N)}(t)$ and its derivatives $y^{(N+1)}(t)$ at these two points. From these, 5 feasible polynomial methods are derived. The interval length Δx and also the method used can be changed from step to step without difficulty. The iterated integrals are multiples of linear combinations of $y_0^{(N)}$, $\Delta = y_0^{(N)} - y_{-1}^{(N)}$, $\cdots$, $y_0^{(N-\nu)}/h^\nu$; $y_0^{(N+1)}h$, with integer coefficients. The multiplying factor is the product of a certain power of h or $h\lambda$ and a rational number which depends (as do the integer coefficients) solely on the method employed and on the (rational) interval ratio. Therefore, they can be computed once and for all and inserted in removable headings for the computation form. The computation procedures described work precisely the same way at the very beginning of the computation. Quantities like Δ which cannot be computed in the routine way have to be computed according to the assumptions made about $y^{(N)}(x)$, and the computation then follows the routine scheme. The flexibility of the interval length facilitates the starting procedure, too. These methods are also applicable to Volterra-type integro-differential equations and to systems of equations.

NAVAL ORDNANCE LABORATORY,
SILVER SPRING, MD., U. S. A.

PARTIAL DIFFERENTIAL EQUATIONS

LE PROBLÈME DE CAUCHY POUR CERTAINS SYSTÈMES D'ÉQUATIONS LINÉAIRES AUX DÉRIVÉES PARTIELLES TOTALEMENT HYPERBOLIQUES

Florent Bureau

L'étude de certains systèmes d'équations linéaires aux dérivées partielles est liée à celle d'équations linéaires aux dérivées partielles en général d'ordre supérieur au second. On peut en déduire pour ces systèmes d'équations, des solutions élémentaires et éventuellement des solutions auxiliaires permettant l'application de la méthode des singularités.

Application à quelques exemples et en particulier à l'intégration des équations de propagation des ondes lumineuses dans les milieux cristallins uniaxes.

University of Liège,
Liège, Belgium.

ON A QUASI-LINEAR SYSTEM OF HYPERBOLIC DIFFERENTIAL EQUATIONS WITH A PARAMETER AND A SINGULARITY

Yu Why Chen

Consider a system of four equations for u, v, x, and y in two characteristic variables σ and τ:

$$(*) \quad \begin{aligned} x_\sigma - Sy_\sigma = 0, \qquad x_\tau + Ty_\tau = 0, \qquad -Pv_\sigma + u_\sigma + Quy_\sigma y^{-1} = 0, \\ Pv_\tau + u_\tau + Ruy_\tau y^{-1} = 0. \end{aligned}$$

P, Q, R, S, and T are functions of u and v with continuous second derivatives in a domain which contains a segment $u = 0$, $a \leqq v \leqq b$. We assume that (1) $Q = k + uQ^*(u, v)$, $R = k + uR^*(u, v)$ where k is a constant parameter, (2) $P \neq 0$, $S + T \neq 0$. Such a system of equations is given, for example, with $k = 1$ by axial symmetric, supersonic flows and by cylindrical waves, and with $k = 2$ by spherical waves. Two problems are treated and their dependence on k investigated.

Problem I. Given data along two half-characteristics C ($\sigma \geqq 0$, $\tau = 0$) and D ($\tau \geqq 0$, $\sigma = 0$) with $y = 0$ at the origin, to find the solution of (*) in Δ_ρ : $0 \leqq \sigma + \tau \leqq \rho$, $\sigma \geqq 0$, $\tau \geqq 0$ for sufficiently small ρ. Let $y = \sigma$, $u = f_1(\sigma)$, $v = v_0 + f_2(\sigma)$ along C, and $y = \tau$, $u = g_1(\tau)$, $v = v_0 + g_2(\tau)$ along D. v_0 is a constant and f_i and g_i are continuous differentiable except possibly at the origin. Depending on the behavior of the data at $\sigma = 0$ and $\tau = 0$ we have two types of solutions. Denote $\gamma = \text{Max}\,(0, 1 - k/2, -k)$. Type A: $f_i(\sigma) = O(\sigma^{\gamma+\epsilon})$, $g_i(\tau) = O(\tau^{\gamma+\epsilon})$

with $\epsilon > 0$. Solution u, v, and $y/(\sigma + \tau)$ possess continuous first partial derivatives $= O((\sigma + \tau)^{\gamma+\epsilon-1})$. Type B: $0 < |k| < 2$. $f_i(\sigma) = \text{const} \cdot \sigma^\gamma + O(\sigma^{\gamma+\epsilon})$, $g_i(\tau) = O(\tau^{\gamma+\epsilon})$. The first partial σ-derivatives of u, v, and $y/(\sigma + \tau)$ behave like $O((\sigma + \tau)^{\gamma-1})$ while their τ-derivatives are $O((\sigma + \tau)^{\gamma-1} (\log (\tau/(\sigma + \tau)) + 1))$. $\partial(x, y)/\partial(\sigma, \tau)$ is $\neq 0$.

Problem II. We consider only $k > 0$. Along a segment s: $\sigma + \tau = 0$, $-c \leqq \tau \leqq 0$, $0 \leqq \sigma \leqq c$ we assign initial values $x = h_1(\sigma)$, $y = 0$, $u = 0$, $v = v_0 + h_2(\sigma)$ with $h_2(0) = 0$. h_1 is continuous and h_2 is continuous differentiable except possibly at $\sigma = 0$. We again find two types of solutions of the initial value problem for (*) in a domain $\Omega_\rho : 0 \leqq \sigma + \tau \leqq \rho$, $-c \leqq \tau \leqq 0$, $0 \leqq \sigma \leqq c$ for sufficiently small ρ. Depending on whether $h_2(\sigma) = O(\sigma^{\gamma+\epsilon})$ or $h_2(\sigma) = \text{const} \cdot \sigma^\gamma + O(\sigma^{\gamma+\epsilon})$ we have (A): the first derivatives of u, v, and $y/(\sigma + \tau)$ are $O(\sigma^{\gamma-1+\epsilon})$, or (B): the first σ-derivatives are $O(\sigma^{\gamma-1})$ while the τ-derivatives are $O(\sigma^{\gamma-1}(\log(-\tau/\sigma) + 1))$; $\partial(x, y)/\partial(\sigma, \tau) \neq 0$.

Series expansion of the solution in proper variables and Goursat's problems are discussed. The method is also applied to treat supersonic flows beyond a sonic line.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

MULTIPLE REPRESENTATIONS FOR GREEN'S FUNCTIONS OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

BERNARD FRIEDMAN

Let $L[\varphi(\xi, \eta)]$ be an elliptic, linear self-adjoint differential operator of order two on a function $\varphi(\xi, \eta)$. Assume that L can be separated in the (ξ, η) coordinate system, that is, there exist two ordinary differential operators M and N such that if $M[u(\xi)] = 0$, $N[v(\eta)] = 0$ and if $\varphi(\xi, \eta) = u(\xi)v(\eta)$, then $L[\varphi(\xi, \eta)] = 0$. Consider the problem of finding a Green's function for (*) $L(\varphi) = 0$, that is, a function $\varphi(\xi, \eta)$ satisfying (*) in the region $\alpha < \xi < \beta, \gamma < \eta < \infty$ except at the point ξ_0, η_0 where $\varphi(\xi, \eta)$ has an appropriate singularity, and satisfying the following boundary conditions: $\varphi(\alpha, \eta) = \varphi(B, \eta) = \varphi(\xi, \gamma) = 0$ and a "Sommerfeld radiation" condition as η approaches infinity. The classical solution of (*) is obtained by solving the following eigenvalue problem: $M[u_n(\xi)] = 0$, $u_n(\alpha) = u_n(\beta) = 0$, and then expressing $\varphi(\xi, \eta)$ in an eigenfunction expansion: $\varphi(\xi, \eta) = \sum c_n u_n(\xi) v_n(\eta)$, where $v_n(\eta)$ is a continuous function with a discontinuous derivative at η_0, and such that $N[v(\eta)] = 0$, $v(\gamma) = 0$, and $v(\eta)$ satisfies the Sommerfeld radiation condition as η approaches infinity.

Using an idea due to Watson (Proc. Roy. Soc. London Ser. A vol. 95 (1918)) we may express $\varphi(\xi, \eta)$ as a contour integral in the n-plane and by an appropriate shifting of the contour of integration we find that $\varphi(\xi, \eta) = \sum c_m \bar{v}_m(\eta) \bar{u}_m(\xi)$

where $\bar{v}_m(\eta)$ is an eigenfunction of the following problem: $(\overset{*}{*})$ $N[\bar{v}_m(\eta)] = 0$, $\bar{v}_m(\gamma) = 0$, and $\bar{v}_m(\eta)$ satisfies the Sommerfeld radiation condition at infinity, while $\bar{u}_m(\xi)$ is a continuous function with a discontinuous derivative at ξ_0 and such that $M[\bar{u}(\xi)] = 0$, $\bar{u}(\alpha) = \bar{u}(\gamma) = 0$. It is to be noted that $(\overset{*}{*})$ may have a continuous spectrum so that $\varphi(\xi, \eta)$ will actually be represented by an integral.

The method outlined above may be applied to the discussion of the scattering of acoustic or electro-magnetic waves from a sphere or a cylinder and the second representation of the Green's function is found to be the one more suitable for practical computation. In other cases the second representation gives proofs similar to those of Titchmarsh "Eigenfunction Expansions" for the validity of the usual integral transform theorems, such as the Fourier, Mellin, or Hankel transforms.

An interesting unsolved question in connection with $(\overset{*}{*})$ is to find conditions under which it has a discrete spectrum. Because of the Sommerfeld radiation condition the problem is not self-adjoint and so can not be handled by the usual methods.

New York University,
New York, N. Y., U. S. A.

THE GENERAL INTEGRATION OF A QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER COMPOSED OF SYMMETRIC CARTESIAN INVARIANTS

Robert D. Gordon

Given a differential equation of the type described in the title, in two independent variables, which is quasi-linear, let u be the unknown function, a second unknown v is introduced which is orthogonal to u, and the orthogonality relation is expressed through a third unknown w. If (u, v) are now introduced as a coordinate system, one obtains a linear differential relation of pfaffian form which does not involve v. Minor group-theoretical considerations permit the general integration of this relation in a form which permits direct fitting to an arbitrary Cauchy boundary problem. Trigonometric considerations result finally in a first order partial differential equation of form determined by the boundary conditions, whose characteristics determine uniquely the final solution.

Necessary and sufficient restrictions on the boundary conditions are discussed. At the time of writing, the problem of Dirichlet has not been studied. Results are applicable to plane gas dynamics.

University of Buffalo,
Buffalo, N. Y., U. S. A.

THE HYPERCIRCLE METHOD OF APPROXIMATION TO THE SOLUTION OF A GENERAL CLASS OF BOUNDARY-VALUE PROBLEMS

A. J. McCONNELL

The hypercircle method of approximation, first used by Prager and Synge (Quarterly of Applied Mathematics, 1947) as a special method for the elastostatic problem, is now known to be a general method applicable to any boundary-value problem which leads to a system of linear second-order differential equations, derivable from a variational principle; it can also be applied to the solution of the general linear fourth-order differential equation with given boundary conditions.

The method is expressed in geometrical language by representing the solution of our boundary-value problem as a vector (or point) $\mathbf{U}$ in a certain function space, for which a scalar product $\mathbf{U}' \cdot \mathbf{U}''$, and therefore also a metric, can be defined. The first step consists in splitting the original problem into two complementary relaxed problems, solutions of which are easily obtained. These solutions define two linear subspaces of the function space which are mutually orthogonal to each other. If $\mathbf{U}$ is the solution of our original problem, then $\mathbf{U} - \mathbf{U}_1$ belongs to the first subspace and $\mathbf{U} - \mathbf{U}_2$ belongs to the second subspace, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are two specified points of the function space. From this we conclude that (i) $\mathbf{U}$ lies on the hypersphere $(\mathbf{U} - \mathbf{U}_1) \cdot (\mathbf{U} - \mathbf{U}_2) = 0$, (ii) $\mathbf{U}$ lies on the hyperplane $\mathbf{I}_2 \cdot (\mathbf{U} - \mathbf{U}_1) = 0$, where $\mathbf{I}_2$ is any vector of the second subspace, (iii) $\mathbf{U}$ lies on the hyperplane $\mathbf{I}_1 \cdot (\mathbf{U} - \mathbf{U}_2) = 0$, where $\mathbf{I}_1$ is any vector of the first subspace. Our solution therefore lies on the intersection of a hypersphere and an indefinite number of hyperplanes, that is, on a hypercircle, whose centre and radius can be easily obtained. Moreover the radius of the hypercircle can be made progressively smaller by taking an increasing number of vectors of the two fundamental subspaces. We take the centre of the hypercircle as an approximation (with an arbitrarily close degree of accuracy) to the solution of the boundary-value problem, and the radius gives a measure of the error involved in the approximation.

UNIVERSITY OF DUBLIN,
DUBLIN, IRELAND.

COORDINATE SYSTEMS PERMITTING SEPARATION OF THE LAPLACE AND HELMHOLTZ EQUATIONS

DOMINA EBERLE SPENCER

It is well known that the Laplace and Helmholtz equations separate in only a limited number of coordinate systems. Both simple separability and R-separability will be considered. Two sets of conditions must be satisfied by the metric coefficients: the Stäckel conditions, which give necessary and sufficient conditions for separability, and the Lamé conditions, which insure that the metric coefficients can be associated with an orthogonal family of surfaces in Euclidean three-space. The former were considered in a paper presented last year [D. E. Spencer, *Separability conditions for some equations of mathematical physics*, Bull. Amer. Math. Soc. Abstract 55-11-566]. The problem treated in this paper is to find all possible solutions of both the Stäckel and the Lamé equations.

This leads to a specification of the coordinate systems that allow separation. Included are the eleven coordinate systems made famous by Eisenhart [L. P. Eisenhart, *Separable systems of Stäckel*, Ann. of Math. vol. 35 (1934) p. 284]. In the solution of the Laplace equation it is also possible to have R-separability which holds for certain families of cyclids. The results will be compared with the work of Levinson, Bogert, and Redheffer [N. Levinson, B. Bogert, and R. M. Redheffer, *Separation of Laplace's equation*, Quarterly of Applied Mathematics vol. 7 (1949) p. 241] and attention will be drawn to several important discrepancies in the latter work.

BROWN UNIVERSITY,
PROVIDENCE, R. I., U. S. A.

AN INVERSION FORMULA

GÖRAN BORG

In the last few years some problems centering around the following question have been studied. Let two differential equations

$$(\mathrm{L}) \quad y'' + (\lambda + q(x))y = 0, \qquad (\mathrm{L}_1) \quad y'' + (\lambda + q_1(x))y = 0$$

have the properties (1) $q(x), q_1(x)$ continuous for $0 < x < \infty$, $\int_1^\infty | q^i (x) | \, dx < \infty$, and either (2) $\int_0^\infty x \, | q^i_{\cdot} (x) | \, dx < \infty$ $(i = 0, 1)$ or (2′) $q(x) \to \infty$ when $x \to 0$. Then those solutions $\phi(x, k)$, which have the property (3) $\phi(x, k) \to 0$ when $x \to 0$, can be written $\phi(x, k) = A \sin (kx - \eta_1) + o(1)$ for x large, and the relation

$$k \sin (\eta - \eta_1) = \int_0^\infty (q - q_1)\phi(x, k)\phi_1(x, k)A^{-1}A_1^{-1} \, dx \qquad (k = \lambda^{1/2})$$

will hold. The problem is to invert this relation, i.e., to solve for $q - q_1$ if this be possible. As far as I know, it has not yet been done rigorously.

The question is to find a suitable singular kernel by means of linear combinations of the products $\phi(x, k) \cdot \phi_1(x, k)$. This singular kernel can easily be constructed from the Green functions G and $G_1(x, t)$ of (L) and (L_1), corresponding to end conditions (M): (3) and $\phi(x, k) \subset \mathrm{L}^2(1, \infty)$, $\operatorname{Im} \lambda \neq 0$. If further $g(x, t)$ and $g_1(x, t)$ are solutions, satisfying $g(t, t) = 0$, $g_x'(x, t) = 1$ for $x = t$, then $\Gamma(x, t, k) = GG_1$ for $x > t$ and $\Gamma = GG_1 + gG_1 + g_1G$ for $x < t$ will be the kernel in question.

Integrating $\int d\lambda \int_0^\infty \Gamma_t' (x, t, k)f(t) \, dt = I$ around a contour, composed of $| \lambda | = R$, $\operatorname{Im} \lambda \geqq \delta$ and the straight line $\operatorname{Im} \lambda = \delta$, $-R \leqq \operatorname{Re} \lambda \leqq + R$ ($f(t)$ being two times differentiable, $= 0$ for $t < c$ and for $t > b$, $0 < c < b < \infty$), we shall get $I = 0$. The part of the integral, which is taken over the half of a circle is equal to $-\pi i f(x)/2 + o(1)$, $o(1)$ tending to 0 when R tends to ∞, independently of δ. Then, after putting $q(x)$ and $q_1(x) = 0$ for $x > b$, letting $\delta \to 0$ and $R \to \infty$, we get a formula, which can be extended to hold true for more general functions $f(x)$ than the ones above, such that we can put $f(x) = \int_x^\infty (q - q_1) \, dt$. Suitable passages to the limit will then give the following *inversion formula*

$$\int_x^\infty - (q - q_1) \, dt = \frac{4}{\pi} \int_0^\infty \frac{dk}{AA_1} [\phi_1(x, k)\theta(x, k) + \phi\theta_1] \sin (\eta - \eta_1)$$
$$- 2 \sum_n \left\{ \phi_n(x)\bar{\phi}_{1n}(x) \int_0^\infty \phi_n(t)\phi_1(t, \lambda_n)(q - q_1) \, dt \right.$$
$$\left. + \bar{\phi}_n\phi_{1n} \int_0^\infty \phi(t, \lambda_{1n})\phi_{1n}(t)(q - q_1) \, dt \right\}.$$

$\phi_n(x)$ is the nth eigenfunction of (L, M), $\bar{\phi}_{1n}(x)$ is a corresponding solution of (L_1) for $\lambda = \lambda_n$. $\theta(x, k)$ is a solution defined by $\theta(x, k) = \mu \sin kx - \nu \cos kx + o(1)$ if μ and ν are defined by $\phi(x, k) = \mu \cos kx + \nu \sin kx + o(1)$.

The formula is certainly valid and the number of terms in the series is finite, if in addition to (1), (2), (2′) the following conditions hold: (4) $\int_1^\infty x \mid q(x) \mid dx < \infty$, $\int_1^\infty x \mid q_1(x) \mid dx < \infty$, $q - q_1 = o(1)$, $x \to 0$, (5) (L, M) and (L_1, M) have no eigenvalues in common.

If (5) does not hold, we shall get more terms in the series. If (4) does not hold, there can be infinitely many eigenvalues. Anyhow, the right-hand member of the formula, constructed for functions $q(x)$ and $q_1(x)$, which are $= 0$ for $x > b$, will tend to $\int_x^\infty (q - q_1)\, dt$ when $b \to \infty$.

If there is no point spectrum, we get a complete inversion of the original formula.

Ref. C. E. Fröberg, Arkiv för Matematik, Astronomi och Fysik 36A no. 11 (1948). Cf. also the uniqueness theorems of Levinson, Det Kgl. Danske Videnskabernes Selskab. Matematisk-Fysiske Meddelelser vol. 25 no. 9 (1949).

Matematiska Institutionen,
Uppsala, Sweden.

THE MATHEMATICAL NATURE OF LAMARCK'S HYPOTHESIS THAT A BIOLOGICAL SPECIES TENDS TO INCREASE IN SIZE

Oliver E. Glenn

Biology furnishes the following systematization of real numbers. An individual organism is a structure of characters each of which increases (or develops), numerically, from the time of its appearance in the embryo. If we plot this numerical value (r), using time (φ) as the independent variable, for any character, we obtain a curve k_1. Repeating this, for the same character, in each individual of a line of heredity, we obtain a field F of curves k_j. A simple construction by means of F gives a more expressive field F_1 of space-curves c_j. The c_j are permuted by a group G of Lie's type defined by,

$$S = S(u_1, v_1): \{r' = r + u_1 p(r), \theta' = \theta + v_1 q(\theta), \varphi' = \varphi, (\mid u_1 \mid, \mid v_1 \mid \doteq 0)\};$$

$$S^2 = S(u_1, v_1)S(u_2, v_2) = S(u_1 + u_2, v_1 + v_2).$$

We regard the c_j as integral curves of an invariant, ordinary differential equation in (r, θ, φ), whence they are either a determinate single infinity on an arbitrary surface or arbitrary curves on a single surface, viz.,

$$f = \int (dr)/p(r) + K \int (d\theta)/q(\theta) + \omega(\varphi) = 0, \qquad (\omega \text{ arbitrary}).$$

Problems of biological significance are then:

(a) To find, in F_1, the curves, or segments, which remain invariant under G. This is the problem of biological stabilization, or checkmate of evolution.

(b) To study the variations, in the individual, of any curve that is checkmated in the line of heredity. For example, the nearly circular curve of blood temperature varies, during a fever, according to a cyclical case of G, viz., when, for some N, $u_1 + \cdots + u_N = 0$, $v_1 + \cdots + v_N = 0$.

(c) Analysis of the noncyclical case of G, assuming that natural selection is operative in the line of heredity. Here there is an absolute minimal (extremal), in F, and a succession of relative minima which expand as time increases. Their expansion requires that F should diverge as a system. Thus the characters of a species (and hence the typical organism itself) gradually increase in size. Lamarck formally assumed this.

LANSDOWNE, PA., U. S. A.

STRUCTURE OF RANDOM NETS

ANATOL RAPOPORT AND RAY SOLOMONOFF

Consider an aggregate of n points, from each of which issue a outwardly directed lines, each of which terminates upon another point at random. The resulting configuration constitutes a random net. Problems arising in connection with such nets have applications in mathematical theories of the central nervous system, in probabilistic theories of epidemics, and in mathematical sociology.

The existence of a "path" in a random net from a point A to another point B implies the possibility of tracing directed lines from A, through any number of intermediate points upon which they terminate, to B. One problem to be considered is the following: given a pair of arbitrarily selected points A and B, to find the expression for the probability that a path exists from A to B in terms of n and a (weak connectivity). Another problem concerns the probability that there exist paths from an arbitrarily selected point A to all the other points in the aggregate (strong connectivity). If n and a are small, the problems can be directly attacked by the method of Markoff chains. For large n, however, the amount of computation becomes prodigious and approximation methods must be used. These methods lead to either the differential-difference equation,

$$\frac{dx}{dt} = k[x(t) - x\,(t - \tau)]\,[n - x(t)],$$

or the difference equation,

$$x(t + 1) - x(t) = [n - x(t)]\,[1 - \exp\{-a[x(t) - x(t - 1)]/n\}].$$

Actually, only the asymptotic value $X = x(\infty)$ is required. For large n, this value can be approximated by the solution of the transcendental equation

$$X = n - (n - 1)\exp\{-\,aX/n\}.$$

If in the embryonic development of a nervous system the neuroblasts are represented by the points of our aggregate, and the axones by the directed lines, then the weak and strong connectivities measure the number of associations which may be learned by the resulting neural net.

If the points are individuals in a closed community, in which an epidemic is spreading, where a stricken individual is capable of infecting others only during a limited time and thereupon becomes non-infectious and immune (or dead), then the weak connectivity measures the expected total number of infections by the time the epidemic has passed, and strong connectivity is the probability that everyone will succumb to the disease.

Similar considerations may be applied to the spread of rumors, panics, etc.

University of Chicago,
Chicago, Ill., U. S. A.

THE USE OF THE NULL-UNIT FUNCTION IN GENERALIZED INTEGRATION

J. J. Smith and P. L. Alger

This paper develops a procedure for nth order integration, including negative and fractional, as well as integral values of n for a certain class of functions. This class of functions embraces all the usual mathematical functions, with the provision that each one is multiplied by the null-unit function $H(x, x_0)$, so that the resulting function is zero when $x < x_0$ and takes its usual value when $x > x_0$. By limiting the study to this type of function it is found that the difficulties encountered in the present theory when the null-unit function is not used are avoided. The development proceeds in three steps.

A definition of generalized integration proposed by Liouville is taken as the starting point, since this reduces to the standard definition for an integral when n is 1 and to the definition for a differential coefficient when n is -1. It is found that, while this definition is satisfactory for a function such as e^x, it gives indeterminate results for functions such as x^m.

The second step is to modify the definition by limiting the integration to a given interval, from x_0 to x. This overcomes the difficulty in integrating x^m. However, it is found that nth order differentiation so defined (n negative) is not the inverse of a succeeding nth order integration (n positive). With this non-commutative process, the so-called constants of integration must be found, offering serious limitations to the solution of practical problems.

The third step, leaving the definition unchanged, is to introduce the null-unit function, $H(x, x_0)$ as a factor, so that the integrated function is equal to zero when $x < x_0$. It is then found that the processes of integration and differentiation become commutative, for all values of n, in the same way as shown in a previous paper for integral values of n.

The nth order derivatives of x which tend to infinity as $x \doteq 0$ are handled by the method of Schwartz using their functional representation in mass space. These are particularly difficult to handle by the usual methods, as appears from the limitations that are always placed on the accepted formulas for generalized integrations, when dealing with functions having derivatives that become infinite. The Schwartz method shows that there is a representation in mass space of the derivative in this region although the derivative itself does not exist.

The null-unit function, therefore, enables the generalized processes of integration and differentiation to be carried out reversibly, for fractional as well as integral orders, and without the need for separate determination of any constants of integration.

General Electric Company,
Schenectady, N. Y., U. S. A.

SECTION VI

LOGIC AND PHILOSOPHY

SECTION VI. LOGIC AND PHILOSOPHY

RECURSIVE FUNCTIONS AND INTUITIONISTIC MATHEMATICS

S. C. KLEENE[1]

The intuitionistic mathematics (cultivated by Brouwer and his school since 1908) and the theory of general recursive functions (developed since 1934 from notions of Herbrand and Gödel, Church and Kleene, and Turing and Post) are both concerned with reasonings or operations which can be carried out constructively.

However the two theories remained unrelated in their details until a connection was established by the speaker [12] in collaboration with David Nelson [16] between the intuitionistic logic and number theory and the theory of recursive functions. The present paper is a report on studies by the speaker, still in a preliminary stage, which aim to extend the connection to the intuitionistic set theory and function theory.

Brouwer [2; 4] defines a "set" as a certain kind of a "law" (Gesetz), by which it seems clear that he means an effective process or algorithm. In [3; 5] he does use the word "algorithm." But according to the thesis put forward first by Church [7] and also by Turing [19] and by Post [17] and now generally accepted, an algorithm is always represented by a general recursive function.

This suggests that we attempt to identify laws or algorithms as used by Brouwer with algorithms in the sense studied in the theory of general recursive functions. If this identification is correct, then it should be possible to give a version of Brouwer's theory in which the theory of general recursive functions is used as the basis for proving or disproving propositions about sets. This would give a new avenue for penetrating into Brouwer's theory, which some persons including myself have found difficult of access. Beth [1] makes related suggestions for the elucidation of the foundations of intuitionistic mathematics. If the identification of law for Brouwer with general recursive function is not correct, it will be desirable to learn wherein Brouwer's handling of his notion of a law or algorithm in his set theory differs from the treatment of algorithms in the theory of recursive functions. In either case, the attempt to correlate the two theories will demand a close scrutiny of the fundamentals of both, and this it is hoped may produce results of mathematical interest.

1. Under Brouwer's definition, a "set" (Menge) or "spread" (in the terminology he is now adopting) arises in the following way. Let an object be chosen arbitrarily from a certain fixed enumerable class C_0 of objects, then another, then another, and so on. These choices constitute a "free choice sequence"

[1] The work reported in §§2 and 3 was supported by the John Simon Guggenheim Foundation and the Graduate School of the University of Wisconsin. Numbers in brackets refer to the bibliography at the end.

(freie Wahlfolge). After each choice, one of three things may happen, depending by a law upon the choices up to that one inclusive. A certain object from another enumerable class C_1 is correlated, without terminating the process, i.e., another choice is to follow (Case 0). Similarly, but terminating the process (Case 1). The sequence of the choices up to the one in question inclusive is said to be "sterilized" (gehemmt), in which case no object from C_1 is correlated, the result of previous correlations is discarded, and no further choice is to follow (Case 2). It is stipulated that after each unterminated and unsterilized sequence of $n - 1$ choices, another choice can be made so that the resulting sequence of n choices is not sterilized. The "set" is the law which determines, after each choice, which of Cases 0, 1, or 2 applies, and in Cases 0 and 1 what object from C_1 is correlated. The "set elements" are the finite or infinite sequences of objects from C_1 which arise from the various (unsterilized) terminated or infinite choice sequences. Set elements which arise from infinite choice sequences are not considered as given in their entirety, but only by stages consisting of the first n members (corresponding to the first n choices, for some n) with "possibility of continuation" (Fortsetzungsmöglichkeit) by new choices made either freely or under restriction by some law (Verengerungszusatz).

We shall take C_0 to be the natural numbers 0, 1, 2, $\cdots$. (Brouwer took 1, 2, 3, $\cdots$ (Nummern).)

What the class C_1 is may depend on the set. This flexibility is allowed to permit treating, e.g., sets whose elements are points defined as sequences of nested intervals $(a_n 2^{-n}, (a_n + 1)2^{-n})$. However, since C_1 is enumerable, it can (if it is not already the natural numbers) be replaced by the natural numbers, or some subclass of the natural numbers can be correlated to it by Gödel numbering [8].

Similarly we can represent a sequence of n natural numbers

$$\alpha(0), \cdots, \alpha(n-1),$$

constituting the first n choices of a choice sequence α, by a single number $\bar{\alpha}(n)$ (or $\tilde{\alpha}(n)$) defined thus,

$$(1) \qquad \bar{\alpha}(n) = \prod_{i<n} p_i^{\alpha(i)+1}, \qquad \tilde{\alpha}(n) = \prod_{i<n} p_i^{\alpha(i)},$$

where $p_i = \{$the ith prime number, counting 2 as the 0th$\}$. Here $\bar{\alpha}(n)$ has the advantage over $\tilde{\alpha}(n)$ that n is a function of $\bar{\alpha}(n)$ but not of $\tilde{\alpha}(n)$; in fact $n = \lambda(\bar{\alpha}(n))$ where $\lambda(w)$ is primitive recursive.

Now we can interpret a particular set as consisting of a pair of general recursive functions $\rho(w)$ and $\sigma(w)$ such that for every $n \geqq 1$:

$\rho(\bar{\alpha}(n)) =$ 0, 1, or 2, according as Case 0, 1, or 2 applies to the sequence of choices $\alpha(0), \cdots, \alpha(n-1)$,

(2) $\sigma(\bar{\alpha}(n)) = \{$the number, as a member or Gödel number of a member of C_1, correlated to the sequence of choices $\alpha(0), \cdots, \alpha(n-1)\}$, in Case 0 or 1.

The speaker took this step (apart from minor details) in 1941, when he first considered the problem of interpreting intuitionistic set theory in terms of recursive functions; but he was held up then by failure to understand one of Brouwer's examples [4, p. 246], and consequently turned to the problem of interpreting intuitionistic number theory. The same step (apart from details) is proposed in Beth's paper [1]. The speaker returned to the problem briefly in 1948 with a better understanding of intuitionistic logic, and found the example clear enough.

However, to carry through the undertaking one must interpret not only the definition of set but the various logical operations applied to predicates of set elements. For example, what shall the existential statement "$(E\eta)A(\xi, \eta)$" mean where ξ and η are variables over elements of a given set? Since ξ is never given in its entirety (in case it is an infinite sequence), it should apparently suffice that, for each n (if η is also an infinite sequence), there should be some m such that the nth member $\eta(n)$ of η can be determined from $\bar{\xi}(m)$, so that $A(\xi, \eta)$ holds. Here we need to consider what kind of predicate $A(\xi, \eta)$ is. I shall not go further with this analysis, except to remark that the attempt to establish an interpretation for all of a suitable class of predicates closed under the logical operations of the predicate calculus, in such a way that the laws of the intuitionistic logic of Heyting [9] are satisfied but not also laws which hold only in the classical logic, by using only the ideas indicated above, including Brouwer's notion of a restriction on freedom of choice, led into various complications.

2. During the speaker's third examination of the problem (in 1950), it has come to appear quite promising that a solution will be obtained by using two further ideas. The first of these ideas is the notion of a function general recursive in other functions, or briefly the idea of "relative general recursiveness." The notion was defined briefly in the author's paper [10], and in another form even in Turing's [20], and was applied by Post in [18]. But much of the theory, e.g., the normal form theorem, was worked out only recently by the speaker [15] and I understand also by Martin Davis (in his dissertation, Princeton).

A function φ is general recursive in functions $\alpha_1, \cdots, \alpha_l$, if there is a system E of equations which defines φ recursively from $\alpha_1, \cdots, \alpha_l$, in the sense that equations expressing the correct and only the correct values of φ can be deduced by a substitution rule R1 and a replacement rule R2 from E and equations giving the values of $\alpha_1, \cdots, \alpha_l$ (Kleene [10] or [15]).

For our purpose now it is important that the relationship of φ to $\alpha_1, \cdots, \alpha_l$ be "uniform," in the sense that the E can be chosen to be independent of $\alpha_1, \cdots, \alpha_l$ (Kleene [15]). For the present application, we take the case that $\alpha_1, \cdots, \alpha_l$ (or α, β, etc.) are functions of one variable.

When $\varphi(x_1, \cdots, x_n)$ is general recursive (uniformly) in $\alpha_1, \cdots, \alpha_l$, we can regard $\varphi(x_1, \cdots, x_n)$ as a function of the $l + n$ variables $\alpha_1, \cdots, \alpha_l, x_1, \cdots, x_n$. We write it then $\varphi(\alpha_1, \cdots, \alpha_l, x_1, \cdots, x_n)$, and say that this function is "general recursive." Thus now we have general recursive functions having independent variables of either or both of two types, namely, number variables ($x_1, \cdots, x_n, x, y$, etc.) and variables for number-theoretic functions of one variable ($\alpha_1, \cdots, \alpha_l, \alpha, \beta$, etc.).

This notion, which arose in the development of the theory of recursive functions without conscious reference to the intuitionistic concept of a choice sequence, is very well in keeping with the latter. This is emphasized by the speaker's normal form for a general recursive function $\varphi(\alpha, x)$, as follows,

$$(3) \qquad \varphi(\alpha, x) = U(\mu n T_1^1(\bar{\alpha}(n), e, x, n)) = U(\mu n T_1^1(\bar{\alpha}(n), e, x)),$$

where U is a particular primitive recursive function, $T_1^1(w, z, x, n)$ or $T_1^1(w, z, x)$ a particular primitive recursive predicate, "μn" means "the least n such that," and e is a number said to "define φ recursively" or to be a "Gödel number of φ." (The first form appears in [15]; the second follows using $n = \lambda(\bar{\alpha}(n))$. The uniformity makes e independent of α as well as of x.) Thus, given a fixed x, the function value $\varphi(\alpha, x)$ for any α is determined after choosing a sufficient number n of initial values $\alpha(0), \cdots, \alpha(n - 1)$ of α, the number n depending on the values chosen by the formula $n = \mu n T_1^1(\bar{\alpha}(n), e, x)$. When these n values have been chosen, the value of φ is determined from $\bar{\alpha}(n)$ by the formula $\varphi(\alpha, x) = U(\lambda(\bar{\alpha}(n)))$. (The fact that $\varphi(\alpha, x) = U(n)$ is a peculiarity arising from the proof I have used for the normal form theorem. This proof employs a larger n than the least for which $\varphi(\alpha, x)$ could be determined from $\bar{\alpha}(n)$, in order to make this determination by a function $U(\lambda(w))$ independent of x and φ or e.) Similarly for $\varphi(\alpha_1, \cdots, \alpha_l, x_1, \cdots, x_n)$.

In fact, we have exactly the situation envisaged by Brouwer in [3] or [5] when he speaks of the correlation of a number φ to each element of a set (here $l = 1$, $n = 0$). Thus the extension of Church's thesis to the case of effective calculation of a function φ uniformly from a function α (i.e., of a function $\varphi(\alpha)$) enables it to be deduced that an algorithm for this purpose must behave in the manner Brouwer stated.

From a preliminary examination, it appears that a vocabulary consisting of "0," "+1", number variables, function variables, a suitable list of symbols standing for general recursive functions $\varphi(\alpha_1, \cdots, \alpha_l, x_1, \cdots, x_n)$ (for various l and n), similarly for general recursive predicates $P(\alpha_1, \cdots, \alpha_l, x_1, \cdots, x_n)$, Church's λ-operator [6] (where, e.g., for each fixed α and x, "$\lambda y \varphi(\alpha, x, y)$" stands for $\varphi(\alpha, x, y)$ as a function of y), and the logical symbols of the predicate calculus (with the two sorts of variables), subject to appropriate syntactical rules, will be adequate for expressing a substantial part of the intuitionistic set theory, at least as long as consideration is limited to properties of sets and set elements (called by Brouwer "species of first order").

The other idea which we propose to apply to the interpretation of intuitionistic set theory is that of "realizability," in which the logical operators are given meanings referring to recursive functions. The definition, as it was given for number-theoretic formulas (Kleene [12; 13]), should be extended to formulas in the larger vocabulary just described. In doing so, the notion of relative recursiveness should play a part. For example, $(E\beta)A(\alpha, \beta)$ should be realizable, for a given interpretation of α as a number-theoretic function of one variable, only if there is a function β general recursive in α for which $A(\alpha, \beta)$ is realizable. The extended realizability notion should have, in particular, the property that realizability of formulas is preserved under deduction in a corresponding extension of the intuitionistic number-theoretic formalism.

3. In the vocabulary just described no particular number-theoretic function of one variable can be expressed which is not general recursive. But quantification is allowed with respect to function variables α, β, etc. considered as ranging over all possible choice sequences. Do we get more in this way than if $\alpha, \beta, \cdots$ range over only choice sequences constituting the values of general recursive functions? If not, then the function variables could be eliminated.

A "finite set" (or in Brouwer's recent terminology a "bounded spread") is a set for which the unsterilized choice sequences contain only numbers $\leqq$ some fixed number b; i.e., Case 2 arises, whenever a number $> b$ is chosen.

A theorem which is fundamental for Brouwer's function theory states that, if to each element ξ of a finite set M a natural number φ_ξ is correlated, then there is a number z such that φ_ξ is completely determined by the first z of the choices generating ξ ([3, Theorem 2] or [5, Theorem 2]).

We shall show by an example that this theorem would fail, if the variable α (for the choice sequence generating ξ) were restricted to range over recursive functions only. In our example we take the set M to be the infinite sequences of 0's and 1's, and identify each set element ξ with its choice sequence α. Let us then write α in place of ξ, $\varphi(\alpha)$ in place of φ_ξ, and understand henceforth that α is a number-theoretic function taking only 0's and 1's as values.

We use two primitive recursive predicates $W_0(x, y)$ and $W_1(x, y)$ which were discussed recently by the speaker [14]. We correlate numbers $\varphi(\alpha)$ to (some) choice sequences α by the formula

$$\varphi(\alpha) = \mu n (Ex)_{x<n}(Ey)_{y<n \dot{-} x} W_{\alpha(x)}(x, y), \tag{4}$$

which can be put in the form $\varphi(\alpha) = \mu n A(\bar{\alpha}(n))$ where $A(w)$ is primitive recursive. The condition on α that a number $\varphi(\alpha)$ is correlated to α is then $(En)A(\bar{\alpha}(n))$. Let us say that $\bar{\alpha}(n)$ is "secured," if the choices $\alpha(0), \cdots, \alpha(n-1)$ determine[2] a

[2] We are using "determine" here in an effective sense to mean that the algorithm for calculating numbers $\phi(\alpha)$ from choice sequences α already produces the number $\phi(\alpha)$ when the choices $\alpha(0), \ldots, \alpha(n-1)$ are supplied to it, without making any use of further choices. The algorithm referred to is the obvious one for calculating the $\phi(\alpha)$ of (4). Brouwer proves his theorem for this effective sense of "determine"; but our result that the theorem fails when α ranges only over recursive sequences holds likewise when "determine" in the theorem is read in the weaker sense to mean the number ϕ_ξ is the same for all choice sequences agreeing in the first z choices, though the given algorithm might need to operate with further choices in producing ϕ_ξ. For this case, $(Ez)(\alpha)A(\bar{\alpha}(z))$ in the last paragraph below is obtained by using the weakened theorem to obtain a z_1, and then taking as z the maximum of the values of $\widetilde{\phi}(\alpha)$, which is a finite collection because they are determined by the first z_1 choices, which can be made in only 2^{z_1} ways.

correlated number. The condition that $\bar{\alpha}(n)$ is secured is $A(\bar{\alpha}(n))$. For example, suppose $\alpha(0) = 1$, $\alpha(1) = \alpha(2) = 0$, etc. Then $\bar{\alpha}(1)$ is secured (and $\varphi(\alpha) = 1$), if $W_1(0, 0)$; otherwise not, but then $\bar{\alpha}(2)$ is secured (and $\varphi(\alpha) = 2$), if $W_1(0, 1)$ or $W_0(1, 0)$; otherwise not, but then $\bar{\alpha}(3)$ is secured (and $\varphi(\alpha) = 3$), if $W_1(0, 2)$ or $W_0(1, 1)$ or $W_0(2, 0)$; otherwise not, etc. For $m > n$, $A(\bar{\alpha}(n))$ implies $A(\bar{\alpha}(m))$.

For each x, $(Ey)W_0(x, y)$ and $(Ey)W_1(x, y)$ are not both true (Kleene [14, (1)]). Hence, classically, we get an α such that $\bar{\alpha}(n)$ is unsecured for every n, i.e., no number is correlated, thus

$$\alpha(x) = \begin{cases} 1 & \text{if} \quad (Ey)W_0(x, y), \\ 0 & \text{if} \quad (Ey)W_1(x, y), \\ \text{either 1 or 0 otherwise.} \end{cases} \tag{5}$$

(Except for this remark, our discussion is intuitionistic.)

Now we show, first, that to every recursive choice sequence α a number $\varphi(\alpha)$ is correlated, i.e., $(\alpha)(En)A(\bar{\alpha}(n))$ is true when α ranges only over general recursive functions. Take $\alpha(x) = 1$ as the $R_0(x, y)$ and $\alpha(x) = 0$ as the $R_1(x, y)$ of Kleene [14, (2) and (3)]. Now either $\alpha(f) = 1$ or $\alpha(f) = 0$. If $\alpha(f) = 1$, then $(Ey)R_0(f, y)$, and (since $\alpha(f) \neq 0$) $\overline{(Ey)}R_1(f, y)$; whence it follows as before that $(Ey)W_1(f, y)$, i.e., (since $\alpha(f) = 1$) $(Ey)W_{\alpha(f)}(f, y)$. But if $W_{\alpha(f)}(f, y)$, then $\bar{\alpha}(f + y + 1)$ is secured. Similarly, if $\alpha(f) = 0$.

Second, we show that a number is not correlated to every choice sequence α, i.e., $(\alpha)(En)A(\bar{\alpha}(n))$ is false when α ranges over all number-theoretic functions (and that Brouwer's theorem does not hold for correlations only to all recursive choice sequences). Supposing $(\alpha)(En)A(\bar{\alpha}(n))$ to be true with the unrestricted range of α (or Brouwer's theorem to be true with the restricted), then, by the theorem, $(Ez)(\alpha)A(\bar{\alpha}(z))$. But replacing "$(Ey)$" by "$(Ey)_{y<z\dot{-}x}$" in (5), we get a recursive α (e.g., taking $\alpha(x) = 0$ for all $x \geqq z$) such that $\bar{\alpha}(z)$ is not secured, i.e., $\bar{A}(\bar{\alpha}(z))$, contradicting $(\alpha)A(\bar{\alpha}(z))$.

Bibliography

1. E. W. Beth, *Semantical considerations on intuitionistic mathematics*, Neder. Akad. Wetensch. vol. 50 (1947) pp. 1246–1251 (also Indagationes Mathematicae vol. 9 (1947) pp. 572–577).
2. L. E. J. Brouwer, *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten, 1. Teil*, Verhandelingen K. Akademie van Wetenschappen, Amsterdam vol. 12 no. 5 (1918). See errata on last page of 2. Teil, ibid. vol. 12 no. 7 (1919).
3. ———, *Beweis, dass jede volle Funktion gleichmässig stetig ist*, Neder. Akad. Wetensch. vol. 27 (1924) pp. 189–193.
4. ———, *Zur Begründung der intuitionistischen Mathematik I*, Math. Ann. vol. 93 (1924–1925) pp. 244–257.
5. ———, *Über Definitionsbereiche von Funktionen*, Math. Ann. vol. 97 (1927) pp. 60–75.
6. Alonzo Church, *A set of postulates for the foundation of logic*, Ann. of Math. ser. 2 vol. 33 (1932) pp. 346–366.
7. ———, *An unsolvable problem of elementary number theory*, Amer. J. Math. vol. 58 (1936) pp. 345–363.

8. KURT GÖDEL, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik vol. 38 (1931) pp. 173–198.
9. AREND HEYTING, *Die formalen Regeln der intuitionistischen Logik*, and *Die formalen Regeln der intuitionistischen Mathematik*, Sitzungsberichte Akademie des Wissenschaften, Berlin, phys.-math. Kl. (1930) pp. 42–71, 158–169.
10. S. C. KLEENE, *Recursive predicates and quantifiers*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 41–73. See the erratum under the next title.
11. ———, *On the forms of the predicates in the theory of constructive ordinals*, Amer. J. Math. vol. 66 (1944) pp. 41–58. Erratum: On p. 46, (18) is not simply another way of writing the inductive definition, but can have other solutions for $P(a)$ besides "a is provable", e.g., "a is a formula". The technique described hence requires a supplemental argument. This is easily supplied when only an existential quantifier is involved, but not when there is a generality quantifier. Thus Theorem 1, the first half of Theorem 2, and Theorem IX of the preceding paper, are not established. The author plans to discuss the situation in a second paper under this title.
12. ———, *On the interpretation of intuitionistic number theory*, J. Symbolic Logic vol. 10 (1945) pp. 109–124.
13. ———, *On the intuitionistic logic*, Proceedings of the Tenth International Congress of Philosophy (Amsterdam, Aug. 11–18, 1948) pp. 741–743.
14. ———, *A symmetric form of Gödel's theorem*, Neder Akad. Wetensch. vol. 53 (1950) pp. 800–802 (also Indagationes Mathematicae vol. 12 (1950) pp. 244–246).
15. ———, *Introduction to Metamathematics*. A book intended for publication by 1952 (publisher undetermined).
16. DAVID NELSON, *Recursive functions and intuitionistic number theory*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 307–368.
17. EMIL POST, *Finite combinatory processes—formulation 1*, J. Symbolic Logic vol. 1 (1936) pp. 103–105.
18. ———, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284–316.
19. A. M. TURING, *On computable numbers, with an application to the Entscheidungsproblem*, Proc. London Math. Soc. ser. 2 vol. 42 (1937) pp. 230–265. A correction, ibid. vol. 43 (1937) pp. 544–546.
20. ———, *Systems of logic based on ordinals*, Ibid. vol. 45 (1939) pp. 161-228.

UNIVERSITY OF WISCONSIN,
MADISON, WIS., U. S. A.

ON THE APPLICATION OF SYMBOLIC LOGIC TO ALGEBRA[1]

ABRAHAM ROBINSON

1. Symbolic logic has been used to clarify a variety of subjects in philosophy and epistemology; it has permeated directly or indirectly some of the new "operational" sciences; the calculi of classes and of relations which, at least historically, were evolved as branches of logic, have been applied to various combinatorial problems and to biology. However, in the present paper we shall be concerned with the effective application of symbolic logic to mathematics proper, more particularly, to abstract algebra. Thus, we may hope to find the answer to a genuine mathematical problem by applying a decision procedure to a certain formalised statement. While the practical possibilities in this direction, though limited, may be quite real, the present paper will be concerned with applications which are rather less intimately connected with any particular deductive procedure, and rather more with the general relations between a system of formal statements and the mathematical structures which it describes. The argument will be developed from the point of view of a fairly robust philosophical realism in mathematics, and it is left to those to whom this point of view is unacceptable to interpret our undoubtedly positive results according to their individual outlook. Thus we shall attribute full "reality" to any given mathematical structure, and we shall use our formal language merely to describe the structure, but not to justify its "reality" or "existence" which is taken for granted.

There is no room here for a detailed historical survey, but perhaps we may mention the names of K. Gödel, L. Henkin, and A. Tarski as representative of those who either directly or indirectly contributed towards the establishment of symbolic logic as an effective tool in mathematical research. In particular, it is understood that the theorem on algebraically closed fields which is proved below and which was stated by the present author in 1948 (see J. Symbolic Logic vol. 14 (1949) p. 74) has also been found independently by Professor Tarski.

2. The particular mathematical structures which we shall consider are exemplified by the ordered field of all real numbers, any specific group, or any specific ring, and are, more generally, sets of objects a, b, c, $\cdots$ and of relations $R(,\cdots)$, $S(,\cdots)$, $T(,\cdots)$, and (possibly) of functors $\phi(,\cdots)$, $\psi(,\cdots)$ such that any particular $R(a, b, \cdots)$, etc., either holds or does not hold, and such that there is just one object a which is the functional value of any given $\phi(b, \cdots)$.

Our formal language, on the other hand, will coincide roughly with the lower predicate calculus, that is to say, in more detail, it will contain the following atomic symbols:

(i) Object symbols a, b, c, $\cdots$ and dummy symbols u, v, w, $\cdots$ where the former are used as "free variables" and the latter as "bound variables." The rigid distinction between the two classes is important in the present context.

[1] This address was listed in the printed program under the title *Applied symbolic logic*.

(ii) Relative symbols $R(, \cdots)$, $S(, \cdots)$, $T(, \cdots)$, and functor symbols $\phi(, \cdots)$, $\psi(, \cdots)$.

We use the word "symbol" in (i) and (ii) in order to differentiate between the entities in question which belong to our formal language and the corresponding entities of specific mathematical structures. Though not unavoidable, this distinction is very convenient.

(iii) Propositional copulae $\sim$, $\wedge$, $\vee$, $\supset$, $\equiv$, and quantifiers (x), $(\exists x)$.

(iv) Brackets.

Well formed formulae are produced in the usual way, and statements are defined as well formed formulae which do not include unbound dummy symbols. All other well formed formulae are called predicates. In order to have in mind a concrete example we may write down the commutative law of addition for an abelian group. Its form still depends on the choice of our relative symbols and, possibly, functor symbols. For instance, if we introduce the relative symbol $S(x, y, z)$ ("z is the sum of x and y") as well as a relative symbol for equality, $E(x, y)$, the commutative law may be expressed as

$$(x)(y)(z)(t)[S(x, y, z) \wedge S(y, x, t) \supset E(z, t)].$$

On the other hand, if we replace S by the functor $\sigma(x, y)$ (i.e., "sum of x and y"), then the law in question may be written as

$$(x)(y)[E(\sigma(x, y), \sigma(y, x))].$$

There is no fundamental distinction in this formulation between the relative symbol of equality and any other relative symbol.

Having established the domain of well formed formulae, we may then develop the deductive calculus of "valid" or "analytic" statements, basing it on a set of axioms and rules of inference in the usual way, except that we have to take the inclusion of functor symbols into account. On the other hand, we still have to interpret our statements, which so far are merely sequences of symbols, i.e., we have to define recursively under what conditions a statement will be said to hold in a given structure. It is clear that the question whether a particular statement holds in a specified structure may still depend on the correspondence between the relative symbols of the language and the relations of the structure; and some reflection shows that it may also depend on the correspondence between the object symbols and objects of language and structure respectively.

Finally, we have to establish the correlation between deductive and semantic concepts as given above. Thus, it must and can be shown that a set of statements K which is deductively contradictory cannot possess a model, i.e., a structure in which all its statements hold. But the converse also is true, i.e., any set K that is deductively consistent (not contradictory) possesses a model. This is the extended "completeness theorem" which holds whatever the cardinal number of the set K and of the object, relative, and functor symbols contained in it. It follows that if a statement X holds in all structures in which a set of statements

K holds, then X can be deduced from K; and by the rules of the deductive calculus this implies that X can even be deduced from a finite subset of K.

3. A convenient starting point for the application of the logical framework sketched above is obtained when we reflect that in "orthodox" mathematics it is our normal business to prove a particular statement or theorem either for a specific structure, e.g., for the ring of rational integers, or for all structures which obey a specific set of axioms, e.g., for all groups. It is now natural to try and establish principles which do not refer to any specific theorem, but apply to all theorems or statements which belong to a certain class. For instance, we may try to prove that any statement of a certain class which is true for one particular type of mathematical structure is also true for another type. Such a metamathematical theorem may be called a "transfer principle." It is exemplified by the classical principle of duality in projective geometry which is logically so simple that no formal apparatus was required to establish it. However, in other cases, the use of symbolic logic becomes indispensible, if only in order to delimit the class of theorems to which the principle applies. We shall now state three transfer principles which belong to different types, sketch their proofs, and apply one of them to the demonstration of an actual mathematical theorem whose proof by more conventional methods is not apparent.

"Any statement X, formulated in the lower predicate calculus in terms of the relation of equality and the operations of addition and multiplication, which is true for all commutative fields of characteristic 0, is true for all commutative fields of characteristic $p \geqq p_0$ where p_0 is a constant depending on the statement X."

To prove the theorem, let K be a (finite) set of axioms for the concept of a commutative field, formulated after the manner indicated above, in terms of $E(x, y)$ ("x equals y"), $S(x, y, z)$ ("z is the sum of x and y"), and $P(x, y, z)$ ("z is the product of x and y"). It is easy to construct such a set K, and we may or may not include object symbols for 0 and 1, and functor symbols for sum and product, having already introduced S and P. Furthermore, it is not difficult to formulate a sequence of statements Y_n such that any Y_n is interpreted semantically as:

"There exists an element x such that $x + \cdots + x$ (p_n times) is different from 0, where p_n is the nth prime number."

A set of axioms for the concept of a commutative field of characteristic 0 is now given by $H = K \cup \{Y_1, Y_2, \cdots\}$, so that X holds in all models of H, by assumption. It then follows from the extended completeness theorem that X can be deduced from H, and hence that it can be deduced from some finite subset of H. Thus, X can be deduced from some set $H' = K \cup \{Y_1, Y_2, \cdots, Y_{n'}\}$, for some n'. But H' is satisfied by all commutative fields of characteristic greater than $p_{n'}$ and so X holds in all such fields.

In spite of the simplicity of its proof the above principle can be used to establish mathematical theorems whose demonstration by other methods is not apparent. For example,

"Let $q_i(x_1, \cdots, x_n) = 0$, $i = 1, 2, \cdots, k$, be a system of polynomials with integral coefficients which have no more than m roots $(\xi_1, \cdots, \xi_n)$ in common in any extension in the field of rational numbers. Then if we take the coefficients of the q_i modulo p, the resulting polynomials cannot have more than m roots in common in any field of characteristic $p \geqq p_0$, where p_0 depends on the given system."

Taking into account the above transfer principle, we have to show only that the statement, "There cannot be more than m different solutions to the system of equations $q_i(x_1, \cdots, x_n) = 0$" can be formalised within our language, where the integral coefficients are taken as operators indicating continued addition.

A transfer principle of a different type is the following:

"Any statement formulated in terms of the relations of equality, addition, and multiplication, as above, that holds in the field of all complex numbers, holds in any other algebraically closed commutative field of characteristic 0."

This is equivalent to the statement that an axiomatic system corresponding to the concept of an algebraically closed field of characteristic 0 (or indeed of any other characteristic) is complete in Tarski's sense, as defined in his calculus of systems. However, the statement of this fact as a transfer principle is more suggestive from the point of view of a working mathematician. In the proof, considerable use will be made of the results of Steinitz' field theory and this may be regarded as an interesting example of what can be done by the close integration of logic and algebra, or as a flaw in purity, according to taste.

Let X be any statement formulated in terms of equality, addition, and multiplication, as above, and let H be a set of axioms for the concept of a commutative field of characteristic 0, as before. It is not difficult to construct statements Z_n which assert that "every equation of nth degree whose highest coefficient does not vanish possesses a root," $n = 2, 3, \cdots$. Then $J = H \cup \{Z_2, Z_3, \cdots\}$ is a set of axioms for the concept of an algebraically closed field of characteristic 0, where we may assume for simplicity that J does not contain any object symbols.

Let $a_1, a_2, \cdots, a_n, \cdots$ be a countable sequence of object symbols, and let F be a set of statements affirming that no polynomial relation with integral coefficients, $p(a_1, \cdots, a_n) = 0$, exists between these a_n, or rather, between the objects corresponding to them. Then F is countable and so therefore is the set $G = F \cup J$. G expresses the concept of an algebraically closed field whose degree of transcendence is $\geqq \aleph_0$, or more precisely, every model of G must be such a field while every such field becomes a model of G if we match (let correspond) the $a_1, a_2, \cdots$ with algebraically independent numbers of the field.

If the set $G' = G \cup \{X\}$ is consistent, then it follows from the theorem of Skolem and Löwenheim (or from the construction required for the proof of the completeness theorem), that G' is satisfied in a countable field M', whose degree of transcendence therefore is exactly $\aleph_0$. Similarly, if the set $G'' = G \cup \{\sim X\}$ is consistent, it possesses a model M'' whose degree of transcendence is exactly $\aleph_0$. But by a theorem due to Steinitz any two algebraically closed fields of equal characteristic and of equal degree of transcendence are isomorphic, e.g., M' and M'' are isomorphic, and this would imply that both G' and G'' hold in both M' and M''. This is impossible and so either G' or G'' is contradictory, i.e., either $\sim X$ or X is deducible from G, more precisely it is deducible from a finite subset of G. Assuming for the sake of argument that the second alternative applies, this signifies that X holds in all algebraically closed fields of characteristic 0 which contain elements satisfying a *finite* number of inequalities $p_i(a_1, \cdots, a_k) \neq 0$. It is easy to find elements in any algebraically closed field of characteristic 0 which satisfy this condition. Thus either X holds in all algebraically closed fields of characteristic 0, or the same applies to $\sim X$, and this is equivalent to the theorem stated above.

Next we come to a transfer principle of yet another type, which requires a more difficult method for its proof.

"Let R be the field of rational numbers, and let $S = R[x_1, x_2, \cdots]$ be the field obtained by adjoining to R a sequence of indeterminate elements (S is the field of all rational functions of $x_1, x_2, \cdots$ with rational coefficients). Let K be the set of all statements formulated in terms of equality, addition, and multiplication, as above, which hold in R. Then there exists a field $S' \supseteq S$ such that the elements of $S' - S$ (if any) are transcendental with respect to S, and such that all the statements of K hold in S'."

It may be conjectured that we might take $S' = S$, but the available proof appears to be inadequate for establishing this.[2]

We observe that we are now referring to a specific field (for which an explicit set of axioms is unknown) and not, as formerly, to all the models of a given set of axioms. It is for this reason that the theorem cannot be obtained by the direct application of a completeness theorem. The proof depends on the introduction of additional functor symbols by means of which the statements of K are replaced by statements containing universal quantifiers only. Use is made of a purely algebraic theorem, due to Skolem, which states that if a polynomial $p(x, y, z, \cdots; t)$ with integral coefficients has a rational integral root t for integral values of $x, y, z, \cdots$ which are "dense" in a certain sense, then p possesses a linear factor $t - q(x, y, \cdots)$.

Let $p(x_1, x_2, \cdots; y_1, y_2, \cdots)$ be a polynomial with integral coefficients, where the $y_1, y_2, \cdots$ are regarded as variables and the $x_1, x_2, \cdots$ as parameters. The property of any such problem to be (relatively) irreducible in R, S, or S' may be regarded as a predicate of the $x_1, x_2, \cdots$. Using this fact, we may show by means of a simple application of the above transfer principle that if p—taken as a polynomial of the variables $y_1, y_2, \cdots$—is irreducible for indeterminate $x_1, x_2, \cdots$, then it must be irreducible for an infinite number of rational values of $x_1, x_2, \cdots$. This result is slightly weaker than Hilbert's well-known irreducibility theorem, according to which integral values may be

[2] *Added in proof*. More precisely it can be shown that S' must be different from S.

taken for $x_1, x_2, \cdots$, but it is all that is needed for the algebraic applications. However, the elegance of our proof is reduced somewhat by the fact that we have to show first that if p is irreducible in S, then it must be irreducible in S'. Also, it should be mentioned that the above mentioned lemma was formulated by Skolem precisely in order to prove Hilbert's theorem, or rather to improve it.

Let α be any irrational algebraic number. Then it is not difficult to show that all the statements formulated as above that hold in $R(\alpha)$ also hold in $S'(\alpha)$. From this fact we may again deduce a modified version of the counterpart of Hilbert's theorem for algebraic extensions of the field of rational numbers.

4. Another train of thoughts leading to the application of symbolic logic to mathematics is prompted by the idea that instead of discussing the properties of algebraic structures defined by specific sets of axioms, we may consider structures given by different sets of axioms simultaneously. We define an "algebra of axioms" as any set of axioms formulated in the lower predicate calculus which includes a relative symbol of equality, i.e., a binary relative symbol which satisfies axioms of equivalence as well as substitutivity. A model of an algebra of axioms will be called an "algebraic structure". We may compare these definitions with the concept of a general algebra as given by G. Birkhoff. Birkhoff's definition includes only axioms of an equational type, such as the associative and distributive laws. We may extend the domain of admissible axioms slightly by including "relations" of constants, e.g., $ab^3 = e$, although it is customary, in algebra, to differentiate between these "relations" and the "axioms". Even so, it can be shown that no system of axioms for a Birkhoff algebra as defined can realize the concept of an algebraic field. The proof of this assertion depends on the fact that one of the ordinary field axioms, that which stipulates the existence of an inverse for multiplication, includes a disjunction, and for that reason cannot be deduced from any set of axioms for a Birkhoff algebra as defined above. Being more restricted, the concept of a Birkhoff algebra is correspondingly more definite, but the fact that it does not include the concept of a field shows that a more general theory is justified. Since an algebra of axioms contains a relative symbol for equality, it is clear that all functor symbols can be replaced by relative symbols and the former may therefore be omitted for the sake of simplicity.

The question now arises whether the concept of an algebra of axioms as defined above is sufficiently definite to permit the development of a general theory. More specifically, we may take a standard concept of algebra and may ask whether it can be defined jointly for all algebras of axioms (and algebraic structures) in such a way that it is not merely analogous to the particular concept from which it is taken, but that it actually reduces to its prototype for the particular case in question. For example, let us consider the concept of a polynomial ring of n variables adjoined to a given ring. A close investigation of the concept of a polynomial in algebra shows—in contradistinction to the homonymous concept in the theory of functions—that it cannot be regarded simply as a certain type of function in a specific structure. In fact, two polynomials with coefficients in a field of characteristic p may well take the same values in that field without being identical. Accordingly, we prefer to define polynomials as constructs of a formal language. Thus, given any algebra of axioms K (e.g., formulated in terms of a relative symbol for equality $E(,)$ and of two ternary relative symbols $S(,,)$ and $P(,,)$, for the case of a commutative ring), we may consider all predicates $Q(x_1, \cdots, x_n, y)$ formulated in terms of the constants of K, such that y is "uniquely" determined by $x_1, \cdots, x_n$, i.e., such that

$$(x_1) \cdots (x_n)(\exists y)(z)[Q(x_1, \cdots, x_n, y) \wedge [Q(x_1, \cdots, x_n, z) \supset E(y, z)]]$$

can be deduced from K. We now define an algebraic *structure* M_K whose objects are the predicates just selected, and whose relations are determined in the following way. If $R(,,)$ is any relative symbol in K (ternary, say), then a relation $R^*(Q_1, Q_2, Q_3)$ shall hold between predicates Q_1, Q_2, Q_3 as defined above whenever the statement

$$(x_1) \cdots (x_n)(y_1)(y_2)(y_3)[Q_1(x_1, \cdots, x_n, y_1) \wedge Q_2(x_1, \cdots, x_n, y_2) \\ \wedge Q_3(x_1, \cdots, x_n, y_3) \supset R(y_1, y_2, y_3)]$$

is deducible from K. The structure M_K thus defined does not always satisfy all the statements of K (in a natural correspondence), but it does so for wide classes of algebras. In particular, if K is a set of axioms for the concept of a commutative ring, M_K also is a commutative ring. M_K is not itself isomorphic to the ring of polynomials of n variables with integral coefficients but it contains a substructure M_K' which possesses this property and which can again be characterised in a perfectly general formal way.

Another important concept which can be generalised in the sense detailed above is that of an ideal. It would appear natural to take the homomorphisms of any algebraic structure as the counterparts of the ideals of a ring, but it is difficult to formulate some of the familiar notions of ideal theory in that framework, e.g., the notion of a basis. Here again it appears to be more convenient to formulate the generalised concept by means of deductive considerations within a formal language, with respect to a given algebra of axioms, K. For instance, we may define an ideal as the set of statements J which can be deduced from K together with an additional set of statements which "identify" (i.e., make equal) certain individual constants, or objects, which were previously supposed different, $E(a, a')$, $E(b, b')$, etc. This yields a general concept which reduces to the notion of an ideal, or of a normal subgroup, for rings and groups respectively.

Other examples of concepts which can be so generalised are: the concept of an algebraic number and, more generally, of a number which is algebraic with respect to a given commutative field; the distinction between a separable and an inseparable extension of a given field; the concept of a power series ring associated with a given ring. Thus, the generalisation of the last concept is based on the following definitions.

Let $J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n \supseteq \cdots$ be a fixed descending chain of sets of statements formulated in the lower predicate calculus. A sequence of statements $\{X_n\} = \{X_0, X_1, \cdots\}$ will be said to be deducible from $\{J_n\}$, if for every n, X_p is deducible from J_n for sufficiently high p. The sequence will be called convergent with respect to $\{J_n\}$ if for every n, $X_p \equiv X_q$ is deducible from J_n for sufficiently high p and q. There are similar definitions for sets and sequences of predicates, and from one of those the concept of a power series ring adjoined to a given ring can be obtained by specialization (and not merely by analogy). As in the algebraic case, the general theory indicated here can be expounded conveniently by the introduction of a valuation.

5. Since all the above concepts were formulated in terms of statements, or axioms, in a formal language, the question may be asked how we can relate them to particular structures. Let K be a set of axioms as above, e.g., for the notion of a commutative ring, and let M be one of its models. We now add to the symbols of K, object symbols corresponding to all the objects of M which are not already designated by object symbols in K, and we add to the statements of K, all statements of the form $R(a_1, a_2, \cdots)$ or $\sim R(a_1, a_2, \cdots)$ according as the corresponding relation does or does not hold in M. The set of such statements may be called a "diagram" D of M. Every model of M' of $K' = K \cup D$ is an extension of M, or more precisely it contains a substructure isomorphic to M, and if we construct the structures $M_{K'}$ and $M'_{K'}$ as above, then $M'_{K'}$ turns out to be the polynomial ring of n variables adjoined to M.

The concept of a diagram, in conjunction with the completeness theorem also permits us to establish algebraic theorems whose proof by conventional methods is not apparent. A simple example is as follows.

"Let $P_0, P_1, P_2, \cdots$ be an infinite sequence of finite sets of polynomials of n variables in a skew field F (where the variables do not necessarily commute with the elements of F or with each other). Then if in every skew extension F' of F, in which all the polynomials of P_0 have a joint zero, all the polynomials of at least one P_i, $i \geqq 1$, have a joint zero, it follows that there exists a finite set $P_1, \cdots, P_k$, such that in every $F' \supseteq F$ in which all the polynomials of P_0 have a joint zero, all the polynomials of at least one P_i, $1 \leqq i \leqq k$, have a joint zero also."

There is a similar theorem for nonassociative fields. It was stated at the beginning that we attribute an equal degree of reality to a mathematical structure and to the language within which it is described. Accordingly we may introduce notions which are defined partly with reference to a given algebra of axioms, and partly with reference to its models. For example, an algebra of axioms will be called convex if, roughly speaking, the intersection of any number of its models is again a model. A number of interesting theorems can be proved concerning such algebras; thus it can be shown that the union of an increasing chain of models $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots$ of a convex algebra of axioms K is again a model of K.

A considerable number of the results of standard algebra can be extended to the more general concepts mentioned above, as shown in a 1949 London thesis which will be published shortly. However, the concrete examples produced in the present paper will have shown that contemporary symbolic logic can produce useful tools—though by no means omnipotent ones—for the development of actual mathematics, more particularly for the development of algebra and, it would appear, of algebraic geometry. This is the realisation of an ambition which was expressed by Leibnitz in a letter to Huyghens as long ago as 1679.

College of Aeronautics,
Cranfield, England.

SOME REMARKS ON THE FOUNDATION OF SET THEORY

Th. Skolem

To begin with I would like to express my best thanks to the chairman of this section, Professor Tarski, for the kindness and the great honour he has shown me by allowing me to give an address to you on the present occasion. I accepted the invitation only reluctantly, because I was afraid that it was too great an honour. My fear in this respect was chiefly due to the fact that in recent years I have worked more as an ordinary mathematician than as a logician. But nevertheless I decided to deliver this address, because I should really like to make some remarks concerning the logical foundation of mathematics even if they scarcely contain any real novelty. But my points of view are not quite in accordance with the current ones I think. In the last part of my lecture I shall indeed express a view of the logical development of mathematics that is perhaps rather subjective, but I think that there are good reasons to support it.

I shall make six different remarks. The first gives a unifying point of view for diverse possible set theories. The second deals with the axiom of infinity. The third remark concerns the illusory character of extensions of a set theory which is already sufficiently extensive. The fourth is a survey of the chief desires among people concerning the foundation of mathematics. The fifth remark concerns the ramified theory of types. The sixth remark is a proposal to make a serious attempt to build up mathematics in a strictly finitary way.

1. My first remark deals in the first instance with the so-called naive reasoning with sets. This reasoning is very clearly exposed in Dedekind: "Was sind und was sollen die Zahlen". Let us call this the first set theory, abbreviated FST. If we should formalize FST—which by the way we know is inconsistent but of this let us for the moment pretend ignorance—, it would have to be done as follows: We start with the fundamental relation $x \in y$, x is an element of y, which is a propositional function of two variables which run through all elements of the considered domain, D. From this propositional function others are built up using the operations of the lower predicate calculus. Let P be the totality of these propositional functions. We build expressions of the form $\hat{x}A(x)$ meaning "the class of all the x for which $A(x)$ is valid", where $A(x)$ is a propositional function containing x as a free variable. If x is the only free variable, $\hat{x}A(x)$ is called a set. Otherwise $\hat{x}A(x)$ is a set function of the other free variables in $A(x)$. In FST every set is again an element of D. We have the equivalence

$$(y \in \hat{x}A(x)) \rightleftarrows A(y). \tag{1}$$

Further the identity of sets is defined thus

$$(x = y) \rightleftarrows (z)((z \in x) \rightleftarrows (z \in y)), \tag{2}$$

and we have the axiom

$$(x = y) \rightarrow (z)((x \in z) \rightarrow (y \in z)). \tag{3}$$

Now I assert that in FST all definable sets will be given by the expression $\hat{x}X(x)$, if here X runs through all propositional functions in P. To prove this I shall first show that every propositional function which can be built when we use the symbols $\hat{x}A(x)$ is always equivalent to some function belonging to P, or in other words: The symbols $\hat{x}A(x)$ can be eliminated. Indeed besides (1) we have

$$(\hat{x}A(x) \in y) \rightleftarrows (z)((u)((u \in z) \rightleftarrows A(u)) \rightarrow (z \in y)) \tag{4}$$

and

$$(\hat{x}A(x) \in \hat{y}B(y)) \rightleftarrows (z)((u)((u \in z) \rightleftarrows A(u)) \rightarrow B(z)). \tag{5}$$

Hence every propositional function $\Phi(x, y, \cdots)$ which is built up by use of the set symbols is equivalent to a function $F(x, y, \cdots)$ in P. Therefore, if $\Phi(x)$ is an arbitrary propositional function, $\hat{x}\Phi(x) = \hat{x}F(x)$, where $F(x)$ belongs to P. Thus it is proved that all definable sets in FST are of the form $\hat{x}X(x)$, where X runs through all elements in P with one free variable.

Now, using the natural numbers, it is very easy to enumerate all elements of P. Hence it follows that it is possible to enumerate all definable sets in FST. The natural numbers can be defined as sets in D. The enumeration of the sets yields a class of ordered pairs (ν, n), ν running through all elements of D, n running through all natural numbers. This class is defined syntactically, i.e., in quite a new manner so that it cannot be said to be a set or, in other words, be an element of D. Hence all logically, not syntactically, definable sets in FST can be syntactically enumerated. If Dedekind, Cantor, and others had been aware of this fact, they might have wondered, and perhaps some doubt as to the transfinite of Cantor would have resulted.

We know now that FST is inconsistent, because, for example, Russell's antinomy can be deduced in it. However, it is quite clear that the syntactical enumerability will remain valid in every set-theory which arises from FST by restricted use of the requirement that the classes $\hat{x}X(x)$ shall be sets, i.e., again belong to the considered domain D as individuals therein. The characteristic feature of FST is the unrestricted requirement that every class of individuals in D is again an individual in D or, in other words: Every symbol $\hat{x}X(x)$ is one of the values the variable x can take. This demand is called impredicative, as well as every demand that a logical expression containing a variable shall be one of the values of the variable. Such impredicative principles can easily lead to antinomies. On the other hand it is certain that it is not necessary to avoid completely the impredicative demands in order to obtain a consistent theory. There may therefore be many consistent theories arising from FST by stronger or weaker restrictions in the use of the expressions $\hat{x}A(x)$ as being both individuals and sets of individuals. The strongest restriction would be to declare that the classes $\hat{x}A(x)$ are altogether new objects, not belonging to the original domain D. Such is the case in the theory of types. Between FST and STT, meaning the simple theory of types, we can imagine a great variety of set theories differing from one another in as far as some classes $\hat{x}A(x)$ are still elements of D and are called sets, whereas the remaining classes do not belong to D. Naturally there will be other differences, too. When the classes mostly belong to D, it will perhaps be sufficient to take only the elements of D into account; whereas when the classes do not or only to a small extent do belong to D, it will be necessary to take into account also the classes not belonging to D and even perhaps the classes of classes and so on. We obtain in this way a certain survey of the logical systems which can be used.

2. My second remark concerns the best known axiomatic theory of sets, namely Zermelo's. In his formulation there was an obscure point, the notion "definite Aussage". An improvement was given by A. Fraenkel, and independently I gave a precise definition of "definite Aussage" in an address at the Fifth Congress of Scandinavian Mathematicians in Helsinki in 1922. I identify "definite Aussage" and "propositional function belonging to P" as just explained. This definition will be convenient for my considerations now. It is easy to see that all the axioms of Zermelo except the axioms of infinity and of choice can immediately be formulated either in the demand that $\hat{x}A(x)$ shall belong to the considered set theoretic domain D, where $A(x)$ is a certain propositional function, or in the demand $(\hat{x}A(x) \in D) \rightarrow (\hat{x}B(x) \in D)$, $B(x)$ denoting a propositional function. For example the existence of the null set in D means that

$$\hat{x}((x \in x) \,\&\, (x \notin x))$$

belongs to D, x having D as its range of variation. Similarly the existence of the "elementary" sets $\{m\}$ and $\{m, n\}$ can be formulated. The axiom of separation takes the pretty form

$$(\hat{x}A(x) \in D) \rightarrow (\hat{x}(A(x) \,\&\, B(x)) \in D).$$

The axiom stating the existence of union is

$$(\hat{x}A(x) \in D) \rightarrow (\hat{y}(Ez)((y \in z) \,\&\, A(z)) \in D).$$

The axiom asserting the existence of the power set, i.e., set of all subsets, is

$$(\hat{x}A(x) \in D) \rightarrow (\hat{y}(x)((x \in y) \rightarrow A(x)) \in D).$$

The "Ersetzungsaxiom", or axiom of replacement, does not belong to Zermelo's axioms, but may be added with advantage. It is of the form $(\hat{x}A(x) \in D) \rightarrow (\hat{x}B(x) \in D)$. It may be stated in the form:

$$(\hat{x}A(x) \in D) \rightarrow \hat{x}(Ey)(F(x, y) \,\&\, A(y) \,\&\, (z)(F(z, y) \rightarrow (z = x))) \in D.$$

I leave the axiom of choice quite out of account, because it is of a different character from the other axioms. But the axiom of infinity can be put in the form $\hat{x}A(x) \in D$ for a certain $A(x)$ in P. This is not so in the ordinary formulation, namely: There exists a set Z containing 0 as one of its elements, and whenever $x \in Z$, we have $\{x\} \in Z$. It is evident that this axiom is not of the form $\hat{x}A(x) \in D$, because an axiom of the latter kind uniquely determines a set relative to D. I should like to show, however, that an improvement is possible here. The previous axioms yield among others such sets as

$$\{0\}, \qquad \{0, \{0\}\}, \qquad \{0, \{0\}, \{\{0\}\}\}, \cdots.$$

We can call an element b of the set m an i-element (initial element) of m, if no element c of m exists such that $b = \{c\}$. Similarly we call $s \in m$ an f-element (final element) of m, if $\{s\}$ is not $\in m$. Some sets m (see above) will have the following property:

1) $0 \in m$; 2) m contains a single f-element; 3) every subset of m contains an f-element; 4) every subset of m containing 0 as element and containing the same single f-element as m is identical with m. This property can be expressed as a propositional function $I(m)$ belonging to P. Then the axiom of infinity can be written in the form

$$\hat{m}I(m) \in D.$$

Indeed the elements of $\hat{m}I(m)$, i.e., the sets having the property $I(m)$, are the sets $\{0\}$, $\{0, \{0\}\}$, $\{0, \{0\}, \{\{0\}\}\}$, $\cdots$ ad inf. However, it may be preferred to introduce the Zermelo number series $\{0, \{0\}, \{\{0\}\}, \cdots\}$. This is the union

$$\hat{n}(Em)((n \in m) \,\&\, I(m)).$$

I do not think this is the right occasion to go into details of a proof of this. I shall mention only that it can be proved by the aid of the following lemmas:

1) $I(m)$ & $(m_1 \subset m)$ & $(0 \in m_1)$ & (m_1 containing a single f-element) $\rightarrow I(m_1)$.
2) $I(m) \rightarrow$ (m contains no i-element $\neq 0$).
3) $I(m_1)$ & $I(m_2) \rightarrow (m_1 \subset m_2) \vee (m_2 \subset m_1)$.
4) $I(m)$ & (s is f-element of m) $\rightarrow I(m + \{s\})$.
5) $(0 \in n)$ & (n contains no f-element) & $I(m) \rightarrow (m \subset n)$.

The proofs of these lemmas are straightforward.

It would seem possible to introduce infinite sets in easier ways. We could for example set up the axiom

$$\hat{x}(Ey)(x = \{y\})) \in D.$$

The set thus introduced would contain $\{0\}$, $\{\{0\}\}$, $\cdots$ ad inf. as elements, but it is an element of an element of itself, and it would be possible to deduce a contradiction.

As I already said above, we can have many different set theories: from FST at one extremity to the type theories, especially STT, the simple type theory, and RTT, the ramified type theory, at the opposite extremity. Somewhere between we have ZST, meaning Zermelo's set theory without the axiom of choice, but with addition of the axiom of replacement.

As to STT a reinterpretation is possible which shows that it is a weakened form of ZST. For greater simplicity I shall take the part of STT we obtain when apart from the $\in$-relation only relations $R(x, y)$, $R(x, y, z)$, $\cdots$ are allowed, where $x, y, z, \cdots$ are all of the same type. Then we can conceive the ordered pair (x, y) as the set $\{\{x, y\}, \{x\}\}$, the ordered triplet as $\{\{x, y, z\}, \{x, y\}, \{x\}\}$, and so on. The advantage of this is that we can conceive the relations also as sets. Then in the domain D we must suppose given the fundamental relations $\in$, the identity relation $=$, and the equivalence relation $\sim$ meaning "of the same type as", together with the usual axioms for the identity and for this special equivalence relation. Further it is assumed that some of the individuals in D have no elements, i.e., if a is such an element, we have in D, $(x)(x \mathrel{\overline{\in}} a)$. Then the following axioms shall be valid:

$$(x)(x \mathrel{\overline{\in}} a) \mathbin{\&} (y)(y \mathrel{\overline{\in}} b) \rightarrow (a \sim b), \qquad (x)(x \mathrel{\overline{\in}} a) \mathbin{\&} (Ey)(y \in b) \rightarrow (a \not\sim b)$$

$$(a \in m) \mathbin{\&} (b \in n) \rightarrow ((a \sim b) \rightleftarrows (m \sim n))$$

$$(Ex)((x \sim a) \mathbin{\&} A(x)) \rightarrow (\hat{x}((x \sim a) \mathbin{\&} A(x)) \in D).$$

Here $A(x)$ means an arbitrary propositional function which can be derived from $\in$, $=$, and $\sim$ by the lower predicate calculus. All this still leaves undetermined the set of all individuals without elements. To make the development of mathematics possible, it is then suitable to let this be the set N of natural numbers, setting up the Peano axioms.

However, the relation $\sim$ can be omitted without destroying the deducibility of ordinary mathematics. Indeed, it will be sufficient to start with the natural number series N characterized by the Peano axioms and then presuppose the following two axioms:

$$(Ex)((x \in a) \mathbin{\&} A(x)) \rightarrow \hat{x}((x \in a) \mathbin{\&} A(x)) \in D.$$

$$\hat{x}((Ey)(y \in x) \mathbin{\&} (y)((y \in x) \rightarrow (y \in a)) \in D.$$

A void class is here never a set. I scarcely believe that a simpler formal theory, equivalent to the STT with the axiom of infinity for its original individuals, is possible. It is easily seen that any set which can be construoted by the two axioms has only elements of the same type, if we define the type of an arbitrary individual a in D as meaning the number of terms in a decreasing $\in$ series starting from a and ending with a_ν, ν a natural number, viz.,

$$a_\nu \in \cdots \in a_2 \in a_1 \in a.$$

I must be content with this hint.

3. Now I come to my third remark. Let us consider the ZST. Let Z denote Zermelo's number series, and let Z^2 be the set of all ordered pairs from it. Then assuming the consistency of ZST, it can be proved that a subset S of Z^2 must be definable such that the axioms of the domain D will be valid for S, if everywhere $(x, y) \in S$ is written instead of $x \in y$. More generally, if A is a consistent system of axioms for a set theoretic domain D_A, there exists a definable subset S_A of Z^2 for which A is valid in the same sense. Whether all this is important or not will of course depend on the answer to the question whether ZST is consistent or not. If ZST is consistent, the mentioned theorem shows that it is in a certain sense logically closed.

Because of the uncertainty with regard to consistency, I shall on this occasion be content with giving the hint concerning the proof that it is only a proof of the theorem of Löwenheim adapted to ZST. The essential thing is that a *set* S can be found satisfying the axioms A in the mentioned sense. That a subclass of D could have this property without being a set would not be so astonishing.

4. My fourth remark refers to the question: Which one of the different theories shall we prefer? That depends on the desires we have in the foundation of mathematics. There are, I believe, at present chiefly three different sorts of desire.

1) One desires only to have a foundation which makes it possible to develop present day mathematics, and which is consistent so far as is known yet. Should any contradiction occur, we may try to make such restrictions in the underlying postulates that the deduction of the contradiction proves impossible. This may perhaps be called the opportunistic standpoint. It is a very practical one. See for example N. Bourbaki, J. Symbolic Logic vol. 14 pp. 1–18. He says: "Let the rules be so formulated, the definitions so laid out, that every contradiction may most easily be traced back to its cause, and the latter either removed or so surrounded by warning signs as to prevent serious trouble". But this standpoint has the unpleasant feature that we can never know when we have finished the foundation of mathematics. We are not only adding new floors at the top of our building, but from time to time it may be necessary to make changes in the basis.

If one agrees in this standpoint, the best thing to do will be to use STT or the axiomatic set theory as it is proposed by Zermelo, Fraenkel, or von Neumann, because this will be most convenient for the development of present day mathematics.

2) One desires to obtain a way of reasoning which is logically correct so that it is clear and certain in advance that contradictions will never occur, and what we prove are truths in some sense. This standpoint might be called the natural one or perhaps the logicistic one. Indeed, it was the generally assumed point of view before the discovery of the set theoretic antinomies. These, however, scattered the conviction that it was possible to find logical principles which were reliable. But, certainly, the mistake that the naive set theory was reliable does not prove that it should not be possible to detect the error in the classical set theoretic thinking and perhaps formulate a really correct reasoning. It was emphasized by Poincaré that the error in the classic set theory was the impredicative definitions. B. Russell has agreed in this and to avoid the impredica-

tive reasoning he invented the theory of types. But there are different type theories. Especially well known are the STT and the RTT. Of these two, only RTT is really free from impredicative notions. Therefore it seems to me that if we will put ourselves on the natural standpoint, and if we believe that the impredicative definitions constitute the error in classical set theory, the only thing to do is to use the RTT. Now most logicians prefer STT. We can use STT putting ourselves on the opportunistic standpoint, but then also the ZST could be convenient. From the natural point of view we have to choose RTT. Only then could we be justified in believing that our reasoning is logically sound. Now Professor F. B. Fitch has given a consistency proof of RTT or perhaps rather a certain modification of it. (J. Symbolic Logic vol. 3 pp. 140–149). However, taking the natural standpoint we cannot need any such proof, because when we are sure that our reasoning is correct, we are also sure in advance that paradoxes cannot occur. But of course it may be doubted, if there are not other weak points in the classic reasoning than the impredicative definitions. Indeed the use of the unrestricted quantifiers may be a weak point. I shall return to this.

3) The Hilbert program. This is the result of the giving up of the logicistic standpoint and not being content with the opportunistic one. We may distinguish the original and the modified Hilbert program. According to both of them we have to formalize mathematics and then prove that the formalism is consistent. Such a proof requires the proof of numerically general theorems, viz., theorems valid for an arbitrary number of applications of axioms or rules of procedure. To prove numerically general theorems we must at least use ordinary complete induction. Now the original Hilbert program takes into account only what is called finitary reasoning, and the ordinary complete induction in the intuitive sense belongs hereto. However a result of Gödel is known showing that this sort of reasoning is not sufficient to enable us to prove the consistency of the usual formal systems of mathematics. Thus it is well known that the proof of the consistency of number theory (say, the system Z in Hilbert-Bernays, *Grundlagen der Mathematik* vol. 1 p. 371) can be performed only by use of a certain form of transfinite induction, i.e., a higher form of induction than that occurring in number theory. If we would formalize a theory containing this transfinite induction, we could again prove the consistency of the latter theory only by using a still higher form of induction, and so on. This is as if we should hang up the ground floor of a building to the first floor, this again to the second floor, etc. I cannot understand the enthusiasm with which these ideas have been met among so many mathematicians. To me it seems that a more natural foundation ought to be tried, and again I think that it is the use of the unrestricted quantifiers which makes all the trouble.

5. My fifth remark concerns mostly RTT. As I just explained, one should from the logicistic standpoint have strong reason to be interested in RTT. However it has not been treated very much. Apart from some studies of Chwistek, which I must confess I have never read, the previously mentioned paper by F. B. Fitch, and the second edition of *Principia Mathematica*, there seems only little to be found in the literature concerning RTT. I have been somewhat astonished by the general lack of interest in this theory. In the book *Theoretische Logik*, 2d ed., by Hilbert and Ackermann it is said on p. 122 that the ramified theory of types is unnecessary, and that there is no reason to take it into account. Further it is said that the STT is proved consistent. However, as far as I know, the consistency proof for STT has been set up only when the number of individuals of the lowest type is finite, and this is insufficient for mathematics. The necessity of an axiom of infinity is a weakness of both STT and RTT, because a logical reason for it can scarcely be found. I had many years ago made an attempt to base arithmetic on RTT but did not succeed very well at that time. Recently I tried again with better results. These investigations have not been published. In his review of Fitch's paper Professor Bernays also asserts that arithmetic can be developed within RTT. This utterance of Bernays makes it doubtful whether I shall continue my investigations, because the matter is perhaps sufficiently treated already. As to the second edition of *Principia Mathematica*, I have seen only the first volume of it. It is well known that traditional analysis in its whole extent cannot be founded on RTT, but a great part of it remains valid. We have surely no right to condemn RTT, because it does not yield the whole of ordinary analysis. We ought not to regard all that is written in the traditional textbooks as something sacred.

6. Now I come to my sixth and last remark which I shall emphasize most of all. As I already said, the difficult thing in the logical development of mathematics is the use of quantifiers. However, it is to a great extent possible to develop mathematics without use of quantifiers. Indeed, I showed in a paper published in 1923 (Skrifter Norske Videnskaps-akademi, Oslo. 1923, I, no. 6b.) that ordinary arithmetic could be established by the aid of definition by recursion and proof by complete induction without use of quantifiers. By the way, it can be remarked that the use of restricted quantifiers, i.e., restricted to a finite range of variation, does not matter. In the same paper I also gave a hint of the fact that arbitrary arithmetical functions could be treated in the same way, and this makes also a sort of analysis possible. Indeed, the treatment of arbitrary functions or arbitrary series of integers according to these principles will be almost the same thing as the theory of progressions of choice, in the German of Brouwer's terminology "Wahlfolgen". Apart from my own paper which I wrote 30 years ago I have found this kind of mathematics treated in only two places, namely Hilbert-Bernays, *Grundlagen der Mathematik* vol. 1 pp. 286–346, and in a paper by H. B. Curry, *A formalization of recursive arithmetic*, Amer. J. Math. vol. 63 (1941) pp. 263–282.

Further, this finitistic way of foundation has been briefly mentioned by the Hungarian logician R. Péter in one of her articles *Über den Zusammenhang der verschiedenen Begriffe der rekursiven Function*, Math. Ann. vol. 110 pp. 612–632. But on the whole there has been among logicians a conspicuous lack of interest in it, which has been a disappointment to me. When I wrote my article I hoped that the very natural feature of my considerations would convince people that this finististic treatment of mathematics was not only a possible one but *the* true or correct one—at least for arithmetic. Now I can well understand that the lack of interest in the finitism is due to the circumstance that most logicians and mathematicians do not believe that it will be sufficient for mathematics. Of course this is also true, if mathematics shall mean present day mathematics altogether, as we find it in textbooks and scientific journals. It is trivial to say that present day mathematics, which is partially built up by the aid of transfinite ideas, cannot without change be based on finitary reasoning. The question is, however, what we shall lose or gain by such a change. As to clearness and security we certainly can only gain much. As pointed out in Hilbert-Bernays, *Grundlagen der Mathematik* vol. 1, the reasoning without quantifiers only yields propositions which are true in the sense of being finitarily interpretable and verifiable. As a special consequence of this no inconsistency can occur. However, it can be asked whether the change will not have the effect that we get a form of analysis which is more cumbersome and less effective than the classic one. This is generally believed to be an inevitable effect, I think. However, I should like to express some doubt in this respect. Indeed, we are certainly too much bound by tradition in considering the possibilities of establishing mathematical theories. That this is true we have had an excellent example of quite recently. In the theory of numbers we have a famous theorem called the prime-number theorem asserting that the quotient between $x/\log x$ and the number of primes up to x approaches 1 when x tends to ∞. It was maintained by the greatest experts in this field, E. Landau and G. H. Hardy, that an elementary proof of this theorem, i.e., a proof without use of the theory of analytic functions, was impossible. Quite recently, however, A. Selberg and P. Erdös established such an elementary proof. This urges us to be careful not to believe that what has been done before and has become classical mathematics is the only possible thing. I think that the fear that mathematics will be crippled by the restriction to the use of only free logical variables is exaggerated. I am aware that it may look different to mathematicians accustomed to analysis—to the theory of functions say—and those only working in the theory of numbers, but there are certainly many more ways of treating mathematics than we know today.

Now I will not be misunderstood. I am no fanatic, and it is not my intention to condemn the nonfinitistic ideas and methods. But I should like to emphasize that the finitistic development of mathematics as far as it may be carried out has a very great advantage with regard to clearness and security. Further it may be good reason to conjecture that it can be carried out very far, if one would make serious attempts in that direction. I would also like to say that the more importance we can attach to the purely finitistic development of mathematics, the less we need to be worried by the difficult consistency proofs of logical systems containing quantifiers. The great ingenuity in the setting up of these consistency proofs, especially the proof by Gentzen and Ackermann of the consistency of formalized number theory, must be admired. But it has happened before that products of great ingenuity have lost their interest, because simpler ways of thinking have been found. Another thing which can properly be mentioned in this connection is the general lack of interest among ordinary mathematicians with regard to symbolic logic. This is I think mostly due to the circumstance that the mathematicians believe—or have an instinctive feeling—that a very great part of the problems treated in symbolic logic are rather irrelevant to mathematics. As to the proof of the consistency of number theory, for example, there is a characteristic of which Gentzen himself is well aware, expressing it as follows: (*Die Widerspruchsfreiheit der reinen Zahlentheorie*, Math. Ann. vol. 112 p. 533) "The problem to prove the consistency of Number Theory is more a foundation of *possible* conclusions than of *actually used* ones". Of course mathematicians will be content if the conclusions actually used are tenable.

MATEMATISKE INSTITUT,
OSLO, NORWAY.

SOME NOTIONS AND METHODS ON THE BORDERLINE OF ALGEBRA AND METAMATHEMATICS

ALFRED TARSKI

The content of this paper is an outline of the general *theory of arithmetical classes* and a discussion of some of its applications.[1] Roughly speaking, an arithmetical class is any set of algebraic systems whose definition involves no set-theoretical terms; thus, e.g., the set of all groups and that of all lattices are arithmetical classes, while the set of all simple groups and that of all denumerable lattices are not. The tendency to distinguish between those constructions and derivations which involve set-theoretical notions from those which do not involve them is undoubtedly a pronounced trend of modern algebraic research. The theory of arithmetical classes provides this trend with a theoretical framework and exhibits its reach and limitations.

The fact that we shall concern ourselves with those algebraic notions which involve no set-theoretical elements by no means implies that we shall avoid set-theoretical apparatus in studying these notions. On the contrary, it will be seen that set-theoretical constructions and methods play an essential part in the development of the general theory of arithmetical classes.

The notion of an arithmetical class is of a metamathematical origin; whether or not a set of algebraic systems is an arithmetical class depends upon the form in which its definition can be expressed. However, it has proved to be possible to characterize this notion in purely mathematical terms and to discuss it by means of normal mathematical methods. The theory of arithmetical classes has thus become a mathematical theory in the usual sense of this term, and in fact it can be regarded as a chapter of universal algebra.

The theory concerns arbitrary algebraic systems formed by a nonempty set A, some operations O_0, O_1, $\cdots$ under which A is closed, some relations R_0, R_1, $\cdots$ between elements of A, and possibly some distinguished elements c_0, c_1, $\cdots$ of A.[2] In the present paper we are interested exclusively in systems with finitely many operations, relations, and distinguished elements (though the set A of all elements of the system may be infinite); this restriction, however, is not essential for the major part of our discussion. To simplify the discussion, we formulate definitions and theorems exclusively for systems $\mathfrak{A} = \langle A, +\rangle$

[1] Some fundamental facts concerning the theory of arithmetical classes have been stated in [11]. The ideas which have been systematically developed in this theory can already be found in earlier papers of the author; see [13, second part, pp. 298 ff.] and [14]. The numbers in brackets refer to the bibliography at the end of the paper.

[2] As is known, we could restrict ourselves without loss of generality to systems formed by a set and certain relations between its elements.

formed by a set A and one binary operation $+$. Such systems will be referred to for brevity as *algebras*, and the set of all algebras will be denoted by $\mathfrak{A}$.[3, 4]

A property of algebras is called arithmetical if it is expressed by a sentence of the *arithmetic of algebras*, i.e., of that part of the general theory of algebras which can be formalized within elementary logic (the lower predicate calculus). In such a sentence no set-theoretical terms occur; the symbol $+$ is the only non-logical constant; and all variables are assumed to range over elements of an algebra (and not, e.g., over sets of, or relations between, such elements). Thus, e.g., the sentence *for all x and y*, $x + y = y + x$ is an arithmetical sentence, and the property expressed by this sentence—the commutativity of an algebra—is an arithmetical property. Instead of arithmetical properties we shall speak of arithmetical classes. By an arithmetical class we understand the set of all algebras which have a certain arithmetical property in common, i.e., in which a certain arithmetical sentence holds. Thus the set of all commutative algebras is an arithmetical class.

To obtain a precise and purely mathematical definition of an arithmetical class we imitate the metamathematical definition of a sentence in a formalized arithmetic of algebras.[5] Sentences are particular cases of sentential functions; in fact, a sentence is a sentential function without free variables. Sentential functions are defined recursively in terms of simplest, so-called atomic sentential functions, which are described explicitly; a sentential function in general is obtained from atomic sentential functions with the help of certain operations which consist in combining expressions by means of sentential connectives (*or*, *and*, *not*, etc.) and quantifiers (*for every x*, *for some x*). It is convenient to assume that the only variables used in the formalized arithmetic are the symbols x, x', x'', $\cdots$, $x^{(k)}$, $\cdots$, and the only atomic sentential functions are the expressions $x^{(k)} = x^{(l)}$ and $x^{(k)} + x^{(l)} = x^{(m)}$. (Thus, the expression $x' + x'' = x'' + x'$ is not regarded as a sentential function; it is replaced by the expression *for some* x, $x' + x'' = x$ *and* $x'' + x' = x$.)

In a given algebra $\mathfrak{A} = \langle A, +\rangle$, every sentential function Φ determines the set of all those finite sequences, with terms in A, which satisfy Φ. The number of terms in these sequences varies dependent on the number of free variables in Φ. For instance, the sentential function $x + x' = x''$ determines the set of all three-termed sequences (ordered triples) $\langle y_0, y_1, y_2\rangle$ such that y_0, y_1, y_2 are members

[3] To avoid any appearance of antinomial constructions we can agree to restrict ourselves to algebras $\langle A, +\rangle$ in which A is a subset of a certain infinite set U fixed in advance. With this restriction, the set $\mathfrak{A}$ and all other sets involved in the discussion become legitimate mathematical entities whose existence can be derived from axioms of set theory.

[4] $\mathfrak{A}$ is what is called in [2] a *species* of algebraic systems; it can be replaced in the whole discussion by any other species. Every set of algebraic systems considered in the theory of arithmetical classes is assumed to be a species or a subset of a species, i.e., to consist of similar algebraic systems.

[5] The method applied here was developed in [14]; for its geometric interpretation see [4]. A closely related method was used in [12] to define the notion of truth and other semantical notions.

of A ($y_0, y_1, y_2 \in A$) and $y_0 + y_1 = y_2$. To simplify the construction we replace finite sequences by infinite ones. The set of all natural numbers is denoted by ω; and the set of all infinite sequences $y = \langle y_0, y_1, \cdots \rangle$ whose terms are in A, i.e., the set of all functions on ω to A, is denoted by A^ω. The atomic sentential functions $x^{(k)} = x^{(l)}$ and $x^{(k)} + x^{(l)} = x^{(m)}$ determine the sets $I_{k,l}$ and $S_{k,l,m}$ of all those sequences $y \in A^\omega$ for which $y_k = y_l$ and $y_k + y_l = y_m$, respectively. There are operations on subsets X of A^ω ($X \subseteq A^\omega$) corresponding to those metamathematical operations which are used in constructing sentential functions. These are, first, the familiar operations of set addition $X \cup Y$, set multiplication $X \cap Y$, and complementation $\bar{X} = A^\omega - X$. (The symbol $\bar{X}$ is used in the present paper in those cases in which it appears clear from the context with respect to which set the complement of X is taken.) Secondly, we have two sequences of operations: the operations $\nabla_0, \nabla_1, \cdots$ of outer cylindrification and the operations $\Delta_0, \Delta_1, \cdots$ of inner cylindrification. Given any $X \subseteq A^\omega$, the set $\nabla_k X$ (or $\Delta_k X$) consists by definition of all sequences $y \in A^\omega$ such that $x \in X$ for some (or for every) $x \in A^\omega$ which differs from y at most in its kth term. The reasons for the choice of the term *cylindrification* become clear if we think of A^ω as an abstract infinite-dimensional analytic space over A. We now define the family of all *arithmetical sets* $X \subseteq A^\omega$ (i.e., all sets determined by arithmetical sentential functions) as the smallest family which contains the sets $I_{k,l}$ and $S_{k,l,m}$ and is closed under the operations $\cup$, $\cap$, $^-$, ∇_k, and Δ_k ($k, l, m = 0, 1, \cdots$; the operations $\cap$ and Δ_k could be omitted in this definition).

If we consider a sentential function Φ as referring to, not a particular algebra $\mathfrak{A} = \langle A, + \rangle$, but the totality $\mathfrak{a}$ of such algebras, then what is determined by Φ is not a set $X \subseteq A^\omega$, but a function $\mathfrak{F}$ (in the mathematical sense) whose domain is $\mathfrak{a}$ (in symbols, $D(\mathfrak{F}) = \mathfrak{a}$) and which correlates a set $\mathfrak{F}(\mathfrak{A}) \subseteq A^\omega$ with every algebra $\mathfrak{A} = \langle A, + \rangle$; $\mathfrak{F}(\mathfrak{A})$ consists, of course, of all those sequences $y \in A^\omega$ which satisfy Φ. A function $\mathfrak{F}$ determined in this way by some arithmetical sentential function Φ is called an arithmetical function. We arrive at a mathematical definition of this notion by means of a construction entirely analogous to that which was outlined above for the notion of an arithmetical set of sequences in a particular algebra. The construction will be described here in a formal way.

DEFINITION 1. (i) *By* **F** *we denote the set of all functions* $\mathfrak{F}$ *such that* $D(\mathfrak{F}) = \mathfrak{a}$, *and* $\mathfrak{F}(\mathfrak{A}) \subseteq A^\omega$ *for every* $\mathfrak{A} = \langle A, + \rangle \in \mathfrak{a}$.

(ii) $\mathcal{I}_{k,l}$ *and* $\mathcal{S}_{k,l,m}$ *for* $k, l, m = 0, 1, \cdots$ *are the functions defined by the formulas*:

$$D(\mathcal{I}_{k,l}) = D(\mathcal{S}_{k,l,m}) = \mathfrak{a}$$

and, for every $\mathfrak{A} = \langle A, + \rangle$,

$$\mathcal{I}_{k,l}(\mathfrak{A}) = \{x \mid x \in A^\omega, x_k = x_l\},$$

$$\mathcal{S}_{k,l,m}(\mathfrak{A}) = \{x \mid x \in A^\omega, x_k + x_l = x_m\}.$$

(iii) *The functions* $\mathcal{I}_{k,l}$ *and* $\mathcal{S}_{k,l,m}$ *for* $k, l, m = 0, 1, \cdots$ *are called* ELEMENTARY FUNCTIONS; *the set of all elementary functions is denoted by* **EF**.

A symbol of the form $\{x \mid \cdots\}$ denotes the set of all elements x which satisfy the formulas to the right of the stroke $|$.

DEFINITION 2. *Let* $\mathfrak{F}$, $\mathfrak{G} \in \mathbf{F}$ *and let* $k = 0, 1, \cdots$.

(i) *The* UNION $\mathfrak{F} \cup \mathfrak{G}$ *is the function* $\mathfrak{H}$ *defined by the conditions*: $D(\mathfrak{H}) = \mathfrak{A}$, *and* $\mathfrak{H}(\mathfrak{A}) = \mathfrak{F}(\mathfrak{A}) \cup \mathfrak{G}(\mathfrak{A})$ *for every* $\mathfrak{A} \in \mathfrak{A}$.

(ii) *Analogously we define, in terms of operations on sets of sequences, the* INTERSECTION $\mathfrak{F} \cap \mathfrak{G}$, *the* COMPLEMENT $\bar{\mathfrak{F}}$, *as well as the* UNION $\bigcup(\mathfrak{H}_i \mid i \in I)$ *and the* INTERSECTION $\bigcap(\mathfrak{H}_i \mid i \in I)$ *of an arbitrary system of functions* $\mathfrak{H}_i \in \mathbf{F}$ *correlated with elements* i *of a set* I.

(iii) *The* OUTER CYLINDRIFICATION (*or* EXISTENTIAL QUANTIFICATION) $\nabla_k\mathfrak{F}$ *is the function* $\mathfrak{H}$ *defined by the conditions*: $D(\mathfrak{H}) = \mathfrak{A}$, *and, for every* $\mathfrak{A} = \langle A, +\rangle$, $\mathfrak{H}(\mathfrak{A})$ *is the set of all sequences* $y \in A^\omega$ *such that* $x \in \mathfrak{F}(\mathfrak{A})$ *for some sequence* $x \in A^\omega$ *which differs from* y *at most in its* k*th term. Similarly we define the* INNER CYLINDRIFICATION (*or* UNIVERSAL QUANTIFICATION) *by changing "for some sequence" to "for every sequence".*

Thus the symbols previously introduced for operations on sets of sequences will also be used to denote the corresponding operations on functions; it will always be clear from the context which meaning of the symbols is intended.

DEFINITION 3. (i) $\mathfrak{Z}$ *and* $\mathfrak{U}$ *are functions defined by the conditions*: $D(\mathfrak{Z}) = D(\mathfrak{U}) = \mathfrak{A}$, $\mathfrak{Z}(\mathfrak{A}) = \Lambda$ *and* $\mathfrak{U}(\mathfrak{A}) = A^\omega$ *for every* $\mathfrak{A} = \langle A, +\rangle$.

(ii) *Given* $\mathfrak{F}$, $\mathfrak{G} \in \mathbf{F}$ *we say that* $\mathfrak{F}$ *is* INCLUDED *in* $\mathfrak{G}$, $\mathfrak{F} \subseteq \mathfrak{G}$, *if* $\mathfrak{F}(\mathfrak{A}) \subseteq \mathfrak{G}(\mathfrak{A})$ *for every* $\mathfrak{A} \in \mathfrak{A}$.

Λ denotes as usual the empty set.

All the theorems will be stated in this paper without proof; most of them can easily be derived from the definitions of the notions involved.

THEOREM 4. (i) *The set* $\mathbf{F}$ *together with the operations* $\cup$, $\cap$, *and* $^-$ *forms a complete atomistic Boolean algebra.*

(ii) *In this algebra* $\mathfrak{Z}$ *and* $\mathfrak{U}$ *are respectively the zero and the unit elements*; $\subseteq$ *is the inclusion relation*; *and* $\bigcup$ *and* $\bigcap$ *are the join and the meet operations on arbitrary systems of elements.*

THEOREM 5. *For any* $\mathfrak{F}$, $\mathfrak{G} \in \mathbf{F}$ *and* $k, l = 0, 1, \cdots$ *we have*:

(i) $\nabla_k\mathfrak{Z} = \mathfrak{Z}$ *and* $\Delta_k\mathfrak{U} = \mathfrak{U}$.

(ii) $\nabla_k\mathfrak{F} \in \mathbf{F}$ *and* $\mathfrak{F} \subseteq \nabla_k\mathfrak{F}$; $\Delta_k\mathfrak{F} \in \mathbf{F}$ *and* $\Delta_k\mathfrak{F} \subseteq \mathfrak{F}$.

(iii) $\nabla_k(\mathfrak{F} \cap \nabla_k\mathfrak{G}) = \nabla_k\mathfrak{F} \cap \nabla_k\mathfrak{G}$, $\Delta_k(\mathfrak{F} \cup \Delta_k\mathfrak{G}) = \Delta_k\mathfrak{F} \cup \Delta_k\mathfrak{G}$.

(iv) $\overline{\nabla_k\mathfrak{F}} = \Delta_k\bar{\mathfrak{F}}$ *and* $\overline{\Delta_k\mathfrak{F}} = \nabla_k\bar{\mathfrak{F}}$.

(v) $\nabla_k\nabla_l\mathfrak{F} = \nabla_l\nabla_k\mathfrak{F}$ *and* $\Delta_k\Delta_l\mathfrak{F} = \Delta_l\Delta_k\mathfrak{F}$.

THEOREM 6. *For any* $\mathfrak{F} \in \mathbf{F}$ *and* $k, l, m = 0, 1, \cdots$ *we have*:

(i) $\mathfrak{I}_{k,l} \in \mathbf{F}$.

(ii) $\mathcal{I}_{k,k} = \mathfrak{U}$.

(iii) *If* $k \neq m$ *and* $l \neq m$, *then* $\mathcal{I}_{k,l} = \nabla_m(\mathcal{I}_{k,m} \cap \mathcal{I}_{l,m})$.

(iv) *If* $k \neq l$, *then* $\nabla_k(\mathcal{I}_{k,l} \cap \mathfrak{F}) \cap \nabla_k(\mathcal{I}_{k,l} \cap \overline{\mathfrak{F}}) = \mathfrak{Z}$.

It is known that all the equations which involve the operation symbols $\cup$, $\cap$, $^-$, ∇_k, Δ_k, the special function symbols $\mathfrak{Z}$, $\mathfrak{U}$, $\mathcal{I}_{k,l}$, and some variables $\mathfrak{F}$, $\mathfrak{G}$, $\cdots$, and which are identically satisfied when the variables $\mathfrak{F}$, $\mathfrak{G}$, $\cdots$ range over arbitrary functions in **F**, can be derived in a purely formal way from the identities stated (explicitly or implicitly) in Theorems 4–6.[6] As examples the following equations may be mentioned (where k and l range over arbitrary natural numbers):

$$\nabla_k\nabla_k\mathfrak{F} = \nabla_k\mathfrak{F}.$$

$$\nabla_k(\mathfrak{F} \cup \mathfrak{G}) = \nabla_k\mathfrak{F} \cup \nabla_k\mathfrak{G}.$$

$$\nabla_k\Delta_l\mathfrak{F} \cap \nabla_l\Delta_k\overline{\mathfrak{F}} = \mathfrak{Z}.$$

$$\nabla_k(\mathcal{I}_{k,l} \cap \mathfrak{F} \cap \mathfrak{G}) = \nabla_k(\mathcal{I}_{k,l} \cap \mathfrak{F}) \cap \nabla_k(\mathcal{I}_{k,l} \cap \mathfrak{G}) \qquad \text{for } k \neq l.$$

Various identities are known which, in addition to the operations defined in Definition 2 and the functions $\mathfrak{Z}$, $\mathfrak{U}$, $\mathcal{I}_{k,l}$, involve also the function $\mathcal{S}_{k,l,m}$; for instance:

$$\mathcal{S}_{k,l,m} \cap \mathcal{S}_{k,l,n} \cap \overline{\mathcal{I}_{m,n}} = \mathfrak{Z}.$$

THEOREM 7. *For every function* $\mathfrak{F} \in$ **F** *the following two conditions are equivalent*:

(i) $\Delta_k\mathfrak{F} = \mathfrak{F}$ (*or, what amounts to the same*, $\nabla_k\mathfrak{F} = \mathfrak{F}$) *for every natural number* k;

(ii) *there is no algebra* $\mathfrak{A} = \langle A, +\rangle$ *for which* $\mathfrak{F}(\mathfrak{A}) \neq \Lambda$ *and* $\mathfrak{F}(\mathfrak{A}) \neq A^\omega$.

DEFINITION 8. *The set of the* ARITHMETICAL FUNCTIONS, *in symbols* **AF**, *is the intersection of all sets* $\mathbf{X} \subseteq \mathbf{F}$ *which include* **EF** *and are closed under the operations* $\cup$, $^-$, *and* ∇_k *for* $k = 0, 1, \cdots$.

THEOREM 9. (i) $\mathbf{EF} \subseteq \mathbf{AF}$.

(ii) *The set* **AF** *is closed under the operations* $\cup$, $\cap$, $^-$, *as well as* ∇_k *and* Δ_k *for* $k = 0, 1, \cdots$.

THEOREM 10. *The set* **AF** *is denumerable* (*i.e., of the power* $\aleph_0$).

THEOREM 11. *In order that* $\mathfrak{F} \in$ **AF** *it is necessary and sufficient that* $\mathfrak{F}$ *be representable in the form*

$$\mathfrak{F} = O_1O_2 \cdots O_n(\mathfrak{G})$$

[6] This statement is based on some unpublished work by Professor L. H. Chin, Mr. F. B. Thompson, and the author.

where each of the operations $O_i (i = 1, 2, \cdots, n)$ coincides with one of the operations $\nabla_0, \nabla_1, \cdots$ and $\Delta_0, \Delta_1, \cdots$, and where $\mathfrak{G}$ is a finite union of finite intersections of elementary functions and their complements.

This is the so-called *theorem on canonical representation.*

THEOREM 12. *For every $\mathfrak{F} \in \mathbf{AF}$ there exist only finitely many natural numbers k for which $\Delta_k\mathfrak{F} \neq \mathfrak{F}$ (or $\nabla_k\mathfrak{F} \neq \mathfrak{F}$).*

The set of all natural numbers k for which $\Delta_k\mathfrak{F} \neq \mathfrak{F}$ can be referred to as the *dimension index of the function* $\mathfrak{F}$.

Much deeper than all the preceding is the *compactness theorem for arithmetical functions*:

THEOREM 13. *If $\mathbf{K} \subseteq \mathbf{AF}$ and $\bigcap(\mathfrak{F} \mid \mathfrak{F} \in \mathbf{K}) = \mathfrak{z}$, then there is a finite set $\mathbf{L} \subseteq \mathbf{K}$ for which $\bigcap(\mathfrak{F} \mid \mathfrak{F} \in \mathbf{L}) = \mathfrak{z}$.*

A mathematical proof of Theorem 13 is rather involved. On the other hand, this theorem easily reduces to a metamathematical result which is familiar from the literature, in fact to Gödel's completeness theorem for elementary logic.[7]

We proceed to defining the notion of an arithmetical class.

DEFINITION 14. (i) *For every $\mathfrak{F} \in \mathbf{F}$ we put*

$$\mathcal{C}\mathcal{L}(\mathfrak{F}) = \{\mathfrak{A} \mid \mathfrak{A} = \langle A, + \rangle \in \mathfrak{a}, \mathfrak{F}(\mathfrak{A}) = A^{\omega}\}.$$

(ii) *A set $\mathcal{S} \subseteq \mathfrak{a}$ is called an* ARITHMETICAL CLASS *if $\mathcal{S} = \mathcal{C}\mathcal{L}(\mathfrak{F})$ for some $\mathfrak{F} \in \mathbf{AF}$. The family of all arithmetical classes is denoted by* **AC**.

THEOREM 15. (i) *For every $\mathfrak{F} \in \mathbf{F}$ and $k = 0, 1, \cdots$ we have $\mathcal{C}\mathcal{L}(\mathfrak{F}) = \mathcal{C}\mathcal{L}(\Delta_k\mathfrak{F})$.*

(ii) *For every $\mathfrak{F} \in \mathbf{AF}$ there exists a $\mathfrak{G} \in \mathbf{AF}$ such that $\mathcal{C}\mathcal{L}(\mathfrak{F}) = \mathcal{C}\mathcal{L}(\mathfrak{G})$, and $\Delta_k\mathfrak{G} = \mathfrak{G}$ for every natural number k.*

A function $\mathfrak{F} \in \mathbf{F}$ which satisfies one of the two equivalent conditions 7(i) and 7(ii) (i.e., whose dimension index is empty) is called a *simple function.* In view of Theorem 15(ii) we can equivalently transform Definition 14(ii) by assuming that $\mathfrak{F}$ is not an arbitrary but a simple arithmetic function. The content of Definition 14(ii) thus transformed is in perfect agreement with the intuitive notion of an arithmetical class. In fact, while arbitrary arithmetical functions correspond to arbitrary sentential functions, simple arithmetical functions correspond to sentences, i.e., sentential functions without free variables; $\mathfrak{F}$ being a simple function correlated with an arithmetical sentence Φ, $\mathcal{C}\mathcal{L}(\mathfrak{F})$ is clearly the set of all those algebras $\mathfrak{A} \in \mathfrak{a}$ in which Φ holds. To give an example, consider the set $\mathcal{S}$ of all commutative algebras. Let

$$\mathfrak{F} = \Delta_0\Delta_1\nabla_2(\mathcal{S}_{0,1,2} \cap \mathcal{S}_{1,0,2}).$$

[7] For a proof of Gödel's theorem see, e.g., [3].

As is easily seen, $\mathfrak{F}$ is a simple arithmetical function and $\mathcal{CL}(\mathfrak{F}) = \mathcal{S}$; hence $\mathcal{S}$ is an arithmetical class.

THEOREM 16. (i) **AC** *is a field of subsets of* $\mathcal{Q}$; *in other words, the system* $\langle \mathbf{AC}, \cup, \cap, {}^{-}\rangle$ (*where* $\bar{\mathfrak{X}} = \mathcal{Q} - \mathfrak{X}$ *for every* $\mathfrak{X} \in \mathbf{AC}$) *is a Boolean algebra.*

(ii) **AC** *is denumerable.*

THEOREM 17. *If* $\mathbf{K} \subseteq \mathbf{AC}$ *and* $\bigcap(\mathfrak{X} \mid \mathfrak{X} \in \mathbf{K}) = \Lambda$, *then there is a finite family* $\mathbf{L} \subseteq \mathbf{K}$ *such that* $\bigcap(\mathfrak{X} \mid \mathfrak{X} \in \mathbf{L}) = \Lambda$.

This theorem—the *compactness theorem for arithmetical classes*—is clearly a corollary of Theorem 13. As an improvement of Theorem 17 we obtain

THEOREM 18. *If* $\mathbf{K} \subseteq \mathbf{AC}$ *and* $\bigcap(\mathfrak{X} \mid \mathfrak{X} \in \mathbf{K}) \in \mathbf{AC}$, *then there is a finite family* $\mathbf{L} \subseteq \mathbf{K}$ *such that* $\bigcap(\mathfrak{X} \mid \mathfrak{X} \in \mathbf{L}) = \bigcap(\mathfrak{X} \mid \mathfrak{X} \in \mathbf{K})$.

An especially important particular case of Theorems 17 and 18 is given in the following theorem.

THEOREM 19. *If* $\mathcal{S}_n \in \mathbf{AC}$, $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$ *and* $\mathcal{S}_{n+1} \neq \mathcal{S}_n$ *for every* $n \in \omega$, *then* $\bigcap(\mathcal{S}_n \mid n \in \omega) \neq \Lambda$ *and, more generally,* $\bigcap(\mathcal{S}_n \mid n \in \omega) \notin \mathbf{AC}$.

In view of the fact that the family **AC** is closed under complementation, we can derive from Theorems 17–19 their duals in which $\bigcap$ and Λ are replaced by $\bigcup$ and $\mathcal{Q}$. Theorems 18 and 19 are frequently applied to show that various sets of algebras are not arithmetical classes. As an application of this kind we give

THEOREM 20. (i) *A set* $\mathcal{S} \subseteq \mathcal{Q}$ *of finite algebras is in* **AC** *if and only if, together with every algebra* $\mathfrak{A}$, $\mathcal{S}$ *contains all algebras* $\mathfrak{B}$ *isomorphic to* $\mathfrak{A}$ ($\mathfrak{A} \cong \mathfrak{B}$), *and there is a number* n *such that every algebra* $\mathfrak{A} \in \mathcal{S}$ *has at most* n *elements.*

(ii) *In particular, the set of all finite algebras and hence also the set of all infinite algebras* $\mathfrak{A} \in \mathcal{Q}$ *are not in* **AC**.

When saying that an algebra $\mathfrak{A} = \langle A, +\rangle$ is finite, or has n elements, etc., we have of course in mind that the set A is finite, or has n elements, etc.

We are often led to consider families which are more comprehensive than **AC**; in fact, the families $\mathbf{AC}_\sigma$, $\mathbf{AC}_\delta$, $\mathbf{AC}_{\sigma\delta}$, $\mathbf{AC}_{\delta\sigma}$, etc. ($K$ being a family of sets, K_σ and K_δ are respectively the families of all denumerable unions and all denumerable intersections of sets in K.) Among these families, $\mathbf{AC}_\delta$ is especially important from the viewpoint of the intuitions underlying our whole discussion. A set $\mathcal{S}$ of algebras is in **AC** if it can be characterized axiomatically by means of a single arithmetical sentence or—what amounts to the same—by means of a finite system of arithmetical sentences; it is in $\mathbf{AC}_\delta$ if it can be characterized axiomatically by means of any finite or infinite system of arithmetical sentences.

For instance, the set of all infinite algebras is easily seen to be in $\mathbf{AC}_\delta$ (though, by Theorem 20(ii), not in $\mathbf{AC}$), and the set of all finite algebras is in $\mathbf{AC}_\sigma$. Theorem 17 remains valid if $\mathbf{AC}$ is replaced by $\mathbf{AC}_\delta$. The following important result essentially due to G. Birkhoff should be mentioned here: If $\mathcal{S}$ is a set of algebras which (i) together with any algebra $\mathfrak{A}$ contains every subalgebra of $\mathfrak{A}$ and every algebra isomorphic to $\mathfrak{A}$, and (ii) together with any algebras $\mathfrak{A}_i$, $i \in I$, contains their cardinal (direct) product, then $\mathcal{S} \in \mathbf{AC}_\delta$.[8]

DEFINITION 21. *Two algebras* $\mathfrak{A}, \mathfrak{B} \in \mathfrak{A}$ *are said to be* ARITHMETICALLY EQUIVALENT, *in symbols* $\mathfrak{A} \equiv \mathfrak{B}$, *if every set* $\mathcal{S} \in \mathbf{AC}$ *contains either both of these algebras or neither of them.*

Obviously, $\equiv$ is an equivalence relation on the set $\mathfrak{A}$.

THEOREM 22. *For any algebras* $\mathfrak{A}, \mathfrak{B} \in \mathfrak{A}$,

(i) *the formula* $\mathfrak{A} \cong \mathfrak{B}$ *implies* $\mathfrak{A} \equiv \mathfrak{B}$;

(ii) *if* $\mathfrak{A}$ *is finite, then the two formulas are equivalent.*

THEOREM 23. *For every algebra* $\mathfrak{A} \in \mathfrak{A}$ *of an infinite power* α *and for every infinite cardinal* β *there is an algebra* $\mathfrak{B} \in \mathfrak{A}$ *of the power* β *such that* $\mathfrak{A} \equiv \mathfrak{B}$. *If in addition* $\beta \leqq \alpha$, *then such an algebra* $\mathfrak{B}$ *can be found among the subalgebras of* $\mathfrak{A}$.

The mathematical proof of this theorem is related to that of Theorem 13 and is rather involved. On the other hand, Theorem 23 can easily be recognized as a mathematical translation of a familiar metamathematical result—in fact, of an extension of the Löwenheim-Skolem theorem.[9]

By Theorems 22(ii) and 23, the equivalence of the formulas $\mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$ for every algebra $\mathfrak{B}$ is a characteristic property of finite algebras $\mathfrak{A}$.

The following theorem is a rather special consequence of Theorem 13; as will be seen later, it has many interesting applications.

THEOREM 24. *If* $\mathfrak{A} \in \mathfrak{A}$, *and if* $\mathfrak{F}_n \in \mathbf{AF}$ *and* $\mathfrak{F}_{n+1}(\mathfrak{A}) \subseteq \mathfrak{F}_n(\mathfrak{A}) \neq \Lambda$ *for every natural* n, *then there is an algebra* $\mathfrak{B} \in \mathfrak{A}$ *such that* $\mathfrak{A} \equiv \mathfrak{B}$ *and* $\bigcap(\mathfrak{F}_n(\mathfrak{B}) \mid n \in \omega) \neq \Lambda$.

DEFINITION 25. *A set* $\mathcal{S} \subseteq \mathfrak{A}$ *is called an* ARITHMETICAL TYPE *if* $\mathcal{S} = \{\mathfrak{B} \mid \mathfrak{A} \equiv \mathfrak{B}\}$ *for some algebra* $\mathfrak{A} \in \mathfrak{A}$. *The family of all arithmetical types is denoted by* $\mathbf{AT}$.

Thus, arithmetical types are partition sets under the equivalence relation $\equiv$. In metamathematical terminology, a set of algebras is an arithmetical type if it is axiomatically characterized by means of a complete and consistent system of arithmetical sentences.[10]

[8] Cf. [2].

[9] This extension of the Löwenheim-Skolem theorem follows from the results in [5]; see also [3] and [11].

[10] The term "complete" is used here in the sense of [13, second part, p. 283].

THEOREM 26. (i) $\mathcal{S} \in \mathbf{AT}$ *if and only if* $\mathcal{S} = \bigcap(\mathfrak{X} \mid \mathfrak{A} \in \mathfrak{X} \in \mathbf{AC})$ *for some* $\mathfrak{A} \in \mathfrak{a}$.
(ii) $\mathbf{AT}$ *has the power* $2^{\aleph_0}$.

By Theorem 26(i), every arithmetical type is in $\mathbf{AC}_\delta$. By Theorems 20(i) and 22(ii), the arithmetical types of finite algebras are in $\mathbf{AC}$; this property, however, is not characteristic for finite algebras.

THEOREM 27. (i) *If* $\mathbf{I}$ *is a prime ideal in the Boolean algebra* $\langle \mathbf{AC}, \cup, \cap, {}^- \rangle$ *of Theorem* 16 *and* $\mathcal{S} = \bigcap(\mathfrak{X} \mid \mathfrak{X} \in \mathbf{AC} - \mathbf{I})$, *then* $\mathcal{S} \in \mathbf{AT}$.
(ii) *If* $\mathcal{S} \in \mathbf{AT}$, *then there is exactly one prime ideal* $\mathbf{I}$ *in* $\langle \mathbf{AC}, \cup, \cap, {}^- \rangle$ *for which* $\mathcal{S} = \bigcap(\mathfrak{X} \mid \mathfrak{X} \in \mathbf{AC} - \mathbf{I})$, *in fact* $\mathbf{I} = \{\mathfrak{X} \mid \mathfrak{X} \in \mathbf{AC}, \mathcal{S} \cap \mathfrak{X} = \Lambda\}$.

Theorem 27(i) is a consequence of Theorem 17.

DEFINITION 28. *A set* $\mathcal{S} \subseteq \mathfrak{a}$ *is said to be* ARITHMETICALLY CLOSED *if, together with every algebra* $\mathfrak{A}$, *it contains all algebras* $\mathfrak{B}$ *such that* $\mathfrak{A} \equiv \mathfrak{B}$. *The family of all arithmetically closed sets is denoted by* $\mathbf{ACL}$.

THEOREM 29. (i) $\mathcal{S} \in \mathbf{ACL}$ *if and only if* $\mathcal{S} = \bigcup(\mathfrak{X} \mid \mathfrak{X} \in \mathbf{K})$ *for some* $\mathbf{K} \subseteq \mathbf{AT}$.
(ii) $\mathbf{ACL}$ *has the power* $2^{2^{\aleph_0}}$.

THEOREM 30. $\mathbf{ACL}$ *is a complete field of subsets of* $\mathfrak{a}$, *and hence* $\langle \mathbf{ACL}, \cup, \cap, {}^- \rangle$ *is a complete and atomistic Boolean algebra.* $\mathbf{AC}$ *is a subfield of* $\mathbf{ACL}$, *and* $\mathbf{AT}$ *is the set of all atoms of* $\mathbf{ACL}$.

From Theorem 30 it follows that not only $\mathbf{AC}$ but also all the families $\mathbf{AC}_\sigma$, $\mathbf{AC}_\delta$, $\mathbf{AC}_{\sigma\delta}$, $\mathbf{AC}_{\delta\sigma}$, $\cdots$ are included in $\mathbf{ACL}$; by Theorem 29(ii), $\mathbf{ACL}$ is much more comprehensive than all these families. On the other hand, simple examples of sets of algebras are known which are not in $\mathbf{ACL}$. E.g., by Theorem 23, the isomorphism type of an infinite algebra $\mathfrak{A}$ (i.e., the set of all algebras $\mathfrak{B}$ with $\mathfrak{A} \cong \mathfrak{B}$) as well as the set of all algebras of any given infinite power α are not arithmetically closed.

By Theorem 30, the Boolean algebra $\mathfrak{A} = \langle \mathbf{AC}, \cup, \cap, {}^- \rangle$ is a subalgebra of the Boolean algebra $\mathfrak{B} = \langle \mathbf{ACL}, \cup, \cap, {}^- \rangle$; by Theorem 27, $\mathfrak{B}$ proves to be isomorphic to the Boolean algebra formed by all the sets of prime ideals in $\mathfrak{A}$. Hence, by means of familiar results from Stone's representation theory for Boolean algebras, the theory of arithmetical classes acquires a topological interpretation.[11] To obtain this interpretation directly, we proceed as follows. With every set $\mathfrak{X} \subseteq \mathfrak{a}$ we correlate the set $\mathfrak{C}(\mathfrak{X})$ defined by the formula

$$\mathfrak{C}(\mathfrak{X}) = \bigcap(\mathfrak{Y} \mid \mathfrak{X} \subseteq \mathfrak{Y} \in \mathbf{AC}).$$

Using exclusively Theorem 16(i), we easily show that the set $\mathfrak{a}$ of all algebras

[11] For Stone's representation theory and the notions involved in the following discussion see, e.g., [1].

$\langle A, \dotplus \rangle$ is a topological space with $\mathfrak{C}$ as closure operation. Since, however, $\{\mathfrak{A}\} \neq \mathfrak{C}(\{\mathfrak{A}\})$ for any $\mathfrak{A} \in \mathfrak{a}$ ($\{\mathfrak{A}\}$ denoting as usually the set containing $\mathfrak{A}$ as the only element), it is not a topological space in the narrower sense (i.e., not a T_1-space). On the other hand, Theorem 16(i) also implies that $\mathfrak{C}(\{\mathfrak{A}\}) = \mathfrak{C}(\{\mathfrak{B}\})$ or $\mathfrak{C}(\{\mathfrak{A}\}) \cap \mathfrak{C}(\{\mathfrak{B}\}) = \Lambda$ for arbitrary $\mathfrak{A}, \mathfrak{B} \in \mathfrak{a}$. Hence we can transform this space into a topological space in the narrower sense by "identifying" two points $\mathfrak{A}, \mathfrak{B} \in \mathfrak{a}$ in case $\mathfrak{C}(\{\mathfrak{A}\}) = \mathfrak{C}(\{\mathfrak{B}\})$. The new space can be referred to as the *arithmetical space over* $\mathfrak{a}$. The definition of closure in the new space remains unchanged. As is easily seen, the formula $\mathfrak{C}(\{\mathfrak{A}\}) = \mathfrak{C}(\{\mathfrak{B}\})$ is equivalent to $\mathfrak{A} \equiv \mathfrak{B}$; hence the points of the new space, i.e., the sets $\mathfrak{X}$ of the form $\mathfrak{X} = \mathfrak{C}(\{\mathfrak{A}\})$ for some $\mathfrak{A} \in \mathfrak{a}$, are simply arithmetical types, and the power of the space is $2^{\aleph_0}$ (see Theorem 26). The point sets of the space are arbitrary (not sets, but) unions of points; by Theorem 27(i), they coincide with arithmetically closed sets. $\mathbf{AC}_\delta$ and $\mathbf{AC}_\sigma$ are respectively the families of all closed and all open sets; by the formula $\mathbf{AC} = \mathbf{AC}_\delta \cap \mathbf{AC}_\sigma$, which is a consequence of Theorem 17, arithmetical classes coincide with those point sets which are both closed and open. The families $\mathbf{AC}_\delta$, $\mathbf{AC}_\sigma$, $\mathbf{AC}_{\delta\sigma}$, $\mathbf{AC}_{\sigma\delta}$, $\cdots$ are what are called Borelian classes of point sets. The arithmetical space over $\mathfrak{a}$ is easily seen to be totally disconnected; by Theorem 16(ii) it is separable and by Theorem 17 it is bicompact. (Thus, the bicompactness of the space discussed is a consequence of the completeness of elementary logic.) The theory of totally disconnected, separable, and bicompact spaces is a well developed branch of topology; and various results obtained in this theory can be automatically carried over to the theory of arithmetical classes.[12] In this way we arrive, e.g., at the conclusion that every arithmetical class is a union of finitely many or denumerably many or $2^{\aleph_0}$ arithmetical types.

Within the framework of the ideas outlined in the present paper various special studies are carried on. In order to obtain stronger and deeper results we usually concentrate in these studies upon algebraic systems of some special kind, e.g., upon Abelian groups, arbitrary groups, or—turning now to algebraic systems with more than one operation—upon rings, fields, lattices, Boolean algebras, etc. To apply the results of our general discussion to such special sets of algebraic systems, we first subject all the notions involved to a process of relativization. Thus, $\mathfrak{V}$ being the set of all algebraic systems in which we are interested, we introduce the notions of an arithmetical function, arithmetical class, etc., *relative to* $\mathfrak{V}$, in symbols $\mathbf{AF}(\mathfrak{V})$, $\mathbf{AC}(\mathfrak{V})$, etc. In case $\mathfrak{V}$ is a subset of $\mathfrak{a}$, the definitions of relativized notions are obtained in the following way: by modifying Definition 1(i), we agree to denote by $\mathbf{F}(\mathfrak{V})$ the set of all functions $\mathfrak{F}$ such that $D(\mathfrak{F}) = \mathfrak{V}$, and $\mathfrak{F}(\mathfrak{A}) \subseteq A^\omega$ for every $\mathfrak{A} = \langle A, \dotplus \rangle \in \mathfrak{V}$; in all the subsequent definitions we replace $\mathbf{F}$ by $\mathbf{F}(\mathfrak{V})$. It turns out, in particular, that arithmetic classes relative to $\mathfrak{V}$ are simply intersections of $\mathfrak{V}$ with arithmetical

[12] For related applications of topology to metamathematics see [6].

classes in the absolute sense, i.e., relative to $\mathfrak{A}$;[13] and if $\mathfrak{V}$ is itself an arithmetical class in the absolute sense, then arithmetical classes relative to $\mathfrak{V}$ coincide with those arithmetical classes in the absolute sense which are subsets of $\mathfrak{V}$. Most theorems stated in this paper automatically extend to relativized notions. In a few important cases, however, some restrictive assumptions concerning $\mathfrak{V}$ are necessary. For instance, Theorems 17–19 do not apply to arithmetical classes relative to an arbitrary set $\mathfrak{V}$, but they prove to hold under the assumption that $\mathfrak{V}$ is itself a member of **AC** or, more generally, of $\mathbf{AC}_\delta$ (in the absolute sense).

Whenever a set $\mathfrak{V}$ of algebras is studied from the viewpoint of the theory of arithmetical classes, the main problem is that of giving an exhaustive description of all arithmetical classes relative to $\mathfrak{V}$. The solution of the problem usually consists in (i) singling out certain special arithmetical classes referred to as basic classes; (ii) showing that the family of all arithmetical classes (relative to $\mathfrak{V}$) coincides with the field of subsets of $\mathfrak{V}$ generated by the basic classes—or, in other words, that every arithmetical class can be represented as a finite union of finite intersections of basic classes and their complements (to $\mathfrak{V}$); (iii) establishing a criterion which permits us to decide in each particular case whether or not two representations of the type just mentioned yield the same arithmetical class. As a rule, such a description of arithmetical classes is preceded by and derived from an analogous description of arithmetical functions. In obtaining the latter description the crucial point consists in showing that, $\mathfrak{B}$ being the Boolean algebra generated by basic functions, the operation ∇_k (or Δ_k) performed on an arbitrary function in $\mathfrak{B}$ yields a new function in $\mathfrak{B}$. Hence the method applied is referred to as the *method of eliminating quantifiers*.[14]

A detailed description of arithmetical classes has many important consequences. It gives us a clear insight into the structure of arithmetical classes and, speaking more precisely, permits us to determine the isomorphism type of the Boolean algebra formed by these classes. It enables us to decide in various special cases whether or not a given subset of $\mathfrak{V}$ is an arithmetical class. It leads to a description of all arithmetical types (relative to $\mathfrak{V}$) and provides us with a criterion for arithmetical equivalence of any two algebras in $\mathfrak{V}$.

So far, an exhaustive description of arithmetical classes has been obtained for a few special sets of algebras; the most important among them are the sets of all Abelian groups, all algebraically closed fields, all Boolean algebras, and all well ordered systems (i.e., systems formed by a set A and a binary relation R which establishes a well-ordering in A).[15] The results are especially simple in the case of the set $\mathfrak{V}$ of all algebraically closed fields. In fact, p being 0 or a prime number, let $\mathfrak{C}_p$ be the set of all algebraically closed fields with characteristic p. All these sets $\mathfrak{C}_p$, with the exception of $\mathfrak{C}_0$, are easily seen to be in $\mathbf{AC}(\mathfrak{V})$. They are chosen as basic classes, and it is shown that every set in $\mathbf{AC}(\mathfrak{V})$ can

[13] Or, more generally, to the species which includes $\mathfrak{V}$ as a subset; cf. footnote 4.

[14] For information concerning this method see [10, pp. 15, 50].

[15] See [7], [9], and [11].

be represented in a unique way either as a union of finitely many sets $\mathcal{C}_p$ ($p \neq 0$) or as an intersection of finitely many complements of these sets. In consequence, the family $\mathbf{AT}(\mathcal{V})$ proves to consist of all the basic classes $\mathcal{C}_p$ and, in addition, of the set $\mathcal{C}_0$; two algebraically closed fields prove to be arithmetically equivalent if and only if they have the same characteristic.

The method of eliminating quantifiers which is used in deriving results of this type is of metamathematical origin; it has been frequently applied to establish a decision procedure for various theories formalized within elementary logic. In fact, whenever the decision problem for the arithmetic of algebraic systems of a set $\mathcal{V}$ has been solved in an affirmative way, a detailed description of arithmetical classes relative to $\mathcal{V}$ has also been obtained. If, conversely, the solution of the decision problem for the arithmetic of $\mathcal{V}$ is known to be negative, the task of exhaustively describing arithmetical classes relative to $\mathcal{V}$ appears to be hopeless; this applies, e.g., to the set of all algebras $\langle A, +\rangle$, or all groups, or all fields, or all lattices.[16]

However, even in those cases in which the structure of the family of arithmetical classes is not known, various special problems involving this family can be successfully discussed. We can ask, e.g., the question whether a given special set $\mathcal{S}$ of algebraic systems is an arithmetical class. If an arithmetical definition of $\mathcal{S}$ is known, it obviously implies an affirmative answer to this question. If, on the other hand, no such definition of $\mathcal{S}$ is available, the conjecture that $\mathcal{S}$ is not an arithmetical class arises in a natural way; often, however, a confirmation of this conjecture presents considerable difficulties. Sometimes an application of Theorem 18, 19, or 24 leads to the desirable result. Let, for instance, $\mathcal{S}$ be the set of all fields of characteristic 0, and $\mathcal{S}_n$ be the set of all fields whose characteristic is not a prime number $\leqq n$. Clearly, all the sets $\mathcal{S}_n$ for $n = 0, 1, 2, \cdots$ are in **AC**, and

$$\mathcal{S} = \bigcap(\mathcal{S}_n \mid n \in \omega).$$

Therefore $\mathcal{S}$ is in $\mathbf{AC}_\delta$ and hence also in **ACL**; on the other hand, Theorem 19 implies that $\mathcal{S}$ is not in **AC**. The same holds, e.g., for the set of all algebraically closed fields or all groups without elements of finite order >1; see also Theorem 20. On the other hand, let $\mathfrak{J}$ be the set of all groups without elements of infinite order. From Theorem 24 we easily conclude that, for every group $\mathfrak{G} \in \mathfrak{J}$ in which the orders of elements are not bounded above, there is an arithmetically equivalent group $\mathfrak{H}$ which is not in $\mathfrak{J}$. Hence $\mathfrak{J}$ is not in **ACL** and a fortiori not in **AC**. In an entirely analogous way we can show by means of Theorem 24 that the set of all well ordered systems (or, more generally, of all simply ordered systems without densely ordered subsystems) and the set of all ordered rings with Archimedean order are neither in **ACL** nor in **AC**. A different argument leads to the conclusion that the set of all directly indecomposable groups is not in **ACL** or **AC**. Various problems in this domain remain open. Thus, it is known that

[16] See [8] and [15].

the set of all simple groups (or all simple algebras) is not in **AC**; it is not known, however, whether this set is in **ACL**.[17]

When new notions are introduced in mathematics, the question of their usefulness and applicability is often raised. Mathematicians want to know whether the discussion of the new notions leads to interesting results whose significance is not restricted to the intrinsic development of the theory of these notions. We believe that the theory of arithmetical classes has good chances to pass the test of applicability. To support this statement we want to discuss some applications of this theory which may be of general interest to mathematicians and especially to algebraists; in some of these applications the notions of the theory itself are not involved at all.

Theorem 13 and its consequences (Theorems 17, 19, 24) provide us with a rather general method of constructing (or, at least, proving the existence of) algebraic systems with some properties prescribed in advance. Consider, for instance, the problem of the existence of non-Archimedean ordered fields. Let $\mathfrak{A}'$ denote the set of all algebraic systems $\mathfrak{A} = \langle A, +, \cdot, <, 0, 1\rangle$ with two binary operations $+$ and $\cdot$, a binary relation $<$, and two distinguished elements 0 and 1; all the notions of the theory of arithmetical classes are understood to refer to $\mathfrak{A}'$ instead of to $\mathfrak{A}$.[4] For any given natural number n, let $\mathfrak{F}_n$ be the function in **F** defined as follows: for $\mathfrak{A} = \langle A, +, \cdot, <, 0, 1\rangle$, $\mathfrak{F}_n(\mathfrak{A})$ is the set of all sequences $x \in A^\omega$ such that $x_1 = 0$, $x_{k+1} = x_k + 1$ for $k = 1, 2, \cdots, n$, and $x_{n+1} < x_0$. Taking for $\mathfrak{A}$ the field of all real numbers, we see that the hypothesis of Theorem 24 is satisfied. Hence there is an algebraic system $\mathfrak{B}$ such that $\mathfrak{A} \equiv \mathfrak{B}$ and $\bigcap(\mathfrak{F}_n(\mathfrak{B}) \mid n \in \omega) \neq \Lambda$; a moment's reflection shows that $\mathfrak{B}$ is a real closed field with a non-Archimedean order. Of course, as is well known, such fields can be constructed by means of direct, specific methods; still it seems interesting that their existence is an almost immediate consequence of a general theorem from the theory of arithmetical classes. Moreover, by analyzing the argument just outlined, we notice that it leads to a stronger result, which—neither in its general form nor in various particular cases—seems to be easily obtainable by means of direct construction methods: For every ordered ring $\mathfrak{R}$ there exists an arithmetically equivalent ring $\mathfrak{S}$ with a non-Archimedean order. (Hence, as was mentioned above, the set of all rings with Archimedean order is not in **ACL**.) The following theorem obtained in an entirely analogous way may also be of interest: For every integral domain $\mathfrak{D}$ which is not a field there exists an arithmetically equivalent integral domain $\mathfrak{E}$ containing two elements x and y such that x is not a unity element, $y \neq 0$, and x^n divides y for every natural number n. We should like to mention some further results in the same general

[17] Among the results mentioned in the last paragraph, those concerning various sets of groups have been communicated to the author by Dr. W. Szmielew; they were originally obtained as consequences of her results in [9], without the help of Theorems 18, 19, or 24. The remaining results are due to the author; for well ordered systems see [13, second part, p. 301]. *Added in proof*: The answer to the question whether the set of all simple groups is arithmetically closed has recently been found to be negative.

direction whose proofs, based on Theorem 19, are even simpler: I. If $\mathcal{S} \in \mathbf{AC}$ (or, more generally, $\mathcal{S} \in \mathbf{AC}_\delta$) and if, for every n, there is a group in $\mathcal{S}$ no element of which is of order $\leqq n$ (except, of course, the unit), then there is a group in $\mathcal{S}$ all elements of which are of infinite order. II. Dually, if $\mathcal{S} \in \mathbf{AC}$ (or, more generally, $\mathcal{S} \in \mathbf{AC}_\sigma$), and if $\mathcal{S}$ contains every group all elements of which are of infinite order, then, for some natural number n, $\mathcal{S}$ contains every group no element of which is of order $\leqq n$. I and II remain valid if groups without elements of order $\leqq n$ are replaced by fields of characteristic $\geqq n$, and groups with all elements of infinite order by fields of characteristic 0.[18]

Applications of a different type can be obtained from various special results by which any two algebraic systems of a certain set are arithmetically equivalent. It has been shown, for instance, that any two real closed fields are arithmetically equivalent (and that, consequently, the set of all real closed fields is an arithmetical type). Hence, if an arithmetical class $\mathcal{S}$ contains some real closed field, e.g., the field of all real numbers, then it contains every real closed field as a member; and, what is very important, the same still holds if $\mathcal{S}$ is not necessarily an arithmetical class but an arithmetically closed set. Let now, for every natural number n, $\mathcal{S}_n$ be the set of all ordered fields $\mathfrak{A} = \langle A, +, \cdot, <, 0, 1\rangle$ satisfying the following condition: every polynomial in two variables over $\mathfrak{A}$ of degree n reaches a maximum on every rectangle, i.e., on every set of couples $\langle x, y\rangle$ such that $a \leqq x \leqq b$ and $c \leqq y \leqq d$ for some $a, b, c, d \in A$, $a < b$, $c <$ d. Let $\mathcal{S} = \bigcap(\mathcal{S}_n \mid n \in \omega)$; i.e., $\mathcal{S}$ is the set of all ordered fields $\mathfrak{A}$ such that every polynomial in two variables over $\mathfrak{A}$ of an arbitrary degree reaches a maximum on every rectangle. Clearly, all the sets $\mathcal{S}_n$ for $n = 0, 1 \cdots$ are in $\mathbf{AC}$; hence, $\mathcal{S}$ is in $\mathbf{AC}_\delta$ and therefore in $\mathbf{ACL}$. Obviously, the field of real numbers belongs to $\mathcal{S}$. Consequently, every real closed field belongs to $\mathcal{S}$—a result which can also be obtained directly, without the help of the theory of arithmetic classes, but which is not quite trivial. In exactly the same way many different theorems which have been established for the field of real numbers by essentially applying the continuity of this field, and sometimes by using difficult topological methods, automatically extend to all real closed fields.[19]

[18] The method discussed in the last paragraph was mentioned by the author in his address at the Princeton University Bicentennial Conference on the Problems of Mathematics, 1946; some of its applications, e.g., to the problem of the existence of non-Archimedean ordered fields, were explicitly pointed out. The problem of the existence of integral domains with two elements x, y such that x is not a unity, $y \neq 0$, and x^n divides y for $n = 0, 1, \cdots$ was formulated by Professor R. M. Robinson and solved by the author with the help of the same general method. Some other applications and an extension of the same method can be found in the Princeton University doctoral dissertation of L. Henkin, *The completeness of formal systems*, 1947. The question whether the method discussed provides an effective construction of algebraic systems with prescribed properties or merely proves their existence is somewhat involved and will not be discussed here. *Added in proof*: For a direct construction of integral domains mentioned above see the recent paper of R. M. Robinson, *Undecidable rings*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 149–150.

[19] The arithmetical equivalence of all real closed fields was established in [10, p. 54]. The far-reaching applicability of this result to non-arithmetical theorems became clear

In connection with the last remarks the following observation seems to be appropriate. The statement that any two real closed fields are arithmetically equivalent is a mathematical translation of the metamathematical result by which the arithmetic of real closed fields is a complete (and consistent) theory.[10] This result implies as an immediate consequence that every arithmetical statement which holds in a particular real closed field, e.g., in the field of real numbers, automatically holds in any other real closed field. As we have seen above, the consequence just mentioned extends to a comprehensive set of statements which cannot be formulated in the arithmetic of real closed fields; in fact, to every statement such that the set of all ordered fields in which this statement holds is arithmetically closed, without being an arithmetical class. This extension is immediate once the completeness theorem for real closed fields has been translated into the language of arithmetic classes; however, it could hardly be derived in a purely metamathematical (syntactical) way from the completeness theorem itself—unless we allow ourselves to apply some rather intricate semantical notions and methods. At first sight the mathematical theory of arithmetical classes seems to be merely a translation of the metamathematics of arithmetical formalisms; actually this theory paves the way for constructions and derivations which go far beyond purely metamathematical procedures.

Bibliography

1. G. Birkhoff, *Lattice Theory*, Rev. ed., Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1948.

2. ———, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. vol. 31 (1935) pp. 433–454.

3. L. Henkin, *The completeness of the first-order functional calculus*, J. Symbolic Logic vol. 14 (1949) pp. 159–166.

4. C. Kuratowski and A. Tarski, *Les opérations logiques et les ensembles projectifs*, Fund. Math. vol. 17 (1931) pp. 240–248.

5. A. Malcev, *Untersuchungen aus dem Gebiete der mathematischen Logik*, Rec. Math. (Mat. Sbornik) N. S. vol. 1 (1936) pp. 323–336.

6. A. Mostowski, *Abzählbare Boolesche Körper und ihre Anwendung auf die allgemeine Metamathematik*, Fund. Math. vol. 29 (1937) pp. 34–53.

7. A. Mostowski and A. Tarski, *Arithmetical classes and types of well ordered systems*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 65, 1192.

8. J. Robinson, *Definability and decision problems in arithmetic*, J. Symbolic Logic vol. 14 (1949) pp. 98–114.

9. W. Szmielew, *Arithmetical classes and types of Abelian groups*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 65, 1192.

10. A. Tarski, *A decision method for elementary algebra and geometry*. Prepared for publication by J. C. C. McKinsey. U. S. Air Force Project RAND, R-109. Santa Monica, Calif., 1948.

11. ———, *Arithmetical classes and types of mathematical systems. Metamathematical aspects of arithmetical classes and types. Arithmetical classes and types of Boolean algebras.*

---

to the author during a stimulating conversation with Professor H. Hopf. *Added in proof*: For further applications and bibliographical references see the new edition of [10] (University of California Press, 1951), supplementary note 7.

Arithmetical classes and types of algebraically closed and real closed fields, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 63–64, 1192.

12. ———, *Der Wahrheitsbegriff in den formalisierten Sprachen*, Studia philosophica vol. 1 (1936) pp. 261–405.

13. ———, *Grundzüge des Systemenkalküls*, First part, Fund. Math. vol. 25 (1935) pp. 503–526. Second part, Fund. Math. vol. 26 (1936) pp. 283–301.

14. ———, *Sur les ensembles définissables de nombres réels*. I. Fund. Math. vol. 17 (1931) pp. 210–239.

15. ———, *Undecidability of group theory*, J. Symbolic Logic vol. 14 (1949) pp. 76–77. *Undecidability of the theory of lattices and projective geometries*, J. Symbolic Logic vol. 14 (1949) pp. 77–78.

University of California,
Berkeley, Calif., U. S. A.

INCOMPLETABILITY, WITH RESPECT TO VALIDITY IN EVERY FINITE NONEMPTY DOMAIN, OF FIRST ORDER FUNCTIONAL CALCULUS

WILLIAM CRAIG

First order functional calculus is complete with respect to validity in every nonempty domain, i.e., every formula which is valid in every nonempty domain is a theorem. Moreover, given any positive integer k, first order functional calculus can, by additional axioms or rules of inference, be made complete, in a similar sense, with respect to validity in every nonempty domain of not more than k individuals. In contrast to these well-known results, a proof is given of the following *Theorem*: There exists no constructive logistic system whose theorems are exactly those formulas of first order functional calculus which are valid in every finite nonempty domain. By a constructive logistic system is meant a logistic system such that there is a general recursive process whereby, given any sequence of formulas, it can be determined whether or not the sequence constitutes a proof in the system.

Introducing the notion of a *general model*, Henkin has proved that, for $n = 1$, $n = 2$, and $n = \omega$, the customary systems of functional calculus of order n are complete in the following sense: Every formula which, roughly speaking, is true for any assignment with respect to any nonempty general model is a theorem. [Leon Henkin, *Completeness in the theory of types*, J. Symbolic Logic vol. 15 (1950) pp. 81–91.] In contrast, from the *Theorem* the following result is derived: There exists no constructive logistic system whose theorems are exactly those formulas of first order functional calculus which, roughly speaking, are true for any assignment with respect to any finite nonempty general model.

The *Theorem* is shown to follow readily from the following weaker *Lemma*: There is no general recursive process whereby, given any formula of first order functional calculus, it can be determined whether or not there exists a finite nonempty domain in which the formula is satisfiable. The *Lemma* is proved by methods similar to those which Turing [A. M. Turing, *On computable numbers, with an application to the entscheidungsproblem*, Proc. London Math. Soc. (2) vol. 42 (1936) pp. 230–265, especially pp. 261–262] uses to demonstrate that there is no general recursive (computable) process for determining provability in first order functional calculus. (A theorem which is equivalent to the *Lemma* has recently been proved by a different method. See B. A. Trahténbrot, *Névozmožnost' algorifma dlá problémy razréšimosti na konéčnyh klassah* (Impossibility of an algorithm for the decision problem of finite classes), C. R. (Doklady) Acad. Sci. URSS. vol. 70 (1950) pp. 569–572. The work of Trahténbrot was called to my attention only after I had completed the proof of the *Theorem*.)

PRINCETON UNIVERSITY,
PRINCETON, N. J., U. S. A.

THE INFERENTIAL THEORY OF NEGATION

HASKELL B. CURRY

In lectures delivered in 1948 (see *A theory of formal deducibility*, Notre Dame University, Mathematical Lectures, no. 6, 1950) I proposed what is essentially a semantical definition of the propositions formed from the elementary propositions of a formal system by propositional connectives and quantifiers. The theory was based on Gentzen's inferential rules. So far as negation was concerned, four different systems were considered, viz.: *LM*, the minimal system; *LJ*, the intuitionist system; *LD*, the minimal system with excluded middle; and *LK*, the classical system. Each of these systems had a semantical interpretation: *LM*, when negation is interpreted as refutability in the sense of Carnap, i.e., as implying one of a given set of "directly refutable" propositions; *LJ* for absurdity, i.e., implying every proposition; *LD* for refutability with excluded middle, leading to a system of strict implication; and *LK* for the interpretation by truth tables. This paper reports—using the previous notations—some recent improvements in the theory of negation as follows:

1. If $\mathfrak{X}$, $\mathfrak{Y}$, and A are positive (i.e., formed wholly without negation), and if $\mathfrak{X}, \neg \mathfrak{Y} \Vdash A$ holds in LD^*, then $\mathfrak{X} \Vdash A$ holds in LA^*. Hence the positive propositions A such that $\Vdash A$ holds in LD^* are the same as those for LA^*; and if such an A is elementary, it is derivable in the original system $\mathfrak{S}$.

2. If $\mathfrak{X} \Vdash A$ is derivable in *LD* (*LK*), then $\mathfrak{X}, \neg A \Vdash A$ is derivable in *LM* (*LJ*). This is an extension of the Glivenko theorem. It shows that *LD* is decidable. The proof does not hold if quantifiers are present.

3. If $\neg A$ is defined as $A \supset F$, where F is a new primitive proposition, and blank right prosequences are replaced by F, then each system *LX* is transformed into a system *LXF*, and every proposition, prosequence, etc. of *LX* is changed into an "F-transform" indicated by a subscript F. A necessary and sufficient condition that $\mathfrak{X} \Vdash \mathfrak{Y}$ hold in *LX* is that $\mathfrak{X}_F \Vdash \mathfrak{Y}_F$ hold in *LXF*. Further, the elimination theorem holds for the system *LXF* if it holds for *LX*.

4. The possibility of other formulations of *LD*, *LK* has been examined. Reasons for the failure of a multiple-consequent formulation of *LD* are given. A single-consequent formulation of *LK* (and of the positive system *LC*) has been found and proved equivalent to the original one.

These results are established in the basic L-systems without recourse to the H-systems, or to technical restrictions necessary in the lectures.

PENNSYLVANIA STATE COLLEGE,
STATE COLLEGE, PA., U. S. A.

RELATIVELY RECURSIVE FUNCTIONS AND THE EXTENDED KLEENE HIERARCHY

MARTIN DAVIS

The properties of Kleene's notion of a function being recursive in other functions is developed in a manner entirely analagous to his *Recursive predicates and quantifiers*, Trans. Amer. Math. Soc. vol. 53, pp. 41–73. In particular direct analogues of Kleene's Theorems I, II, IV, VII, and VIII are proved. It is proved that the set S is recursive in the set T if and only if both S and $\bar{S}$ are T-canonical in the sense of Post, thus correlating our point of view with Post's. (Cf. Bull. Amer. Math. Soc. Abstract 54-7-269.)

Next, making use of Post's idea of representing an entire family of predicates by a single "complete" set, a generalized theory of the Kleene hierarchy of predicate forms is developed. The results of Post's abstract cited above are then easily obtainable by specialization. The precise degree of unsolvability of certain actual decision problems is determined. The generalized theory of the Kleene hierarchy is then used to extend the ordinary Kleene hierarchy into the constructive transfinite. It is shown that for ordinals $<\omega^2$ the same predicates are obtained from different integers representing the same ordinal. Moreover the predicates corresponding to ordinals $<\omega^2$ are all shown to be definable in a second order functional calculus with Peano's postulates.

UNIVERSITY OF ILLINOIS,
URBANA, ILL., U. S. A.

ON DISPLACEMENTS OF SYSTEMS OF DATA AND STRUCTURES

MARIO DOLCHER

A first outline of the foundations which I have posed for a general structure theory is given in the paper: M. Dolcher, *Nozione generale di struttura per un insieme* (Rendiconti del Seminario Matematico della Università di Padova (1949) pp. 265–291). My program, aiming to reach a well-founded systematics of mathematical theories by means of an intrinsic (group-theoretic) characterization of these theories, has been exposed at the Innsbruck Mathematical Congress (resumed in M. Dolcher, *Sur l'axiomatisation de la systématique des théories mathématiques par une théorie générale des structures*, Nachrichten der Oesterr. Mathematischen Gesellschaft, Dez. (1949) p. 33).

Starting from a *system of data* S (i.e. a set of "sentences" as the definition of operations, relations, $\cdots$) on a set I (briefly: a *system* $\{I \mid S\}$), we consider the *autogroup* Γ of the system, i.e., the [sub]group [of the symmetric group $(I!)$] of the *automorphisms* of $\{I \mid S\}$. We assume Γ to characterize S on I, assuming as *equivalent* two systems S', S'' such that $\Gamma' = \Gamma''$; this equivalence implies the possibility of converting the data of S' into terms of S''. Then, we assume the classes of similar groups in $(I\,!)$ as the different *structures* which are possible on I; we say, the system S is a *representation* of the corresponding structure on I.

Besides the autogroup Γ, it is important to consider the [sub]group Δ (including Γ) [of $(I\,!)$] of the substitutions which take S into a system equivalent to S (*displacements* allowed by S). It is easy to establish that Δ is the largest group in which Γ is invariant.

The extreme cases are $\Delta = (I\,!)$ (*invariant* system) and $\Delta = \Gamma$(*fixed* system). To be mentioned is the case in which Γ is a (invariant) subgroup of index 2; then $\{I \mid S\}$ may be said to be an *orientable* system; the various cases of "orientability" fall under the above general definition.

Furthermore, we can avail ourselves of the notions given for a *system* in order to get at analoguous ones for *structures*.

On the other hand, we can arrive at "relative" (i.e. immersion-) notions by considering instead of $(I\,!)$ a subgroup of $(I\,!)$.

It is easy to give examples of the described notions by assuming for I a set of 4, 5, or 6 elements. More interesting are the ones we can easily find in classical geometry: the similitudes of a space are "displacements" even if they are not automorphisms (=congruences); the affine geometry is a fixed structure if conceived as "relative" to the projective structure. The investigation of classical geometry seems to be fairly interesting from this point of view; philosophical considerations are possible.

I expect such considerations would be of interest for topology, in connection with Wiener's problem and the research of topological invariants.

For the ideal-theoretic investigation of algebraic$_a$ geometry, the considerations of the autogroup and of the displacement group of the lattice of ideals seems to lead to a criterion—perhaps the only one—in order to ascertain the correspondence of ideal-lattice-theoretic propositions to the geometrical ones.

UNIVERSITY OF TRIESTE,
TRIESTE.

ONTOLOGICAL POSITIVISM

JAMES K. FEIBLEMAN

The aim of the positivists is to systematize scientific knowledge. For this they need not only logic, mathematics, and the empirical sciences but also metaphysics and particularly ontology. Their anti-metaphysical thesis has been aimed at the uncontrolled extrapolation of transcendental metaphysics. They have identified metaphysics with the unbridled invention of metaphysical entities which it was not considered necessary to submit to the test of sense experience. The possibility of metaphysics was condemned for the errors of some metaphysicians. But a finite ontology consistent with the aims of positivism is possible. Accordingly, the postulate-set of logical positivism is adopted, with two exceptions: the anti-metaphysical postulate is denied and one more postulate is added. The result is a new postulate-set from which we can deduce a finite ontology. Examples from the latter are given. Universals, no longer claimed to be ubiquitous and eternal, are nevertheless retained as widespread and persistent. They are not created at will but must be discovered by means of the mathematical and empirical sciences. Ontology is defined by consistency-rules between divergent sets of empirical data. Thus, as in transcendental metaphysics, two orders are required; but in place of realms of essence and of existence, we have persistent and transient orders; and in place of destiny, intent. The remainder of the paper is concerned with a reinterpretation of the old postulates which are retained in the new postulate-set, and with the establishment of ontology as a speculative field of operations.

Tulane University of Louisiana,
New Orleans, La., U. S. A.

SUR LES BASES PHILOSOPHIQUES DE LA FORMALISATION

Félix Fiala

Les méthodes formelles des algébristes contemporains constituent un remarquable instrument d'analyse et d'expression des démarches fondamentales du mathématicien et semblent parfois en épuiser l'essentiel. La technique du calcul formel est en tout cas suffisamment élaborée pour qu'il puisse être utile d'en dégager les bases méthodologiques, ou ce que nous appellerons la *métaphysique*. Or si les exigences techniques du formalisme mathématique sont clairement explicitées, ses prétentions métaphysiques, souvent encore implicites, ne sont pas moins grandes; elles sont de plus souvent exclusives et peuvent devenir un obstacle au développement des mathématiques.

Dans un récent congrès de philosophie des sciences (Paris, octobre 1949), on entendit, par la voix de M. Dieudonné, le formalisme bourbakien exclure de son rayon d'intelligibilité l'intuitionisme brouwerien, représenté par M. Heyting, et affirmer l'existence d'une cloison étanche entre deux manières de penser incommunicables. Ceci nous apparut l'expression même d'une métaphysique exclusive, *forte* et *fermée*, plutôt qu'une exigence strictement technique. L'origine philosophique de l'incompréhension réciproque fut d'ailleurs immédiatement dénoncée par M. Bernays.

Les mathématiciens ont pourtant déjà eu à eclaircir de semblables situations. Lors de l'invention des géométries non-euclidiennes, la synthèse de deux théories apparemment irréductibles exigea un effort sur deux plans: sur le plan technique (surfaces à courbure négative constante, modèles de Poincaré et de Klein) et sur le plan philosophique (abandon de la métaphysique de l'évidence et de l'adéquation totale de la géométrie à l'espace réel ou de celle des jugements synthétiques a priori).

La réconciliation entre les points de vue formaliste et intuitioniste, désirable pour la cohérence de la science mathématique et de la connaissance en général, doit être poursuivie sur deux plans: sur le plan technique (correspondances entre logiques classique et intuitioniste) et sur le plan philosophique.

C'est là que certaines philosophies à base métaphysique *ouverte* et *affaiblie* (philosophie du non, de Bachelard, dialectique idonéiste de Gonseth, etc.) jouent un rôle efficace. En reconnaissant la légitimité de différents niveaux de formalisation, la relativité d'une démonstration et peut-être même la contingence de la méthode hypothético-déductive, ces philosophies ne s'opposent nullement à l'exigence technique de formalisation. Elles cherchent au contraire à en exprimer mieux les principes méthodologiques, tout en garantissant la plus vaste extension de son champs d'action. L'affaiblissement de postulats métaphysiques peut être la source de progrès analogues à ceux qui ont parfois accompagné l'abandon de positions techniques trop fortement déterminées.

(Résumé d'un travail à paraître dans *Les études de philosophie des sciences*, Ed. du Griffon, Neuchâtel, 1950.)

UNIVERSITY OF NEUCHÂTEL,
NEUCHATEL, SWITZERLAND.

ARE WE LACKING WORDS?

JACQUES HADAMARD

Humanity does not cease to gain new ideas. How will it find words to express them?

In many cases, new words are created, which, however, does not take place without bringing some scandal among our grammatists and other purists.

More frequently, scientists and especially mathematicians prefer using a known word in a new meaning. Even if exceptionally useful, it takes place too often and the number of meanings attributed to the same word increases in an excessive and disquieting way. "Conjugate" has at least three meanings in geometry, two in algebra. The case is the same for "pole."

"Class" is most unfortunate from that point of view. It is applied to algebraic curves; but its use is also fundamental in logistics; and how many kinds of "classes" are there in the theory of real functions! There are Baire's classes; but there are also quasi-analytic classes, and indefinitely differentiable functions are also distributed in classes according to the magnitude of their derivatives of an increasing order.

Those who endow a word with such various meanings are confident that they belong to separate chapters of science and, therefore, are not liable to come in conflict with each other; but, precisely, the progress of science more and more frequently puts separate chapters in mutual relation, and unexpected conflicts become rather frequent. The author has often been impeded by them.

A useful thing, in order to avoid lack of words, would be not to waste them; I mean, not to use several words where only one is needed. Why speak of the "roots" of an equation or of a polynomial while an equation has solutions and the polynomial has zeros. There is also no reason to denote the solutions of a differential system as "integrals", an expression which, moreover, is needed in a different meaning.

College of France and Polytechnic School,
Paris, France.

KONSTRUKTIVE BEGRÜNDUNG DER KLASSISCHEN MATHEMATIK

Paul Lorenzen

Praktisch brauchbare Konstruktionsregeln, wie

(1) 1

(2) $x \to x + 1$

zur Konstruktion der "Zahlen" $1, 1 + 1, 1 + 1 + 1, \cdots$ (Variable $x, y, \cdots$) oder

(3) $x + 1 \neq 1$

(4) $1 \neq x + 1$

(5) $x \neq y \to x + 1 \neq y + 1$

zur Konstruktion von "Aussagen" $1 + 1 \neq 1, 1 \neq 1 + 1, 1 + 1 + 1 \neq 1 + 1, \cdots$ (Variable $A, B, \cdots$) bilden den Ausgangspunkt der konstruktiven Mathematik.

Zur Erleichterung des Ableitens, d.h., des Konstruierens von Zeichen nach einem System S von Regeln, werden "eliminierbare" Regeln konstruiert. Eine Regel R heißt eliminierbar (bzgl. S), wenn ein Verfahren bekannt ist, jede Ableitung, die außer S noch R benutzt, umzuformen in eine Ableitung, die nur S benutzt—aber noch dasselbe Zeichen konstruiert. Ist R eliminierbar bzgl. $R_1, \cdots, R_n$ dann wird $R_1, \cdots, R_n \to R$ als Metaregel zur Konstruktion von Regeln gebraucht, z.B., $A_1 \to A_2 ; A_2 \to A_3 \dot{\to} A_1 \to A_3$. So entsteht die sog. positive Implikationslogik. Wird zur Zusammensetzung von Aussagen die "Disjunktion" eingeführt durch

(6) $A \to A \vee B$

(7) $B \to A \vee B$

(8) $A(x) \to \bigvee x A(x)$

so entsteht die sog. positive Logik.

Nach Einführung der Ungleichheit $\neq$ für Aussagen (entsprechend zu (3) – (5)) wird die Unableitbarkeit von A (bzgl. S) definiert durch $A \neq B$ für alle bzgl. S ableitbaren Aussagen B. Die Negation $\neg A$ wird definiert durch die

Eliminierbarkeit von $A \to B$ für eine unableitbare Aussage B. So entsteht die intuitionistische Logik. Mit dieser ist die zweiwertige Logik leicht als zweckmäßige Fiktion zu begründen (Kolmogoroff).

Um über die Arithmetik hinaus zu einer Analysis zu kommen, muß zunächst eine Konstruktion für "alle" arithmetischen Aussagen angegeben werden (Weyl). Hierzu wird außer der Zusammensetzung durch die logischen Operationen $\to$, $\wedge$, $\bigwedge_x$, $\vee$, $\bigvee_x$, $\neg$ jedes System

$$\begin{array}{c} A_1 \to \rho(x_1) \\ \cdot \\ \cdot \\ \cdot \\ A_n \to \rho(x_n) \end{array}$$

als Definition einer Aussage $\rho(x)$—entsprechend für mehrstellige Relationen ρ—zugelassen. Mit diesen "Aussagen 1. Schicht" werden "Mengen 1. Schicht" gebildet (die Menge der x mit $A(x)$) und dann "reelle Zahlen 1. Schicht". Durch Iteration der Schichtenbildung bis ω erhält man das Vollständigkeitsprinzip der klassischen Analysis: "Zu jeder Menge von reellen Zahlen gibt es eine reelle Zahl als untere Grenze." Diese Abgeschlossenheit der klassischen Analysis entsteht also durch die Beschränkung auf endliche Schichten. Durch geeignete Definition von "reelle Funktion", "Stetigkeit", usw. läßt sich die Gültigkeit alle Fundamentalsätze der klassischen Analysis erzwingen.

Die Frage, ob die benutzte Konstruktion "alle" reellen Zahlen der klassischen Analysis liefert, ist sinnlos, da die klassische Analysis "reelle Zahl" ja nur mit undefinierten Begriffen wie "unendliche Folge", "Einteilung der rationalen Zahlen", usw. definiert.

Die Widerspruchsfreiheit der konstruktiven Mathematik ergibt sich aus der Widerspruchsfreiheit der einzelnen benutzten Konstruktionsschritte, sie erfordert keine eigene sog. Metamathematik. Ebenso entfällt das Scheinproblem der Gegenüberstellung von "inhaltlichen" und "formalen" Schließen, da der Aufbau der konstruktiven Mathematik sprachunabhängig ist, d.h. an keiner Stelle die Kenntnis einer der traditionellen Umgangssprachen voraussetzt.

University of Bonn,
Bonn, Germany.

EXISTENTIAL DEFINABILITY IN ARITHMETIC

Julia Robinson

A relation $\rho(x_1, \cdots, x_n)$ among integers is said to be existentially definable in terms of certain given relations and operations if there is a formula containing the free variables $x_1, \cdots, x_n$, any number of bound variables, and symbols for particular integers, involving only existential quantifiers, conjunction, disjunction, equality, and the given relations and operations, which holds if and only if the relation $\rho(x_1, \cdots, x_n)$ is satisfied. The range of the quantifiers is to be the set of all integers. The relation $\rho(x_1, \cdots, x_n)$ will be called existentially definable if it is existentially definable in terms of addition and multiplication. In this case, there is a polynomial $R(x_1, \cdots, x_n; y_1, \cdots, y_k)$ with integer coefficients which vanishes for some $y_1, \cdots, y_k$ if and only if $x_1, \cdots, x_n$ satisfy the relation $\rho(x_1, \cdots, x_n)$. Alfred Tarski has raised the question whether every general recursive relation is existentially definable, but this problem remains unsolved. It may be remarked that no existential definition of the set of powers of 2 or the set of primes has yet been found, though the complementary sets are readily seen to be existentially definable.

Let $\phi(u, v)$ be any relation with the following properties: If $\phi(u, v)$, then $u > 0$ and $0 < v < u^u$, but there is no integer n such that $v < u^n$ whenever $\phi(u, v)$. The principal result of this paper is that the relation among x, y, z, defined by $y \geqq 0$ and $z = x^y$, is existentially definable in terms of $\phi(u, v)$. Application is made of the fact that the quadratic unit $\beta = b + (b^2 - 1)^{1/2}$ is a power of $\alpha = a + (a^2 - 1)^{1/2}$ if and only if $b^2 - 1 = (a^2 - 1)c^2$ for some integer c. The chief difficulty is in finding what power β is of α, and it is in identifying this power that use is made of the relation $\phi(u, v)$. It is not known whether there is a relation $\phi(u, v)$ with the desired properties which is itself existentially definable, although this appears probable. At present, too little is known about the solutions of Diophantine equations, so that it is not even possible to decide whether the condition that u and v are positive integers for which $u + v^3$ is a perfect square does or does not define a suitable relation. If there is a suitable existentially definable relation $\phi(u, v)$, Fermat's equation $x^n + y^n = z^n$ can be replaced by a Diophantine equation of fixed degree in x, y, z, n, and some additional variables. Also, it can be shown that the set of primes is existentially definable in terms of exponentiation, so that under the same hypothesis, the set of primes as well as the set of powers of 2 would be existentially definable.

BERKELEY, CALIF., U. S. A.

AN ESSENTIALLY UNDECIDABLE AXIOM SYSTEM

RAPHAEL M. ROBINSON

Mostowski and Tarski have found a finite axiom system, consisting of true formulas of the arithmetic of natural numbers, which is essentially undecidable (that is, no consistent extension is decidable). [See J. Symbolic Logic vol. 14 (1949) p. 76.] In this paper, a simpler axiom system is given with the same properties, and a simpler method of showing the essential undecidability is found. [Concerning the relation between the two axiom systems, see the following abstract of Szmielew and Tarski, *Mutual interpretability of some essentially undecidable theories*.]

The new axiom system has the primitive concepts 0, S, $+$, $\cdot$, and consists of the following seven axioms: If $Sa = Sb$, then $a = b$; $0 \neq Sb$; if $a \neq 0$, then $a = Sb$ for some b; $a + 0 = a$; $a + Sb = S(a + b)$; $a \cdot 0 = 0$; $a \cdot Sb = a \cdot b + a$. (If any one of these axioms is omitted, the resulting system is no longer essentially undecidable.) Putting $1 = S0$, $2 = S1$, $3 = S2$, $\cdots$, it is readily seen that all true formulas of the arithmetic of natural numbers not involving any variables can be proved from the given axioms. Furthermore, if $a \leqq b$ is defined to mean that there is an x such that $x + a = b$, then the following statements can be proved for $\alpha = 0, 1, 2, 3, \cdots$: For every x, $x \leqq \alpha$ or $\alpha \leqq x$; $x \leqq \alpha$ if and only if $x = 0$ or $x = 1$ or $\cdots$ or $x = \alpha$. From these results, it is possible to prove that general recursive functions and sets are formally definable, and this is all that is needed in the following argument. (On the other hand, many simple formulas, such as $0 + a = a$ and $a \leqq a$, are not provable from the given axioms.)

To show that the above axiom system is essentially undecidable, it must be shown that any decidable extension is inconsistent, and it is sufficient to consider extensions which are logically closed. Suppose that $\mathcal{P}$ is such an extension. The Gödel numbers of the sentences of $\mathcal{P}$ form a general recursive set. It follows that there is a formula $\Phi(x)$, such that for $\alpha = 0, 1, 2, 3, \cdots$, $\Phi(\alpha)$ or its negation belongs to $\mathcal{P}$, according as the sentence whose number is α is in $\mathcal{P}$ or not. Now let $T\alpha$ be the number of the formula obtained from formula number α by replacing the variable x (wherever it is free) by α. Then there is a formula $\Theta(x, y)$, such that for $\alpha = 0, 1, 2, 3, \cdots$, the sentence expressing the equivalence of $\Theta(\alpha, y)$ and $y = T\alpha$ belongs to $\mathcal{P}$. If ν denotes the number of the sentence, "For every y, if $\Theta(x, y)$, then not $\Phi(y)$", then both $\Phi(T\nu)$ and its negation must be in $\mathcal{P}$, hence $\mathcal{P}$ is inconsistent.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF., U. S. A.

A LOGISTIC PROOF OF A THEOREM RELATED TO LANDAU'S THEOREM 4

IRA ROSENBAUM

Peano's approach to the arithmetic of the positive integers is familiar. Starting with the undefined concepts, 1, successor of, and natural number, the notions of addition, multiplication, etc. are introduced recursively and their properties established. Landau remarks, "On the basis of his five axioms, Peano defined $x + y$ for fixed x and all y as follows: $x + 1 = x'$, $x + y' = (x + y)'$, and he and his successors then think: $x + y$ is defined generally $\cdots$." But, as the objections of Landau's colleague, Grandjot, indicated, "$x + y$ has not been defined."

A resolution of the difficulty to which attention had been called by Grandjot was sought by Landau, in collaboration with von Neumann, and although obtained, was put aside in favor of a simpler solution suggested by Kalmar, which is embodied in Landau's proof, in the *Grundlagen der analysis*, of his Theorem 4. Because Landau's proof introduces, without comment, entities other than the undefined concepts, specifically the *functions* a_y and b_y, his argument acquires a certain informality, which it is the purpose of the theorem of the present paper to correct.

The Peano postulates are used in the forms, $1 \; e \; N$, $(y)E\,!\,S'y$, $\sim(Ex)(1Sx)$, $S \; e \; Cls \rightarrow 1$, and $1 \; eM.S''M \subset M. \rightarrow .M = N$. Here $S'x$ replaces Landau's x'. Next in place of the usual recursive relations, use is made of a relation Σ pairing positive integers with ordered couples of positive integers in accordance with the two conditions: 1) $z\Sigma x1. \rightleftarrows .z = S'x$, and 2) $z\Sigma xS'y. \rightleftarrows .z \; e \; S''\Sigma'^{\rightarrow}(xy)$. The second condition avoids the assumption, which vitiates the usual discussion, that the couple x, y has a unique sum, so that the successor of this sum is well-defined. A logistic proof is then given,—using quantification theory, the theory of the definite descriptive operator, and portions of the general theory of classes and relations,—that $(x)(y)E\,!\Sigma'(xy)$. The proof requires the two lemmas, 1) $\Sigma \; e \; 1 \rightarrow Cls$, (i.e., $(x)(y)(z\Sigma(xy).\, w\Sigma(xy).\rightarrow .z = w)$), and 2) $(x)(y)(Ez)(z\Sigma(xy))$. These two lemmas are a sufficient basis, in accordance with Hilbert and Bernays' theory of the descriptive operator and Russell and Whitehead's theory of descriptions, to validate the theorem of the present paper, namely, $(x)(y)E\,!\,\Sigma(xy)$.

UNIVERSITY OF MIAMI,
CORAL GABLES, FLA., U. S. A.

TRANSFINITE CARDINAL ARITHMETIC IN QUINE'S NEW FOUNDATIONS

J. BARKLEY ROSSER

In order to deal adequately with the arithmetic of transfinite cardinals in Quine's New Foundations, it appears to be necessary to have an ordered pair which is of the same type (in the sense of type theory) as its constituents. Quine has shown that if the axiom of infinity be assumed, then such an ordered pair can be defined. We show that if such an ordered pair is available (either by definition, or assumed as an undefined term), then the arithmetic of transfinite cardinals is forthcoming in practically its classical form.

This result has two interesting consequences. The first is that the axiom of infinity suffices for the development of a full fledged theory of transfinite cardinals. The second is that, since the axiom of infinity is an easy consequence of the theory of transfinite cardinals, it follows that one can infer the axiom of infinity from the assumption that there is an ordered pair which is of the same type as its constituents. Since this assumption about ordered pairs seems to be a purely logical assumption, it would appear that in Quine's New Foundations the axiom of infinity can be derived from purely logical considerations. Quine had originally thought otherwise.

An interesting feature of the cardinal arithmetic is that the universe has a cardinal number, which is necessarily the greatest cardinal. The Cantor paradox is avoided because Cantor's theorem that 2^n is a greater cardinal than n is apparently available only in case n has at least one member whose members are all unit classes. This additional hypothesis is proved for all the cardinals in common use in mathematics, so that the classical arithmetic of cardinals breaks down only for extremely large cardinals of a sort which have no practical use in mathematics.

National Bureau of Standards,
Los Angeles, Calif., U. S. A.

APPLIED LOGIC AND MODERN PROBLEMS

E. R. Stabler

The main object of this paper is to point out the need of applied logic as an aid to the solution of current national and international problems.

To emphasize and illustrate the need, a brief postulational study is presented of a formal statement (S) issued by a group of physical scientists concerning the problem of control of a new type of weapon. The verdict is reached that the logical structure of S is seriously deficient, and bordering on inconsistency. It is noted that the authors of S are presumably in the habit of obtaining assistance in their own research from applied mathematicians; and that similarly, in formulating S, they might well have profited by advice from logicians acting in an applied capacity.

It may be that responsibility lies partially with the logicians themselves, whose best efforts seem to remain centered on problems of an ultra-pure nature. It is suggested that some logicians might devote more attention to applications, and attempt to create a demand for their services as consultants on problems of the type under consideration. A few directions in which their help might be especially useful are the following: investigation of postulational questions and deduction of theorems in connection with formation of new policies or evaluation of policies already in operation; introduction of abstraction and symbolism as a means of neutralizing preconceived judgments; detecting pitfalls due to elementary, but common, logical fallacies; and, in general, creating more precision and consistency in the use of language.

In conclusion, the following hypotheses are presented for consideration: 1. That the often mentioned lag in the social-political-economic development of civilization behind its scientific-technological-military development is explainable in part by the real discrepancy between the intellectual resources which have been evoked or mobilized for use in the respective areas. 2. That the lag can be considerably narrowed by cutting down this discrepancy, especially through the systematic application of logic to crucial problems in the former area. 3. That a new and remarkable logical paradox will occur if the time ever arrives when methods of destruction have been so perfected by aid of logic and mathematics, and have been confirmed empirically on such a large-scale basis, that further intellectual activity in the realm of logic and mathematics becomes paralyzed or impossible.

HOFSTRA COLLEGE,
HEMPSTEAD, N. Y., U. S. A.

ON THE MATHEMATICAL EXISTENCE

ZYOITI SUETUNA

According to the formalistic point of view, the mathematical existence is what can be apprehended without contradiction. The existence of something does not however follow from the mere logical consistency without contradiction. From the intuitionistic point of view man asserts that the mathematical existence is what we can construct by our finite acts. But the infinite can never be grasped adequately by mere finite acts. Indeed to all human knowledge there underlies our act as foundation. But the real ground of our knowledge is not our act itself; it is rather our intuition based on acts. The object of our knowledge is formed and grasped by such a positive, not fictitious, intuition.

The notion of natural numbers is formed by repeatedly adding 1. Every new addition gives rise to a new number, a process which repeats itself without end; and since we have the insight into the whole of such processes, the notion "totality of natural numbers" is formed as mathematical object. Herein lies the real ground why the notion "arbitrary natural number", which is not a definite particular number and represents as concrete-universal element all natural numbers, has a precise mathematical meaning. The mathematical existence par excellence, I think, is what can be grasped by our positive intuition based on acts by means of the totality of natural numbers and linear continuum. With respect to the continuum I lay stress on the fact that the linear continuum as notion can never be cut off from the continuum of real numbers. I don't think that the so-called freie Wahlfolge could be a mathematical object. In order to reconstruct the ordinary analysis as mathematics which is of intuitive significance, we must have the insight into the "contradictory self-identity" of both continua.

UNIVERSITY OF TOKYO,
TOKYO, JAPAN.

MUTUAL INTERPRETABILITY OF SOME ESSENTIALLY UNDECIDABLE THEORIES

WANDA SZMIELEW AND ALFRED TARSKI

The theories discussed are formalized within elementary logic. (For notation see Tarski, J. Symbolic Logic vol. 14, p. 75.) Each theory $\mathfrak{T}$ has its own (non-logical) constants and axioms—$\mathfrak{T}$-constants and $\mathfrak{T}$-axioms. The $\mathfrak{T}$-variables range over elements of a fixed set, which is represented by a special $\mathfrak{T}$-constant, say U (the *universe predicate*). Consider a constant C not occurring in $\mathfrak{T}$, e.g., the operation symbol $+$. A *possible definition of C in $\mathfrak{T}$* is any expression of the form

$$\mathbf{A}x, y, z[U(x) \wedge U(y) \wedge U(z) \rightarrow (x + y = z \leftrightarrow \Phi)]$$

where Φ represents a formula in $\mathfrak{T}$ with the free variables x, y, z; the sentence stating that for any x, y there is just one z satisfying Φ is assumed to be provable in $\mathfrak{T}$. A theory $\mathfrak{T}'$ is (*strictly*) *interpretable in* $\mathfrak{T}$ if a theory $\mathfrak{T}''$ can be constructed for which (i) the set of $\mathfrak{T}''$-constants consists of all $\mathfrak{T}$-constants and $\mathfrak{T}'$-constants; (ii) the set of $\mathfrak{T}''$-axioms consists of all $\mathfrak{T}$-axioms and of possible definitions of $\mathfrak{T}'$-constants in $\mathfrak{T}$ (one definition for each $\mathfrak{T}'$-constant); (iii) all $\mathfrak{T}'$-axioms are provable in $\mathfrak{T}''$. (If any $\mathfrak{T}'$-constants are also $\mathfrak{T}$-constants, we first replace them in $\mathfrak{T}'$ by symbols not occurring in $\mathfrak{T}$.) If $\mathfrak{T}$ is interpretable in $\mathfrak{T}'$ and conversely, the theories are called *mutually interpretable.*

The theories described below are finitely axiomatizable. $\mathfrak{F}$ is a theory of finite sets. The $\mathfrak{F}$-constants are the universe predicate U and the membership symbol $\in$. Roughly speaking, the $\mathfrak{F}$-axioms are obtained from Bernays' axiom system (J. Symbolic Logic vol. 2, pp. 65 ff; ibid. vol. 6, pp. 1 ff.) by identifying classes with sets and by eliminating or restricting all axioms which imply the existence of infinite classes. $\mathfrak{F}'$ is a small fragment of $\mathfrak{F}$ with three axioms only —Bernays' axioms I(1), II(1), II(2). $\mathfrak{J}$ is a fragmentary theory of integers. The $\mathfrak{J}$-constants are U, 0, 1, $<$, $+$, $\cdot$. The $\mathfrak{J}$-axioms characterize the set of integers as an ordered ring in which 1 immediately follows 0. $\mathfrak{N}$ is a small fragment of the theory of non-negative integers. It is essentially the theory discussed in a preceding abstract (Robinson, *An essentially undecidable axiom system*), but with a universe predicate explicitly introduced.

Theories $\mathfrak{J}$ and $\mathfrak{N}$ are mutually interpretable; they are both interpretable in $\mathfrak{F}'$ and hence also in $\mathfrak{F}$. $\mathfrak{J}$ is known to be essentially undecidable (Mostowski and Tarski, J. Symbolic Logic vol. 14, p. 76). A theory $\mathfrak{T}$ being essentially undecidable, the same applies to every theory in which $\mathfrak{T}$ is interpretable. Hence *all the four theories are essentially undecidable.* (For $\mathfrak{N}$ this has been proved directly in the preceding abstract of Robinson.) Many other theories are known whose essential undecidability can be analogously derived from that of $\mathfrak{J}$; e.g., some fragmentary theories of concatentation.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF., U. S. A.

AN INQUIRY INTO THE NATURE OF KNOWLEDGE

GEORGE C. VEDOVA

In these days of startling and profuse scientific discoveries, partial and conflicting theories, and obliterations of old distinctions, the seeker of knowledge is often confused and discouraged. This inquiry is intended to provide some help to such a seeker.

An inquiry into the nature of knowledge must first take into account the view of "reality" taken by the inquirer. The view of "scientific realism" is adopted here. This step taken, the next step must be the adoption of a "suitable" logic. The traditional logic, refined by modern standards, and retaining the "law of the excluded middle"—for Brouwer's views are not generally accepted—is the logic used in science. Adopting this logic we lay down the definition:

Definition. A thing, x, may be said to be *known* if the truth-value of every proposition $p_i(x)$ can be determined in this logic.

Gödel's theorem on completability assures us beforehand that no x can be fully known under this definition. We resign ourselves to this fact since there are, besides, additional reasons why no x will ever be fully known.

But in science, knowledge is sought not of an individual thing but rather of a set, C_i, of physical entities $c_{i1}, c_{i2}, \cdots$ connected by a set, R_i, of physical relations $r_{i1}, r_{i2}, \cdots$ apparently obeying a set, L_i, of physical laws $l_{i1}, l_{i2}, \cdots$.

Let this system be denoted by $P_i(C_i, R_i, L_i)$, and let $M(C, R, A)$ denote the mathematical system consisting of a class, C, of undefined terms $c_1, c_2, \cdots$, a class, R, of undefined relations $r_1, r_2, \cdots$, and a class, A, of consistent assumptions $a_1, a_2, \cdots$.

Definition. Let $P_i(C_i, R_i, L_i)$ be an interpretation (in the usual sense) of $M(C, R, A)$ and let $p_j(c_j, r_j)$ be a true (or false) proposition of $M(C, R, A)$; then $p_j(c_{ij}, r_{ij})$ is a *true (or false)* proposition of $P_i(C_i, R_i, L_i)$.

Definition. The system $P_i(C_i, R_i, L_i)$ may be said to be *known* if the truth-value of all its propositions $p_j(c_{ij}, r_{ij})$ can be determined in this way.

THEOREM. If P_1 and P_2 are interpretations of M, and if the class of assumptions A is complete, then the systems P_1 and P_2 are isomorphic (in the usual sense).

THEOREM. If all interpretations of M are isomorphic, then the class A is complete.

NEWARK COLLEGE OF ENGINEERING,
NEWARK, N. J., U. S. A.

Section VII

HISTORY AND EDUCATION

SECTION VII. HISTORY AND EDUCATION

ON PLAUSIBLE REASONING

G. Pólya

1. Why should a mathematician care for plausible reasoning? His science is the only one that can rely on demonstrative reasoning alone. The physicist needs inductive evidence, the lawyer has to rely on circumstantial evidence, the historian on documentary evidence, the economist on statistical evidence. These kinds of evidence may carry strong conviction, attain a high level of plausibility, and justly so, but can never attain the force of a strict demonstration. Our decisions in everyday life are sometimes based on reasoning, but then merely on plausible reasoning. Trying to use a demonstrative argument in everyday affairs would look silly. Perhaps it is silly to discuss plausible grounds in mathematical matters. Yet I do not think so. Mathematics has two faces. Presented in a finished form, mathematics appears as a purely demonstrative science, but mathematics in the making is a sort of experimental science. A correctly written mathematical paper is supposed to contain strict demonstrations only, but the creative work of the mathematician resembles the creative work of the naturalist: observation, analogy, and conjectural generalizations, or mere guesses, if you prefer to say so, play an essential rôle in both. A mathematical theorem must be guessed before it is proved. The idea of a demonstration must be guessed before the details are carried through.

I wish to discuss the nondemonstrative grounds that underlie such guesses. Let us get down to concrete examples from which we may reascend to more distinct general ideas.

2. Non-mathematical induction is important in all branches of mathematics. Its rôle, however, is the most conspicuous in the theory of numbers. Many theorems of this theory were first stated as mere conjectures, supported mainly by inductive evidence, and were proved afterwards. Some such conjectures still await proof or disproof. Such is the following little known conjecture of Euler:[1] *Any integer of the form* $8n + 3$ *is the sum of a square and of the double of a prime.* Euler, of course, could not prove this conjecture, and the difficulty of a proof appears perhaps even greater today than in Euler's time. Yet Euler verified his statement for all integers of the form $8n + 3$ under 200; for $n = 1, 2, \cdots, 10$ see the following table:

Table I

$$11 = 1 + 2 \times 5$$
$$19 = 9 + 2 \times 5$$

[1] *Opera omnia* ser. 1 vol. 4 pp. 120–124. The square is necessarily odd and the prime of the form $4n + 1$. Moreover, in this context, Euler regards 1 as a prime; this is needed to account for the case $3 = 1 + 2 \times 1$.

$$
\begin{aligned}
27 &= 1 + 2 \times 13 \\
35 &= 1 + 2 \times 17 = 9 + 2 \times 13 = 25 + 2 \times 5 \\
43 &= 9 + 2 \times 17 \\
51 &= 25 + 2 \times 13 \\
59 &= 1 + 2 \times 29 = 25 + 2 \times 17 = 49 + 2 \times 5 \\
67 &= 9 + 2 \times 29 \\
75 &= 1 + 2 \times 37 = 49 + 2 \times 13 \\
83 &= 1 + 2 \times 41 = 9 + 2 \times 37 = 25 + 2 \times 29 = 49 + 2 \times 17
\end{aligned}
$$

Such empirical work can be easily carried further; no exception has been found in numbers under 1000.[2] Does this prove Euler's conjecture? By no means; even verification up to 1,000,000 would prove nothing. Yet each verification renders the conjecture somewhat more credible, and we can see herein a general pattern.

Let A denote some conjecture. (For instance, A may be Euler's conjecture that, for $n = 1, 2, 3, \cdots$,

$$8n + 3 = x^2 + 2p$$

where x is an integer and p a prime.) Let B denote some consequence of A. (For instance, B may be the first particular case of Euler's conjecture not listed in the table which asserts that $91 = x^2 + 2p$.) For the moment we do not know whether A or B is true. We do know, however, that

$$A \text{ implies } B.$$

Now, we undertake to check B. (A few trials suffice to find out whether the assertion about 91 is true or not.) If it turned out that B is false, we could conclude that A also is false. This is completely clear. We have here a classical elementary pattern of reasoning, the "modus tollens" of the so-called hypothetical syllogism:

$$
\begin{array}{c}
A \text{ implies } B \\
B \text{ false} \\
\hline
A \text{ false}
\end{array}
$$

The horizontal line separating the two premises from the conclusion stands as usual for the word "therefore." We have here *demonstrative inference* of a well known type.

What happens if B turns out to be true? (Actually, $91 = 9 + 2 \times 41 = 81 + 2 \times 5$.) There is no demonstrative conclusion: the verification of its consequence B does not prove the conjecture A. Yet such verification renders A more credible. (Euler's conjecture, verified in one more case, becomes somewhat more credible.) We have here a pattern of *plausible inference*:

[2] Communication of Professor D. H. Lehmer.

$$\frac{\begin{array}{c} A \text{ implies } B \\ B \text{ true} \end{array}}{A \text{ more credible}}$$

The horizontal line again stands for "therefore." With a little attention, we can observe that countless reasonings in everyday life, in the law courts, in science, etc., conform to this pattern.

3. The idea that led Euler to his conjecture also deserves mention. Euler devoted much of his work to those celebrated propositions of number theory that Fermat has stated without proof. One of these (we call it B for the moment) says that any integer is the sum of three trigonal numbers. Euler observed (see loc. cit.[1]) that *if* his conjecture $8n + 3 = x^2 + 2p$ (which we keep on calling A) were true, Fermat's conjecture would easily follow (A implies B). Bent on proving Fermat's assertion (B), Euler naturally desired that his conjecture (A) should be true. Is this mere wishful thinking? I do not think so; the relations considered yield some weak but not unreasonable ground for believing Euler's conjecture (A) according to the following scheme:

$$\frac{\begin{array}{c} B \text{ credible} \\ A \text{ implies } B \end{array}}{A \text{ (somewhat) credible}}$$

Here is another pattern of plausible inference. I can not here enter into a more detailed discussion, which would show that the present pattern is essentially a shaded, weakened form of the pattern encountered above (in §2).

4. In passing to another example, I quote a curious passage of Descartes:[3] "In order to show by enumeration that the perimeter of a circle is less than that of any other figure of the same area, we do not need a complete survey of all the possible figures, but it suffices to prove this for a few particular figures, whence we can conclude the same thing, by induction, for all the other figures."

In order to understand the meaning of this puzzling passage let us actually perform what Descartes suggests. We compare the circle to a few other figures, triangles, rectangles, and circular sectors. We take two triangles, the equilateral and the isosceles right triangle (with angles 60°, 60°, 60° and 90°, 45°, 45°, respectively). The shape of a rectangle is characterized by the ratio of its width to its height; we choose the ratios 1:1 (square), 2:1, 3:1, and 3:2. The shape of a sector of the circle is determined by the angle at the center; we choose the angles 180°, 90°, and 60° (semicircle, quadrant, and sextant). We assume that all these figures have the same area; let us say, 1 square inch. Then we compute

[3] *Oeuvres de Descartes*, edited by Adam and Tannery, vol. 10, 1908, p. 390. The passage is altered, but not essentially: the property of the circle under consideration is stated here in a different form.

the length of the perimeter of each figure in inches. The numbers obtained are collected in the following table II; the order of the figures is so chosen that the perimeters increase as we read them down.

Table II

Perimeters of figures of equal area

Circle	3.55
Square	4.00
Quadrant	4.03
Rectangle 3:2	4.08
Semicircle	4.10
Sextant	4.21
Rectangle 2:1	4.24
Equilateral triangle	4.56
Rectangle 3:1	4.64
Isosceles right triangle	4.84

Of the ten figures listed, which are all of the same area, the circle, listed at the top, has the shortest perimeter. Can we conclude hence by induction, as Descartes seems to suggest, that the circle has the shortest perimeter not only among the ten figures listed but among all possible figures? By no means. But it cannot be denied that our relatively short list very strongly suggests the general theorem. So strongly, indeed, that if we added one or two more figures to the list, the suggestion could not be made much stronger.

I am inclined to believe that Descartes, in writing the passage quoted, thought of this last, more subtle point. He intended to say, I think, that prolonging the list would not have much influence on our belief. Yet we can perceive herein still another pattern of plausible inference:

$$\begin{array}{c} A \text{ implies } B \\ B \text{ is similar to the formerly verified consequences } B_1, B_2, \cdots, B_n \text{ of } A \\ B \text{ true} \\ \hline A \text{ just a little more credible} \end{array}$$

This pattern appears as a modification or sophistication of our first pattern; see §2.

5. A little more than two hundred years after the death of Descartes, the physicist Lord Rayleigh investigated the tones of membranes. The parchment stretched over a drum is a "membrane" (or, rather, a reasonable approximation to the mathematical idea of a membrane) provided that it is very carefully made and stretched so that it is uniform throughout. Drums are usually circular in shape, but, after all, we could make drums of an elliptical, or polygonal, or any other shape. A drum of any form can produce different tones, of which usually the deepest tone, called the principal tone, is much the strongest. Lord Rayleigh compared the principal tones of membranes of different shapes but of equal area and subject to the same physical conditions. He constructed the following table III which is very similar to our table II. Table III lists the same shapes as table II, but in somewhat different order, and gives for each shape the pitch (the frequency) of the principal tone.[4]

Table III

Principal frequencies of membranes of equal area

Circle	4.261
Square	4.443
Quadrant	4.551
Sextant	4.616
Rectangle 3:2	4.624
Equilateral triangle	4.774
Semicircle	4.803
Rectangle 2:1	4.967
Isosceles right triangle	4.967
Rectangle 3:1	5.736

Of the ten membranes listed, which are all of the same area, the circular membrane, listed at the top, has the deepest principal tone. Can we conclude hence by induction that the circle has the lowest principal tone of *all* shapes? Of course, we can not; induction is never conclusive. Yet, the suggestion is very strong, still stronger than in the foregoing case. We know (and Lord Rayleigh and his contemporaries also knew) that of all figures with a given area the circle has the minimum perimeter, and that this theorem can be demonstrated mathematically. With this geometrical minimum property of the circle in our mind, we are inclined to believe that the circle has also the physical minimum property suggested by table III. Our judgment is influenced by analogy, and analogy has a deep influence. In fact, we have before us still another pattern of plausible inference:

$$\frac{\begin{array}{c} A \text{ analogous to } B \\ B \text{ true} \end{array}}{A \text{ more credible}}$$

The comparison of tables II and III can yield several further instructive suggestions, but I must refrain from discussing them here.

6. In a little known short note[5] Euler considers, for positive values of the parameter n, the series

[4] Lord Rayleigh, *The theory of sound*, 2d ed., vol. 1, p. 345.
[5] *Opera omnia* ser. 1 vol. 16 sec. 1 pp. 241–265.

$$(1)\quad 1-\frac{x^2}{n(n+1)}+\frac{x^4}{n(n+1)(n+2)(n+3)}-\frac{x^6}{n\cdots(n+5)}+\cdots$$

which converges for all values of x. He observes the sum of the series and its zeros for $n = 1, 2, 3, 4$.

$n = 1$: sum $\cos x$, zeros $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \cdots$

$n = 2$: sum $(\sin x)/x$, zeros $\pm\pi, \pm 2\pi, \pm 3\pi, \cdots$

$n = 3$: sum $2(1\text{-}\cos x)/x^2$, zeros $\pm 2\pi, \pm 4\pi, \pm 6\pi, \cdots$

$n = 4$: sum $6(x\text{-}\sin x)/x^3$, no real zeros.

Euler observes a difference: in the first three cases all the zeros are real, in the last case none of the zeros is real. Euler notices a more subtle difference between the first two cases and the third case: for $n = 1$ and $n = 2$, the distance between two consecutive zeros is π (provided that we disregard the zeros next to the origin in the case $n = 2$), but for $n = 3$ the distance between consecutive zeros is 2π (with a similar proviso). This leads him to a striking observation: in the case $n = 3$ all the zeros are double zeros. "Yet we know from Analysis," says Euler, "that two roots of an equation always coincide in the transition from real to imaginary roots. Thus we may understand, why all the zeros suddenly become imaginary when we take for n a value exceeding 3." On the basis of these observations he states a surprising conjecture: the function defined by the *series* (1) *has only real zeros, and an infinity of them, when* $0 < n \leqq 3$, *but has no real zero at all when* $n > 3$. In this statement he regards n as a continuously varying parameter.

In Euler's time questions about the reality of the zeros of transcendental equations were absolutely new, and we must confess that even today we possess no systematic method to decide such questions. (For instance, we cannot prove or disprove Riemann's famous hypothesis.) Therefore, Euler's conjecture appears extremely bold. I think that the courage and clearness with which he states his conjecture are admirable.

Yet Euler's admirable performance is understandable to a certain extent. Other experts perform similar feats in dealing with other subjects, and each of us performs something similar in everyday life. In fact, Euler *guessed the whole from a few scattered details*. Quite similarly, an archaeologist may reconstitute with reasonable certainty a whole inscription from a few scattered letters on a worn-out stone. A paleontologist may describe reliably the whole animal after having examined a few of its petrified bones. When a person whom you know very well starts talking in a certain way, you may predict after a few words the whole story he is going to tell you. Quite similarly, Euler guessed the whole story, the whole mathematical situation, from very few clearly recognized points.

It is still remarkable that he guessed it from so few points, by considering just four cases, $n = 1, 2, 3, 4$. We should not forget, however, that circumstantial evidence may be very strong. A defendant is accused of having blown up the yacht of his girl friend's father, and the prosecution produces a receipt signed by the defendant acknowledging the purchase of such and such an amount of dynamite. Such evidence strengthens the prosecution's case immensely. Why? Because the purchase of dynamite by an ordinary citizen is a very unusual event in itself, but such a purchase is completely understandable if the purchaser intends to blow up something or somebody. Please, observe that this court case is very similar to the case $n = 3$ of Euler's series. That all roots of an equation written at random turn out to be double roots is a very unusual event in itself. Yet it is completely understandable that in the transition from two real roots to two imaginary roots a double root appears. The case $n = 3$ is the strongest piece of circumstantial evidence produced by Euler and we can perceive herein a general pattern of plausible inference:

$$\begin{array}{c} A \text{ implies } B \\ B \text{ very improbable in itself} \\ B \text{ true} \\ \hline A \text{ very much more credible} \end{array}$$

Also this pattern appears as a modification or a sophistication of the fundamental pattern that we have encountered first (in §2).

By the way, Euler was right: 150 years later, his conjecture has been completely proved.[6]

7. At this stage, I wish you to observe the bearing of our discussion on a much agitated philosophical problem, the problem of induction. In fact, inductive reasoning is a particular case of plausible reasoning. Older writers, as Euler and Laplace, did not fail to notice that the role of inductive evidence in mathematical investigation is similar to its role in physical research, but more modern writers seem to have forgotten this remark almost completely. For this reason, and for many others, I think that mathematical examples of inductive reasoning such as the foregoing are most instructive. The philosophical discussion of induction produced many contradictory opinions. If you wish to see clearly the inconsistencies of such opinions, the best thing may be to test them on well chosen mathematical examples, as the naturalist tests his theories on well chosen specimens; in short, to investigate *induction inductively*.

It would be more philosophical, I think, to consider the more general idea of plausible reasoning instead of the particular case of inductive reasoning. The patterns discussed in the foregoing express, I think, essential aspects of plausible inference. These aspects become clearer if we try to systematize our patterns, to derive them from each other, to characterize them by ideas of probability.

[6] See the author's paper: *Sopra una equazione transcendente trattata da Eulero*, Bolletino dell'Unione Matematica Italiana vol. 5 (1926) pp. 64–68.

I cannot here enter into details of the work that I undertook in this direction,[7] but I wish to sketch roughly the general aspect to which these details seem to lead. From the outset, the two sorts of reasoning appear very different: demonstrative reasoning as definite, final, "machinelike" and plausible reasoning as vague, provisional, specifically "human." Let us compare the two patterns:

Demonstrative	*Plausible*
A implies B	A implies B
B false	B true
A false	A more credible

In the demonstrative inference, the conclusion is fully determined by the two premises. In the plausible inference the "weight" of the conclusion remains indeterminate: A can become only a little more credible or much more credible, and how much more credible it does become depends not only on the clarified grounds expressed in the premises, but on unclarified, unexpressed grounds somewhere in the background of the person who draws the conclusion. A person has a background, a machine has not. Indeed, you can build a machine that draws demonstrative conclusions for you, but you can never delegate to a machine the drawing of plausible inferences.

8. I wish you to observe also the bearing of the foregoing discussion on the teaching of mathematics. It has been said, often enough and certainly with good reason, that teaching mathematics affords a unique opportunity to teach demonstrative reasoning. I wish to add that teaching mathematics also affords an excellent opportunity to teach plausible reasoning. There is little doubt that both opportunities should be used. A student of mathematics should learn, of course, demonstrative reasoning: it is his profession and the distinctive mark of his science. Yet he should also learn plausible reasoning: this is the kind of reasoning on which his creative work will mainly depend. The general student should get a taste of demonstrative reasoning; he may have little opportunity to use it directly, but he should acquire a standard with which he can compare alleged evidence of all sorts aimed at him in modern life. He needs, however, in all his endeavors plausible reasoning. At any rate, an ambitious teacher of mathematics should teach both kinds of reasoning to both kinds of students. He should, more than any particular facts, teach his students two things:

First, to distinguish a valid demonstration from an invalid attempt, a proof from a guess.

[7] For further indications see the last pages of the author's book, *How to Solve It*, 5th printing, Princeton, 1948, and three papers: *Heuristic reasoning and the theory of probability*, Amer. Math. Monthly. vol. 48 (1941) pp. 450–465; *On patterns of plausible inference*, Studies and Essays presented to R. Courant, 1948, pp. 277–288; and *Preliminary remarks on a logic of plausible inference*, Dialectica vol. 3 (1949) pp. 28–35.

Second, to distinguish a more reasonable guess from a less reasonable guess. I say that it is desirable to distinguish between guesses and guesses; I do not say that such distinction is easy to learn or to teach. I think, however, that examples such as the foregoing and due emphasis on the underlying patterns of plausible inference may help. I undertook a collection of problems so introduced and so grouped that they may lend some little help also. At any rate, we should not forget an important opportunity of our profession: *Let us teach guessing!*

STANFORD UNIVERSITY,
STANFORD, CALIF., U. S. A.

HISTORY

THE FOREMOST TEXTBOOK OF MODERN TIMES

CARL B. BOYER

The most influential mathematics textbook of ancient times is easily named, for the *Elements* of Euclid has set the pattern in elementary geometry ever since. The most effective textbook of the medieval age is less easily designated; but a good case can be made out for the *Al-jabr* of Al-Khowarizmi, from which algebra arose and took its name. Is it possible to indicate a modern textbook of comparable influence and prestige? Some would mention the *Géométrie* of Descartes or the *Principia* of Newton or the *Disquisitiones* of Gauss; but in pedagogical significance these classics fell short of a work by Euler titled *Introductio in analysin infinitorum.* Here in effect Euler accomplished for analysis what Euclid and Al-Khowarizmi had done for synthetic geometry and elementary algebra respectively. Coordinate geometry, the function concept, and the calculus had arisen by the seventeenth century; yet it was the *Introductio* which in 1748 fashioned these into the third member of the triumvirate comprising geometry, algebra, and analysis.

Euler was not the first to use the word analysis, but he gave it a new emphasis. Plato's analysis had reference to the logical order of steps in geometrical reasoning, and the analytic art of Viète was akin to algebra; but the first volume of the *Introductio* resembles analysis in the current orthodox sense—the study of functions by means of infinite processes.

Euler avoided the phrase analytic geometry, probably to obviate confusion with the older Platonic usage; yet the second volume of the *Introductio* has been referred to, appropriately, as the first textbook on the subject. It contains the earliest systematic graphical study of functions of one and two independent variables, including the recognition of the quadrics as constituting a single family of surfaces. The *Introductio* was first also in the algorithmic treatment of logarithms as exponents and in the analytic treatment of the trigonometric functions as numerical ratios.

The *Introductio* does not boast an impressive number of editions, yet its influence was pervasive. In originality and in the richness of its scope it ranks among the greatest of textbooks; but it is outstanding also for clarity of exposition. Published two hundred and two years ago, it nevertheless possesses a remarkable modernity of terminology and notation, as well as of viewpoint. Imitation is indeed the sincerest form of flattery.

BROOKLYN COLLEGE,
BROOKLYN, N. Y., U. S. A.

THE ORIGIN OF POLAR COORDINATES

JULIAN L. COOLIDGE

Polar coordinates were used for special purposes and for the study of particular curves before they were appreciated as a general geometrical tool. The first writer to employ them was Bonaventura Cavalieri, who used them to find the area within an Archimedian spiral by relating it to that outside a parabola. Pascal used the same transformation to calculate the length of a parabolic arc, a problem previously solved by Roberval, but his solution was not universally accepted as valid. James Gregory had a similar transformation between two individual curves, where the areas were related, while Pierre Varignon used a slightly different transformation for the study of spirals.

The first writer who looked on polar coordinates as a means of fixing any point in the plane was Newton. He, however, considered them alongside Cartesian, bipolar, and other systems, his only interest at that point being to show how the tangent could be determined when the equation of the curve was given in the one or the other system. A deeper interest was shown by James Bernoulli, who went so far as to write the expression for the radius of curvature when the equation of the curve was given in polar form.

The first writer to think of polar coordinates in 3-space was Clairaut, but he merely mentions the possibility of such things. The first to develop them was Euler to whom we owe both polar and radio-angular coordinates. An interesting modification of the latter was developed by Ossian Bonnet.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., U. S. A.

BROOK TAYLOR AND THE MATHEMATICAL THEORY OF LINEAR PERSPECTIVE, HIS CONTRIBUTIONS AND INFLUENCE

PHILLIP S. JONES

The last of four periods in the development of the theory of linear perspective began in the eighteenth century when Willem J. Gravesande, Brook Taylor, and J. H. Lambert constructed a general and abstract mathematical theory.

The only original English work prior to Taylor's *Linear Perspective* (London, 1715) was that of Humphrey Ditton published in 1712. The only mathematically significant later English writer was John Hamilton who admittedly based his work on Taylor's but added much to it.

Taylor's book is a concise, generalized, mathematical treatment formulated in terms of axioms and theorems. The concepts of horizon line and principal point are generalized to vanishing lines and vanishing points for all planes and lines. The object, eye, and picture are considered to be free to assume all possible relative positions. Use is made of the invariance of a ratio which is essentially the cross ratio of four points so projected that one of them goes to infinity. Taylor's basic theorem is equivalent to Desargues' theorem although phrased in his own perspective terminology with no mention of Desargues.

In his first edition Taylor gave an interesting construction for the perspective of a circle. He explained it by translating standard Euclidean geometric constructions into perspective rather than by constructing the original circle and then drawing its perspective. The procedure actually amounts to the construction of a conic given a point of the curve and a point and line which are pole and polar with respect to it. Taylor gave "perspective" proofs, terming lines "parallel" if they met on the vanishing line of their common plane. He also gave the first constructions for a line from a point to the inaccessible intersection of two given lines. One procedure is equivalent to using harmonic sets and complete quadrangles.

A final unique feature of Taylor's work is his treatment of inverse problems. Although the first mention of such problems was in Monte's *Perspectivae Libri Sex* (1600), little had been done prior to Taylor. Later Lambert further expanded the treatment and so earned credit for originating the modern science of photogrammetry. Lambert also carried Taylor's concept of doing geometric constructions and proofs directly in perspective much farther. Max Steck lists Jacquier's French translation of Taylor's work among the books in Lambert's library.

Taylor's work appeared in four English editions and three translations. Nine other authors published books appearing in 22 editions from 1738 to 1888 which claimed explicitly to be based on Taylor's work and to give his methods. These facts alone would establish Taylor's wide influence and justify having made a special study of his work.

University of Michigan,
Ann Arbor, Mich., U. S. A.

THE SCOTTISH CONTRIBUTION TO THE EARLY HISTORY OF THE CALCULUS

H. W. Turnbull

Recent examination of original and mostly unpublished manuscripts and letters has thrown new light upon the early developments of the Calculus by James Gregory (1638–1675) and Isaac Newton (1642–1727). Interpolation formulae involving successive order of finite differences as well as the power series, involving successive derivatives and found by Taylor and Maclaurin, were used over forty years earlier (1670–1671) by Gregory. The differential equation $y^2(1 + (dy/dx)^2) = f(x)$ in geometrical guise was discussed by Gregory and Barrow. There are examples of independent and almost simultaneous discoveries by Gregory and Newton. Light is thrown on the problem of dating accurately certain results of Newton which were published only many years later. David Gregory (1661–1710) left unpublished notes including those of a projected History of Fluxions, and of discussions with Newton, 1690 onwards, on mathematical and physical problems. The notes provide evidence of work by Hudde at Amsterdam prior to 1660 on the logarithmic series, antedating Newton and Mercator. Letters and manuscripts of Colin Campbell bridging the years between J. Gregory and Maclaurin give a vivid picture of scientific activity in Edinburgh and in the Highlands.

University of St. Andrews,
St. Andrews, Scotland.

EDUCATION

MATHEMATICS FOR THE MILLION, OR FOR THE FEW?

William Betz

The publications of the International Commission on the Teaching of Mathematics, dating back to the Congress of 1908, made known the mathematical curricula of the participating nations. They showed that practically everywhere the schools provided one kind of education for "the many" and a radically different kind of education for "the few." In the United States alone, it seems, every type of public elementary and secondary school has been open to "all the children of all the people." Hence these American schools have become the world's unique laboratory in mass education. For more than three decades there have been ever-renewed efforts to adapt the curricula to the real or alleged "needs" of all the pupils. Thousands of mathematical courses of study have emerged thus far, and the end is not yet. Throughout this period of curriculum revision, two grave mistakes have recurred with fatal regularity. The first consisted in toning down virtually all mathematical instruction to the level of mediocrity. The second was that of postponing or completely eliminating essential types of mathematical training, and of making optional all offerings in secondary mathematics.

However, a new orientation is now taking place, partly as a reaction against the errors of the past, and partly because of the influence of recent, authoritative committee reports. We have come to the parting of the ways. There is no desire to copy or imitate the dual system of the European schools. But two ideas are being urged with great definiteness. First, we must continue to offer adequate sequential courses for those who need a strong mathematical foundation for their future professional work. Second, we must also build "for the many" a second-track program in mathematics, geared in simpler terms and emphasizing immediate life application, without sacrificing mathematical soundness and thoroughness.

The task of creating a really functioning and up-to-date mathematical program, suitable "for the million," is now engaging the attention of our American schools. But can there be any reasonable doubt, in this scientific and technological age, that such a program is also a world-wide necessity? Hence it would seem to merit the careful attention of this Congress, or of the proposed International Mathematical Union. Is it too much to expect that a work which was begun so auspiciously in 1908, at the instance of our own David Eugene Smith, be resumed and eventually brought to more complete consummation?

Rochester, N. Y., U. S. A.

A NOTE ON THE TEACHING OF GENERAL MATHEMATICS

LEE EMERSON BOYER

The emergence of general mathematics was greatly stimulated by a growing conviction on the part of American educators that specialization should not start before a broad background in the various fields of knowledge was acquired. Although for years most American college students were required to enroll in some of the conventional mathematics courses evidence accumulated that this practice was not satisfactory for the background purposes of general education. Thus, general mathematics is a year's effort by the nonspecialist student at acquiring a comprehension of the nature of mathematics and its methods. General mathematics is not a fragmentary survey course. It is a carefully planned course aimed at the study of mathematical concepts and processes basic to many fields of study.

General mathematics teaching facilitates the student's grasp of the significance mathematics has for civilization and present day society. It makes special efforts to vitalize the subject matter.

Too often in the past the success of general mathematics was jeopardized by poor teaching. As a group, mathematicians seemed to be unusually subject centered. They showed few fascinating applications of topics in the several subjects.

The criticism that the course is only "about mathematics" or that it is not challenging to students does not concern the writer so much now as it did ten years ago. There is so much substantial problem material available. A much more crucial issue is how to present clear cut ideas and get students to assimilate them. In addition, there are the important but difficult tasks of establishing certain helpful habits, appreciations, and a substantial mathematics vocabulary. If these tasks are achieved to only a limited extent, the effort may still be outstandingly worthwhile.

It is the teacher's broadmindedness, patience, enthusiasm and willingness to meet the students where they are and fit the course to their needs and interests that are relied upon to set off and nurture the growth of ideas.

The size of the class should be limited. Students should be sectioned. Since ideas, habits, and appreciations are so difficult to develop and mature so slowly, it is necessary to adjust the character and nature of the topics to be covered, the number of objectives to be covered, the speed of covering them, the depth to which they shall be probed, and the character of performance for which passing grades are assigned on the several section levels.

The teaching of general mathematics is difficult but it is new; it is growing; it is significant; and it is challenging. There is nothing on the educational horizon that dims the social value of its objectives.

MILLERSVILLE STATE TEACHERS COLLEGE,
MILLERSVILLE, PA., U. S. A.

UNIFYING CONCEPTS IN MATHEMATICS

HAROLD P. FAWCETT

A steadily increasing emphasis on the importance of general education has led, in mathematics, to what has been called "the double track program". One track is designed to meet the mathematical needs of general education, while the purpose of the other is to provide for the technical demands of special education. To assume, however, that these two tracks are distinct and separate is to make a highly questionable assumption. A mathematical program which provides those skills and understandings essential for general education does not differ in kind from that which satisfies the demands of special education. It differs only in degree.

Such fundamental concepts as number, measurement, operation, symbolism, relationship, and proof permeate the fabric of any mathematics program. The growing concept of number, for example, begins with the idea of a whole number and is extended to include fractions, signed numbers, and on through complex, and transfinite numbers. It includes conceptual understandings desirable for general education out of which come those major understandings so necessary for the specialist. So it is with other major concepts. In the long history of man's continued effort to understand and control his environment there has never been a time when he was not struggling with problems of mesaurement. The crude units of antiquity have now been replaced by standards which provide an accuracy of one part in a hundred million and the development of mathematics made this refining process possible. Through steady and continuous development the fundamental operation of counting has led to addition, subtraction, and on through differentiation and integration.

From the specific and well defined symbols of arithmetic to the generalized symbols of later years the language of mathematics provides an increasingly powerful means for dealing with ideas. Elementary number relationships are extended and generalized, reflecting the idea that mathematics is indeed "the science of necessary relationships". Freedom of individual judgment is associated with understandings related to the nature of proof and from the checking of results in the early grades, to the organizing of authorities in the deductive proofs of more advanced mathematics, there should be a continuing emphasis on this important concept.

These six major concepts, permeating the study of mathematics, are unifying concepts which tend to eliminate the dual characteristics of the so-called "double track" program. They provide a continuum along which the special flows out of the general and ever serves to enrich it. There is really no "double track". There is no difference in kind but only in degree.

OHIO STATE UNIVERSITY,
COLUMBUS, OHIO, U. S. A.

FURTHER EXPERIENCE WITH UNDERGRADUATE MATHEMATICAL RESEARCH

Frank L. Griffin

At the 1941 summer meeting of the M.A.A., Professor Griffin mentioned the experience of 16 colleges and universities with undergraduate mathematical research, and described the experience at Reed College with the senior theses [Amer. Math. Monthly vol. 49 (1942) pp. 379–385]. Since then 41 more theses in mathematics have been written, 34 of which have been of a research type, getting some apparently new and sometimes substantial results. Instances of this sort are mentioned.

The average level of recent research seems possibly a little higher than that reported on in 1941. Of 13 theses in the field of analysis, five were on problems in the calculus of variations relating to geodesics or motion; three dealt with approximate methods of solving differential equations, some of the new formulas being remarkably accurate; four others dealt with special functions and series; and one with hypercomplex function theory. In geometry there were 12 theses, three of which studied transformations and projective correspondences; another investigated some new porisms; three others the geometry of certain surfaces, one of which is related to the Möbius strip; and five studied other topics. Of four theses on questions relating to abstract logic or to the foundations of mathematics, one was a particularly mature study in the algebra of logic. Two theses dealt with algebra and number theory; four with actuarial theory and the mathematics of finance; two with mathematical statistics; two with applications of statistical method, one with a physical problem on light, and one with the more intricate techniques of sound-ranging in warfare.

The long-term program of undergraduate mathematical research may be partly responsible for the unusually large number of Reed graduates who have gone on to the doctorate and for the substantial number who have become Fellows of the Actuarial Society.

Reed College,
Portland, Ore., U. S. A.

GESTALT THEORY IN THE TEACHING OF MATHEMATICS

Bernard H. Gundlach

The argument presented takes as a starting position the complaint by teachers of mathematics that lack of time and/or deficiencies in the earlier training of students impose class-room conditions such that only definite "techniques" can be taught, but little or nothing of the "true spirit" of mathematical reasoning. The thesis is presented that these "techniques" intrinsically contain the essence of mathematical reasoning and can thus be used to bring it out clearly if problem situations are consistently presented in a certain way.

This way is approached from the viewpoint of gestalt theory in the light of *that* character of mathematics known as transformability.

Transformations are considered (a) *within* one system of mathematics (transformations of equivalence), (b) *between* two or more mathematical systems (transformations of isomorphism).

It is shown that the essence of mathematical reasoning is directly related to the more general transformation: unfamiliar gestalt $\rightarrow$ familiar gestalt, and conditions are investigated to determine the degree with which mathematical situations satisfy the conditions of gestalt-transformability.

Constructive (creative) situations are recognized as vector-field functions (dynamically structured), and an attempt is presented to build a gestalt function $\vec{G}(P)$ in which components are defined which represent distinctness and intensity of transform direction. A scalar factor, the "readiness factor", is introduced which determines the degree of correspondence between familiar "type-forms" already existent in the student's mind and the desired situation transform.

The readiness factor is also investigated as to implications for the *early* (primary and secondary) mathematics training of the student.

An "ultimate" situation transform (the completely unproblematic situation) is postulated as transcendental.

A number of "good" and "bad" mathematical problem situations from various fields of mathematics are analyzed by means of the gestalt function and, as a summary, some recommendations are put forth for the construction of well-structured problem situations and the gradual awakening (or reawakening) of transform-consciousness in students.

UNIVERSITY OF ARKANSAS,
FAYETTEVILLE, ARK., U. S. A.

THE INTRODUCTION OF APPLIED PROBLEMS FOR THE ENRICHMENT OF CLASSROOM INSTRUCTION IN THE SCHOOLS AND COLLEGES

MAX S. KRAMER

This is a summary of the major considerations for the inclusion of applied material in curricula and a synopsis of the types of problems in current use. A new field of application is offered, that of meteorology, which has developed into a mathematical science in recent years. The vital role which mathematics plays in this study is presented and a variety of problems suitable for classroom instruction in the high schools and colleges are discussed. A report is made of the author's survey of the mathematics used by the average meteorologist.

NEW MEXICO COLLEGE OF AGRICULTURE AND MECHANIC ARTS,
STATE COLLEGE, N. M., U. S. A.

SIMPLIFICATION OF RIGOROUS LIMIT PROOFS

KENNETH MAY AND KIRK MCVOY

Many rigorous limit proofs are characterized by a sequence of apparently unmotivated choices of constants and related deltas which finally yield precisely the epsilon demanded by the epsilon-delta definition of limit. The paper suggests a form of rigorous proof which replaces these choices by a brief and intuitively appealing manipulation of a rather standard nature. The idea behind the method is suggested by the following lemmas, whose proofs are immediate: (1) If there exists a function $\phi(\epsilon)$ which takes each positive value for some positive ϵ and such that for any $\epsilon > 0$ there exists a δ for which

$$0 < |x - a| < \delta$$

implies $|f(x) - A| < \phi(\epsilon)$, then $\lim_{x \to a} f(x) = A$. (2) If there exists a constant $e > 0$ and a function $\phi(\epsilon)$ which takes each positive value for some positive $\epsilon < e$ and such that given any positive $\epsilon < e$ there exists a δ for which

$$0 < |x - a| < \delta$$

implies $|f(x) - A| < \phi(\epsilon)$, then $\lim_{x \to a} f(x) = A$. Examples of simplified proofs, including those of the theorems on limits of sums, products, and quotients, are given. The functions ϕ emerge naturally from the heuristic manipulations necessary to construct the usual proofs and are seen by elementary algebra to have the required property. The general applicability and heuristic advantages of the method are indicated.

CARLETON COLLEGE,
NORTHFIELD, MINN., U. S. A.

CURRENT TRENDS IN THE TEACHING OF PLANE TRIGONOMETRY

ORLANDO E. OVERN

The most conspicuous current trends in the teaching of plane trigonometry may be listed as follows:

1. To begin with the general angle and to define the six functions from the start in terms of x, y, and r, postponing until later the special consideration of acute angles and the definition of their functions in terms of the sides of a right triangle; emphasizing the meaning of functional relations, and featuring graphs of trigonometric functions.

2. To precede the definitions of the six functions by a complete discussion of angles and their measurement, emphasizing the angle as produced by rotation, and introducing the radian and the mil early as angle measures, using them throughout the course instead of in a single unit.

3. To give less attention to logarithms, particularly to the theory, and to

postpone their treatment until near the middle of the course, when the student has learned by experience the value of some device for shortening computation.

4. To put more emphasis on approximate computation, significant figures, and the number of accurate figures in the solution of a problem corresponding to the accuracy of the given data.

5. To treat trigonometric arguments as numbers rather than as angles, a trend to be encouraged since in science and advanced mathematics the independent variable is very often not an angle.

State Teachers College,
Milwaukee, Wis., U. S. A.

FUNDAMENTALS IN THE TEACHING OF UNDERGRADUATE MATHEMATICS

Moses Richardson

Every course in mathematics touches to some extent each of the following three aspects of the subject: T. techniques; A. applied problems; F. fundamentals. Under F are included fundamental concepts, careful reasoning, logical structure, historical growth, interrelations with other subjects, cultural significance, justification of the position of mathematics at the base of the tree of knowledge.

In the practice of teaching, T is too often stressed to the detriment of A and the virtual exclusion of F. Recently, progress has been made in terminal courses for non-science and non-mathematics majors which stress A and F while limiting T. However, for science and mathematics majors, including prospective teachers of mathematics in secondary schools, the problem of teaching F without diminishing the desired attention to T and A remains a difficult but important one. Too many in this group emerge from undergraduate instruction equipped with a collection of heterogeneous techniques, a smattering of applications, and little grasp of fundamentals.

Reasons why F is understressed appear to be: 1) a few teachers believe F should not be taught to most undergraduates; 2) some believe it should but cannot; 3) it is more difficult to teach F than either T or A; 4) it is difficult to test results of time spent on F; 5) T and A leave no time free for F.

Recommendations for the partial amelioration of this situation: 1) Time should be allowed for F in every course—students will learn what the teacher stresses; 2) effort should be made to include some of F on examinations; 3) supplementary reading is not used sufficiently in mathematics courses; 4) special courses in fundamental concepts, perhaps at an elementary level, may be of value especially for prospective secondary teachers who ordinarily do not get much graduate study of mathematics.

Brooklyn College,
Brooklyn, N. Y., U. S. A.

THE TEACHING OF MATHEMATICS IN DENMARK

Erling Rossing

Danish elementary education (7–14 years) is compulsory, but after 5 years the pupils may go on to a so-called Middle School (4 years), from which again they pass on to the Gymnasium (High School), which is divided up into 2 language lines and a mathematical line, all leading up to matriculation.

Thus the teaching of mathematics in the Middle School forms the basis of all further study comprising the elements of geometry and algebra up to, respectively, the theory of similarity and the existence of irrational numbers, from which basis we develop, in the mathematical line, a suitable education aiming at forming a basis for further mathematical and technical education involving considerable cultural values.

For a long time past it has been attempted to settle what would be the subjects most suitable for study—and in our present curriculum are required the knowledge of irrational numbers as a basis for the study of functions, analytical geometry comprising differentiation and integration of simple functions of algebraical and transcendental character, a fairly comprehensive treatment of conic sections (analytical as well as stereometrical), complex numbers, the most common equations and less comprehensive subjects; further, stereometry, comprising common polyhedra and bodies with curved surfaces, and spherical geometry. This course covers 6 out of 35 weekly periods.

The choice of subject matter for this double-purpose matter has often been discussed and some changes have taken place. The volume of subject matter is often thought to be a hindrance to that more intensive study which most teachers aim at in order to give prominence to the cultural values, and a reduction of inessentials has been proposed, without a corresponding reduction of the present number of lessons.

On the language lines the scope is considerably narrower; some simple graphs, extension of the concept of power, logarithms, calculation of interest, and trigonometry. In recent years it has been proposed to discontinue completely the teaching of mathematics on the language lines in favor of languages against which the mathematicians have proposed a reconstruction of the curriculum along these lines: less emphasis on acquisition of definite knowledge but an effort—within the scantily allotted time—to underline the cultural values, through, for instance, a brief survey of the history of mathematics, approach to famous uncomplicated problems and the concept of infinity.

Tönder Statsskole,
Tönder, Denmark.

CONDENSED GRAPHS

R. L. Swain

Let $\varphi(t)$ be a single-valued, twice differentiable, increasing function, defined for $0 \leqq t \leqq +\infty$, with $\varphi''(t) < 0$ for $t \neq 0$, $\varphi'(0) > 0$, $\varphi(0) = 0$, $\varphi(+\infty) = 1$. Let the plane be mapped into the interior of the square of side 2 with center at the origin according to the equations

$$X = \begin{cases} \varphi(x) & \text{if } x \geqq 0 \\ -\varphi(-x) & x \leqq 0 \end{cases} \qquad Y = \begin{cases} \varphi(y) & \text{if } y \geqq 0 \\ -\varphi(-y) & y \leqq 0. \end{cases}$$

Each interior point of the square whose coordinates are (X, Y) is now considered as having coordinates (x, y) in a *new scale*. The boundary of the square is appropriately scaled (using $\pm\infty$ as a coordinate) and adjoined to the interior, the resulting closed region being called the (rectangular) *condensed plane* generated by the mapping function $\varphi(t)$.

The graph of a function may be plotted in the condensed plane, and the "complete" behavior of the function, including that for large $|x|$ and large $|y|$, exhibited to advantage. Many examples of such graphs are shown, using as the mapping function $\varphi(t) = t/1 + t$. Little violence is done to our customary notions about the graphs of common functions. If $\varphi(t)$ satisfies the identity $\varphi(t) + \varphi(1/t) = 1$, then: (a) the graph of a function $y = f(x)$, its inverse function $y = f^{-1}(x)$, the function of reciprocal x, $y = f(1/x)$, the reciprocals of each of these, and their negatives—in all, twelve graphs—are each congruent quadrantwise; (b) if $f(x)$ is differentiable and $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} f'(x) = m > 0$, then the "drawn slope" of the graph of $y = f(x)$ at $(+\infty, +\infty)$ in the condensed plane is $1/m$.

Some decisions with regard to the convergence or divergence of infinite series and of improper integrals may be made directly by inspection of graphs in the condensed plane.

Ohio State University,
Columbus, Ohio, U. S. A.

TABLE OF CONTENTS

VOLUME I

INDEX OF AUTHORS

Volume I